

# ON THE DOMAIN OF ODDNESS OF AN INFIMAL CONVOLUTION

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ABSTRACT. To illustrate a recent uniqueness result on the Hahn-Banach extensions, we show that if  $p$  is a sublinear functional of a vector space  $X$ ,  $\varphi$  is a linear functional of a subspace  $V$  of  $X$  such that  $\varphi(v) \leq p(v)$  for all  $v \in V$ , and

$$q(x) = \inf_{v \in V} (p(x - v) + \varphi(v))$$

for all  $x \in X$ , then the domain

$$W = \{ x \in X : q(-x) = -q(x) \}$$

of the oddness of the infimal convolution  $q$  of  $p$  and  $\varphi$  is a subspace of  $X$  such that any of the three possible cases  $V = W$ ,  $V \neq W \neq X$  and  $W = X$  can naturally occur even if, in particular,  $X$  is an Euclidean space and  $p$  is a norm on  $X$ .

## 1. INTRODUCTION

Suppose that  $X$  is a vector space over  $\mathbb{R}$  and  $p$  is a sublinear (positively homogeneous and subadditive) function of  $X$  to  $\mathbb{R}$ .

Moreover, assume that  $V$  is a subspace of  $X$  and  $\varphi$  is a linear function of  $V$  to  $\mathbb{R}$  such that  $\varphi$  is dominated by  $p$  on  $V$  in the sense that  $\varphi(v) \leq p(v)$  for all  $v \in V$ .

Now, because of the classical Hahn-Banach theorem [1, 4], a linear function  $f$  of  $X$  to  $\mathbb{R}$ , that extends  $\varphi$  and is dominated by  $p$  on  $X$ , may be called a Hahn-Banach extension of  $\varphi$  to  $X$ .

Moreover, in accordance with the more general definitions of Moreau [3] and Strömberg [5], the function  $q = p * \varphi$ , defined by

$$q(x) = \inf_{v \in V} (p(x - v) + \varphi(v))$$

for all  $x \in X$ , may be called the infimal convolution of  $p$  and  $\varphi$ .

In this notation, the following two theorems have been proved in [2].

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**Theorem 1.1.** *If  $f$  is a Hahn-Banach extension of  $\varphi$  to  $X$ , then for any  $x \in X$  we have*

$$-q(-x) \leq f(x) \leq q(x).$$

**Theorem 1.2.** *The following assertions are equivalent:*

- (1)  $q$  is odd on  $X$ ;
- (2)  $q$  is a Hahn-Banach extension of  $\varphi$  to  $X$ ;
- (3) there exists a unique Hahn-Banach extension  $f$  of  $\varphi$  to  $X$ ;
- (4) there exists at most one Hahn-Banach extension  $f$  of  $\varphi$  to  $X$ .

The latter theorem has motivated us to investigate the domain

$$W = \{x \in X : q(-x) = -q(x)\}$$

of the oddness of the function  $q$ .

By using some basic properties of  $q$ , we shall show that  $W$  is always a subspace of  $X$  containing  $V$ . Moreover, by using a particular infimal value result, we shall show that any of the three possible cases  $V = W$ ,  $V \neq W \neq X$ , and  $W = X$  can naturally occur even if, in particular,  $X$  is an Euclidean space and  $p$  is a norm on  $X$ .

## 2. LINEARITY PROPERTIES OF $q$ AND $W$

The following two theorems have also been proved in [2].

**Theorem 2.1.**  *$q$  is a real-valued, sublinear function of  $X$ .*

**Corollary 2.2.**  *$q(0) = 0$  and  $\lambda q(x) \leq q(\lambda x)$  for all  $\lambda \in \mathbb{R}$  and  $x \in X$ .*

*Remark 2.3.* Hence, one can easily derive that  $q(\lambda x) = \lambda q(x)$  for all  $\lambda \geq 0$  and  $x \in X$ .

**Theorem 2.4.**  *$q(x + v) = q(x) + \varphi(v)$  for all  $x \in X$  and  $v \in V$ .*

**Corollary 2.5.**  *$q$  is an extension of  $\varphi$  and  $q(x + v) = q(x) + q(v)$  for all  $x \in X$  and  $v \in V$ .*

Now, by using the above results, we can easily prove the following

**Theorem 2.6.**  *$W$  is a subspace of  $X$  containing  $V$ .*

*Proof.* Since  $\varphi$  is odd and  $q$  is an extension of  $\varphi$ , it is clear that  $V \subset W$ . Therefore, we need only show that  $W$  is closed under the linear operations in  $X$ .

For this, note first that if  $x \in W$ , then  $q(-x) = -q(x)$ . Hence, it is clear that

$$q(-(-x)) = q(x) = -(-q(x)) = -q(-x).$$

Therefore,  $-x \in W$  also holds.

Moreover, if  $\lambda \geq 0$  and  $x \in W$ , then by Remark 2.3 and  $q(-x) = -q(x)$ , we have

$$q(-\lambda x) = q(\lambda(-x)) = \lambda q(-x) = \lambda(-q(x)) = -\lambda q(x) = -q(\lambda x).$$

Therefore,  $\lambda x \in W$  also holds.

While, if  $\lambda < 0$  and  $x \in W$ , then  $-\lambda > 0$  and  $-x \in W$ . Therefore, by the above observation, we also have

$$\lambda x = (-\lambda)(-x) \in W.$$

Finally, we note that if  $x, y \in W$ , then by the subadditivity of  $q$ , the equalities  $q(-x) = -q(x)$  and  $q(-y) = -q(y)$ , and the  $\lambda = -1$  particular case of Corollary 2.2 we have

$$\begin{aligned} q(-(x+y)) &= q(-x + (-y)) \leq q(-x) + q(-y) \\ &= -q(x) + (-q(y)) = -(q(x) + q(y)) \leq -q(x+y) \leq q(-(x+y)). \end{aligned}$$

Therefore,

$$q(-(x+y)) = -q(x+y),$$

and thus  $x+y \in W$  also holds.  $\square$

*Remark 2.7.* To feel the importance of  $W$ , note that by definition  $W$  is the largest subset of  $X$  such that  $q$  is odd on  $W$ .

Hence, by Theorems 2.6 and 1.2, we can see that  $W$  is the largest subspace of  $X$  such that  $\varphi$  has a unique Hahn-Banach extension  $\psi$  to  $W$ . Moreover, we necessarily have  $\psi(w) = q(w)$  for all  $w \in W$ .

### 3. A PARTICULAR INFIMAL VALUE RESULT

In the proof of our subsequent infimal value proposition, we shall use the next plausible minimal value lemma.

**Lemma 3.1.** *If  $f$  is a continuous function of  $\mathbb{R}$  to itself such that*

$$\lim_{t \rightarrow -\infty} f(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} f(t) = +\infty,$$

*then there exists  $\tau \in \mathbb{R}$  such that*

$$f(\tau) = \min_{t \in \mathbb{R}} f(t).$$

*Proof.* By using the above limit conditions, we can see that there exists  $r > 0$  such that

$$f(0) \leq f(t)$$

whenever either  $t < -r$  or  $r < t$ . Now, by a basic theorem on continuous functions, we can state that there exists  $\tau \in [-r, r]$  such that

$$f(\tau) \leq f(t)$$

for all  $t \in [-r, r]$ . Now, since  $f(\tau) \leq f(0)$ , it is clear that the above inequality also holds for all  $t \in \mathbb{R} \setminus [-r, r]$ .  $\square$

Our colleague, Mihály Bessenyei has observed that the  $|\beta| \neq 1$  particular case of the following proposition can be proved in a more elementary way, without using the standard methods of analysis for finding local and global minima.

However, our subsequent proof can be well used to illustrate the above mentioned methods in class rooms. Note that, by giving particular values of the parameters  $\alpha$  and  $\beta$ , we can get several useful test problems for undergraduate students.

**Proposition 3.2.** *For any  $\alpha, \beta \in \mathbb{R}$ , with  $\alpha \geq 0$  and  $|\beta| \leq 1$ , we have*

$$\inf_{t \in \mathbb{R}} \left( \sqrt{t^2 + \alpha} + \beta t \right) = \sqrt{\alpha(1 - \beta^2)}.$$

*Proof.* Define

$$f_{\alpha\beta}(t) = \sqrt{t^2 + \alpha} + \beta t$$

for all  $t \in \mathbb{R}$ . Then, in particular

$$f_{0\beta}(t) = \sqrt{t^2} + \beta t = |t| + \beta t = \begin{cases} (\beta + 1)t & \text{for } t \geq 0, \\ (\beta - 1)t & \text{for } t < 0. \end{cases}$$

Hence, since  $\beta + 1 \geq 0$  and  $\beta - 1 \leq 0$ , we can see that  $f_{0\beta}(t) \geq 0$  for all  $t \in \mathbb{R}$ . Now, since  $f_{0\beta}(0) = 0$ , it is clear that

$$\min_{t \in \mathbb{R}} f_{0\beta}(t) = 0 = \sqrt{0(1 - \beta^2)}.$$

This shows that the required equality is true in the particular case  $\alpha = 0$ . Therefore, in the sequel, we may assume that  $\alpha \neq 0$ , and thus  $\alpha > 0$ .

Now, we can also note that

$$f_{0\beta}(t) = \sqrt{t^2} + \beta t < \sqrt{t^2 + \alpha} + \beta t = f_{\alpha\beta}(t)$$

for all  $t \in \mathbb{R}$ . Hence, by the nonnegativity of the function  $f_{0\beta}$ , we can see that  $f_{\alpha\beta}(t) > 0$  for all  $t \in \mathbb{R}$ . Moreover, we can note that

$$\begin{aligned} \lim_{t \rightarrow -\infty} f_{\alpha 1}(t) &= \lim_{t \rightarrow -\infty} \left( \sqrt{t^2 + \alpha} + t \right) \\ &= \lim_{t \rightarrow -\infty} \frac{\left( \sqrt{t^2 + \alpha} \right)^2 - t^2}{\sqrt{t^2 + \alpha} - t} = \lim_{t \rightarrow -\infty} \frac{\alpha}{\sqrt{t^2 + \alpha} - t} = 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow +\infty} f_{\alpha(-1)}(t) &= \lim_{t \rightarrow +\infty} \left( \sqrt{t^2 + \alpha} - t \right) \\ &= \lim_{t \rightarrow +\infty} \frac{\left( \sqrt{t^2 + \alpha} \right)^2 - t^2}{\sqrt{t^2 + \alpha} + t} = \lim_{t \rightarrow +\infty} \frac{\alpha}{\sqrt{t^2 + \alpha} + t} = 0. \end{aligned}$$

Hence, it is clear that

$$\inf_{t \in \mathbb{R}} f_{\alpha 1}(t) = 0 = \sqrt{\alpha(1-1^2)} \quad \text{and} \quad \inf_{t \in \mathbb{R}} f_{\alpha(-1)}(t) = 0 = \sqrt{\alpha(1-(-1)^2)}.$$

This shows that the required equality is true in the particular case  $|\beta| = 1$ . Therefore, in the sequel, we may assume that  $|\beta| \neq 1$ , and thus  $|\beta| < 1$ .

Now, since  $\beta - 1 < 0$  and  $\beta + 1 > 0$ , we can also note that

$$\lim_{t \rightarrow -\infty} f_{0\beta}(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} f_{0\beta}(t) = +\infty.$$

Hence, by using that  $f_{0\beta} \leq f_{\alpha\beta}$ , we can infer that

$$\lim_{t \rightarrow -\infty} f_{\alpha\beta}(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} f_{\alpha\beta}(t) = +\infty$$

also hold. Moreover, we can note that  $f_{\alpha\beta}$  is continuous. Thus, by Lemma 3.1, there exists  $\tau \in \mathbb{R}$  such that

$$f_{\alpha\beta}(\tau) = \min_{t \in \mathbb{R}} f_{\alpha\beta}(t).$$

Moreover, since  $f_{\alpha\beta}$  is differentiable, we can also state that  $f'_{\alpha\beta}(\tau) = 0$ . Hence, by using that

$$f'_{\alpha\beta}(\tau) = \frac{1}{2\sqrt{\tau^2 + \alpha}} 2\tau + \beta = \frac{\tau + \beta\sqrt{\tau^2 + \alpha}}{\sqrt{\tau^2 + \alpha}},$$

we can infer that  $\tau + \beta\sqrt{\tau^2 + \alpha} = 0$ , and thus

$$\tau = -\beta\sqrt{\tau^2 + \alpha}.$$

This implies that  $\operatorname{sgn}(\tau) = -\operatorname{sgn}(\beta)$ . Moreover, we can also easily see that  $\tau^2 = \beta^2(\tau^2 + \alpha)$ , whence

$$\tau^2 = \frac{\alpha\beta^2}{1-\beta^2}, \quad \text{and thus} \quad |\tau| = \frac{\sqrt{\alpha}|\beta|}{\sqrt{1-\beta^2}}.$$

Hence, by using that

$$|\tau| = \operatorname{sgn}(\tau)\tau \quad \text{and} \quad |\beta| = \operatorname{sgn}(\beta)\beta = -\operatorname{sgn}(\tau)\beta,$$

we can infer that

$$\operatorname{sgn}(\tau)\tau = -\operatorname{sgn}(\tau)\frac{\sqrt{\alpha}\beta}{\sqrt{1-\beta^2}}, \quad \text{and thus} \quad \tau = -\frac{\sqrt{\alpha}\beta}{\sqrt{1-\beta^2}}.$$

Namely, if  $\operatorname{sgn}(\tau) = 0$ , then because of  $\operatorname{sgn}(\tau) = -\operatorname{sgn}(\beta)$  we also have  $\operatorname{sgn}(\beta) = 0$ . Therefore, not only  $\tau = 0$ , but also  $\beta = 0$ . Now, to complete the proof, it remains only to note that

$$\begin{aligned} f_{\alpha\beta}(\tau) &= \sqrt{\tau^2 + \alpha} + \beta\tau = \sqrt{\frac{\alpha\beta^2}{1-\beta^2} + \alpha} - \frac{\sqrt{\alpha}\beta^2}{\sqrt{1-\beta^2}} \\ &= \frac{\sqrt{\alpha\beta^2 + \alpha - \alpha\beta^2}}{\sqrt{1-\beta^2}} - \frac{\sqrt{\alpha}\beta^2}{\sqrt{1-\beta^2}} = \sqrt{\alpha}\frac{1-\beta^2}{\sqrt{1-\beta^2}} = \sqrt{\alpha(1-\beta^2)}. \end{aligned}$$

□

*Remark 3.3.* Note that if in particular  $|\beta| \neq 1$ , then we can write minimum instead of infimum in the above proposition.

#### 4. THE FIRST ILLUSTRATING EXAMPLE

The following example is a little more difficult than that given in [2].

*Example 4.1.* Take

$$X = \mathbb{R}^2 \quad \text{and} \quad V = \mathbb{R}(1, 1).$$

Moreover, define

$$\varphi(r, r) = r \quad \text{and} \quad p(r, s) = \sqrt{r^2 + s^2}$$

for all  $r, s \in \mathbb{R}$ . Then, it is clear that  $V$  is a subspace of  $X$ ,  $\varphi$  is a linear function of  $V$ , and  $p$  is a norm on  $X$  such that

$$|\varphi(r, r)| = |r| \leq \sqrt{2}|r| = \sqrt{r^2 + r^2} = p(r, r)$$

for all  $r \in \mathbb{R}$ . Thus, in particular,  $\varphi$  is dominated by  $p$  on  $V$ .

Moreover, by using Proposition 3.2, we can see that

$$\begin{aligned}
q(r, s) &= \inf_{u \in \mathbb{R}} (p((r, s) - (u, u)) + \varphi(u, u)) \\
&= \inf_{u \in \mathbb{R}} (p(r - u, s - u) + \varphi(u, u)) = \inf_{u \in \mathbb{R}} \left( \sqrt{(r - u)^2 + (s - u)^2} + u \right) \\
&= \inf_{u \in \mathbb{R}} \sqrt{2} \left( \sqrt{\frac{(r - u)^2 + (s - u)^2}{2}} + \frac{u}{\sqrt{2}} \right) \\
&= \sqrt{2} \inf_{u \in \mathbb{R}} \left( \sqrt{\frac{(r - u)^2 + (s - u)^2}{2}} + \frac{u}{\sqrt{2}} \right) \\
&= \sqrt{2} \inf_{u \in \mathbb{R}} \left( \sqrt{\left(u - \frac{r + s}{2}\right)^2 + \frac{(r - s)^2}{4}} + \frac{1}{\sqrt{2}} \left(u - \frac{r + s}{2}\right) + \frac{r + s}{2\sqrt{2}} \right) \\
&= \sqrt{2} \left( \inf_{u \in \mathbb{R}} \left( \sqrt{\left(u - \frac{r + s}{2}\right)^2 + \frac{(r - s)^2}{4}} + \frac{1}{\sqrt{2}} \left(u - \frac{r + s}{2}\right) \right) + \frac{r + s}{2\sqrt{2}} \right) \\
&= \sqrt{2} \inf_{t \in \mathbb{R}} \left( \sqrt{t^2 + \frac{(r - s)^2}{4}} + \frac{1}{\sqrt{2}} t \right) + \frac{r + s}{2} \\
&= \sqrt{2} \sqrt{\frac{(r - s)^2}{4} \left(1 - \frac{1}{2}\right)} + \frac{r + s}{2} = \frac{|r - s|}{2} + \frac{r + s}{2} = \max\{r, s\}
\end{aligned}$$

for all  $r, s \in \mathbb{R}$ .

Hence, it is clear that, for any  $r, s \in \mathbb{R}$ , we have

$$\begin{aligned}
(r, s) \in W &\iff q(-(r, s)) = -q(r, s) \iff q(-r, -s) = -q(r, s) \\
&\iff \max\{-r, -s\} = -\max\{r, s\} \iff -\max\{-r, -s\} = \max\{r, s\} \\
&\iff \min\{r, s\} = \max\{r, s\} \iff r = s \iff (r, s) \in V.
\end{aligned}$$

Therefore, in the present particular case,  $W = V$  holds.

*Remark 4.2.* Note that now we have

$$(r, s) = \frac{s - r}{2} (-1, 1) + \frac{r + s}{2} (1, 1)$$

for all  $r, s \in \mathbb{R}$ .

Therefore, if  $f$  is a Hahn-Banach extension of  $\varphi$  to  $X$ , then

$$\begin{aligned} f(r, s) &= \frac{s-r}{2} f(-1, 1) + \frac{r+s}{2} f(1, 1) \\ &= \frac{s-r}{2} f(-1, 1) + \frac{r+s}{2} \varphi(1, 1) = \frac{s-r}{2} f(-1, 1) + \frac{r+s}{2} \\ &= \frac{1-f(-1, 1)}{2} r + \frac{1+f(-1, 1)}{2} s \end{aligned}$$

for all  $r, s \in \mathbb{R}$ . Moreover, by Theorem 1.1, we have

$$-1 = -q(1, -1) \leq f(-1, 1) \leq q(-1, 1) = 1.$$

Hence, by taking

$$\lambda = \frac{1-f(-1, 1)}{2},$$

we can see that  $\lambda \in [0, 1]$  such that

$$f(r, s) = \lambda r + (1 - \lambda) s$$

for all  $r, s \in \mathbb{R}$ . Thus, we can obtain all Hahn-Banach extensions  $f$  of  $\varphi$  to  $X$ .

## 5. THE SECOND AND THIRD ILLUSTRATING EXAMPLES

*Example 5.1.* Let  $X$ ,  $V$  and  $\varphi$  be as in Example 4.1. Moreover, define

$$p(r, s) = \sqrt{\frac{r^2 + s^2}{2}}$$

for all  $r, s \in \mathbb{R}$ . Then, it is clear that  $p$  is a norm on  $X$  such that

$$|\varphi(r, r)| = |r| = \sqrt{\frac{r^2 + r^2}{2}} = p(r, r)$$

for all  $r \in \mathbb{R}$ . Thus, in particular,  $\varphi$  is dominated by  $p$  on  $V$ .

Moreover, quite similarly as in Example 4.1, we can see that

$$\begin{aligned}
q(r, s) &= \inf_{u \in \mathbb{R}} (p(r - u, s - u) + \varphi(u, u)) = \\
&= \inf_{u \in \mathbb{R}} \left( \sqrt{\frac{(r - u)^2 + (s - u)^2}{2}} + u \right) \\
&= \inf_{u \in \mathbb{R}} \left( \sqrt{\left(u - \frac{r + s}{2}\right)^2 + \frac{(r - s)^2}{4}} + \left(u - \frac{r + s}{2}\right) \right) + \frac{r + s}{2} \\
&= \inf_{t \in \mathbb{R}} \left( \sqrt{t^2 + \frac{(r - s)^2}{4}} + t \right) + \frac{r + s}{2} \\
&= \sqrt{\frac{(r - s)^2}{4} (1 - 1)} + \frac{r + s}{2} = \frac{r + s}{2}
\end{aligned}$$

for all  $r, s \in \mathbb{R}$ .

Hence, it is clear that

$$q(-r, -s) = q(-r, -s) = \frac{-r - s}{2} = -\frac{r + s}{2} = -q(r, s)$$

for all  $r, s \in \mathbb{R}$ , and thus  $q$  is an odd function of  $X$ . Therefore, in the present particular case,  $W = X$  holds.

*Remark 5.2.* Now, by Theorem 1.2, we can state that  $q$  is the unique Hahn-Banach extension of  $\varphi$  to  $X$ .

*Example 5.3.* Take

$$X = \mathbb{R}^3 \quad \text{and} \quad V = \mathbb{R} (1, 1, 0).$$

Moreover, define

$$\varphi(r, r, 0) = r \quad \text{and} \quad p(r, s, t) = \sqrt{\frac{r^2 + s^2}{2}} + |t|$$

for all  $r, s, t \in \mathbb{R}$ . Then, it is clear that  $V$  is a subspace of  $X$ ,  $\varphi$  is a linear function of  $V$ , and  $p$  is a norm on  $X$  such that

$$|\varphi(r, r, 0)| = |r| = \sqrt{\frac{r^2 + r^2}{2}} + |0| = p(r, r, 0)$$

for all  $r \in \mathbb{R}$ . Thus, in particular,  $\varphi$  is dominated by  $p$  on  $V$ .

Moreover, by making use of the computation given in Example 5.1, we can see that

$$\begin{aligned}
q(r, s, t) &= \inf_{u \in \mathbb{R}} (p((r, s, t) - (u, u, 0)) + \varphi(u, u, 0)) \\
&= \inf_{u \in \mathbb{R}} (p(r - u, s - u, t) + \varphi(u, u, 0)) \\
&= \inf_{u \in \mathbb{R}} \left( \sqrt{\frac{(r - u)^2 + (s - u)^2}{2}} + |t| + u \right) \\
&= \inf_{u \in \mathbb{R}} \left( \sqrt{\frac{(r - u)^2 + (s - u)^2}{2}} + u \right) + |t| = \frac{r + s}{2} + |t|
\end{aligned}$$

for all  $r, s, t \in \mathbb{R}$ .

Hence, it is clear that, for any  $r, s, t \in \mathbb{R}$ , we have

$$\begin{aligned}
(r, s, t) \in W &\iff q(-(r, s, t)) = -q(r, s, t) \\
&\iff q(-r, -s, -t) = -q(r, s, t) \\
&\iff \frac{-r - s}{2} + |-t| = -\left(\frac{r + s}{2} + |t|\right) \\
&\iff |-t| = -|t| \iff 2|t| = 0 \iff |t| = 0 \iff t = 0.
\end{aligned}$$

Therefore, in the present particular case,  $W = \mathbb{R} \times \mathbb{R} \times \{0\}$ , and thus  $V \neq W \neq X$  holds.

*Remark 5.4.* Note that now we have

$$(r, s, t) = t(0, 0, 1) + (r, s, 0)$$

for all  $r, s, t \in \mathbb{R}$ .

Therefore, if  $f$  is a Hahn-Banach extension of  $\varphi$  to  $X$ , then by Theorem 1.2 we have

$$\begin{aligned}
f(r, s, t) &= t f(0, 0, 1) + f(r, s, 0) = t f(0, 0, 1) + q(r, s, 0) \\
&= t f(0, 0, 1) + \frac{r + s}{2} = \frac{1}{2} r + \frac{1}{2} s + f(0, 0, 1) t
\end{aligned}$$

for all  $r, s, t \in \mathbb{R}$ . Moreover, by Theorem 1.1, we have

$$-1 = -q(0, 0, -1) \leq f(0, 0, 1) \leq q(0, 0, 1) = 1.$$

Hence, by taking

$$\lambda = f(0, 0, 1),$$

we can see that  $\lambda \in [-1, 1]$  such that

$$f(r, s, t) = \frac{1}{2} r + \frac{1}{2} s + \lambda t$$

for all  $r, s, t \in \mathbb{R}$ . Thus, we can obtain all Hahn-Banach extensions  $f$  of  $\varphi$  to  $X$ .

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