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On a generalized Hosszú functional equation

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Dedicated to the 70th birthday of Professor Zoltán Daróczy

Abstract. In this note we discuss a general pexiderized version of the Hosszú functional equation.

1. Introduction

The functional equation

$$f(x+y-xy)+f(xy)=f(x)+f(y) \tag{1}$$

was first considered by M. Hosszú in 1967. In the real case the general solution was given by Blanuša [2] and Daróczy [3]. In [4] Fenyő studied the following generalization of (1)

$$f[r_0+(r_1x+r_2)(r_3y+r_4)]+g[s_0+(s_1x+s_2)(s_3y+s_4)]=h(x)+k(y) \tag{2}$$

where r_i and s_i $(i=0,1,2,3,4)$ are fixed real numbers such that $r_1r_3s_1s_3\neq 0$. If this condition is not satisfied then (2) reduces to a much simpler equation, namely to the logarithmic Pexider equation $p(xy)=q(x)+r(y)$.

Fenyő supposed (2) to hold for all x, $y\in\mathbb{R}$ (the set of real numbers) and he discussed also the case when (2) holds only for the pairs $(x,y)\in D$ where

$$D=\left\{(x,y)\in\mathbb{R}^2:(r_1x+r_2)(r_3y+r_4)(s_1x+s_2)(s_3y+s_4)\neq 0\right\}.$$

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In both cases he determined the locally integrable solutions. Introducing the notations:

$$\frac{s_1}{r_1}=\alpha,\quad \frac{r_2s_1-r_1s_2}{r_1}=\beta,\quad \frac{s_3}{r_3}=\gamma,\quad \frac{r_4s_3-r_3s_4}{r_3}=\delta,$$

we find that $\alpha\gamma\neq 0$, furthermore we have the following possible cases

$$(a)\quad \beta=\delta=0,\quad (b)\quad \beta\delta\neq 0,\quad (c)\quad \beta\neq 0,\ \delta=0,\quad (d)\quad \beta=0,\ \delta\neq 0.$$

In case (a) equation (2) can easily be reduced to the logarithmic Pexider equation again. In case (b) we are not able to give the general solution, however the measurable solutions can be obtained by using the ideas of Lajkó [6].

In this note we deal with the cases (c) and (d) and give the general solution supposing (2) to hold for all $(x,y)\in D$ or $(x,y)\in\mathbb{R}^2$ with unknown functions f, g, h, k having suitable domains.

2. Preliminary results

Our basic tool is the following theorem which is an easy consequence of the main results of Baker [1] and Lajkó [5].

Theorem 1. *The functions F G, K $Q:\,]0,+\infty[\,\to\mathbb{R}$ satisfy the functional equation*

$$F(x)+G(y)=K(x+y)+Q\left(\frac{x}{y}\right) \tag{3}$$

for all $x,y\in\,]0,+\infty[$ if, and only if, there exist $a:\mathbb{R}\to\mathbb{R}$, $l_1,l_2:\,]0,+\infty[\,\to\mathbb{R}$ and $a_0,b_0,c_0,d_0\in\mathbb{R}$ such that a is additive, i.e.,

$$a(x+y)=a(x)+a(y),\qquad (x,y\in\mathbb{R})$$

l_i is logarithmic, i.e.,

$$l_i(xy)=l_i(x)+l_i(y),\qquad (x,y\in\,]0,+\infty[,\,i=1,2)$$

$a_0+b_0=c_0+d_0$ and, for all $x\in\,]0,+\infty[$,

$$F(x)=a(x)+l_1(x)+a_0,\quad G(x)=a(x)+l_2(x)+b_0,$$

$$K(x)=a(x)+l_1(x)+l_2(x)+c_0,\quad Q(x)=l_1\left(\frac{x}{1+x}\right)+l_2\left(\frac{1}{1+x}\right)+d_0.$$

In what follows we need the solutions F, G, K and Q of (3) defined not only on the set of positive reals but the sets $\mathbb{R}_0:=\mathbb{R}\setminus\{0\}$ and $\mathbb{R}_0\setminus\{-1\}$, respectively. Therefore first we prove the following extension result.

Lemma 1. *The functions α, β, $\gamma : \mathbb{R}_0 \to \mathbb{R}$ satisfy the equation*

$$\alpha(x+y) = \beta(x) + \gamma(y) \tag{4}$$

for all $(x,y) \in D_0 := \left\{(x,y) \in \mathbb{R}^2 : xy \neq 0,\ x \neq 1\right\}$ if, and only if, there exist additive function $A : \mathbb{R} \to \mathbb{R}$ and real numbers β_0, γ_0 such that, for all $x \in \mathbb{R}_0$,

$$\alpha(x) = A(x) + \beta_0 + \gamma_0, \quad \beta(x) = A(x) + \beta_0, \quad \gamma(x) = A(x) + \gamma_0. \tag{5}$$

Proof. The set D_0 is the disjoint union of the following six open and connected sets

$$D_1 = \{(x,y) \in \mathbb{R}^2 \mid x,\ y > 0\}, \quad D_2 = \{(x,y) \in \mathbb{R}^2 \mid x < 0,\ y, x+y > 0\},$$

$$D_3 = \{(x,y) \in \mathbb{R}^2 \mid x,\ x+y < 0,\ y > 0\}, \quad D_4 = \{(x,y) \in \mathbb{R}^2 \mid x < 0,\ y < 0\},$$

$$D_5 = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y, x+y < 0\}, \quad D_6 = \{(x,y) \in \mathbb{R}^2 \mid x,\ x+y > 0,\ y < 0\}.$$

Applying the result of Rimán [7] on each D_k $(k = 1,2,3,4,5,6)$ separately, we get that the functions α, β, γ have the following form on the corresponding projections of D_k

$$\alpha(x) = A_k(x) + \beta_{0k} + \gamma_{0k}, \quad \beta(x) = A_k(x) + \beta_{0k}, \quad \gamma(x) = A_k(x) + \gamma_{0k}$$

with some additive function A_k and real numbers β_{0k}, γ_{0k}. It is not difficult to show that all the additive functions $A_1, \ldots, A_6$ coincide, moreover $\beta_{01} = \ldots = \beta_{06}$ and $\gamma_{01} = \ldots = \gamma_{06}$. Thus we have (5) with some additive function A and real numbers β_0, γ_0. The converse is obvious. □

Now we are ready to give all the functions $F, G, K : \mathbb{R}_0 \to \mathbb{R}$, $Q : \mathbb{R}_0 \setminus \{-1\} \to \mathbb{R}$ satisfying (3) on $E := \left\{(x,y) \in \mathbb{R}^2 : x,\ y, x+y \in \mathbb{R}_0\right\}$.

Theorem 2. *The functions F, G, $K : \mathbb{R}_0 \to \mathbb{R}$ and $Q : \mathbb{R}_0 \setminus \{-1\} \to \mathbb{R}$ satisfy (3) for all $(x,y) \in E$ if, and only if, there exist an additive function $a : \mathbb{R} \to \mathbb{R}$, logarithmic functions l_1, $l_2 :]0, +\infty[\to \mathbb{R}$ and real numbers a_0, b_0, c_0, d_0 with $a_0 + b_0 = c_0 + d_0$ such that*

$$F(x) = a(x) + l_1(|x|) + a_0, \qquad G(x) = a(x) + l_2(|x|) + b_0,$$
$$K(x) = a(x) + l_1(|x|) + l_2(|x|) + c_0 \quad (x \in \mathbb{R}_0) \tag{6}$$

and

$$Q(x) = l_1\left(\left|\frac{x}{x+1}\right|\right) + l_2\left(\frac{1}{|x+1|}\right) + d_0. \quad (x \in \mathbb{R}_0 \setminus \{-1\}). \tag{7}$$

Proof. It follows from Lemma 1 that equations (2) and (7) hold for all $x > 0$, instead of $x \in \mathbb{R}_0$ and $x \in \mathbb{R}_0 \setminus \{-1\}$, respectively, with some additive function $a : \mathbb{R} \to \mathbb{R}$, logarithmic functions $l_1, l_2 :]0, \infty[\to \mathbb{R}$ and real numbers a_0, b_0, c_0, d_0 with $a_0 + b_0 = c_0 + d_0$. We show that (2) and (7) hold also for $x < 0$

and $x<0$, $x\neq -1$, respectively. Indeed, replacing x and y in (3) by $-x$ and $-y$, respectively and subtracting the equation so obtained from (3) we get that

$$K(x+y)-K(-(x+y))=F(x)-F(-x)+G(y)-G(-y)\,. \quad ((x,y)\in D_0) \tag{8}$$

Define the functions α,β,γ on $\mathbb{R}_0$ by

$$\alpha(x)=K(x)-K(-x)\,, \quad \beta(x)=F(x)-F(-x)\,, \quad \gamma(x)=G(x)-G(-x) \tag{9}$$

to get (4) from (8). Applying Lemma 1 and Theorem 1, by (5) and (9), after some calculation we have that

$$\begin{aligned} F(x)&=-a(x)+A(x)+l_1(|x|)+a_0+\beta_0, && (x<0)\\ G(x)&=-a(x)+A(x)+l_2(|x|)+b_0+\gamma_0, && (x<0)\\ K(x)&=-a(x)+A(x)+l_1(|x|)+l_2(|x|)+c_0+\beta_0+\gamma_0, && (x<0) \end{aligned} \tag{10}$$

where $a,A:\mathbb{R}\to\mathbb{R}$ are additive functions, $l_1,l_2:\,]0,\infty[\,\to\mathbb{R}$ are logarithmic functions and a_0, b_0, c_0 are real numbers. Now we show that $A=2a$ on $\mathbb{R}$. To prove this, let $x>0$, $y<0$ such that $x+y>0$ in (3). Then, by (2) (which holds for $x>0$) and (10), after some calculation, we obtain that

$$\begin{aligned} &l_1(x)+l_2(|y|)+A(x)-a(y)+a_0+b_0+\gamma_0\\ &\qquad\qquad=a(y)+l_1(|x+y|)+l_2(|x+y|)+c_0+Q\left(\frac{x}{y}\right). \end{aligned}$$

Let $0<t\in\mathbb{R}$ be arbitrary and replace here x and y by tx and ty, respectively, and compare the equation so obtained with the above one. Then an easy calculation shows that $A(ty)-2a(ty)=A(y)-2a(y)$ holds for all $t>0$ and $y<0$. This implies $A=2a$. Thus (10) can be written in the form

$$\begin{aligned} F(x)&=a(x)+l_1(|x|)+a_0+\beta_0 && (x<0)\\ G(x)&=a(x)+l_2(|x|)+b_0+\gamma_0 && (x<0)\\ K(x)&=a(x)+l_1(|x|)+l_2(|x|)+c_0+\beta_0+\gamma_0 && (x<0)\,. \end{aligned} \tag{11}$$

In what follows, using that (2) holds for all $x>0$, (11) holds for all $x<0$, and (3) holds for all $(x,y)\in D_0$, it can easily be proved that $\beta_0=\gamma_0=0$ in (11), that is, (2) holds also for all $x\in\mathbb{R}_0$.

Finally, we get (7) to hold for $x<-1$, and for $0>x>-1$, respectively. □

3. The main result

In the following theorem we determine the general solution of (2) on D in case (c). The suitable domains of the unknown functions f, g, h and k will be

$$D_f = \{r_0 + (r_1x + r_2)(r_3y + r_4) : (x, y) \in D\},$$

$$D_g = \{s_0 + (s_1x + s_2)(s_3y + s_4) : (x, y) \in D\},$$

$$D_h = \{x : (x, y) \in D\} \quad \text{and} \quad D_k = \{y : (x, y) \in D\},$$

respectively.

Theorem 3. *Under the conditions $\beta \neq 0$, $\delta = 0$, the functions f, g, h, k satisfy (2) for all $(x, y) \in D$ if, and only if,*

$$f(x) = a\left(\frac{\alpha}{\beta}(x - r_0)\right) + l_1\left(\left|\frac{\alpha}{\beta}(x - r_0)\right|\right) + a_0, \qquad (x \in D_f)$$

$$g(x) = a\left(-\frac{1}{\beta\gamma}(x - s_0)\right) + l_2\left(\left|-\frac{1}{\beta\gamma}(x - s_0)\right|\right) + b_0, \qquad (x \in D_g)$$

$$h(x) = l_1\left(\left|\frac{\alpha}{\beta}(r_1x + r_2)\right|\right) + l_2\left(\left|-\frac{1}{\beta}(s_1x + s_2)\right|\right) + c_0, \qquad (x \in D_h)$$

$$k(x) = a(r_3x + r_4) + l_1(|r_3x + r_4|) + l_2(|r_3x + r_4|) + d_0, \qquad (x \in D_k) \tag{12}$$

with some additive function $a : \mathbb{R} \to \mathbb{R}$, logarithmic functions $l_1, l_2 :]0, +\infty[\to \mathbb{R}$ and real numbers a_0, b_0, c_0, d_0 satisfying $a_0 + b_0 = c_0 + d_0$.

Proof. Let $(u, v) \in D_0$ and $x = \frac{\beta}{s_1}u - \frac{r_2}{r_1}$, $y = \frac{v - r_4}{r_3}$. Then $(x, y) \in D$ and, with the definitions

$$F(x) = f\left(r_0 + \frac{\beta}{\alpha}x\right), \qquad G(x) = g(s_0 - \beta\gamma x),$$

$$H(x) = h\left(\frac{\beta}{s_1}x - \frac{r_2}{r_1}\right), \qquad K(x) = k\left(\frac{x - r_4}{r_3}\right), \quad (x \in \mathbb{R}_0) \tag{13}$$

equation (2) and the condition $\delta = 0$ imply that

$$F(uv) + G((1 - u)v) = H(u) + K(v) \tag{14}$$

for all $(u, v) \in D_0$. Define the function Q on $\mathbb{R}_0 \setminus \{-1\}$ by

$$Q(x) = H\left(\frac{x}{1 + x}\right). \tag{15}$$

Then (14) implies that (3) holds on E. Thus Theorem 2 can be applied. Finally, (12) follows from the definitions (13) and (15). The converse is a simple computation. □

Remark 1. If (2) holds for all $(x,y)\in\mathbb{R}^2$ with $f,\ g,\ h,\ k:\mathbb{R}\to\mathbb{R}$ then (14) holds for all $(u,v)\in\mathbb{R}^2$. The substitution $v=0$ shows that H is constant thus (14) becomes a Pexider equation and, by (13), h is constant, as well. Therefore the logarithmic functions l_1 and l_2 vanish in the general solution.

Remark 2. The case (d) $(\beta=0,\ \delta\neq 0)$ can be handled in a quite similar way. Here the definitions

$$F(x)=f\left(r_0+\frac{\delta}{\gamma}x\right),\qquad G(x)=g\left(s_0-\alpha\delta x\right),$$

$$H(x)=h\left(\frac{x-r_2}{r_1}\right),\qquad K(x)=k\left(\frac{\delta}{s_3}x-\frac{r_4}{r_3}\right)$$

lead to equation (14) and, after all, to the general solution of (14).

Remark 3. The regular solutions (say the locally integrable or measurable solutions) can easily be obtained from the general ones in the cases we investigated.

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