ON THE DECOMPOSABILITY OF LINEAR COMBINATIONS OF BERNOULLI POLYNOMIALS

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Dedicated to the 75th birthday of Kálmán Győry

Abstract. In the present paper we describe the complete decomposition (over $\mathbb{C}$) of linear combinations of the form

$$R_n(x) = B_n(x) + cB_{n-2}(x)$$

of Bernoulli polynomials, where c is an arbitrary rational number.

1. Introduction

Let $\mathbb{K}$ be a field. A *decomposition* of a polynomial $F(x) \in \mathbb{K}[x]$ is defined as $F_1(F_2(x))$, where $F_1(x)$, $F_2(x) \in \mathbb{K}[x]$. The decomposition is *nontrivial* if $\deg F_1(x)$, $\deg F_2(x) > 1$. The polynomial $F(x)$ is called *decomposable* if it has at least one nontrivial decomposition, and *indecomposable* otherwise. Two decompositions $F(x) = F_1(F_2(x))$ and $F(x) = G_1(G_2(x))$ are said to be *equivalent* over the field $\mathbb{K}$, written $F_1 \circ F_2 \sim_{\mathbb{K}} G_1 \circ G_2$, if there exists a linear polynomial $l(x) \in \mathbb{K}[x]$ such that

$$F_1(x) = G_1(l(x)) \text{ and } G_2(x) = l(F_2(x)).$$

For a given $F(x) \in \mathbb{K}[x]$ with $\deg F(x) > 1$, a *complete decomposition* of $F(x)$ over $\mathbb{K}$ is a decomposition $F = F_1 \circ \cdots \circ F_k$, where the polynomials $F_i \in \mathbb{K}[x]$ are indecomposable over $\mathbb{K}$ for $i = 1, \ldots, k$. A polynomial of degree greater than 1 always has a complete decomposition. However, it is not unique even up to equivalence.

The nth Bernoulli polynomial $B_n(x)$ is defined by the following generating function:

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{tx}}{e^t - 1}.$$

In 2002, Bilu, Brindza, Kirschenhofer, Pintér and Tichy [1] gave the complete decomposition of the nth Bernoulli polynomial over $\mathbb{C}$ up to equivalence. They proved that the polynomial $B_n(x)$ is indecomposable for odd

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n. Further, if $n = 2m$ is even, then any nontrivial decomposition of $B_n(x)$ is equivalent to

$$B_n(x) = \widetilde{B}_m\left(\left(x - \frac{1}{2}\right)^2\right), \tag{1}$$

where $\widetilde{B}_m(x) \in \mathbb{Q}[x]$ is a polynomial of degree m. In particular, the polynomial $\widetilde{B}_m(x)$ is indecomposable for any m. In 2013, Kreso and Rakaczki [7] gave an analogous decomposition result for Euler polynomials. For further decomposition theorems for certain infinite families of polynomials satisfying second-order linear recurrences we refer to [5] and [4].

The main purpose of the present paper is to consider the effect of addition on decomposability. More precisely, we give the complete decomposition of linear combinations of the form

$$R_n(x) = B_n(x) + cB_{n-2}(x)$$

of Bernoulli polynomials, where c is an arbitrary rational number. Our theorem will be presented in Section 2. In Section 3 we collect the auxiliary results and lemmas which we need for the proofs. The proof of our theorem is in Section 4. More precisely, we prove in Section 4 that apart from three exceptional cases the polynomial $R_n(x)$ is indecomposable over $\mathbb{C}$. Section 5 contains the investigation of these exceptional cases. In that section we give the complete decomposition of $R_n(x)$ up to equivalence. The Diophantine consequences of the main result will be shown in a forthcoming paper.

2. The New Result

Theorem. *Let $R_n(x) = B_n(x) + cB_{n-2}(x)$, where c is an arbitrary rational number, $n \geq 3$ is an integer and $B_n(x)$ denotes the nth Bernoulli polynomial. Then the polynomials $R_n(x)$ are indecomposable over $\mathbb{C}$ for all odd n. If $n = 2m$ is even, then apart from the cases*

$$(n, c) \in \{(6, -1/4), (6, 3/4), (8, -2/3)\},$$

every nontrivial decomposition of $R_n(x)$ over $\mathbb{C}$ is equivalent to

$$R_n(x) = \widetilde{R}_m\left(\left(x - \frac{1}{2}\right)^2\right), \tag{2}$$

where $\widetilde{R}_m(x) \in \mathbb{Q}[x]$ is an indecomposable polynomial over $\mathbb{C}$ of degree $m \geq 2$. Further, up to equivalence

$$B_6(x) - \frac{1}{4}B_4(x) = f_1(g_1(x)) = F_1(G_1(x)), \text{ where } f_1(x) = x^2 - \frac{17}{560},$$

$$g_1(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, F_1(x) = x^3 - \frac{3}{2}x^2 + \frac{9}{16}x - \frac{17}{560}, G_1(x) = \left(x - \frac{1}{2}\right)^2;$$

$$B_6(x) + \frac{3}{4}B_4(x) = f_2(g_2(x)) = F_2(G_2(x)), \text{ where } f_2(x) = x^2 - \frac{1}{840},$$

$$g_2(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, F_2(x) = x^3 - \frac{1}{2}x^2 + \frac{1}{16}x - \frac{1}{840}, G_2(x) = \left(x - \frac{1}{2}\right)^2;$$

$$B_8(x) - \frac{2}{3}B_6(x) = f_3(g_3(h_3(x))), \text{ where } f_3(x) = x^2 - \frac{31}{630},$$

$$g_3(x) = x^2 - \frac{3}{2}x + \frac{5}{16}, h_3(x) = \left(x - \frac{1}{2}\right)^2.$$

We remark that the Theorem is a generalization of the result of Bilu et al. [1] since we get back their result taking $c = 0$ in our theorem. The decomposition of the infinite families of polynomials mentioned Section 1 is consistent, the corresponding polynomial $F(x)$ is indecomposable if its degree is odd, and $F(x) = F_1(F_2(x))$ for even degree, where $F_2(x)$ is a quadratic polynomial. In contrast to these cases, we have some exceptional cases. For $n = 6$, one can see that our decompositions are not equivalent. In the case $n = 8$, the complete decomposition consists of three polynomials.

3. Auxiliary Results

Let

$$S^+ = \{f(x) \in \mathbb{C}[x] \mid \quad f(x) = f(1-x)\}$$

and

$$S^- = \{f(x) \in \mathbb{C}[x] \mid \quad f(x) = -f(1-x)\}.$$

Lemma 1. *Let $f(x) = b_k x^k + \cdots + b_1 x + b_0 \in S^+ \cup S^-$. Then*

$$b_{k-1} = -\frac{k}{2}b_k \quad \text{and } b_{k-3} = \frac{k(k-1)(k-2)}{24}b_k - \frac{k-2}{2}b_{k-2}. \tag{3}$$

Further, if $f(x) \in S^+$, then $\deg f(x)$ is even and $f'(x) \in S^-$, while if $f(x) \in S^-$ then $\deg f(x)$ is odd, $f'(x) \in S^+$ and $f(1/2) = 0$.

Proof. Compare the coefficients of x^k, x^{k-1} and x^{k-3} in the equality $f(x) = \pm f(1-x)$. □

We recall two general theorems from the theory of decomposability.

Lemma 2 (Kreso and Rakaczki [7])**.** *Let $F(x) \in \mathbb{K}[x]$ be a monic polynomial such that $\deg F(x)$ is not divisible by the characteristic of the field $\mathbb{K}$. Then for every nontrivial decomposition $F = F_1 \circ F_2$ over any field extension $\mathbb{L}$ of $\mathbb{K}$, there exists a decomposition $F = \tilde{F}_1 \circ \tilde{F}_2$ such that the following conditions are satisfied*

- *$\tilde{F}_1 \circ \tilde{F}_2$ and $F_1 \circ F_2$ are equivalent over $\mathbb{L}$: $F_1 \circ F_2 \sim_{\mathbb{L}} \tilde{F}_1 \circ \tilde{F}_2$,*
- *$\tilde{F}_1(x)$ and $\tilde{F}_2(x)$ are monic polynomials with coefficients in $\mathbb{K}$,*
- *the coefficient of $x^{\deg \tilde{F}_1(x)-1}$ in $\tilde{F}_1(x)$ is 0.*

Moreover, such decomposition $\tilde{F}_1 \circ \tilde{F}_2$ is unique.

Lemma 3 (Gusić [6])**.** *Let $\mathbb{K}$ be a field of characteristic zero. Suppose that the nonconstant polynomials $g, h, G, H \in \mathbb{K}[x]$ satisfy $g \circ h = G \circ H$ and $\deg h = \deg H$. Then there exists a, $b \in \mathbb{K}$ such that*

$$H(x) = ah(x) + b, \;\; G(x) = g\left(\frac{1}{a}x - b\right).$$

Lemma 4. *Let $P(x) \in \mathbb{Q}[x]$ be a monic polynomial. Assume that $P(x) \in S^-$ and $P(x) = f(g(x))$, where $f(x)$, $g(x) \in \mathbb{Q}[x]$ and $\deg f(x), \deg g(x) > 1$. Then we can assume that $f(x)$, $g(x)$ are monic, the coefficient of $x^{\deg f(x)-1}$ in $f(x))$ is 0, $g(x) \in S^-$ and $f(x) = -f(-x)$.*

Proof. By Lemma 2 we can assume that $f(x), g(x) \in \mathbb{Q}[x]$ are monic and the coefficient of $x^{\deg f(x)-1}$ in $f(x)$ is 0. Let $f(x) = x^k + b_{k-2}x^{k-2} + \cdots + b_1x + b_0$ and $g(x) = x^t + c_{t-1}x^{t-1} + \cdots + c_1x + c_0$. Since

$$f(g(x)) = P(x) = -P(1-x) = -f(g(1-x)),$$

from Lemma 3 we obtain that there exist a, $b \in \mathbb{Q}$ such that

$$g(x) = ag(1-x) + b, \;\; f(x) = -f\left(\frac{1}{a}x - b\right). \tag{4}$$

It follows from $\deg P(x) = kt$ and Lemma 1 that k and t are odd. A comparison of the coefficients of x^t in $g(x)$ and $ag(1-x)+b$ shows that $1 = (-1)^t a = -a$. Further, on comparing the coefficients of x^{k-1} in $f(x)$ and $-f(-x-b)$ we obtain that $0 = kb$, that is $b = 0$. This proves our assertion. ☐

Lemma 5. *Let $P(x) \in \mathbb{Q}[x]$ be a monic polynomial. Assume that $P(x) \in S^+$ and $P(x) = f(g(x))$, where $f(x)$, $g(x) \in \mathbb{Q}[x]$ and $\deg f(x), \deg g(x) > 1$. Then we can assume that $f(x)$, $g(x)$ are monic, the coefficient of $x^{\deg f(x)-1}$ in $f(x)$ is 0 and one of the following two assertions is satisfied:*

(a) *$g(x) \in S^+$,*
(b) *$g(x) \in S^-$ and $f(x) = f(-x)$.*

Proof. By Lemma 2 we can assume again that $f(x), g(x) \in \mathbb{Q}[x]$ are monic and the coefficient of $x^{\deg f(x)-1}$ in $f(x)$ is 0. Let $f(x) = x^k + b_{k-2}x^{k-2} + \cdots + b_1x + b_0$ and $g(x) = x^t + c_{t-1}x^{t-1} + \cdots + c_1x + c_0$. Since

$$f(g(x)) = P(x) = P(1-x) = f(g(1-x)),$$

Lemma 3 says that there exist a, $b \in \mathbb{Q}$ such that

$$g(x) = ag(1-x) + b, \;\; f(x) = f\left(\frac{1}{a}x - b\right). \tag{5}$$

Comparing the coefficients of x^t in $g(x)$ and $ag(1-x)+b$ we get that $1 = (-1)^t a$. Now there are two cases according as t is even or odd. If t is even then $a = 1$ and computing the coefficients of x^{k-1} in the equality $f(x) = f(x-b)$ we obtain that $0 = k(-b)$. This means that $b = 0$ and $g(x) = g(1-x)$, that is $g(x) \in S^+$.

In the case when t is odd we have $a = -1$. Comparing again the coefficients x^{k-1} in the equality $f(x) = f(-x-b)$, one can deduce that $b = 0$ and so $f(x) = f(-x)$ and $g(x) = -g(1-x)$. ☐

Lemma 6. *Let* $f(x)$, $g(x) \in \mathbb{Q}[x]$ *be monic polynomials with* $\deg f(x)$, $\deg g(x) \geq 3$. *Denote by* $\mathrm{mult}(\alpha, f(x))$ *the multiplicity of the zero* α *of the polynomial* $f(x)$. *Suppose that*

$$\mathrm{mult}(0, g(x) - g(-x)) < \mathrm{mult}(0, f(g(x)) - f(g(-x))).$$

Then $f'(g(0)) = 0$.

Proof. Let $f(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0$. Obviously $g(0) - g(-0) = 0 = f(g(0)) - f(g(-0))$. By expanding $f(g(x)) - f(g(-x))$ we get

$$f(g(x)) - f(g(-x)) = (g(x) - g(-x))H(x), \tag{6}$$

where

$$\begin{aligned} H(x) = {} & g(x)^{k-1} + g(x)^{k-2}g(-x) + \cdots + g(x)g(-x)^{k-2} + g(-x)^{k-1} + \\ & b_{k-1}\left(g(x)^{k-2} + g(x)^{k-3}g(-x) + \cdots + g(x)g(-x)^{k-3} + g(-x)^{k-2}\right) + \cdots + \\ & + b_3\left(g(x)^2 + g(x)g(-x) + g(-x)^2\right) + b_2(g(x) + g(-x)) + b_1. \end{aligned} \tag{7}$$

We know that $H(0) = 0$ by the condition of the Lemma. But from (7) we get that $H(0) = kg(0)^{k-1} + (k-1)b_{k-1}g(0)^{k-2} + \cdots + 3b_3g(0)^2 + 2b_2g(0) + b_1 = f'(g(0))$.

☐

Lemma 7. *Let*

$$f(x) = x^k + b_{k-2}x^{k-2} + b_{k-3}x^{k-3} + \cdots + b_1x + b_0,$$
$$g(x) = x^t + c_{t-1}x^{t-1} + \cdots + c_1x + c_0 \in \mathbb{Q}[x].$$

If k, $t \geq 2$ *then the coefficient of* x^{kt-2} *in the polynomial* $f(g(x))$ *is*

$$kc_{t-2} + \binom{k}{2}c_{t-1}^2.$$

Proof. It is easy to see that x^{kt-2} occurs only in the term $g(x)^k$. Expanding $g(x)^k$, the assertion easily follows. ☐

Lemma 8 (Faà di Bruno formula [3])**.** *If* f *and* g *are functions with a sufficient number of derivatives, then*

$$\frac{d^n}{dx^n}f(g(x)) = \sum \frac{n!}{k_1!k_2!\cdots k_n!}f^{(r)}(g(x))\left(\frac{g'(x)}{1!}\right)^{k_1}\left(\frac{g''(x)}{2!}\right)^{k_2}\cdots\left(\frac{g^{(n)}(x)}{n!}\right)^{k_n}, \tag{8}$$

where the sum is taken over all different solutions in nonnegative integers $k_1, \ldots, k_n$ *of* $k_1 + 2k_2 + \cdots + nk_n = n$, *and* $r = k_1 + k_2 + \cdots + k_n$.

Finally, we need the following lemma collecting some well-known properties of Bernoulli polynomials which will be used in the sequel, sometimes without particular reference.

Lemma 9. *For the Bernoulli polynomials we have the following.*

(a) $B_n(x) = (-1)^n B_n(1-x)$;
(b) $B_n(x+1) - B_n(x) = nx^{n-1}$;
(c) $B'_n(x) = nB_{n-1}(x)$;
(d) $B_n\left(\frac{1}{2}\right) = (2^{1-n}-1)B_n$, *where* $B_n = B_n(0)$ *is the nth Bernoulli number;*
(e) $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$;
(f) $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^{n-k}$;
(g) $B_{2n+1}(0) = B_{2n+1}(1) = B_{2n+1}\left(\frac{1}{2}\right) = 0$ *for* $n \in \mathbb{N}$;
(h) $B_{4n} < 0, B_{4n-2} > 0$ *for* $n \in \mathbb{N}$.

Proof. See ([2]). □

4. Proof of the Theorem

First let n be odd. Then by (a) of Lemma 9 we have $R_n(x) \in S^-$. Suppose that $R_n(x) = f(g(x))$. By Lemma 4 we can assume that $f(x)$, $g(x) \in \mathbb{Q}[x]$ are monic, and $g(x) \in S^-$, $f(x) = -f(-x)$. Let

$$f(x) = x^k + b_{k-2}x^{k-2} + b_{k-4}x^{k-4} \cdots + b_3x^3 + b_1x,$$
$$g(x) = x^t + c_{t-1}x^{t-1} + \cdots + c_1x + c_0.$$

Since n is odd, obviously t and k are also odd. Using (b) of Lemma 9 one can deduce that

(9) $f(g(x+1)) - f(g(x)) = R_n(x+1) - R_n(x) = nx^{n-1} + c(n-2)x^{n-3}.$

Since $g(x) \in S^-$ thus $g(x+1) = -g(-x)$. From (9) we infer that the polynomial $-g(-x) - g(x)$ divides the polynomial $nx^{n-1} + c(n-2)x^{n-3}$ and so one of the cases

(10)
$$-g(-x) - g(x) = \begin{cases} (i) & dx^s; \\ (ii) & dx^s(x-u), \text{ where } u \in \left\{\pm\sqrt{-c(n-2)/n}\right\}; \\ (iii) & dx^s(nx^2 + c(n-2)), \end{cases}$$

holds, where d is a non-zero rational number. We know that

(11) $\quad -g(-x) - g(x) = -2c_{t-1}x^{t-1} - 2c_{t-3}x^{t-3} - \cdots - 2c_2x^2 - 2c_0.$

Consider the above three cases in (10). *The case (ii)* is simple. In this case $x = u$ is a zero of the even polynomial $-g(-x) - g(x)$, but then $x = -u$ must be also a zero which only possible if $c = 0$. If $c = 0$ we get back the case (i).

In the case (i) we have $d = t$ and $s = t-1$ by Lemma 1. Further, it follows from (11) that $c_{t-3} = c_{t-5} = \cdots = c_2 = c_0 = 0$. This yields that

(12) $\quad g(0) = g''(0) = \cdots = g^{(t-5)}(0) = g^{(t-3)}(0) = 0, \quad \text{but } g^{(t-1)}(0) \neq 0.$

Take the $(t-1)$-th derivative of the both sides of the equation $R_n(x) = f(g(x))$:

$$\frac{d^{t-1}}{dx^{t-1}}(B_n(x) + cB_{n-2}(x)) = \frac{d^{t-1}}{dx^{t-1}} f(g(x)). \tag{13}$$

Using (c) of Lemma 9 one can deduce that the left side of (13) is

$$\frac{n!}{(n-(t-1))!} B_{n-(t-1)}(x) + c\frac{(n-2)!}{(n-2-(t-1))!} B_{n-2-(t-1)}(x). \tag{14}$$

On the other hand, we obtain from the Faà di Bruno formula that

$$\frac{d^{t-1}}{dx^{t-1}} f(g(x)) = \tag{15}$$
$$\sum \frac{(t-1)!}{k_1!k_2!\cdots k_{t-1}!} f^{(r)}(g(x)) \left(\frac{g'(x)}{1!}\right)^{k_1} \left(\frac{g''(x)}{2!}\right)^{k_2} \cdots \left(\frac{g^{(t-1)}(x)}{(t-1)!}\right)^{k_{t-1}},$$

where the sum is taken over all different solutions in nonnegative integers $k_1, \ldots, k_{t-1}$ of $k_1 + 2k_2 + \cdots + (t-1)k_{t-1} = t-1$, and $r = k_1 + k_2 + \cdots + k_{t-1}$. Substituting $x = 0$ into (14) we get 0 by (g) of Lemma 9. Now write $x = 0$ into (15). Because of (12) we can assume that $k_2 = k_4 = \cdots k_{t-3} = 0$, otherwise we get 0 here in the sum (15). If $k_{t-1} = 1$ then $r = 1$ and in the sum (15) we obtain the term

$$(t-1)! f'(g(0)) \frac{g^{(t-1)}(0)}{(t-1)!} = f'(g(0)) g^{(t-1)}(0).$$

But, in the case $k_{t-1} = 0$ and $k_2 = k_4 = \cdots = k_{t-3} = 0$ we get from the equalities

$$k_1 + 3k_3 + 5k_5 + \cdots + (t-2)k_{t-2} = t-1,$$

$$k_1 + k_3 + k_5 + \cdots + k_{t-2} = r$$

that $r = t - 1 - (2k_3 + 4k_5 + \cdots + (t-3)k_{t-2})$ is even. From (12) and the fact that $f(x)$ is odd we can deduce that $f^{(r)}(g(0)) = f^{(r)}(0) = 0$. Substituting $x = 0$ into (15) it follows that $0 = f'(g(0))g^{(t-1)}(0)$. Together with (12) this gives that $f'(g(0)) = f'(0) = b_1 = 0$. We know that

$$nB_{n-1}(x) + c(n-2)B_{n-3}(x) = f'(g(x))g'(x). \tag{16}$$

Taking $x = 0$ we obtain that

$$nB_{n-1} + c(n-2)B_{n-3} = 0. \tag{17}$$

On the other hand, $g(x)^3$ divides $f(g(x)) = g(x)^k + b_{k-2}g(x)^{k-2} + \cdots + b_3 g(x)^3$. Hence $g(x)$ divides $(B_n(x) + cB_{n-2}(x))' = nB_{n-1}(x) + c(n-2)B_{n-3}(x)$. Using the fact that $g(x) \in S^-$, from Lemma 1 we get that $g(1/2) = 0$, thus by (16)

$$nB_{n-1}\left(\frac{1}{2}\right) + c(n-2)B_{n-3}\left(\frac{1}{2}\right) = 0. \tag{18}$$

Combining (d) of Lemma 9, (17) and (18) one can infer that

$$-\frac{nB_{n-1}}{(n-2)B_{n-3}} = -\frac{n(2^{2-n}-1)B_{n-1}}{(n-2)(2^{4-n}-1)B_{n-3}}, \tag{19}$$

which is a contradiction.

Consider now the case (iii). Then $d = t/n$ and $s = t - 3$. If we assume that $t > 3$ then from (11) we obtain that

$$g(0) = g''(0) = \cdots = g^{(t-7)}(0) = g^{(t-5)}(0) = 0, \quad \text{but } g^{(t-3)}(0) \neq 0. \tag{20}$$

We can now proceed in a similar way as in the case (i). After taking the $(t-3)$-th derivative of the both sides of equality $R_n(x) = f(g(x))$, the left side will be

$$\frac{n!}{(n-(t-3))!}B_{n-(t-3)}(x) + c\frac{(n-2)!}{(n-2-(t-3))!}B_{n-2-(t-3)}(x), \tag{21}$$

while the right side will be

(22)

$$\sum \frac{(t-3)!}{k_1!k_2!\cdots k_{t-3}!} f^{(r)}(g(x)) \left(\frac{g'(x)}{1!}\right)^{k_1} \left(\frac{g''(x)}{2!}\right)^{k_2} \cdots \left(\frac{g^{(t-3)}(x)}{(t-3)!}\right)^{k_{t-3}},$$

where the sum is taken over all different solutions in nonnegative integers $k_1, \ldots, k_{t-3}$ of $k_1 + 2k_2 + \cdots + (t-3)k_{t-3} = t-3$, and $r = k_1 + k_2 + \cdots + k_{t-3}$. Substituting $x = 0$ into (21) and (22) we get similarly as above that $f'(g(0)) = f'(0) = b_1 = 0$. Now repeat the demonstration from (16) we obtain again a contradiction.

It remains to study the case when $\deg g(x) = t = 3$. Since $g(x) \in S^-$, we can deduce from Lemma 1 and (11) that

$$g(x) = x^3 - \frac{3}{2}x^2 + \left(\frac{1}{2} + \frac{3c(n-2)}{n}\right)x - \frac{3c(n-2)}{2n}. \tag{23}$$

Differentiate $n-2$ times the equality $R_n(x) = f(g(x))$ and then substitute $x = 1/2$ into both sides. On the left side we get

$$-\frac{1}{24}n! + c(n-2)!. \tag{24}$$

After derivation of the right side we obtain

$$\frac{d^{n-2}}{dx^{n-2}}f(g(x)) = \tag{25}$$

$$\sum \frac{(n-2)!}{k_1!k_2!\cdots k_{n-2}!} f^{(r)}(g(x)) \left(\frac{g'(x)}{1!}\right)^{k_1} \left(\frac{g''(x)}{2!}\right)^{k_2} \cdots \left(\frac{g^{(n-2)}(x)}{(n-2)!}\right)^{k_{n-2}},$$

where the sum is taken over all different solutions in nonnegative integers $k_1, \ldots, k_{n-2}$ of $k_1 + 2k_2 + \cdots + (n-2)k_{n-2} = n-2$, and $r = k_1 + k_2 + \cdots + k_{n-2}$. Now $\deg g(x) = 3$, hence $g^{(4)}(x) = 0$ and so we can assume that $k_4 = k_5 = \cdots = k_{n-2} = 0$. By $g''(1/2) = 0$ we can also suppose that $k_2 = 0$. Thus

$$k_1 + 3k_3 = n - 2 = 3k - 2 \quad \text{and } r = k_1 + k_3. \tag{26}$$

In (26), $k_3 \le k-1$. Let $k_3 = k-i$ for some $1 \le i \le k$. Then $3k-2 = k_1 + 3k - 3i$ and so $k_1 = 3i-2$. From this we can infer that $r = k_1 + k_3 = k + 2i - 2 > k = \deg f(x)$ if $i \ge 2$, and $r = k$ if $i = 1$. After these observations we deduce that the right side will be

$$\frac{(n-2)!}{(k-1)!} f^{(k)}\left(g\left(\frac{1}{2}\right)\right) \cdot \left(\frac{g'\left(\frac{1}{2}\right)}{1!}\right)^1 \cdot \left(\frac{g'''\left(\frac{1}{2}\right)}{3!}\right)^{k-1}. \tag{27}$$

It follows from (23), (24) and (27) that

$$-\frac{n(n-1)}{24} + c = k\left(-\frac{1}{4} + \frac{3c(n-2)}{n}\right). \tag{28}$$

One can deduce in the same way that if we differentiate $n-4$ times the equality $R_n(x) = f(g(x))$ and substitute $x = 1/2$ into both sides, then we get the following equation
(29)

$$\frac{7n(n-1)(n-2)(n-3)}{240} - c(n-2)(n-3) = 12k(k-1)\left(-\frac{1}{4} + \frac{3c(n-2)}{n}\right)^2.$$

On noting that $n = 3k$, the equations (28) and (29) yields that

$$\frac{27}{40}k^4 - \frac{153}{80}k^3 + \frac{81}{80}k^2 + \frac{9}{40}k = 0, \tag{30}$$

which is true only if $k \in \{0, 1, 2, -1/6\}$. But this contradicts the fact that $k = \deg f(x) \ge 3$.

In the case when n is even we have $R_n(x) = B_n(x) + cB_{n-2}(x) \in S^+$. Suppose that $R_n(x) = f(g(x))$. By Lemma 5 we can assume that $f(x)$, $g(x) \in \mathbb{Q}[x]$ are monic and the coefficient of $x^{\deg f(x)-1}$ in $f(x))$ is 0. Further, $g(x) \in S^+$ or $g(x) \in S^-$ and $f(x) = f(-x)$. Let

$$f(x) = x^k + b_{k-2}x^{k-2} + b_{k-3}x^{k-3} + \cdots + b_1 x + b_0,$$
$$g(x) = x^t + c_{t-1}x^{t-1} + \cdots + c_1 x + c_0.$$

Using (b) of Lemma 9 one can deduce that

$$f(g(x+1)) - f(g(x)) = R_n(x+1) - R_n(x) = nx^{n-1} + c(n-2)x^{n-3}. \tag{31}$$

First study the case when $g(x) \in S^+$. Then $g(1+x) = g(-x)$ and so $g(x+1) - g(x) = g(-x) - g(x)$ divides the polynomial $nx^{n-1} + c(n-2)x^{n-3}$.

It is easy to see that one of the cases (i), (ii), (iii)
(32)

$$g(-x) - g(x) = \begin{cases} (i) & dx^s; \\ (ii) & dx^s(x-u), \text{ where } u \in \left\{\pm\sqrt{-c(n-2)/n}\right\}; \\ (iii) & dx^s(nx^2 + c(n-2)), \end{cases}$$

where d is a non-zero rational number.

In the case (ii) we get a contradiction since, if $x = u$ is a zero of the polynomial $g(-x) - g(x)$ then $x = -u$ must be also a zero of this polynomial, which is possible only if $c = 0$. If $c = 0$ we get back the case (i).

In the case (i) assume that $t \geq 4$. We study the case $t = 2$ in Section 5. From

$$g(-x) - g(x) = -2c_{t-1}x^{t-1} - 2c_{t-3}x^{t-3} - \cdots - 2c_1 x = dx^s \tag{33}$$

we get that $d = t$, $s = t-1$ and $c_1 = g'(0) = 0$. Further,

$$\mathrm{mult}(0, g(x) - g(-x)) = t - 1 < n - 3 = \mathrm{mult}(0, f(g(x)) - f(g(-x))).$$

Applying Lemma 6 one can deduce that $f'(g(0)) = 0$. Take the $(t+1)$-th derivative of both sides of the equation $R_n(x) = f(g(x))$.

$$\frac{d^{t+1}}{dx^{t+1}}(B_n(x) + cB_{n-2}(x)) = \frac{d^{t+1}}{dx^{t+1}} f(g(x)). \tag{34}$$

Using (c) of Lemma 9 one can deduce that the left side of (34) is

$$\frac{n!}{(n-(t+1))!} B_{n-(t+1)}(x) + c\frac{(n-2)!}{(n-2-(t+1))!} B_{n-2-(t+1)}(x). \tag{35}$$

On the other hand, we obtain from the Faà di Bruno formula that

$$\frac{d^{t+1}}{dx^{t+1}} f(g(x)) = \tag{36}$$
$$\sum \frac{(t+1)!}{k_1!k_2!\cdots k_{t+1}!} f^{(r)}(g(x)) \left(\frac{g'(x)}{1!}\right)^{k_1} \left(\frac{g''(x)}{2!}\right)^{k_2} \cdots \left(\frac{g^{(t+1)}(x)}{(t+1)!}\right)^{k_{t+1}},$$

where the sum is taken over all different solutions in nonnegative integers $k_1, \ldots, k_{t+1}$ of $k_1 + 2k_2 + \cdots + (t+1)k_{t+1} = t+1$, and $r = k_1 + k_2 + \cdots + k_{t+1}$.

Substituting $x = 0$ into (35) we get 0 by (g) of Lemma 9. Now write $x = 0$ into (36). By (33) we know that $g'(0) = c_1 = g'''(0) = 3!c_3 = \cdots = g^{(t-3)}(0) = (t-3)!c_{t-3} = 0$ and $g^{(t-1)}(0) = (t-1)!c_{t-1} \neq 0$. But then we can assume that $k_1 = k_3 = \cdots = k_{t-3} = 0$, otherwise we get 0 here in the sum (36). Moreover, if $k_{t-1} = k_t = k_{t+1} = 0$, then we can infer that $2k_2 + 4k_4 + \cdots + (t-2)k_{t-2} = t+1$, which is not possible because t is even. The following table contains the possible values of $k_1, k_2, \ldots, k_{t+1}$ and r.

	k_1	k_2	k_3	$\cdots$	k_{t-2}	k_{t-1}	k_t	k_{t+1}	r
(a)	0	0	0	$\cdots$	0	0	0	1	1
(b)	1	0	0	$\cdots$	0	0	1	0	2
(c)	2	0	0	$\cdots$	0	1	0	0	3
(d)	0	1	0	$\cdots$	0	1	0	0	2

Since we assumed that $k_1 = 0$, we obtain from the table that substituting $x = 0$ into (36)

$$(t+1)!f''(g(0))\frac{g''(0)}{2!}\frac{g^{(t-1)}(0)}{(t-1)!} = 0. \tag{37}$$

By $g^{(t-1)}(0) \neq 0$ we have that

$$f''(g(0))g''(0) = 0. \tag{38}$$

Now, taking the fourth derivative of $R_n(x) = f(g(x))$ and substituting $x = 0$ we deduce that

$$R_n^{(4)}(0) = 4!f'(g(0))\frac{g^{(4)}(0)}{4!} + 4!f''(g(0))\frac{g'(0)}{1!}\frac{g'''(0)}{3!} + \tag{39}$$

$$+\frac{4!}{2!}f'''(g(0))\left(\frac{g'(0)}{1!}\right)^2\frac{g''(0)}{2!} + \frac{4!}{2!}f''(g(0))\left(\frac{g''(0)}{2!}\right)^2 + \frac{4!}{4!}f^{(4)}(g(0))\left(\frac{g'(0)}{1!}\right)^4.$$

It follows from $f'(g(0)) = 0 = g'(0)$, $f''(g(0))g''(0) = 0$ and (39) that

$$R_n^{(4)}(0) = 0.$$

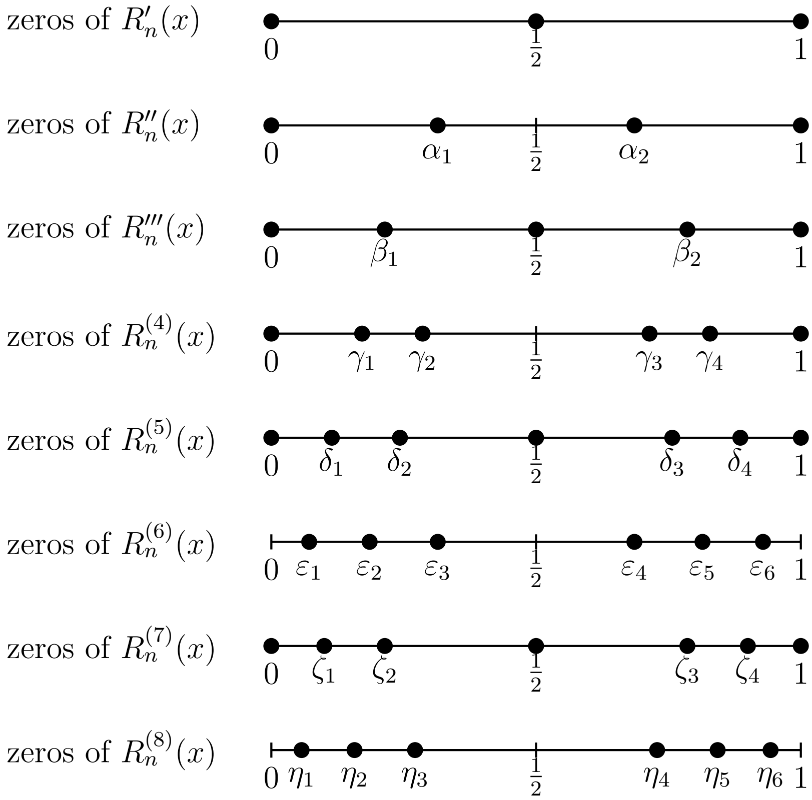
On the other hand, from $R_n'(x) = f'(g(x))g'(x)$ and $f'(g(0)) = g'(0) = 0$ we get that

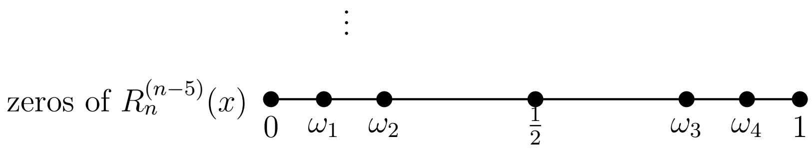
$$R_n''(0) = 0.$$

Further, by Lemma 9

$$R_n^{(m)}(0) = R_n^{(m)}\left(\frac{1}{2}\right) = R_n^{(m)}(1) = 0, \text{ if } m \text{ is odd and } n - m \geq 5.$$

Applying the above, we can investigate the number of zeros of the polynomials $R_n^{(m)}(x)$ in the interval [0,1] for $m = 1, 2, \ldots n - 5$. In the following table we use only the Rolle's theorem.





But this is a contradiction because the polynomial of $R_n^{(n-5)}(x)$ of degree 5 has at least 7 different zeros.

Investigating (iii) one can infer that $d = t/n$, $s = t - 3$ and

$$\text{mult}(0, g(x) - g(-x)) = t - 3 < n - 3 = \text{mult}(0, f(g(x)) - f(g(-x))).$$

By Lemma 6 we get again that $f'(g(0)) = 0$. If $t > 4$ then $c_1 = g'(0) = 0$ and it follows from the above and

$$R_n'(x) = nB_{n-1}(x) + c(n-2)B_{n-3}(x) = f'(g(x))g'(x) \tag{40}$$

that 0 is at least double zero of $R_n'(x)$. Thus, apart from the case $t = 4$,

$$R_n''(0) = n(n-1)B_{n-2} + c(n-2)(n-3)B_{n-4} = 0 \tag{41}$$

and so

$$c = -\frac{n(n-1)B_{n-2}}{(n-2)(n-3)B_{n-4}} > 0. \tag{42}$$

On the other hand, in the case (iii), from (32) and Lemma 1 we infer that

$$-2c_{t-3} = c(n-2)\frac{t}{n}, \quad c_{t-3} = \frac{1}{4}\binom{t}{3} - \frac{t-2}{2}c_{t-2}. \tag{43}$$

Hence

$$k(t-2)c_{t-2} = \frac{1}{2}\binom{t}{3}k + c(n-2)\frac{kt}{n} = \frac{1}{2}\binom{t}{3}k + c(n-2). \tag{44}$$

If we compare the coefficient of x^{n-2} in the equality

$$R_n(x) = B_n(x) + cB_{n-2}(x) = g(x)^k + b_{k-2}g(x)^{k-2} + \cdots + b_1 g(x) + b_0$$

we obtain, using Lemmas 1 and 7, that

$$kc_{t-2} + \binom{k}{2}\frac{t^2}{4} = \binom{n}{2}\frac{1}{6} + c. \tag{45}$$

One can deduce from equations (44) and (45) that

$$c = \frac{k(t-2)}{24(n-t)}t^2(1-k) < 0 \tag{46}$$

which contradicts (42). The case $t = 4$ is treated in the next section.

It remains the case when $g(x) \in S^-$ and $f(x) = f(-x)$. Then

$$f(x) = x^k + b_{k-2}x^{k-2} + \cdots + b_2x^2 + b_0,$$

where $k = \deg f(x)$ is even. If $k > 2$ let

$$h(x) = x^{\frac{k}{2}} + b_{k-2}x^{\frac{k-2}{2}} + \cdots + b_2x + b_0.$$

Then obviously we have

$$f(g(x)) = h(r(x)),$$

where $r(x) = g(x)^2 \in S^+$ and $\deg h(x)$, $\deg r(x) = 2t > 1$. This case was studied before.

In the case when $k = 2$ we have $f(x) = x^2 + b_0$, and since $g(x+1) = -g(-x)$ we get from (31) that

(47)
$$(-g(-x))^2 - g(x)^2 = (g(-x) - g(x))(g(-x) + g(x)) = nx^{n-1} + c(n-2)x^{n-3}.$$

It is easy to see that

$$\text{mult}(0, g(x) - g(-x)) \leq t < \text{mult}(0, f(g(x)) - f(g(-x)) = n - 3 = 2t - 3,$$

provided that $t > 3$, that is $n > 6$. The case $t = 3$ is investigated in the next section. Now, $f'(g(0))/2 = g(0) = c_0 = 0$ by Lemma 6. From $g(x) \in S^-$, $g(0) = 0$ and $B_n(x) + cB_{n-2}(x) = g(x)^2 + b_0$ we can compute that

$$R_n(0) = B_n(0) + cB_{n-2}(0) = b_0 \tag{48}$$

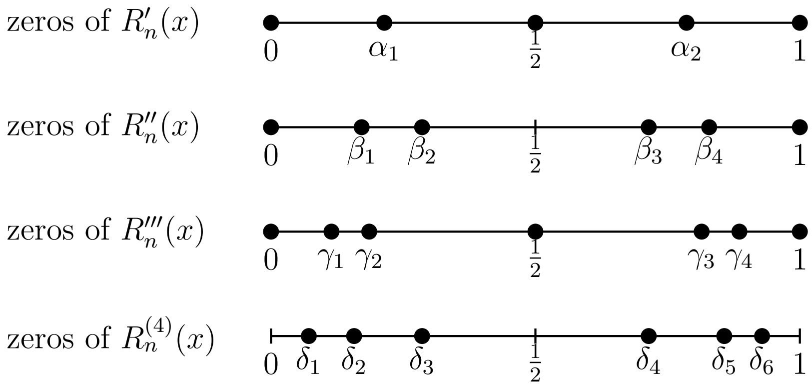
and

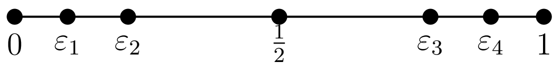
$$R_n\left(\frac{1}{2}\right) = B_n\left(\frac{1}{2}\right) + cB_{n-2}\left(\frac{1}{2}\right) = b_0. \tag{49}$$

From the Rolle's Theorem we get that there exists a zero α_1 of the polynomial $R_n'(x)$ in the interval $]0, 1/2[$. On the other hand, (47) yields that

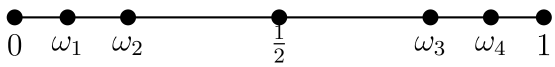
(50)
$$(-2x^t - 2c_{t-2}x^{t-2} - \cdots - 2c_3x^3 - 2c_1x)(2c_{t-1}x^{t-1} + 2c_{t-3}x^{t-3} + \cdots + 2c_2x^2) = nx^{n-1} + c(n-3)x^{n-3}.$$

We know that $t < n - 3$ provided that $t > 3$. Let $0 < i \leq (t-1)/2$ be the smallest index for which $c_{2i} \neq 0$. Comparing the coefficients of x^{2i+1} in (50) one can obtain that $c_1 = g'(0) = 0$. Together with $f'(g(0)) = 0$ we obtain that $R_n''(0) = 0$. Similar to the case (i) when n is even we can make a table using only the Rolle's theorem and the fact that 0, 1/2, 1 are zeros of the polynomials $R_n^{(m)}(x)$ provided that m is odd and $n - m \geq 5$.



zeros of $R_n^{(5)}(x)$ 

$\vdots$

zeros of $R_n^{(n-5)}(x)$ 

But this means that the polynomial $R_n^{(n-5)}(x)$ has at least seven different zeros, which is impossible.

5. Exceptional cases when n is even

We have only three cases which were not treated in the previous section:

(e1) (i) and $t = 2$;
(e2) (iii) and $t = 4$;
(e3) $g(x) \in S^-$, $f(x) = f(-x)$, $k = 2$ and $t = 3$.

First suppose that $t = 4$ and

$$g(-x) - g(x) = dx^s(nx^2 + c(n-2)) = \frac{4}{4k}x(4kx^2 + c(4k-2)). \tag{51}$$

Since $g(x) \in S^+$, applying Lemma 1 we have that

$$g(x) = x^4 - 2x^3 + c_2x^2 + (1-c_2)x + c_0 = \left(\left(x - \frac{1}{2}\right)^2 + a\right)^2 + b, \tag{52}$$

where

$$a = \frac{2c_2 - 3}{4}, \quad b = \frac{16c_0 - (2c_2 - 2)^2}{16}. \tag{53}$$

One can deduce from (51) and (52) that

$$c_2 - 1 = \frac{c(2k-1)}{k}. \tag{54}$$

Moreover, it is easy to see that we can write $R_n(x) = f(g(x))$ in the from $F(G(x))$, where $F(x) = f(x^2 + b)$ and $G(x) = (x - 1/2)^2 + a$.

Now

$$R_n(x) = R_{4k}(x) = \sum_{j=0}^{2k} \binom{4k}{2j} B_{2j}\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^{4k-2j} + \\ + c\sum_{i=0}^{2k-1} \binom{4k-2}{2i} B_{2i}\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^{4k-2-2i}. \tag{55}$$

If we define

$$H(x) := \sum_{j=0}^{2k} \binom{4k}{2j} B_{2j}\left(\frac{1}{2}\right) x^{2k-j} + c\sum_{i=0}^{2k-1} \binom{4k-2}{2i} B_{2i}\left(\frac{1}{2}\right) x^{2k-1-i},$$

then

$$R_{4k}(x) = H\left(\left(x-\frac{1}{2}\right)^2\right) = F\left(\left(x-\frac{1}{2}\right)^2 + a\right). \tag{56}$$

The above equality yields that $F(x) = H(x-a)$, where we know that $F(x)$ is an even polynomial. This means that $H(x-a)$ must be an even polynomial and so the coefficients of x^{2k-1} and x^{2k-3} are 0.

If we compute the coefficient of x^{2k-1} we get

$$-2ak - \frac{1}{12}\binom{4k}{2} + c = 0.$$

Using (53) and (54) we obtain

$$c = -\frac{k}{3}. \tag{57}$$

Computing the coefficients of x^{2k-3} one can check that

(58)

$$-\binom{2k}{3}a^3 - \frac{1}{12}\binom{4k}{2}\binom{2k-1}{2}a^2 - \frac{7}{240}\binom{4k}{4}(2k-2)a - \frac{31}{1344}\binom{4k}{6} +$$
$$+ c\binom{2k-1}{2}a^2 + \frac{1}{12}c\binom{4k-2}{2}(2k-2)a + c\frac{7}{240}\binom{4k-2}{4} = 0.$$

It follows from (53), (54), (57) and (58)

$$k(k-1)(k-2)(2k-1)(32k^2 - 92k - 61) = 0. \tag{59}$$

So we obtain $k = 2$, $t = 4$, $n = kt = 8$ and $c = -2/3$. A simple calculation shows

$B_8(x) - \frac{2}{3}B_6(x) = f(g(x))$, where $f(x) = x^2 - \frac{31}{630}$ and $g(x) = x^4 - 2x^3 + x$.

When $t = 2$ in (i) we get $g(x) = x^2 - x + c_0 = (x-1/2)^2 + c_0 - 1/4$. We know from Lemma 9 that

$$R_n(x) = \sum_{j=0}^{n/2} \binom{n}{2j} B_{2j}\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^{n-2j} + $$
$$+ c\sum_{i=0}^{n/2-1} \binom{n-2}{2i} B_{2i}\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^{n-2-2i}. \tag{60}$$

If we define

$$H(x) := \sum_{j=0}^{n/2} \binom{n}{2j} B_{2j}\left(\frac{1}{2}\right) x^{n/2-j} + c\sum_{i=0}^{n/2-1} \binom{n-2}{2i} B_{2i}\left(\frac{1}{2}\right) x^{n/2-1-i},$$

then

$$R_n(x) = H\left(\left(x-\frac{1}{2}\right)^2\right) = f(g(x)), \tag{61}$$

where $f(x) = H(x - c_0 + 1/4)$.

In the last case (e3) a direct computation yields that $c = -\frac{1}{4}$ or $\frac{3}{4}$ and

$$B_6(x) - \frac{1}{4}B_4(x) = f(g(x)), \text{ where } f(x) = x^2 - \frac{17}{560}, g(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4};$$

$$B_6(x) + \frac{3}{4}B_4(x) = f(g(x)), \text{ where } f(x) = x^2 - \frac{1}{840}, g(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

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