

BALANCING NUMBERS WHICH ARE PRODUCTS OF CONSECUTIVE INTEGERS

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ABSTRACT. In 1999 A. Behera and G. K. Panda defined balancing numbers as follows. A positive integer n is called a balancing number if $1+2+\dots+(n-1) = (n+1)+(n+2)+\dots+(n+k)$ for some $k \in \mathbb{N}$. The sequence of balancing numbers is denoted by B_m for $m \in \mathbb{N}$. In this paper we show that the Diophantine equation $B_m = x(x+1)(x+2)(x+3)(x+4)$ has no solution with $m \geq 0$ and $x \in \mathbb{Z}$. We follow ideas described in [11], that is we combine Baker's method and the so-called Mordell-Weil sieve to obtain all solutions.

1. INTRODUCTION

Behera and Panda [3] proved many interesting results related to the sequence B_m . They showed that the balancing numbers fulfill the following recurrence relation

$$B_{m+1} = 6B_m - B_{m-1} \quad (m > 1)$$

where $B_0 = 1$ and $B_1 = 6$. Later several authors investigated balancing numbers and their various generalizations. In [21] Liptai proved that there are no Fibonacci balancing numbers and in [22] he showed that there are no Lucas balancing numbers. He used a method by Baker and Davenport [2]. Szalay [31] obtained the same results by using different techniques. In [25] Panda introduced sequence balancing numbers. Let $\{s_m\}_{m=1}^\infty$ be a sequence such that $s_m \in \mathbb{R}$. An element s_m of this sequence is called sequence balancing number if

$$s_1 + s_2 + \dots + s_{m-1} = s_{m+1} + s_{m+2} + \dots + s_{m+k}$$

for some $k \in \mathbb{N}$. Further generalization in this direction is due to Bérczes, Liptai and Pink [4]. Let a, b be non-negative coprime integers. The following definition is from [19]. A positive integer $an + b$ is called (a, b) -type balancing number if

$$(a+b) + (2a+b) + \dots + (a(n-1)+b) = (a(n+1)+b) + \dots + (a(n+k)+b)$$

for some $k \in \mathbb{N}$. Denote by $B_m^{(a,b)}$ the m -th positive integer $an + b$ among the (a, b) -type balancing numbers. Kovács, Liptai and Olajos proved some general finiteness results concerning the equation

$$B_m^{(a,b)} = f(x),$$

where f is a monic polynomial with integral coefficients. They also resolved some related Diophantine equations. Liptai, Luca, Pintér and Szalay [23] introduced the concept of (k, l) -power numerical center as follows. Let y, k, l be fixed positive integers with $y \geq 2$. A positive integer x with $x \leq y - 2$ is called a (k, l) -power numerical center for y if

$$1^k + \dots + (x-1)^k = (x+1)^l + \dots + (y-1)^l.$$

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In [23] the authors obtained certain effective and ineffective finiteness results for (k, l) -power numerical centers.

For positive integers k, x let

$$\Pi_k(x) = x(x+1) \cdots (x+k-1).$$

That is, $\Pi_k(x)$ is a polynomial in x of degree k . In [19] it was proved that the equation

$$B_m = \Pi_k(x)$$

for fixed $k \geq 2$ has only finitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. We note that in [19] the "small" solutions of the above equation with $k \in \{6, 8\}$ were also computed.

In this paper we deal with the case $k = 5$. That is we consider the equation

$$B_m = x(x+1)(x+2)(x+3)(x+4).$$

We prove the following theorem.

Theorem 1. *The Diophantine equation*

$$B_m = x(x+1)(x+2)(x+3)(x+4) \quad m \geq 0, x \in \mathbb{Z}$$

has no solution.

2. AUXILIARY RESULTS

Consider the hyperelliptic curve

$$(1) \quad \mathcal{C}: \quad y^2 = F(x) := x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0,$$

where $b_i \in \mathbb{Z}$. Let α be a root of F and $J(\mathbb{Q})$ be the Jacobian of the curve \mathcal{C} . We have that

$$x - \alpha = \kappa \xi^2$$

where $\kappa, \xi \in K = \mathbb{Q}(\alpha)$ and κ comes from a finite set. By knowing the Mordell-Weil group of the curve \mathcal{C} it is possible to provide a method to compute such a finite set. To each coset representative $\sum_{i=1}^m (P_i - \infty)$ of $J(\mathbb{Q})/2J(\mathbb{Q})$ we associate

$$\kappa = \prod_{i=1}^m (\gamma_i - \alpha d_i^2),$$

where the set $\{P_1, \dots, P_m\}$ is stable under the action of Galois, all $y(P_i)$ are non-zero and $x(P_i) = \gamma_i/d_i^2$ where γ_i is an algebraic integer and $d_i \in \mathbb{Z}_{\geq 1}$. If P_i, P_j are conjugate then we may suppose that $d_i = d_j$ and so γ_i, γ_j are conjugate. We have the following lemma (Lemma 3.1 in [11]).

Lemma 1. *Let \mathcal{K} be a set of κ values associated as above to a complete set of coset representatives of $J(\mathbb{Q})/2J(\mathbb{Q})$. Then \mathcal{K} is a finite subset of \mathcal{O}_K and if (x, y) is an integral point on the curve (1) then $x - \alpha = \kappa \xi^2$ for some $\kappa \in \mathcal{K}$ and $\xi \in K$.*

As an application of his theory of lower bounds for linear forms in logarithms, Baker [1] gave an explicit upper bound for the size of integral solutions of hyperelliptic curves. This result has been improved by many authors (see e.g. [5], [6], [8], [12], [27], [29], [30] and [33]).

In [11] an improved completely explicit upper bound were proved combining ideas from [12], [13], [14], [20], [24], [26], [33], [32]. Now we will state the theorem which gives the improved bound. We introduce some notation. Let K be a number

field of degree d and let r be its unit rank and R its regulator. For $\alpha \in K$ we denote by $h(\alpha)$ the logarithmic height of the element α . Let

$$\partial_K = \begin{cases} \frac{\log 2}{d} & \text{if } d = 1, 2, \\ \frac{1}{4} \left(\frac{\log \log d}{\log d} \right)^3 & \text{if } d \geq 3 \end{cases}$$

and

$$\partial'_K = \left(1 + \frac{\pi^2}{\partial_K^2} \right)^{1/2}.$$

Define the constants

$$\begin{aligned} c_1(K) &= \frac{(r!)^2}{2^{r-1}d^r}, & c_2(K) &= c_1(K) \left(\frac{d}{\partial_K} \right)^{r-1}, \\ c_3(K) &= c_1(K) \frac{d^r}{\partial_K}, & c_4(K) &= rdc_3(K), \\ c_5(K) &= \frac{r^{r+1}}{2\partial_K^{r-1}}. \end{aligned}$$

Let

$$\partial_{L/K} = \max \left\{ [L : \mathbb{Q}], [K : \mathbb{Q}] \partial'_K, \frac{0.16[K : \mathbb{Q}]}{\partial_K} \right\},$$

where $K \subseteq L$ are number fields. Define

$$C(K, n) := 3 \cdot 30^{n+4} \cdot (n+1)^{5.5} d^2 (1 + \log d).$$

The following theorem will be used to get an upper bound for the size of the integral solutions of our equation. It is Theorem 3 in [11].

Theorem 2. *Let α be an algebraic integer of degree at least 3 and κ be an integer belonging to K . Denote by $\alpha_1, \alpha_2, \alpha_3$ distinct conjugates of α and by $\kappa_1, \kappa_2, \kappa_3$ the corresponding conjugates of κ . Let*

$$K_1 = \mathbb{Q}(\alpha_1, \alpha_2, \sqrt{\kappa_1 \kappa_2}), \quad K_2 = \mathbb{Q}(\alpha_1, \alpha_3, \sqrt{\kappa_1 \kappa_3}), \quad K_3 = \mathbb{Q}(\alpha_2, \alpha_3, \sqrt{\kappa_2 \kappa_3}),$$

and

$$L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \sqrt{\kappa_1 \kappa_2}, \sqrt{\kappa_1 \kappa_3}).$$

In what follows R stands for an upper bound for the regulators of K_1, K_2 and K_3 and r denotes the maximum of the unit ranks of K_1, K_2, K_3 . Let

$$c_j^* = \max_{1 \leq i \leq 3} c_j(K_i)$$

and

$$N = \max_{1 \leq i, j \leq 3} |\text{Norm}_{\mathbb{Q}(\alpha_i, \alpha_j)/\mathbb{Q}}(\kappa_i(\alpha_i - \alpha_j))|^2$$

and

$$H^* = c_5^* R + \frac{\log N}{\min_{1 \leq i \leq 3} [K_i : \mathbb{Q}]} + h(\kappa).$$

Define

$$A_1^* = 2H^* \cdot C(L, 2r+1) \cdot (c_1^*)^2 \partial_{L/L} \cdot \left(\max_{1 \leq i \leq 3} \partial_{L/K_i} \right)^{2r} \cdot R^2,$$

and

$$A_2^* = 2H^* + A_1^* + A_1^* \log\{(2r+1) \cdot \max\{c_4^*, 1\}\}.$$

If $x \in \mathbb{Z} \setminus \{0\}$ satisfies $x - \alpha = \kappa \xi^2$ for some $\xi \in K$ then

$$\log|x| \leq 8A_1^* \log(4A_1^*) + 8A_2^* + H^* + 20 \log 2 + 13 h(\kappa) + 19 h(\alpha).$$

To obtain a lower bound for the possible unknown integer solutions we are going to use the so-called Mordell-Weil sieve. The Mordell-Weil sieve has been successfully applied to prove the non-existence of rational points on curves (see e.g. [9], [10], [18] and [28]).

Let C/\mathbb{Q} be a smooth projective curve (in our case a hyperelliptic curve) of genus $g \geq 2$. Let J be its Jacobian. We assume the knowledge of some rational points on C , so let D be a fixed rational point on C and let j be the corresponding Abel-Jacobi map:

$$j : C \rightarrow J, \quad P \mapsto [P - D].$$

Let W be the image in J of the known rational points on C and D_1, \dots, D_r generators for the free part of $J(\mathbb{Q})$. By using the Mordell-Weil sieve we are going to obtain a very large and smooth integer B such that

$$j(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q}).$$

Let

$$\phi : \mathbb{Z}^r \rightarrow J(\mathbb{Q}), \quad \phi(a_1, \dots, a_r) = \sum a_i D_i,$$

so that the image of ϕ is the free part of $J(\mathbb{Q})$. The variant of the Mordell-Weil sieve explained in [11] provides a method to obtain a very long decreasing sequence of lattices in \mathbb{Z}^r

$$B\mathbb{Z}^r = L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \dots \supsetneq L_k$$

such that

$$j(C(\mathbb{Q})) \subset W + \phi(L_j)$$

for $j = 1, \dots, k$.

The next lemma [11, Lemma 12.1] gives a lower bound for the size of rational points whose image are not in the set W .

Lemma 2. *Let W be a finite subset of $J(\mathbb{Q})$ and L be a sublattice of \mathbb{Z}^r . Suppose that $j(C(\mathbb{Q})) \subset W + \phi(L)$. Let μ_1 be a lower bound for $h - \hat{h}$ and*

$$\mu_2 = \max \left\{ \sqrt{\hat{h}(w)} : w \in W \right\}.$$

Denote by M the height-pairing matrix for the Mordell-Weil basis D_1, \dots, D_r and let $\lambda_1, \dots, \lambda_r$ be its eigenvalues. Let

$$\mu_3 = \min \left\{ \sqrt{\lambda_j} : j = 1, \dots, r \right\}$$

and $m(L)$ the Euclidean norm of the shortest non-zero vector of L . Then, for any $P \in C(\mathbb{Q})$, either $j(P) \in W$ or

$$h(j(P)) \geq (\mu_3 m(L) - \mu_2)^2 + \mu_1.$$

3. PROOF OF THEOREM 1

It was shown by Liptai that the integers B_m satisfy the following equation

$$z^2 - 8y^2 = 1$$

for some integer z . So one has to determine all solution of the equation

$$z^2 = 8(x(x+1)(x+2)(x+3)(x+4))^2 + 1.$$

Rewrite the latter equation as follows

$$z^2 = 8(x^2 + 4x)^2(x^2 + 4x + 3)^2(x^2 + 4x + 4) + 1.$$

Let $X = 2x^2 + 8x$. We obtain that

$$(2) \quad \mathcal{C} : Y^2 = X^2(X+6)^2(X+8) + 4,$$

where $Y = 2z$. It remains to find all integral points on \mathcal{C} . The rank of the Jacobian of \mathcal{C} is 3, so classical Chabauty's method [15],[16],[17] cannot be applied. In this paper we combine Baker's method and the so-called Mordell-Weil sieve to obtain all integral solutions of equation (2).

Lemma 3. *The only integral solutions to the equation (2) are*

$$(0, \pm 2), (-6, \pm 2), (-8, \pm 2).$$

Proof. Let $J(\mathbb{Q})$ be the Jacobian of the genus two curve (2). Using MAGMA [7] we determine a Mordell-Weil basis which is given by

$$\begin{aligned} D_1 &= (0, 2) - \infty, \\ D_2 &= (-6, 2) - \infty, \\ D_3 &= (\omega, -\omega - 10) + (\bar{\omega}, -\bar{\omega} - 10) - 2\infty, \end{aligned}$$

where ω is a root of the polynomial $x^2 + 7x + 4$. Let $f = x^2(x+6)^2(x+8) + 4$ and α be a root of f . We will choose for coset representatives of $J(\mathbb{Q})/2J(\mathbb{Q})$ the linear combinations $\sum_{i=1}^3 n_i D_i$, where $n_i \in \{0, 1\}$. We have

$$x - \alpha = \kappa \xi^2,$$

where κ belongs to a finite set (having 8 elements). This set can be constructed as described in Lemma 1. Many values can be eliminated by local computations in \mathbb{Q}_p , for some small primes p . We apply Theorem 2 to get a large upper bound for $\log |x|$. A MAGMA code were written to obtain the bounds appeared in [11], it can be found at <http://www.warwick.ac.uk/~maseap/progs/intpoint/bounds.m>. We used the above MAGMA functions to compute an upper bound for $\log |x|$, the results are summarized in the following table

κ	bound for $\log x $
1	$1.51 \cdot 10^{246}$
$-\alpha$	$3.21 \cdot 10^{512}$
$-6 - \alpha$	$6.25 \cdot 10^{531}$

The set of known rational points on the curve (2) is $\{\infty, (0, \pm 2), (-6, \pm 2), (-8, \pm 2)\}$. Let W be the image of this set in $J(\mathbb{Q})$. Applying the Mordell-Weil sieve implemented by Bruin and Stoll we obtain that

$$j(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q}),$$

where

$$B = 2^6 \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19^2 \cdot 29 \cdot 31 \cdot 41 \cdot 43 \cdot 47 \cdot 61 \cdot 67 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 107 \cdot 109 \cdot 113,$$

that is

$$B = 46247720065121846143591520774334300410472000.$$

Now we use an extension of the Mordell-Weil sieve due to Samir Siksek to obtain a very long decreasing sequence of lattices in \mathbb{Z}^3 . After that we apply Lemma 2 to obtain a lower bound for possible unknown rational points. We get that if (x, y) is an unknown integral point, then

$$\log |x| \geq 1.03 \times 10^{580}.$$

This contradicts the bound for $\log |x|$ we obtained by Baker's method. \square

Now we prove Theorem 1.

Proof of Theorem 1. By Lemma 3 we have that

$$X \in \{0, -6, -8\}$$

and we also have that $X = 2x^2 + 8x$. That is it remains to solve three quadratic equations in x . We obtain that $x \in \{-4, -3, -2, -1, 0\}$. Therefore the equation

$$B_m = x(x+1)(x+2)(x+3)(x+4)$$

has no solution with $m \geq 0$ and $x \in \mathbb{Z}$. \square

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