ON FIBONACCI NUMBERS WITH BOUNDED PRIME GAPS

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Abstract. In this paper, we look at terms of Lucas sequences whose prime factors have indices with bounded gaps in the sequence of all prime numbers. Some of our results depend on certain widely believed conjectures. In our proofs we combine various tools, including Baker's method, the subspace theorem, and results of Stewart, and Murty and Wong.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence. While perfect powers in the Fibonacci sequence are well-understood (see [6]), much less is known about prime values. Since $F_a \mid F_b$ whenever $a \mid b$ holds, it follows that if F_n is prime, then either $n = 4$ or n is prime itself. It is believed that there are infinitely many primes in the Fibonacci sequence but proving this is almost certainly as hard as proving that there are infinitely many Mersenne primes, that is primes of the form $2^p - 1$ for some prime number p. In this paper, we take a look at terms of Lucas sequences (which are generalizations of Fibonacci numbers) all whose prime divisors coincide with all the primes in an interval.

To state our results, let us begin with some terminology. Given a positive integer C, we say that a positive integer m has prime factors with gaps bounded by C if m is either prime, or m is composite and by writing

$$m = p_{i_1}^{\alpha_1} \cdots p_{i_k}^{\alpha_k}$$

for indices $1 \leq i_1 < i_2 < \cdots < i_k$, where p_t is the t-th prime, and $\alpha_1, \ldots, \alpha_k$ are positive integers, then $i_j - i_{j-1} \leq C$ for $j = 2, \ldots, k$. When $C = 1$, having prime factors with gaps bounded by C means that the distinct prime factors of m cover an interval of primes. Note that $F_{12} = 2^4 \cdot 3^2$ has prime gaps bounded by 1, whereas $F_8 = 3 \cdot 7$ and $F_{10} = 5 \cdot 11$ have prime gaps bounded by 2.

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Further, recall that a Lucas sequence $(U_n)_{n\geq 0}$ is a binary recurrent sequence satisfying $U_0 = 0,\ U_1 = 1$ and

$$U_{n+2} = rU_{n+1} + sU_n \qquad \text{for all} \qquad n \geq 0 \tag{1}$$

for some fixed integers $r,\ s$. We assume that $X^2 - rX - s = (X - \gamma)(X - \delta)$, where $\gamma \neq \delta$ are both nonzero and γ/δ is not a root of unity. We label the roots γ, δ such that $|\gamma| \geq |\delta|$. In general, one assumes that $\gcd(r, s) = 1$, but for our problem, the situation is much easier if in fact $\gcd(r, s) > 1$. We state our first result.

Theorem 1. *Assume that $(U_n)_{n\geq 0}$ is a Lucas sequence satisfying recurrence (1) where $\gcd(r, s) > 1$. Then for all C there exist a computable constant $f_0(r, s, C)$ such that if $|U_n|$ has prime factors with gaps bounded by C then $n < f_0(r, s, C)$.*

From now on, we will assume that $\gcd(r, s) = 1$. Our next results only apply to Lucas sequences with real roots $\gamma,\ \delta$, so we assume that $\Delta = r^2 + 4s > 0$. In this case, $|\gamma| > |\delta|$. Also $|\gamma| \geq (1 + \sqrt{5})/2$. Furthermore, up to replacing (r, s) by $(-r, s)$ (which has as effect replacing (γ, δ) by $(-\gamma, -\delta)$ and U_n by $(-1)^{n-1}U_n$), we may assume that $\gamma > |\delta|$. In this case, U_n is positive for all $n \geq 1$.

We prove three additional theorems about such Lucas sequences. Here is our next theorem. For a positive integer m, we write $\omega(m)$ for the number of distinct prime divisors of m.

Theorem 2. *Let $(U_n)_{n\geq 0}$ be a Lucas sequence with real roots where the dominant root γ is positive and irrational. Let C and L be given. There exists a computable constant $f_1(\gamma, C, L)$ such that if U_n has $\omega(U_n) \leq L$ and its prime factors have gaps bounded by C, then either $n \leq f_1(\gamma, C, L)$, or n is prime.*

One may ask wheather at this stage it is possible to replace the conclusion of the Theorem 2 that n is prime by the stronger statement that U_n is prime. We do not know how to do this, but see Theorem 3 below. Since when n is prime it might be that U_n is also prime, we cannot hope for a better conclusion since we already mentioned that it is believed that there are infinitely many primes in the Fibonacci sequence and a similar belief holds for other Lucas sequences for which there is no algebraic reason why U_n should not be a prime when n is prime (for example, when γ and δ are squares in their field of definition). For a positive integer m, we write $\Omega(m)$ for the total number of prime factors in the prime factorization of m. If instead of asking that $\omega(U_n) \leq L$, where L is given before hand, we require that $\Omega(U_n) \leq L$, then we show conditionally upon a result concerning primes in short intervals that we get the best possible answer. Here is the result we need.

Conjecture 1. *For $\nu > 0$ there exists $x_0 := x_0(\nu)$ such that for $x > x_0$, the interval $[x, x + x^\nu]$ contains a prime.*

The Riemann Hypothesis implies that Conjecture 1 holds with $\nu = 1/2+\varepsilon$ for any $\varepsilon > 0$. Perhaps Conjecture 1 holds with any $\nu > 0$. Indeed, such a statement is implied by Cramér's conjecture (see [8] and e.g. [10] for some related discussion), which we recall below.

Conjecture 2 (Cramér's conjecture)**.** *There are positive constants x_0, c_0 such that for $x > x_0$ the interval $[x, x + c_0(\log x)^2]$ contains a prime.*

Unconditionally, the only known results about primes in short intervals assert that for $x > x_0$, the interval $[x, x + x^{1/2+\eta}]$ contains a prime for some $\eta > 0$. The value $\eta = 0.025$ is acceptable and follows from work of Baker, Harman and Pintz [1].

Now we can formulate our next result.

Theorem 3. *Let $(U_n)_{n\geq 0}$ be a Lucas sequence with real roots where the dominant root γ is positive and irrational. Let C and L be given, and assume that Conjecture 1 holds with any $\nu > 0$. Then there exists a constant $f_2(\gamma, C, L)$ such that if U_n has $\Omega(U_n) \leq L$ and its prime factors have gaps bounded by C, then either $n \leq f_2(\gamma, C, L)$, or U_n is prime.*

The proof of Theorem 3 uses the Subspace theorem so $f_2(\gamma, C, L)$ is not computable, although in principle one could obtain a bound for the number of exceptional n's but not on their sizes. One may ask what can we say if we do not bound the number of prime factors of U_n, but still keep the requirement that its prime factors have gaps bounded by C. Our last theorem is conditional on the *abc* conjecture as well, which we now recall. For a nonzero integer m we write

$$N(m) := \prod_{p\mid m} p,$$

and call it the *algebraic radical of m*.

Conjecture 3 (*abc* conjecture)**.** *For every $\varepsilon > 0$ there is a constant C_ε such that whenever a, b, c are coprime nonzero integers with $a + b = c$, then*

$$\max\{|a|, |b|, |c|\} \leq C_\varepsilon N(abc)^{1+\varepsilon}.$$

We can now formulate our last result.

Theorem 4. *Let $(U_n)_{n\geq 0}$ be a Lucas sequence with real roots where the dominant root γ is positive and irrational. Assume the abc conjecture and that Conjecture 1 holds with $\nu = 1/2 - \eta$ for some $\eta > 0$. Then there exists a constant C_0 depending on γ (and η) only such that if U_n has prime gaps bounded by C then either n is bounded by some number depending on C, or $\Omega(n) \leq C_0$.*

Note that apart from proving that $C_0 = 1$ in the case of the Fibonacci sequence, we cannot expect a better conclusion of Theorem 4 in light of the fact that if F_n is prime, then it certainly has bounded gaps by C for any $C \geq 1$, and we already said that likely there are infinitely many prime Fibonacci numbers all of which except for $F_4 = 3$ have a prime index.

The method of proof of Theorem 4 has the following corollary which may be interesting in its own.

Theorem 5. *Let $(U_n)_{n\geq 0}$ be a Lucas sequence with real roots where the dominant root γ is positive. Assume the abc conjecture and Conjecture 1 holds with any $\nu = 1/2 - \eta$ for some $\eta > 0$. If U_n has prime factors with bounded gaps by C and $\omega(U_n) \leq L$, then U_n is squarefree if n is sufficiently large with respect to γ, C, L.*

2. Proofs

We give the proofs of our results in separate subsections. We note that in what follows, the implied constants in $\gg$ and $\ll$, and in O are absolute. If they depend on some parameters A, B, etc, we indicate this as $\gg_A$, $\ll_B$, $O_{A,B}$, etc.

2.1. The proof of Theorem 1.

Let $p \mid \gcd(r, s)$. It follows by induction that $p \mid U_n$ for all $n \geq 2$. Write

$$U_n = p_{i_1}^{\alpha_1} \cdots p_{i_k}^{\alpha_k},$$

where $i_1 < i_2 < \cdots < i_k$ and $i_j - i_{j-1} \leq C$ for $j = 2, 3, \ldots, k$. Clearly, $i_1 \leq \pi(p) = \kappa$. It then follows that $i_j \leq \kappa + (j-1)C$ for $j = 1, 2, \ldots, k$. Note that

$$k = \omega(|U_n|) \ll \frac{\log |U_n|}{\log\log |U_n|} \ll_{r,s} \frac{n}{\log n},$$

since $|U_n| \leq 2|\gamma|^n$. By the prime number theorem, it follows that

$$p_{i_k} \ll_{r,s,C} n.$$

However, by a result of Stewart [17], there exists an absolute constant c_1 such that for $n > n_0$, U_n has a prime factor at least as large as

$$M(n) := n \exp\left(c_1 \frac{\log n}{\log\log n}\right).$$

Stewart shows that $c_1 := 1/104$ is acceptable. Bilu, Gun and Hong [5] verify that one can take $n_0 := \exp(10^6)$ and $c_1 := 0.0005$. This yields that if $n \geq n_0$ then

$$n \exp\left(c_1 \frac{\log n}{\log\log n}\right) = M(n) \leq p_{i,k} \ll_{r,s,C} n,$$

which implies that $n \leq f_0(r, s, C)$.

2.2. **The proof of Theorem 2.** For the proof of Theorem 2 we shall need a deep result of Matveev [14], concerning linear forms in logarithms. For this we need to introduce some notation.

Let β be an algebraic number of degree t with minimal polynomial

$$f(x) = a_0 \prod_{i=1}^{t} (x - \beta_i)$$

over $\mathbb{Z}$, where $a_0 > 0$. The absolute logarithmic height of β is defined by

$$h(\beta) := \frac{1}{t} \left(\log a_0 + \sum_{i=1}^{t} \log \max\{|\beta_i|, 1\} \right).$$

Let $K \subset \mathbb{R}$ be an algebraic number field and let d be the degree of K. Let $\tau_1, \ldots, \tau_\ell$ be nonzero elements of K and $b_1, \ldots, b_\ell$ be integers. Set

$$\Lambda := \prod_{i=1}^{\ell} \tau_i^{b_i} - 1, \tag{2}$$

and

$$B \geq \max\{|b_1|, \ldots, |b_\ell|\}. \tag{3}$$

Let $A_1, \ldots, A_\ell$ be positive integers such that

$$A_i \geq h'(\tau_j) := \max\{dh(\tau_i), |\log \tau_i|, 0.16\} \quad \text{for} \quad i = 1, \ldots \ell.$$

The following statement is contained in Theorem 9.4 of [6], which is a consequence of a result due to Matveev [14].

Lemma 1. *If* $\Lambda \neq 0$ *then*

$$\log |\Lambda| > -C(\ell, d)(1 + \log B) A_1 \cdots A_\ell$$

with

$$C(\ell, d) := 1.4 \cdot 30^{\ell+3} \ell^{4.5} d^2 (1 + \log d).$$

Now we turn to the proof of Theorem 2. Assume that $n \geq n_0$ and

$$U_n = p_{i_1}^{\alpha_1} \cdots p_{i_k}^{\alpha_k},$$

where $i_j - i_{j-1} \leq C$ for $j = 2, \ldots, k$ and $k \leq L$. By Stewart's result [17], using the notation from the proof of Theorem 1 we have $p_{i_k} \geq M(n)$. Let $P := p_{i_k}$. We next bound p_{i_j} and α_j in terms of P. For α_j, we use the Binet formula for U_n, namely

$$U_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, \tag{4}$$

to write that

$$\left| \frac{\gamma^n}{\gamma - \delta} - p_{i_1}^{\alpha_1} \cdots p_{i_k}^{\alpha_k} \right| = \frac{|\delta|^n}{\gamma - \delta},$$

from where we deduce that

$$\left|1-\gamma^{-n}(\gamma-\delta)p_{i_1}^{\alpha_1}\cdots p_{i_k}^{\alpha_k}\right|=\left(\frac{|\delta|}{\gamma}\right)^n.$$

Observe that as the right–hand side above is nonzero, also the left–hand side is nonzero. So we may apply Lemma 1 to obtain

$$\exp(-c_2(c_3\log P)^L\log n)<\left|1-\gamma^{-n}(\gamma-\delta)p_{i_1}^{\alpha_1}\cdots p_{i_k}^{\alpha_k}\right|=\left(\frac{|\delta|}{\gamma}\right)^n,$$

where c_2 and c_3 are some positive constants which can be computed explicitly. Since $|\delta|<\gamma$, we get

$$n\ll(c_3\log P)^L\log n,$$

which implies that $n\ll(c_3\log P)^L\log\log P$. Since

$$n\log\gamma>\log U_n=\sum_{j=1}^k\alpha_j\log p_{i_j},$$

we get that

$$\alpha_j\ll(c_3\log P)^L\log\log P\qquad\text{for}\qquad j=1,\ldots,k. \tag{5}$$

For p_{i_j} we use the result of Baker, Harman and Pintz [1] to the effect that for $x>x_0$ the interval $[x-x^{0.525},x]$ contains a prime, to deduce that $p_{i_j}\geq P-C(k-j)P^{0.525}$ holds for all $j=1,2,\ldots,k$. In particular, writing

$$p_{i_j}=P-\delta_j\qquad\text{for}\qquad j=1,\ldots,k,$$

we get that

$$\delta_j\leq(k-j)CP^{0.525}\qquad\text{for}\qquad j=1,\ldots,k. \tag{6}$$

We now write $a:=\alpha_1+\cdots+\alpha_j$ and

$$\begin{aligned}U_n &= P^a\left(\frac{p_{i_1}}{P}\right)^{\alpha_1}\cdots\left(\frac{p_{i_{k-1}}}{P}\right)^{\alpha_{k-1}}\\ &= P^a\exp\left(\sum_{j=1}^{k-1}\alpha_j\log\left(1-\frac{\delta_j}{P}\right)\right)\\ &= P^a\exp\left(-\sum_{j=1}^{k-1}\frac{\alpha_j\delta_j}{P}+O\left(\frac{a}{P^2}\sum_{j=1}^{k-1}\delta_j^2\right)\right). \qquad (7)\end{aligned}$$

From (5) and (6), we have

$$\begin{aligned}\sum_{j=1}^{k-1}\frac{\alpha_j\delta_j}{P} &\ll \frac{(c_3\log P)^L(\log\log P)P^{0.525}}{P}C\sum_{j=1}^k j\\ &\ll \frac{(c_3\log P)^L(\log\log P)CL^2}{P^{0.475}}\\ &< \frac{1}{P^{0.4}}.\end{aligned}$$

Next,

$$\begin{aligned}\frac{a}{P^2}\sum_{j=1}^{k}\delta_j^2 &\ll \frac{(c_3\log P)^L(\log\log P)P^{1.05}}{P^2}C^2\sum_{j=1}^{k}j^2\\ &\ll \frac{(c_3\log P)^L(\log\log P)C^2L^3}{P^{0.95}}\\ &< \frac{1}{P^{0.4}},\end{aligned}$$

where the last inequality holds when P is sufficiently large with respect to C and L. Thus, (7) now implies

$$U_n = P^a\left(1+O\left(\frac{1}{P^{0.4}}\right)\right).$$

Using the Binet formula in the left–hand side, this can be rearranged as

$$\left|\gamma^n(\gamma-\delta)^{-1}P^{-a}-1\right| \ll \frac{1}{P^{0.4}}+\frac{1}{\gamma^n} \ll \frac{1}{P^{0.4}},$$

which can be put in logarithmic form

$$|n\log\gamma-\log(\gamma-\delta)-a\log P| \ll \frac{1}{P^{0.4}}. \tag{8}$$

Assume now that n is not prime and let $m \ge \sqrt{n}$ be a proper divisor of n. Then U_m is a divisor of U_n. The above argument implies that for large n we also have

$$|m\log\gamma-(\gamma-\delta)-b\log P| \ll \frac{1}{P^{0.4}}, \tag{9}$$

where now $b := \Omega(U_m)$. From inequalities (8) and (9), we get

$$|(nb-ma)\log\gamma-(a-b)\log(\gamma-\delta)| \ll \frac{a+b}{P^{0.4}} \ll \frac{(c_3\log P)^L(\log\log P)}{P^{0.4}} < \frac{1}{P^{0.3}},$$

where the last inequality holds if P (so n) is sufficiently large with respect to C and L. The left–hand side above is not zero since $b < a$ and γ and $\gamma-\delta=\sqrt{\Delta}=\sqrt{r^2+4s}$ are multiplicatively independent since r and s are coprime and $\Delta > 1$ is not a square. A lower bound for the linear form in logarithms in the left–hand side obtained through Lemma 1 gives that the left–hand side above is bounded from below by

$$\exp(-c_4\log(2na)),$$

where c_4 is some explicit constant. Thus, using that by $F_n \le P^a$ we have $n \ll a\log P$, we get

$$0.3\log P < c_4\log(2na) \ll c_4\log(c_5(c_3\log P)^{2L}(\log\log P)^2) \ll L\log\log P,$$

which shows that P is bounded in terms of L. Thus, n is also bounded. This finishes the proof of Theorem 2.

2.3. **The proof of Theorem 3.** To prove Theorem 3, we shall need Schmidt's subspace theorem. We shall use a simplified version due to Evertse [9]. For history and related versions, see, e.g. [9] and the references there. We need to introduce several notions and notation.

Let K be an algebraic number field of degree d with ring of integers $\mathcal{O}_K$. Write M_K for the set of places (equivalence classes of absolute values) of K. For $v \in M_K$ and $x \in K$, define the absolute value $|x|_v$ by

(i) $|x|_v = |\sigma(x)|^{1/d}$ if v corresponds to an infinite real embedding σ of K into $\mathbb{C}$,
(ii) $|x|_v = |\sigma(x)|^{2/d}$ if v corresponds to an infinite non-real embedding σ of K into $\mathbb{C}$,
(iii) $|x|_v = (N(\mathcal{P}))^{-\mathrm{ord}_{\mathcal{P}}(x)/d}$ if v corresponds to the prime ideal $\mathcal{P}$ of $\mathcal{O}_K$.

Here, $N(\mathcal{P})$ is the norm of $\mathcal{P}$, and $\mathrm{ord}_{\mathcal{P}}(x)$ is the exponent of $\mathcal{P}$ in the prime ideal factorization of (x), with the convention $\mathrm{ord}_{\mathcal{P}}(0) = \infty$. For a tuple $\mathbf{x} = (x_1, \ldots, x_n) \in K^n$ with $\mathbf{x} \neq \mathbf{0}$ set

$$|\mathbf{x}|_v := \begin{cases} \left(\sum\limits_{i=1}^{n} |x_i|_v^{2d}\right)^{1/2d}, & \text{if } v \text{ is infinite and real,} \\ \left(\sum\limits_{i=1}^{n} |x_i|_v^{d}\right)^{1/d}, & \text{if } v \text{ is infinite and non-real,} \\ \max\{|x_1|_v, \ldots, |x_n|_v\}, & \text{if } v \text{ is finite.} \end{cases}$$

The height of $\mathbf{x}$ is defined as

$$H(\mathbf{x}) := \prod_{v \in M_K} |\mathbf{x}|_v.$$

When $\mathbf{x} \in \mathcal{O}_K^n$, then

$$H(\mathbf{x}) \le n \prod_{\substack{v \in M_K \\ v \text{ infinite}}} \max\{|x_1|_v, \ldots, |x_n|_v\}^{r_v/d},$$

where $r_v = 1$ if v corresponds to an infinite real embedding and $r_v = 2$ if v corresponds to an infinite non-real embedding. We extend in some way all places of M_K to $\overline{\mathbb{Q}}$. The following statement is a reformulation of the Theorem of [9].

Lemma 2. *Let S be a finite set of M_K containing all infinite places. Let $0 < \vartheta < 1$. For each $v \in S$, let $l_{1,v}(\mathbf{x}), \ldots, l_{n,v}(\mathbf{x})$ be n linearly independent linear forms in $\mathbf{x} = (x_1, \ldots, x_n)$ with algebraic coefficients. Then the solutions $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{O}_K^n$ of the inequality*

$$\prod_{v \in S} \prod_{i=1}^{n} |l_{iv}(\mathbf{x})| < \left(\prod_{\substack{v \in M_K \\ v \text{ infinite}}} \max\{|x_1|_v, \ldots, |x_n|_v\}^{r_v/d} \right)^{-\vartheta}$$

are contained in at most c proper subspaces of K^n, where c is an explicitly computable constant depending only on $\vartheta, n, |S|$ and the linear forms $l_{i,v}$.

Now we turn to the proof of Theorem 3. We keep the notation of the proof of Theorem 2. In particular, we write

$$p_{i_j} = P - \delta_j \qquad \text{for all} \qquad j = 1, \ldots, k.$$

However, since we assume Conjecture 1 with some $\nu \in (0,1)$, by the argument used in the proof of Theorem 2 we now get

$$\delta_j \leq c_0(k-j)P^{\nu} \qquad \text{for} \qquad j = 1, \ldots, k.$$

We have

$$U_n = \prod_{j=1}^{k}(P-\delta_j)^{\alpha_j} = P^a + c_1 P^{a-1} + c_2 P^{a-2} + \cdots + c_a,$$

where

$$c_1 = -\sum_{j=1}^{k} \alpha_j \delta_j, \qquad c_2 = \sum_{j=1}^{k} \binom{\alpha_j}{2} \delta_j^2 + 2 \sum_{1 \leq i < j \leq k} \alpha_i \alpha_j \delta_i \delta_j, \qquad \text{etc.}$$

Note that $c_i = O_{C,L}(P^{\nu i})$ for $i = 1, \ldots, a$. We assume $a \geq 2$, and write $Q := P + c_1/a$. We then get

$$U_n = Q^a + d_1 Q^{a-1} + \cdots + d_a, \tag{10}$$

where $d_1 = 0$ and $d_i = O_{C,L}(P^{\nu i})$ for $i = 2, \ldots, a$. Using the Binet formula, this gives

$$|\gamma^n/\sqrt{\Delta} - Q^a| = O_{C,L}(Q^{a-2+2\nu}).$$

Let $n = am + t$, where $t \in \{0, 1, \ldots, a-1\}$. We may assume that n is large enough so that $t \neq 0$ by Theorem 2. We then get

$$|\gamma^m(a\gamma^{t/a}/\Delta^{1/(2a)}) - aQ| = O_{C,L}(Q^{-1+2\nu}). \tag{11}$$

Fix a. Let $K := \mathbb{Q}(\sqrt{\Delta})$ of degree $d = 2$ with its two infinite valuations given by $|x|_{\infty_i} = |\sigma_i(x)|^{1/2}$, where $\{\sigma_1, \sigma_2\}$ is the Galois group of K over $\mathbb{Q}$. Write $\mathbf{x} = (x_1, x_2, x_3) \in K^3$, $\eta := a\gamma^{t/a}/(\gamma - \delta)^{1/(2a)}$, and take the linear forms

$$\begin{aligned} L_{1,1}(\mathbf{x}) &= x_1, & L_{2,1}(\mathbf{x}) &= x_2, & L_{3,1}(\mathbf{x}) &= \eta x_1 - x_3; \\ L_{1,2}(\mathbf{x}) &= x_1, & L_{2,2}(\mathbf{x}) &= x_2, & L_{3,2}(\mathbf{x}) &= \eta x_2 - x_3. \end{aligned}$$

Take $S = \{\infty_1, \infty_2\}$. A quick computation shows that for $\mathbf{x} = (\gamma^m, \delta^m, aQ) \in \mathcal{O}_K^3$, we get

$$\prod_{i=1}^{2}\prod_{j=1}^{3} |L_{j,i}(\mathbf{x})|_{\infty_i} = O_{C,L}(Q^{-1+2\nu})$$

and

$$\prod_{i=1}^{2} \max\{|x_1|_{\infty_i}, |x_2|_{\infty_i}, |x_3|_{\infty_i}\}^{1/2} \ll_L Q.$$

Thus, assuming that $\nu < 1/2$, in view of (10), Lemma 2 leads to the conclusion that the vector (γ^m, δ^m, aQ) lives on finitely many proper subspaces of K^3. In particular, satisfies one of finitely many equations of the type

$$C_1\gamma^m + C_2\delta^m + C_3Q = 0,$$

where not all coefficients $C_1, C_2, C_3 \in K$ are zero. If $C_3 = 0$, then m is unique. If $C_3 \neq 0$, then $Q = C_1'\gamma^m + C_2'\delta^m$, where $C_i' := -C_i/C_3$ for $i = 1, 2$. Since aQ is an integer and γ and δ are conjugated in K, it follows that C_1' and C_2' are conjugated in K. In particular, none of them is zero.

Then equation (11) shows that the expression

$$|\gamma^m(\gamma^{t/a}/\Delta^{1/(2a)} - C_1') - C_2'\delta^m| \ll_{C,L} Q^{-1+2\nu} \ll_{C,L} \gamma^{m(-1+2\nu)}.$$

Since $|\delta| < \gamma$ and $-1 + 2\mu < 1$, this can happen for large n only if

$$C_1' = \gamma^{t/a}/\Delta^{1/(2a)}.$$

In particular, $C_1' \in K$. Note that $C_1'^a = \gamma^t/\sqrt{\Delta}$. Since C_1' and C_2' are conjugated in K it follows that $C_2'^a = \delta^t/\sqrt{\Delta}$. In particular, $C_2' = \delta^{t/a}\Delta^{1/(2a)}\eta$, where η is a root of unity. Thus, $C_1'C_2' = \eta s^{t/a}/\Delta^{1/a} \in \mathbb{Q}$. Since η is a root of unity, it follows that $\eta = \pm 1$. Further, since $s^{t/a}/\Delta^{1/a}$ is rational and s and r are coprime, we get that $\Delta^{1/a} \in \mathbb{N}$. As an aside, note that if $U_n = F_n$ is the nth Fibonacci number then the above equation leads to a contradiction for $a \geq 1$ since $\Delta = 5$ without any additional assumptions on ν.

Next, we also have that

$$|\delta|^m \ll_{C,L} (\gamma^m)^{-1+2\nu},$$

which gives

$$|s|^m = |\gamma\delta|^m \ll (\gamma^{2\nu})^m.$$

Assuming as we did in the hypothesis that Conjecture 1 holds for any $\nu > 0$, we may choose $\nu := \log(1.1)/\log\gamma$. We get that $|s|^m \ll_{C,L} (1.21)^m$, which shows that m is bounded in terms of C and L except if $|s| = 1$. From now on we work under the condition that $s = \pm 1$. We recall that we must have that $\Delta^{1/a}$ is an integer and furthermore that $\gamma^{t/a} \in K$. Since n is prime, we can replace a by any prime factor q of a and then conclude that both conditions $\Delta^{1/q} \in \mathbb{N}$ and $\gamma^{1/q} \in K$ (because $q \nmid n$, therefore $q \nmid t$). We shall show that this cannot happen. Write

$$\Delta_1^q = \Delta = r^2 + 4s = r^2 \pm 4 \qquad \text{where} \qquad \Delta_1 \in \mathbb{N}.$$

If $q = 2$, we get that $4 = |\Delta_1^2 - r^2|$ which is clearly impossible. So, $q \geq 3$. Assume next that r is odd. The only solution of the equation

$$r^2 + 4 = \Delta_1^q$$

with $q \geq 3$ is $(r, \Delta_1, q) = (11, 5, 3)$ (see [12]). This is not convenient for us since in this case

$$\gamma^{1/q} = \left(\frac{11 + 5\sqrt{5}}{2}\right)^{1/3} = \left(\frac{1 + \sqrt{5}}{2}\right)^{5/3} \notin K = \mathbb{Q}(\sqrt{5}).$$

The equation $r^2 - 4 = \Delta_1^q$ is also impossible for odd r. Indeed, the left-hand side factors as $(r-2)(r+2)$, which implies that $\Delta = uv$, where $r - 2 = u^q$, $r + 2 = v^q$. Thus, $4 = v^q - u^q$. However, as $v \geq u + 1$, it is not possible. Suppose next that r is even. In this case $r = 2r_0$, $\Delta_1 = 2\Delta_2$ and we get the equation

$$r_0^2 \pm 1 = 2^{q-2}\Delta_2^q$$

Since $q \geq 3$, it follows that r_0 is odd. In the equation $r_0^2 + 1 = 2^{q-2}\Delta_2^q$, the left–hand side is congruent to $2 \pmod 4$ showing that $q = 3$. Thus,

$$r_0^2 + 1 = 2\Delta_2^3$$

and the only solution of this last equation is $(r_0, \Delta_2) = (1, 1)$. This leads to $r = 2$, $\gamma = 1 + \sqrt{2}$ which is not convenient for us since

$$\gamma^{1/3} = (1 + \sqrt{2})^{1/3} \notin K = \mathbb{Q}(\sqrt{2}).$$

Finally, for the equation $r_0^2 - 1 = 2^{q-2}\Delta_2^q$, the left–hand side factors as $(r_0 - 1)(r_0 + 1)$. It follows that there are positive integers u, v such that $\Delta_2 = uv$ and

$$\{r_0 - 1, r_0 + 1\} = \{2u^q, 2^{q-3}v^q\}.$$

This leads to

$$u^q \pm 2^{q-4}v^q = \pm 1. \tag{12}$$

The only solution for $q = 5$ is $1^5 - 2 \cdot 1^5 = -1$ (see [2]), which leads to $\Delta_2 = 1$, so $\Delta = 2$, $r_0 = 3$ so $r = 6$. We thus get $\gamma = 3 + 2\sqrt{2}$ and this is not convenient for us since

$$\gamma^{1/5} = (3 + 2\sqrt{2})^{1/5} = (1 + \sqrt{2})^{2/5} \notin K = \mathbb{Q}(\sqrt{2}).$$

Finally, (12) has no solutions for $q \geq 7$ prime (see [3]).

Remark. We saw in the proof of Theorem 3 that for the particular Fibonacci sequence the validity of Conjecture 1 with some $\nu < 1/2$ suffices, This is maybe the case for all Lucas sequences with real irrational roots except that we have to rule our the case $\gamma^{1/a} \in K$ and $\Delta = \Delta_1^a$ for some integer Δ_1. Note that these two conditions lead to the conclusion that $\gamma = \gamma_1^a$, $\delta = \delta_1^a$ and

$$\gamma_1^a - \delta_1^a = \gamma - \delta = (\sqrt{\Delta_1})^a$$

which is a Fermat-type equation in $K = \mathbb{Q}(\sqrt{\Delta})$. Writing $\Delta = d\Box$, it is known that such equations have no solutions for $a \geq 4$ if K is a small quadratic field (say $K = \mathbb{Q}(\sqrt{d})$ with some squarefree $d \leq 100$ (see [11] and [16])) but we do not know of a general result in this direction.

2.4. **The proof of Theorem 4.** We keep the notation of the proof of Theorem 2. In particular, we have that (4) holds with $p_{i_k} = P \geq M(n)$ for $n \geq n_0$. But we do not have any additional information about k.

2.4.1. *Bounds for* α_j. Here is where we use the *abc* conjecture. Consider the well-known identity

$$V_n^2 - \Delta U_n^2 = \pm 4s^n \quad (n \geq 0),$$

where $(V_n)_{n\geq 0}$ is the companion sequence of general term $V_n = \gamma^n + \delta^n$. Let $d := \gcd(U_n, V_n) \in \{1,2\}$, and put $a := (V_n/d)^2$, $b := -\Delta(U_n/d)^2$, $c := \pm 4s^n/d^2$. Then the *abc* conjecture implies

$$U_n^2 \leq \max\{|a|,|b|,|c|\} \leq C_\varepsilon N(abc)^{1+\varepsilon} \leq C_\varepsilon (2\Delta|s|V_n p_{i_1}\cdots p_{i_k})^{1+\varepsilon}.$$

Using $U_n > \gamma^{n-2}$ and $V_n < 2\gamma^n$, we get that

$$\gamma^{2n-4} < U_n^2 \leq (4\Delta|s|C_\varepsilon \gamma^n p_{i_1}\cdots p_{i_k})^{1+\varepsilon}.$$

This gives that

$$p_{i_1}\cdots p_{i_k} \gg_\varepsilon \gamma^{\frac{2n}{1+\varepsilon}-n} = \gamma^{\frac{n(1-\varepsilon)}{1+\varepsilon}} > \gamma^{n(1-2\varepsilon)}.$$

This shows that

$$p_{i_1}^{\alpha_1-1}\cdots p_{i_k}^{\alpha_k-1} = \frac{U_n}{p_{i_1}\cdots p_{i_k}} \ll_\varepsilon \gamma^{2n\varepsilon}.$$

Letting $K(\varepsilon)$ be the multiplicative constant implied by the above Vinogradov symbol, we get that

$$\sum_{j=1}^{k}(\alpha_j - 1)\log p_{i_j} \leq 2n\varepsilon \log\gamma + \log K(\varepsilon). \tag{13}$$

Thus, we have

$$\sum_{j=1}^{k}(\alpha_{i_j} - 1)\log p_{i_j} \ll \varepsilon n \qquad \text{for} \qquad n > n(\varepsilon). \tag{14}$$

2.4.2. *The interval* $[p_{i_1}, P]$. Let us show first that $p_{i_1} > P/2$. Assume this is not so. Since U_n has prime factors with bounded gap C, we get, by the Prime Number Theorem,

$$F_n \geq p_{i_1}\cdots p_{i_k} \geq \left(\prod_{P/2<p\leq P} p\right)^{1/C+o(1)} = \exp(P(1/(2C)+o(1))),$$

giving that $P \ll Cn$. On the other hand, Stewart's result implies that $P \geq M(n)$, which puts a bound on n in terms of C. So, we may assume that $p_{i_1} > P/2 > n$ for $n > n_1$.

2.4.3. *The smallest prime factor of* n. Let q_1 be the smallest prime factor of n. Since $U_{q_1} \mid U_n$, it follows that $U_{q_1} \geq p_{i_1} > P/2$. Thus, $q_1 \gg \log n$.

2.4.4. *The size of P.* We are now ready to show that in fact $P \gg n^2$. This is an argument borrowed from Murty and Wong [15]. We start with the cyclotomic part of U_n, which is

$$\Phi_n := \prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}} (\gamma - e^{2\pi i k/n}\delta).$$

It is known that

$$\Phi_n = \Delta' \prod_{i=1}^{S} P_i^{\beta_i},$$

where $\Delta' = O(n)$ (for $n > 12$, it is a prime factor of n or 1)[1] and the P_i are the primitive prime factors of U_n, that is prime factors of U_n that do not divide U_m for any $m < n$. In particular, the prime powers $P_i^{\beta_i}$ for $i = 1, \ldots, S$ are among $p_{i_j}^{\alpha_{i_j}}$ for $j = 1, \ldots, k$. On the other hand,

$$\Phi_n = \prod_{d|n} U_{n/d}^{\mu(d)}.$$

Here, μ is the Möbius function. Using the inequalities $\gamma^{m-2} \le U_m \le 2\gamma^m$ valid for all positive integers m, we get

$$\begin{aligned}\Phi_n &= \gamma^{\sum_{d|n}\mu(d)\frac{n}{d} + O(\sum_{d|n}|\mu(d)|)}\\ &= \gamma^{\phi(n)+O(2^{\omega(n)})}.\end{aligned}$$

Thus, taking logarithms we get that

(15)

$$\phi(n)\log\gamma + O(2^{\omega(n)}) = \log(\Phi(n)) = \log\Delta' + \sum_{i=1}^{S}(\beta_i - 1)\log P_i + \sum_{i=1}^{S}\log P_i.$$

Since the smallest prime factor q_1 of n satisfies $q_1 \gg \log n$ and $\omega(n) = O(\log n/\log\log n)$, we get that

$$\begin{aligned}\phi(n) &= n\prod_{q|n}\left(1-\frac{1}{q}\right) = n\exp\left(O\left(\sum_{q|n}\frac{1}{q}\right)\right)\\ &= n\exp\left(O\left(\frac{1}{\log\log n}\right)\right) \ge n/2\end{aligned}$$

for large n. In the above, we used the fact that

$$\sum_{q|n}\frac{1}{q} \le \frac{\omega(n)}{q_1} = O\left(\frac{1}{\log\log n}\right).$$

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[1]The authors of [4] refer to this as the "Stewart criterion" although it certainly appeared earlier, like in the work of Carmichael [7]. In fact, Stewart himself (in "On divisors of Fermat, Fibonacci, Lucas and Lehmer numbers", Proc. London Math. Soc. **35** (1977), 425–447) says that it may even go as far back as E. Lucas "Théorie des fonctions numériques simplement periodiques", Amer. J. Math. **1** (1878), 184–240, 289–321.

So, in equation (15), in the left–hand side the error term is $n^{o(1)}$, as $n \to \infty$, thus its left–hand side is $\gg n$. In the right–hand side, we have that $\log \Delta' \le \log n$ and

$$\sum_{i=1}^{S} (\beta_i - 1) \log P_i \ll \varepsilon n$$

by (14). Taking ε to be sufficiently small, we get that

$$\sum_{i=1}^{S} \log P_i \gg n.$$

We may assume that $P \le n^3$, otherwise we are done. Since $P_i \le P$, we get that

$$S \gg n/\log n.$$

Now since P_i are primitive prime factors of U_n for $i = 1, \dots, S$, they are all of the form $\ell_i n \pm 1$ for some integers ℓ_i and $i = 1, \dots, S$. Thus, we get that

$$\pi(P, n, 1) + \pi(P, n, -1) \ge S \gg n/\log n.$$

Here, $\pi(x, u, v)$ counts the number of primes $p \le x$ in the progression $p \equiv v \pmod u$. Since certainly $\pi(P, n, \pm 1) \le P/n$, we get that $P \gg n^2/\log n$. Next, by the Brun-Titchmarsh theorem,

$$\pi(P, n, \pm 1) \ll \frac{P}{\phi(n) \log(P/n)} \ll \frac{P}{n \log n}.$$

For the last inequality we used that $\phi(n) \gg n$ and $P \gg n^2/\log n$, so $\log(P/n) \gg \log n$. Thus, we get that

$$\frac{P}{n \log n} \gg \pi(P, n, 1) + \pi(P, n - 1) \gg \frac{n}{\log n},$$

giving $P \gg n^2$.

2.4.5. *The interval* $[p_{i_1}, P]$ *revisited.* Now that we know that $P \gg n^2$, we can get a more precise information about the interval $[p_{i_1}, P]$. Namely, first $k = \omega(U_n) \ll \log U_n/\log\log U_n \ll n/\log n$. Next all prime factors of U_n have bounded gap difference by C, they are at most $O(n/\log n)$ of them and the largest one is $P \gg n^2$. Since we are assuming Conjecture 1 holds with $\nu = 1/2 - \eta$, it follows that by denoting

$$p_{i_j} = P - \delta_j \qquad \text{for} \qquad j = 1, \dots, k,$$

we have that

$$\delta_j \le C(k - j) P^{1/2-\eta} \qquad \text{for} \qquad j = 1, \dots, k.$$

In particular, we have

$$[p_{i_1}, P] \subseteq [P - CnP^{1/2-\eta}, P].$$

2.4.6. *The conclusion.* Assume now that $\Omega(n) \ge t$, where t is sufficiently large to be made more precise later. Let $n = q_1 q_2 \cdots q_{\Omega(n)}$ where $q_1 \le q_2 \le \cdots \le q_{\Omega(n)}$. Then $q_1 \le n^{1/t}$, $q_2 \le n^{1/(t-1)}$, so $q_1 q_2 < n^{2/(t-1)}$. Let $m \in \{q_1, q_1 q_2\}$. We write

$$U_m = \prod_{j \in I(m)} p_{i_j}^{\beta_j},$$

where I_m is some subset of $\{1, \ldots, k\}$ and $1 \le \beta_j \le \alpha_j$ for $j \in I_m$. Putting $b(m) := \sum_{j \in I_m} \beta_j$, we have a formula analogous to (7), namely

$$\begin{aligned} U_m &= P^{b(m)} \prod_{j \in I_m} \left(\frac{p_{i_j}}{P} \right)^{\beta_j} \\ &= P^{b(m)} \exp \left(\sum_{j \in I_m} \beta_j \log \left(1 - \frac{\delta_j}{P} \right) \right) \\ &= P^{b(m)} \exp \left(- \sum_{j \in I_m} \frac{\beta_j \delta_j}{P} + O \left(\frac{b(m)}{P^2} \sum_{j \in I_m} \delta_j^2 \right) \right). \end{aligned} \tag{16}$$

Since

$$\sum_{j \in I_m} \beta_j \ll m \ll n^{2/(t-1)},$$

we get that

$$\sum_{j \in I_m} \frac{\beta_j \delta_j}{P} \ll \frac{1}{P} (\sum_{j \in I_m} \beta_j) \max_{1 \le j \le k} \delta_j \ll \frac{C n^{2/(t-1)+1} P^{1/2-\eta}}{P} \le \frac{1}{P^{\eta/3}},$$

provided $t \ge 1 + 2/\eta$ and n is sufficiently large with respect to C and η. The last inequality follows since $P \gg n^2$. Similarly,

$$\frac{b(m)}{P^2} \sum_{j \in I_m} \delta_j^2 \ll \frac{C n^{4/(t-1)} n^2 P^{1-2\eta}}{P^2} < \frac{1}{P^\eta},$$

provided again that n is large enough with respect to C and η and $t \ge 1 + 2/\eta$. In the above, we used that $k \ll n/\log n$ and $P \gg n^2$. Thus, we get that

$$U_m = P^{b(m)} \left(1 + O \left(\frac{1}{P^{\eta/3}} \right) \right).$$

As in the proof of Theorem 2 this leads to

$$|m \log \gamma - \log(\gamma - \delta) - b(m) \log P| = O \left(\frac{1}{P^{\eta/3}} \right).$$

We apply this for $m = q_1$ and $m = q_1q_2$ and use the two relations to eliminate the term depending on $\log P$ and we get

$$\begin{aligned} |(q_1 b(q_1q_2) - (q_1q_2 b(q_1))) \log\gamma - (b(q_1q_2) - b(q_1)) \log(\gamma-\delta)| &= O\left(\frac{b(q_1) + b(q_1q_2)}{P^{\eta/3}}\right) \\ &= O\left(\frac{n^{2/(t-1)}}{P^{\eta/3}}\right) \\ &= O\left(\frac{1}{P^{\eta/6}}\right) \end{aligned}$$

assuming $t \geq 1 + 6/\eta$. In the above, we used again that $P \gg n^2$. The left–hand side above is not zero, since if it were zero, we would first get that $b(q_1) = b(q_1q_2)$, next that $q_1 = q_1q_2$, which is false. Applying Lemma 1 for the left–hand side above we get that it is at least

$$\exp(-c_6 \log q_1q_2) \geq \exp\left(-\frac{2c_6 \log n}{t-1}\right),$$

where c_6 is an absolute constant. Thus, we obtain that

$$(\eta/3)\log n + O(1) \leq (\eta/6)\log P \ll \frac{\log n}{t-1},$$

so $t = O(1 + 1/\eta)$, which is what we wanted.

2.5. **The proof of Theorem 5.** We keep the notations from the proof of Theorem 4. Since U_n has prime gaps bounded by C, it follows that for n sufficiently large with respect to C, we have that $p_{i_1} \geq P/2$. Let $\alpha := \max\{\alpha_j : 1 \leq j \leq k\}$ and assume that $\alpha \geq 2$. We write j for the index in $\{1, \ldots, k\}$ such that $\alpha_{i_j} = \alpha$. Equation (13) implies that

$$(\alpha/2)\log(P/2) \leq (\alpha_{i_j} - 1)\log p_{i_j} \leq 2\varepsilon n \log\gamma + \log K(\varepsilon).$$

This shows that

$$\alpha \log P \leq 8\varepsilon n \log\gamma + 4\log K(\varepsilon),$$

holds once n is sufficiently large with respect to C and ε. In particular,

$$\begin{aligned} (n-2)\log\gamma \leq \log U_n &= \sum_{j=1}^{k} \alpha_j \log p_{i_j} \leq \omega(F_n)(\alpha \log P) \\ &\leq 8L\varepsilon n \log\gamma + 4L\log K(\varepsilon). \end{aligned}$$

Choosing $\varepsilon = \varepsilon_0 := 1/(16L)$, we see that

$$n \leq 4 + 8L\log K(\varepsilon_0)/\log\gamma,$$

which shows that n is bounded in terms of C alone.

3. Comments

This paper started from an informal discussion about which Fibonacci numbers are products of an initial interval of primes, that is have prime factors with bounded gap by 1 and their smallest prime is 2. By the Primitive Divisor theorem, for $n \geq 13$, F_n has a prime factor which is at least as large as $n-1$. Thus, with $\gamma = (1+\sqrt{5})/2$, we have

$$\begin{aligned} \exp(0.482(n-1)) &\geq \gamma^{n-1} \geq F_n \geq \prod_{2 \leq p \leq n-1} p = \exp(\sum_{p \leq n-1} \log p) \\ &= \exp(\theta(n-1)) \geq \exp(0.6628(n-1)), \end{aligned}$$

where θ is the Chebyshev function and the last inequality holds for $n \geq 21$ by Proposition 2.5 in [13]. This shows that $n \leq 20$ and now a short calculations gives the solutions

$$F_1 = F_2 = 1, \quad F_3 = 2, \quad F_6 = 2^3 \qquad \text{and} \qquad F_{12} = 2^4 \cdot 3^2.$$

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