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CONVERGENCE OF MATRIX TRANSFORM MEANS WITH RESPECT TO VILENKIN SYSTEMS

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Abstract. In this paper we discuss the convergence of matrix transform means of Vilenkin-Fourier series of integrable functions in Vilenkin-Lebesgue points, where the elements of the matrix rows are monotonic. We also prove convergence in $L_1(G_m)$ norm concerning these means.

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1. NOTATIONS AND DEFINITIONS

We give a brief introduction to the theory of Vilenkin-Fourier analysis (for more details see [1, 24]). Denote by $\mathbb{P}$ the set of the positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ be a sequence of positive integers not less than 2. Denote by $\mathbb{Z}_{m_n} := \{0, 1, \dots, m_n - 1\}$ the additive group of integers modulo m_n. Define the group G_m as the complete direct product of the groups $\mathbb{Z}_{m_n}$ with the product of the discrete topologies of $\mathbb{Z}_{m_n}$'s.

The direct product μ of the measures

$$\mu_n(\{j\}) := 1/m_n \ (j \in \mathbb{Z}_{m_n})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence m is bounded, then G_m is called a bounded Vilenkin group, otherwise, it is called an unbounded one. In this paper, we only deal with the bounded situation. For results on unbounded Vilenkin groups see e.g. [5, 15–17].

We mention, that in case of $m = (2, 2, \dots)$ we get G, the so-called Walsh group. The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_n, \dots) \ (x_n \in \mathbb{Z}_{m_n}).$$

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It is easy to give a base for the neighbourhoods of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \ (x \in G_m, n \in \mathbb{P}).$$

We define the so-called generalized number system based on sequence m in the following way:

$$M_0 := 1, \ M_{n+1} := m_n M_n \ (n \in \mathbb{N}).$$

Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in \mathbb{Z}_{m_k}$ $(k \in \mathbb{N})$ and only a finite number of n_k's differ from zero. Let us denote by $|n|$ a natural number such that $M_{|n|} \leq n < M_{|n|+1}$.

Let $L_p(G_m)$ denote the usual Lebesgue spaces on G_m with corresponding norms $\|.\|_p$ and $C(G_m)$ denote the space of continuous functions on G_m with the norm $\|f\|_\infty := \sup\{|f(x)| : x \in G_m\}$.

Next, we introduce on G_m an orthonormal system which is called Vilenkin system. At first, we define the complex-valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi \imath x_k / m_k) \ \left(\imath^2 = -1, \ x \in G_m, \ k \in \mathbb{N}\right).$$

Let us define the Vilenkin system $\psi := \{\psi_n : n \in \mathbb{N}\}$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \ (n \in \mathbb{N}, \ x \in G_m).$$

Specifically, we call this system the Walsh-Paley system, when $m = (2, 2, \dots)$. The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [25]). Exactly, the elements of the Vilenkin system are the characters of G_m. Namely, $f : G_m \to \mathbb{C}$ continuous,

$$f(x+y) = f(x)f(y)$$

and $|f(x)| = 1$ for all $x, y \in G_m$. Moreover, it holds if and only if $f(x) = \psi_n(x)$ for some $n \in \mathbb{N}$ (see [24]).

Let $\mathcal{P}_n$ be the collection of Vilenkin polynomials of order less than n, that is, functions of the form

$$P(x) = \sum_{k=0}^{n-1} a_k \psi_k(x),$$

where $n \in \mathbb{P}$ and $\{a_k\}$ is a sequence of complex numbers. Let $\mathcal{P}$ denote the set of Vilenkin polynomials.

The nth Vilenkin-Fourier-coefficient, the nth partial sum of the Vilenkin-Fourier series, the nth Vilenkin-Fejér mean and the nth Vilenkin-Dirichlet kernel is defined by

$$\hat{f}(n) := \int_{G_m} f \bar{\psi}_n \mathrm{d}\mu, \ \ S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \ \ \sigma_n(f) := \frac{1}{n} \sum_{k=1}^{n} S_k(f), \ \ D_n := \sum_{k=0}^{n-1} \psi_k,$$

where $n \in \mathbb{P}$, and $D_0 := 0$. It is well-known, that

$$S_n(f;x) = \int_{G_m} f(u)D_n(x-u)\mathrm{d}\mu(u). \tag{1.1}$$

Fejér kernels are defined as the arithmetical means of Dirichlet kernels, that is,

$$K_n := \frac{1}{n}\sum_{k=1}^{n} D_k.$$

Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-negative numbers. The nth Nörlund mean of the Vilenkin-Fourier series is defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k}S_k(f;x),$$

where $Q_n := \sum_{k=0}^{n-1} q_k$ $(n \in \mathbb{P})$. It is always assumed that $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty.$$

In this case, the summability method generated by $\{q_k\}$ is regular (see [28]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

Let $\{p_k : k \in \mathbb{P}\}$ be a sequence of nonnegative numbers. The nth weighted mean T_n of Vilenkin-Fourier series is defined by

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k S_k(f;x),$$

where $P_n := \sum_{k=1}^{n} p_k$ $(n \in \mathbb{P})$. It is always assumed that $p_1 > 0$ and

$$\lim_{n\to\infty} P_n = \infty,$$

which is the condition for regularity.

In particular case Nörlund and weighted means are Fejér ones (for all k set $p_k := 1$ and $q_k := 1$).

Let $T := (t_{i,j})_{i,j=1}^{\infty}$ be a doubly infinite matrix of numbers. It is always supposed that matrix T is triangular. Let us define the nth linear mean (or matrix transform mean) determined by the matrix T

$$\sigma_n^T(f;x) := \sum_{k=1}^{n} t_{k,n}S_k(f;x).$$

Since, the nth column of the matrix T determines the linear mean σ_n^T and its definition contains only finite number of entries, for the simplicity we say $\{t_{k,n} : 1 \le k \le n,\ k \in \mathbb{P}\}$ is a finite sequence of numbers for each $n \in \mathbb{P}$. We also use the notation

$$\Delta t_{k,n} := t_{k,n} - t_{k+1,n}.$$

In the further part of this paper, let $\{t_{k,n} : 1 \leq k \leq n,\ k \in \mathbb{P}\}$ be a finite sequence of non-negative numbers for each $n \in \mathbb{P}$. The nth matrix transform kernel is defined by

$$K_n^T(x) := \sum_{k=1}^{n} t_{k,n} D_k(x).$$

Based on (1.1) it is easily seen that

$$\sigma_n^T(f;x) = \int_{G_m} f(u) K_n^T(x-u) \mathrm{d}\mu(u). \tag{1.2}$$

2. Historical overview

Matrix transform means are common generalizations of several well-known summation methods. It follows by simple consideration that the Nörlund and weighted means, the Fejér (or the $(C,1)$) and the (C,α) means are special cases of the matrix transform summation method introduced above. For matrix transform means with respect to trigonometric system see e.g. results of Chandra [13] and Leindler [20], to Vilenkin system see paper of Blyumin [12]. This paper [12] of Blyumin gives an estimate of the approximation by general linear means of Vilenkin-Fourier series in terms of best approximations and some integrals similar to Lebesgue constants.

In recent years several papers were published with respect to the matrix transform means based on work of Móricz and Siddiqi [22] on the Walsh-Nörlund summation method and the result of Móricz and Rhoades [21] on the weighted mean method. A common generalization of these two results of Móricz and Siddiqi [22] and Móricz and Rhoades [21] was given by K. Nagy and the first author [7]. For other results on Walsh-Paley system see [10, 11], on bounded Vilenkin systems [8, 9]. In their paper [19], Goginava and K. Nagy improved these results (for the Walsh-system), introducing the definition of the so-called matrix transform variation.

Other aspects of these methods with respect to Walsh-Fourier series are treated in the papers [14, 27]. Avdispahić and Pepić proved some results also for Vilenkin system in their paper [2].

In their paper [18] Goginava and Gogoladze introduced the concept of Vilenkin-Lebesgue point generalized the definition of Walsh-Lebesgue point which was introduced by Weisz [26]. A point $x \in G_m$ is a Vilenkin-Lebesgue point of $f \in L_1(G_m)$, if

$$\lim_{A\to\infty} \sum_{s=0}^{A-1} M_s \sum_{r_s=1}^{m_s-1} \int_{I_A(x-r_s e_s)} |f(t)-f(x)| \mathrm{d}\mu(t) = 0,$$

where $e_s := (0,\dots,0,1,0,\dots) \in G_m$ (only the sth coordinate is 1, the others are 0).

An important result of the Vilenkin-Fourier analysis, that if $f \in L_1(G_m)$, then $S_{M_n}(f) \to f$ and $\sigma_n(f) \to f$, as $n \to \infty$ almost everywhere on G_m. Moreover, it

is known for any integrable function that almost every point of G_m is a Vilenkin-Lebesgue point and for any such point x

$$S_{M_n}(f;x) \to f(x) \tag{2.1}$$

as $n \to \infty$ for any $f \in L_1(G_m)$.

Also with respect to Vilenkin systems D. Baramidze, Dvalashvili and Tutberidze [3] for Nörlund means, D. Baramidze, Gogolashvili and Nadirashvili [4] for weighted means proved theorems connected to Lebesgue points. Our main aim is to generalize statements of papers of D. Baramidze, Dvalashvili, Tutberidze [3] and of D. Baramidze, Gogolashvili, Nadirashvili [4], to matrix transform means.

3. Auxiliary results

It is well-known that if G_m is a bounded Vilenkin group, the $L_1(G_m)$ norm of the Fejér kernels are uniformly bounded. Namely, there exists a positive constant c that depends only on $\sup_{n\in\mathbb{N}} m_n$ such that

$$\|K_n\|_1 \leq c \tag{3.1}$$

holds for every $n \in \mathbb{P}$.

It is also a famous result, that the M_nth Dirichlet kernel has the following closed form

$$D_{M_n}(x) = \begin{cases} 0, & \text{if } x \notin I_n(0), \\ M_n, & \text{if } x \in I_n(0). \end{cases} \tag{3.2}$$

To prove our corollaries we also need the following result.

Theorem 1 (Goginava and Gogoladze [18], see also Nadirashvili, Persson, Tephnadze and Weisz [23])**.** *If $f \in L_1(G_m)$, then*

$$\lim_{n\to\infty} \sigma_n(f;x) = f(x)$$

holds for all Vilenkin-Lebesgue points of f.

For the Walsh version of the next Lemma (more general, with $\sum_{k=1}^{n} t_{k,n} = O(1)$ assumption instead of (3.3)) see [6]. See also [19].

Lemma 1. *For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \leq k \leq n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{3.3}$$

is satisfied.

a) Let the finite sequences $\{t_{k,n} : 1 \leq k \leq n\}$ of nonnegative numbers be non-increasing for all n, or

b) let the finite sequences $\{t_{k,n}\colon 1 \le k \le n\}$ *of nonnegative numbers be non-decreasing for all n and we suppose that*

$$t_{n,n} = O\left(\frac{1}{n}\right) \tag{3.4}$$

holds.

Then in both cases there exists absolute constant c, for which

$$\left\|K_n^T\right\|_1 < c$$

holds for every $n \in \mathbb{P}$.

Proof. From Abel transformation we get

$$K_n^T = \sum_{k=1}^{n} t_{k,n} D_k = \sum_{k=1}^{n-1} (t_{k,n} - t_{k+1,n}) k K_k + t_{n,n} n K_n,$$

so using (3.1)

$$\left\|K_n^T\right\|_1 \le \sum_{k=1}^{n-1} |\Delta t_{k,n}| k \|K_k\|_1 + t_{n,n} n \|K_n\|_1 \le c \left(\sum_{k=1}^{n-1} |\Delta t_{k,n}| k + t_{n,n} n \right).$$

It is easy to see that

$$\sum_{k=1}^{n-1} \Delta t_{k,n} k = \sum_{k=1}^{n} t_{k,n} - n t_{n,n}. \tag{3.5}$$

In case a) from (3.3) and (3.5)

$$\sum_{k=1}^{n-1} |\Delta t_{k,n}| k + t_{n,n} n = \sum_{k=1}^{n} t_{k,n} = 1.$$

In case b), using conditions (3.3), (3.4) and (3.5) we have

$$\sum_{k=1}^{n-1} |\Delta t_{k,n}| k + t_{n,n} n = 2 n t_{n,n} - \sum_{k=1}^{n} t_{k,n} = 2 n t_{n,n} - 1 \le 2c - 1 \le c.$$

□

4. Convergence in continuity or Vilenkin-Lebesgue points, almost everywhere convergence

Theorem 2. *For every* $n \in \mathbb{P}$, $\{t_{k,n}\colon 1 \le k \le n\}$ *be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{4.1}$$

is satisfied. Suppose that $f \in L_1(G_m)$ *and for some* $x \in G_m$ $\lim_{n\to\infty} \sigma_n(f;x) = f(x)$.

a) Let the finite sequences $\{t_{k,n}\colon 1 \le k \le n\}$ *of nonnegative numbers be non-increasing for all n and condition*

$$\lim_{n\to\infty} t_{1,n} = 0 \tag{4.2}$$

is satisfied, or

b) let the finite sequences $\{t_{k,n}\colon 1 \le k \le n\}$ *of nonnegative numbers be non-decreasing for all n. We also suppose that*

$$t_{n,n} = O\left(\frac{1}{n}\right) \tag{4.3}$$

holds.

Then in both cases

$$\lim_{n\to\infty} \sigma_n^T(f;x) = f(x).$$

Proof. From Abel transformation it is easy to get equality

$$\sum_{k=1}^{l} t_{k,n} = \sum_{k=1}^{l-1} \Delta t_{k,n} k + t_{l,n} l, \tag{4.4}$$

where $1 \le l \le n$ and a connection between T and Fejér means

$$\sigma_n^T(f;x) = \sum_{k=1}^{n-1} \Delta t_{k,n} k \sigma_k(f;x) + t_{n,n} n \sigma_n(f;x). \tag{4.5}$$

Using (4.1), (4.4) and (4.5) we can conclude

$$\sigma_n^T(f;x) - f(x) = \sum_{k=1}^{n-1} \Delta t_{k,n} k (\sigma_k(f;x) - f(x)) + t_{n,n} n (\sigma_n(f;x) - f(x)).$$

Fix $f \in L_1(G_m)$ and $x \in G_m$, for which $\lim_{n\to\infty} \sigma_n(f;x) = f(x)$.

Let us see case a). Then $|\Delta t_{k,n}| = \Delta t_{k,n}$.

$$\left|\sigma_n^T(f;x) - f(x)\right| \le \sum_{k=1}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| + t_{n,n} n |\sigma_n(f;x) - f(x)| =: I_n + II_n.$$

Since $\lim_{n\to\infty} |\sigma_n(f;x) - f(x)| = 0$, for any $\varepsilon > 0$ there exits $n_{a,1} \in \mathbb{P}$ such that

$$|\sigma_n(f;x) - f(x)| < \frac{\varepsilon}{2}, \tag{4.6}$$

when $n > n_{a,1}$, and there exists $A > 0$ constant, that

$$|\sigma_n(f;x) - f(x)| \le A \tag{4.7}$$

for every n. Then

$$I_n = \sum_{k=1}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)|$$

$$= \sum_{k=1}^{n_{a,1}-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| + \sum_{k=n_{a,1}}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)|$$
$$=: I_{n,1} + I_{n,2}.$$

For the first part using (4.4), (4.7) and the non-increasing monotony of sequence

$$I_{n,1} \le A \left(\sum_{k=1}^{n_{a,1}} t_{k,n} - t_{n_{a,1},n} n_{a,1} \right) \le A n_{a,1} t_{1,n}.$$

From condition (4.2) it follows that for any $\varepsilon > 0$ there exits $n_{a,2} \in \mathbb{P}$ such that $t_{1,n} < \varepsilon/(2An_{a,1})$ as $n_{a,2} < n$, so if $n_a := \max(n_{a,1}, n_{a,2}) < n$, then $I_{n,1} \le \varepsilon/2$.

Using (4.4), (4.6) and condition (4.1)

$$\begin{aligned} I_{n,2} &= \sum_{k=n_{a,1}}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| \\ &\le \frac{\varepsilon}{2} \sum_{k=n_{a,1}}^{n-1} \Delta t_{k,n} k \\ &\le \frac{\varepsilon}{2} \sum_{k=1}^{n-1} \Delta t_{k,n} k \\ &= \frac{\varepsilon}{2} (1 - t_{n,n} n) \le \frac{\varepsilon}{2}. \end{aligned}$$

Summarising our result we get $0 \le I_n \le \varepsilon$, if $n_a < n$, so $\lim_{n\to\infty} I_n = 0$.

On the other side, since the sequence is non-increasing, we obtain from condition (4.1), that

$$0 \le II_n \le \sum_{k=1}^{n} t_{k,n} |\sigma_n(f;x) - f(x)| = |\sigma_n(f;x) - f(x)| \to 0$$

as $n \to \infty$, so $\lim_{n\to\infty} II_n = 0$.

It yields that $I_n + II_n = \left|\sigma_n^T(f;x) - f(x)\right|$ also tends to zero in every point $x \in G_m$, for which $\lim_{n\to\infty} \sigma_n(f;x) = f(x)$.

Let us see case b). Then $|\Delta t_{k,n}| = -\Delta t_{k,n}$.

Assumption (4.3) means, that there exists absolute constant c, for which $nt_{n,n} \le c$.

Now since $\lim_{n\to\infty} |\sigma_n(f;x) - f(x)| = 0$, for any $\varepsilon > 0$ there exits $n_{b,1} \in \mathbb{P}$ such that

$$|\sigma_n(f;x) - f(x)| < \frac{\varepsilon}{2c}, \tag{4.8}$$

when $n > n_{b,1}$.

$$\left|\sigma_n^T(f;x) - f(x)\right| \le -\sum_{k=1}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| + t_{n,n} n |\sigma_n(f;x) - f(x)| =: III_n + IV_n.$$

Then

$$\begin{aligned} III_n &= -\sum_{k=1}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| \\ &= -\sum_{k=1}^{n_{b,1}-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| + \left(-\sum_{k=n_{b,1}}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| \right) \\ &=: III_{n,1} + III_{n,2}. \end{aligned}$$

In this case b) $\Delta t_{k,n} \leq 0$ is satisfied for all $k \in \{1,\dots,n\}$, therefore $0 \leq III_{n,1}$ and $0 \leq III_{n,2}$.

For the first part using (4.3), (4.4), (4.7) and the non-decreasing monotony of sequence $t_{k,n}$

$$III_{n,1} \leq A \left(t_{n_{b,1},n} n_{b,1} - \sum_{k=1}^{n_{b,1}} t_{k,n} \right) \leq A n_{b,1} t_{n_{b,1},n} \leq A n_{b,1} t_{n,n} \leq \frac{c}{n} A n_{b,1}.$$

It means, that for any $\varepsilon > 0$ let $n_b := \max(n_{b,1}, n_{b,2})$, where $n_{b,2} := [2cAn_{b,1}/\varepsilon] + 1$. (Here $[y]$ denotes the integer part of the real number y.) Then $III_{n,1} \leq \varepsilon/2$ holds for every $n_b < n$.

Using (4.3), (4.4) and (4.8) we get

$$\begin{aligned} III_{n,2} &= -\sum_{k=n_{b,1}}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| \\ &\leq -\frac{\varepsilon}{2c} \sum_{k=n_{b,1}}^{n-1} \Delta t_{k,n} k \\ &\leq -\frac{\varepsilon}{2c} \sum_{k=1}^{n-1} \Delta t_{k,n} k \\ &= \frac{\varepsilon}{2c} (t_{n,n} n - 1) \leq \frac{\varepsilon}{2c} \frac{c}{n} n = \frac{\varepsilon}{2}. \end{aligned}$$

Summarising our result we get $0 \leq III_n \leq \varepsilon$, if $n_b < n$, so $\lim_{n\to\infty} III_n = 0$.

On the other side, since the sequence is non-decreasing satisfying (4.3), it yields

$$0 \leq IV_n = t_{n,n} n |\sigma_n(f;x) - f(x)| \leq c |\sigma_n(f;x) - f(x)| \to 0$$

as $n \to \infty$, so $\lim_{n\to\infty} IV_n = 0$.

Consequently $III_n + IV_n = \left|\sigma_n^T(f;x) - f(x)\right|$ also tends to zero in every appropriate points x. □

Corollary 1. *For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \leq k \leq n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1$$

is satisfied.

a) Let the finite sequences $\{t_{k,n}\colon 1 \le k \le n\}$ *of nonnegative numbers be non-increasing for all n and condition*

$$\lim_{n\to\infty} t_{1,n} = 0$$

is satisfied, or

b) let the finite sequences $\{t_{k,n}\colon 1 \le k \le n\}$ *of nonnegative numbers be non-decreasing for all n. We also suppose that*

$$t_{n,n} = O\left(\frac{1}{n}\right)$$

holds.

Then in both cases

$$\lim_{n\to\infty} \sigma_n^T(f;x) = f(x)$$

for all continuity or Vilenkin-Lebesgue points x of $f \in L_1(G_m)$.

Proof. This statement is a simple consequence of Theorem 1 and Theorem 2. □

Remark 1. We get immediately, that with the conditions of Corollary 1

$$\sigma_n^T(f) \to f$$

almost everywhere.

5. Convergence in norm

Theorem 3. *Suppose that* $f \in L_1(G_m)$ *and for every* $n \in \mathbb{N}$, $\{t_{k,n}\colon 1 \le k \le n\}$ *be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1$$

is satisfied.

a) If the finite sequence $\{t_{k,n}\colon 1 \le k \le n\}$ *is non-increasing as a function of k for all fixed n and we also have*

$$\lim_{n\to\infty} t_{1,n} = 0$$

is satisfied, or

b) if the finite sequence $\{t_{k,n}\colon 1 \le k \le n\}$ *is non-decreasing as a function of k for all fixed n and the condition*

$$t_{n,n} = O\left(\frac{1}{n}\right),$$

then we have the L_1*-norm convergence* $\sigma_n^T(f) \to f$.

Proof. The proof is a simple consequence of Lemma 1 and the usual density argument.

The $\mathcal{P}$ set of Vilenkin polynomials is dense in $L_p(G_m)$ for each $1 \leq p < \infty$. Besides, for any Vilenkin polynomial $P \in \mathcal{P}$ we have $S_n(P) = P$ for sufficiently large n, say for $n \geq k_0$.

In both cases we have

$$\lim_{n\to\infty} t_{k,n} = 0$$

for any fixed k, so $t_{k,n} \to 0$ if $k < k_0$ and

$$\lim_{n\to\infty} \sum_{k=k_0}^{n} t_{k,n} = 1.$$

It follows

$$\sigma_n^T(P) = \sum_{k=1}^{k_0-1} t_{k,n} S_k(P) + \sum_{k=k_0}^{n} t_{k,n} P \to 0 + P = P$$

as $n \to \infty$, in norm and also everywhere. □

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