



# **Generalized fractals in generalized contexts**

Thesis for the Degree of Doctor of Philosophy (PhD)

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Debrecen, 2026.



Hereby I declare that I prepared this thesis within the Doctoral Council for Natural Sciences and Engineering, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at Debrecen University.

The results published in the thesis are not reported in any other PhD theses.

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I also declare that the results published in the thesis are not reported in any other theses.

I support the acceptance of the thesis.

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# Contents

|                                                      |    |
|------------------------------------------------------|----|
| Introduction                                         | 1  |
| Chapter 1. Invariant sets of monotone maps           | 7  |
| 1.1. Preliminaries                                   | 7  |
| 1.2. The main results                                | 11 |
| 1.3. Applications, examples and remarks              | 11 |
| Chapter 2. Fractals obtained by condensing families  | 15 |
| 2.1. Preliminaries                                   | 15 |
| 2.2. Auxiliary tools                                 | 16 |
| 2.3. The main results                                | 20 |
| 2.4. Applications, examples and remarks              | 22 |
| Chapter 3. Generalized fractals in semimetric spaces | 27 |
| 3.1. Preliminaries                                   | 27 |
| 3.2. Auxiliary tools                                 | 31 |
| 3.3. The main results                                | 34 |
| 3.4. Applications, examples and remarks              | 36 |
| Chapter 4. Dimension of generalized fractals         | 39 |
| 4.1. Preliminaries                                   | 39 |
| 4.2. Auxiliary tools                                 | 39 |
| 4.3. The main results                                | 45 |
| 4.4. Applications, examples and remarks              | 46 |
| Chapter 5. An axiomatic treatment                    | 47 |
| 5.1. Preliminaries                                   | 47 |
| 5.2. Auxiliary tools                                 | 48 |
| 5.3. The main results                                | 48 |
| 5.4. Applications, examples and remarks              | 52 |
| Summary                                              | 53 |
| Összefoglalás                                        | 59 |
| Appendix                                             | 65 |
| List of publications                                 | 71 |
| List of talks                                        | 73 |
| Bibliography                                         | 75 |



## Introduction

*"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."  
(Benoit B. Mandelbrot)*

**The fundamentals: Mandelbrot and Hutchinson.** Mandelbrot, the "Father of fractals" emphasizes in the above quotation, that our world is full of irregular shapes. The well-known regular geometrical forms of mathematics, which are easy to handle, smooth or even, are actually quite rare in nature. The mentioned clouds, mountains, coastlines, barks and lightnings can be considered as fractals from a mathematical view.

Thus, we can meet with fractals in physical, chemical and biological processes. Moreover, fractals can be found even in art, for instance in the illustrations of Escher or in the music of composers. Although fractals are everywhere and definitely belong to the most popular mathematical inventions, they have no unified definition in the technical literature.

Two of the most important properties of fractals are *self-similarity* and their *fragmented* nature. That is, the enlarged parts of the fractal are similar to the whole shape or to other parts. On the other hand, the boundary lines or surfaces of the fractal are infinitely fragmented and crinkled.

Measure theorists prefer to capture fractals' fragmented nature, which is implied by the Latin expression "fragmentus". Initially, Mandelbrot defined fractals through the notion of fractal dimension, which intends to describe the measure of discontinuity. He considered them as sets *for which the Hausdorff–Besicovitch dimension strictly exceeds the topological dimension*, see his pioneer work [36], or its English version [37]. The book of Barnsley [8] also gives a great summary from this point of view. However, later Mandelbrot found this definition too strict and proposed to use 'fractal' without a pedantic definition, since we can see *fractals everywhere* [8].

Fixed point theorists, on the other hand, primarily focus on the self-similarity of fractals. This approach and its interaction with the Hausdorff–Besicovitch dimension was elaborated by Hutchinson [27]. He considered fractals via solutions of fixed point equations, which consist of self-similar mappings. His pioneering

work meant a great breakthrough in the development of fractal theory and paved the way for studying fractals in greater depth. In this thesis, we follow the fixed point approach to fractals initiated by Hutchinson [27] and Barnsley [8].

**The aim of the thesis.** In what follows, we present the aims of this thesis and provide an overview of its chapters. We also refer to the corresponding literature.

The main objective of this thesis is to investigate fractals in different contexts through the framework of fixed point theory. The monographs of Granas and Dugundji [24], Shapiro [46], and Zeidler [50] provide comprehensive accounts of fixed point theory and its numerous applications. Our work follows the fixed point approach to fractals; however, the methods developed here are independent of and substantially different from the classical Hutchinson framework. In particular, we extend the underlying setting by considering more general classes of mappings and/or more general spaces than those employed in Hutchinson's classical results. That is, we are investigating generalized fractals in generalized contexts through an alternative method.

**CHAPTER 1.** First, we study fractals in an elementary way, that is by completely avoiding advanced analysis and using only naive set theory. We introduce the fundamental notion of this thesis, the *invariance operator*, which is examined throughout the thesis in different contexts. In the first Chapter, we study isotone and continuous maps. We give a slightly improved version of the well-known fixed point result of Knaster and Tarski [32], [47] by adopting the idea of Kantorovich [31]. That is, we represent the smallest fixed point of isotone mappings.

In the main results, we give the minimal solution of the invariance equation in a naive set theoretical setting. As the closing of this Chapter, we show through some classical examples how one can find the minimal solution of the invariance operator. It turns out that the Kantorovich-iteration is important not only in theoretical aspects: Implementing this method we can easily approximate fractals. To illustrate this phenomena we present the approximations of the classical Sierpeński-carpet and our own “Wave” fractal. The results of this Chapter are based on the paper of Bessenyei and Péntes [3].

**CHAPTER 2.** We introduce fractals in metric space and present the classical results of Hutchinson [27]. His fundamental theorem gives an existence and uniqueness property for fractals in metric space obtained by a family of contractions. His approach is based on the fact that the invariance operator is a contraction in the fractal space [26], [41]. Then the Banach Contraction Principle [1] is applied. At this point of the argument he uses the Blaschke Completeness Theorem [9]. However, we note there is a known alternative way to the proof without the Banach Contraction Principle and Blaschke Completeness Theorem, using a

construction with Cantor's Intersection Theorem and the properties of Hausdorff–Pompeiu, see [17]. We also note that a wide range of extensions in metric spaces via various fixed point theorems can be obtained by following the original ideas of Hutchinson [27]. An excellent summary of the topic (with further important developments) is provided in the survey by Leśniak, Snigireva, and Strobin [35]. The precise connections between different fixed point theorems can be found in the papers by Rhoades [42] and in those by Jachymski and Jóźwik [28].

We give another alternative way to fractals, which avoids the Blaschke Theorem. Our method is based on the concept of Hausdorff measure of noncompactness [26]. Its properties with nice applications can be found in the monograph of Granas and Dugundji [24]. Moreover, we use the ideas of the previous Chapter and extend the existence part of the Hutchinson Theorem in two senses. Firstly, instead of contractions, we prove fractal results for generalized contractions by the Matkowski Fixed Point Theorem [38], [10]. Secondly, we use condensing maps and prove fractal theorems by the Darbo–Sadovskii Fixed Point Theorem [15], [44].

We conclude the Chapter by showing some fractal examples where the classical Hutchinson Theorem cannot be applied. Instead of rigorous mathematical proof, we accept this fact using numerical methods. As previously, we implement the Kantorovich-iteration in order to approximate our fractals. The results of this Chapter are based on the paper of Bessenyei and Péntzes [4].

**CHAPTER 3.** We further develop our alternative method to fractals by generalizing the underlying structure, using semimetric spaces instead of metric spaces. These are obtained by dropping the third axiom, the triangle inequality, from the usual definition of metric spaces due to Fréchet [18]. A variant for a certain kind of semimetric space appears in the paper by Jachymski, Matkowski and Świątkowski [29]. The work of Chrząszcz, Jachymski, and Turoboś [14] also gives a nice overview of this topic. We apply the fundamental result of this paper, which states that certain semimetrics are uniformly equivalent to metrics. We also use the fixed point theorem of Bessenyei and Páles [2], which is an extension of the Banach Contraction Principal in semimetric space. We further study the notions of the previous Chapter in semimetric, namely the Hausdorff measure and its properties, condensing maps, and the fractal space over a semimetric.

In the main results we solve the invariance equation in full generality: we show fractal results obtained by condensing maps of semimetric space. Furthermore, we completely extend Hutchinson's result by using generalized contractions of semimetric. We also present some applications for the main results and recall some examples from Rhodes [42]. The results of this Chapter are based on the paper by Bessenyei and Péntzes [5]. We also refer to the papers by Leśniak [34] and by Leśniak, Snigireva, and Strobin [35] who elaborate the methods of Hutchinson in the hyperspace. However, their aims are the same as ours: to cover in a unified way the existence of fractals for a wide range of multifunctions.

**CHAPTER 4.** Hutchinson studied not only the existence and uniqueness of fractals, but also their dimension [27]. In this Chapter, we introduce the Hausdorff measure in semimetric space. Formally, it is the same as in metric space, see Rogers [43] and the book of J  rai [30].

The main result of this Chapter gives a condition for the Hausdorff dimension of generalized fractals of a semimetric space in terms of an exponential expression. However, in the formula we use the right-upper Dini-derivatives of the comparison functions related to the generalized contractions. We note that our theorem is only a "weak" dimension formula, as it is a necessary condition. Whereas Hutchinson's classical result is a sufficient one. Thus, the question remains open whether the complete extension of Hutchinson's fractal dimensional formula is valid for generalized fractals of a regular semimetric space. These results are developed in the paper of Bessenyei and P  nzes [5].

**CHAPTER 5.** In the last Chapter, we revisit the main results of the thesis again, and make it even more general. As we have seen, one of the main tools of this thesis is the Hausdorff measure of noncompactness. However, it turns out that this measure does not have such a distinguished role as it may seem at first glance. Requiring quite simple properties on a set-function, we can prove analogous results. The common idea behind these results is an abstract domain invariance property which can be justified with the Knaster–Tarski and the Kantorovich Fixed Point Theorems. This is also highlighted in the well-known topological fixed point results of Brouwer [11], Schauder [45], Darbo [15] and Sadovskii [44]. We also refer to the paper by Reich [40].

As applications of this axiomatic approach, we present the concept of degree of nondensifiability, which is studied by Garc  a and Mora [21]. We show a correspondence with the Sadovskii Fixed Point Theorem and present how this concept can be applied in fractal theory as it is done by Garc  a and Mora [22], [23] and Mena, Mart  n, and Sadarangani [39]. Furthermore, we may obtain similar results for the Hausdorff measure of noncompactness or in various similar situations. The monograph [48] by Toledano, Benavides, and Acedo covers several such settings. The results of this Chapter are based on the paper by Bessenyei and P  nzes [6].

**Notations.** We set up some basic notions and notations here which are used throughout the thesis.

| Notation                                     | Meaning                                                                     |
|----------------------------------------------|-----------------------------------------------------------------------------|
| $\mathbb{R}_+$                               | Set of nonnegative real numbers                                             |
| $\overline{\mathbb{R}}_+$                    | Set of extended nonnegative real numbers                                    |
| $\mathbb{N}$                                 | Set of positive integers                                                    |
| $X$                                          | An arbitrary nonempty set                                                   |
| $\mathcal{P}(X)$                             | Power set of $X$                                                            |
| $F: X \rightrightarrows X$                   | Self-map of the power set $\mathcal{P}(X)$                                  |
| $\text{id}$                                  | Identity map on $X$                                                         |
| $F^0 = \text{id}$                            | Zeroth composite iterate of $F$                                             |
| $F^{n+1} := F \circ F^n$                     | Composite iterate of $F$ whenever $n \in \mathbb{N}$                        |
| $U(x_0, r) = \{x \in X \mid d(x_0, x) < r\}$ | Open ball with center $x_0$ and radius $r$ in a (semi)metric space $(X, d)$ |
| $\mathcal{F}$                                | Family of self-maps of $\mathcal{P}(X)$                                     |
| $\Phi$                                       | Invariance operator associated with $\mathcal{F}$                           |
| $\mathcal{L}$                                | Countable chain in $\mathcal{P}(X)$                                         |
| $\mathcal{K}(X)$                             | Family of nonempty, compact subsets of $X$                                  |
| $d_{HP}$                                     | Hausdorff–Pompeiu metric on $\mathcal{K}(X)$                                |
| $\varphi$                                    | Comparison function for a Matkowski contraction                             |
| $\chi$                                       | Hausdorff noncompactness measure                                            |
| $\Delta$                                     | Basic triangle function of a semimetric space                               |
| $\Delta^*$                                   | Triangle function of a semimetric space                                     |
| $\text{diam}(H)$                             | Diameter of the set $H$                                                     |
| $\chi^r$                                     | $r$ -dimensional Hausdorff measure                                          |
| $\mathcal{H}^r$                              | $\sigma$ -algebra of Hausdorff-measurable sets                              |
| $\dim_{\mathcal{H}}$                         | Hausdorff dimension                                                         |
| $D\varphi(0)$                                | Right-upper Dini derivative of $\varphi$ at 0                               |

TABLE 1. List of frequently used notations.



## CHAPTER 1

### Invariant sets of monotone maps

The aim of this Chapter is to present an elementary way to fractals which completely avoids advanced analysis and uses purely naive set theory. The approach relies on fixed point methods, where the role of the Banach Contraction Principle is replaced by a slightly improved version of the Knaster–Tarski Fixed Point Theorem [32]. Our key tool is the Kantorovich-iteration [31].

We provide a minimalist fractal theory in a double sense: A minimal theoretical setup leads to the minimum solution of the invariance equation. The main results of this Chapter are based on the paper by Bessenyei and Péntes [3].

#### 1.1. Preliminaries

**Invariance operator.** Let  $\mathcal{F}$  be a family of self-maps of the power set  $\mathcal{P}(X)$ . We say that the map  $\Phi: X \rightrightarrows X$  defined by

$$(1.1) \quad \Phi(H) := \bigcup_{F \in \mathcal{F}} F(H)$$

is the *invariance operator* of  $\mathcal{F}$ . Moreover, the set  $H \subseteq X$  is  $\mathcal{F}$ -*invariant*, if

$$H = \Phi(H)$$

is satisfied, that is  $H$  is a *fixed point* of the invariance operator.

In the following, we use both alternative terms: in the context of fractal theory, the term *invariant set* is used, while in the context of fixed point theory, the term *fixed point* is used, respectively.

Similarly, in the case where  $H \subseteq \Phi(H)$  holds, we say  $H$  is a *subinvariant set* or *half fixed point* of  $\Phi$ .

**Isotone and continuous maps.** A mapping  $F: X \rightrightarrows X$  is called *isotone*, or alternatively *monotone*, if it preserves the set-theoretic inclusion, that is  $F(A) \subseteq F(B)$  holds provided that  $A \subseteq B$ .

Isotone maps have several obvious properties. For instance, the images of the emptyset and the basic set under an isotone map  $F$  are

$$F(\emptyset) = \bigcap \{F(A) \mid A \in \mathcal{P}(X)\} \quad \text{and} \quad F(X) = \bigcup \{F(A) \mid A \in \mathcal{P}(X)\}.$$

Note also that isotone maps are superadditive with respect to union, while subadditive with respect to intersection. That is, for any isotone map  $F: X \rightrightarrows X$ , we have

$$\bigcup_{A \in \mathcal{A}} F(A) \subseteq F\left(\bigcup \mathcal{A}\right) \quad \text{and} \quad F\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} F(A),$$

where  $\mathcal{A} \in \mathcal{P}(X)$  is arbitrary.

Another important mapping class (which turns out to be a subfamily of isotone maps) is the class of continuous maps. As usual, under a chain  $\mathcal{L}$  we mean a family of sets on which the inclusion is an ordering. In the power set  $\mathcal{P}(X)$ , it is useful to consider countable chains as sequences and the union of the chain as the limit. Guided by this, a map  $F: X \rightrightarrows X$  is said to be *continuous* (in the set theoretical sense) if for every countable chain  $\mathcal{L}$ ,

$$F\left(\bigcup \mathcal{L}\right) = \bigcup_{L \in \mathcal{L}} F(L).$$

In the latter, we use both terms isotone and monotone alternately, depending on whether we want to emphasize a set theoretical or an analytical property. The same applies for continuity: We don't pronounce if we mean it in the sense of set theory or analytically, as it is obvious from the given setting.

Further obvious properties play a distinguished role in our investigations. Therefore we formulate them as Propositions. The first one highlights the precise relation of isotone and continuous self-maps, while the second and third ones show how to generate isotone and continuous maps using these kind of maps.

**PROPOSITION 1.1.** *Any continuous self-map of a power set is isotone. Furthermore, any self-map of a finite power set is isotone if and only if it is continuous.*

**PROOF.** Let  $F: X \rightrightarrows X$  be a continuous map, and assume that  $A \subseteq B$ . Consider the chain  $\mathcal{L} = \{A, B\}$ . Then,

$$F(A) \subseteq F(A) \cup F(B) = \bigcup_{L \in \mathcal{L}} F(L) = F\left(\bigcup \mathcal{L}\right) = F(B),$$

resulting in the first statement.

To prove the second one, it suffices to prove only that any isotone map  $F$  on a finite power set is continuous. Observe first that, any countable chain in a finite set is finite. For a finite chain  $\mathcal{L}$ , the set  $\{F(L) \mid L \in \mathcal{L}\}$  is a chain again, since  $F$  is isotone. Moreover, if  $L$  represents the greatest member of  $\mathcal{L}$ , then  $F(L)$  becomes the greatest member in the corresponding chain. Therefore,  $F(\bigcup \mathcal{L}) = F(L) = \bigcup F(\mathcal{L})$  holds, yielding continuity.  $\square$

**PROPOSITION 1.2.** *The finite composition of isotone (respectively, continuous) maps is isotone (respectively, continuous). In particular, the iterates of an isotone (respectively, continuous) map are also isotone (respectively, continuous).*

**PROPOSITION 1.3.** *If  $\mathcal{F}$  is an isotone (respectively: continuous) family, then the invariance operator given by (1.1) is also isotone (respectively, continuous).*

PROOF. We restrict the proof only to the second statement. Let  $\mathcal{L}$  be a countable chain. Then, by the continuity of the members,

$$\begin{aligned}\Phi\left(\bigcup \mathcal{L}\right) &= \bigcup_{F \in \mathcal{F}} F\left(\bigcup_{L \in \mathcal{L}} L\right) = \bigcup_{F \in \mathcal{F}} \bigcup_{L \in \mathcal{L}} F(L) \\ &= \bigcup_{L \in \mathcal{L}} \bigcup_{F \in \mathcal{F}} F(L) = \bigcup_{L \in \mathcal{L}} \Phi(L).\end{aligned}$$

This verifies the continuity of  $\Phi$ .  $\square$

A simple example for continuous maps can be given by the standard interpretation of the image of sets.

EXAMPLE 1.4. Let  $f: X \rightarrow X$  be an arbitrary function, and define the map  $F_f: X \rightrightarrows X$  via the standard way of “image of sets”

$$F_f(H) := \{f(x) \mid x \in H\}.$$

Then,  $F_f$  is continuous.

**Fixed point theorems.** A nontrivial property of isotone maps, the well-known result of Knaster and Tarski [32], states that “each isotone map has a fixed point.”

The next theorem slightly improves this statement, guaranteeing the existence of a *smallest* fixed point. Hence, in this form, it may be considered as an existence and uniqueness result.

THEOREM 1.5. If  $F: X \rightrightarrows X$  is isotone, then it has a smallest fixed point.

PROOF. Let  $X$  be an arbitrary set and  $F: X \rightrightarrows X$  be isotone. Denote by  $\mathcal{H}$  the family of those sets  $H \subseteq X$ , for which  $F(H) \subseteq H$  is satisfied. This family  $\mathcal{H}$  is nonempty, because it contains the set  $X$ , as  $F(X) \subseteq X$  holds. Let the joint parts of the sets in  $\mathcal{H}$  denoted by  $H_\infty$ :

$$H_\infty = \bigcap \mathcal{H} = \bigcap \{H \subseteq X \mid F(H) \subseteq H\}.$$

If  $H$  is an arbitrary set in  $\mathcal{H}$ , then  $F(H_\infty) \subseteq F(H)$  holds because of  $H_\infty \subseteq H$  and  $F$  is isotone. Furthermore,  $F(H_\infty) \subseteq H$  follows from  $F(H) \subseteq H$ . That is, the set  $F(H_\infty)$  is contained by all sets in  $\mathcal{H}$ . Likewise, the intersection  $H_\infty$  of  $\mathcal{H}$  contains  $F(H_\infty)$ , too. This means  $F(H_\infty) \subseteq H_\infty$ .

Anyway, from  $F(H_\infty) \subseteq H_\infty$  one can conclude  $F(F(H_\infty)) \subseteq F(H_\infty)$ . That is, the set  $F(H_\infty)$  is also in  $\mathcal{H}$ . So  $H_\infty \subseteq F(H_\infty)$  also holds, as  $H_\infty$  is part of all sets of  $\mathcal{H}$ .

Altogether,  $F(H_\infty) \subseteq H_\infty$  and  $H_\infty \subseteq F(H_\infty)$  hold simultaneously, so  $F(H_\infty) = H_\infty$ . That is, we found the desired subset, which is a fixed point of  $F$ . We shall prove it is the smallest one.

If now  $H$  is an arbitrary fixed point, then  $F(H) \subseteq H$  is also satisfied. So  $H_\infty \subseteq H$ , based on the above definition. Namely,  $H_\infty$  is part of all other fixed points. Consequently, it is the smallest fixed point of the map  $F$ .  $\square$

We note that the standard proof of the Knaster–Tarski Theorem, for example, as Shapiro [46] presents, is to show that

$$H^\infty = \bigcup \{H \subseteq X \mid H \subseteq F(H)\}.$$

is a fixed point of  $F$ . The proof follows similar arguments as we did in Theorem 1.5. Moreover, one can show that the set  $H^\infty$  is the greatest fixed point of  $F$ : Indeed, if now  $H$  is an arbitrary fixed point, then  $H \subseteq F(H)$  is also satisfied. So  $H \subseteq H^\infty$ , based on the above definition. Namely,  $H^\infty$  contains all other fixed points. Consequently, it is the greatest fixed point of the map  $F$ .

Another representation of the smallest fixed point can be given by adopting the idea of Kantorovich [31].

Assume that  $F: X \rightrightarrows X$  and let  $H_1 \subseteq X$  be a given set. The below recursion is called *Kantorovich-iteration*, while the union of the sets is the *limit of the iteration*:

$$(1.2) \quad H_{m+1} := F(H_m), \quad H_\infty = \bigcup_{m \in \mathbb{N}} H_m.$$

**THEOREM 1.6.** *If  $F: X \rightrightarrows X$  is continuous, then the Kantorovich-iteration expands all half fixed points of  $F$  to the smallest fixed point which contains this half fixed point.*

**PROOF.** If  $H_1$  is a half fixed point of  $F$ , then  $H_1 \subseteq F(H_1) = H_2$ . Due to Proposition 1.1,  $F(H_1) \subseteq F(H_2)$  also holds. So  $H_2 \subseteq H_3$ . By induction, we can conclude  $(H_m)$  is an expanding chain. Using these arguments,

$$F(H_\infty) = F\left(\bigcup_{m \in \mathbb{N}} H_m\right) = \bigcup_{m \in \mathbb{N}} F(H_m) = \bigcup_{m \in \mathbb{N}} H_{m+1} = \bigcup_{m \in \mathbb{N}} H_m = H_\infty$$

follows. That is,  $H_\infty$  is a fixed point of  $F$ .

Obviously,  $H_1 \subseteq H_\infty$  holds. Now let  $A$  be a fixed point such that, it contains the set  $H_1$ . As  $H_1$  is a half fixed point and  $F$  is isotone, one can conclude

$$H_2 = F(H_1) \subseteq F(A) = A.$$

That is,  $H_2 \subseteq A$ . By induction,  $H_m \subseteq A$  is satisfied for all  $m \in \mathbb{N}$ . Therefore,  $H_\infty \subseteq A$ . This shows that  $H_\infty$  is the smallest fixed point, which contains the half fixed point  $H_1$ .  $\square$

Finally, we present some examples of isotone and continuous maps and describe their fixed point properties, as well. The details of the proofs are omitted.

**EXAMPLE 1.7.** *If  $A \subseteq X$  is arbitrary, then the map  $F: X \rightrightarrows X$  defined by  $F(H) = A$  is continuous. Its smallest and greatest fixed point is  $A$ .*

**EXAMPLE 1.8.** *The identity map of any power set is continuous, and each subset is a fixed point. The smallest fixed point is the emptyset, while the greatest one is the basic set.*

**EXAMPLE 1.9.** Let  $X$  be an infinite set, and define  $F: X \rightrightarrows X$  in the following way. Let  $F(H) = \emptyset$  if  $H$  is finite and  $F(H) = X$  if  $H$  is infinite. Then  $F$  is isotone, but not a continuous map. Its smallest fixed point is the emptyset, while the greatest one is  $X$ . No more fixed points exist.

**EXAMPLE 1.10.** Let  $X$  be a vector space, and fix  $x_0 \in X$  and  $\lambda \in \mathbb{R}$  arbitrarily. Then the mapping  $F: X \rightrightarrows X$  given by  $F(H) = \{\lambda x + x_0 \mid x \in H\}$  is continuous, and its smallest fixed point  $H_\infty$  which contains  $x_0 \in X$  can be represented as

$$H_\infty = \left\{ \left( \sum_{k=1}^n \lambda^{k-1} \right) x_0 \mid n \in \mathbb{N} \right\}.$$

## 1.2. The main results

**Minimalist fractals.** Using the fixed point properties of isotone and continuous mappings, we study the invariance equation  $\Phi(H) = H$ . We formulate a fractal theorem purely in naive set theoretical context. The above Kantorovich-iteration and Knaster–Tarski Fixed Point Theorem lead to the minimal solution of the invariance equation.

**THEOREM 1.11.** *If  $\mathcal{F}$  is a family of isotone maps, then there exists a smallest  $\mathcal{F}$ -invariant set. Moreover, if  $\mathcal{F}$  is a continuous family, then the Kantorovich-iteration expands a given subinvariant set to the smallest invariant set which contains this subinvariant set.*

**PROOF.** Due to Proposition 1.3, the invariance operator inherits the isotonicity and continuity of  $\mathcal{F}$ . Keeping this in mind, the first statement is a consequence of Theorem 1.5, while the second one follows immediately from Theorem 1.6.  $\square$

## 1.3. Applications, examples and remarks

**Classical examples.** In the following, we present some classical examples of fractals, such as the Cantor-set, Sierpiński-carpet and Menger-sponge. We show how the main results can be applied to find the minimal solution of the corresponding invariance equations. It may occur that the minimalist solution is the emptyset, but we can avoid trivial cases by starting the iteration with a given point of the original set. The next Corollaries explain this phenomenon.

COROLLARY 1.12. *The smallest invariant set  $H_\infty$  which contains  $0 \in \mathbb{R}$  and satisfies the invariance equation*

$$H = \frac{1}{3}H \cup \left(\frac{1}{3}H + \frac{2}{3}\right)$$

*is a proper subset of the Cantor-set, and consists of exactly those real numbers whose ternary expansions are finite, and the digits belong to the set  $\{0, 2\}$ . Moreover, the closure of  $H_\infty$  is exactly the Cantor-set.*

HINT. By Example 1.14, the right-hand side of the invariance equation above contains continuous maps. Using induction and the fact that  $\{0\}$  is a subinvariant set of the induced invariance operator  $\Phi$ , one can easily prove that

$$\Phi^{n+1}(\{0\}) = \{(0, x_1 \dots x_n)_3 \mid x_k \in \{0, 2\}, k = 1, \dots, n\},$$

where  $(\cdot)_3$  stands for the basic ternary expansion of reals. Hence (1.1) reduces to

$$H_\infty = \{(0, x_1 \dots x_n)_3 \mid x_k \in \{0, 2\}, k = 1, \dots, n; n \in \mathbb{N}\},$$

as it was stated. Therefore, by Theorem 1.11, there exists a smallest fractal satisfying the invariance equation, indeed.

Denote the Cantor-set by  $C$ . As the Cantor-set includes such real numbers whose ternary expansions contain the digits 0 or 2, all members of the Cantor-set can be constructed as the limit of a series from  $H_\infty$ . That is,  $C \subseteq \overline{H_\infty}$ .

Obviously,  $H_\infty \subseteq C$ . Thus,  $\overline{H_\infty} \subseteq \overline{C} = C$  since the Cantor-set is closed.  $\square$

One can conclude the next Corollaries similarly as above, so the proofs are omitted.

COROLLARY 1.13. *For all  $i, j \in \{0, 1, 2\}$  and  $(i, j) \neq (1, 1)$ , define the mappings  $f_{ij}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by*

$$f_{ij}(x, y) = \left(\frac{x+i}{3}, \frac{y+j}{3}\right).$$

*The smallest planar invariant set which contains the origin in  $\mathbb{R}^2$  and satisfies the invariance equation*

$$H = \bigcup \{f_{ij}(H) \mid i, j \in \{0, 1, 2\}, (i, j) \neq (1, 1)\}$$

*is a proper subset of the Sierpiński-carpet, and consists of exactly those points whose components' ternary expansions are finite, and have no position in which both numbers have the digit 1. Moreover, its closure is exactly the Sierpiński-carpet.*

Figure 1 shows two subsets of the Sierpiński-carpet. We started the Kantorovich-iteration from two different fixed points of the invariance equation of Corollary 1.13. The implementation of the iteration can be found in the Appendix.

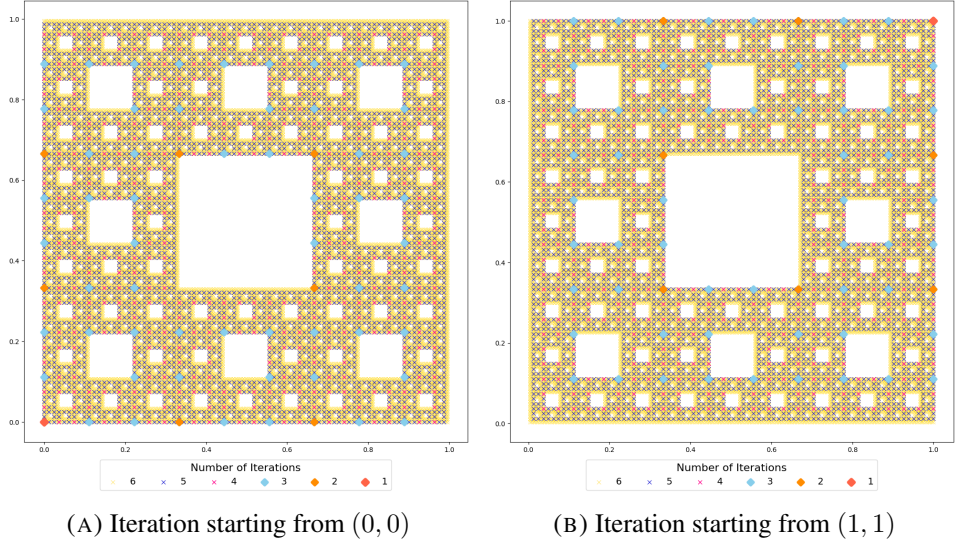


FIGURE 1. The Sierpiński-carpet

**COROLLARY 1.14.** *For all  $i, j \in \{0, 1, 2, 3\}$ ,  $(1, 1) \notin \{(i, j), (j, k), (k, i)\}$ , define the mappings  $f_{i,j,k}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by*

$$f_{ijk}(x, y, z) = \left( \frac{x+i}{3}, \frac{y+j}{3}, \frac{z+k}{3} \right).$$

*The smallest 3D invariant set which contains the origin in  $\mathbb{R}^3$  and satisfies the invariance equation*

$$H = \bigcup \left\{ f_{ijk}(H) \mid i, j, k \in \{0, 1, 2\}, \right. \\ \left. (1, 1) \notin \{(i, j), (j, k), (k, i)\} \right\}$$

*is a proper subset of the Menger-sponge, and consists of exactly those points whose components' ternary expansions are finite, and have no position in which each number has the digit 1. Moreover, its closure is exactly the Menger-sponge.*

As it can be seen from the previous examples, starting the iteration from different fixed points of the invariance equation, one may get different minimal solutions. However, the closure of any minimal solution results in the same.

Let's consider the minimal and the classical solutions of the above invariance equations. In arithmetical sense they are different: the minimalist solutions are countably infinite, while the classical sets are continuum. Nevertheless, in the sense of topology the solutions are not different: the closure of any minimal solution is the classical one. Note that these observations are essential in the forthcoming sections.

**Implementing the Kantorovich-iteration.** We also remark that the Kantorovich-iteration can be well implemented and we may get many different examples of fractals by applying the iteration on invariance equations defined on  $\mathbb{R}^2$  - similarly as we did in Corollary 1.13. Not claiming completeness, we may present only one more example to conclude this Chapter. We emphasize that we do not want to decide at this point whether the Hutchinson Theorem covers our example or not. Our main aim is to illustrate the Kantorovich-iteration again. The Python implementation of our example can be found in the Appendix.

EXAMPLE 1.15. Let the mappings  $f_{ij}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be

$$f_{ij}(x, y) = \left( \left( \frac{x+i}{3} \right)^{\exp(\frac{y+i}{3})}, \left( \frac{y+j}{3} \right)^{\exp(\frac{x+i}{3})} \right).$$

for all  $i, j \in \{0, 1, 2\}$  and  $(i, j) \neq (1, 1)$ .

It is immediate to see that  $(0, 0)$  is a fixed point of  $f_{00}$  and  $(1, 1)$  is a fixed point of  $f_{22}$ . Thus we may start the Kantorovich-iteration from any of the found fixed points.

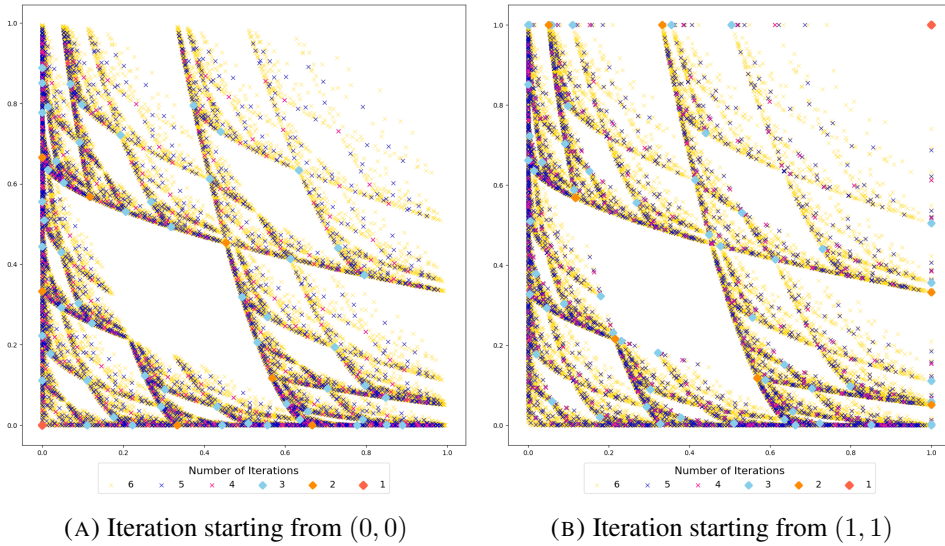


FIGURE 2. "Wave" fractals

## CHAPTER 2

### Fractals obtained by condensing families

In this Chapter we shall study the  $\Phi$  invariance operator in metric spaces. The original approach of Hutchinson [27] to fractals applies the Banach Contraction Principle [1] on the invariance equation. To do this, he uses the Blaschke Completeness Theorem [9]. Avoiding Blaschke's result, this Chapter presents an alternative via the Hausdorff noncompactness measure [26].

Moreover, our technique uses the ideas of the previous Chapter and extends the existence part of the Hutchinson Theorem. First, we use generalized contractions by the Matkowski Fixed Point Theorem [38], then condensing maps by the Darbo–Sadovskii Fixed Point Theorem [15, 44] instead of contractions. The main results of this Chapter are based on the paper by Bessenyei and Péntzes [4].

#### 2.1. Preliminaries

**Fractals defined in metric space.** Throughout this Chapter let  $(X, d)$  be a metric space. We can define the invariance equation the same way as in (1.1). However, in this Chapter the family  $\mathcal{F}$  contains finitely many self-maps of the metric space  $X$ . Hence, we shall use the notation  $\mathcal{F} = \{f_1, \dots, f_n\}$ , where  $f_1, \dots, f_n: X \rightarrow X$ .

In this setting, the invariance operator  $\Phi: X \rightrightarrows X$  of  $\mathcal{F}$  is defined by

$$(2.1) \quad \Phi(H) := \bigcup \mathcal{F}(H) = f_1(H) \cup \dots \cup f_n(H).$$

Moreover, the set  $H \subseteq X$  is  $\mathcal{F}$ -invariant, if

$$H = \Phi(H)$$

is satisfied.

Under an  $\mathcal{F}$ -fractal, we mean a nonempty, compact,  $\mathcal{F}$ -invariant subset of the metric space  $X$ .

**The classical results of Hutchinson.** For completeness, first we represent Hutchinson's classical fractal result [27] and recall the leading ideas of his proof.

**THEOREM. (Hutchinson)** *If  $\mathcal{F}$  is a finite family of contractions in a complete metric space, then there exists precisely one  $\mathcal{F}$ -fractal.*

Let  $(X, d)$  be a metric space and denote by  $\mathcal{K}(X)$  the family of nonempty and compact subsets of  $X$ . For arbitrary sets  $A, B \in \mathcal{K}(X)$  let

$$d_{HP}(A, B) := \inf \left\{ \varepsilon > 0 \mid A \subseteq \bigcup_{b \in B} U(b, \varepsilon), B \subseteq \bigcup_{a \in A} U(a, \varepsilon) \right\}.$$

The above defined value is called the *Hausdorff–Pompeiu distance* of the sets  $A$  and  $B$ . The metric space  $(\mathcal{K}(X), d_{HP})$  is often called *fractal space*, see [26] and [41].

LEMMA 2.1. *Under the previous notations and conventions,  $(\mathcal{K}(X), d_{HP})$  is a metric space.*

Blaschke [9] proved that the fractal space inherits the completeness of the original space.

THEOREM. (Blaschke) *Using the earlier notions, if  $(X, d)$  is a complete metric space, then  $(\mathcal{K}(X), d_{HP})$  is also complete.*

We can consider the invariance equation as a fixed point problem such that the desired fixed point belongs to the family  $\mathcal{K}(X)$ . It turns out, if  $\mathcal{F}$  is a finite family of contractions, then the  $\Phi$  invariance operator is also a contraction in the fractal space. Moreover, if the original space is complete, then the fractal space is also complete by the Blaschke theorem. So we can apply the Banach Contraction Principle [1].

THEOREM. (Banach) *Any contraction of a complete metric space has exactly one fixed point.*

Altogether, the found fixed point is a nonempty, compact,  $\mathcal{F}$ -invariant set. In other words, a fractal.

In the following, we prove analogous fractal results for classes of mappings that are more general than contractions. Furthermore, we develop an alternative approach to fractal theorems that differs from and is independent of Hutchinson's framework.

## 2.2. Auxiliary tools

**Matkowski contractions.** Under a *comparison function* we mean an increasing (that is monotone nondecreasing), right-continuous function  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  fulfilling  $\varphi(t) < t$  for  $t > 0$ . Clearly, any comparison function vanishes at zero:  $\varphi(0) = 0$ . Since the composition of nondecreasing, right-continuous functions remains nondecreasing and right-continuous, the iterates of comparison functions are comparison functions, as well.

Let  $(X, d)$  be an arbitrary metric space. We say that the map  $f: X \rightarrow X$  is a *Matkowski contraction* [38] (or shortly, a *generalized contraction*) with comparison function  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , if, for all elements  $x, y \in X$ , the next inequality holds:

$$(2.2) \quad d(f(x), f(y)) \leq \varphi(d(x, y)).$$

See also Boyd and Wong in [10], who studied similar generalized contractions, but they had slightly different assumptions about  $\varphi$ . As we follow Matkowski's assumptions, we simply refer to generalized contractions as Matkowski contractions in this Thesis.

For  $q \in ]0, 1[$ , the particular choice  $\varphi(t) = qt$  shows that usual contractions are special Matkowski contractions. Therefore, it is obvious that this notion of contraction is an extension of the classic Banach-type one. Moreover, the statement of the Banach Contraction Principle is also valid for these generalized contractions, which is proved by Matkowski. For further details see [38].

**THEOREM.** (Matkowski) *Each Matkowski contraction of a complete metric space has exactly one fixed point.*

The proof of the above theorem uses a basically different approach than the Banach Contraction Principle's. However, they have a common idea: the composite iterates of the mappings have a distinguished role. That is, let us consider the composite iterates of Matkowski contractions. As in the classical case, if one evaluates any member along the sequence, then it results in a Cauchy sequence.

Note also that the same fixed point property remains true if we drop the assumption on right-continuity from the definition of comparison functions. However, in our aspect, this property turns out to be crucial: Besides several technical reasons, right-continuity of the comparison function provides the continuity of Matkowski contractions.

**LEMMA 2.2.** *If  $\mathcal{F}$  is a finite family of Matkowski contractions of a metric space, then the invariance operator (2.1) is a Matkowski contraction in the Hausdorff–Pompeiu metric. In particular, its composite iterates create a Cauchy sequence.*

**PROOF.** Let  $f_1, \dots, f_n: X \rightarrow X$  be Matkowski contractions of a metric space  $X$  with comparison functions  $\varphi_1, \dots, \varphi_n$ . First we show that  $\varphi := \max\{\varphi_1, \dots, \varphi_n\}$  is a comparison function, as well. Clearly,  $\varphi$  is right-continuous and

$$\varphi(t) = \max\{\varphi_1(t), \dots, \varphi_n(t)\} < \max\{t, \dots, t\} = t.$$

In the second step, we show that the invariance operator  $\Phi$  is a Matkowski contraction with comparison function  $\varphi$ . Let  $A, B \in \mathcal{K}(X)$ . Choose  $\varepsilon > 0$  such that  $\varepsilon > d_{HP}(A, B)$  and fix  $\delta > 0$ . Then for any  $a \in A$  there exists  $b \in B$  such that  $a \in U(b, \varepsilon)$ . Hence

$$d(f_k(a), f_k(b)) \leq \varphi_k(d(a, b)) \leq \varphi(d(a, b)) \leq \varphi(\varepsilon) < \varphi(\varepsilon) + \delta.$$

This yields

$$f_k(a) \in U(f_k(b), \varphi(\varepsilon) + \delta) \subseteq \bigcup_{y \in \Phi(B)} U(y, \varphi(\varepsilon) + \delta),$$

and consequently

$$\Phi(A) \subseteq \bigcup_{y \in \Phi(B)} U(y, \varphi(\varepsilon) + \delta).$$

Similar arguments result in

$$\Phi(B) \subseteq \bigcup_{y \in \Phi(A)} U(y, \varphi(\varepsilon) + \delta).$$

Thus  $d_{HP}(\Phi(A), \Phi(B)) \leq \varphi(\varepsilon) + \delta$ . Taking the limits  $\varepsilon \downarrow d_{HP}(A, B)$  and  $\delta \downarrow 0$ , we arrive at the desired contraction property of  $\Phi$ , since  $\varphi$  is right-continuous.

The second statement about the Cauchy-property can be proved with an elegant domain invariance method developed by Matkowski. For the precise details, we recommend his paper [38].  $\square$

**Hausdorff noncompactness measure.** The following notion and its properties play a crucial role in our investigations.

Let  $(X, d)$  be a metric space. For an arbitrary set  $H \subseteq X$ , the (extended) real number

$$\chi(H) = \inf \{ \varepsilon > 0 \mid \exists x_1, \dots, x_n \in X : H \subseteq U(x_1, \varepsilon) \cup \dots \cup U(x_n, \varepsilon) \}$$

is called the *Hausdorff noncompactness measure* of  $H$ . Clearly,  $\chi(H) < +\infty$  if and only if  $H$  is bounded, and  $\chi(H) = 0$  if and only if  $H$  is totally bounded. In our investigations, we need three additional properties of  $\chi$ :

- ✧ If  $A, B$  are arbitrary subsets of  $X$ , then  $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$ .
- ✧ If  $H \subseteq X$ , then  $\chi(H) = \chi(\overline{H})$ .
- ✧ If  $X$  is complete, then  $\chi(H) = 0$  if and only if  $H$  is relatively compact.

The first and second property can be easily concluded by the interpretation of the Hausdorff noncompactness measure. The third one is a consequence of the Hausdorff Compactness Theorem and the second property. Further important properties of the Hausdorff noncompactness measure [26] with the hints of their proof can be found, for example, in the monograph of Dugundij and Granas [24].

**Condensing maps.** We introduce a closely related notion to the Hausdorff noncompactness measure.

The map  $f: X \rightarrow X$  is called *condensing* if it is continuous and decreases the Hausdorff noncompactness measure of bounded, but not totally bounded sets. That is,  $\chi(f(H)) < \chi(H)$  holds if  $H \subseteq X$  bounded and not totally bounded.

For condensing maps the Darbo–Sadovskii Fixed Point Theorem [15, 44] holds true, which is analogous to Schauder's Fixed Point Theorem [45].

**THEOREM.** (Darbo–Sadovskii) *If  $X$  is a Banach-space,  $K$  is a nonempty, bounded, closed, convex subset, then each condensing map  $f: K \rightarrow K$  has at least one fixed point.*

Now we formulate two auxiliary lemmas, which are closely related to condensing maps and their properties.

**LEMMA 2.3.** *If  $\mathcal{F}$  is a finite family of such self-maps of a metric space which decrease the Hausdorff noncompactness measure, then any bounded,  $\mathcal{F}$ -invariant set is totally bounded.*

**PROOF.** Let  $\mathcal{F} = \{f_1, \dots, f_n\}$ . Assume to the contrary that an  $\mathcal{F}$ -invariant set  $H$  is bounded, but not totally bounded. Then,  $\chi(H)$  is positive and finite. Therefore, using the properties of the Hausdorff noncompactness measure,

$$\begin{aligned} \chi(H) &= \chi(\Phi(H)) = \chi(f_1(H) \cup \dots \cup f_n(H)) \\ &= \max\{\chi(f_1(H)), \dots, \chi(f_n(H))\} \\ &< \max\{\chi(H), \dots, \chi(H)\} = \chi(H) \end{aligned}$$

follows, which is a contradiction.  $\square$

**LEMMA 2.4.** *If  $\mathcal{F}$  is a finite family of continuous self-maps of a metric space and  $H$  is relatively compact  $\mathcal{F}$ -invariant set, then  $\overline{H}$  is  $\mathcal{F}$ -invariant, as well.*

**PROOF.** Let  $y \in \Phi(\overline{H})$  be arbitrary. Then,  $y \in f(\overline{H})$  for some  $f \in \mathcal{F}$ . That is,  $y = f(x)$ , where  $x \in \overline{H}$ . Consider a sequence  $(x_m)$  from  $H$  such that  $x_m \rightarrow x$ . Since  $H$  is  $\mathcal{F}$ -invariant,  $f(x_m) \in H \subseteq \overline{H}$  holds. The continuity of  $f$  guarantees that  $y = f(x) \in \overline{H}$ . This results in the inclusion  $\Phi(\overline{H}) \subseteq \overline{H}$ . By the continuity again and the compactness of  $\overline{H}$ , the set  $\Phi(\overline{H})$  is compact, as well. In particular, it is closed.

On the other hand,  $H = \Phi(H) \subseteq \Phi(\overline{H})$  shows that  $H$  is a subset of the closed set  $\Phi(\overline{H})$ . Therefore, we arrive at the reversed inclusion  $\overline{H} \subseteq \Phi(\overline{H})$ .  $\square$

Finally, we show the connection between generalized contractions and condensing maps.

**LEMMA 2.5.** *Any Matkowski contraction of a metric space is a condensing map.*

**PROOF.** Let  $(X, d)$  be a metric space and let  $f: X \rightarrow X$  be a Matkowski contraction with comparison function  $\varphi$ .

If  $H \subseteq X$  is a bounded subset and  $\varepsilon > \chi(H)$  is fixed, then there exists a finite  $\varepsilon$ -net  $E \subseteq X$  for  $H$ . If  $x \in H$  is arbitrary, then there exists  $h \in E$  fulfilling  $d(x, h) < \varepsilon$ . Therefore,

$$d(f(x), f(h)) \leq \varphi(d(x, h)) \leq \varphi(\varepsilon),$$

yielding that  $\{f(h) \mid h \in E\}$  is a finite  $\varphi(\varepsilon)$ -net for  $f(H)$ . Thus,  $\chi(f(H)) \leq \varphi(\varepsilon)$ . Taking the limit  $\varepsilon \downarrow \chi(H)$  and using the right-continuity of  $\varphi$ , we arrive at

the desired estimation

$$\chi(f(H)) \leq \varphi(\chi(H)).$$

In particular, the properties of comparison functions guarantees  $\varphi(\chi(H)) < \chi(H)$ . Thus  $f$  decreases the Hausdorff noncompactness measure. On the other hand, any Matkowski contraction is continuous. Altogether, any Matkowski contraction is condensing.  $\square$

### 2.3. The main results

**A new approach to fractals.** In the main results we conclude the structured solutions of the  $\Phi(H) = H$  invariance equation. These are generalizations of Hutchinson's Theorem. Moreover, we use a different approach than the classical one. The basics of our methodology are inspired by the structure-free fractal theorem covered in Chapter 1.

Now the strategy of the alternative way to fractals is the following. Assume that  $\mathcal{F}$  is finite and consists of contractions. Take the fixed point of any member. Then, using the Kantorovich-process (in exactly the same way as in Chapter 1), we can produce a nonempty  $\mathcal{F}$ -invariant set. Moreover, this set is bounded, since the Kantorovich-iteration creates a Cauchy-sequence. Applying the properties of the Hausdorff noncompactness measure, the invariant set turns out to be a relatively compact one. Finally we show that the closure of the invariant set (which is hence a nonempty and compact one), is  $\mathcal{F}$ -invariant, as well. This results in the existence part of Hutchinson's Theorem. Uniqueness is an immediate consequence of the facts that the invariance operator is a contraction in the fractal space and that a contraction can have at most one fixed point.

We show in the following subsection that this method can be applied on families containing generalized contractions or condensing maps, as well.

For completeness, we note there is a known alternative way to obtain fractal results without the Banach Contraction Principle and Blaschke Completeness Theorem, using a construction with Cantor's Intersection Theorem and the properties of Hausdorff–Pompeiu, see [17].

**Generalized fractal theorems in metric space.** In the first result we formulate a fractal theorem using condensing maps.

**THEOREM 2.6.** *If  $\mathcal{F}$  is a finite family of condensing self-maps of a nonempty, bounded, closed, convex subset in a Banach space, then there exists at least one  $\mathcal{F}$ -fractal.*

**PROOF.** Let  $K$  be a nonempty, bounded, closed, convex subset of a Banach space  $X$  and let  $\mathcal{F} = \{f_1, \dots, f_n\}$  be the family of condensing self-maps of  $K$ . According to the Darbo–Sadovskii Fixed Point Theorem, there exists a fixed

point for each member of  $\mathcal{F}$ . Having such a fixed point  $x_0$ , the set  $H_1 = \{x_0\}$ , which is a half fixed point of  $\mathcal{F}$ , generates a nonempty  $\mathcal{F}$ -invariant set  $H_\infty$  via the Kantorovich-iteration (1.2) according to Theorem 1.6.

Obviously,  $H_\infty$  is bounded since  $H_\infty \subseteq K$ . Thus by Lemma 2.3  $H_\infty$  is relatively compact, as the members of  $\mathcal{F}$  decrease the Hausdorff noncompactness measure. Otherwise,  $\overline{H}_\infty$  is invariant by Lemma 2.4, as the members of  $\mathcal{F}$  are continuous, too. Therefore,  $\overline{H}_\infty$  is a nonempty, compact,  $\mathcal{F}$ -invariant set, that is an  $\mathcal{F}$ -fractal.  $\square$

In case of a compact domain, condensing and compact mappings are the same. Hence, the next statement is an immediate consequence of the previous one.

**THEOREM 2.7.** *If  $\mathcal{F}$  is a finite family of continuous self-maps of a nonempty, compact, convex subset in a Banach space, then there exists at least one  $\mathcal{F}$ -fractal.*

If the invariance operator consists of generalized contractions, then we can prove not only the existence of fractals, but also uniqueness.

**THEOREM 2.8.** *If  $\mathcal{F}$  is a finite family of Matkowski contractions of a complete metric space, then there exists precisely one  $\mathcal{F}$ -fractal.*

**PROOF.** In what follows, let  $(X, d)$  be the underlying complete metric space and  $\mathcal{F} = \{f_1, \dots, f_n\}$  be the family of Matkowski contractions with comparison functions  $\varphi_1, \dots, \varphi_n$ . By the Matkowski Fixed Point Theorem, each member of  $\mathcal{F}$  has exactly one fixed point in  $X$ . Let  $x_0$  be an arbitrary one, and let  $H_1 = \{x_0\}$ . Then,  $H_1$  is a nonempty  $\mathcal{F}$ -subinvariant set, and hence the Kantorovich-process produces a nonempty  $\mathcal{F}$ -invariant limit  $H$  by Theorem 1.6.

Consider the sequence of sets  $(H_m)$  defined in (1.2) produced by the Kantorovich-iteration. By Lemma 2.2, this is a Cauchy sequence, and hence it is bounded in the Hausdorff–Pompeiu metric. In particular, there exists  $r > 0$  such that for all  $m \in \mathbb{N}$ , the inequality  $d_{HP}(H_1, H_m) < r$  holds. That is,

$$H_m \subseteq \bigcup_{x \in H_1} U(x, r) = U(x_0, r).$$

This implies  $H_\infty \subseteq U(x_0, r)$ , showing the boundedness of  $H_\infty$  given in (1.2).

Thus, by the statement of Lemma 2.5, Lemma 2.3 and Lemma 2.4, the set  $\overline{H}_\infty$  is a nonempty, compact,  $\mathcal{F}$ -invariant set.

The uniqueness is a direct consequence of Lemma 2.2 and the fact that a Matkowski contraction may have at most one fixed point. This completes the proof.  $\square$

## 2.4. Applications, examples and remarks

It is obvious that Hutchinson's classical result is an immediate consequence of Theorem 2.8.

Observe that Theorem 2.8 can also be proved via the original approach of Hutchinson. Indeed, Lemma 2.2 guarantees that the invariance operator induces a Matkowski contraction in the space  $(\mathcal{K}(X), d_{HP})$ . This space becomes complete if the underlying space is complete, by the Blaschke Theorem. These facts enable us to use the Matkowski Fixed Point Theorem directly, from which we can conclude the uniqueness and existence of a nonempty, bounded and closed invariant set. After some extra effort, this invariant set also turns out to be compact.

However, the classical approach cannot be followed to prove Theorem 2.6, since the continuous image of a closed set is not necessarily closed. Therefore, the invariance operator may not be a self-map of the space of nonempty, closed, bounded subsets.

We further note that the Kantorovich-iteration enables the approximation of fractals if one point of the fractal is known. This approximation can be applied even if the usual Banach–Picard-iteration cannot be used.

Moreover, Theorem 2.8 may work even in very simple cases where the Hutchinson Theorem does not. We illustrate this with the examples below by modifying the invariance equation of the Cantor-set. In the examples, the metric space is the set of nonnegative reals  $\mathbb{R}_+$  equipped with the Euclidean metric.

EXAMPLE 2.9. Let  $\mathcal{F} = \{f_1, f_2\}$  where  $f_2(x) = \frac{1}{3}x + \frac{2}{3}$ .

✧ If  $f_1(x) = x$ , then the smallest  $\mathcal{F}$ -fractal containing zero is the set

$$\left\{ \frac{2}{3} + \cdots + \frac{2}{3^n} \mid n \in \mathbb{N} \right\} \cup \{0, 1\}.$$

✧ The existence of a smallest  $\mathcal{F}$ -fractal containing zero also follows if  $f_1(x) = \arctan(x)$ .

The map defined above,  $f_1(x) = x$ , is not a Banach contraction, and thus Hutchinson's result cannot be applied. What is more, the interval  $[0, 1]$  is a fractal solution in this example. Therefore, uniqueness cannot be guaranteed in general.

The same holds for the mapping  $Tx := \arctan x$ : it cannot be a Banach contraction on any neighborhood of its unique fixed point  $p = 0$ , which can be shown by means of the Lagrange Mean Value Theorem. If we add to this mapping the transformation obtained by shifting the one-third contraction by two-thirds, then we obtain a condensing but non-contractive family. Hence, Hutchinson's theorem cannot be applied. However, the members of the family map the interval  $[0, 1]$  into itself, and therefore the Kantorovich iteration starting from the fixed point  $p = 0$  is necessarily bounded. Consequently, by Theorem 2.8, there exists a smallest fractal containing the point  $p = 0$ .

In the following example, we show a Matkowski contraction which is not a Banach contraction. Moreover, as it turns out, we cannot use the Banach Contraction Principle while the Matkowski Fixed Point Theorem still works.

EXAMPLE 2.10. Let  $X = [0, 1]$  and let  $d$  be the Euclidean metric. Furthermore, define the function  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and the mapping  $T: X \rightarrow X$  as follows:

$$\varphi(t) := \frac{t}{t+1} \quad \text{and} \quad Tx := \frac{1}{2-x}.$$

Then  $\varphi$  is a comparison function, and  $T$  is a Matkowski contraction with respect to  $\varphi$ .

The latter property is (by a straightforward computation) equivalent to the following inequality:

$$1 + |x - y| \leq |4 - 2(x + y) + xy|.$$

In the second term, the absolute value may be omitted due to the assumption  $x, y \in [0, 1]$ . After this, we may assume that  $x \geq y$ . Then the inequality equivalent to the one to be proved is

$$0 \leq 3 - 3x - y + xy = 2 - 2x + (1 - x)(1 - y),$$

which is obviously true for  $x, y \in [0, 1]$ . It is clear that the unique fixed point of  $T$  is the point  $p = 1$ . Moreover,

$$\sup_{x \neq p} \frac{|Tx - Tp|}{|x - p|} = \sup_{x \neq 1} \frac{|\frac{1}{2-x} - 1|}{|x - 1|} = \sup_{x \neq 1} \frac{|x - 1|}{|x - 1|} = 1.$$

This means that  $T$  is not a Banach contraction on any neighbourhood of the fixed point  $p$ . Therefore,  $T$  can behave as a Banach contraction only on a metric subspace that does not contain the fixed point  $p$ , and consequently such a subspace must necessarily be fixed-point free. In summary, the Banach Fixed Point Theorem is not applicable in this case, whereas Matkowski's theorem is.

To end this Chapter, we present here two examples that cannot be handled by the Hutchinson Theorem. In both examples, the involved families contain such members that have more than one fixed point. Evidently, such mappings cannot be contractions even if we anyhow change the underlying metric. The implementations are based on the Kantorovich-iteration as in Example 1.15. The source Python codes can be found in the Appendix. We do not solve the fixed point equations exactly. We may rely on just a numerical optimization algorithm as its accuracy is convincing enough. We placed its source code in the Appendix as well.

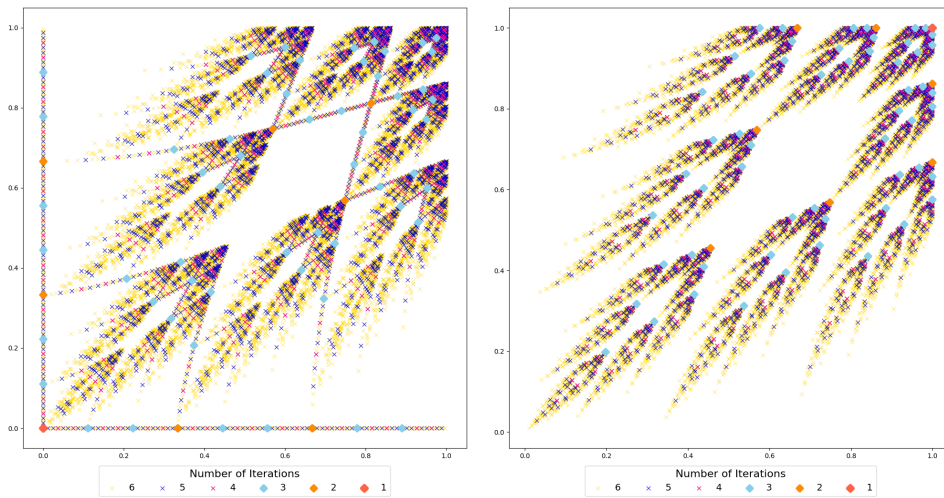
EXAMPLE 2.11. Let the mappings  $f_{ij}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be

$$f_{ij}(x, y) = \left( \left( \frac{x+i}{3} \right)^{\frac{1}{\exp(\frac{y+i}{3})}}, \left( \frac{y+j}{3} \right)^{\frac{1}{\exp(\frac{x+j}{3})}} \right).$$

for all  $i, j \in \{0, 1, 2\}$  and  $(i, j) \neq (1, 1)$ .

The next table contains the fixed points of the mappings (rounded to two digits) and the figures show the fractal approximations using two different fixed points in the Kantarovich-iteration (in six steps).

| Fixed point              | $(i, j)$ |
|--------------------------|----------|
| $(0, 0)$                 | $(0, 0)$ |
| $(0, 0.5); (0.19, 0.53)$ | $(0, 1)$ |
| $(0, 1); (0.53, 1)$      | $(0, 2)$ |
| $(0.5, 0); (0.53, 0.19)$ | $(1, 0)$ |
| $(0.83, 1)$              | $(1, 2)$ |
| $(1, 0); (1, 0.53)$      | $(2, 0)$ |
| $(1, 0.83)$              | $(2, 1)$ |
| $(1, 1)$                 | $(2, 2)$ |



(A) Iteration starting from  $(0, 0)$

(B) Iteration starting from  $(1, 1)$

FIGURE 1. "Falling stars" fractals

EXAMPLE 2.12. Let the mappings  $f_{ij}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be

$$f_{ij}(x, y) = \left( (\sin(x + i))^{\frac{1}{\exp(y+i)}}, (\sin(y + j))^{\frac{1}{\exp(x+i)}} \right).$$

for all  $i, j \in \{0, 1, 2\}$  and  $(i, j) \neq (1, 1)$ .

The next table contains the fixed points of the mappings (rounded to two digits) and the figures show the fractal approximations using two different fixed points in the Kantarovich-iteration (in six steps).

| Fixed point               | $(i, j)$ |
|---------------------------|----------|
| $(0, 0); (0.91, 0.91)$    | $(0, 0)$ |
| $(0, 0.93); (0.97, 0.97)$ | $(0, 1)$ |
| $(0, 0.55); (0.99, 0.72)$ | $(0, 2)$ |
| $(0.93, 0); (0.97, 0.97)$ | $(1, 0)$ |
| $(0.99, 0.85)$            | $(1, 2)$ |
| $(0.55, 0); (0.72, 0.99)$ | $(2, 0)$ |
| $(0.85, 0.99)$            | $(2, 1)$ |
| $(0.92, 0.92)$            | $(2, 2)$ |

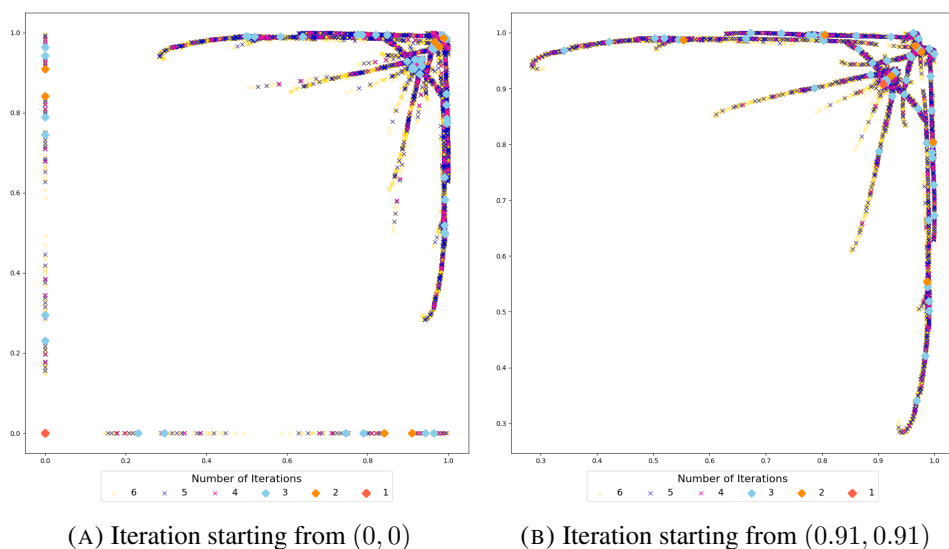


FIGURE 2. "Branchfall" fractals

Observe that the fractal figures in Example 2.11 and Example 2.12 depend on the choice of the starting point. This phenomenon suggests that in general uniqueness cannot be guaranteed if the family contains non-contractive members. The problem of uniqueness has been studied in detail in a recent paper by Bessenyei and Szabó [7]. In this paper one can find further examples of non-contractive iterated function systems.



## CHAPTER 3

### Generalized fractals in semimetric spaces

The aim of this Chapter is to develop a general method that guarantees the resolvability of the  $\Phi(H) = H$  invariance equation among proper compact sets. The extension now concerns both the underlying space and the mapping: our main result is an existence theorem for those fractals that are generated by condensing maps of complete and regular semimetric spaces.

Semimetric spaces are intensively studied by Jachymski, Matkowski, Świątkowski in [29] and by Chrząszcz, Jachymski, Turoboś in [14]. The key fixed point theorem of the Chapter is by Bessenyei and Páles [2], which is an extension of the Matkowski Fixed Point Theorem in semimetric spaces. The main results are based on the paper by Bessenyei and Péntes [5].

#### 3.1. Preliminaries

**Topology of semimetric spaces.** Under a *semimetric space* we mean a pair  $(X, d)$  where  $X$  is a nonempty set, and  $d: X^2 \rightarrow \mathbb{R}_+$  is a positive definite, symmetric function, that is, fulfils the first two axioms of metric spaces by Fréchet [18].

In a semimetric space, the notions of *open ball*, *Cauchy sequences*, *convergent sequences*, and *continuity* of functions between semimetric spaces can be introduced similarly as in metric spaces. A semimetric space is *complete* if every Cauchy sequence is convergent.

Having these notions, we have essentially two possibilities to introduce topology:

- ✧ The *neighbourhood topology* is generated by the *open sets*. A set is open if it contains all its points together with an open ball centred at the point. As usual, a set is closed if its complement is open.
- ✧ The *sequential topology* is defined in the following way. A *closure point* of a set is a point that arises as the limit of a sequence drawn from the set. A set is *closed* if it contains all its closure points.

In metric spaces, sequentially closed sets coincide with topologically closed sets. However, in semimetric spaces this property does not hold in general: the neighbourhood topology and the sequential topology may differ. In particular, the

neighbourhood topology need not be Hausdorff; limits of sequences may be non-unique; and a convergent sequence need not be Cauchy. The paper [14] by Chrzszcz, Jachymski, and Turoboř and the references therein give a nice and detailed overview of this topic.

Following [14], let  $\mathcal{T}_d$  denote the neighbourhood topology and let  $\mathcal{T}_d^{\text{seq}}$  be the sequential topology obtained from the limit-operator/sequence-closure approach. Introduce also  $\mathcal{T}_d^\wedge$ , the topology generated by *finite intersections* of open balls. A central structural statement of Chrzszcz–Jachymski–Turoboř is

$$\mathcal{T}_d^\wedge = \mathcal{T}_d^{\text{seq}} \subseteq \mathcal{T}_d$$

and the inclusion may be proper. Thus, outside the regular regime, the neighbourhood topology can strictly dominate the sequential one. The next example – based exactly on the construction in [14] – illustrates that the gap can be extreme even under robust “polygonal” control.

**EXAMPLE 3.1.** *The  $c$ -relaxed polygonal inequality and a topology gap: A semimetric  $d$  satisfies the  $c$ -relaxed polygonal inequality if there exists  $c \geq 1$  such that, for every finite chain*

$$x_0, x_1, \dots, x_n \in X \quad (n \geq 1),$$

*one has*

$$d(x_0, x_n) \leq c \sum_{k=0}^{n-1} d(x_k, x_{k+1}).$$

Using a general construction based on Hewitt’s resolvability theorem, Chrzszcz, Jachymski and Turoboř [14, Thm. 5.6] are building semimetrics satisfying a  $c$ -relaxed polygonal inequality for which *no* open ball is open and, simultaneously, *no* closed ball is closed. Combining this with the previously mentioned result of [14], one obtains  $\mathcal{T}_d^\wedge = \mathcal{T}_d^{\text{seq}} \subsetneq \mathcal{T}_d$ , so the neighbourhood topology is strictly finer than the sequential one even under the  $c$ -relaxed polygonal control.

In summary, open balls of a semimetric space indeed induce a topology in the usual way, though the induced topology does not necessarily contain the open balls. In semimetric spaces we can only say that the centre of an open ball is an inner point. However, in this Thesis we do not rely on the topological properties of open balls, only on the induced topology itself.

**The triangle function.** In spite of these deviations, semimetric spaces have a tight relation with metric spaces as the next construction illustrates.

Let  $(X, d)$  be a semimetric space. We say that  $\Delta^*: \overline{\mathbb{R}}_+^2 \rightarrow \overline{\mathbb{R}}_+$  is a *triangle function* for  $d$  if  $\Delta^*$  is symmetric and monotone increasing in both of its arguments, satisfies  $\Delta^*(0, 0) = 0$ , and, for all  $x, y, z \in X$ ,

$$d(x, y) \leq \Delta^*(d(x, z), d(y, z)).$$

Obviously, metric spaces are semimetric spaces with  $\Delta^*(u, v) := u + v$ . Ultrametric spaces are also semimetric spaces if we choose  $\Delta^*(u, v) := \max\{u, v\}$ . Not claiming completeness, we present here some intensively studied cases that can be interpreted in this framework:

- ✧  $\Delta^*(u, v) = c(u + v)$   
induces the *c-relaxed triangle inequality*, where  $c \geq 1$ ;
- ✧  $\Delta^*(u, v) = c \max\{u, v\}$   
induces the *c-inframetric inequality*, where  $c \geq 1$ ;
- ✧  $\Delta^*(u, v) = (u^p + v^p)^{1/p}$   
induces the *p-th-order triangle inequality*, where  $p > 0$ .

It turns out that each semimetric space can be equipped with a triangle function  $\Delta: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  lying below every triangle function  $\Delta^*$  of the space, that is  $\Delta(u, v) \leq \Delta^*(u, v)$  for all  $(u, v) \in \mathbb{R}_+^2$ . This optimal triangle function  $\Delta$  is called the *basic triangle function*. For details, see [2].

**Normal and regular semimetric spaces.** Having the basic triangle function, we introduce two further important properties. A semimetric space is

- ✧ *normal* if its basic triangle function is finite-valued;
- ✧ *regular* if its basic triangle function is continuous at the origin.

Clearly, the examples above are both normal and regular semimetric spaces. Throughout this thesis, we focus on regular semimetrics.

Regular semimetric spaces can be characterized in terms of uniform equivalence. We say that the semimetrics  $d_1$  and  $d_2$  on  $X$  are *uniformly equivalent*, if the identity map  $\text{id}: (X, d_1) \rightarrow (X, d_2)$  is uniformly bi-continuous. That is,

- (i) for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $d_2(x, y) < \varepsilon$  if  $d_1(x, y) < \delta$ ; and
- (ii) for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $d_1(x, y) < \varepsilon$  if  $d_2(x, y) < \delta$ .

Chrząszcz, Jachymski and Turoboś studied normal and regular semimetric spaces, too, and their fundamental result from [14] reads as follows.

**PROPOSITION 3.2.** *A semimetric is regular if and only if it is uniformly equivalent to a metric.*

Uniform equivalence in itself guarantees that the neighbourhood topology, the sequential topology, continuity, and all notions related to sequences are invariants of such semimetrics. Moreover, in view of this theorem the above-mentioned pathologies disappear. In regular semimetric spaces, the neighbourhood and sequential topologies coincide and the Hausdorff property holds. In other words, the topological behaviour of regular semimetric spaces and metric spaces cannot be distinguished.

**EXAMPLE 3.3.** *If a semimetric  $d$  on a set  $X$  satisfies the c-relaxed or the c-inframetric or the p-th-order triangle inequality, then  $(X, d)$  is a regular semimetric space.*

A key takeaway from [14] is that all spaces in Example 3.3 are uniformly metrizable. In particular, the neighbourhood, finite-intersection, and sequential topologies coincide and the usual metric-type properties (Hausdorff, uniqueness of limits, relationship of convergent and Cauchy sequences) are recovered.

In any semimetric space  $(X, d)$ , a set  $H$  is *bounded* by definition if there exists  $x_0 \in X$  and  $r > 0$  such that  $H \subseteq U(x_0, r)$ . In normal semimetric spaces, boundedness can be characterized in the same way as in metric spaces:

**PROPOSITION 3.4.** *If  $(X, d)$  is a normal semimetric space and  $H \subseteq X$ , then the following statements are equivalent:*

- (i) *for all  $x_0 \in X$  there exists  $r > 0$  such that  $H \subseteq U(x_0, r)$ ;*
- (ii) *there exists  $x_0 \in X$  and  $r > 0$  such that  $H \subseteq U(x_0, r)$ ;*
- (iii)  $\sup\{d(x, y) \mid x, y \in H\} := \text{diam}(H) < +\infty$ .

**PROOF.** Denote the basic triangle function of  $(X, d)$  by  $\Delta$ . Implication (i)  $\Rightarrow$  (ii) is trivial. To prove (ii)  $\Rightarrow$  (iii), let  $a, b \in H$  be arbitrary. Then, by partial monotonicity of  $\Delta$ ,

$$d(a, b) \leq \Delta(d(x_0, a), d(x_0, b)) \leq \Delta(r, r) =: r_0.$$

Thus  $\text{diam}(H) \leq r_0 < +\infty$  since  $d$  is normal. For the implication (iii)  $\Rightarrow$  (i), assume that  $r_0 := \text{diam}(H) < +\infty$  and let  $x_0 \in X$  be arbitrary. Fix  $y \in H$ , and let  $x \in H$ . Then  $d(x, y) \leq r_0$  and hence

$$d(x_0, x) \leq \Delta(d(x_0, y), d(y, x)) \leq \Delta(d(x_0, y), r_0) =: r.$$

Normality ensures that  $r < +\infty$  and the inequality above shows that  $H \subseteq U(x_0, r)$ . That is, the proof is completed.  $\square$

**Fixed point results in semimetric spaces.** Besides the above generalizations of metric spaces, we shall need an appropriate extension of the Banach Contraction Principle [1].

In an arbitrary semimetric space, generalized contractions can be defined similarly as in metric spaces, see Section 2.2. A variant of fixed point results for a certain kind of semimetric space appears in the paper by Jachymski, Matkowski, and Świątkowski [29]. Our key tool is the fixed point theorem of Bessenyei and Páles [2], which is an extension of the Matkowski Fixed Point Theorem [38]. It states the existence of a unique fixed point of generalized contractions in semimetric spaces. The Cauchy-property in the second part of the Proposition will be fundamental as well. It can be proved with an appropriate domain invariance method; for details see [2].

**PROPOSITION 3.5.** *The composite iterates of a generalized contraction in a regular semimetric space generate Cauchy sequences pointwise. Each generalized contraction of a complete and regular semimetric space has a unique fixed point.*

### 3.2. Auxiliary tools

**Hausdorff noncompactness measure.** Notice that a similar noncompactness measure can be introduced in semimetric spaces, to the one invented by Hausdorff [26] in the metric setting.

Let  $(X, d)$  be a semimetric space and  $H \subseteq X$ . Then the (extended) nonnegative real number

$$\chi(H) = \inf \{ \varepsilon > 0 \mid \exists x_1, \dots, x_n \in X : H \subseteq U(x_1, \varepsilon) \cup \dots \cup U(x_n, \varepsilon) \}$$

is the *noncompactness measure* of  $H$ , while the finite set  $E = \{x_1, \dots, x_n\}$  is an  $\varepsilon$ -net for  $H$ . The next lemma summarizes the noncompactness measure's properties in semimetric context.

LEMMA 3.6. *If  $(X, d)$  is a regular semimetric space, then*

- (i)  $\chi(A) \leq \chi(B)$  whenever  $A \subseteq B \subseteq X$ ;
- (ii)  $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$  for all  $A, B \subseteq X$ ;
- (iii)  $\chi(A) = 0$  if and only if  $\chi(\overline{A}) = 0$ .

PROOF. Statement (i) is obvious, since each finite  $\varepsilon$ -net of  $B$  is a finite  $\varepsilon$ -net for  $A$  provided that  $A \subseteq B$ .

To prove (ii), let  $\varepsilon > \max\{\chi(A), \chi(B)\}$ . Then there exist  $E_A$  and  $E_B$  finite  $\varepsilon$ -nets for  $A$  and  $B$ , respectively. Then  $E_A \cup E_B$  is a finite  $\varepsilon$ -net for  $A \cup B$ , resulting in

$$\chi(A \cup B) \leq \varepsilon.$$

Taking the limit  $\varepsilon \rightarrow \max\{\chi(A), \chi(B)\}$ , we arrive at  $\chi(A \cup B) \leq \max\{\chi(A), \chi(B)\}$ . The other inequality follows from the property (i).

Therefore, to verify statement (iii), we have to show only that  $\chi(A) = 0$  implies  $\chi(\overline{A}) = 0$ . Let  $\varepsilon > 0$  be arbitrary. By regularity, the basic triangle function  $\Delta$  is continuous at the origin. Thus there exists  $\delta > 0$  such that  $\Delta(\delta, \delta) < \varepsilon$ . Since  $\chi(A) = 0$ , there exists a finite  $\delta$ -net for  $A$ :

$$A \subseteq \bigcup_{x \in E} U(x, \delta).$$

Regularity guarantees that the topological and analytical closure of  $A$  coincide by Proposition 3.2. Therefore

$$\overline{A} \subseteq \bigcup_{a \in A} U(a, \delta).$$

If  $a^* \in \overline{A}$  is arbitrary, then there exists  $a \in A$  such that  $a^* \in U(a, \delta)$  and there exists  $x \in E$  such that  $a \in U(x, \delta)$ . Thus

$$d(x, a^*) \leq \Delta(d(x, a), d(a^*, a)) \leq \Delta(\delta, \delta) < \varepsilon.$$

This shows that  $E$  is a finite  $\varepsilon$ -net for  $\overline{A}$ . Since  $\varepsilon > 0$  was arbitrary,  $\chi(\overline{A}) = 0$  follows.  $\square$

As we saw in Chapter 2, the third property can be improved in metric spaces so that  $\chi(\overline{A}) = \chi(A)$ . However, the present relaxed form, combined with Proposition 3.2, is still strong enough to prove the semimetric version of the Hausdorff Compactness Theorem.

We note here that a subset of a semimetric space is *totally bounded* if its non-compactness measure is zero.

**LEMMA 3.7.** *In a regular semimetric space, a subset is compact if and only if it is complete and totally bounded. In a complete and regular semimetric space, a subset is relatively compact if and only if its Hausdorff measure of noncompactness is zero.*

**PROOF.** Let  $(X, d)$  be a regular semimetric space. By Proposition 3.2, there exists a metric  $\rho$  uniformly equivalent to  $d$ . We note first that compactness, completeness, and totally boundedness are invariants of uniform equivalence. That is, a set  $H \subseteq X$  is

- (i) compact in  $d$  if and only if it is compact in  $\rho$ ;
- (ii) complete in  $d$  if and only if it is complete in  $\rho$ ;
- (iii) totally bounded in  $d$  if and only if it is totally bounded in  $\rho$ .

The first and second properties are immediate consequences of Proposition 3.2. Now assume that  $H$  is totally bounded in  $\rho$  and let  $\varepsilon > 0$  be arbitrary. Choose  $\delta > 0$  such that  $d(x, y) < \varepsilon$  whenever  $\rho(x, y) < \delta$ . Since  $H$  is totally bounded, there exists a finite  $\delta$ -net  $E$  for  $H$ : If  $x \in H$  is arbitrary, we can find  $y \in E$  such that  $\rho(x, y) < \delta$ . In this case,  $d(x, y) < \varepsilon$ , showing that  $E$  is a finite  $\varepsilon$ -net for  $H$  in  $d$ . The converse statement can be proved similarly. Thus the first part of the lemma is a consequence of the Hausdorff Compactness Theorem.

For the second part, assume that  $H$  is relatively compact. Then  $\overline{H}$  is totally bounded by the first part. That is,  $\chi(\overline{H}) = 0$ ; in particular,  $\chi(H) = 0$  by the first property of Lemma 3.6. Conversely, assume that  $\chi(H) = 0$ . Then  $\chi(\overline{H}) = 0$  by the third property of Lemma 3.6. Thus  $\overline{H}$  is totally bounded. On the other hand, it is a closed subset in a complete and regular semimetric space and hence  $\overline{H}$  is complete. By the first part again,  $\overline{H}$  must be compact.  $\square$

**Condensing maps.** Similarly as in metric space, a map  $f: X \rightarrow X$  of a semimetric space  $(X, d)$  is called *condensing* if it is continuous and decreases the noncompactness measure of bounded, but not totally bounded sets. That is,  $\chi(f(H)) < \chi(H)$  provided that  $H \subseteq X$  is bounded and not totally bounded.

In the following lemma we can see that the class of generalized contractions is one of the most important subclasses of condensing maps. For the same result in metric space refer to Lemma 2.5 and to Granas and Dugundji [24].

**LEMMA 3.8.** *Each generalized contraction of a complete and regular semimetric space is a condensing map.*

HINT. Use the same arguments as in the metric space case Lemma 2.5 with some extra considerations on the properties of noncompactness measure. Namely, the previous Lemma 3.7.  $\square$

**The fractal space.** Given a semimetric space  $(X, d)$ , denote the family of nonempty and compact subsets by  $\mathcal{K}(X)$ . Notice that the Hausdorff–Pompeiu distance [41] can be introduced as in a metric space. That is, for  $A, B \in \mathcal{K}(X)$ , define the (extended) real number

$$d_{HP}(A, B) := \inf \left\{ r > 0 \mid A \subseteq \bigcup_{b \in B} U(b, r), \ B \subseteq \bigcup_{a \in A} U(a, r) \right\}.$$

The next lemma shows that  $(\mathcal{K}(X), d_{HP})$  is a semimetric space under suitable assumptions. This space is the *fractal space* over  $(X, d)$ . Note that the proof is similar as in Lemma 2.1, completing with the properties of triangle function.

LEMMA 3.9. *The fractal space over a normal and regular semimetric space is a regular semimetric space.*

PROOF. Let  $(X, d)$  be a normal and regular semimetric space with basic triangle function  $\Delta$  and consider its fractal space  $(\mathcal{K}(X), d_{HP})$ . Normality and regularity ensure that compact sets, in particular: fractals are bounded and closed. Observe first that  $d_{HP}$  has finite values. Indeed, for arbitrary  $A, B \in \mathcal{K}(X)$ , there exist positive numbers  $\alpha, \beta$  and there exist  $x, y \in X$  such that

$$A \subseteq U(x, \alpha) \quad \text{and} \quad B \subseteq U(y, \beta)$$

by boundedness of compact sets. Thus, for all  $a \in A$  and for all  $b \in B$ ,

$$\begin{aligned} d(a, b) &\leq \Delta(d(x, a), d(x, b)) \\ &\leq \Delta(d(x, a), \Delta(d(x, y), d(y, b))) \\ &\leq \Delta(\alpha, \Delta(d(x, y), \beta)) =: r \end{aligned}$$

by the monotonicity properties of  $\Delta$ . Moreover,  $r < +\infty$  since  $\Delta$  is real valued. Therefore,

$$a \in U(b, r) \subseteq \bigcup_{b \in B} U(b, r) \quad \text{and} \quad b \in U(a, r) \subseteq \bigcup_{a \in A} U(a, r).$$

Hence  $d_{HP}(A, B) \leq r < +\infty$ . If  $A = B$ , then  $d_{HP}(A, B) = 0$  obviously holds.

Conversely, assume that  $d_{HP}(A, B) = 0$  for some  $A, B \in \mathcal{K}(X)$ . Let  $a \in A$  be fixed. Then, for all  $n \in \mathbb{N}$ , there exists  $b_n \in B$ , such that  $d(a, b_n) < 1/n$ . Thus  $(b_n)$  tends to  $a \in A$ . Since  $B$  is closed and the space is regular,  $a \in B$ . However,  $a \in A$  is arbitrary, consequently  $A \subseteq B$ . The other inclusion can be proved similarly, resulting in  $A = B$ .

The symmetry follows directly from the definition. These arguments prove that the fractal space is a semimetric space, indeed.

Finally, for fixed  $A, B, C \in \mathcal{K}(X)$ , let  $d_{HP}(A, B) < r_1$  and let  $d_{HP}(B, C) < r_2$ . If  $a \in A$  is arbitrary, then there exists  $b \in B$  and  $c \in C$ , such that  $d(a, b) < r_1$  and  $d(b, c) < r_2$ . Thus  $d(a, c) < \Delta(r_1, r_2)$ . Interchanging the role of  $A$  and  $C$ , we can conclude that

$$A \subseteq \bigcup_{c \in C} U(c, \Delta(r_1, r_2)) \quad \text{and} \quad C \subseteq \bigcup_{a \in A} U(a, \Delta(r_1, r_2)).$$

In other words,  $d_{HP}(A, C) \leq \Delta(r_1, r_2)$  whenever  $d_{HP}(A, B) < r_1$  and  $d_{HP}(B, C) < r_2$ . Consider now the function  $\Delta^*: \mathbb{R}_+^2 \rightarrow \mathbb{R}$  defined by

$$\Delta^*(u, v) = \lim_{n \rightarrow \infty} \Delta\left(u + \frac{1}{n}, v + \frac{1}{n}\right).$$

The partial monotonicity of  $\Delta$  ensures that  $\Delta^*$  is well-defined, while the previous observation guarantees that  $\Delta^*$  is a triangle function for  $d_{HP}$ . Thus  $\Psi \leq \Delta^*$  where  $\Psi$  stands for the basic triangle function of the fractal space.

Now we prove that  $\Delta^*$  is continuous at the origin. Let  $\varepsilon > 0$  be arbitrary. By the regularity of the underlying space,  $\Delta$  is continuous at the origin, thus there exists  $\delta > 0$  such that  $\Delta(u, v) < \varepsilon$  if  $0 \leq u, v < \delta$ . If  $n \in \mathbb{N}$  is sufficiently large,

$$0 \leq u + \frac{1}{n}, v + \frac{1}{n} < \delta \quad \text{hence} \quad \Delta\left(u + \frac{1}{n}, v + \frac{1}{n}\right) < \varepsilon.$$

Taking the limit  $n \rightarrow \infty$ , we arrive at  $\Delta^*(u, v) \leq \varepsilon$ . The inequality  $0 \leq \Psi \leq \Delta^*$  implies that  $\Psi$  is continuous at the origin, as well.  $\square$

We note here the classical metric space case in Lemma 2.1 also follows from the above result.

### 3.3. The main results

**Invariant sets of semimetric spaces.** Let us recall the definition of the invariance operator  $\Phi: X \rightrightarrows X$  defined in (2.1),

$$\Phi(H) := \bigcup \mathcal{F}(H),$$

where the family  $\mathcal{F}$  of self-maps of a set  $X$  induces a set-valued operator via the right-hand side of the invariance equation.

Evidently, a subset of  $X$  is  $\mathcal{F}$ -invariant if and only if it is a fixed point of the invariance operator. This easy observation enables us to use the standard methods of fixed point theory for fractals, the same way as in the previous sections. We note the Kantorovich-iteration ((1.2)) and the Kantorovich Fixed Point Theorem (Theorem 1.6) are important in our further investigations, as well. That is, we use the fact that the Kantorovich-iteration extends each  $\mathcal{F}$ -subinvariant set to the smallest  $\mathcal{F}$ -invariant set.

In the next results, the roles of the two defining properties of condensing maps are enlightened. Note that we presented the same results in Lemma 2.3 and Lemma 2.4 in metric space.

LEMMA 3.10. *If  $\mathcal{F}$  is a finite family of self-maps of a complete and regular semimetric space whose members decrease the Hausdorff noncompactness measure of bounded, but not totally bounded sets, then any bounded,  $\mathcal{F}$ -invariant set is relatively compact.*

HINT. Use the same ideas as in Lemma 2.3 and the properties of noncompactness measure in semimetric space, as presented in Lemma 3.6.  $\square$

LEMMA 3.11. *If  $\mathcal{F}$  is a finite family of continuous self-maps of a regular semimetric space, then the closure of any relatively compact  $\mathcal{F}$ -invariant set is  $\mathcal{F}$ -invariant, as well.*

HINT. By Proposition 3.2, we can apply topological arguments in the standard way as in Lemma 2.4.  $\square$

As we saw in the Preliminaries of Chapter 2, Hutchinson's original idea is to handle the invariance equation as a fixed point problem of the invariance operator (2.1) in the fractal space. His key observation is that the invariance operator inherits the contraction properties of the generating family  $\mathcal{F}$ . Our last auxiliary result corresponds to this idea. However, we shall need it only for proving the uniqueness of generalized fractals. For the same result in metric space refer to Lemma 2.2.

THEOREM 3.12. *If  $\mathcal{F}$  is a finite family of generalized contractions of a normal and regular semimetric space, then the invariance operator (2.1) is a generalized contraction in the fractal space, and the sequence of its composite iterates generates Cauchy sequences pointwise.*

HINT. Use the arguments of the proof of Lemma 2.2. Additionally, for the second statement consider Proposition 3.5.  $\square$

**Fractal theorems in semimetric space.** The essence of the previous lemmas is our first main result on the existence of generalized fractals in semimetric space. We may regard it as the topological correspondence of Theorem 1.6. The proof uses the same method to approximate fractals as in previous sections: The Kantorovich-iteration plays the role of the Banach–Picard one. Let us emphasize that the method avoids completely the hyperspace of fractals.

THEOREM 3.13. *If  $\mathcal{F}$  is a finite family of condensing self-maps of a normal, complete and regular semimetric space such that the invariance operator generates a bounded Kantorovich-iteration on a nonempty  $\mathcal{F}$ -subinvariant set  $H$ , then there exists a smallest  $\mathcal{F}$ -fractal containing  $H$ .*

PROOF. Consider the invariant set  $H_\infty$  generated by  $H$  and guaranteed by Theorem 1.6. Then  $H_\infty$  is bounded and hence  $\overline{H}_\infty$  is compact by Lemma 3.10. Moreover, it is also invariant by Lemma 3.11. Thus  $\overline{H}_\infty$  is a fractal since  $H \subseteq \overline{H}_\infty$  and  $H$  is nonempty. Finally,  $H_\infty \subseteq K$  whenever  $K$  is a fractal containing  $H$  by Theorem 1.6. Therefore  $\overline{H}_\infty \subseteq \overline{K} = K$  and the proof is completed.  $\square$

In general, we cannot state uniqueness in Theorem 3.13 even if the underlying space is a metric space (see Example 2.9). Furthermore, the assumptions on the existence of a subinvariant set and the boundedness of the Kantorovich iteration may seem to be sophisticated. However, these critical observations lose their validity if we impose suitable restrictions on the family: It turns out that, using Theorem 3.13, we can generalize completely Hutchinson's result for generalized contractions semimetric spaces:

**THEOREM 3.14.** *If  $\mathcal{F}$  is a finite family of generalized contractions of a complete, normal and regular semimetric space, then there exists a unique  $\mathcal{F}$ -fractal.*

PROOF. Clearly, Proposition 3.5 provides a nonempty  $\mathcal{F}$ -subinvariant set using the fixed point of the generalized contraction. (The singleton containing the fixed point is  $\mathcal{F}$ -subinvariant.) Thus there exists a nonempty and smallest  $\mathcal{F}$ -invariant set by Theorem 1.6. Furthermore, the Kantorovich-iteration produces a Cauchy sequence in the fractal space by Theorem 3.12. Using Proposition 3.4 and the same arguments as in the proof of Theorem 2.8 one can prove that the members of the Cauchy-sequence are allocated in the same ball. Hence their union, our  $\mathcal{F}$ -invariant set is bounded. Applying Theorem 3.13 we arrive at the existence part. In other words, we found a fixed point of the invariance operator. However, the invariance operator is a generalized contraction in the fractal space by Lemma 3.9 and a generalized contraction has at most one fixed point. This yields uniqueness.  $\square$

A complete metric space is a complete, normal and regular semimetric space. Thus Theorem 2.8 and hence in particular Hutchinson's result is a direct consequence of Theorem 3.14. Moreover, relaxed metric spaces, inframetric spaces, and  $p$ th order spaces mentioned in the previous subsection are normal and regular semimetric spaces, as well. Therefore Theorem 3.14 applies for these settings. We omit the explicit formalizations of these corollaries.

### 3.4. Applications, examples and remarks

Although the first main result of this Chapter, Theorem 3.13, may seem quite technical, we can apply it very effectively. We used it to extend Hutchinson's result to generalized contractions of semimetric spaces (see Theorem 3.14). Moreover, it implies immediately Theorem 2.6, as well. Indeed, if  $x_0$  is a fixed point of some

map in  $\mathcal{F}$ , then the singleton  $\{x_0\}$  serves as an  $\mathcal{F}$ -subinvariant set. On the other hand, the theorem of Darbo and Sadovskii guarantees the existence of a fixed point in this case. In general, we may use Theorem 3.13 whenever we can provide at least one fixed point.

Now we present a general method which allows us to construct several examples for Matkowski contractions. Let  $(X, d)$  be a metric space, and let  $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous, strictly increasing, surjective function. Under these assumptions,  $g(t) = 0$  if and only if  $t = 0$ . Hence the function  $\rho := g \circ d$  is a semimetric on the underlying set  $X$ , and it is typically not a metric. Moreover, as can be readily verified,  $\rho$  and  $d$  are uniformly equivalent; consequently,  $\rho$  is regular. Therefore, the analytic properties induced by  $\rho$  and  $d$  (completeness, compactness, continuity) coincide. In particular, if we consider a family of continuous self-mappings defined on a nonempty, compact, convex subset of a Euclidean space, and then modify the metric in the above manner, we obtain an example satisfying the required conditions.

Following the above construction, it turns out that every Matkowski contraction with respect to the original metric is also a Matkowski contraction with respect to the new semimetric. Assume that  $T: X \rightarrow X$  is a Matkowski contraction in the metric  $d$  with comparison function  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . Let  $\psi := g \circ \varphi \circ g^{-1}$ . Then  $\psi$  is increasing. If  $t > 0$ , then  $\varphi(t) < t$ , and therefore

$$\psi(t) = g(\varphi(g^{-1}(t))) < g(g^{-1}(t)) = t.$$

Thus  $\psi$  is a comparison function. Observe that  $T$  is also a Matkowski contraction with respect to the semimetric  $\rho$ :

$$\rho(Tx, Ty) = g(d(Tx, Ty)) \leq g(\varphi(d(x, y))) = \psi(g(d(x, y))) = \psi(\rho(x, y)).$$

A drawback of the above construction is that one could immediately pass from the semimetric back to the metric. However, in view of the applications, it may occur that the semimetric obtained by the "deforming" process is more convenient than the original metric.

We can justifiably object that this Chapter does not present convincing examples and applications that clearly demonstrate the effectiveness of the main results. In our opinion, addressing these shortcomings could be the subject of further research.



## CHAPTER 4

### Dimension of generalized fractals

In this Chapter we shall study the dimension of generalized fractals. We present a variant of the fractal dimension formula for fractals obtained by generalized contractions of semimetric spaces. Hutchinson gave a dimension formula with the help of the factors of contractions [27]. In our results, these factors are replaced with the right-upper Dini-derivatives of the comparison functions related to the generalized contractions.

Our main reference of measure theory is the book of J  rai [30]. The main results are based on the paper by Bessenyei and P  nzes [5].

#### 4.1. Preliminaries

**The classical results of Hutchinson.** Hutchinson studied not only the existence and uniqueness of fractals, but also their dimension. He proved a dimension formula [27] for contractive similarities that satisfy the open set condition. Under a *contractive similarity* we mean a special contraction which fulfils its defining inequality with equality. The family  $\mathcal{F} = \{f_1, \dots, f_n\}$  satisfies the *open set condition*, if there exists a nonempty, open set  $U \subseteq X$  such that

$$\bigcup_{i=1}^n f_i(U) \subseteq U \quad \text{and} \quad f_i(U) \cap f_j(U) = \emptyset, \quad \text{if } i \neq j.$$

**THEOREM.** (Hutchinson) *If a finite family of contractive similarities  $\mathcal{F}$  of a complete metric space fulfils the open set condition and the ratios of the members are  $q_1, \dots, q_n$ , then the Hausdorff dimension of the induced fractal is obtained as the unique solution of the exponential equation*

$$(4.1) \quad q_1^r + \dots + q_n^r = 1.$$

#### 4.2. Auxiliary tools

**The Hausdorff measure.** The Hausdorff measure on a semimetric space can be introduced formally as in metric spaces. For details, we refer to Rogers [43],

but for convenience we sketch it briefly. We are going to construct the Hausdorff measure in the same way as in the book of J  rai [30] which serves as our measure-theoretic reference.

Having a set function  $\mu: \mathcal{P}(X) \rightarrow [0, +\infty]$  on a nonempty set  $X$ , the definition

$$\mu^*(A) := \inf \left\{ \sum_{C \in \mathcal{C}} \mu(C) \mid \text{card}(\mathcal{C}) \leq \mathbb{N}, C \subseteq X, A \subseteq \bigcup \mathcal{C} \right\}$$

results in a nonnegative valued,  $\sigma$ -subadditive set function  $\mu^*: \mathcal{P}(X) \rightarrow [0, +\infty]$  or briefly: in an *outer measure* on  $X$ . We say that  $A \subseteq X$  is  $\mu^*$ -measurable if, for all  $T \subseteq X$ ,

$$\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \setminus A).$$

The Carath  odory Theorem states that the  $\mu^*$ -measurable subsets of  $X$  form a  $\sigma$ -algebra on which  $\mu^*$  is a measure. To construct the Hausdorff measure in semimetric spaces, we apply the Carath  odory construction using a suitable set function. If  $\Gamma: \mathbb{R}_+ \rightarrow \mathbb{R}$  is the gamma function and  $r \geq 0$ , then define

$$\alpha(r) = \frac{\Gamma^r(\frac{1}{2})}{\Gamma(1 + \frac{r}{2})}.$$

Let  $(X, d)$  be a semimetric space. From now on, we are going to use the notion of diameter as it is introduced in (iii) of Proposition 3.4. Denote the family of subsets with diameter at most  $\delta$  by  $\mathcal{B}_\delta(X)$  and define  $\nu_\delta^r$  by

$$\nu_\delta^r(A) := \begin{cases} \frac{\alpha(r)}{2^r} \text{diam}^r(A) & \text{if } A \in \mathcal{B}_\delta(X), \\ +\infty & \text{if } A \notin \mathcal{B}_\delta(X). \end{cases}$$

Now we can introduce a set function  $\chi^r: \mathcal{P}(X) \rightarrow [0, +\infty]$  with the outer measure  $\chi_\delta^r = (\nu_\delta^r)^*$  via the formula

$$\chi^r(A) = \lim_{\delta \rightarrow 0+0} \chi_\delta^r(A).$$

Using the same arguments as in the metric case, one can prove that this function is well-defined and it is an outer measure on the underlying semimetric space. In the following, we refer to it as the  *$r$ -dimensional outer Hausdorff measure*. We denote the  $\sigma$ -algebra of the  $\chi^r$ -measurable sets by  $\mathcal{H}^r$ . The  *$r$ -dimensional Hausdorff measure* is the restriction of  $\chi^r$  to this  $\sigma$ -algebra, and since it is not confusing, we denote it by  $\chi^r$ , as well.

Finally, the *Hausdorff dimension* of a set  $A \subseteq X$  is the (extended) real number

$$\dim_{\mathcal{H}}(A) := \sup\{r \geq 0 \mid \chi^r(A) > 0\}.$$

We note that if  $r$  is an integer, than the  $r$ -dimensional Lebesgue and Hausdorff measures differ only by a constant scalar. By incorporating the  $\Gamma$  function, one can arrange that the volume of the  $r$ -dimensional unit ball takes its usual form. This

rescaling clearly changes the measure of sets, but it has no effect on the Hausdorff dimension itself.

The most important properties of the Hausdorff dimension are still valid if we use semimetrics:

- ✧ If  $\chi^r(A) < +\infty$  and  $t > r$ , then  $\chi^t(A) = 0$ .
- ✧ If  $\chi^r(A) > 0$  and  $t < r$ , then  $\chi^t(A) = +\infty$ .
- ✧ If  $t > \dim_{\mathcal{H}}(A)$ , then  $\chi^t(A) = 0$ .
- ✧ If  $t < \dim_{\mathcal{H}}(A)$ , then  $\chi^t(A) = +\infty$ .

To prove these properties, we can use the inequality (2.10) in [17, Section 2.2] as it does not rely on the triangle inequality and hence it is valid even in semimetric spaces.

In the first auxiliary result we show that, in certain cases, outer measures inherit the relation of the generating set functions.

**LEMMA 4.1.** *If  $\mu: \mathcal{P}(X) \rightarrow \overline{\mathbb{R}}_+$  and  $\nu: \mathcal{P}(Y) \rightarrow \overline{\mathbb{R}}_+$  are given and  $f: X \rightarrow Y$  satisfies  $\nu \circ f \leq \mu$  on  $\mathcal{P}(X)$ , then  $\nu^* \circ f \leq \mu^*$ . If  $f$  is a bijection and  $\nu \circ f = \mu$  on  $\mathcal{P}(X)$ , then  $\nu^* \circ f = \mu^*$ .*

**PROOF.** We may assume that  $\mu^*(A)$  is finite. Let  $c > \mu^*(A)$  and let  $\mathcal{C}$  be a countable cover for  $A$  such that  $\sum_{C \in \mathcal{C}} \mu(C) < c$ . Obviously  $\{f(C) \mid C \in \mathcal{C}\}$  is a countable cover for  $f(A)$ . Thus

$$\nu^*(f(A)) \leq \sum_{C \in \mathcal{C}} \nu(f(C)) \leq \sum_{C \in \mathcal{C}} \mu(C) < c.$$

Taking the limit  $c \rightarrow \mu^*(A)$ , we arrive at the first statement. The second statement follows from the first one applying the above proved inequality for  $f$  and its inverse.  $\square$

**Lipschitz maps.** In what follows, we shall need the next concepts. Let  $H \subseteq \mathbb{R}$  be a nonempty set and let  $t_0 \in H$  be a right limit point of  $H$ . The *right-upper Dini-derivative* of a function  $\varphi: H \rightarrow \mathbb{R}$  at  $t_0$  is the (extended) real number

$$D\varphi(t_0) := \limsup_{t \rightarrow t_0+0} \frac{\varphi(t) - \varphi(t_0)}{t - t_0}.$$

Let  $(X, d)$  be a semimetric space. We say that a function  $f: X \rightarrow X$  is  $\varphi$ -Lipschitz if it fulfils (2.2) with  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\varphi(0) = 0$  and  $D\varphi(0) < +\infty$ . We point out that each generalized contraction is a  $\varphi$ -Lipschitz map. Additionally, if  $\varphi$  is strictly increasing, differentiable at zero,  $\varphi'(0) > 0$ , and (2.2) holds with equality, then we speak about  $\varphi$ -similarity. The next lemma clarifies the connection of the Hausdorff measure and  $\varphi$ -Lipschitz maps.

**LEMMA 4.2.** *If  $f$  is a  $\varphi$ -Lipschitz self-map of a semimetric space  $(X, d)$  and  $A \subseteq X$ , then*

$$\chi^r(f(A)) \leq (D\varphi(0))^r \cdot \chi^r(A).$$

Moreover, if  $f$  is a surjective  $\varphi$ -similarity, then this inequality is valid with equality, and  $A \subseteq X$  is Hausdorff-measurable if and only if  $f(A)$  is Hausdorff-measurable.

PROOF. If  $\varepsilon > D\varphi(0)$ , then there exists a right neighborhood  $I_\varepsilon$  of zero such that  $\varphi(t) \leq \varepsilon t$  whenever  $t \in I_\varepsilon$  since  $\varphi(0) = 0$ . Let  $\delta \in I_\varepsilon$  be fixed and let  $A$  be such that  $\text{diam}(A) < \delta$ . If  $u, v \in f(A)$ , then  $u = f(x)$  and  $v = f(y)$  where  $x, y \in A$ . Furthermore

$$d(u, v) = d(f(x), f(y)) \leq \varphi(d(x, y)) \leq \varepsilon \text{diam}(A).$$

Hence the diameter of  $f(A)$  is at most  $\varepsilon \text{diam}(A)$ . This property ensures that

$$\nu_{\varepsilon\delta}^r(f(A)) = \frac{\alpha(r)}{2^r} \text{diam}^r(f(A)) \leq \frac{\alpha(r)}{2^r} \varepsilon^r \text{diam}^r(A) = \varepsilon^r \nu_\delta^r(A)$$

holds for all  $\delta \in I_\varepsilon$  and for all  $A \in \mathcal{B}_\delta(X)$ . Thus by Lemma 4.1,

$$\chi_{\varepsilon\delta}^r(f(A)) = (\nu_{\varepsilon\delta}^r)^*(f(A)) \leq (\varepsilon^r \nu_\delta^r)^*(A) = \varepsilon^r \chi_\delta^r(A).$$

Taking first the limit  $\delta \rightarrow 0$  and then  $\varepsilon \rightarrow D\varphi(0)$ , we arrive at the first statement.

Now assume that  $f$  is a surjective  $\varphi$ -similarity. Since  $\varphi^{-1}(\{0\}) = \{0\}$ , it must be a bijection. Then  $f^{-1}$  is a  $\psi$ -similarity with  $\psi := \varphi^{-1}$ . Therefore the inequality of the first part results in

$$\chi^r(A) = \chi^r(f^{-1}(f(A))) \leq (D\psi(0))^r \cdot \chi^r(f(A)).$$

Observe that  $\varphi'(0) > 0$  implies  $D\psi(0) = 1/\varphi'(0) > 0$ . Rearranging the inequality above, we get the reversed one.

Finally, assume that  $A \subseteq X$  is Hausdorff-measurable and let  $q = \varphi'(0)$ . Then, since  $f$  is a bijection, for all  $T \subseteq X$ ,

$$\begin{aligned} \chi^r(f(T)) &= q^r \chi^r(T) = q^r (\chi^r(T \cap A) + \chi^r(T \setminus A)) \\ &= \chi^r(f(T \cap A)) + \chi^r(f(T \setminus A)) \\ &= \chi^r(f(T) \cap f(A)) + \chi^r(f(T) \setminus f(A)). \end{aligned}$$

Since  $f$  is a bijection, each subset  $S$  of  $X$  can be obtained as  $S = f(T)$  with suitable  $T \subseteq X$ . Thus the calculation above ensures that  $f(A)$  is Hausdorff-measurable. The converse statement follows similarly.  $\square$

**Borel sets.** It turns out that the fractals of a regular semimetric space are Hausdorff measurable. We shall prove a more general result by showing that even Borel sets have this feature. As in the standard setting, first we characterize the measurability of Borel-sets. We adopt some well-known techniques from the book of J  rai [30]. However, the proof contains several independent movements, thus we share its details. We define the *distance* among subsets of a semimetric space by  $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$ .

LEMMA 4.3. *Let  $\mu$  be an outer measure on a regular semimetric space  $(X, d)$ . The Borel sets of  $X$  are  $\mu$ -measurable if and only if all nonempty sets  $A, B \subseteq X$  of positive distance satisfy*

$$(4.2) \quad \mu(A \cup B) = \mu(A) + \mu(B).$$

PROOF. For necessity of (4.2), assume that  $d(A, B) > 0$ . We show that  $\overline{A} \cap B = \emptyset$ . Indeed, assume to the contrary that this is not the case. Then there exists  $b \in B$  and  $a_n \in A$  such that  $a_n \rightarrow b$  by Proposition 3.2. Thus  $0 < d(A, B) \leq d(a_n, b) \rightarrow 0$  as  $n \rightarrow \infty$ , which is a contradiction. Now let  $T = A \cup B$ . Since  $\overline{A}$  is  $\mu$ -measurable,

$$\begin{aligned} \mu(A \cup B) &= \mu(T) = \mu(\overline{A} \cap T) + \mu(T \setminus \overline{A}) \\ &= \mu(A \cup (\overline{A} \cap B)) + \mu(B \setminus \overline{A}) \\ &= \mu(A) + \mu(B). \end{aligned}$$

For sufficiency of (4.2), denote the basic triangle function of the semimetric space by  $\Delta$ , and define  $\alpha^*$  attached to  $\alpha > 0$  by

$$\alpha^* := \lim_{\delta \rightarrow 0+0} \Delta(\alpha, \delta).$$

The partial monotonicity of  $\Delta$  ensures that the definition is correct. Furthermore, the continuity of  $\Delta$  at the origin implies that, for all  $\varepsilon > 0$  there exists  $\alpha$  such that  $\alpha^* < \varepsilon$ . This property ensures that we can construct a strictly decreasing null sequence  $(\alpha_n)$  satisfying  $\alpha_{n+1}^* < \alpha_n$ .

Let  $U \subseteq X$  be an open set and let  $T \subseteq X$  be such that  $\mu(T) < +\infty$ . Consider the sequences of sets  $(A_n)$  and  $(B_n)$  defined by

$$A_n = \{x \in T \cap U \mid d(x, X \setminus U) \geq \alpha_n\}$$

and

$$B_1 = A_1, \quad B_{n+1} = A_{n+1} \setminus A_n.$$

Clearly,  $(A_n)$  is an ascending chain. Moreover, the odd subsequence of  $(B_n)$  consists of sets with positive distance. To verify this property, we shall check it only for consecutive members. Assume to the contrary that  $d(B_{2n-1}, B_{2n+1}) = 0$ . Then, for all  $\delta > 0$  we can find elements  $b_{2n-1} \in B_{2n-1}$  and  $b_{2n+1} \in B_{2n+1}$  such that  $d(b_{2n-1}, b_{2n+1}) < \delta$ . If  $y \in X \setminus U$ ,

$$\alpha_{2n-1} \leq d(y, b_{2n-1}) \leq \Delta(d(y, b_{2n+1}), d(b_{2n-1}, b_{2n+1})) \leq \Delta(\alpha_{2n}, \delta).$$

Taking the limit  $\delta \rightarrow 0$ , we arrive at the contradiction  $\alpha_{2n-1} \leq \alpha_{2n}^* < \alpha_{2n-1}$ .

Thus, applying induction and (4.2), we arrive at

$$\mu\left(\bigcup_{k=1}^n B_{2k-1}\right) = \sum_{k=1}^n \mu(B_{2k-1}); \quad \text{hence} \quad \mu\left(\bigcup_{k=1}^{\infty} B_{2k-1}\right) \geq \sum_{k=1}^{\infty} \mu(B_{2k-1}).$$

The same argument results in the corresponding estimation for even indices. Therefore,

$$+\infty > 2\mu(T) \geq \mu\left(\bigcup_{k=1}^{\infty} B_{2k-1}\right) + \mu\left(\bigcup_{k=1}^{\infty} B_{2k}\right) \geq \sum_{k=1}^{\infty} \mu(B_k).$$

Let  $\varepsilon > 0$  be arbitrary. Since  $T \cup U = A_n \cup B_{n+1} \cup B_{n+2} \cup \dots$  and  $\sum \mu(B_k)$  is convergent, there exists  $n \in \mathbb{N}$  such that

$$\varepsilon > \sum_{k=n}^{\infty} \mu(B_k) \geq \mu((T \cap U) \setminus A_n).$$

Thus, by (4.2) again,

$$\begin{aligned} \mu(T \cap U) + \mu(T \setminus U) &\leq \mu((T \cap U) \setminus A_n) + \mu(A_n) + \mu(T \setminus U) \\ &\leq \varepsilon + \mu((T \setminus U) \cup A_n) \\ &\leq \varepsilon + \mu(T). \end{aligned}$$

Since  $\varepsilon > 0$  has been chosen arbitrarily, this means that  $U$  is  $\mu$ -measurable.  $\square$

**COROLLARY 4.4.** *The Borel sets of a regular semimetric space are Hausdorff-measurable.*

**PROOF.** Using the notations of the previous proof, let  $A$  and  $B$  be subsets of positive distance. By the continuity of  $\Delta$ , we can choose  $\delta > 0$  such that  $\Delta(\delta, \delta) < d(A, B)$ . Let  $\mathcal{C}$  be a countable cover for  $A \cup B$  which consists of sets belonging to  $\mathcal{B}_\delta(X)$ . Define

$$\mathcal{C}_A = \{C \in \mathcal{C} \mid C \cap A \neq \emptyset\} \quad \text{and} \quad \mathcal{C}_B = \{C \in \mathcal{C} \mid C \cap B \neq \emptyset\}.$$

Then  $\mathcal{C}_A$  and  $\mathcal{C}_B$  are countable covers for  $A$  and  $B$ , respectively. Moreover, these covers do not overlap. Indeed, assume to the contrary that this is not the case. Then there exist  $C_A \in \mathcal{C}_A$  and  $C_B \in \mathcal{C}_B$  such that  $x \in C_A \cap C_B$ . If  $a \in A$  and  $b \in B$ , we arrive at the contradiction

$$d(A, B) \leq d(a, b) \leq \Delta(d(a, x), d(b, x)) \leq \Delta(\delta, \delta) < d(A, B).$$

These properties of  $\mathcal{C}_A$  and  $\mathcal{C}_B$  result in

$$\chi_\delta^r(A \cup B) \geq \chi_\delta^r(A) + \chi_\delta^r(B).$$

However, this inequality implies that  $\mu = \chi^r$  fulfils (4.2). Thus Borel sets are Hausdorff-measurable, indeed.  $\square$

### 4.3. The main results

**The dimension formula in semimetric space.** The main result of this Chapter is just an initial step toward the Hausdorff dimension of generalized fractals in semimetric spaces. In the second part of its statement, we use a special kind of comparison functions. We say that a function  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a *strong comparison function* if it fulfils the following requirements:

- ✧  $\varphi$  is strictly monotone increasing;
- ✧  $\varphi(0) = 0$  and  $\varphi(t) < t$  whenever  $t > 0$ ;
- ✧  $\varphi$  is differentiable at  $t = 0$  and  $0 < \varphi'(0)$ .

In other words, a strong comparison function is a comparison function which is strictly increasing and its derivative at zero is positive.

**THEOREM 4.5.** *If  $f_1, \dots, f_n$  are generalized contractions of a complete, normal and regular semimetric space with comparison functions  $\varphi_1, \dots, \varphi_n$  and  $0 < \chi^r(H) < +\infty$  holds for the induced fractal  $H$  with some  $r > 0$ , then the Dini-derivatives of the comparison functions cannot be zero simultaneously,  $\dim_{\mathcal{H}}(H) = r$ , and*

$$1 \leq (D\varphi_1(0))^r + \dots + (D\varphi_n(0))^r.$$

*Moreover, if  $\chi^r(f_k(H) \cap f_l(H)) = 0$  whenever  $k \neq l$  and each generalized contraction is a similarity with strong comparison function, then all derivatives at zero are strictly smaller than one and the inequality above is valid with equality.*

**PROOF.** Note first that the induced fractal  $H$  exists and it is unique by Theorem 3.14. The assumption  $0 < \chi^r(H) < +\infty$  implies immediately that  $\dim_{\mathcal{H}}(H) = r$ . The compactness of the sets  $H$  and  $f_1(H), \dots, f_n(H)$  provides their Hausdorff-measurability by Corollary 4.4. Applying (2.1), the properties of the measure, and Lemma 4.2,

$$\chi^r(H) = \chi^r\left(\bigcup_{k=1}^n f_k(H)\right) \leq \sum_{k=1}^n \chi^r(f_k(H)) \leq \sum_{k=1}^n (D\varphi_k(0))^r \chi^r(H).$$

Observe that the Dini-derivatives cannot be zero simultaneously, or else we would arrive at the contradiction  $\chi^r(H) = 0$ . Moreover, since  $\chi^r(H)$  is positive and finite, this estimation reduces to the inequality of the theorem after simplifying. To prove the second statement, note first that the inequalities above are, in fact, equations by Lemma 4.2. Since we have strong comparison functions, all derivatives at zero are positive. Thus, if we had an index  $k$  such that  $\varphi'_k(0) = 1$ , then the contradiction  $\chi^r(H) = 0$  would follow from this identity. To complete the proof, we have to simplify our identity again.  $\square$

If our system contains a comparison function whose Dini-derivative at zero is exactly one, then the inequality of Theorem 4.5 holds trivially and hence it is not

informative. As we have seen in the statement, this case is excluded a priori in case of  $\varphi$ -similarities. However, the set of ordinary similarities is a proper subset of  $\varphi$ -similarities having derivative at zero strictly smaller than one. Thus Theorem 4.5 may have impact for such cases when the classical theory does not work.

#### 4.4. Applications, examples and remarks

The right-upper Dini-derivative of the comparison function of a classical contraction coincides with the factor of the contraction. Thus if the underlying family in Theorem 4.5 consists of contractions, the involved inequality reduces to (4.1). Therefore, our theorem just a weak dimension formula for generalized fractals: While Hutchinson's classical result is a sufficient condition, Theorem 4.5 is a necessary one. The question of whether the complete extension of Hutchinson's fractal dimensional formula is valid for generalized fractals of regular semimetric spaces remains open and could be the subject of further studies.

Now we present a method to construct  $\varphi$ -isometries. However, as we shall see, this construction is not really appropriate to illustrate Theorem 4.5.

EXAMPLE 4.6. Let  $X = [0, 1]$  and define  $d: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  in the following way:

$$d(x, y) := \begin{cases} 0 & \text{if } x = y, \\ \max\{x, y\} & \text{if } x \neq y. \end{cases}$$

Then,  $(X, d)$  is a metric space. If  $\varphi$  is strictly increasing, differentiable at zero,  $\varphi'(0) > 0$ , and  $T: X \rightarrow X$  is defined by  $Tx = \varphi(x)$ , then  $T$  is a  $\varphi$ -isometry.

We focus only on the second statement. If  $x \neq y$ , then  $\varphi(x) \neq \varphi(y)$ , and hence

$$d(Tx, Ty) = \max\{\varphi(x), \varphi(y)\} = \varphi(\max\{x, y\}) = \varphi(d(x, y)).$$

If  $x = y$ , then  $d(Tx, Ty) = 0$ , and  $\varphi(0) = 0$  implies  $\varphi(d(x, y)) = 0$ . That is,  $T$  is an isometry with respect to  $\varphi$ . Unfortunately, if  $\mathcal{F}$  contains such isometries, then the only common fixed point is  $p = 0$ . Thus, the fractal obtained by the Kantorovich-iteration is the singleton  $H = \{0\}$ . That is, the positivity assumption on the Hausdorff-measure in Theorem 4.5 is valid only with  $r = 0$ . Consequently, the inequality of the theorem holds trivially, while the assumption on the measure of the intersection is violated.

The last Chapter is open to the same criticism as the previous one. The results on Hausdorff dimension presented in this thesis represent only the initial steps of a broader research project that requires substantially more in-depth investigation. Stronger theoretical results and convincing examples are still lacking, and developing these will be the subject of future research.

## CHAPTER 5

### An axiomatic treatment

The Hausdorff measure of noncompactness [26] provides a direct approach to the Darbo–Sadovskii Fixed Point Theorem [15, 44] or to Hutchinson’s fundamental result in fractal theory, as we have seen in the previous Chapters. It turns out that the Hausdorff measure does not have a particular role among other measures: requiring quite simple properties on a set-function, we can prove analogous results. The method of the main result of this Chapter is a reducing principle which allows us to shrink bounded and closed domains to compact ones. The Knaster–Tarski [32] and the Kantorovich Fixed Point Theorems [31] play a key role.

As applications of this axiomatic approach, we present the concept of degree of nondensifiability, which is studied by García and Mora [21]. The main results of this Chapter are based on the paper by Bessenyei and Péntes [6].

#### 5.1. Preliminaries

**Topological fixed point results.** Although the classical result of Brouwer [11] might seem to be the simplest topological fixed point result, it plays a highly distinguished role in this theory. Indeed, many extensions apply Brouwer’s result by reducing the situation to an essentially finite dimensional setting. For example, Schauder [45] completes the reduction with his famous approximation technique and proves a fixed point result for compact maps. Darbo [15] uses a Cantor-type intersection theorem, while Sadovskii [44] applies the Hausdorff measure of noncompactness [26] and both of them extend independently Brouwer’s result to condensing maps. Therefore in the technical literature we are speaking about Darbo–Sadovskii Fixed Point Theorem. In each case, the common idea behind them is the construction of an invariant compact domain which allows the use of the Brouwer Theorem.

The most general result among the mentioned ones is due to Sadovskii [44]. The reason for the effectiveness of his approach is the Hausdorff measure of noncompactness. However, the role of this measure is not as distinguished as it seems at first glance. The aim of this Chapter is to illustrate that some simple properties of a set-function provide the existence of a suitable “small” invariant domain,

moreover: a “small” self-invariant subset in the invariant domain. The first statement is a consequence of the Knaster–Tarski Fixed Point Theorem [32] while the second one follows from a result of Kantorovich [31].

## 5.2. Auxiliary tools

**Set functions, hull operators.** A set-function  $\chi$  on  $X$ , that is, a function  $\chi: \mathcal{P}(X) \rightarrow [0, +\infty]$  is called

- ✧ *non-atomic* if  $\chi(\{p\}) = 0$  for all  $p \in X$ ;
- ✧ *finite* if it takes finite values, that is,  $\chi(A) < +\infty$  for all  $A \subseteq X$ .

Based on the monograph of van de Vel [49], we recall the notion of convex hulls. Observe that the convex hull in vector spaces or the closure operator in metric spaces fulfils all requirements.

A map  $\Phi: X \rightrightarrows X$  is a *hull operator* on the nonempty set  $X$ , if it is

- ✧ *monotone*:  $\Phi(A) \subseteq \Phi(B)$  whenever  $A \subseteq B \subseteq X$ ;
- ✧ *idempotent*:  $\Phi(\Phi(A)) = \Phi(A)$  for all  $A \subseteq X$ ;
- ✧ *extensive*:  $A \subseteq \Phi(A)$  for all  $A \subseteq X$ .

We have to link set-functions and hull operators properly and (motivated by the notion of condensing maps) clarify the mapping class we shall deal with:

A set-function  $\chi: \mathcal{P}(X) \rightarrow [0, +\infty]$  is compatible with the hull operator  $\Phi: X \rightrightarrows X$ , or in short:  *$\Phi$ -compatible* if, for all  $A, B \subseteq X$ ,

$$\chi(\Phi(A \cup B)) \leq \max\{\chi(A), \chi(B)\}.$$

A map  $F: X \rightrightarrows X$  is *decreasing* with respect to the set function  $\chi$  (or simply:  $\chi$ -decreasing) if  $\chi(F(A)) < \chi(A)$  whenever  $A \subseteq X$  and  $0 < \chi(A) < +\infty$ .

## 5.3. The main results

**Monotone maps revisited.** Now we can formulate our main result. It states that, under suitable requirements, we can change the domain of an invariance operator to a “small” one such that it inherits the properties of the original domain and on which the invariance operator is still a self-map. In the proof, we apply again the Knaster–Tarski Fixed Point Theorem (Theorem 1.5) and the Kantorovich-iteration (1.2) from Chapter 1.

**THEOREM 5.1.** *Assume that  $\Phi: X \rightrightarrows X$  is a hull operator,  $D \subseteq X$  is nonempty,  $\Phi$ -invariant, and  $\chi$  is  $\Phi$ -compatible, finite, non-atomic on  $D$ . If  $\mathcal{F}$  is a finite set of  $\chi$ -decreasing self-maps of  $D$  and  $F: D \rightrightarrows D$  is the induced invariance operator, then there exists a nonempty  $\Phi$ -invariant set  $H \subseteq D$  such that  $F: H \rightrightarrows H$  and  $\chi(H) = 0$ .*

PROOF. Fix an arbitrary point  $p \in D$ . For a set  $A \subseteq D$ , define  $T(A) \subseteq X$  as

$$T(A) = \Phi(F(A) \cup \{p\}).$$

Then  $T: D \rightrightarrows D$  since  $F(A) \cup \{p\} \subseteq D$  and  $D$  is  $\Phi$ -invariant. Furthermore,  $T$  inherits the monotonicity of  $\Phi$  and  $F$ . By the Knaster–Tarski Fixed Point Theorem (Theorem 1.5), there exists a subset  $H$  of  $D$  such that  $T(H) = H$ . We prove that  $H$  is the set we are looking for. Since  $\Phi$  is extensive,  $p \in H$  yielding  $H \neq \emptyset$ . Using that  $\Phi$  is idempotent, we arrive at

$$\Phi(H) = \Phi(T(H)) = T(H) = H.$$

Thus  $H$  is  $\Phi$ -invariant. On the other hand,  $\Phi$  is monotone and extensive; hence

$$H = T(H) = \Phi(F(H) \cup \{p\}) \supseteq \Phi(F(H)) \supseteq F(H).$$

Therefore  $F: H \rightrightarrows H$ . Finally, assume to the contrary that  $\chi(H) > 0$  and let  $\mathcal{F} = \{f_1, \dots, f_n\}$ . Then  $\chi(f_k(H)) < \chi(H)$  for all  $k \in \{1, \dots, n\}$  since  $\chi$  is finite valued. Using that  $\chi$  is  $\Phi$ -compatible and non-atomic,

$$\begin{aligned} \chi(H) &= \chi(T(H)) = \chi(\Phi(F(H) \cup \{p\})) \\ &= \chi(\Phi(f_1(H) \cup \dots \cup f_n(H) \cup \{p\})) \\ &\leq \max\{\chi(f_1(H)), \dots, \chi(f_n(H)), \chi(p)\} < \chi(H). \end{aligned}$$

This contradiction shows that  $\chi(H) = 0$  and the proof is completed.  $\square$

**The measure of noncompactness.** First of all, we show some applications when the set-function  $\chi_k$  is the Hausdorff measure of noncompactness and the underlying mappings are condensing (as defined in Chapter 2). Applying Theorem 5.1, we show a slightly simplified proof of the Darbo–Sadovskii Fixed Point Theorem [44].

In our argument, the closure-convex operator plays a key role. Although the composition of two hull operators is monotone and extensive, but need not to be idempotent. However, the closure of a convex set is convex, and this property ensures that the composition in our setting remains a hull operator, indeed.

**COROLLARY 5.2.** *If  $D$  is a bounded, convex and closed subset of a Banach space  $X$  and  $f: D \rightarrow D$  is condensing, then  $f$  has a fixed point in  $D$ .*

PROOF. Let  $\Phi := \overline{\text{Conv}}$  and let  $\chi_k$  be the Hausdorff measure of noncompactness. Then  $\chi_k$  is non-atomic and finite (being  $D$  bounded), while  $\Phi$  is a hull operator on  $D$  (as we have just mentioned). Moreover, the properties of  $\chi_k$  show that

$$\chi_k(\overline{\text{Conv}}(A \cup B)) = \chi_k(A \cup B) = \max\{\chi_k(A), \chi_k(B)\}.$$

Thus  $\chi_k$  is  $\Phi$ -compatible. If  $\mathcal{F} = \{f\}$ , then it consists of a  $\chi_k$ -decreasing member and the induced invariance operator  $F: D \rightrightarrows D$  is evidently monotone. By the main result, there exists a nonempty subset  $H$  of  $D$  such that  $H$  is  $\Phi$ -invariant,

that is, convex and closed in  $D$ ; moreover,  $F: H \rightrightarrows H$  and  $\chi_k(H) = 0$ . Equivalently,  $H$  is convex and closed in  $X$  since  $D$  is closed,  $f: H \rightarrow H$ , and  $H$  is totally bounded. In particular,  $H$  is compact by the completeness of  $X$ . Hence the statement follows from the Schauder Fixed Point Theorem.  $\square$

The main result has impacts in Fractal Theory, as well. Thus, now we revisit Theorem 2.6 which is one of the main results of Chapter 2 and present its proof via Theorem 5.1.

**COROLLARY 5.3.** *If  $D$  is a bounded, convex and closed subset of a Banach space  $X$  and  $\mathcal{F}$  is a finite family of condensing self-maps of  $D$ , then there exists an  $\mathcal{F}$ -fractal.*

**PROOF.** Let  $\Phi := \overline{\text{Conv}}$  and let  $\chi_k$  be the Hausdorff measure of noncompactness. As we have seen in the previous proof,  $\Phi$  and  $\chi_k$  fulfill the requirements of the main result. Consider the invariance operator  $F: D \rightrightarrows D$  induced by  $\mathcal{F}$ . The main result ensures that there exists a nonempty,  $\Phi$ -invariant (that is: closed and convex) set  $H \subseteq D$  such that  $F: H \rightrightarrows H$  and  $\chi_k(H) = 0$ . Thus  $H$  is totally bounded. The completeness of the underlying space results in the compactness of  $H$ .

The invariance operator commutes with countable chain union and Corollary 5.2 provides a fixed point of a member in  $\mathcal{F}$ . By Theorem 1.6, there exists a smallest  $\mathcal{F}$ -invariant subset  $H$  containing that fixed point. The closure of the  $\mathcal{F}$ -invariant set is compact (being a closed subset of the compact set  $H$ ) and it is also  $\mathcal{F}$ -invariant by Lemma 2.4. In other words: it is an  $\mathcal{F}$ -fractal.  $\square$

Contractions are condensing, therefore Corollary 5.3 extends the existence part of Hutchinson's result in Banach spaces. However, uniqueness cannot be guaranteed now.

**The degree of nondensifiability.** Based on the concepts given by García and Mora [21], we present another branch of applications of our main results.

Let  $H \neq \emptyset$  be a bounded subset in a metric space  $(X, d)$  and let  $\alpha \geq 0$ . A continuous curve  $c: [0, 1] \rightarrow H$  is  $\alpha$ -dense in  $H$  if, for all  $x \in H$ , there exists a parameter  $t \in [0, 1]$  such that  $d(x, c(t)) \leq \alpha$ . Denote the set of  $\alpha$ -dense curves in  $H$  by  $\Gamma_\alpha(H)$ . The *degree of nondensifiability* of  $H$  is given by the real number

$$\chi_d(H) := \inf\{\alpha \geq 0 \mid \Gamma_\alpha(H) \neq \emptyset\}.$$

If  $H \neq \emptyset$  is bounded, then  $\chi_d(H)$  is finite. If additionally  $H$  is arc-connected, then  $\chi_d(H) = 0$  if and only if  $H$  is totally bounded. Contrary to the Hausdorff measure of noncompactness, the degree of nondensifiability is not a monotone set-function. However, some characteristic properties coincide or are similar. Let  $A, B \subseteq X$  be nonempty, bounded sets. Then

$$\diamond \chi_d(A \cup B) \leq \max\{\chi_d(A), \chi_d(B)\} \text{ if } A, B \text{ are arc-connected, and intersecting;}$$

- ✧  $\chi_d(A) = \chi_d(\overline{A})$ ;
- ✧  $\chi_d(\text{Conv}(A)) \leq \chi_d(A)$  provided  $X$  is a normed space;
- ✧  $\chi_d(\text{Conv}(A \cup B)) \leq \max\{\chi_d(\text{Conv}(A)), \chi_d(\text{Conv}(B))\}$  provided  $X$  is a normed space.

In complete metric spaces,  $\chi_d(H) = 0$  if and only if  $H$  is relatively compact, provided that  $H \neq \emptyset$  is bounded and arc-connected. This measure induces a condensing property, as well: We say that the map  $f: X \rightarrow X$  is  $\chi_d$ -condensing, if it is continuous and  $\chi_d(f(H)) < \chi_d(H)$  holds for all nonempty, arc-connected, bounded, but not totally bounded subsets  $H$ .

The first consequence of the main result in this setting is an analogue of the Darbo–Sadovskii Fixed Point Theorem obtained by García and Mora [22]. Since there are a few subtle differences in the proof compared to the proof of Corollary 5.2, we present the arguments behind.

**COROLLARY 5.4.** *If  $D$  is a bounded, convex and closed subset of a Banach space  $X$  and  $\chi_d$  is the degree of nondensifiability on  $X$ , and  $f: D \rightarrow D$  is  $\chi_d$ -condensing, then  $f$  has a fixed point in  $D$ .*

**PROOF.** Let  $\Phi := \overline{\text{Conv}}$  and let  $\chi_d$  be the degree of nondensifiability. Since the subsets of a bounded subset are bounded,  $\chi_d$  is finite and clearly nonatomic on  $D$ . The properties of  $\chi_d$  directly show that  $\chi_d$  is compatible with respect to the hull operator  $\Phi$ :

$$\begin{aligned} \chi_d(\overline{\text{Conv}}(A \cup B)) &= \chi_d(\text{Conv}(A \cup B)) \\ &\leq \max\{\chi_d(\text{Conv}(A)), \chi_d(\text{Conv}(B))\} \\ &\leq \max\{\chi_d(A), \chi_d(B)\}. \end{aligned}$$

The same arguments as in the proof of Corollary 5.2 result in the existence of a nonempty, closed, convex set  $H \subseteq D$  such that  $f: H \rightarrow H$  and  $\chi_d(H) = 0$ . The underlying space is complete and  $H$  is arc-connected by its convexity, thus the last property means the total boundedness of  $H$ . But  $H$  is also complete, therefore it is compact. Hence the statement follows from the Schauder Fixed Point Theorem.  $\square$

Recently García and Mora [23] and Mena, Martín, and Sadarangani [39] have demonstrated that the degree of nondensifiability is applicable in Fractal Theory. One can prove their result (see below) with appropriate modifications on the proof of Corollary 5.3, thus we omit the details.

**COROLLARY 5.5.** *If  $D$  is a bounded, convex and closed subset of a Banach space  $X$  and  $\chi_d$  is the degree of nondensifiability on  $X$ , and  $\mathcal{F}$  is a finite family of  $\chi_d$ -condensing self-maps of  $D$ , then there exists an  $\mathcal{F}$ -fractal.*

### 5.4. Applications, examples and remarks

The monograph [48] by Toledano, Benavides, and Acedo presents an axiomatic approach to measures of noncompactness. They assume that  $(X, d)$  is a complete metric space. They require, among others, the following properties on the abstract measure  $\chi: X \rightarrow \mathbb{R}_+$ .

- ✧  $\chi(A) = 0$  if and only if  $A$  is precompact;
- ✧  $\chi(A) = \chi(\overline{A})$ ;
- ✧  $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$ ;
- ✧  $\chi(A) = \chi(\text{Conv}(A))$  provided that  $X$  is a Banach space.

These properties guarantee that an abstract measure of noncompactness is always compatible with the closed-convex hull operator (for details, see the proof of Corollary 5.2). Thus, the main result Theorem 5.1 can be applied in such a situation, including for example, the Hausdorff measure of noncompactness.

The degree of nondensifiability, as we have mentioned it, was introduced and studied by García and Mora [21]. The basic construction behind it, namely the notion of  $\alpha$ -dense curves, is older. These curves allow us to obtain an approximation method to turn certain global optimization problems in Banach spaces into optimization problems on the real line. For details, see [21] and the relevant references therein. Generally speaking, the degree of nondensifiability plays the same role and has the same applications in functional analysis as the measure of noncompactness: We can prove topological fixed point theorems and then we can apply them to show the existence of the solutions of nonlinear equations in Banach algebras. For example, García and Mora [22] study a quadratic integral equation which generalizes the Volterra-, Uryshon-, and Ambartsumian-Chandrasekhar-type integral equations simultaneously.

## Summary

In this Thesis we are following the fixed point approach to fractals initiated by Hutchinson [27] and Barnsley [8].

Our main concept throughout the Thesis is the invariance operator. Let  $\mathcal{F}$  be a family of self-maps of the power set  $\mathcal{P}(X)$ . We say that the map  $\Phi: X \rightrightarrows X$  defined by

$$\Phi(H) := \bigcup_{F \in \mathcal{F}} F(H)$$

is the *invariance operator* of  $\mathcal{F}$ . Moreover, the set  $H \subseteq X$  is  $\mathcal{F}$ -invariant, if

$$H = \Phi(H)$$

is satisfied, that is  $H$  is a *fixed point* of the invariance operator.

In a metric space  $(X, d)$  let  $\mathcal{F} = \{f_1, \dots, f_n\}$ , where  $f_1, \dots, f_n: X \rightarrow X$ . Under an  $\mathcal{F}$ -fractal we mean a nonempty, compact,  $\mathcal{F}$ -invariant subset of the metric space  $X$ .

Hutchinson's classical fractal result [27] states that: *If  $\mathcal{F}$  is a finite family of contractions in a complete metric space, then there exists precisely one  $\mathcal{F}$ -fractal.*

Let  $(X, d)$  be a metric space and denote by  $\mathcal{K}(X)$  the family of nonempty and compact subsets of  $X$ . Then for arbitrary sets  $A, B \in \mathcal{K}(X)$ , we can define the *Hausdorff–Pompeiu distance* of the sets  $A$  and  $B$ , which we denote by  $d_{HP}$ . The metric space  $(\mathcal{K}(X), d_{HP})$  is often called *fractal space*, see [26] and [41].

Blaschke [9] proved that the fractal space inherits the completeness of the original space. That is *if  $(X, d)$  is a complete metric space, then  $(\mathcal{K}(X), d_{HP})$  is also complete.*

Therefore, Hutchinson's approach to fractals can be summarized as follows: We can consider the invariance equation as a fixed point problem such that the desired fixed point belongs to the family  $\mathcal{K}(X)$ . It turns out that if  $\mathcal{F}$  is a finite family of contractions, then the  $\Phi$  invariance operator is also a contraction in the fractal space. Moreover, if the original space is complete, then so is the fractal space by the Blaschke theorem. So we can apply the Banach Contraction Principle [1]: *Any contraction of a complete metric space has exactly one fixed point.*

Altogether, the found fixed point is a nonempty, compact,  $\mathcal{F}$ -invariant set. In other words, a fractal.

Throughout this Thesis, we present an alternative approach to fractals, that is essentially different from Hutchinson's framework. We give solutions to Hutchinson's fixed point problem in different settings, using generalized mappings. In this way, we study generalized fractals through generalized contexts.

**CHAPTER 1.** This Chapter presents an elementary way to fractals and uses only naive set theory. Our focus is isotone and continuous mappings.

In the main results of Chapter 1, we formulate a fractal theorem purely in naive set theoretical context. The Kantorovich-iteration [31] and Knaster–Tarski Fixed Point Theorem [32] lead to the minimal solution of the invariance equation.

**THEOREM.** *If  $\mathcal{F}$  is a family of isotone maps, then there exists a smallest  $\mathcal{F}$ -invariant set. Moreover, if  $\mathcal{F}$  is a continuous family, then the Kantorovich-iteration expands a given subinvariant set to the smallest invariant set which contains this subinvariant set.*

We show through some classical examples how one can find the minimal solution of the invariance operator. It turns out that the Kantorovich-iteration is important not only in theoretical aspects: Implementing this method we can easily approximate fractals. To illustrate this phenomena we present the approximations of the classical Sierpeński-carpet and our own “Wave” fractal. Moreover, we conclude that the closure of any minimal solution yields the same result. This idea plays a main role in the following sections. The main results of this Chapter are based on the paper of Bessenyei and Péntzes [3].

**CHAPTER 2.** In this Chapter we consider the above invariance operator in metric space. So the family  $\mathcal{F}$  contains finitely many self maps of the metric space. Under an  $\mathcal{F}$ -fractal we mean a nonempty, compact,  $\mathcal{F}$ -invariant subset of the metric space.

We generalize the classical results of Hutchinson [27] in the sense that instead of contractions we use condensing maps and Matkowski contractions.

Our strategy for constructing fractals is the following. According to the Darbo–Sadovskii Fixed Point Theorem [15, 44], there exists a fixed point for each member of  $\mathcal{F}$ . Take the fixed point of any of them. Then, using the Kantorovich-process [31], we can produce a nonempty  $\mathcal{F}$ -invariant set. Moreover, this set is bounded, since the Kantorovich-iteration creates a Cauchy-sequence. Applying the properties of the Hausdorff noncompactness measure, the invariant set turns out to be a relatively compact one. Finally, we show that the closure of the invariant set (which is hence a nonempty and compact one), is  $\mathcal{F}$ -invariant, as well. This results in the existence of an  $\mathcal{F}$ -fractal.

**THEOREM.** *If  $\mathcal{F}$  is a finite family of condensing self-maps of a nonempty, bounded, closed, convex subset in a Banach space, then there exists at least one  $\mathcal{F}$ -fractal.*

If the invariance operator consists of Matkowski contractions, then we can prove not only the existence of a fractal but also its uniqueness. It is an immediate consequence of the facts that the invariance operator is a Matkowski contraction in the fractal space and that a Matkowski contraction can have at most one fixed point by the Matkowski Fixed Point Theorem [38].

**THEOREM.** *If  $\mathcal{F}$  is a finite family of Matkowski contractions of a complete metric space, then there exists precisely one  $\mathcal{F}$ -fractal.*

We conclude the Chapter by showing some fractal examples where the classical Hutchinson Theorem cannot be applied. Instead of rigorous mathematical proof we accept this fact using numerical methods. As previously, we implement the Kantorovich-iteration in order to approximate our fractals. The main results are based on the paper of Bessenyei and Péntzes [4].

**CHAPTER 3.** In this Chapter we introduce fractals in semimetric spaces. Under a semimetric space we mean a pair  $(X, d)$  where  $X$  is a nonempty set, and  $d: X^2 \rightarrow \mathbb{R}_+$  is a positive definite, symmetric function, that is, fulfils the first two axioms of metric spaces by Fréchet. Such spaces are intensively studied by Jachymski, Matkowski, Świątkowski in [29] and Chrząszcz, Jachymski, Turoboś in [14]. In this Chapter, the mappings of the invariance operator consist of finitely many condensing maps or Matkowski contractions of a semimetric space.

**THEOREM.** *If  $\mathcal{F}$  is a finite family of condensing self-maps of a normal, complete and regular semimetric space such that the invariance operator generates a bounded Kantorovich-iteration on a nonempty  $\mathcal{F}$ -subinvariant set  $H$ , then there exists a smallest  $\mathcal{F}$ -fractal containing  $H$ .*

**THEOREM.** *If  $\mathcal{F}$  is a finite family of Matkowski contractions of a complete, normal and regular semimetric space, then there exists a unique  $\mathcal{F}$ -fractal.*

The ideas behind the proofs are the same as in the previous chapter, but we need some extra considerations of the properties of semimetric spaces. We use the connection between semimetric and metric spaces through uniform equivalency. We introduce an appropriate fixed point theorem of Matkowski contractions of semimetric spaces by Bessenyei and Páles [2]. Moreover, we need to investigate the properties of the Hausdorff noncompactness measure in semimetric space. A correspondence of the Hausdorff Compactness Theorem is essential. Namely, in a complete and regular semimetric space, a subset is relatively compact if and only if its Hausdorff measure of noncompactness is zero.

The main results are based on the paper of Bessenyei and Péntzes [5].

**CHAPTER 4.** In this Chapter we introduce the Hausdorff measure in semimetric spaces and study the dimension of generalized fractals. Our main reference of measure theory is the book of Járαι [30].

Hutchinson studied not only the existence and uniqueness of fractals, but also their dimension. He proved a dimension formula [27] for contractive similarities that satisfy the open set condition.

*If a finite family of contractive similarities  $\mathcal{F}$  of a complete metric space fulfils the open set condition and the ratios of the members are  $q_1, \dots, q_n$ , then the Hausdorff dimension of the induced fractal is obtained as the unique solution of the exponential equation*

$$q_1^r + \dots + q_n^r = 1.$$

In the main results, we present a variant of the fractal dimension formula for fractals obtained by Matkowski contractions of semimetric spaces in terms of an exponential expression. The factors of contractions are replaced with the right-upper Dini-derivatives of the comparison functions related to the Matkowski contractions.

**THEOREM.** *If  $f_1, \dots, f_n$  are Matkowski contractions of a complete, normal and regular semimetric space with comparison functions  $\varphi_1, \dots, \varphi_n$  and  $0 < \chi^r(H) < +\infty$  holds for the induced fractal  $H$  with some  $r > 0$ , then the Dini-derivatives of the comparison functions cannot be zero simultaneously,  $\dim_{\mathcal{H}}(H) = r$ , and*

$$1 \leq (D\varphi_1(0))^r + \dots + (D\varphi_n(0))^r.$$

*Moreover, if  $\chi^r(f_k(H) \cap f_l(H)) = 0$  whenever  $k \neq l$  and each Matkowski contraction is a similarity with strong comparison function, then all derivatives at zero are strictly smaller than one and the inequality above is valid with equality.*

The main results of this Chapter are based on the paper of Bessenyei and Péntes [5].

**CHAPTER 5.** We revisit the main results of this Thesis again, and show that the role of Hausdorff measure of noncompactness is not as distinguished, when we talk about topological fixed point results or fractal theory. We use the concepts of set functions and hull operators to arrive at even more general fractal theorems.

A map  $\Phi: X \rightrightarrows X$  is a *hull operator* on the nonempty set  $X$ , if it is monotone, idempotent and extensive.

We have to link set-functions and hull operators properly and (motivated by the notion of condensing maps) clarify the mapping class we shall deal with:

A set-function  $\chi: \mathcal{P}(X) \rightarrow [0, +\infty]$  is  $\Phi$ -compatible if, for all  $A, B \subseteq X$ ,

$$\chi(\Phi(A \cup B)) \leq \max\{\chi(A), \chi(B)\}.$$

A map  $F: X \rightrightarrows X$  is  $\chi$ -decreasing if  $\chi(F(A)) < \chi(A)$  whenever  $A \subseteq X$  and  $0 < \chi(A) < +\infty$ .

In the main results of this Chapter, we illustrate that we can change the domain of an invariance operator to a “small” one such that it inherits the properties of

the original domain and on which the invariance operator is still a self-map. The Knaster–Tarski Fixed Point Theorem and the Kantorovich-iteration process play a key role again.

**THEOREM.** *Assume that  $\Phi: X \rightrightarrows X$  is a hull operator,  $D \subseteq X$  is nonempty,  $\Phi$ -invariant, and  $\chi$  is  $\Phi$ -compatible, finite, non-atomic on  $D$ . If  $\mathcal{F}$  is a finite set of  $\chi$ -decreasing self-maps of  $D$  and  $F: D \rightrightarrows D$  is the induced invariance operator, then there exists a nonempty  $\Phi$ -invariant set  $H \subseteq D$  such that  $F: H \rightrightarrows H$  and  $\chi(H) = 0$ .*

Applying the main results, we get another branch of fractal theorems, using different measures than the Hausdorff one. Based on the concepts introduced by García and Mora [21], we present applications for the so-called degree of nondensifiability. The main results of this Chapter are based on the paper of Bessenyei and Péntzes [6].



## Összefoglalás

Az értekezésben Hutchinson [27] és Barnsley [8] munkásságát követve fixpontelméleti megközelítésből tanulmányozzuk a fraktálokat.

Értekezésünk központi fogalma az invarianciaoperátor. Tegyük fel, hogy  $\mathcal{F}$  a  $\mathcal{P}(X)$  hatványhalmazon értelmezett önmagukba ható leképezések családja. Azt mondjuk, hogy a

$$\Phi(H) := \bigcup_{F \in \mathcal{F}} F(H)$$

módon értelmezett  $\Phi: X \rightrightarrows X$  leképezés az  $\mathcal{F}$  család *invariancia operátora*. Továbbá, a  $H \subseteq X$  halmaz  $\mathcal{F}$ -*invariáns*, ha

$$H = \Phi(H)$$

teljesül, azaz  $H$  *fixpontja* az invariancia operátornak.

Az  $(X, d)$  metrikus térben legyen  $\mathcal{F} = \{f_1, \dots, f_n\}$ , ahol  $f_1, \dots, f_n: X \rightarrow X$ . Egy  $\mathcal{F}$ -*fraktál* alatt az  $X$  metrikus tér egy nemüres, kompakt,  $\mathcal{F}$ -invariáns részhalmazát értjük.

Hutchinson klasszikus fraktálméleti tétele [27] azt állítja, hogy ha  $\mathcal{F}$  *kontrakciók véges családja egy teljes metrikus térben, akkor pontosan egy  $\mathcal{F}$ -fraktál létezik.*

Legyen  $(X, d)$  metrikus tér, és jelölje  $\mathcal{K}(X)$  az  $X$  nemüres és kompakt részhalmazainak családját. Ekkor tetszőleges  $A, B \in \mathcal{K}(X)$  halmazokra definiálhatjuk az  $A$  és  $B$  halmazok *Hausdorff–Pompeiu-távolságát*, amelyet jelöljön  $d_{HP}$ . A  $(\mathcal{K}(X), d_{HP})$  metrikus teret gyakran *fraktáltérnek* nevezik, lásd [26] és [41].

Blaschke [9] belátta, hogy a fraktáltér örökli az alaptér teljességét. Azaz, ha  $(X, d)$  *teljes metrikus tér, akkor  $(\mathcal{K}(X), d_{HP})$  is teljes.*

Hutchinson fraktálokhöz vezető útját tehát a következőképpen foglalhatjuk össze: Az invarianciaegyenletet felfoghatjuk úgy, mint egy fixpontproblémát, ahol a keresett fixpont a  $\mathcal{K}(X)$  család elemei közül kerül ki. Kiderül, hogy ha  $\mathcal{F}$  kontrakciók véges családja, akkor a  $\Phi$  invarianciaoperátor maga is kontrakció a fraktáltéren. Továbbá, ha az eredeti tér teljes, akkor a Blaschke-tétel alapján a fraktáltér is teljes. Ezért alkalmazhatjuk a Banach-féle fixponttételt [1]: *Teljes metrikus tér bármely kontrakciója pontosan egy fixponttal rendelkezik.*

Összességében a kapott fixpont egy nemüres, kompakt,  $\mathcal{F}$ -invariáns halmaz. Más szóval: egy fraktál.

Jelen értekezés során a fraktálok egy alternatív megközelítését mutatjuk be, ami teljes mértékben különbözik a Hutchinsoni úttól. Hutchinson fixpontproblémájára különböző környezetekben adunk megoldásokat, az eredetinel általában sabb leképezéseket használva. Tehát így módon általánosított fraktálokat tanulmányozunk általánosított kontextusokban.

**1. FEJEZET.** Ez a fejezet egy elemi utat mutat a fraktálokhoz és csupán a naiv halmazelméletre támaszkodik. A monoton és folytonos leképezések állnak vizsgálataink középpontjában.

Az első fejezet fő eredményeiben megfogalmazunk egy fraktáltételt tisztán naiv halmazelméleti kontextusban. A Kantorovics-iteráció [31] és a Knaster–Tarski fixponttétel [32] segítségével megadjuk az invariancia egyenlet minimális megoldását.

*TÉTEL. Ha  $\mathcal{F}$  monoton leképezéscsalád, akkor létezik egy legszűkebb  $\mathcal{F}$ -invariáns halmaz. Továbbá, ha  $\mathcal{F}$  folytonos család, akkor a Kantorovics-iteráció az adott szubinvariáns halmazt a legszűkebb invariáns halmazzá bővíti, ami ezt a szubinvariáns halmazt tartalmazza.*

Megmutatjuk néhány klasszikus példán keresztül, hogyan találhatjuk meg az invariancia egyenlet minimális megoldását. Az is kiderül, hogy a Kantorovics-iteráció nem csupán elméleti szempontból fontos: Ugyanis a módszer implementálásával könnyen approximálhatunk fraktálokat. Ennek illusztrálásaként bemutatjuk a klasszikus Sierpeński-szőnyeg és egy saját “Wave” fraktál approximációját. Azt is láthatjuk továbbá, hogy bármelyik minimális megoldás lezártja épp a kapcsolódó klasszikus fraktál. Ez a megfigyelés kiemelkedő szerepet játszik a későbbi fejezetekben is. A fejezet fő eredményei Bessenyei és Pénzes [3] cikkén alapulnak.

**2. FEJEZET.** Ebben a fejezetben metrikus terekben vizsgáljuk a fenti invariancia operátort. Tehát az  $\mathcal{F}$  család a metrikus tér véges sok önmagába ható leképezését tartalmazza. Azaz,  $\mathcal{F}$ -fraktál alatt a metrikus tér egy nemüres, kompakt,  $\mathcal{F}$ -invariáns részalmazát értjük.

Hutchinson [27] klasszikus eredményeit általánosítjuk abban az értelemben, hogy kontrakciók helyett kondenzáló leképezéseket valamint Matkowski kontrakciókat használunk.

A stratégiánk a fraktáltételek bizonyításához a következő. A Darbo–Sadovszkij fixponttétel [15, 44] szerint az  $\mathcal{F}$  minden elemének létezik legalább egy fixpontja. Vegyük bármelyiknek egy tetszőleges fixpontját. A Kantorovics-iterációt [31] alkalmazva egy nemüres  $\mathcal{F}$ -invariáns halmazhoz jutunk. Továbbá, ez a halmaz korlátos, mivel a Kantorovics-iteráció Cauchy-sorozatot származtat. A Hausdorff nemkompaktsági mérték tulajdonságait alkalmazva kiderül, hogy az

invariáns halmaz relatív kompakt. Végezetül megmutatjuk, hogy az invariáns halmaz lezártja (ami tehát nemüres és kompakt) is  $\mathcal{F}$ -invariáns. Ez pedig épp egy  $\mathcal{F}$ -fraktál létezését jelenti.

*TÉTEL. Ha  $\mathcal{F}$  egy Banach-tér nemüres, korlátos, zárt, konvex részhalmazán értelmezett kondenzáló leképezések véges családja, akkor létezik legalább egy  $\mathcal{F}$ -fraktál.*

Ha az invariancia operátor Matkowski kontrakciókat tartalmaz, akkor nem csupán a fraktál létezését, hanem egyértelműségét is tudjuk bizonyítani. Ez azonnali következménye annak a ténynek, hogy maga az invariancia operátor Matkowski kontrakció a fraktáltérben és a Matkowski kontrakciónak legfeljebb egy fixpontja van a Matkowski fixponttétel [38] szerint.

*TÉTEL. Ha  $\mathcal{F}$  egy teljes metrikus téren értelmezett Matkowski kontrakciók véges családja, akkor létezik pontosan egy  $\mathcal{F}$ -fraktál.*

A fejezet lezárásaként olyan fraktál példákat mutatunk be, amikor a klasszikus Hutchinson Tételt nem lehet alkalmazni. Ezt a tényt csupán numerikus módszerekkel támasztjuk alá szigorú matematikai bizonyítás helyett. Ahogyan korábban is tettük, implementáljuk a Kantorovics-iterációt, hogy fraktálok approximációjára alkalmazzuk. A fő eredmények Bessenyei és Péntes [4] cikkén alapulnak.

**3. FEJEZET.** Ebben a fejezetben szemimetrikus terekben értelmezett fraktálokat vezetünk be. Szemimetrikus tér alatt egy olyan  $(X, d)$  párt értünk, ahol  $X$  nemüres halmaz, míg  $d: X^2 \rightarrow \mathbb{R}_+$  pozitív definit és szimmetrikus függvény, azaz teljesíti a Fréchet-féle metrikus terek első két axiómáját. Az ilyen tereket részletesen vizsgálták Jachymski, Matkowski és Świątkowski [29], valamint Chrzęszcz, Jachymski és Turoboś [14]. Ebben a fejezetben az invarianciaoperátor leképezései egy szemimetrikus tér véges sok kondenzáló leképezéséből, illetve Matkowski kontrakciójából állnak.

*TÉTEL. Ha  $\mathcal{F}$  egy normális, teljes és reguláris szemimetrikus téren értelmezett kondenzáló leképezések véges családja úgy, hogy az invariancia operátor korlátos Kantorovics-iterációt származtat egy nemüres  $\mathcal{F}$ -szubinvariáns  $H$  halmazból, akkor létezik legkisebb  $\mathcal{F}$ -fraktál, ami tartalmazza  $H$ -t.*

*TÉTEL. Ha  $\mathcal{F}$  egy normális, teljes és reguláris szemimetrikus téren értelmezett Matkowski kontrakciók véges családja, akkor létezik egyértelmű  $\mathcal{F}$ -fraktál.*

A bizonyítások alapgondolatai megegyeznek az előző fejezetben alkalmazottakkal, azonban szükség van a szemimetrikus terek tulajdonságainak mélyebb vizsgálatára. Ehhez felhasználjuk a reguláris szemimetrikus és metrikus terek között főnálló uniform ekvivalenciát [14], továbbá Bessenyei és Páles szemimetrikus

terek Matkowski kontrakcióira vonatkozó fixponttételét [2]. Emellett megvizsgáljuk a Hausdorff-féle nemkompaktsági mérték tulajdonságait szemimetrikus térben. Lényeges szerepet játszik a Hausdorff-féle kompaktsági tétel szemimetrikus térbeli megfelelője. Eszerint teljes és reguláris szemimetrikus terekben egy részhalmaz pontosan akkor relatív kompakt, ha Hausdorff-féle nemkompaktsági mértéke nulla.

A fő eredmények Bessenyei és Péntes [5] cikkén alapulnak.

**4. FEJEZET.** Ebben a fejezetben bevezetjük a Hausdorff-mértéket szemimetrikus terekben, és az általánosított fraktálok dimenzióját vizsgáljuk. Mértékelméleti alapforrásunk Járai könyve [30].

Hutchinson nemcsak a fraktálok létezését és egyértelműségét tanulmányozta, hanem azok dimenzióját is. Egy dimenzióképletet adott meg [27] olyan kontraktív hasonlóságokra, amelyek teljesítik a nyílt halmaz feltételt.

*Ha  $\mathcal{F}$  kontraktív hasonlóságok véges családja egy teljes metrikus térben, teljesíti a nyílt halmaz feltételt, és a leképezések arányai rendre  $q_1, \dots, q_n$ , akkor a generált fraktál Hausdorff-dimenziója az*

$$q_1^r + \dots + q_n^r = 1$$

*exponenciális egyenlet egyértelmű megoldásaként adódik.*

Fő eredményeinkben a fraktáldimenzió-képlet egy változatát mutatjuk be olyan fraktálokra, amelyeket szemimetrikus terek Matkowski kontrakciói generálnak. A képlet exponenciális alakú, azonban a kontrakciós tényezőket a Matkowski kontrakciókhoz tartozó összehasonlító függvények jobb felső Dini-deriváltjai helyettesítik.

**TÉTEL.** *Legyenek  $f_1, \dots, f_n$  egy teljes, normális és reguláris szemimetrikus tér Matkowski kontrakciói a hozzájuk tartozó  $\varphi_1, \dots, \varphi_n$  összehasonlító függvényekkel. Ha a generált  $H$  fraktálra valamely  $r > 0$  esetén  $0 < \chi^r(H) < +\infty$  teljesül, akkor az összehasonlító függvények Dini-deriváltjai nem lehetnek egyszerre nullák,  $\dim_{\mathcal{H}}(H) = r$ , valamint*

$$1 \leq (D\varphi_1(0))^r + \dots + (D\varphi_n(0))^r.$$

*Továbbá, ha  $\chi^r(f_k(H) \cap f_l(H)) = 0$  minden  $k \neq l$  esetén és minden Matkowski kontrakció hasonlóság erős összehasonlító függvényvel, akkor az összes nullabeli derivált szigorúan kisebb egynél és a fenti egyenlőtlenség egyenlőségként teljesül.*

A fejezet fő eredményei Bessenyei és Péntes [5] cikkén alapulnak.

**5. FEJEZET.** Ebben a fejezetben ismét visszatérünk az értekezés fő eredményeihez, és megmutatjuk, hogy a Hausdorff-mérték szerepe nem olyan kitüntetett, mint amilyennek első látásra tűnhet a topológiai fixponttételek vagy a fraktálelmélet szempontjából. Halmazfüggvények és burokoperátorok fogalmát használjuk fel ahhoz, hogy még általánosabb fraktáltételekhez jussunk.

Az  $X$  nemüres halmazon értelmezett  $\Phi: X \rightrightarrows X$  leképezést *burokoperátor*-nak nevezzük, ha monoton, idempotens és extenzív.

A halmazfüggvények és a burokoperátorok közötti megfelelő kapcsolat megteremtéséhez, illetve a kondenzáló leképezések fogalma által motiválva, pontosítanunk kell a vizsgált leképezésoosztályt:

A  $\chi: \mathcal{P}(X) \rightarrow [0, +\infty]$  halmazfüggvényt  $\Phi$ -kompatibilisnek nevezzük, ha minden  $A, B \subseteq X$  esetén

$$\chi(\Phi(A \cup B)) \leq \max\{\chi(A), \chi(B)\}.$$

Továbbá az  $F: X \rightrightarrows X$  leképezést  $\chi$ -redukálónak nevezzük, ha  $\chi(F(A)) < \chi(A)$  teljesül minden olyan  $A \subseteq X$  halmazra, amelyre  $0 < \chi(A) < +\infty$ .

A fejezet fő eredményeiben megmutatjuk, hogy egy invarianciaoperátor értelmezési tartományát lecserélhetjük egy "kicsi" tartományra oly módon, hogy az örökölje az eredeti tartomány tulajdonságait, és az invarianciaoperátor továbbra is önmagába ható leképezés maradjon rajta. Ebben ismét kulcsszerepet játszik a Knaster–Tarski-féle fixponttétel és a Kantorovics-iterációs eljárás.

**TÉTEL.** *Tegyük fel, hogy  $\Phi: X \rightrightarrows X$  burok operátor, a  $D \subseteq X$  nemüres,  $\Phi$ -invariáns halmaz, valamint  $\chi$   $\Phi$ -kompatibilis, véges, nematomos  $D$ -n. Ha  $\mathcal{F}$  a  $D$ -n értelmezett önmagukba ható,  $\chi$ -redukáló leképezések véges halmaza és  $F: D \rightrightarrows D$  az indukált invariancia operátor, akkor létezik egy nemüres  $\Phi$ -invariáns halmaz  $H \subseteq D$  úgy, hogy  $F: H \rightrightarrows H$  és  $\chi(H) = 0$ .*

A fő eredmény alkalmazásával olyan fraktáltételt igazolhatunk, amely a Hausdorff-féle nemkompaktsági mértéktől különböző, a García és Mora [21] által bevezetett, úgynevezett nemsűrűségi mérték fogalmára épít. A fejezet fő eredményei Bessenyei és Péntes [6] cikkén alapulnak.



## Appendix

In Section 1 and 2 we present some examples how the Kantorovich-iteration can be used in practice, while plotting images of fractals. Below is the code used, which is written in Python.

```
1 # import base packages
2 import numpy as np
3 import pandas as pd
4 import matplotlib.pyplot as plt
5 import time
6 from tqdm import tqdm
7
8 # import solver package
9 from scipy.optimize import fsolve
10 from sympy import symbols, Eq, exp, nsolve, Rational
11 from itertools import product
12 import warnings
13 warnings.filterwarnings("ignore", category=RuntimeWarning)
14
15 # define fractal set functions
16 def Sierpinski_set(SetNum: int, type: int) -> dict:
17     """
18     inputs:
19         SetNum: int
20             Number of iteration
21         type: int
22             Fractal set type (1: square, 2: diamond, 3: wave)
23     output:
24         FractalSet: dict
25             key: number of iteration, elements: x,y coordinates
26     for the set
27     """
28     # create initial set without members
29     FractalSet = dict.fromkeys(range(1, SetNum+1), [])
30     # fill initial member
31     FractalSet[1] = [[1,1]]
32     # iterate calculation
33     sub_set = []
```

```

33     i = [1,2,0]
34     j = [1,2,0]
35
36     if type == 1: # square
37
38         for o in tqdm(range(2,SetNum+1)):
39             for g in range(len(FractalSet[o-1])):
40                 for p in range(len(i)):
41                     for e in range(len(j)):
42                         if (i[p]!=1)|(j[e]!=1):
43                             x_n = (FractalSet[o-1][g][0]+i[p])*1/3
44                             y_n = (FractalSet[o-1][g][1]+j[e])*1/3
45                             coordinate = [x_n,y_n]
46                             sub_set.append(coordinate)
47                             x_n = 0
48                             y_n = 0
49             FractalSet[o] = sub_set
50             sub_set = []
51
52     if type == 2: # diamond
53
54         for o in tqdm(range(2,SetNum+1)):
55             for g in range(len(FractalSet[o-1])):
56                 for p in range(len(i)):
57                     for e in range(len(j)):
58                         if (i[p]!=1)|(j[e]!=1):
59                             x_n = ((FractalSet[o-1][g][0]+i[p])
60 *1/3)**(1/np.exp(((FractalSet[o-1][g][1]+i[e])*1/3)))
61                             y_n = ((FractalSet[o-1][g][1]+j[e])
62 *1/3)**(1/np.exp(((FractalSet[o-1][g][0]+j[p])*1/3)))
63                             coordinate = [x_n,y_n]
64                             sub_set.append(coordinate)
65                             x_n = 0
66                             y_n = 0
67             FractalSet[o] = sub_set
68             sub_set = []
69
70     if type == 3: # wave
71
72         for o in tqdm(range(2,SetNum+1)):
73             for g in range(len(FractalSet[o-1])):
74                 for p in range(len(i)):
75                     for e in range(len(j)):
76                         if (i[p]!=1)|(j[e]!=1):
77                             x_n = ((FractalSet[o-1][g][0]+i[p])
78 *1/3)**(np.exp(((FractalSet[o-1][g][1]+i[e])*1/3)))

```

---

```

76             y_n = ((FractalSet[o-1][g][1]+j[e])
*1/3)**(np.exp(((FractalSet[o-1][g][0]+j[p])*1/3)))
77             coordinate = [x_n,y_n]
78             sub_set.append(coordinate)
79             x_n = 0
80             y_n = 0
81             FractalSet[o] = sub_set
82             sub_set = []
83
84     return FractalSet
85
86 def Sinus_set(SetNum: int) -> dict:
87     """
88     input:
89         SetNum: int
90             Number of iteration
91     output:
92         FractalSet: dict
93             key: number of iteration, elements: x,y coordinates
94     for the set
95     """
96     # create initial set without members
97     FractalSet = dict.fromkeys(range(1,SetNum+1), [])
98     # fill initial member
99     FractalSet[1] = [[0,0]]
100    # iterate calculatlon
101    sub_set = []
102    i = [1,2,0]
103    j = [1,2,0]
104
105    for o in tqdm(range(2,SetNum+1)):
106        for g in range(len(FractalSet[o-1])):
107            for p in range(len(i)):
108                for e in range(len(j)):
109                    if (i[p]!=1)|(j[e]!=1):
110                        x_n = np.sin(FractalSet[o-1][g][0]+i[p])
111                        y_n = np.sin(FractalSet[o-1][g][1]+j[e])
112                        coordinate = [x_n,y_n]
113                        sub_set.append(coordinate)
114                        x_n = 0
115                        y_n = 0
116                    FractalSet[o] = sub_set
117                    sub_set = []

```

```

118     return FractalSet
119
120 # define fractal set plotter
121 def FractalSetPlotter(FractalSet: dict, iterationNum: list, filename:
    str, saveplot: bool):
122     """
123     input:
124         FractalSet: dict
125             key: number of iteration, elements: x,y coordinates
126         for the set
127             iterationNum: list
128                 Iteration plotted to teh figure
129             filename: str
130                 name of the saved file
131             saveplot: bool
132                 True or False
133     output:
134         figure in png format
135     """
136     # marker list
137     markerList = ['x', 'd', 'o', '.', '8', '>', '<', 'v', '^', 's']
138     markerList = ['x', 'x', 'x', 'D', 'D', 'D', 'D']
139     # linewidth list
140     lineWidhtsList = [0.4, 0.7, 1, 4, 4, 5, 5]
141     # color list
142     colorList = ['gold', 'mediumblue', 'deeppink', 'skyblue', '
    darkorange', 'tomato', 'dodgerblue', 'red']
143     plt.figure(figsize=(12, 12))
144     for g in range(len(iterationNum)):
145         temp_df = pd.DataFrame(FractalSet[iterationNum[g]], columns
    =['x', 'y'])
146         plt.scatter(
147             temp_df.x, temp_df.y, c=colorList[g], label=iterationNum[
    g],
148             marker=markerList[g], linewidths=lineWidhtsList[g])
149     plt.legend(
150         loc="lower center", bbox_to_anchor=(0.5, -0.12),
151         title="Number of Iterations", ncol=len(iterationNum),
152         fontsize=14, title_fontsize=16)
153     # save plot
154     if saveplot == True:
155         plt.savefig(f'{filename}.png', dpi=400)
156
157 # define the function of the fractal sets
158 def G1(v: float, i: int, j: int) -> list:
    """

```

---

```

159     input:
160         v: float, i:int, j: int
161     output:
162         x,y coordinates in list format
163     """
164     x, y = v
165     return [
166         ((x+i)/3)**(1/np.exp((y+j)/3)) - x,
167         ((y+j)/3)**(1/np.exp((x+i)/3)) - y]
168
169 def G2(v: float, i:int, j: int)->list:
170     """
171     input:
172         v: float, i:int, j: int
173     output:
174         x,y coordinates in list format
175     """
176     x, y = v
177     return [
178         ((x+i)/3)**(np.exp((y+j)/3)) - x,
179         ((y+j)/3)**(np.exp((x+i)/3)) - y]
180
181 def G3(v: float, i:int, j: int)->list:
182     """
183     input:
184         v: float, i:int, j: int
185     output:
186         x,y coordinates in list format
187     """
188     x, y = v
189     return [
190         (np.sin(x+i))*(1/np.exp(y+j)) - x,
191         (np.sin(y+j))*(1/np.exp(x+i)) - y]
192
193
194 def fixed_point_solutions_i_j(G, grid: tuple = (0, 1, 100)) ->
    None:
195     """
196     input:
197         G: function
198             the function used to search the fix-points
199         grid: tuple = (0, 1, 100)
200     output:
201         printed fix-point solution
202     """
203     ij_tuple = product([0,1,2], [0,1,2])

```

```

204     for ij in ij_tuple:
205         solutions = []
206         if (ij[0] != 1) | (ij[1] != 1):
207             for gx in np.linspace(grid[0], grid[1], grid[2]):
208                 for gy in np.linspace(grid[0], grid[1], grid[2]):
209                     try:
210                         solution, info, ier, _ = fsolve(G, (gx, gy
211 ), args=(ij[0], ij[1]), full_output = True)
212                         if ier == 1:
213                             res = G(solution, ij[0], ij[1])
214                             if (np.all(np.isfinite(res)) and np.
allclose(res, 0, atol=1e-5) and solution[0] >= 0 and solution
[1] >= 0):
215                                     if not any(np.allclose(solution, s
, atol= 1e-5) for s in solutions):
216                                         solutions.append(solution)
217                             except Exception:
218                                 pass
219                             solution_txt = f"Found {len(solutions)} solution(s)"
220                             for solution in solutions:
221                                 solution_temp_txt = f"    (x, y) = ({solution
[0]:.10g}, {solution[1]:.10g}); (i, j) = ({ij[0]}, {ij[1]})"
222                                 solution_txt = solution_txt + "\n" +
solution_temp_txt
223                             print(solution_txt)

```

## List of publications

- [1] M. Bessenyei and E. Péntes, *Separation problems in the context of  $h$ -convexity*, J. Convex Anal. **25** (2018), no. 3, 1033–1043., ISSN: 0944-6532, IF: 0.794
- [2] M. Bessenyei and E. Péntes, *Higher-order quasimonotonicity and integral inequalities*, Math. Inequal. Appl. **21** (2018), no. 3, 897–909., ISSN: 1331-4343, IF: 1.083
- [3] M. Bessenyei and E. Péntes, *Fractals for minimalists*, Aequat. Math. **94** (2020), no. 3, 595–603., ISSN: 0001-9054, IF: 0.832
- [4] M. Bessenyei and E. Péntes, *Monoton leképezések fixpontjai 1*, KöMaL **70** (2020), no. 3, 141–146., ISSN: 1215-9247
- [5] M. Bessenyei and E. Péntes, *Monoton leképezések fixpontjai 2*, KöMaL **70** (2020), no. 6, 328–333., ISSN: 1215-9247
- [6] M. Bessenyei and E. Péntes, *Hutchinson without Blaschke: an alternative way to fractals*, Expo. Math. **39** (2021), 25–32., ISSN: 0723-0869, IF: 1.339
- [7] M. Bessenyei and E. Péntes, *Generalized fractals in semimetric spaces*, Nonlinear Anal. **220** (2022), Article no. 112853, ISSN: 0362-546X, IF: 1.4
- [8] M. Bessenyei and E. Péntes, *Support theorems for generalized monotone functions*, J. Convex Anal. **30** (2023), no. 1, 359–369., ISSN: 0944-6532, IF: 0.6
- [9] M. Bessenyei and E. Péntes, *Shrinking bounded domains to totally bounded ones*, J. Math. Anal. Appl. **552** (2025) art no. 129808., ISSN: 0022-247X, IF: 1.3



## List of talks

- [1] *Separation problems in context of  $h$ -convexity*, International Student's Conference on Analysis'13 Ustroń (Poland), February 4–7, 2017.
- [2] *Szeparációs problémák  $h$ -konvex függvényekre*, Research Seminar, University of Miskolc, May 3, 2017.
- [3] *Separation problems in context of  $h$ -convexity*, Research Seminar on Analysis, University of Debrecen, May 31, 2017.
- [4] *Magasabbrendű kvázimonotonitás és integrálegyenlőtlenségek*, Research Seminar, University of Miskolc, November 8, 2017.
- [5] *Magasabbrendű kvázimonotonitás és integrálegyenlőtlenségek*, KTDK, University of Debrecen, November 24, 2017.
- [6] *Higher-order quasimonotonicity and integral inequalities*, Research Seminar on Analysis, University of Debrecen, December 20, 2017.
- [7] *Magasabbrendű kvázimonotonitás és integrálegyenlőtlenségek*, ÚNKP conference, University of Debrecen, January 29, 2018.
- [8] *Higher-order quasimonotonicity and integral inequalities*, International Student's Conference on Analysis'14 Sífőrkút (Hungary), February 3–6, 2018.
- [9] *A  $h$ -konvexitás a konvex geometria tükrében*, KTDK, University of Debrecen, November 30, 2018.
- [10] *Fractals for minimalists*, International Student's Conference on Analysis'15 Ustroń (Poland), February 2–5, 2019.
- [11] *A  $h$ -konvexitás a konvex geometria tükrében*, OTDK, Eger, April 25, 2019.
- [12] *A fraktálmélet alaptétele*, Research Seminar, University of Miskolc, May 8, 2019.
- [13] *Hutchinson without Blaschke: An alternative way to fractals*, Research Seminar on Analysis, University of Debrecen, May 30, 2019.
- [14] *Konvexitás és fraktálmélet*, ÚNKP conference, University of Debrecen, June 17, 2019.
- [15] *An alternative way to fractals*, International Student's Conference on Analysis'16 Sífőrkút, February 1–4, 2020.
- [16] *Fixponttételek és fraktálmélet*, ÚNKP conference, University of Debrecen, June 6, 2020.
- [17] *Általánosított fraktálok*, KTDK, University of Debrecen, November 20, 2020.
- [18] *Általánosított fraktálok*, OTDK, Szeged, April 6, 2021.
- [19] *Support theorems for generalized monotone functions*, Katowice-Debrecen Winter Seminar'21 Brenna (Poland), February 2-5, 2022.
- [20] *Generalized fractals in semimetric spaces*, Research Seminar on Analysis, University of Debrecen, May 4, 2022.
- [21] *Generalized fractals in semimetric spaces*, International Symposium of Functional Equations'58 Innsbruck (Austria), June 19-26, 2022.
- [22] *Dimension of generalized fractals in semimetric spaces*, Katowice-Debrecen Winter Seminar'22 Hajdúszoboszló, February 1-4, 2023.
- [23] *Invariant subsets of invariant domains*, International Symposium of Functional Equations'59, Hajdúszoboszló, June 18-25, 2023.



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- [6] M. Bessenyei and E. Péntzes, *Shrinking bounded domains to totally bounded ones*, J. Math. Anal. Appl. **552** (2025) art no. 129808.
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