Inequalities and limit theorems for random allocations

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Abstract. Random allocations of balls into boxes are considered. Properties of the number of boxes containing a fixed number of balls are studied. A moment inequality is obtained. A merge theorem with Poissonian accompanying laws is proved. It implies an almost sure limit theorem with a mixture of Poissonian laws as limiting distribution. Almost sure versions of the central limit theorem are obtained when the parameters are in the central domain.

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1 Introduction

Let n balls be placed successively and independently into N boxes. Let $\mu_r(n, N)$ denote the number of boxes containing r balls. There are several theorems concerning the limit laws of $\mu_r(n, N)$ when the parameters belong to certain domains (see, e.g., Weiss (1958), Rényi (1962), Békéssy (1963), and the monograph Kolchin-Sevast'yanov-Chistyakov (1978)). It is known that if $n, N \to \infty$ in the central domain, then the limit of the standardized $\mu_r(n, N)$ is standard normal. In the left-hand and in the right-hand r-domains the limit of $\mu_r(n, N)$ is Poisson distribution. Strong laws of large numbers are obtained for $\mu_r(n, N)$ in Chuprunov-Fazekas (2005). Concerning the generalized allocation scheme see Kolchin (2003).

In this paper the most general result is the inequality in Theorem 2.1. It gives an upper bound for the L_2-distance of $\mu_r(n, N)$ and its conditional expectation given the last $n-k$ allocations.

Then asymptotic results are considered. The most interesting case is the Poisson-type limiting distribution. In that case we do not have one single limiting distribution.

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Instead of a limit theorem we can prove a merge theorem, i.e. we can give a family of accompanying Poissonian laws being close to the original distributions (Theorem 2.2).

Then we obtain almost sure (a.s.) versions of the limit theorems for $\mu_r(n, N)$. The general form of the a.s. limit theorem is the following. Let Y_n, $n \in \mathbb{N}$, be a sequence of random elements defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A.s. limit theorems state that

$$\frac{1}{D_n} \sum\nolimits_{k=1}^{n} d_k \delta_{Y_k(\omega)} \Rightarrow \nu\,, \quad \text{as} \quad n \to \infty, \quad \text{for almost every} \quad \omega \in \Omega\,, \tag{1.1}$$

where δ_x is the unit mass at point x and $\Rightarrow \nu$ denotes weak convergence to the probability measure ν. In the simplest form of the a.s. CLT $Y_k = (X_1 + \cdots + X_k)/\sqrt{k}$, where $X_1, X_2, \ldots,$ are i.i.d. real random variables with mean 0 and variance 1, $d_k = 1/k$, $D_n = \log n$, and ν is the standard normal law $\mathcal{N}(0, 1)$; see Berkes (1998) for an overview. Recently, several papers are devoted to the background, the general forms and certain special cases of the a.s. limit theorem, see e.g. Berkes-Csáki (2001), Fazekas and Rychlik (2002), Matula (2005), Hörmann (2006), Orzóg-Rychlik (2007).

The present paper can be considered as an extension of some results in the paper Fazekas-Chuprunov (2005) where a.s. limit theorems were obtained for the number of empty boxes (see also Becker-Kern (2007)). In Section 2, we consider an appropriate representation of $\mu_r(n, N)$ in terms of independent, uniformly distributed random variables in order to handle the dependence structure inside the array $\mu_r(n, N)$, $n, N = 1, 2, \ldots.$ As $\mu_r(n, N)$ depends on two parameters, we consider a.s. limit theorems of the form

$$\frac{1}{D_n} \sum\nolimits_{(k,K)\in T_n} d_{kK} \delta_{Y_{kK}(\omega)} \Rightarrow \nu\,, \quad \text{as} \quad n, N \to \infty, \quad \text{for almost every} \quad \omega \in \Omega \tag{1.2}$$

where T_n denotes a twodimensional domain. To prove the above type theorems, we apply a general a.s. limit theorem, i.e. Theorem 2.1 of Fazekas-Chuprunov (2005). We quote it in Theorem 4.1. This result is an extension of known general a.s. limit theorems. (see, e.g., Fazekas and Rychlik (2002)). We remark that multiindex versions of a.s. limit theorems were obtained in Fazekas-Rychlik (2003). However, as the weights there are of product type, we can not apply those results for domains like $\{(k, i) : \alpha_1(k) \leq i \leq \alpha_2(k),\ k \in \mathbb{N}\}$.

In this paper we use the general theorem to obtain Theorems 2.3, 2.4, 2.5, and 2.6. Among them Theorems 2.5, 2.6 concern the central domain (i.e. when $0 < \alpha_1 \leq n/N \leq \alpha_2 < \infty$) and the limiting distribution is standard normal. In Theorem 2.4 the parameters can vary in a domain not included in the central domain but the limiting distribution is again standard normal. The most interesting case is the Poisson-type limiting distribution (Theorem 2.3). The limiting distribution in the almost sure limit theorem (i.e. in Theorem 2.3) will be a mixture of the accompanying laws in the usual limit theorem (i.e. in Theorem 2.2). In almost sure limit theory the above situation is well-known (see Fazekas-Chuprunov (2007) for semistable laws, see also Theorems 2.10, 2,11, 2.12 in Fazekas-Chuprunov (2005) for random allocations).

2 Main results

Random allocations. Let ξ, ξ_j, $j \in \mathbb{N}$, be independent random variables uniformly distributed on $[0,1]$. Let $N \in \mathbb{N}$. Consider the subdivision of the interval $[0,1)$ into the subintervals $\Delta_i = \Delta_{Ni} = [\frac{i-1}{N}, \frac{i}{N})$, $1 \le i \le N$.

We consider the intervals Δ_i, $i = 1, \dots, N$, as a row of boxes. Random variables ξ_j, $j = 1, 2, \dots$, are realizations of ξ. Each realization of ξ we treat as a random allocation of one ball into the N boxes. The event $\xi_j \in \Delta_i$ means that the jth ball falls into the ith box. Let $n \in \mathbb{N}$, $A_{(0)} = \{1, 2, \dots, n\}$.

$$\mu_r(n, N) = \sum_{i=1}^{N} \sum_{|A|=r, A \subset A_{(0)}} \prod_{j \in A} I_{\{\xi_j \in \Delta_i\}} \prod_{j \in A_{(0)} \setminus A} I_{\{\xi_j \notin \Delta_i\}} \tag{2.1}$$

is the number of boxes containing r balls and $N C_n^r \frac{1}{N^r} \left(1 - \frac{1}{N}\right)^{n-r}$ is its expectation. Here $C_n^r = \binom{n}{r}$ is the binomial coefficient.

For $n, N \in \mathbb{N}$ we will use the notation $\alpha = \frac{n}{N}$ and $p_r(\alpha) = (\alpha^r / r!) e^{-\alpha}$. It is known (see Kolchin et al. (1978), Ch.2, Sec.1, Theorem 1) that the following limit relations (2.2) and (2.3) hold for any fixed r, t and if $n, N \to \infty$ such that $\alpha = \mathrm{o}(N)$. For the expectation we have

$$\mathbb{E}\mu_r(n, N) = N p_r(\alpha) + p_r(\alpha) \left(r - \alpha/2 - C_r^2/\alpha\right) + \mathrm{O}\left(1/N\right) \tag{2.2}$$

and for the covariances we have

$$\operatorname{cov}(\mu_r(n, N), \mu_t(n, N)) \sim N \sigma_{rt}(\alpha), \tag{2.3}$$

where

$$\sigma_{rr}(\alpha) = p_r(\alpha) \left(1 - p_r(\alpha) - p_r(\alpha)(\alpha - r)^2/\alpha\right),$$
$$\sigma_{rt}(\alpha) = -p_r(\alpha) p_t(\alpha) \left(1 + (\alpha - r)(\alpha - t)/\alpha\right), \quad \text{if} \quad t \ne r.$$

We shall use the notation

$$\mathbb{D}_{n,N}^{(r)} = \sqrt{\mathbb{D}^2 \mu_r(n, N)} = \sqrt{\operatorname{cov}(\mu_r(n, N), \mu_r(n, N))}.$$

We shall need a lower bound for $\mathbb{D}_{n,N}^{(r)}$, therefore the following remark will be useful.

Remark 2.1.

$$1 - p_r(\alpha) - p_r(\alpha) \frac{(\alpha - r)^2}{\alpha} \ge c_r > 0,$$

if $r \ge 2$ is fixed and α is arbitrary, or if $r = 0, 1$ and $\alpha \ge \alpha_0 > 0$. □

As in the theory of random allocations the roles of n and N are fixed, therefore we shall use the following notation for twodimensional indices: $(n, N), (k, K) \in \mathbb{N}^2$.

Let

$$S_{nN}^{(r)} = \frac{\mu_r(n, N) - \mathbb{E}\mu_r(n, N)}{\mathbb{D}_{n,N}^{(r)}}$$

be the standardized variable, where $(n,N)\in\mathbb{N}^2$.

The main inequality. Let $n,N,r\in\mathbb{N}$, $0\le k\le n$. Recall that n is the number of balls, N is the number of boxes. ξ_j denotes the jth ball, Δ_i denotes the ith box. We use the notation $A_{(k)}=\{k+1,\dots,n\}$, $k=0,1,\dots,n-1$. Let

$$\zeta_n=\zeta_{nN}=\sum_{i=1}^{N}\sum_{|A|=r,A\subseteq A_{(0)}}\prod_{j\in A}I_{\{\xi_j\in\Delta_i\}}\prod_{j\in A_{(0)}\setminus A}I_{\{\xi_j\notin\Delta_i\}}-NC_n^r\frac{1}{N^r}\left(1-\frac{1}{N}\right)^{n-r}.$$

We see that $\zeta_n=\mu_r(n,N)-\mathbb{E}\mu_r(n,N)$, c.f. (2.1). We have

$$\zeta_n=\sum\nolimits_{i=1}^{N}\sum\nolimits_{|A|=r,A\subseteq A_{(0)}}(\eta_{iA}-\mathbb{E}\eta_{iA})$$

where

$$\eta_{iA}=\prod\nolimits_{j\in A}I_{\{\xi_j\in\Delta_i\}}\prod\nolimits_{j\in A_{(0)}\setminus A}I_{\{\xi_j\notin\Delta_i\}}$$

is the indicator of the event that the ith box contains the balls with indices in the set A (and it does not contain any other ball). Let $\mathcal{F}_{kn}$ be the σ-algebra generated by $\xi_{k+1},\dots,\xi_n$. We will use the following conditional expectations $\eta_{iA}^{(k)}=\mathbb{E}(\eta_{iA}|\mathcal{F}_{nk})$ and

$$\zeta_n^k=\zeta_{nN}^k=\mathbb{E}(\zeta_n|\mathcal{F}_{kn})=\sum_{i=1}^{N}\sum_{|A|=r,A\subseteq A_{(0)}}(\eta_{iA}^{(k)}-\mathbb{E}\eta_{iA}^{(k)})= \tag{2.4}$$

$$=\sum_{i=1}^{N}\sum_{|A|=r,A\subseteq A_{(0)}}\left(\frac{1}{N^{r-|A\cap A_{(k)}|}}\left(1-\frac{1}{N}\right)^{k-(r-|A\cap A_{(k)}|)}\prod_{j\in A\cap A_{(k)}}I_{\{\xi_j\in\Delta_i\}}\prod_{j\in A_{(k)}\setminus A}I_{\{\xi_j\notin\Delta_i\}}-\right.$$
$$\left.-\frac{1}{N^r}\left(1-\frac{1}{N}\right)^{n-r}\right).$$

The following inequality will play an important role in the proofs of our theorems.

Theorem 2.1. *Let $0<k<n$, $0<r\le n$ and N be fixed. Then we have*

$$\mathbb{E}(\zeta_n-\zeta_n^k)^2\le ck\alpha^{r-1}\left[\left(1-\frac{1}{N}\right)^{n+k}\alpha^r+\left(1-\frac{1}{N}\right)^{n-r}\right](\alpha+1) \tag{2.5}$$

where $c<\infty$ does not depend on n, N, and k, but can depend on r.

Remark 2.2. In Fazekas-Chuprunov (2005) the following inequality was obtained for the number of empty boxes. Let $r=0$. Let $k<n$ and N be fixed. Then we have

$$\mathbb{E}(\zeta_n-\zeta_n^k)^2\le k\left(1-\frac{1}{N}\right)^{n-k} \tag{2.6}$$

and

$$\mathbb{E}(\zeta_n-\zeta_n^k)^2\le\frac{kn}{N}. \tag{2.7}$$

In Chuprunov-Fazekas (2005) a fourth moment inequality was obtained for $\mu_r(n, N)$.

Limit theorems for random allocations for $r \geq 2$. First we consider the Poisson limiting distribution. In that case we do not have one single limiting distribution in the ordinary limit theorem. Instead of a limit theorem we can prove a merge theorem, i.e. we can give a family of accompanying laws being close to the original distributions (Theorem 2.2). The limiting distribution in the almost sure limit theorem (i.e. in Theorem 2.3) will be a mixture of the accompanying laws.

The following result is a version of Theorem 3 in Section 3, Chapter II of Kolchin-Sevast'yanov-Chistyakov (1978). In our theorem the novelty is that we state uniformity with respect to (n, N) in a certain domain, while l remains fixed.

Theorem 2.2. *Let $r \geq 2$ and $l \in \mathbb{N}$ be fixed. Then, as $n, N \to \infty$,*

$$\mathbb{P}(\mu_r(n, N) = l) = \frac{1}{l!}(Np_r)^l e^{-Np_r}(1 + \mathrm{o}(1)) \tag{2.8}$$

uniformly with respect to the domain $T = \{(n, N) \,:\, N \geq n^{(2r-1)/(2r-2)} \log n\}$.

Now turn to the a.s. version of Theorem 2.2.

Theorem 2.3. *Let $r \geq 2$, $0 < \lambda_1 < \lambda_2 < \infty$ be fixed. Let T_n be the following domain in $\mathbb{N}^2$*

$$T_n = \left\{ (k, K) \in \mathbb{N}^2 : k \leq n, \quad \lambda_1 \leq \frac{k}{K^{1-\frac{1}{r}}} \leq \lambda_2 \right\}.$$

Let

$$Q_n(\omega) = \frac{1}{\frac{r}{r-1}(\lambda_2 - \lambda_1)\log n} \sum_{(k,K)\in T_n} \frac{1}{K^{2-\frac{1}{r}}} \delta_{\mu_r(k,K)(\omega)}.$$

Then, as $n \to \infty$,

$$Q_n(\omega) \Rightarrow \mu_\tau$$

for almost all $\omega \in \Omega$, where τ is a random variable with distribution

$$\mathbb{P}(\tau = l) = \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \frac{1}{l!} \left(\frac{x^r}{r!}\right)^l e^{-\frac{x^r}{r!}} dx, \quad l = 0, 1, \ldots \ . \tag{2.9}$$

Now consider the case of the normal limiting distribution.

Theorem B. *Let $r \geq 2$ be fixed. If $n, N \to \infty$ such that $Np_r(\alpha) \to \infty$, then $S_{nN}^{(r)} \Rightarrow \gamma$.* □

Here and in the following γ denotes the standard normal law. The proof of Theorem B can be found in the monograph Kolchin et al. (1978), Ch.2, Sec.3, Theorem 4.

Consider an almost sure version of Theorem B.

Theorem 2.4. *Let $r \geq 2$ be fixed, $0 < \alpha_1, \alpha_2 < \infty$ and*

$$T_n = \left\{ (k, K) \in \mathbb{N}^2 \,:\, k \leq n, \alpha_1 k \leq K \leq \alpha_2 k^{(2r+1)/(2r)} \right\}.$$

Let

$$Q_n^{(r)+}(\omega) = \frac{1}{\log n} \sum_{(k,K)\in T_n} \frac{1}{k(\log\alpha_2 - \log\alpha_1 + (1/2r)\log k)K} \delta_{S_{kK}^{(r)}(\omega)} .$$

Then, as $n \to \infty$, we have

$$Q_n^{(r)+}(\omega) \Rightarrow \gamma, \quad \textit{for almost every} \quad \omega \in \Omega.$$

Almost sure limit theorems for random allocations in the central domain. If $n, N \to \infty$ such that

$$0 < \alpha_1 \le \frac{n}{N} \le \alpha_2 < \infty,$$

where α_1 and α_2 are some constants, then it is said that $n, N \to \infty$ in a **central domain**. In a central domain we have the following central limit theorem.

Theorem A. *Let $0 < \alpha_1 < \alpha_2 < \infty$. If $n, N \to \infty$ such that $\alpha = \frac{n}{N} \in [\alpha_1, \alpha_2]$, then $S_{nN}^{(r)} \Rightarrow \gamma$.* □

The proof of Theorem A can be found in the monograph Kolchin et al. (1978), Ch.2, Sec.2, Theorem 4.

Consider almost sure versions of Theorem A. In the following theorems the domain is narrower than the one in Theorem 2.4, but they are valid for arbitrary $r \ge 0$.

Theorem 2.5. *Let $r \ge 0$ be fixed, $0 < \alpha_1 < \alpha_2 < \infty$ and*

$$Q_n^{(r)}(\omega) = \frac{1}{(\log\alpha_2 - \log\alpha_1)\log n} \sum_{k\le n} \sum_{\{K\,:\,\alpha_1 \le \frac{k}{K} \le \alpha_2\}} \frac{1}{kK} \delta_{S_{kK}^{(r)}(\omega)} .$$

Then, as $n \to \infty$, we have

$$Q_n^{(r)}(\omega) \Rightarrow \gamma, \quad \textit{for almost every} \quad \omega \in \Omega.$$

In the above theorem the limit was considered for $n \to \infty$ (and the indices of the summands were in a fixed central domain). The following theorem is a two-index limit theorem, i.e. $n \to \infty$ and $N \to \infty$. The relation of n and N could be arbitrary however, as the indices of the summands are in a fixed central domain, we assume that (n, N) is in the central domain considered.

Theorem 2.6. *Let $r \ge 0$ be fixed, $0 < \alpha_1 < \alpha_2 < \infty$ and*

$$Q_{nN}^{(r)}(\omega) = \frac{1}{(\log\alpha_2 - \log\alpha_1)\log n} \sum_{k\le n} \sum_{\{K\,:\,K\le N,\, \alpha_1 \le \frac{k}{K} \le \alpha_2\}} \frac{1}{kK} \delta_{S_{kK}^{(r)}(\omega)} .$$

Then, as $n, N \to \infty$ so that $\alpha_1 \le \frac{n}{N} \le \alpha_2$, we have

$$Q_{nN}^{(r)}(\omega) \Rightarrow \gamma, \quad \textit{for almost every} \quad \omega \in \Omega.$$

3 Proof of Theorem 2.1

Since $\mathbb{E}\eta_{iA} = \mathbb{E}\eta_{iA}^{(k)}$ and $\mathbb{E}(\eta_{i_1A_1} - \eta_{i_1A_1}^{(k)})(\eta_{i_2A_2} - \eta_{i_2A_2}^{(k)}) = \mathbb{E}(\eta_{i_1A_1}\cdot\eta_{i_2A_2}) - \mathbb{E}(\eta_{i_1A_1}^{(k)}\cdot\eta_{i_2A_2}^{(k)})$ for any A_1, A_2, we have

$$\mathbb{E}(\zeta_n - \zeta_n^k)^2 = \left[\sum\nolimits_{i=1}^N \sum\nolimits_{|A|=r, A\subseteq A_{(0)}} (\eta_{iA} - \eta_{iA}^{(k)})\right]^2 =$$

$$= \sum_{i_1,i_2=1}^N \left[\sum_{|A_1|=|A_2|=r, A_1,A_2\subset A_{(0)}} \mathbb{E}(\eta_{i_1A_1} - \eta_{i_1A_1}^{(k)})(\eta_{i_2A_2} - \eta_{i_2A_2}^{(k)})\right] =$$

$$= \sum_{i_1\neq i_2} \left[\sum_{A_1\cap A_2\neq\emptyset, |A_1|=|A_2|=r, A_1,A_2\subset A_{(0)}} \Big(\mathbb{E}(\eta_{i_1A_1}\cdot\eta_{i_2A_2}) - \mathbb{E}(\eta_{i_1A_1}^{(k)}\cdot\eta_{i_2A_2}^{(k)})\Big)\right] +$$

$$+ \sum_{i_1\neq i_2} \left[\sum_{A_1\cap A_2=\emptyset, |A_1|=|A_2|=r, A_1,A_2\subset A_{(0)}} \Big(\mathbb{E}(\eta_{i_1A_1}\cdot\eta_{i_2A_2}) - \mathbb{E}(\eta_{i_1A_1}^{(k)}\cdot\eta_{i_2A_2}^{(k)})\Big)\right] +$$

$$+ \sum_{i=1}^N \left[\sum_{A_1\neq A_2, |A_1|=|A_2|=r, A_1,A_2\subset A_{(0)}} \Big(\mathbb{E}(\eta_{iA_1}\cdot\eta_{iA_2}) - \mathbb{E}(\eta_{iA_1}^{(k)}\cdot\eta_{iA_2}^{(k)})\Big)\right] +$$

$$+ \sum_{i=1}^N \left[\sum_{|A|=r, A\subset A_{(0)}} \Big(\mathbb{E}(\eta_{iA})^2 - \mathbb{E}(\eta_{iA}^{(k)})^2\Big)\right] = B_1 + B_2 + B_3 + B_4.$$

First consider B_1. Let $i_1 \neq i_2$, $A_1 \cap A_2 \neq \emptyset$ and $j \in A_1 \cap A_2$. Then $I_{\{\xi_j\in\Delta_{i_1}\}} I_{\{\xi_j\in\Delta_{i_2}\}} = 0$, therefore $\mathbb{E}(\eta_{i_1A_1}\eta_{i_2A_2}) = 0$. So $B_1 \leq 0$.

Now turn to B_3. Now $i_1 = i_2$, $A_1 \neq A_2$. If $j \in A_1 \setminus A_2$ or $j \in A_2 \setminus A_1$, then $I_{\{\xi_j\in\Delta_{i_1}\}} I_{\{\xi_j\notin\Delta_{i_2}\}} = 0$ or $I_{\{\xi_j\notin\Delta_{i_1}\}} I_{\{\xi_j\in\Delta_{i_2}\}} = 0$. So $\mathbb{E}(\eta_{i_1A_1}\cdot\eta_{i_2A_2}) = 0$. Therefore we have $B_3 \leq 0$.

Now consider B_2. Let $i_1 \neq i_2$ and $A_1 \cap A_2 = \emptyset$. It holds that

$$\mathbb{E}(\eta_{i_1A_1}\eta_{i_2A_2}) - \mathbb{E}(\eta_{i_1A_1}^{(k)}\eta_{i_2A_2}^{(k)}) = \frac{1}{N^{2r}}\left(1 - \frac{2}{N}\right)^{n-2r} -$$

$$-\frac{1}{N^{2r}}\left(1 - \frac{1}{N}\right)^{2k-(2r-|A_{(k)}\cap A_1|-|A_{(k)}\cap A_2|)}\left(1 - \frac{2}{N}\right)^{n-k-|A_{(k)}\cap A_1|-|A_{(k)}\cap A_2|} =$$

$$= \frac{1}{N^{2r}}\left(\left(1 - \frac{2}{N}\right)^{n-2r} - \left(1 - \frac{1}{N}\right)^{2k-2r+x}\left(1 - \frac{2}{N}\right)^{n-k-x}\right).$$

Here $x = |A_{(k)} \cap A_1| + |A_{(k)} \cap A_2|$, so we have $0 \leq x \leq 2r, n-k$. Now let $a = \left(1 - \frac{2}{N}\right)$, $b = \left(1 - \frac{1}{N}\right)$. Then $0 < a < b < 1$, moreover $b^2 - a = 1/N^2$. First consider those terms from B_2 in which $x = 2r$. It means that $A_1, A_2 \subset A_{(k)}$. The number of these terms is $N(N-1)(n-k)!/(r!r!(n-k-2r)!)$. The magnitude of these terms is

$$\left|\frac{1}{N^{2r}}\left(\left(1-\frac{2}{N}\right)^{n-2r} - \left(1-\frac{1}{N}\right)^{2k}\left(1-\frac{2}{N}\right)^{n-k-2r}\right)\right| =$$

$$\left|\frac{1}{N^{2r}}a^{n-k-2r}(a^k - b^{2k})\right| \le \frac{1}{N^{2r}}a^{n-k-2r}kb^{2(k-1)}\frac{1}{N^2}.$$

(Above we applied the mean value theorem.) So the contribution of these terms is not greater than

$$N(N-1)\frac{(n-k)!}{r!r!(n-k-2r)!}\frac{1}{N^{2r}}\left(1-\frac{2}{N}\right)^{n-k-2r} k\left(1-\frac{1}{N}\right)^{2(k-1)}\frac{1}{N^2} = B_{21}.$$

Now turn to the remaining terms of B_2, i.e. the terms with $x < 2r$. The number of these terms is

$$N(N-1)\left(\frac{n!}{r!r!(n-2r)!} - \frac{(n-k)!}{r!r!(n-k-2r)!}\right) \le N(N-1)\frac{2rkn^{2r-1}}{r!r!} = B_{221}.$$

(Above we applied the following fact. If $0 \le b_i \le a_i \le c$ and $a_i - b_i \le l$ for $i = 1, 2, \ldots, s$, then $\prod_{i=1}^s a_i - \prod_{i=1}^s b_i \le slc^{s-1}$.) Using the mean value theorem, we obtain for the magnitudes of these terms that

$$\left|\frac{1}{N^{2r}}(a^{n-2r} - b^{2k-2r+x}a^{n-k-x})\right| = \left|\frac{1}{N^{2r}}a^{n-k-2r}\left((a^k - b^{2k}) + b^{2k}\left(1-\left(\frac{a}{b}\right)^{2r-x}\right)\right)\right| \le$$

$$\le \frac{1}{N^{2r}}a^{n-k-2r}\left(\left(1-\frac{1}{N}\right)^{2(k-1)}k\frac{1}{N^2} + b^{2k}(2r-x)\frac{1}{N-1}\right) =$$

$$= \frac{1}{N^{2r}}\left(1-\frac{2}{N}\right)^{n-k-2r}\left(\left(1-\frac{1}{N}\right)^{2(k-1)}k\frac{1}{N^2} + \left(1-\frac{1}{N}\right)^{2k}(2r-x)\frac{1}{N-1}\right) = B_{222}.$$

Therefore we have

$$B_2 \le B_{21} + B_{221}B_{222} \le c\frac{n^{2r}}{N^{2r}}\left(1-\frac{1}{N}\right)^{n+k-2r-2}k + c\frac{n^{2r-1}}{N^{2r}}\left(1-\frac{1}{N}\right)^{n+k-2r-2}k^2 +$$

$$+c\frac{n^{2r-1}}{N^{2r-1}}\left(1-\frac{1}{N}\right)^{n+k-2r}k \le c\alpha^{2r-1}\left(1-\frac{1}{N}\right)^{n+k}k(\alpha+1). \qquad (3.1)$$

Finally, consider B_4. Let $r_1 = |\{1, 2, \ldots, k\} \cap A| = r - |A \cap A_{(k)}|$. We have

$$B_4 = N\sum_{|A|=r, A\subset A_{(0)}}\left(\mathbb{E}(\eta_{iA})^2 - \mathbb{E}(\eta_{iA}^{(k)})^2\right) =$$

$$= N\sum_{|A|=r, A\subset A_{(0)}}\left(\frac{1}{N^r}\left(1-\frac{1}{N}\right)^{n-r} - \frac{1}{N^{2r_1}}\left(1-\frac{1}{N}\right)^{2(k-r_1)}\frac{1}{N^{r-r_1}}\left(1-\frac{1}{N}\right)^{n-k-(r-r_1)}\right)$$

$$= N \sum_{r_1=\max\{r-(n-k),0\}}^{\min\{k,r\}} C_k^{r_1} C_{n-k}^{r-r_1} \left(\frac{1}{N^r} \left(1-\frac{1}{N}\right)^{n-r} - \frac{1}{N^{r+r_1}} \left(1-\frac{1}{N}\right)^{n+k-r-r_1} \right) =$$

$$= N \sum_{r_1=\max\{r-(n-k),0\}}^{\min\{k,r\}} C_k^{r_1} C_{n-k}^{r-r_1} \frac{1}{N^r} \left(1-\frac{1}{N}\right)^{n-r} \left(1 - \frac{1}{N^{r_1}} \left(1-\frac{1}{N}\right)^{k-r_1}\right) \leq$$

$$\leq N \sum_{r_1=\max\{r-(n-k),0\}}^{\min\{k,r\}} \frac{k^{r_1}}{r_1!} \frac{n^{r-r_1}}{(r-r_1)!} \frac{1}{N^r} \left(1-\frac{1}{N}\right)^{n-r} \left(1 - \frac{1}{N^{r_1}} \left(1-\frac{1}{N}\right)^{k-r_1}\right) \leq$$

$$\leq N \sum_{r_1=0}^{r} \frac{k^{r_1}}{r_1!} \frac{n^{r-r_1}}{(r-r_1)!} \frac{1}{N^r} \left(1-\frac{1}{N}\right)^{n-r} \left(1 - \frac{1}{N^{r_1}} \left(1-\frac{1}{N}\right)^{k-\min\{r_1,k\}}\right).$$

Separating the term with $r_1 = 0$, then applying the mean value theorem, we obtain

$$B_4 \leq N \sum_{r_1=1}^{r} \frac{k^{r_1}}{r_1!} \frac{n^{r-r_1}}{(r-r_1)!} \frac{1}{N^r} \left(1-\frac{1}{N}\right)^{n-r} + N \frac{n^r}{r!} \frac{1}{N^r} \left(1-\frac{1}{N}\right)^{n-r} \left(1 - \left(1-\frac{1}{N}\right)^{k}\right) \leq$$

$$\leq k\alpha^{r-1} \left(1-\frac{1}{N}\right)^{n-r} \sum_{r_1=1}^{r} \left(\frac{k}{n}\right)^{r_1-1} \frac{1}{r_1!} + N \frac{\alpha^r}{r!} \frac{k}{N} \left(1-\frac{1}{N}\right)^{n-r} \leq$$

$$\leq k\alpha^{r-1} \left(1-\frac{1}{N}\right)^{n-r} \left(e + \frac{\alpha}{r!}\right). \qquad (3.2)$$

Now, inequalities (3.1) and (3.2) imply (2.5). □

4 Proofs of the limit theorems

Proof of Theorem 2.2. Consider i.i.d. random variables $\eta_1, \eta_2, \dots, \eta_N$ having Poisson distribution with parameter α. Let $\zeta_N = \eta_1 + \dots + \eta_N$. Consider also i.i.d. random variables $\eta_1^{(r)}, \eta_2^{(r)}, \dots, \eta_N^{(r)}$ having the following distribution

$$\mathbb{P}(\eta_i^{(r)} = l) = \mathbb{P}(\eta_i = l \mid \eta_i \neq r).$$

Let $\zeta_N^{(r)} = \eta_1^{(r)} + \dots + \eta_N^{(r)}$. By Lemma 1 at page 60 of Kolchin et al. (1978)

$$\mathbb{P}(\mu_r(n,N) = l) = \binom{N}{l} p_r^l (1-p_r)^{N-l} \frac{\mathbb{P}(\zeta_{N-l}^{(r)} = n - lr)}{\mathbb{P}(\zeta_N = n)} = F\frac{G}{H}, \qquad (4.1)$$

say. On the domain T, as $n, N \to \infty$, we have $\alpha \to 0$ and $p_r(\alpha) \to 0$. Therefore, concerning F, we have

$$\frac{\binom{N}{l} p_r^l (1-p_r)^{N-l}}{\frac{1}{l!}(Np_r)^l e^{-Np_r}} \sim \frac{(1-p_r)^N}{e^{-Np_r}}.$$

Taking logarithm, then applying Taylor's expansion, we obtain that $(1-p_r)^N / e^{-Np_r} \to 1$ (as $n, N \to \infty$) uniformly in T.

To handle G, we need the following result (Theorem 1 on p. 61 of Kolchin et al. (1978)). For $r \geq 2$, as $m \to \infty$ so that $\alpha m \to \infty$, we have

$$\mathbb{P}(\zeta_m^{(r)} = t) = \frac{1}{\sigma_r \sqrt{2\pi m}} e^{\frac{(t-m\alpha_r)^2}{2m\sigma_r^2}} (1 + \mathrm{o}(1))$$

uniformly with respect to $\frac{(t-m\alpha_r)}{\sigma_r\sqrt{m}}$ in any finite interval. Here

$$\alpha_r = \mathbb{E}\eta_i^{(r)} = \frac{\alpha - rp_r}{1-p_r}, \quad \sigma_r^2 = \mathbb{D}^2\eta_i^{(r)} = \frac{\alpha}{(1-p_r)^2}\left(1 - p_r - \frac{(\alpha - p_r)^2}{\alpha} p_r\right).$$

Therefore

$$G = \mathbb{P}(\zeta_{N-l}^{(r)} = n - lr) = \frac{1}{\sigma_r\sqrt{2\pi(N-l)}} e^{\frac{(n-lr-(N-l)\alpha_r)^2}{2(N-l)\sigma_r^2}} (1 + \mathrm{o}(1)).$$

By straightforward calculations we obtain $G \sim 1/\sqrt{2\pi(N-l)\alpha} \sim 1/\sqrt{2\pi n}$ uniformly in T. Finally, turn to H. As ζ_N has Poisson distribution, applying the Stirling formula, we obtain

$$H = \mathbb{P}(\zeta_N = n) = \frac{n^n}{n!} e^{-n} \sim \frac{1}{\sqrt{2\pi n}} \quad \text{uniformly.}$$

Substituting the asymptotic values of F, G, H into (4.1), we obtain (2.8). □

The proofs of our a.s. limit theorems are based on the following general a.s. limit theorem for two-dimensional domains (see Theorem 2.1 of Fazekas-Chuprunov (2005)). Actually the theorem is a version of Theorem 1.1 in Fazekas-Rychlik (2002). Let $\{\alpha_1(k)\}$ and $\{\alpha_2(k)\}$ be given integer valued sequences with $1 \leq \alpha_1(k) \leq \alpha_2(k) < \infty$, for $k \in \mathbb{N}$. Let (B, ϱ) be a complete separable metric space and let ζ_{ki}, $\alpha_1(k) \leq i \leq \alpha_2(k)$, $k \in \mathbb{N}$, be an array of random elements in B. Let μ_ζ denote the distribution of the random element ζ. Let $\log_+ x = \log x$, if $x \geq 1$ and $\log_+ x = 0$, if $x < 1$.

Theorem 4.1. *Assume that there exist $C > 0$, $\varepsilon > 0$; an increasing sequence of positive numbers c_n with $\lim_{n\to\infty} c_n = \infty$, $c_{n+1}/c_n = \mathrm{O}(1)$; and B-valued random elements ζ_{lj}^{ki}, for $k, i, l, j \in \mathbb{N}$, $k < l$, $\alpha_1(k) \leq i \leq \alpha_2(k)$, $\alpha_1(l) \leq j \leq \alpha_2(l)$, such that the random elements ζ_{ki} and ζ_{lj}^{ki} are independent for $k < l$ and for any i, j; and*

$$\mathbb{E}\{\varrho(\zeta_{lj}, \zeta_{lj}^{ki}) \wedge 1\} \leq C\,(c_k/c_l)^\beta\,, \tag{4.2}$$

for $k < l$ and for any i, j, where $\beta > 0$. Let $0 \leq d_k \leq \log(c_{k+1}/c_k)$, assume that $\sum_{k=1}^\infty d_k = \infty$. Assume that

$$d_k = \sum\nolimits_{i=\alpha_1(k)}^{\alpha_2(k)} d_{ki}$$

for each k, with nonnegative numbers d_{ki}. Let $D_n = \sum_{k=1}^n d_k$. Then for any probability distribution μ on the Borel σ-algebra of B the following two statements are equivalent

$$\frac{1}{D_n} \sum_{k=1}^n \sum_{i=\alpha_1(k)}^{\alpha_2(k)} d_{ki}\delta_{\zeta_{ki}(\omega)} \Rightarrow \mu\,, \quad \text{as} \quad n \to \infty, \quad \text{for almost every} \quad \omega \in \Omega\,; \tag{4.3}$$

$$\frac{1}{D_n}\sum_{k=1}^{n}\sum_{i=\alpha_1(k)}^{\alpha_2(k)} d_{ki}\mu_{\zeta_{ki}} \Rightarrow \mu\,, \quad as \quad n\to\infty\,. \tag{4.4}$$

Remark 4.1. If condition (4.2) is valid only for $1<k_0\le k<l$, then Theorem 4.1 remains valid. □

Now we can turn to the proofs of the a.s. limit theorems.

Proof of Theorem 2.3. Let $\zeta_{kK}=\mu_r(k,K)$. For $k<n$ let $\zeta_{nN}^{kK}=\zeta_n^k+\mathbb{E}\zeta_n^k$, where ζ_n^k is defined in (2.4). We show that ζ_{nN}^{kK} satisfies the conditions of Theorem 4.1. ζ_{nN}^{kK} and ζ_{kK} are independent for $k<n$. By Theorem 2.1, we have

$$\mathbb{E}\left(\zeta_{nN}-\zeta_{nN}^{kK}\right)^2 \le c_0 k\left(\frac{n}{N}\right)^{r-1} \le c_0\frac{k}{n}\left(\frac{n}{N^{1-1/r}}\right)^r \le c_0\frac{k}{n}\left(\lambda_2\right)^r$$

because $(n,N)\in T_n$. Therefore $d_k=c\frac{1}{k}$ is an appropriate choice for any positive constant c. Let $d_{kK}=\frac{1}{K^{2-1/r}}$ for (k,K) with $\lambda_1\le\frac{k}{K^{1-1/r}}\le\lambda_2$. Then

$$d_k=\sum d_{kK}=\sum_{\left\{K\,:\,(k/\lambda_2)^{\frac{r}{r-1}}\le K\le(k/\lambda_1)^{\frac{r}{r-1}}\right\}}\frac{1}{K^{2-1/r}}\approx\frac{r}{r-1}(\lambda_2-\lambda_1)\frac{1}{k}\,.$$

Therefore the above choice is possible. So, in Theorem 4.1, we can put

$$D_n=\sum_{k=1}^{n}d_k=\sum_{k=1}^{n}\frac{r}{r-1}(\lambda_2-\lambda_1)\frac{1}{k}\approx\frac{r}{r-1}(\lambda_2-\lambda_1)\log n.$$

Now we remark that we can apply Theorem 2.2 because the domain in that theorem is wider that the one in Theorem 2.3. According to Theorem 4.1 we have to prove that

$$F=\frac{r-1}{r(\lambda_2-\lambda_1)\log n}\sum_{k=1}^{n}\sum_{\left\{K\,:\,\lambda_1\le\frac{k}{K^{1-1/r}}\le\lambda_2\right\}}\frac{1}{K^{2-\frac{1}{r}}}\mathbb{P}(\mu_r(k,K)=l)\to\mathbb{P}(\tau=l) \tag{4.5}$$

where τ is defined in (2.9). It is easier to calculate F in a wider domain and then remove the surplus, that is

$$F=\cdots\sum_{K=1}^{N(n)}\sum_{\left\{k\,:\,\lambda_1\le\frac{k}{K^{1-1/r}}\le\lambda_2\right\}}\cdots-\cdots\sum_{K=(n/\lambda_2)^{\frac{r}{r-1}}}^{(n/\lambda_1)^{\frac{r}{r-1}}}\sum_{k=n}^{K^{\frac{r-1}{r}}\lambda_2}\cdots=A-B,$$

say, where $N(n)=(n/\lambda_1)^{\frac{r}{r-1}}$.

Now consider the following approximations. Since $\alpha=k/K\to0$ if $k,K\to\infty$ so that $\lambda_1\le\frac{k}{K^{1-1/r}}\le\lambda_2$, therefore $e^{-\alpha}\approx e^0=1$. So we have

$$Kp_r=K\frac{\alpha^r}{r!}e^{-\alpha}=K\frac{1}{r!}\left(\frac{k}{K}\right)^r e^{-k/K}\approx\frac{1}{r!}\left(\frac{k}{K^{1-1/r}}\right)^r.$$

Therefore we obtain

$$\frac{1}{l!}\sum_{\{k\,:\,\lambda_1\le \frac{k}{K^{1-1/r}}\le\lambda_2\}} \frac{1}{K^{1-\frac{1}{r}}}(Kp_r)^l e^{-Kp_r}\approx$$

$$\approx \frac{1}{l!}\sum_{\{k\,:\,\lambda_1\le \frac{k}{K^{1-1/r}}\le\lambda_2\}} \frac{1}{K^{1-\frac{1}{r}}}\left(\frac{1}{r!}\left(\frac{k}{K^{1-\frac{1}{r}}}\right)^r\right)^l \exp\left(-\frac{1}{r!}\left(\frac{k}{K^{1-\frac{1}{r}}}\right)^r\right)\approx$$

$$\approx \frac{1}{l!}\int_{\lambda_1}^{\lambda_2}\left(\frac{x^r}{r!}\right)^l e^{-\frac{x^r}{r!}}\,dx.$$

So we have

$$A\approx \frac{1}{\frac{r}{r-1}(\lambda_2-\lambda_1)\ln n}\sum_{K=1}^{N(n)}\frac{1}{K}\frac{1}{l!}\sum_{\{k\,:\,\lambda_1\le \frac{k}{K^{1-1/r}}\le\lambda_2\}} \frac{1}{K^{1-\frac{1}{r}}}(Kp_r)^l e^{-Kp_r}\approx$$

$$\approx \frac{1}{(\lambda_2-\lambda_1)}\frac{1}{l!}\int_{\lambda_1}^{\lambda_2}\left(\frac{x^r}{r!}\right)^l e^{-\frac{x^r}{r!}}\,dx.$$

For B we have

$$0\le B\le \frac{1}{c\log n}\sum_{K=(n/\lambda_2)^{\frac{r}{r-1}}}^{(n/\lambda_1)^{\frac{r}{r-1}}}\ \sum_{k=n}^{K^{\frac{r-1}{r}}\lambda_2}\frac{1}{K^{2-\frac{1}{r}}}\to 0$$

as $n\to\infty$. So the limit of F is the same as the limit of A. It proves (4.5). □

Proof of Theorem 2.4. Let $r\ge 2$ be fixed. Let $\zeta_{kK}=S_{kK}^{(r)}$. For $k<n$ let $\zeta_{nN}^{kK}=\zeta_n^k/\mathbb{D}_{nN}^{(r)}$, where ζ_n^k is defined in (2.4). We show that ζ_{nN}^{kK} satisfies the conditions of Theorem 4.1. ζ_{nN}^{kK} and ζ_{kK} are independent for $k<n$. As $r\ge 2$, by (2.3) and Remark 2.1, $CN\alpha^r e^{-\alpha}\le(\mathbb{D}_{nN}^{(r)})^2$, where $C>0$. Therefore, by Theorem 2.1, we have

$$\mathbb{E}\left(\zeta_{nN}-\zeta_{nN}^{kK}\right)^2\le c_0\frac{k}{n}(\alpha^{r+1}+1)\le c'\frac{k}{n}\,,$$

if $(n,N)\in T_{n,N}$. Therefore $d_k=c\frac{1}{k}$ is an appropriate choice for any positive constant c. Now let

$$d_{k,K}=\frac{1}{k}\frac{1}{\log\alpha_2-\log\alpha_1+\frac{1}{2r}\log k}\frac{1}{K}.$$

Then we have

$$\sum_{\{K\,:\,\alpha_1 k\le K\le \alpha_2 k^{(1+2r)/(2r)}\}} d_{k,K}\approx\frac{1}{k}=d_k\,.$$

So, in Theorem 4.1, we can put $D_n=\log n$.

If $n,N\to\infty$ so that $(n,N)\in T_{n,N}$, then $Np_r(\alpha)\to\infty$. So we can apply Theorem B. We obtain

$$\frac{1}{\log n}\sum_{(k,K)\in T_{n,N}} d_{kK}\,\mu_{S_{kK}^{(r)}}\Rightarrow\gamma\,,$$

as $n \to \infty$. So we can apply Theorem 4.1. □

Proof of Theorem 2.5. For $r = 0$ our result is Theorem 2.4 of Fazekas and Chuprunov (2005). Now let $r \geq 1$. Let $\zeta_{kK} = S_{kK}^{(r)}$. For $k < n$ let $\zeta_{nN}^{kK} = \zeta_n^k / \mathbb{D}_{nN}^{(r)}$, where ζ_n^k is defined in (2.4). We will show that ζ_{nN}^{kK} satisfies the conditions of Theorem 4.1. ζ_{nN}^{kK} and ζ_{kK} are independent for $k < n$. By (2.3) and Remark 2.1, in the central domain $CN \leq (\mathbb{D}_{nN}^{(r)})^2$, where C depends only on α_1 and α_2. Therefore, by Theorem 2.1, we have

$$\mathbb{E}\left(\zeta_{nN} - \zeta_{nN}^{kK}\right)^2 \leq c_0 \frac{k}{(\mathbb{D}_{nN}^{(r)})^2} \leq \frac{c_0}{C}\frac{k}{N} \leq \frac{c_0 \alpha_2}{C}\frac{k}{n}.$$

Therefore $d_k = c\frac{1}{k}$ is an appropriate choice for any positive constant c. Moreover, as

$$d_k = \frac{1}{k} \sum_{\{K : \frac{k}{\alpha_2} \leq K \leq \frac{k}{\alpha_1}\}} \frac{1}{K} \approx \frac{1}{k}(\log \alpha_2 - \log \alpha_1),$$

the above choice is possible. So, in Theorem 4.1, we can put $D_n = (\log \alpha_2 - \log \alpha_1)\log n$. By Theorem A,

$$\frac{1}{(\log \alpha_2 - \log \alpha_1)\log n} \sum_{k \leq n} \sum_{\{K : \alpha_1 \leq \frac{k}{K} \leq \alpha_2\}} \frac{1}{kK} \mu_{S_{kK}^{(r)}} \Rightarrow \gamma,$$

as $n \to \infty$. So we can apply Theorem 4.1. □

Proof of Theorem 2.6. Consider $Q_n^{(r)}$ from Theorem 2.5 and $Q_{nN}^{(r)}$. Their difference is

$$Q_n^{(r)}\omega) - Q_{nN}^{(r)}(\omega) = \frac{1}{(\log \alpha_2 - \log \alpha_1)\log n} \sum_{k \leq n} \sum_{\{K : K > N, \alpha_1 \leq \frac{k}{K} \leq \alpha_2\}} \frac{1}{kK} \delta_{S_{kK}^{(r)}(\omega)}.$$

As the summands are probability measures, we can confine attention to the weights. However, a direct calculation shows that

$$\sum_{k \leq n} \sum_{\{K : K > N, \alpha_1 \leq \frac{k}{K} \leq \alpha_2\}} \frac{1}{kK} \leq c(\log \alpha_2 - \log \alpha_1)^2.$$

Therefore, when for a fixed ω we have $Q_n^{(r)}(\omega) \Rightarrow \gamma$, as $n \to \infty$, then $Q_{nN}^{(r)}(\omega) \Rightarrow \gamma$, as $n, N \to \infty$. □

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