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Symmetrical Solutions for Non-Local Fractional Integro-Differential Equations via Caputo–Katugampola Derivatives

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Abstract: Fractional calculus, which deals with the concept of fractional derivatives and integrals, has become an important area of research, due to its ability to capture memory effects and non-local behavior in the modeling of real-world phenomena. In this work, we study a new class of fractional Volterra–Fredholm integro-differential equations, involving the Caputo–Katugampola fractional derivative. By applying the Krasnoselskii and Banach fixed-point theorems, we prove the existence and uniqueness of solutions to this problem. The modified Adomian decomposition method is used, to solve the resulting fractional differential equations. This technique rapidly provides convergent successive approximations of the exact solution to the given problem; therefore, we investigate the convergence of approximate solutions, using the modified Adomian decomposition method. Finally, we provide an example, to demonstrate our results. Our findings contribute to the current understanding of fractional integro-differential equations and their solutions, and have the potential to inform future research in this area.

Keywords: Katugampola operator; uniqueness of solutions; Banach space; integro-differential equations; existence theorem; Adomian decomposition; fractional operator; fixed point

MSC: 34A12; 34B18; 34A12; 47H10



Citation: Al-Ghafri, K.S.; Alabdala, A.T.; Redhwan, S.S.; Bazighifan, O.; Ali, A.H.; Iambor, L.F. Symmetrical Solutions for Non-Local Fractional Integro-Differential Equations via Caputo–Katugampola Derivatives. *Symmetry* **2023**, *15*, 662. <https://doi.org/10.3390/sym15030662>

Academic Editors: Hassen Fourati, Abdellatif Ben Makhlof and Omar Naifar

Received: 12 February 2023

Revised: 23 February 2023

Accepted: 4 March 2023

Published: 6 March 2023



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1. Introduction

Fractional calculus is a branch of mathematics that has gained significant attention in recent times, due to its wide range of applications in various fields. It deals with the concept of fractional derivatives and integrals, which are generalized versions of the standard notions of derivatives and integrals. These tools are useful for describing phenomena that exhibit memory or non-local behavior, such as those described by differential equations with power law kernels or fractional order operators. In [1,2], a generalized Riemann–Liouville fractional integral and corresponding fractional derivatives were introduced, which generalized the Riemann–Liouville and Hadamard integrals. The properties of the Katugampola fractional derivative (KFD), and its potential applications to quantum mechanics, were studied by Anderson et al. in [3]. Janaki et al. established the existence and uniqueness of solutions to impulsive differential equations with inclusions in [4], and also established conditions for the existence and uniqueness of solutions to a class of fractional

implicit differential equations with KFD, in [5]. Vivek et al. recently investigated the existence and stability of solutions to impulsive type integro-differential equations in [6], and in [7], the existence and Ulam stability of solutions for impulsive type pantograph equations were studied. Fractional differential equations have been widely applied in various fields, due to their ability to capture memory effects often observed in real-world systems. Examples of phenomena that have been modeled using these equations include anomalous diffusion, viscoelastic behavior, and the spread of epidemics. The following articles [8–10] and their references discuss a number of interesting and new findings on the existence of various types of FDEs.

Recently, Basim [11] investigated the following Caputo fractional Volterra–Fredholm integro-differential equation

$$\begin{cases} {}^C D_{0+}^{\nu} \omega(\varkappa) = g(\varkappa) + \Pi_1 \omega(\varkappa) + \Pi_2 \omega(\varkappa), \varkappa \in \hbar = [0, 1], \\ \omega(0) = \omega_0 + y(\omega), \end{cases}$$

where $0 < \nu < 1$, ${}^C D_{0+}^{\nu}$ is the Caputo fractional derivative of order ν , $g : \hbar \rightarrow \mathbb{R}$, $y : \mathbb{C}(\hbar, \mathbb{R}) \rightarrow \mathbb{R}$, $\chi_1, \chi_2 : \hbar \times \hbar \rightarrow \mathbb{R}$ are continuous functions, and $\mathcal{N}_1, \mathcal{N}_2 : \mathbb{R} \rightarrow \mathbb{R}$, $j = 1, 2$ are Lipschitz continuous functions. In short, they put

$$\Pi_1 \omega(\varkappa) := \int_0^{\varkappa} \chi_1(\varkappa, \xi) \mathcal{N}_1(\omega(\xi)) d\xi,$$

and

$$\Pi_2 \omega(\varkappa) := \int_0^1 \chi_2(\varkappa, \xi) \mathcal{N}_2(\omega(\xi)) d\xi.$$

Here, we confirm that the objective of the present study is to investigate the uniqueness and existence of the solution, by applying Banach’s and Krasnoselskii’s fixed-point theorems (FPTs), after which, we use the method of modified Adomian decomposition (MADM) for the Caputo–Katugampola fractional Volterra–Fredholm integro-differential equation (CK fractional VFIDE), which is given by

$$\begin{cases} {}^C D_{0+}^{\nu; \zeta} \omega(\varkappa) = g(\varkappa) + \Pi_1 \omega(\varkappa) + \Pi_2 \omega(\varkappa), \varkappa \in \hbar = [0, 1], \\ \omega(0) = \omega_0 + y(\omega), \end{cases} \tag{1}$$

where $0 < \nu < 1$, ${}^C D_{0+}^{\nu; \zeta}$ is the CK fractional derivative of order ν , with a parameter ζ , $g : \hbar \rightarrow \mathbb{R}$, $y : \mathbb{C}(\hbar, \mathbb{R}) \rightarrow \mathbb{R}$, $\chi_1, \chi_2 : \hbar \times \hbar \rightarrow \mathbb{R}$ are continuous functions, and $\mathcal{N}_1, \mathcal{N}_2 : \mathbb{R} \rightarrow \mathbb{R}$, $j = 1, 2$ are Lipschitz continuous functions. In brief, we put

$$\Pi_1 \omega(\varkappa) := \int_0^{\varkappa} \chi_1(\varkappa, \xi) \mathcal{N}_1(\omega(\xi)) d\xi,$$

and

$$\Pi_2 \omega(\varkappa) := \int_0^1 \chi_2(\varkappa, \xi) \mathcal{N}_2(\omega(\xi)) d\xi.$$

Amongst the other fractional derivatives, this new fractional differential operator (Caputo–Katugampola fractional derivative ${}^C D_{0+}^{\nu; \zeta}$) is advantageous, because it combines and unites the Caputo and Caputo–Hadamard fractional differential operators, and preserves some basic and fundamental properties of the Caputo and Caputo–Hadamard fractional derivatives; therefore, the Caputo–Katugampola fractional derivative is a generalization of the following fractional derivatives: standard Caputo ($\zeta \rightarrow 1$) [1]; Caputo–Hadamard ($\zeta \rightarrow 0^+$) [12]; Caputo–Liouville ($\zeta \rightarrow 1, a \rightarrow 0$) [1]; Caputo–Wey ($\zeta \rightarrow 1, a \rightarrow -\infty$) [13].

Additionally, numerous studies have been conducted, using George Adomian’s approach of Adomian decomposition (AD), to estimate the solution to this type of equation [14], and other numerical methods (see, for instance, [15–24]). The style and simplicity of the AD approach make it attractive. The answer is given as a series, where each equation may be calculated with ease, using Adomian polynomials that are appropriate for

nonlinear components (see [14,25–29]). In [30], Wazwaz introduced the MADM, which entails splitting the first term of the series into two second terms, one of which is kept, to define the second term of the series. This approach’s primary goals are to perform fewer operations, and to accelerate convergence on the precise solution to the stated problem. For instance, we quote [31] when discussing the application of the MADM. Many authors have used fixed-point methods to study findings on the presence of solutions to CK fractional differential equations: recent papers can be found at [32–34]. In this paper, we establish the existence and uniqueness solution of problem (1), using a contemporary methodology. We arrive at a few prerequisites that are necessary for fractional integro-differential equations with non-local conditions to obtain solutions. To acquire a rough solution, the MADM is utilized. The FPTs of Krasnoselskii and Banach are also used, to assess our findings.

1.1. Significance of This Paper

It appears that the issues indicated by the fractional operator are more difficult than those indicated by the ordinary operator. Some authors have recently considered the applications of fractional derivatives in a variety of scientific fields, such as the fractional Volterra–Fredholm integro-differential equation, the fractional quadratic integral equation, and mechanical applications. Among these are numerous works on numerical techniques for a specific class of fractional differential equations and other types of equations, such as [11,35,36].

1.2. Structurization of the Paper

The remainder of the paper is structured as follows. In Section 2, we review some fractional calculus notations, definitions, and lemmas that are relevant to our research. In Section 3, we provide an important lemma that enables us to convert the fractional Volterra–Fredholm integro-differential equation defined in Equation (1) into an equivalent integral equation. This section also contains the primary existence and uniqueness results for the problem (1), attained by applying the Krasnoselskii and Banach fixed-point theorems. In Section 4, we discuss the modified Adomian decomposition method, and prove that the series generated by this method converges to the exact solution of the problem. A numerical example is presented, to demonstrate the result, in Section 5. Concluding remarks are presented in Section 6.

2. Preliminaries

Denoting $C(\hbar, \mathbb{R})$ as the Banach space of all continuous functions on \hbar . For $z \in C(\hbar, \mathbb{R})$, we have

$$\|z\|_C = \sup_{\varkappa \in \hbar} |z(\varkappa)|$$

For $a < b, c \in \mathbb{R}^+$ and $1 \leq p < \infty$, define the function space

$$X_c^p(a, b) = \left\{ z : \hbar \rightarrow \mathbb{R} : \|z\|_{X_c^p} = \left(\int_a^b |\varkappa^c z(\varkappa)|^p \frac{d\varkappa}{\varkappa} \right)^{\frac{1}{p}} < \infty \right\},$$

for $p = \infty$,

$$\|z\|_{X_c^p} = \text{ess sup}_{a \leq \varkappa \leq T} [|\varkappa^c z(\varkappa)|].$$

Definition 1 ([37]). Let $v > 0, \varsigma > 0, c \in \mathbb{R}^+$ and $z \in X_c^p(a, b)$. Then, the definition of the Katugampola fractional integral of order v with parameter ς is given by

$$\mathcal{I}_{a^+}^{v;\varsigma} z(\varkappa) = \int_a^\varkappa \frac{\eta^{\varsigma-1}}{\Gamma(v)} \left(\frac{\varkappa^\varsigma - \eta^\varsigma}{\varsigma} \right)^{v-1} z(\eta) d\eta.$$

Now, when $\varsigma = 0$, we arrive at the standard Riemann–Liouville fractional integral, which is used to define both the Riemann–Liouville and Caputo fractional derivatives [1,2]. Using L’hopital rule, when $\varsigma \rightarrow 0^+$, we have

$$\begin{aligned} & \lim_{\varsigma \rightarrow 0^+} \int_a^{\varkappa} \frac{\eta^{\varsigma-1}}{\Gamma(\nu)} \left(\frac{\varkappa^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} z(\eta) d\eta \\ &= \frac{1}{\Gamma(\nu)} \int_a^{\varkappa} \lim_{\varsigma \rightarrow 0^+} \eta^{\varsigma-1} \left(\frac{\varkappa^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} z(\eta) d\eta \\ &= \frac{1}{\Gamma(\nu)} \int_a^{\varkappa} \left(\log \frac{\varkappa}{\eta} \right)^{\nu-1} \frac{z(\eta)}{\eta} d\eta. \end{aligned}$$

This is the well-known Hadamard fractional integral.

Definition 2 ([38]). Let $n - 1 < \nu < n$, ($n = [\nu] + 1$), $\varsigma > 0$, $c \in \mathbb{R}^+$ and $z \in X_c^p(a, b)$. Then, the definitions of Katugampola and the CK fractional derivative of order ν , with a parameter ς , are given by

$$\mathcal{D}_{a^+}^{\nu;\varsigma} z(\varkappa) = \left(\varkappa^{1-\varsigma} \frac{d}{d\varkappa} \right)^n \mathcal{I}_{a^+}^{n-\nu;\varsigma} z(\varkappa),$$

and

$$\mathcal{D}_{a^+}^{\nu;\varsigma} z(\varkappa) = \mathcal{I}_{a^+}^{n-\nu;\varsigma} z_\varsigma^{(n)}(\varkappa),$$

respectively, where $z_\varsigma^{(n)}(\varkappa) = \left(\varkappa^{1-\varsigma} \frac{d}{d\varkappa} \right)^n z(\varkappa)$. For example, if we take $z(\varkappa) = \varkappa^\tau$ where $\tau \in \mathbb{R}$, then the generalized derivative of the function $z(\varkappa)$ can be found as follows:

$$\mathcal{D}_{a^+}^{\nu;\varsigma} \varkappa^\tau = \frac{(\varsigma + 1)^\nu}{\Gamma(1-\nu)} \frac{d}{d\varkappa} \int_a^{\varkappa} \frac{\eta^\varsigma}{(\varkappa^{\varsigma+1} - \eta^{\varsigma+1})^\nu} \eta^\tau d\eta.$$

To evaluate the inner integral, we use the substitution $u = \frac{\eta^{\varsigma+1}}{\varkappa^{\varsigma+1}}$, to obtain

$$\begin{aligned} \int_a^{\varkappa} \frac{\eta^\varsigma}{(\varkappa^{\varsigma+1} - \eta^{\varsigma+1})^\nu} \eta^\tau d\eta &= \frac{\varkappa^{(\varsigma+1)(1-\nu)+\tau}}{\varsigma + 1} \int_0^1 \frac{u^{\frac{\tau}{\varsigma+1}}}{(1-u)^\nu} du \\ &= \frac{\varkappa^{(\varsigma+1)(1-\nu)+\tau}}{\varsigma + 1} \int_0^1 u^{\frac{\tau+\varsigma+1}{\varsigma+1}-1} (1-u)^{(1-\nu)-1} du \\ &= \frac{\varkappa^{(\varsigma+1)(1-\nu)+\tau}}{\varsigma + 1} B\left(1-\nu, \frac{\tau + \varsigma + 1}{\varsigma + 1}\right), \end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function.

Lemma 1 ([39]). Let $z \in \mathbb{C}^n[a, b]$, $\varsigma > 0$, Then

$$\left(\mathcal{I}_{a^+}^{\nu;\varsigma} \mathcal{C} \mathcal{D}_{a^+}^{\nu;\varsigma} \right) z(\varkappa) = z(\varkappa) - \sum_{k=0}^{n-1} \frac{\varsigma^{-k}}{k!} (\varkappa^\varsigma - a^\varsigma)^k z_\varsigma^{(n)}(a).$$

Lemma 2 ([37]). Let $\nu, \delta, \beta > 0$ and $z \in X_c^p(a, b)$. Then:

1. $\mathcal{I}_{a^+}^{\nu;\varsigma}$ is bounded on the function space $X_c^p(a, b)$;
2. $\mathcal{I}_{a^+}^{\nu;\varsigma} \mathcal{I}_{a^+}^{\beta;\varsigma} z(\varkappa) = \mathcal{I}_{a^+}^{\nu+\beta;\varsigma} z(\varkappa)$;
3. $\mathcal{I}_{a^+}^{\nu;\varsigma} \left(\frac{\varkappa^\varsigma - a^\varsigma}{\varsigma} \right)^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta+\nu)} \left(\frac{\varkappa^\varsigma - a^\varsigma}{\varsigma} \right)^{\nu+\delta-1}$.

3. Existence Result

We start by assuming the following:

(H₁) Let $\mathcal{N}_1(\omega(\varkappa)), \mathcal{N}_2(\omega(\varkappa))$ be continuous nonlinearity terms, and there exist constants $\ell_{\mathcal{N}_1} > 0$ and $\ell_{\mathcal{N}_2} > 0$, such that

$$|\mathcal{N}_j(\omega_1(\varkappa)) - \mathcal{N}_j(\omega_2(\varkappa))| \leq \ell_{\mathcal{N}_j} |\omega_1 - \omega_2|, j = 1, 2, \forall \omega_1, \omega_2 \in \mathbb{R};$$

(H₂) The kernels $\chi_1(\varkappa, \zeta)$ and $\chi_2(\varkappa, \zeta)$ are continuous on $\hbar \times \hbar$, and there exist two positive constants, χ_1^* and χ_2^* , in $\hbar \times \hbar$, such that

$$\chi_j^* = \sup_{\varkappa \in \hbar} \int_0^\varkappa |\chi_j(\varkappa, \zeta)| d\zeta < \infty, j = 1, 2;$$

(H₃) $g : \hbar \rightarrow \mathbb{R}$ is continuous on \hbar ;

(H₄) $y : \mathbb{C}(\hbar, \mathbb{R}) \rightarrow \mathbb{R}$ is continuous on $\mathbb{C}(\hbar)$, and there exists a constant $0 < \ell_y < 1$, such that

$$|y(\omega_1(\varkappa)) - y(\omega_2(\varkappa))| \leq \ell_y |\omega_1 - \omega_2|, \forall \omega_1, \omega_2 \in \mathbb{C}(\hbar, \mathbb{R}), \varkappa \in \hbar.$$

Problem (1) and the integral equation are equivalent, according to the next lemma. The proof for this lemma is disregarded, as it resembles some traditional arguments that are known from the literature.

Lemma 3. *The function $\omega \in \mathbb{C}(\hbar, \mathbb{R})$ is the CK fractional VFIDE’s (1) solution if and only if ω is the integral equation’s solution, which is given by*

$$\begin{aligned} \omega(\varkappa) = & \omega_0 + y(\omega) + \frac{1}{\Gamma(v)} \int_0^\varkappa \eta^{\zeta-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta} \right)^{v-1} g(\eta) d\eta \\ & + \frac{1}{\Gamma(v)} \int_0^\varkappa \eta^{\zeta-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta} \right)^{v-1} \left\{ \int_0^\eta \chi_1(\eta, \zeta) \mathcal{N}_1(\omega(\zeta)) d\zeta \right. \\ & \left. + \int_0^1 \chi_2(\eta, \zeta) \mathcal{N}_2(\omega(\zeta)) d\zeta \right\} d\eta. \end{aligned}$$

Our first result relates to existence based on the Krasnoselkii’s FPT [40].

Theorem 1. *If conditions (H₁)–(H₄) are met, then there is at least one solution on \hbar to the problem defined in Equation (1) if*

$$\Lambda_1 := \left(\ell_y + \frac{\sum_{j=1}^2 \ell_{\mathcal{N}_j} \chi_j^*}{\zeta^v \Gamma(v+1)} \right) < 1. \tag{2}$$

Proof. Think about the ball:

$$\mathcal{S}_\gamma = \{ \omega \in \mathbb{C}(\hbar, \mathbb{R}) : \|\omega\|_\infty \leq \gamma \} \subset \mathbb{C}(\hbar, \mathbb{R}). \tag{3}$$

Apparently, \mathcal{S}_γ is a subset that is closed, convex, non-empty, and of $\mathbb{C}(\hbar, \mathbb{R})$. Select γ in a way where $\gamma \geq \frac{\Lambda_2}{1-\Lambda_1}$, where $\Lambda_1 < 1$,

$$\Lambda_2 := \mu_0 + \frac{\mu_g + \sum_{j=1}^2 \mu_{\mathcal{N}_j} \chi_j^*}{\zeta^v \Gamma(v+1)}, \tag{4}$$

$\mu_g := \sup_{\varkappa \in [0,1]} |g(\varkappa)|$, $\mu_0 := |\omega_0| + \mu_y$, $\mu_y = |y(0)|$, $\mu_{\mathcal{N}_1} := |\mathcal{N}_1(0)|$, and $\mu_{\mathcal{N}_2} := |\mathcal{N}_2(0)|$.

Using Lemma 3, we can express the equivalent fractional integral equation for the problem defined in Equation (1) as an operator equation in the following form:

$$\omega = \mathbb{T}_1 \omega + \mathbb{T}_2 \omega, \quad \omega \in \mathcal{S}_\gamma \subset \mathbb{C}(\hbar, \mathbb{R}), \tag{5}$$

where \mathbb{T}_1 and \mathbb{T}_2 are two operators on \mathcal{S}_γ defined by

$$(\mathbb{T}_1\omega)(\varkappa) = \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\zeta-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta} \right)^{\nu-1} \left\{ \int_0^\eta \chi_1(\eta, \zeta) \mathcal{N}_1(\omega(\zeta)) d\zeta + \int_0^1 \chi_2(\eta, \zeta) \mathcal{N}_2(\omega(\zeta)) d\zeta \right\} d\eta,$$

and

$$(\mathbb{T}_2\omega)(\varkappa) = \omega_0 + y(\omega) + \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\zeta-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta} \right)^{\nu-1} g(\eta) d\eta.$$

Applying the conditions of Theorem 2, we can find the fixed point of the operator Equation (5), in the following way:

Step 1: We claim that $\mathbb{T}_1\omega + \mathbb{T}_2\omega \in \mathcal{S}_\gamma$ for each $\omega, \omega \in \mathcal{S}_\gamma$. By (H_1) , and for any $\omega, \omega \in \mathcal{S}_\gamma$, we have

$$\begin{aligned} |\mathcal{N}_j(\omega(\varkappa))| &\leq |\mathcal{N}_j(\omega(\varkappa)) - \mathcal{N}_j(0)| + |\mathcal{N}_j(0)| \\ &\leq \ell_{\mathcal{N}_j} \|\omega\|_\infty + |\mathcal{N}_j(0)| \\ &\leq \ell_{\mathcal{N}_j} \gamma + \mu_{\mathcal{N}_j}, \text{ for all } (j = 1, 2), \end{aligned}$$

and

$$\begin{aligned} |y(\omega(\varkappa))| &\leq |y(\omega(\varkappa)) - y(0)| + |y(0)| \\ &\leq \ell_y \|\omega\|_\infty + |y(0)| \\ &\leq \ell_y \gamma + \mu_y. \end{aligned}$$

Let $\omega, \omega \in \mathcal{S}_\gamma$. Then,

$$\begin{aligned} &|(\mathbb{T}_1\omega)(\varkappa) + (\mathbb{T}_2\omega)(\varkappa)| \\ &\leq \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\zeta-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta} \right)^{\nu-1} \left\{ \int_0^\eta \chi_1(\eta, \zeta) \mathcal{N}_1(\omega(\zeta)) d\zeta + \int_0^1 \chi_2(\eta, \zeta) \mathcal{N}_2(\omega(\zeta)) d\zeta \right\} d\eta \\ &\quad + |\omega_0| + |y(\omega)| + \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\zeta-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta} \right)^{\nu-1} g(\eta) d\eta \\ &\leq \mu_0 + \ell_y \gamma + \frac{\mu_g + \sum_{j=1}^2 (\ell_{\mathcal{N}_j} \gamma + \mu_{\mathcal{N}_j}) \chi_j^*}{\zeta^\nu \Gamma(\nu + 1)} \varkappa^{\nu\zeta}, \end{aligned}$$

which implies

$$\begin{aligned} &\|\mathbb{T}_1\omega + \mathbb{T}_2\omega\|_\infty \\ &\leq \mu_0 + \frac{\mu_g + \sum_{j=1}^2 \mu_{\mathcal{N}_j} \chi_j^*}{\zeta^\nu \Gamma(\nu + 1)} + \left(\ell_y + \frac{\sum_{j=1}^2 \ell_{\mathcal{N}_j} \chi_j^*}{\zeta^\nu \Gamma(\nu + 1)} \right) \gamma \\ &\leq \Lambda_2 + \Lambda_1 \gamma \leq \gamma. \end{aligned}$$

Consequently,

$$\mathbb{T}_1\omega + \mathbb{T}_2\omega \in \mathcal{S}_\gamma.$$

Step 2: We demonstrate that \mathbb{T}_2 is a contraction on \mathcal{S}_γ .

Let $\omega, \omega^* \in \mathcal{S}_\gamma$. It follows from (H₄) that

$$\begin{aligned} \|\mathbb{T}_2\omega - \mathbb{T}_2\omega^*\|_\infty &= \sup_{\varkappa \in \mathfrak{h}} |\mathbb{T}_2\omega(\varkappa) - \mathbb{T}_2\omega^*(\varkappa)| = \sup_{\varkappa \in \mathfrak{h}} |y(\omega(\varkappa)) - y(\omega^*(\varkappa))| \\ &\leq \ell_y \|\omega - \omega^*\|_\infty. \end{aligned}$$

This implies that \mathbb{T}_2 is a contraction mapping, as $\ell_y < 1$.

Step 3: We claim that \mathbb{T}_1 is completely continuous on \mathcal{S}_γ , which we will prove in three stages.

Stage 1—we prove that \mathbb{T}_1 is continuous. Let (ω_n) be a sequence, such that $\omega_n \rightarrow \omega$ in $\mathbb{C}(\mathfrak{h}, \mathbb{R})$. Then, for any $\varkappa \in \mathfrak{h}$, and for every $\omega_n, \omega \in \mathbb{C}(\mathfrak{h}, \mathbb{R})$, we deduce

$$\begin{aligned} &|(\mathbb{T}_1\omega_n)(\varkappa) - (\mathbb{T}_1\omega)(\varkappa)| \\ &\leq \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\nu-1} \left(\frac{\varkappa^\nu - \eta^\nu}{\nu}\right)^{\nu-1} \left(\int_0^\eta |\chi_1(\eta, \zeta)| |\mathcal{N}_1(\omega_n(\zeta)) - \mathcal{N}_1(\omega(\zeta))| d\zeta \right. \\ &\quad \left. + \int_0^1 |\chi_2(\eta, \zeta)| |\mathcal{N}_2(\omega_n(\zeta)) - \mathcal{N}_2(\omega(\zeta))| d\zeta \right) d\eta \\ &\leq \frac{\sum_{j=1}^2 \ell_{\mathcal{N}_j} \chi_j^*}{\nu \Gamma(\nu + 1)} \|\omega_n - \omega\|_\infty. \end{aligned}$$

As $\omega_n \rightarrow \omega$ as $n \rightarrow \infty$, $\|\mathbb{T}_1\omega_n - \mathbb{T}_1\omega\|_\infty \rightarrow 0$, as $n \rightarrow \infty$, this shows that \mathbb{T}_1 is continuous on $\mathbb{C}(\mathfrak{h}, \mathbb{R})$.

Stage 2—from Step 1, we observe that

$$\begin{aligned} &|(\mathbb{T}_1\omega)(\varkappa)| \\ &\leq \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\nu-1} \left(\frac{\varkappa^\nu - \eta^\nu}{\nu}\right)^{\nu-1} \left(\int_0^\eta |\chi_1(\eta, \zeta)| |\mathcal{N}_1(\omega(\zeta))| d\zeta \right. \\ &\quad \left. + \int_0^1 |\chi_2(\eta, \zeta)| |\mathcal{N}_2(\omega(\zeta))| d\zeta \right) d\eta \\ &\leq \frac{\sum_{j=1}^2 (\ell_{\mathcal{N}_j} \gamma + \mu_{\mathcal{N}_j}) \chi_j^*}{\nu \Gamma(\nu + 1)} \varkappa^{\nu}. \end{aligned}$$

Thus,

$$\|\mathbb{T}_1\omega\|_\infty \leq \frac{\sum_{j=1}^2 (\ell_{\mathcal{N}_j} \gamma + \mu_{\mathcal{N}_j}) \chi_j^*}{\nu \Gamma(\nu + 1)}.$$

This proves that $(\mathbb{T}_1\mathcal{S}_\gamma)$ is uniformly bounded.

Stage 3—we show that $(\mathbb{T}_1\mathcal{S}_\gamma)$ is equicontinuous. Let $\omega \in \mathcal{S}_\gamma$. Then, for $\varkappa_1, \varkappa_2 \in \mathfrak{h}$ with $\varkappa_1 \leq \varkappa_2$, we have

$$\begin{aligned} &|(\mathbb{T}_1\omega)(\varkappa_2) - (\mathbb{T}_1\omega)(\varkappa_1)| \\ &= \left| \frac{1}{\Gamma(\nu)} \int_0^{\varkappa_2} \eta^{\nu-1} \left(\frac{\varkappa_2^\nu - \eta^\nu}{\nu}\right)^{\nu-1} \left(\int_0^\eta |\chi_1(\eta, \zeta)| |\mathcal{N}_1(\omega(\zeta))| d\zeta \right. \right. \\ &\quad \left. \left. + \int_0^1 |\chi_2(\eta, \zeta)| |\mathcal{N}_2(\omega(\zeta))| d\zeta \right) d\eta \right. \\ &\quad \left. - \frac{1}{\Gamma(\nu)} \int_0^{\varkappa_1} \eta^{\nu-1} \left(\frac{\varkappa_1^\nu - \eta^\nu}{\nu}\right)^{\nu-1} \left(\int_0^\eta |\chi_1(\eta, \zeta)| |\mathcal{N}_1(\omega(\zeta))| d\zeta \right. \right. \\ &\quad \left. \left. + \int_0^1 |\chi_2(\eta, \zeta)| |\mathcal{N}_2(\omega(\zeta))| d\zeta \right) d\eta \right| \end{aligned}$$

$$\begin{aligned}
 & |(\mathbb{T}_1\omega)(\varkappa_2) - (\mathbb{T}_1\omega)(\varkappa_1)| \\
 \leq & \frac{1}{\Gamma(\nu)} \left(\int_{\varkappa_1}^{\varkappa_2} \eta^{\varsigma-1} \left(\frac{\varkappa_2^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} \int_0^\eta |\chi_1(\eta, \zeta)| |\mathcal{N}_1(\omega(\zeta))| d\zeta d\eta \right. \\
 & \left. + \int_0^{\varkappa_1} \eta^{\varsigma-1} \left| \left(\frac{\varkappa_2^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} - \left(\frac{\varkappa_1^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} \right| \int_0^\eta |\chi_1(\eta, \zeta)| |\mathcal{N}_1(\omega(\zeta))| d\zeta d\eta \right) \\
 & + \frac{1}{\Gamma(\nu)} \left(\int_{\varkappa_1}^{\varkappa_2} \eta^{\varsigma-1} \left(\frac{\varkappa_2^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} \int_0^\eta |\chi_2(\eta, \zeta)| |\mathcal{N}_2(\omega(\zeta))| d\zeta d\eta \right. \\
 & \left. + \int_0^{\varkappa_1} \eta^{\varsigma-1} \left| \left(\frac{\varkappa_2^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} - \left(\frac{\varkappa_1^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} \right| \int_0^\eta |\chi_2(\eta, \zeta)| |\mathcal{N}_2(\omega(\zeta))| d\zeta d\eta \right)
 \end{aligned}$$

which implies

$$\begin{aligned}
 |(\mathbb{T}_1\omega)(\varkappa_2) - (\mathbb{T}_1\omega)(\varkappa_1)| & \leq \frac{(\ell_{\mathcal{N}_1}\gamma + \mu_{\mathcal{N}_1})\chi_1^*}{\Gamma(\nu)} \left(\int_{\varkappa_1}^{\varkappa_2} \eta^{\varsigma-1} \left(\frac{\varkappa_2^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} d\eta \right. \\
 & \left. + \int_0^{\varkappa_1} \eta^{\varsigma-1} \left| \left(\frac{\varkappa_2^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} - \left(\frac{\varkappa_1^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} \right| d\eta \right) \\
 & + \frac{(\ell_{\mathcal{N}_2}\gamma + \mu_{\mathcal{N}_2})\chi_2^*}{\Gamma(\nu)} \left(\int_{\varkappa_1}^{\varkappa_2} \eta^{\varsigma-1} \left(\frac{\varkappa_2^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} d\eta \right. \\
 & \left. + \int_0^{\varkappa_1} \eta^{\varsigma-1} \left| \left(\frac{\varkappa_2^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} - \left(\frac{\varkappa_1^\varsigma - \eta^\varsigma}{\varsigma} \right)^{\nu-1} \right| d\eta \right) \\
 & \leq \left(\frac{(\ell_{\mathcal{N}_1}\gamma + \mu_{\mathcal{N}_1})\chi_1^*}{\varsigma^\nu \Gamma(\nu + 1)} + \frac{(\ell_{\mathcal{N}_2}\gamma + \mu_{\mathcal{N}_2})\chi_2^*}{\varsigma^\nu \Gamma(\nu + 1)} \right) \\
 & \times \left(\frac{(\varkappa_2^\varsigma - \varkappa_1^\varsigma)^\nu}{\nu} + \frac{\varkappa_1^\varsigma}{\nu} - \frac{\varkappa_2^\varsigma}{\nu} + \frac{(\varkappa_2^\varsigma - \varkappa_1^\varsigma)^\nu}{\nu} \right) \\
 & \leq \frac{2 \sum_{j=1}^2 (\ell_{\mathcal{N}_j}\gamma + \mu_{\mathcal{N}_j})\chi_j^*}{\varsigma^\nu \Gamma(\nu + 1)} (\varkappa_2^\varsigma - \varkappa_1^\varsigma)^\nu,
 \end{aligned}$$

which tends to zero, as $\varkappa_2 - \varkappa_1 \rightarrow 0$. Thus, $\mathbb{T}_1\mathcal{S}_\gamma$ is equicontinuous. Consequently, by the Arzela–Ascoli alternative, the operator \mathbb{T}_1 is continuous and completely continuous. Thus, by Krasnoselskii’s FPT, \mathbb{T}_1 has a fixed point ω in \mathcal{S}_γ which is a solution of the problem (1). End the proof. \square

In the following result, we provide the uniqueness of the solution of our problem (1), and its proof is based on Banach’s FPT [40].

Theorem 2. Suppose that (H_1) – (H_4) hold. If

$$\Lambda_1 < 1, \tag{6}$$

then there is a unique solution to the problem defined in Equation (1) on \mathfrak{h} .

Proof. Using Lemma 3, we can express the equivalent fractional integral equation for the problem defined in Equation (1) as an operator equation in the following form:

$$\omega = \Upsilon\omega, \quad \omega \in \mathbb{C}(\mathfrak{h}, \mathbb{R}),$$

such that the operator $Y : \mathbb{C}(\hbar, \mathbb{R}) \rightarrow \mathbb{C}(\hbar, \mathbb{R})$, is defined by

$$\begin{aligned} (Y\omega)(\varkappa) &= \omega_0 + y(\omega) + \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\nu-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta}\right)^{\nu-1} g(\eta) d\eta \\ &+ \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\nu-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta}\right)^{\nu-1} \left(\int_0^\eta \chi_1(\eta, \zeta) \mathcal{N}_1(\omega(\zeta)) d\zeta\right) d\eta \\ &+ \int_0^1 \chi_2(\eta, \zeta) \mathcal{N}_2(\omega(\zeta)) d\zeta d\eta, \end{aligned}$$

for all $\varkappa \in \hbar$. Let $\omega, \omega^* \in \mathbb{C}(\hbar, \mathbb{R})$. Then, for each $\varkappa \in \hbar$, we have

$$\begin{aligned} &|Y\omega(\varkappa) - Y\omega^*(\varkappa)| \\ \leq &|y(\omega(\varkappa)) - y(\omega^*(\varkappa))| \\ &+ \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\nu-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta}\right)^{\nu-1} \left(\int_0^\eta \chi_1(\eta, \zeta) |\mathcal{N}_1(\omega(\zeta)) - \mathcal{N}_1(\omega^*(\zeta))| d\zeta\right) d\eta \\ &+ \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\nu-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta}\right)^{\nu-1} \left(\int_0^1 \chi_2(\eta, \zeta) |\mathcal{N}_2(\omega(\zeta)) - \mathcal{N}_2(\omega^*(\zeta))| d\zeta\right) d\eta \\ \leq &\ell_y \|\omega - \omega^*\|_\infty + \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\nu-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta}\right)^{\nu-1} \chi_1^* \ell_{\mathcal{N}_1} \|\omega - \omega^*\|_\infty d\eta \\ &+ \frac{1}{\Gamma(\nu)} \int_0^\varkappa \eta^{\nu-1} \left(\frac{\varkappa^\zeta - \eta^\zeta}{\zeta}\right)^{\nu-1} \chi_2^* \ell_{\mathcal{N}_2} \|\omega - \omega^*\|_\infty d\eta \\ \leq &\left(\ell_y + \frac{\chi_1^* \ell_{\mathcal{N}_1} + \chi_2^* \ell_{\mathcal{N}_2}}{\zeta^\nu \Gamma(\nu + 1)} \varkappa^{\zeta\nu}\right) \|\omega - \omega^*\|_\infty, \end{aligned}$$

which implies

$$\|Y\omega - Y\omega^*\|_\infty \leq \left(\ell_y + \frac{\sum_{j=1}^2 \ell_{\mathcal{N}_j} \chi_j^*}{\zeta^\nu \Gamma(\nu + 1)}\right) \|\omega - \omega^*\|_\infty.$$

The inequality (6) shows that Y is a contraction mapping on $\mathbb{C}(\hbar, \mathbb{R})$. As a result of Banach’s FPT, Y will have a unique fixed point that is the solution of the problem (1). □

4. Approximate Solution

In this section, we use the fractional AD technique to derive an approximate solution to the CK fractional VFIDE that is defined in (1). We start by recalling the classical AD technique, which represents the solution to the problem as a series:

$$\omega = \sum_{n=0}^\infty \omega_n, \tag{7}$$

and the nonlinear terms $\mathcal{N}_1, \mathcal{N}_2$, and y are decomposed as

$$\mathcal{N}_1 = \sum_{n=0}^\infty A_n, \quad \mathcal{N}_2 = \sum_{n=0}^\infty B_n, \quad y = \sum_{n=0}^\infty D_n, \tag{8}$$

such that each of A_n, B_n , and D_n are called as Adomian polynomials for every non-negative and non-zero integer n , where this implies that

$$\omega = \omega(\lambda) = \sum_{n=0}^\infty \lambda^n \omega_n = \omega_0 + \lambda \omega_1 + \lambda^2 \omega_2 + \dots + \lambda^k \omega_k + \dots \tag{9}$$

$$\mathcal{N}_1 = \mathcal{N}_1(\lambda) = \sum_{n=0}^\infty \lambda^n A_n = A_0 + \lambda A_1 + \lambda^2 A_2 + \dots + \lambda^k A_k + \dots \tag{10}$$

$$\mathcal{N}_2 = \mathcal{N}_2(\lambda) = \sum_{n=0}^{\infty} \lambda^n B_n = B_0 + \lambda B_1 + \lambda^2 B_2 + \dots + \lambda^k B_k + \dots \tag{11}$$

$$y = y(\lambda) = \sum_{n=0}^{\infty} \lambda^n D_n = D_0 + \lambda D_1 + \lambda^2 D_2 + \dots + \lambda^k D_k + \dots \tag{12}$$

Using the previous formulas (9), (10), (11), and (12), we can conclude that

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\mathcal{N}_1 \sum_{j=0}^{\infty} \lambda^j \omega_j \right) \right]_{\lambda=0},$$

$$B_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\mathcal{N}_2 \sum_{j=0}^{\infty} \lambda^j \omega_j \right) \right]_{\lambda=0},$$

and

$$D_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(y \sum_{j=0}^{\infty} \lambda^j \omega_j \right) \right]_{\lambda=0},$$

where $\omega_0, \omega_1, \omega_2, \dots$ are repeatedly specified by

$$\begin{cases} \omega_0(\varkappa) = \omega_0 + \mathcal{I}_{0+}^{\nu;\zeta}(g(\varkappa)) \\ \omega_{k+1}(\varkappa) = D_k + \mathcal{I}_{0+}^{\nu;\zeta} \left(\int_0^{\varkappa} \chi_1(\varkappa, \xi) A_k d\xi \right) \\ \quad + \mathcal{I}_{0+}^{\nu;\zeta} \left(\int_0^1 \chi_2(\varkappa, \xi) B_k d\xi \right), \quad k \geq 1. \end{cases} \tag{13}$$

Now, we use the modified AD method, and the scheme (13) yields

$$\begin{cases} \omega_0(\varkappa) = \omega_0 + R_1(\varkappa), \\ \omega_1(\varkappa) = R_2(\varkappa) + D_0 + \mathcal{I}_{0+}^{\nu;\zeta} \left(\int_0^{\varkappa} \chi_1(\varkappa, \xi) A_0 d\xi \right) \\ \quad + \mathcal{I}_{0+}^{\nu;\zeta} \left(\int_0^1 \chi_2(\varkappa, \xi) B_0 d\xi \right), \\ \omega_{k+1}(\varkappa) = D_k + \mathcal{I}_{0+}^{\nu;\zeta} \left(\int_0^{\varkappa} \chi_1(\varkappa, \xi) A_k d\xi \right) \\ \quad + \mathcal{I}_{0+}^{\nu;\zeta} \left(\int_0^1 \chi_2(\varkappa, \xi) B_k d\xi \right), \quad k \geq 1. \end{cases} \tag{14}$$

We will now turn our attention to the convergence of the solution, using the modified Adomian decomposition method.

Theorem 3. Suppose that $(H_1) - (H_4)$ and (2) are all satisfied, if the solution $\omega(\varkappa) = \sum_{j=0}^{\infty} \omega_j(\varkappa)$ and $\|\omega\|_{\infty} < \infty$ of the Caputo–Katugampola fractional Volterra–Fredholm integro-differential equation converges, it will converge to the true solution of the equation (1).

Proof. We omit the proof because it is similar to the proof provided in other works, such as [15]. □

5. An Example

Example 1. Take into consideration the following integro-differential equation with the CK fractional derivative,

$$\begin{cases} {}^C\mathcal{D}_{0+}^{\frac{1}{2};\frac{1}{3}} \omega(\varkappa) = \frac{3}{\sqrt{\pi}} \left(\frac{5\varkappa^{\frac{3}{2}}}{\Gamma(\frac{7}{2})} + \varkappa^{\frac{1}{2}} \right) + \frac{\varkappa^3}{\Gamma(8)} + \frac{\varkappa}{\Gamma(9)} \\ \quad + \frac{1}{5} \int_0^{\varkappa} (1 + \varkappa - \eta) \omega(\eta) d\eta + \frac{7}{20} \int_0^1 e^{\eta - \varkappa} \omega^2(\eta) d\eta, \end{cases} \tag{15}$$

with the non-local condition,

$$\omega(0) = \frac{1}{5} \omega\left(\frac{1}{4}\right), \tag{16}$$

where

$$\begin{aligned} v &= \frac{1}{2}, \zeta = \frac{1}{3}, \omega_0 = 0, y(\omega(\varkappa)) = \frac{1}{5}\omega\left(\frac{1}{4}\right), \\ g(\varkappa) &= \frac{3}{\sqrt{\pi}}\left(\frac{5\varkappa^{\frac{3}{2}}}{\Gamma(7)} + \varkappa^{\frac{1}{2}}\right) + \frac{\varkappa^3}{\Gamma(8)} + \frac{\varkappa}{\Gamma(9)}, \\ \chi_1(\varkappa, \xi) &= \frac{1}{5}(1 + \varkappa - \xi), \chi_2(\varkappa, \xi) = \frac{7}{20}e^{\xi - \varkappa}. \end{aligned}$$

Clearly, $\ell_{N_1} = \ell_{N_2} = 1, \ell_y = \frac{1}{5}$.

$$\begin{aligned} \mu_g &: = \sup_{\varkappa \in [0,1]} |g(\varkappa)| = \|g\|_\infty \\ &= \frac{3}{\sqrt{\pi}}\left(\frac{5}{\Gamma(7)} + 1\right) + \frac{1}{\Gamma(8)} + \frac{1}{\Gamma(9)} \\ &= \frac{3\sqrt{\pi} + 40600}{13440\sqrt{\pi}} \end{aligned}$$

and

$$\begin{aligned} \chi_1^* &= \frac{1}{5} \sup_{\varkappa \in \mathfrak{h}} \int_0^\varkappa |1 + \varkappa - \xi| d\xi = \frac{1}{10}, \\ \chi_2^* &= \frac{7}{20} \sup_{\varkappa \in \mathfrak{h}} \int_0^\varkappa |e^{\xi - \varkappa}| d\xi = \frac{7}{20} \sup_{\varkappa \in \mathfrak{h}} e^{-\varkappa} \int_0^\varkappa |e^\xi| d\xi \\ &= \frac{7}{20} \left(1 - \frac{1}{e}\right). \end{aligned}$$

Hence,

$$\Lambda_1 := \left(\ell_y + \frac{\sum_{j=1}^2 \ell_{N_j} \chi_j^*}{\zeta^v \Gamma(v+1)} \right) \approx 0.65670 < 1.$$

As a consequence of Theorem 2, the problem (15)–(16) has a unique solution in $[0, 1]$.

Applying the operator $\mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}}$ to both sides of Equation (15), we get

$$\begin{aligned} \omega(\varkappa) &= \frac{1}{5}\omega\left(\frac{1}{4}\right) + \mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \left(\frac{3}{\sqrt{\pi}}\left(\frac{5\varkappa^{\frac{3}{2}}}{\Gamma(7)} + \varkappa^{\frac{1}{2}}\right) + \frac{\varkappa^3}{\Gamma(8)} + \frac{\varkappa}{\Gamma(9)} \right) \\ &\quad + \mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \left(\frac{1}{5} \int_0^\varkappa (1 + \varkappa - \eta)\omega(\eta) d\eta \right) + \mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \left(\frac{7}{20} \int_0^1 e^{\eta - \varkappa} \omega^2(\eta) d\eta \right). \end{aligned}$$

Suppose

$$\begin{aligned}
 R(\varkappa) &= \mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \left(\frac{3}{\sqrt{\pi}} \left(\frac{5\varkappa^{\frac{3}{2}}}{\Gamma(7)} + \varkappa^{\frac{1}{2}} \right) + \frac{\varkappa^3}{\Gamma(8)} + \frac{\varkappa}{\Gamma(9)} \right) \\
 &= \frac{3}{\sqrt{\pi}} \frac{5}{\Gamma(7)} \left(\mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \eta^{\frac{3}{2}} \right) (\varkappa) + \frac{3}{\sqrt{\pi}} \left(\mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \eta^{\frac{1}{2}} \right) (\varkappa) \\
 &\quad + \frac{1}{\Gamma(8)} \left(\mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \eta^3 \right) (\varkappa) + \frac{1}{\Gamma(9)} \left(\mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \eta \right) (\varkappa) \\
 &= \frac{3}{\sqrt{\pi}} \frac{5}{\Gamma(7)} \left(\frac{\Gamma(\frac{11}{2})}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(6)} \varkappa^{\frac{5}{3}} \right) + \frac{3}{\sqrt{\pi}} \left(\frac{\Gamma(\frac{5}{2})}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(3)} \varkappa^{\frac{2}{3}} \right) \\
 &\quad + \frac{1}{\Gamma(8)} \left(\frac{\Gamma(10)}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(\frac{21}{2})} \varkappa^{\frac{19}{6}} \right) + \frac{1}{\Gamma(9)} \left(\frac{\Gamma(4)}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(\frac{9}{2})} \varkappa^{\frac{7}{6}} \right).
 \end{aligned}$$

Now, we apply the modified AD method,

$$R(\varkappa) = R_1(\varkappa) + R_2(\varkappa),$$

where

$$R_1(\varkappa) = \frac{15}{\sqrt{\pi}} \frac{\Gamma(\frac{11}{2})}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(7)\Gamma(6)} \varkappa^{\frac{5}{3}},$$

and

$$R_2(\varkappa) = \frac{3}{\sqrt{\pi}} \frac{\Gamma(\frac{5}{2})}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(3)} \varkappa^{\frac{2}{3}} + \frac{\Gamma(10)}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(8)\Gamma(\frac{21}{2})} \varkappa^{\frac{19}{6}} + \frac{\Gamma(4)}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(9)\Gamma(\frac{9}{2})} \varkappa^{\frac{7}{6}}.$$

The modified recursive relation

$$\omega_0(\varkappa) = R_1(\varkappa) = \frac{15}{\sqrt{\pi}} \frac{\Gamma(\frac{11}{2})}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(7)\Gamma(6)} \varkappa^{\frac{5}{3}},$$

$$\begin{aligned}
 \omega_1(\varkappa) &= R_2(\varkappa) + \mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \left(\frac{1}{5} \int_0^\varkappa (1 + \varkappa - \eta) A_0(\eta) d\eta \right) \\
 &\quad + \mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \left(\frac{7}{20} \int_0^1 e^{\eta - \varkappa} B_0(\eta) d\eta \right) + D_0(\varkappa) \\
 &= \frac{3}{\sqrt{\pi}} \frac{\Gamma(\frac{5}{2})}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(3)} \varkappa^{\frac{2}{3}} + \frac{\Gamma(10)}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(8)\Gamma(\frac{21}{2})} \varkappa^{\frac{19}{6}} + \frac{\Gamma(4)}{\left(\frac{1}{3}\right)^{\frac{1}{2}} \Gamma(9)\Gamma(\frac{9}{2})} \varkappa^{\frac{7}{6}} \\
 &\quad + \mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \left(\frac{1}{5} \int_0^\varkappa (1 + \varkappa - \eta) \omega_0(\eta) d\eta \right) \\
 &\quad + \mathcal{I}_{0^+}^{\frac{1}{2}, \frac{1}{3}} \left(\frac{7}{20} \int_0^1 e^{\eta - \varkappa} \omega_0(\eta) d\eta \right) + \frac{1}{5} \omega_0\left(\frac{1}{4}\right)
 \end{aligned}$$

which gives

$$\begin{aligned} \omega_1(\varkappa) &= \frac{3}{\sqrt{\pi}} \frac{\Gamma(\frac{5}{2})}{(\frac{1}{3})^{\frac{1}{2}} \Gamma(3)} \varkappa^{\frac{2}{3}} + \frac{\Gamma(10)}{(\frac{1}{3})^{\frac{1}{2}} \Gamma(8) \Gamma(\frac{21}{2})} \varkappa^{\frac{19}{6}} + \frac{\Gamma(4)}{(\frac{1}{3})^{\frac{1}{2}} \Gamma(9) \Gamma(\frac{9}{2})} \varkappa^{\frac{7}{6}} \\ &+ \mathcal{I}_{0+}^{\frac{1}{2}; \frac{1}{3}} \left(\frac{1}{5} \int_0^\varkappa (1 + \varkappa - \eta) \left(\frac{15}{\sqrt{\pi}} \frac{\Gamma(\frac{11}{2})}{(\frac{1}{3})^{\frac{1}{2}} \Gamma(7) \Gamma(6)} \eta^{\frac{5}{3}} \right) d\eta \right) \\ &+ \mathcal{I}_{0+}^{\frac{1}{2}; \frac{1}{3}} \left(\frac{5}{18} \int_0^1 e^{\eta - \varkappa} \left(\frac{15}{\sqrt{\pi}} \frac{\Gamma(\frac{11}{2})}{(\frac{1}{3})^{\frac{1}{2}} \Gamma(7) \Gamma(6)} \eta^{\frac{5}{3}} \right) d\eta \right) \\ &+ \frac{1}{5} \left(\frac{15}{\sqrt{\pi}} \frac{\Gamma(\frac{11}{2})}{(\frac{1}{3})^{\frac{1}{2}} \Gamma(7) \Gamma(6)} \right) \left(\frac{1}{4} \right)^{\frac{5}{3}} \\ &= 0, \end{aligned}$$

$$\begin{aligned} \omega_2(\varkappa) &= 0, \\ &\vdots \\ \omega_n(\varkappa) &= 0. \end{aligned}$$

Therefore, the obtained solution is

$$\omega(\varkappa) = \sum_{j=0}^{\infty} \omega_j(\varkappa) = \frac{15}{\sqrt{\pi}} \frac{\Gamma(\frac{11}{2})}{(\frac{1}{3})^{\frac{1}{2}} \Gamma(7) \Gamma(6)} \varkappa^{\frac{5}{3}}.$$

6. Conclusions

In this work, we examine a fractional Volterra–Fredholm integro-differential equation, involving the Caputo–Katugampola fractional derivative, as shown in Equation (1). We derive a representation of the solution to this equation, and establish the convergence of approximated solutions and the existence of solutions using classic fixed-point theorems, such as Banach and Krasnoselskii, in addition to the fractional AD technique. We also provide an example, to demonstrate the relevance of these results.

Overall, the study of fractional differential equations has become an important area of research, due to their ability to capture memory effects and non-local behavior in the modeling of real-world phenomena. These equations have a wide range of applications in fields such as physics, engineering, and biology, and further research is necessary, to fully understand and utilize their potential in understanding complex systems. In conclusion, our work adds to the current understanding of fractional integro-differential equations and their solutions.

Author Contributions: Conceptualization, K.S.A.-G. and A.T.A.; Data curation, A.H.A.; Formal analysis, S.S.R. and O.B.; Funding acquisition, L.F.I.; Investigation, S.S.R. and O.B.; Methodology, K.S.A.-G. and A.T.A.; Project administration, L.F.I.; Resources, O.B. and A.H.A.; Supervision, L.F.I.; Validation, S.S.R.; Writing—review & editing, A.H.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the University of Oradea.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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