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Preserver problems on structures of positive operators

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Preserver problems on structures of positive operators

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Contents

1

Introduction

This PhD dissertation is devoted to the investigation of transformations on sets of positive operators with given invariance properties. The mentioned investigation belongs to the research field of preserver problems. In a general sense, by such a problem we mean the question of characterizing all mappings of a given structure X which preserve

- a quantity attached to the elements of X, or
- a prescribed set of elements of X, or
- a given relation between the elements of X,

or other similar objects. Preserver problems appear in many parts of mathematics and their investigation forms an extensive research field. We mention two important examples of such problems. The first one is the description of the homomorphisms of a certain algebraic structure. In fact, these transformations preserve the operations defined on the structure in question. The other example is the description of the isometries of a given metric space. Clearly, these maps preserve the distance between the elements of this space. In the majority of preserver problems the structure under consideration is linear and so are the corresponding mappings. Such problems are called linear preserver problems (LPPs) and they were investigated systematically in matrix theory. For surveys of the topic the reader can consult the papers [**15, 16, 37**]. In the last decades the area of LPPs has been investigated also in the infinite dimensional setting, i.e. in the case where the structure under consideration consists of bounded linear operators on Hilbert or Banach spaces. For a corresponding survey we refer to, e.g. [**5**].

To give a particular example of a result on preserver problems we mention the theorem of Frobenius from 1897 which is usually regarded as the first result on LPPs. This statement describes the forms of the determinant preserving linear maps on matrix algebras. Before formulating the theorem we need some notation. Let n be a positive integer and denote by M_n the algebra of $n \times n$ complex matrices. Moreover, for any $A \in M_n$ we write A^{tr} for the transpose of A.

THEOREM 1.1. (Frobenius [**13**])
Let $n \in \mathbb{N}$ and suppose that $\phi\colon M_n \to M_n$ is a linear map which satisfies

$$\det \phi(A) = \det A \quad (A \in M_n). \tag{1.1}$$

Then there exist matrices $M, N \in M_n$ such that $\det MN = 1$ *and ϕ is either of the form*

$$\phi(A) = MAN \quad (A \in M_n),$$

or of the form

$$\phi(A) = MA^{tr}N \quad (A \in M_n).$$

We remark that clearly, any transformation on M_n which is of one of the above forms is linear and satisfies the equation (1.1). Here the point is that all determinant preserving linear maps on M_n must be of one of those particular forms. This phenomenon occurs very often when dealing with preserver problems. Namely, in the majority of these problems it is easy to prove that a transformation of a particular form has the required invariance property and the essential part of them is to determine whether any map with the given property can be written in this form.

In the last few decades the investigation of preserver problems has been extended to researches in which nonlinear underlying structures are involved. In the mathematical formulation of quantum mechanics due to John von Neumann several nonlinear structures of Hilbert space operators appear. These objects are usually called quantum structures. According to the formulation mentioned previously, to each quantum system there corresponds a complex Hilbert space H. The main quantum mechanical objects in the given system are represented by particular classes of bounded linear operators acting on H. Below, we give three important examples of such objects together with their representing operators.

(i) The pure states of the system are represented by rank-one projections on H. The set of these operators will be denoted by $P_1(H)$.
(ii) The mixed states of the system are represented by positive trace-class operators on H with unit trace (see Section 1.1.). These operators are called density operators. The symbol $S(H)$ stands for the collection of the density operators acting on H.
(iii) The observable physical quantities (observables for short) of the system are represented by self-adjoint operators on H. The set of these operators will be signified by $B_s(H)$.

In this work, there are several results about preserver problems on the set of density operators as underlying structure. Concerning the investigation of preserver problems on quantum structures such as $P_1(H)$ or $S(H)$, one of the most fundamental tools is the application of Wigner's celebrated theorem on quantum mechanical symmetry transformations. Before presenting this result we need some notion. For any pair of pure states which are represented by the rank-one projections P, respectively Q on H, the transition probability between them (or between P and Q) is defined as $\operatorname{tr} PQ$, where tr denotes the trace functional (c.f. Section 1.1.). This is a fundamental quantity in quantum information theory which determines the probabilistic structure of a quantum system. Quantum mechanical symmetry transformations play an important role in the mathematical foundations of quantum mechanics. We call a bijection $\phi\colon P_1(H) \to P_1(H)$ a quantum mechanical symmetry transformation if and only if it preserves the transition probability, i.e. ϕ has the property that the equality

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \tag{1.2}$$

is satisfied by all pairs of rank-one projections $P, Q \in P_1(H)$. In addition to the above concept of transition probability and symmetry transformations, we recall the notion of antiunitary operators which are the surjective conjugate-linear isometries on H. Now we are in a position to formulate the fundamental theorem of Wigner which is about a preserver problem on $P_1(H)$. Namely, it describes the structure of quantum mechanical symmetry transformations. We will use this statement in the following form.

Theorem 1.2. (Wigner)
Let H be a complex Hilbert space and assume that ϕ is a map from $P_1(H)$ onto itself. Then ϕ is a quantum mechanical symmetry transformation if and only if there exists either a unitary or an antiunitary operator U on H such that ϕ is of the form

$$\phi(P) = UPU^* \quad (P \in P_1(H)).$$

This result can be found, e.g. in the book [**22**] (see p. 12 in the Introduction). It tells us that every quantum mechanical symmetry transformation is a conjugation by a fixed unitary or antiunitary operator on H. In other terminology, we say that these maps are induced (or, in other words implemented) by such operators. It is a remarkable fact that in many preserver problems concerning quantum structures consisting of bounded linear operators on H, the maps which have the preserver property given in the considered problem are exactly those induced by unitary-antiunitary operators on H. Therefore the theorems describing the structure of these transformations can be regarded as Wigner-type results. This thesis contains several Wigner-type results.

There is a non-surjective version of the above theorem of Wigner which describes the form of the maps of $P_1(H)$ preserving the transition probability. It reads as follows (see [**22**, Theorem 2.1.4] or [**2, 39**]).

THEOREM 1.3. *Let H be a complex Hilbert space and suppose that ϕ is a selfmap of $P_1(H)$ which satisfies the equality* (1.2) *for all pairs of rank-one projections $P, Q \in P_1(H)$. Then we have either a linear or a conjugate-linear isometry V of H such that*

$$\phi(P) = VPV^* \quad (P \in P_1(H)).$$

We remark that in the finite dimensional case (conjugate-)linear isometries are (anti)unitary operators. Therefore, if $\dim H < \infty$, then the mapping ϕ in Theorem 1.3 is implemented by a unitary-antiunitary operator.

The above two theorems are well-known results about preserver problems on $P_1(H)$. As for the set of density operators on H, observe that it is a convex subset of the real linear space $B_s(H)$. Therefore, it is a natural problem to determine the affine bijections of this set. These are exactly the bijective maps of $S(H)$ which respect the convex combinations. In quantum theory, a convex combination of mixed states of H is usually called a mixture. Hence the affine bijections of $S(H)$ are exactly those bijective transformations on the mixed states of H which preserve mixtures. The mixture preserving bijections of $S(H)$ are called mixture automorphisms (or affine automorphisms or S-automorphisms in the terminology of [**9**] or Kadison automorphisms in the terminology of [**40**]). The below statement describes the structure of these transformations (c.f., e.g. [**9, 40**]).

THEOREM 1.4. *Let H be a complex Hilbert space. Then $\phi\colon S(H) \to S(H)$ is a mixture automorphism if and only if it is of the form*

$$\phi(A) = UAU^* \quad (A \in S(H)),$$

with a unitary or an antiunitary operator U on H.

Some words about the content of the dissertation which is mainly based on the papers [**26, 27, 33, 34**]. Chapter 2 is devoted to the description of some bijections on the set of density operators with preserver properties related to commutativity. In the third chapter relative entropy preserving maps on the collection of density operators or of invertible density operators are investigated. Chapter 4 contains results concerning the structure of the isometries of certain spaces of positive operators in the so-called von

Neumann - Schatten classes with respect to metrics which are closely related to the p-norms (see Section 1.1.). Moreover, we present identification lemmas which form the key part of the proof of some of these results. In the fifth chapter, we determine the general form of the surjective isometries of the space of invertible positive operators acting on a 2-dimensional complex Hilbert space endowed with the Thompson metric or the Hilbert projective metric.

1.1. Preliminaries

In this section we collect those notions, basic facts and notation which will be used throughout the dissertation. The definitions and notation which appear only in a particular chapter will be introduced in that chapter.

Throughout the thesis, H denotes a complex Hilbert space with inner product $\langle ., . \rangle$.

In this work, several structures of Hilbert space operators appear. As usual, the complex unital C^*-algebra of all bounded linear operators acting on H is denoted by $B(H)$. The space of the self-adjoint elements of $B(H)$ is signified by $B_s(H)$. We recall that by definition, an operator $A \in B(H)$ is positive if and only if $\langle Ax, x \rangle$ is a nonnegative real number for any $x \in H$. We remark that any positive operator is self-adjoint. The symbol $B(H)^+_{-1}$ stands for the set of all invertible positive operators on H. A self-adjoint element $P \in B(H)$ with the property that $P^2 = P$ is called a projection. The symbol $P_1(H)$ signifies the set of all rank-one projections on H. For any $x, y \in H$, the operator $x \otimes y$ is defined by

$$(x \otimes y)z = \langle z, y \rangle x \quad (z \in H).$$

It is obvious that an operator $P \in B(H)$ is a rank-one projection if and only if it is of the form $P = x \otimes x$ with some unit vector $x \in H$. By an antiunitary operator we mean a surjective conjugate-linear isometry on H.

In what follows we collect some basic facts concerning two important relations on $B_s(H)$. The first one is the usual partial order $\leq$ which is defined by $A \leq B$ if and only if $B - A$ is a positive operator ($A, B \in B_s(H)$). It is well-known that for any projections P and Q on H one has $P \leq Q$ exactly when the range of Q contains that of P. The second relation is orthogonality. The self-adjoint operators A and B on H are said to be orthogonal if and only if $AB = 0$ which is equivalent to the fact that A and B have mutually orthogonal ranges. By the preservation of orthogonality we mean that it is preserved in both directions. Namely, the following convention will be used. If ϕ is a map between subsets of $B_s(H)$, then we say that ϕ preserves orthogonality if and only if for any pair A, B of elements in the domain of ϕ we have $\phi(A)\phi(B) = 0$ exactly when $AB = 0$.

Now we proceed with the notion of von Neumann - Schatten classes and some related concepts. If $A \in B(H)$, then $|A| = \sqrt{A^*A}$ stands for its absolute value. Let $p \geq 1$ be a number. We denote by $C_p(H)$ the set of those elements of $B(H)$ which have the property that for every orthonormal basis $(e_i)_{i\in I}$ of H the series $\sum_{i\in I}\langle |A|^p e_i, e_i\rangle$ is convergent. This set is usually called the von Neumann - Schatten p-class of compact operators (observe that its elements are necessarily compact operators). For details we refer to [**19, 38**]. We remark that the members of $C_1(H)$ are called trace-class operators and that in the case where $\dim H < \infty$ one has $C_p(H) = B(H)$. Before defining the usual norm on $C_p(H)$, we recall the notion of the trace functional. If $A \in C_1(H)$, then

$$\operatorname{tr} A = \sum_{i\in I}\langle Ae_i, e_i\rangle,$$

where $(e_i)_{i\in I}$ is an orthonormal basis in H. We remark that the trace $\operatorname{tr} A$ of A is well-defined, since the latter sum is convergent and its value does not depend on the special choice of the orthonormal basis $(e_i)_{i\in I}$.

For each $A \in C_p(H)$ one defines the p-norm of A by the formula

$$\|A\|_p \doteq (\operatorname{tr}|A|^p)^{\frac{1}{p}} .$$

It is a straightforward consequence of the definition of p-classes that for any $T \in C_p(H)$ we have $|T|^p \in C_1(H)$ and this shows that the p-norm is well-defined. It is well-known that the function $\|.\|_p \colon C_p(H) \to \mathbb{R}$ is a complete norm on the linear space $C_p(H)$ which is actually an ideal of $B(H)$. As for the case $p = \infty$, denote by $C_\infty(H)$ the whole algebra $B(H)$ and let $\|.\|_\infty$ signify the operator norm on $C_\infty(H)$. For any $1 \leq q \leq \infty$ the symbol $C_q(H)^+$ stands for the set of the positive operators in $C_q(H)$. We call an element of $C_1(H)^+$ a density operator if it has unit trace. The set of the density operators acting on H is denoted by $S(H)$. We remark that if $A \in C_p(H)$ is normal, then

$$\|A\|_p = \left(\sum_{i\in I}|\lambda_i|^p\right)^{\frac{1}{p}}, \tag{1.3}$$

where $(\lambda_i)_{i\in I}$ is a sequence consisting of the nonzero eigenvalues of A (counted according to multiplicities).

For any bounded conjugate-linear operator A on H the adjoint $A^*\colon H \to H$ of A is defined as the unique continuous conjugate-linear operator satisfying
$$\langle Ax, y\rangle = \langle A^*y, x\rangle \quad (x, y \in H).$$
We close this section with some miscellaneous notation. If $A \in B(H)$ is an operator, then $\sigma_p(A)$ stands for its point-spectrum and $\operatorname{rng} A$ signifies its range. As usual, the identity operator on H is denoted by I and for any projection P on H we write $P^\perp = I - P$, furthermore $^\perp$ stands also for the orthogonal complement of subspaces in H. The relation of orthogonality between subspaces of H is signified by $\perp$. We denote by $\overline{L}$ the closure of the set $L \subset H$. If V is a linear space and $M \subset V$, then we define $-M = \{-m \mid m \in M\}$. The set of all nonnegative real numbers is signified by $\mathbb{R}_+$.

2

Commutativity preserving maps on density operators

2.1. Introduction and statement of the results

In this chapter we describe those transformations on the space of density operators which preserve commutativity, or a certain measure of it, or commutativity and the fidelity between commuting operators. The obtained results expose how close those maps are to the transformations implemented by unitary or antinuitary operators. The majority of the theorems in this chapter appeared in the paper [**33**].

Commutativity is a very important concept in mathematics, but it is as much important in physics as well. For example, concerning a pair of observables, the commutativity of the corresponding self-adjoint operators means that the observables in question can be measured in every state of the underlying quantum system jointly. In quantum mechanics one usually uses the expression "compatibility" for that case. Commutativity plays an important role in relation with quantum states, too. For example, according to a celebrated result of Barnum, Caves, Fuchs, Jozsa and Schumacher [**3**], a set $\{A, B\}$ of mixed states can be broadcast exactly when A, B as operators are commuting. This statement shows that the commutativity of density operators is closely related to a quantum information theoretical property of quantum states.

Transformations on quantum structures that preserve some relevant physical properties or quantities can be viewed as representatives of certain symmetries of the underlying system. Therefore, to explore them or to determine their structure is a senseful problem that might provide useful information about the quantum structure itself. For a collection of results relating that kind of problems, we refer to the second chapter of the book [**22**].

In the paper [**28**] Molnár and Šemrl described all bijective transformations on $B_s(H)$ which preserve commutativity. Namely, they proved the following result.

Theorem 2.1. (Molnár, Šemrl [**28**])
Assume that H is separable with $\dim H \geq 3$. *Furthermore, suppose that* $\phi\colon B_s(H) \to B_s(H)$ *is a bijection such that for any* $A, B \in B_s(H)$ *we have*

$$AB = BA \iff \phi(A)\phi(B) = \phi(B)\phi(A). \tag{2.1}$$

Then there exists either a unitary or an antiunitary operator U on H and for any $A \in B_s(H)$ we have a real valued bounded Borel function f_A on the spectrum of A such that

$$\phi(A) = U f_A(A) U^*.$$

The main aim of the present chapter is to carry out a similar work concerning transformations on the space of quantum states.

We begin with some conventions that we shall use throughout this chapter. Let $A \in B(H)$ be a positive compact operator. For any $\lambda \in \sigma_p(A)$ we denote by P_λ the projection onto the eigensubspace of A corresponding to λ. By the spectral theorem for compact self-adjoint operators we have that

$$A = \sum_{\lambda \in \sigma_p(A)} \lambda P_\lambda.$$

In the present chapter we call this representation the spectral decomposition of A.

Now we can formulate the first result of this section which describes the bijective maps of $S(H)$ that preserve commutativity. This is a generalization of the assertion [**33**, Theorem 1] of the author, since it does not contain the separability condition which is assumed in the cited statement.

Theorem 2.2. (Nagy)
Let $\dim H \geq 3$. *Moreover, suppose that* $\phi\colon S(H) \to S(H)$ *is a bijective transformation which has the property that* (2.1) *holds for any* $A, B \in S(H)$. *Then there exists either a unitary or an antiunitary operator U on H, and for every $A \in S(H)$ with spectral decomposition*

$$A = \sum_{\lambda \in \sigma_p(A)} \lambda P_\lambda$$

there exists an injective function $f_A\colon \sigma_p(A) \to [0,1]$ *such that*

$$\phi(A) = U\left(\sum_{\lambda\in\sigma_p(A)} f_A(\lambda)P_\lambda\right)U^*. \tag{2.2}$$

We remark that the functions f_A above are not just arbitrary injective functions. Indeed, they are constrained by the fact that $\phi(A) \in S(H)$ must hold which yields that $\sum_{\lambda\in\sigma_p(A)} f_A(\lambda)\,\mathrm{rank}\,P_\lambda = 1$ (in this chapter rank denotes the rank of operators).

We point out that the operator

$$\sum_{\lambda\in\sigma_p(A)} f_A(\lambda)P_\lambda$$

above can also be considered as a bounded Borel function of A in the sense of functional calculus of self-adjoint operators. Hence, it follows from Theorem 2.2 that any transformation ϕ under consideration is, up to a unitary or antiunitary equivalence, equal to a map which sends every element A of $S(H)$ to some injective bounded Borel function of A. Observe that any transformation on $S(H)$ with the latter property clearly preserves commutativity. The content of our result is that the reverse statement is also true: every commutativity preserving bijection of $S(H)$ is necessarily of that particular form.

We recalled one of the results appearing in the paper [**28**] of Molnár and Šemrl. Indeed, in the proof of our present result we shall use some basic ideas of their proof, but due to the fact that the space $S(H)$ is much smaller than $B_s(H)$ in several respects, here we encounter many technical problems that did not appear in [**28**].

In Theorem 2.2 above we see that there is considerable freedom in the structure of commutativity preserving maps on $S(H)$ (we mean the freedom in choosing the functions f_A). In our next two results we pose a bit more assumptions on the transformations under consideration. Beyond supposing that they preserve commutativity, we also assume that they leave a certain measure of commutativity invariant. As we shall see, in that case those maps are necessarily of a more simple and usual form.

Now, a natural candidate to represent the measure of commutativity between density operators is the norm of their commutator. Here we consider unitarily invariant norms on $C_1(H)$, i.e. norms $\|.\|$ satisfying $\|UAV\| = \|A\|$ $(A \in C_1(H))$ for any unitary operators U and V on H. In the rest of this chapter $\|.\|\colon C_1(H) \to \mathbb{R}$ will denote a fixed unitarily invariant norm. Let $A, B \in C_1(H)$. We remark that in this case the commutator $AB - BA$ of A

and B belongs to $C_1(H)$ and hence the quantity $\|AB - BA\|$ is well-defined. In our following two results we determine those bijective transformations ϕ of $S(H)$ which preserve this measure of commutativity, i.e., which have the property that

$$\|\phi(A)\phi(B) - \phi(B)\phi(A)\| = \|AB - BA\| \tag{2.3}$$

holds for all $A, B \in S(H)$.

Before presenting the results, we give an example of a transformation on $S(H)$ which preserves the above measure of commutativity. Let ϕ be a selfmap of $S(H)$ with the following property: there exists either a unitary or an antiunitary operator U on H and a real number α such that for every $A \in S(H)$ we have either

$$\phi(A) = UAU^*$$

or

$$\phi(A) = \alpha I - UAU^*.$$

The second possibility may appear only in the case where H is finite dimensional and then $\alpha = 2/\dim H$. We see that the previous map is the composition of a transformation of the form $A \mapsto UAU^*$ ($A \in S(H)$ and a mapping which sends an operator $A \in S(H)$ to an element of the form "$\pm A$ + scalar". It is clear that the latter map preserves our measure of commutativity and in the proof of Theorem 2.3 and Theorem 2.4 below we will show that the same holds for the former. Therefore the transformation in this example satisfies (2.3) for each $A, B \in S(H)$. In the following two results we show that the converse is also true: every bijection $\phi\colon S(H) \to S(H)$ with the property that (2.3) holds for all $A, B \in S(H)$ is necessarily of the above form. Those special cases of these statements where $\|.\|$ is a p-norm ($1 \le p \le \infty$) have been appeared in the author's paper [**33**] as Theorems 2 and 3.

Theorem 2.3. (Nagy)
Assume that H is infinite dimensional and $\phi\colon S(H) \to S(H)$ is a bijective transformation such that (2.3) *holds for every $A, B \in S(H)$. Then there exists either a unitary or an antiunitary operator U on H such that ϕ is of the form*

$$\phi(A) = UAU^* \quad (A \in S(H)).$$

If the underlying Hilbert space is finite dimensional, then we have the following result.

Theorem 2.4. (Nagy)
Suppose that $3 \leq \dim H < \infty$ *and* $\phi\colon S(H) \to S(H)$ *is a bijective transformation such that* (2.3) *holds for every* $A, B \in S(H)$*. Then there exists either a unitary or an antiunitary operator* U *on* H *such that for every* $A \in S(H)$ *we have either*

$$\phi(A) = UAU^*$$

or

$$\phi(A) = \frac{2}{\dim H} I - UAU^*.$$

We emphasize that in Theorem 2.4 the form of $\phi(A)$ may really vary as A varies.

In the final result of this chapter we consider bijective transformations on $S(H)$ which preserve commutativity and an important numerical quantity, the fidelity between commuting elements. We show that the corresponding transformations are necessarily of the usual most simple form, namely they are implemented by unitary or antiunitary operators. Similar conclusion was obtained in [**20**] (also see [**22**], Section 2.3) for maps which preserve the fidelity of all pairs of density operators.

The notion of fidelity due to Jozsa and Uhlmann (see e.g., [**44**]) is a fundamental concept in quantum information theory. It is defined in the following way. For any $A, B \in S(H)$, the fidelity $F(A, B)$ between them is

$$F(A, B) = \operatorname{tr} \sqrt{\sqrt{A} B \sqrt{A}}. \tag{2.4}$$

We remark that for every $A, B \in S(H)$ we have $0 \leq F(A, B) \leq 1$ and $F(A, B) = F(B, A) = \left\| \sqrt{B} \sqrt{A} \right\|_1$.

Clearly, any transformation of the form $A \mapsto UAU^*$ ($A \in S(H)$) with a unitary or antiunitary operator U preserves commutativity and also preserves the fidelity between commuting density operators (in fact, it preserves this quantity globally). In the last result of Chapter 2 we show that the reverse statement is also true. Namely, we have the following assertion.

Theorem 2.5. (Nagy [**33**])
Suppose that $\dim H \geq 3$*. Moreover, assume that* $\phi\colon S(H) \to S(H)$ *is a bijective transformation which preserves commutativity and also satisfies*

$$F(\phi(A), \phi(B)) = F(A, B)$$

for every pair $A, B \in S(H)$ *with* $AB = BA$*. Then we have either a unitary or an antiunitary operator* U *on* H *such that* ϕ *is of the form*

$$\phi(A) = UAU^* \quad (A \in S(H)).$$

2.2. Proofs

In what follows, we present the proof of the second result in this chapter.

Proof of Theorem 2.2. In this argument we shall use the following notation and notions. For any subset $\mathcal{M} \subset B(H)$ we define

$$\mathcal{M}^K = \mathcal{M}' \cap C_1(H)^+.$$

Here $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $B(H)$, i.e., the set of all elements of $B(H)$ which commute with every element of $\mathcal{M}$. Apparently, $\mathcal{M}^K$ is the set of those operators in $C_1(H)^+$ which commute with any element of $\mathcal{M}$. For simplicity, we use the notation A^K instead of $\{A\}^K$ when we consider a single positive trace-class operator A on H.

We denote by $P(H)$ the set of all projections on H. The collection of the finite rank elements of $P(H)$ is denoted by $P_f(H)$. The relation of orthogonality between projections on H is signified by $\perp$. The symbols $\vee$, respectively $\wedge$ denote the usual operations of supremum, respectively infimum in the lattice $P(H)$. The map $P \mapsto P^\perp$ $(P \in P(H))$ is called orthocomplementation. By a spectral projection of an element A of $B(H)$ in this chapter we mean a projection whose range is an eigensubspace of A. The cardinality of any set K is denoted by card K.

We introduce the following notation. Let A be a positive compact operator. If $f\colon \sigma_p(A) \to \mathbb{C}$ is a bounded function (observe that in this case such a function is necessarily Borel measurable) and the spectral decomposition of A is

$$A = \sum_{\lambda \in \sigma_p(A)} \lambda P_\lambda$$

then define

$$f(A) \doteq \sum_{\lambda \in \sigma_p(A)} f(\lambda) P_\lambda.$$

As already mentioned in Section 2.1, $f(A)$ coincides with a bounded Borel function of A in the sense of functional calculus of self-adjoint operators. By [**10**, 7.12. Theorem of Chapter II], we have

$$A'' = \{f(A) \mid f\colon \sigma_p(A) \to \mathbb{C} \text{ is a bounded function}\}. \tag{2.5}$$

In the course of the proof, we will define a commutativity preserving bijection of $C_1(H)^+$ which extends ϕ. Before doing so, we present some assertions concerning commutants in $C_1(H)^+$ which are of technical importance.

Lemma 2.6. *If $T \in B(H)$ and A is a positive compact operator such that $TP = PT$ for every $P \in A' \cap P_f(H)$, then $TP = PT$ for every $P \in A' \cap P(H)$.*

Proof. To prove this, let

$$A = \sum_{\lambda \in \sigma_p(A)} \lambda P_\lambda$$

be the spectral decomposition of A. It is well-known that a normal operator $N\colon H \to H$ commutes with an operator R if and only if R commutes with every projection in the range of the spectral measure corresponding to N. Consequently $A' = \{P_\lambda |\ \lambda \in \sigma_p(A)\}'$. So to prove the statement of Lemma 2.6, we have to show that if $TP = PT$ for every $P \in \{P_\lambda |\ \lambda \in \sigma_p(A)\}' \cap P_f(H)$, then $TP = PT$ for each $P \in \{P_\lambda |\ \lambda \in \sigma_p(A)\}' \cap P(H)$.

To do this, we verify that for any $Q \in \{P_\lambda |\ \lambda \in \sigma_p(A)\}' \cap P(H)$ there exists a net in $\{P_\lambda |\ \lambda \in \sigma_p(A)\}' \cap P_f(H)$ which converges to Q strongly. So let Q be as above. It is clear that

$$Q\left(\sum_{\lambda \in \sigma_p(A) \setminus \{0\}} P_\lambda\right)^{\perp} \in P(H),$$

so the range of the last displayed operator is a Hilbert space. Pick an arbitrary orthonormal basis of this space, say $(e_i)_{i \in I}$. Let $\mathcal{F} = \{L \subset I \mid \operatorname{card} L < \infty\}$ and $F \in \mathcal{F}$. Define

$$R_F = \sum_{i \in F} e_i \otimes e_i.$$

Denoting by (λ_n) the sequence of the distinct nonzero eigenvalues of A, set $Q_\emptyset = 0$. Moreover, if $F \neq \emptyset$, then let

$$Q_F = Q \sum_{k=1}^{n_F} P_{\lambda_k} + R_F,$$

where $n_F = \min\{\operatorname{card} F, \operatorname{card} \sigma_p(A) \setminus \{0\}\}$.

It is clear that $(Q_F)_{F \in \mathcal{F}}$ is a net in $B(H)$. We are going to show that (Q_F) has the required properties. Apparently, it converges strongly to Q. Now let $F \in \mathcal{F}$ be a nonempty set. It is easy to see that $Q \sum_{k=1}^{n_F} P_{\lambda_k}$ and R_F are mutually orthogonal projections, so $Q_F \in P(H)$. It follows from the properties of the eigensubspaces of compact operators that Q_F is of finite rank. It is clear that for every $n \in \mathbb{N}$ and $\lambda \in \sigma_p(A)$, the operator $Q \sum_{k=1}^{n} P_{\lambda_k}$ commutes with P_λ. If $\lambda \in \sigma_p(A) \setminus \{0\}$, then $P_\lambda \perp R_F$. This implies that for every $\lambda \in \sigma_p(A)$ the projections R_F and P_λ commute. It follows from these observations that $Q_F \in \{P_\lambda |\ \lambda \in \sigma_p(A)\}^{'}$. Therefore, (Q_F) has the required properties.

As $TQ_F = Q_F T$ and (Q_F) converges to Q strongly, we obtain $TQ = QT$. This completes the proof of the lemma. □

Lemma 2.7. *If $A \in C_1(H)^+$, then $A^{KK} = A^{''} \cap C_1(H)^+$.*

Proof. To prove this assertion, we must show that $A^{''} \cap C_1(H)^+ \subset A^{KK}$ and $A^{KK} \subset A^{''} \cap C_1(H)^+$. The first inclusion is obvious. In what follows we show that $A^{KK} \subset A^{''} \cap C_1(H)^+$. Let $B \in A^{KK}$. Then it is trivial that $B \in C_1(H)^+$. We have to show that $B \in A^{''}$ also holds. It is clear that any $P \in A^{'} \cap P_f(H)$ also belongs to A^K, hence $BP = PB$ holds for all such P. By Lemma 2.6, it follows that $BP = PB$ holds for all $P \in A^{'} \cap P(H)$.

Therefore, we know that B commutes with the projections belonging to the von Neumann algebra A'. It is well-known that every von Neumann algebra is the closed linear span of the set of its projections, so we infer that $BT = TB$ for every $T \in A'$. This gives us $B \in A^{''}$ completing the proof of Lemma 2.7. □

Lemma 2.7 has the following useful corollaries.

Corollary 2.8. *If $A, B \in C_1(H)^+$, then $A^{KK} \subsetneq B^{KK}$ holds if and only if there exists a bounded function $f\colon \sigma_p(B) \to \mathbb{R}_+$ which is not injective and satisfies $A = f(B)$.*

Proof. Let $A, B \in C_1(H)^+$ be such that $A^{KK} \subsetneq B^{KK}$. It is clear that $A \in A^{KK}$, so by Lemma 2.7 and the formula (2.5), we get that there exists a bounded function, $f\colon \sigma_p(B) \to \mathbb{R}_+$ such that $A = f(B)$. It is easy to see that if f was injective, then the sets of the spectral projections of A and B would coincide. But this would imply $A^{KK} = B^{KK}$, a contradiction.

To prove the converse, let $f\colon \sigma_p(B) \to \mathbb{R}_+$ be a bounded function such that f is not injective and $f(B) = A$. It is clear that in this case $B^{'} \subset A^{'}$, which implies that $A^{KK} \subset B^{KK}$. To finish the proof we have to show that $A^{KK} \neq B^{KK}$ also holds. For this, let

$$B = \sum_{\lambda \in \sigma_p(B)} \lambda P_\lambda$$

be the spectral decomposition of B. Since f is not injective, there exist numbers $\lambda_1,\ \lambda_2 \in \sigma_p(B)$ such that $\lambda_1 \neq \lambda_2$ but $f(\lambda_1) = f(\lambda_2)$. Consider the projections P_{λ_1} and P_{λ_2}. Since B is compact, at least one of them, say P_{λ_1}, is of finite rank, so $P_{\lambda_1} \in C_1(H)^+$. Therefore, we have $P_{\lambda_1} \in B^{KK}$.

Let $x \in H$ be a nonzero vector and T be a self-adjoint operator. Clearly, T commutes with $x \otimes x$ if and only if x is an eigenvector of T. Pick nonzero vectors e_1 and e_2 in the ranges of P_{λ_1} and P_{λ_2}, respectively. Then it is clear that $e_1 + e_2$ neither belongs to the range of P_{λ_1}, nor orthogonal to this subspace, so it is not an eigenvector of P_{λ_1}. But the vector $e_1 + e_2$ is an eigenvector of A and hence we get that the projection onto the subspace generated by $e_1 + e_2$ is an element of $C_1(H)^+$ which commutes with A but does not commute with P_{λ_1}. This implies that the latter operator does not belong to A^{KK} completing the proof of the asserted equivalence. □

It is worth mentioning an equivalent and more applicable reformulation of the condition for the relation $A^{KK} \subsetneq B^{KK}$ appearing in Corollary 2.8. Namely, the fact that there is a bounded function $f\colon \sigma_p(B) \to \mathbb{R}_+$ which is not injective and satisfies $A = f(B)$ is easily seen to be equivalent to the following assertion. Each eigensubspace of A is generated by the union of some eigensubspaces of B and at least one of these unions has more than one member. When one applies Corollary 2.8, it is useful to have this interpretation in mind. The next consequence of Lemma 2.7 reads as follows.

Corollary 2.9. *If $A, B \in C_1(H)^+$ are such that $A^{KK} = B^{KK}$, then A and B have the same set of spectral projections.*

Proof. Let $\{P_\lambda |\, \lambda \in \sigma_p(A)\}$ and $\{Q_\lambda |\, \lambda \in \sigma_p(B)\}$ be the set of the spectral projections of A and B, respectively. Clearly, it is enough to show that $\{P_\lambda |\, \lambda \in \sigma_p(A) \setminus \{0\}\} = \{Q_\lambda |\, \lambda \in \sigma_p(B) \setminus \{0\}\}$. Suppose that this is not the case. Then at least one of these sets, say $\{P_\lambda |\, \lambda \in \sigma_p(A) \setminus \{0\}\}$, is not contained in the other. Hence there exists a nonzero number $\lambda \in \sigma_p(A)$ such that

$$P_\lambda \notin \{Q_\mu |\, \mu \in \sigma_p(B) \setminus \{0\}\}.$$

Since $A^{KK} \subset B^{KK}$ and $P_\lambda \in A^{KK}$, it is easy to see using Lemma 2.7 and the formula (2.5) that there is a set $L \subset \sigma_p(B)$ such that $P_\lambda = \sum_{\mu \in L} Q_\mu$ and card $L \geq 2$. Similarly, since $B^{KK} \subset A^{KK}$ also holds, for every $\mu \in L$ there exists a set $\emptyset \neq K(\mu) \subset \sigma_p(A)$ such that $Q_\mu = \sum_{\alpha \in K(\mu)} P_\alpha$. So we obtain that P_λ is the sum of at least two spectral projections of A. By the orthogonality of spectral projections this is a contradiction. □

Now we introduce the following notation. Let $A \in C_1(H)^+$ and define

$$\rho(A^{KK}) = \sup\{n \in \mathbb{N} \mid \exists\, A_i \in C_1(H)^+ \ (i = 1, ..., n) :$$

$$A_1^{KK} \subsetneq ... \subsetneq A_n^{KK},\ A_n = A\} \in [1, \infty].$$

This quantity is well-defined since its value does not depend on the special choice of A, more exactly, if $A, B \in C_1(H)^+$ are such that $A^{KK} = B^{KK}$, then $\rho(A^{KK}) = \rho(B^{KK})$. We present the next lemma.

Lemma 2.10. *For any $A \in C_1(H)^+$ we have $\rho(A^{KK}) = \operatorname{card} \sigma_p(A)$.*

Proof. If $A \in C_1(H)^+$ is such that $\operatorname{card} \sigma_p(A) = 1$, then A is a scalar operator, thus it is easy to see that $\rho(A^{KK}) = 1$.

Suppose now that $\operatorname{card} \sigma_p(A) > 1$ and let

$$A = \sum_{\lambda \in \sigma_p(A)} \lambda P_\lambda$$

be the spectral decomposition of A. Denote by (λ_n) the sequence of the nonzero eigenvalues of A. For any $m \in \mathbb{N}$ define

$$H(m) = \{n \in \mathbb{N} \mid m < n \leq \operatorname{card} \sigma_p(A) \setminus \{0\}\}.$$

Let l be an integer with $1 \leq l \leq \operatorname{card} \sigma_p(A) \setminus \{0\}$. We define the following operators:

$$A_{l0} = 0,\ A_{l1} = \lambda_1 \sum_{i=1}^{l} P_{\lambda_i} + \sum_{i \in H(l)} \lambda_i P_{\lambda_i},\ A_{l2} = \lambda_1 \sum_{i=1}^{l-1} P_{\lambda_i} + \sum_{i \in H(l-1)} \lambda_i P_{\lambda_i}, ...,$$

$$A_{ll} = \lambda_1 P_{\lambda_1} + \sum_{i \in H(1)} \lambda_i P_{\lambda_i} (= A).$$

We infer that for any $k \in \{0, 1, ..., l\}$ the operator A_{lk} belongs to $C_1(H)^+$. By Corollary 2.8, it is easy to see that

$$A_{l0}^{KK} \subset A_{l1}^{KK} \subsetneq ... \subsetneq A_{ll}^{KK}. \tag{2.6}$$

Now suppose that $\operatorname{card} \sigma_p(A) = \infty$. Then for any $l \in \mathbb{N}$ we can construct a chain like in (2.6), and this implies that $\rho(A^{KK}) = \infty$. In the case where $1 < \operatorname{card} \sigma_p(A) < \infty$, by Corollary 2.8 and the definition of $\rho(A^{KK})$, it is easy to check that $\rho(A^{KK}) \leq \operatorname{card} \sigma_p(A)$. On the other hand, let $l = \operatorname{card} \sigma_p(A) \setminus \{0\}$ and consider the corresponding chain (2.6). An elementary argument shows that it contains exactly $\operatorname{card} \sigma_p(A) - 1$ proper inclusions. We deduce that $\rho(A^{KK}) = \operatorname{card} \sigma_p(A)$ and this completes the proof of Lemma 2.10. □

This lemma is interesting on its own right, since it shows that the cardinality of the spectrum of elements in $C_1(H)^+$ can be expressed in terms of commutativity.

Now we turn to the essential part of the proof of Theorem 2.2 and define an extension Φ of ϕ onto $C_1(H)^+$ in the following way. Let $\Phi(0) = 0$ and set

$$\Phi(A) = (\operatorname{tr} A)\phi\left(\frac{1}{\operatorname{tr} A}A\right) \quad (0 \neq A \in C_1(H)^+).$$

It is easy to check that Φ is a commutativity preserving bijection. We will prove that – up to unitary-antiunitary equivalence – Φ sends each element of $C_1(H)^+$ to a bounded injective function of it (The reason for investigating Φ instead of ϕ is purely practical). The following lemma establishes two important preserver properties of Φ.

Lemma 2.11. *For any $n \in \mathbb{N}$ and $A \in C_1(H)^+$ we have $\rho(A^{KK}) = n \iff \rho(\Phi(A)^{KK}) = n$, furthermore* $\operatorname{card} \sigma_p(A) = n \iff \operatorname{card} \sigma_p(\Phi(A)) = n$.

Proof. Observe that by the properties of Φ we have that $\Phi(\mathcal{M}^{KK}) = \Phi(\mathcal{M})^{KK}$ holds for every $\mathcal{M} \subset C_1(H)^+$, and this easily gives the first equivalence. The second one then follows by Lemma 2.10. □

Now we prove a simple and useful equality. First, we introduce the following notation. For arbitrary commuting finite rank projections $P, Q \in B(H)$ let

$$g(P, Q) = PQ + 2PQ^\perp + 3P^\perp Q.$$

We assert that $\{P, Q\}^{KK} = g(P, Q)^{KK}$. To see this, it is enough to verify that $\{P, Q\}' = g(P, Q)'$. But observe that PQ, $PQ^\perp$ and $P^\perp Q$ are mutually orthogonal finite rank projections and we clearly have

$$\{P, Q\}' = \{PQ, PQ^\perp, P^\perp Q\}' = g(P, Q)'$$

which proves the assertion.
The following lemma describes the general form of Φ on $P_1(H)$.

Lemma 2.12. *There exist either a unitary or an antiunitary operator U on H and functions $\lambda\colon P_1(H) \to \mathbb{R} \setminus \{0\}$, $\mu\colon P_1(H) \to \mathbb{R}$ such that for every $P \in P_1(H)$ we have $\Phi(P) = U(\lambda(P)P + \mu(P)I)U^*$.*

Proof. We will consider two cases.

CASE I. In this case we suppose that H is infinite dimensional. The argument consists of several steps.

Step I.1. Let $P, Q \in P_f(H)$ be nonzero such that $PQ = QP$. Then we have $\rho\left(g(P, Q)^{KK}\right) \leq 3$ if and only if at least one of the following assertions is true: $P \perp Q$, $P \leq Q$, $Q \leq P$. To see this observe that each of the projections

PQ, $P^\perp Q$ and $PQ^\perp$ has the property that it is zero or it is a spectral projection of $g(P,Q)$. Therefore by Lemma 2.10 the above inequality is equivalent to the fact that at least one of these projections is zero.

Step I.2. Let $0 \neq P \in P_f(H)$. We assert that P is of rank 1 if and only if for every $0 \neq Q \in P_f(H)$ with $PQ = QP$ the operators P and Q are orthogonal or comparable (with respect to $\leq$). To prove this, first suppose that $P \in P_1(H)$. We easily have that for every $0 \neq Q \in P_f(H)$ with $PQ = QP$ either $P \leq Q$ or $P \perp Q$ holds true. Now suppose that $P \notin P_1(H)$. Then we choose two nonzero vectors x_1 and x_2 such that $x_1 \in \operatorname{rng} P$ and $x_2 \in \operatorname{rng} P^\perp$. Let Q be the projection onto the subspace of H generated by x_1 and x_2. It is easy to check that $0 \neq Q \in P_f(H)$ is such that $PQ = QP$ and P and Q are not comparable nor orthogonal. This completes the proof of the asssertion.

Step I.3. Using the previous steps, we deduce the following. If P is a nonzero finite rank projection, then $P \in P_1(H)$ if and only if for every $0 \neq Q \in P_f(H)$ with $PQ = QP$ we have $\rho\left(g(P,Q)^{KK}\right) \leq 3$.

Step I.4. In this step we define a certain map Ψ on $P_f(H) \setminus \{0\}$.

Let $A \in C_1(H)^+$. It is trivial that card $\sigma_p(A) = 2$ holds if and only if there exist a positive number λ and a nonzero projection $Q \in P_f(H)$ such that $A = \lambda Q$. Suppose that $0 \neq P \in P_f(H)$. By the previous observation and Lemma 2.11 it follows that there exist a positive number λ and a nonzero projection $Q \in P_f(H)$ such that $\Phi(P) = \lambda Q$. Define $\Psi(P) = Q$.

In that way we obtain a map $\Psi\colon P_f(H)\setminus\{0\} \to P_f(H)\setminus\{0\}$. It is clear that Ψ is well-defined and preserves commutativity. It follows from Lemma 2.11 that Ψ is surjective. It is easy to see that Ψ is injective, too.

Step I.5. Let $P, Q \in P_f(H)$ be nonzero such that $PQ = QP$. We assert that

$$\rho\left(g(P,Q)^{KK}\right) \leq 3 \iff \rho\left(g(\Psi(P),\Psi(Q))^{KK}\right) \leq 3.$$

To see this, observe that by Lemma 2.11 we have

$$\rho\left(g(P,Q)^{KK}\right) \leq 3 \iff \rho\left(\Phi(g(P,Q))^{KK}\right) \leq 3.$$

Moreover, using the definition of Ψ and the discussion before Lemma 2.12, we infer that

$$\begin{aligned}\Phi(g(P,Q))^{KK} = \Phi\left(\{P,Q\}^{KK}\right) &= \{\Phi(P),\Phi(Q)\}^{KK} = \{\Psi(P),\Psi(Q)\}^{KK}\\ &= g(\Psi(P),\Psi(Q))^{KK}\end{aligned}$$

and obtain the required assertion.

Step I.6. Using Steps I.3, I.4 and I.5, we deduce that Ψ preserves the rank-one projections and acts as a commutativity preserving bijection on $P_1(H)$.

It is easy to check that if $P, Q \in P_1(H)$, then $PQ = QP$ if and only if either $P = Q$ or $P \perp Q$ holds true. Therefore, we infer that the restriction of Ψ onto $P_1(H)$ is an orthogonality preserving bijection. By a famous theorem of Uhlhorn [**43**] the structure of such transformations is known. Applying this result, we obtain that there exists either a unitary or an antiunitary operator U on H such that $\Psi\big|_{P_1(H)}$ is of the form

$$\Psi(P) = UPU^* \quad (P \in P_1(H)).$$

Now it follows that $\Phi(P) = U\lambda(P)PU^*$ with some function $\lambda\colon P_1(H) \to]0, \infty[$. This completes the proof of Lemma 2.12 in CASE I.

CASE II. Here we suppose that $n = \dim H < \infty$. Just as above, we decompose the proof into several steps.

Step II.1. Let P, Q be nontrivial projections such that $PQ = QP$. Using Lemma 2.10 it is easy to see that $\rho\left(g(P, Q)^{KK}\right) \leq 3$ if and only if at least one of the following holds: $P \perp Q,\ P \leq Q,\ Q \leq P,\ P \vee Q = I$.

Step II.2. We show that if P is a nontrivial projection, then the following two assertions are equivalent:

a. P has rank 1 or $n - 1$.
b. If Q is a nontrivial projection such that $PQ = QP$, then at least one of the following holds: $P \perp Q,\ P \leq Q,\ Q \leq P,\ P \vee Q = I$.

To see this, let $P \in P_1(H)$. Similarly to Step I.2, if Q is a nontrivial projection such that $PQ = QP$, then we have $P \leq Q$ or $P \perp Q$. Now suppose that P has rank $n-1$ and let Q be a nontrivial projection such that $PQ = QP$. Then $P^\perp$ has rank 1, thus $P^\perp \perp Q$ or $P^\perp \leq Q$, i.e. $Q \leq P$ or $P \vee Q = I$. If rank $P \notin \{1, n-1\}$, then we can construct a projection Q (which is the same as the one in Step I.2) such that $0 \neq Q \in P(H),\ PQ = QP$ and all of the following assertions are false: $P \perp Q,\ P \leq Q,\ Q \leq P$. Moreover, since $\dim H > 2$ and rank $P < n - 1$, we also have that $Q \neq I$ and $P \vee Q \neq I$. This completes the proof of the above equivalence.

Step II.3. Using Steps II.1 and II.2, we infer that if P is a nontrivial projection, then P has rank 1 or $n - 1$ if and only if for every nontrivial projection Q with $PQ = QP$ we have $\rho\left(g(P, Q)^{KK}\right) \leq 3$.

Step II.4. Let $A \in C_1(H)^+$. It is easy to see that card $\sigma_p(A) = 2$ if and only if there exist a nontrivial projection P and real numbers λ and μ such that $\lambda \neq 0$ and $A = \lambda P + \mu I$. Moreover, the nontrivial projection P in the latter sum is unique up to orthocomplementation.

Let P and Q be nontrivial commuting projections, $\lambda_1, \lambda_2, \mu_1, \mu_2$ be real numbers such that λ_1 and λ_2 are nonzero and suppose that $\lambda_1 P + \mu_1 I, \lambda_2 Q +$

$\mu_2 I \in C_1(H)^+$. By Lemma 2.11 it follows that there exist nontrivial commuting projections $P', Q' \in P(H)$ and real numbers $\lambda_1', \lambda_2', \mu_1', \mu_2'$ such that λ_1' and λ_2' are nonzero and

$$\Phi(\lambda_1 P + \mu_1 I) = \lambda_1' P' + \mu_1' I, \quad \Phi(\lambda_2 Q + \mu_2 I) = \lambda_2' Q' + \mu_2' I.$$

In a similar fashion as in Step I.5, we get that $\Phi(g(P,Q))^{KK} = g(P',Q')^{KK}$. Hence, by Lemma 2.11 we have

$$\rho\left(g(P,Q)^{KK}\right) \leq 3 \iff \rho\left(g(P',Q')^{KK}\right) \leq 3.$$

Step II.5. Let us call a positive operator primitive if it is of the form $\lambda P + \mu I$ where $P \in P_1(H)$, $\lambda, \mu \in \mathbb{R}$ and $\lambda \neq 0$. Since $\dim H > 2$, the rank-one projection P in the above sum is unique. Moreover, for any $A \in C_1(H)^+$ we have that A is primitive if and only if it is of the form $\lambda P + \mu I$, where $P \in P(H)$ has either rank 1 or $n-1$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq 0$.

Applying Lemma 2.11 and Steps II.3 and II.4, it is now easy to verify that for any $A \in C_1(H)^+$ we have that A is primitive if and only if $\Phi(A)$ is primitive.

Step II.6. We define a map Ψ on $P_1(H)$ in the following way. For any $P \in P_1(H)$, let $\Psi(P)$ be the unique rank-one projection which satisfies that there exist real numbers λ and μ such that $\lambda \neq 0$ and $\Phi(P) = \lambda\Psi(P) + \mu I$. By Step II.5, the existence and uniqueness of such a rank-one projection is clear. Clearly, $\Psi\colon P_1(H) \to P_1(H)$ preserves commutativity.

We show that Ψ is surjective. To do this, let $Q \in P_1(H)$. Since every rank-one projection is primitive and Φ preserves the primitive elements, we have that there exist a projection $P \in P_1(H)$, real numbers $\lambda \neq 0$ and μ such that $\lambda P + \mu I \in C_1(H)^+$ and $\Phi(\lambda P + \mu I) = Q$. We assert that $\Psi(P) = Q$. Indeed, observe that $Q \in \Phi(P^{KK}) = \Phi(P)^{KK}$. This gives us that Q is an affine function of $\Phi(P)$. On the other hand, $\Phi(P)$ is an affine function of $\Psi(P)$. Therefore, Q is an affine function of $\Psi(P)$ and, as both of those operators are rank-one projections, it follows that $\Psi(P) = Q$. This completes the proof of the surjectivity of Ψ. The injectivity of Ψ can be verified in a similar but easier way.

Applying Uhlhorn's theorem, just as in Step I.6 we conclude that there exist functions $\lambda\colon P_1(H) \to \mathbb{R} \setminus \{0\}$ and $\mu\colon P_1(H) \to \mathbb{R}$, and either a unitary or an antiunitary operator U on H such that for every $P \in P_1(H)$ we have $\Phi(P) = U(\lambda(P)P + \mu(P)I)U^*$. This completes the proof of Lemma 2.12. □

Now, to finish the proof of Theorem 2.2, define the selfmap $\widetilde{\Phi}$ of $C_1(H)^+$ by $\widetilde{\Phi}(A) = U^*\Phi(A)U$ $(A \in C_1(H)^+)$. It is clear that $\widetilde{\Phi}$ preserves commutativity and for every $P \in P_1(H)$ we have

$$\widetilde{\Phi}(P) = \lambda(P)P + \mu(P)I.$$

Let $x \in H$ be a unit vector and $A \in C_1(H)$ be a positive operator. Clearly, $x \otimes x$ commutes with A if and only if $\widetilde{\Phi}(x \otimes x)$ commutes with $\widetilde{\Phi}(A)$ which is equivalent to the commutativity of $x \otimes x$ and $\widetilde{\Phi}(A)$. It follows that x is an eigenvector of A if and only if x is an eigenvector of $\widetilde{\Phi}(A)$. Therefore, we obtain that the eigensubspaces of A coincide with those of $\widetilde{\Phi}(A)$. It is then apparent that $\widetilde{\Phi}(A)$ is a bounded injective function of A. As $A \in C_1(H)^+$ was arbitrary, going back to Φ and then to $\phi = \Phi|_{S(H)}$ we obtain the statement of Theorem 2.2. □

We now turn to the verification of the results concerning maps which preserve our measure of commutativity.

Proof of Theorem 2.3 and Theorem 2.4. In the following argument, we will use the notation and definitions below. Fix an orthonormal basis $\{e_i\}_{i\in I}$ in H. For any $x \in H$ and $A \in B(H)$ define $\overline{x} = \sum_{i\in I}\langle e_i, x\rangle e_i$ and $\overline{A}x = \overline{A\overline{x}}$. It is clear that $\overline{A} \in B(H)$ $(A \in B(H))$. The Hilbert dimension of a Hilbert space is defined as the common cardinality of its orthonormal bases. By the multiplicity function m_A for the compact operator $A \in B(H)$ we mean the function which assigns to a given number $\lambda \in \mathbb{C}$ the Hilbert dimension of $\ker(T - \lambda I)$, where ker denotes the kernel of operators.

In what follows, we consider Theorem 2.3 and Theorem 2.4 simultaneously, so the only assumption which we suppose concerning the underlying Hilbert space is that $\dim H \geq 3$. First observe that ϕ preserves commutativity. Hence, by Theorem 2.2 there exists either a unitary or an antiunitary operator U on H such that for every $A \in S(H)$ we have a function $f_A\colon \sigma_p(A) \to [0, 1]$ for which $\phi(A) = Uf_A(A)U^*$. We define the transformation $\psi\colon S(H) \to S(H)$ by $\psi(A) = f_A(A)$ $(A \in S(H))$. Observe that $\psi(A) = U^*\phi(A)U$ $(A \in S(H))$.

Now we are going to show that the transformation $A \mapsto U^*AU$ $(A \in S(H))$ preserves our measure of commutativity. Since the commutator of density operators is a normal trace-class operator, it is apparent that the desired assertion will be verified once we have proved that for any normal operator $T \in C_1(H)$ one has

$$\|U^*TU\| = \|T\|. \tag{2.7}$$

It is easy to see that if V is an antiunitary operator on H, then there is a unitary element $W \in B(H)$ such that for each $A \in B(H)$ we have $V^*AV = W^*\overline{A}W$. Using the unitary invariance of $\|.\|$ this immediately yields that for any normal operator $T \in C_1(H)$ the relation (2.7) is equivalent to the equality $\left\|\overline{T}\right\| = \|T\|$. Now let T be as in the latter sentence. One can easily check that

$$\ker\left(T^* - \overline{\lambda}I\right) = \ker(T - \lambda I) = \left\{x \in H \ : \ \overline{x} \in \ker\left(\overline{T} - \overline{\lambda}I\right)\right\}$$

and therefore the Hilbert dimension of $\ker\left(T^* - \overline{\lambda}I\right)$ equals that of the kernel of $\overline{T} - \overline{\lambda}I$ which yields that $m_{\overline{T}} = m_{T^*}$ $(\lambda \in \mathbb{C})$. By [**10**, 8.3. Theorem of Chapter II], two compact normal operators are unitarily equivalent if and only if their multiplicity functions coincide. Now we obtain that there is a unitary operator V_1 on H such that $\overline{T} = V_1T^*V_1^*$. On the other hand, since T is normal, as it is well-known there is a unitary operator W_1 on H such that $T = W_1|T|$, hence $T^* = W_1^*TW_1^*$. By the previous facts it follows that $\overline{T} = (V_1W_1^*)T(W_1^*V_1^*)$. Referring to the unitary invariance of $\|.\|$ we finally get that $\left\|\overline{T}\right\| = \|T\|$ and hence we conclude that (2.7) is valid. As it is already mentioned, this implies that the desired preserver property holds. Now it follows that for any $A, B \in S(H)$ one has

$$\|\psi(A)\psi(B) - \psi(B)\psi(A)\| = \|AB - BA\|.$$

Next let $P \in P_1(H)$ be arbitrary. By the definition of ψ there are real numbers α, β such that $\psi(P) = \alpha P + \beta I$. Moreover, since $\psi(P) \in S(H)$, we deduce that $\beta, \beta + \alpha \in [0, 1]$, therefore $|\alpha| \leq 1$. Now let $Q \in P_1(H)$ be such that $PQ \neq QP$. By the previous observations we have real numbers α', β' such that $|\alpha'| \leq 1$ and $\psi(Q) = \alpha'Q + \beta'I$. Since ψ preserves our measure of commutativity, it follows that

$$\|PQ - QP\| = |\alpha\alpha'|\|PQ - QP\|$$

and thus $|\alpha||\alpha'| = 1$. We easily infer that $|\alpha| = 1$. Now let $A \in S(H)$ be fixed and $x \in \operatorname{rng} P$ be a unit vector. By the discussion above we have

$$\begin{aligned} \|A \cdot x \otimes x - x \otimes x \cdot A\| &= |\alpha|\|\psi(A) \cdot x \otimes x - x \otimes x \cdot \psi(A)\| \\ &= \|\psi(A) \cdot x \otimes x - x \otimes x \cdot \psi(A)\|. \end{aligned} \tag{2.8}$$

We proceed with presenting a formula concerning the norm of the commutator of a density operator and a rank-one projection. Before doing so, we need the next assertion. There is a number $c > 0$ such that for any rank-two normal operator $N \in B(H)$ with spectrum $\{-i, i, 0\}$ one has

$$\|N\| = c. \tag{2.9}$$

For the proof, let N_1 and N_2 be such operators. Since $\operatorname{rng} N_1$ and $\operatorname{rng} N_2$ are two-dimensional subspaces of H, the Hilbert dimensions of $\ker N_1 = (\operatorname{rng} N_1)^\perp$ and $\ker N_2 = (\operatorname{rng} N_2)^\perp$ coincide. Then it follows easily that $m_{N_1} = m_{N_2}$ and therefore by [**10**, 8.3. Theorem of Chapter II] we obtain that $N_2 = VN_1V^*$ with some unitary operator V on H. Using the unitary invariance of $\|.\|$ we conclude that $\|N_1\| = \|N_2\|$ and this completes the proof of (2.9).

Now we are in a position to present the formula mentioned in the preceding paragraph. It reads as follows. Let $A \in S(H)$ be a fixed density operator and $x \in H$ be a unit vector. We have

$$\|A \cdot x \otimes x - x \otimes x \cdot A\| = c\sqrt{\langle A^2 x, x\rangle - \langle Ax, x\rangle^2}, \tag{2.10}$$

where c is the scalar appearing in (2.9). For the proof, first observe that (2.10) is trivially valid in the case where x and Ax are linearly dependent. Now suppose that this does not hold. We will determine the point-spectrum of the operator $T \doteq A \cdot x \otimes x - x \otimes x \cdot A$. It is obvious that x and Ax generate $\operatorname{rng} T$. Denote by T_1 the restriction of T onto its range. One can check that the matrix of T_1 with respect to the basis $\{x, Ax\}$ is

$$\begin{pmatrix} -\langle Ax, x\rangle & -\langle A^2x, x\rangle \\ 1 & \langle Ax, x\rangle \end{pmatrix}.$$

Using this matrix, we easily get that

$$\sigma_p(T) = \left\{0, i\sqrt{\langle A^2x, x\rangle - \langle Ax, x\rangle^2}, -i\sqrt{\langle A^2x, x\rangle - \langle Ax, x\rangle^2}\right\}.$$

By the spectral theorem we deduce that $T = \sqrt{\langle A^2x, x\rangle - \langle Ax, x\rangle^2}N$, where N is a rank-two normal operator with spectrum $\{-i, i, 0\}$. Therefore, referring to (2.9), we get (2.10).

Next let $A \in S(H)$. By (2.8) and (2.10), it follows that

$$\langle A^2x, x\rangle - \langle Ax, x\rangle^2 = \langle \psi(A)^2x, x\rangle - \langle \psi(A)x, x\rangle^2$$

for any unit vector $x \in H$. Referring to [**32**, Proposition], we deduce that

$$\psi(A) = \lambda I + \varepsilon A, \tag{2.11}$$

where $\lambda \in \mathbb{R}$ and $\varepsilon \in \{-1, 1\}$. Now we consider Theorem 2.3 and Theorem 2.4 separately. First let H be infinite dimensional. Then in the equality (2.11) we must have $\psi(A) = A$ for every $A \in S(H)$ yielding the statement of Theorem 2.3. Next assume that H is finite dimensional. As the range of ψ is contained in $S(H)$, it follows that in (2.11) we must have either $\psi(A) = A$ or $\psi(A) = (2/\dim H)I - A$. This verifies Theorem 2.4. □

Finally, we present the proof of Theorem 2.5.

Proof of Theorem 2.5. We start with a simple observation. Easy computation shows that for any $A, B \in S(H)$ we have $F(A, B) = 0$ if and only if $AB = 0$. It follows that ϕ preserves orthogonality.

Observe that if $A, B \in B_s(H)$, then $AB = 0$ if and only if $\overline{\operatorname{rng} A} \perp \overline{\operatorname{rng} B}$. We introduce the following notation. For any subset $\mathcal{L} \subset S(H)$ let

$$\tilde{m}(\mathcal{L}) \doteq \{B \in S(H) | \ \forall A \in \mathcal{L}: \ \overline{\operatorname{rng} B} \perp \overline{\operatorname{rng} A}\}.$$

It follows from the properties of ϕ that $\phi(\tilde{m}(\tilde{m}(\mathcal{L}))) = \tilde{m}(\tilde{m}(\phi(\mathcal{L})))$ $(\mathcal{L} \subset S(H))$. Moreover, it is easy to see that for any $A \in S(H)$ we have

$$\tilde{m}(\tilde{m}(\{A\})) = \{B \in S(H) | \ \overline{\operatorname{rng} B} \subset \overline{\operatorname{rng} A}\}.$$

It is obvious that for each $A \in S(H)$, the operator A is a rank-one projection if and only if the set $\{B \in S(H) | \ \overline{\operatorname{rng} B} \subset \overline{\operatorname{rng} A}\}$ is a singleton. Using this characterization we obtain that ϕ preserves the elements of $P_1(H)$ in both directions.

Putting together all the information we have, one sees that ϕ acts as an orthogonality preserving bijection on $P_1(H)$. So the theorem of Uhlhorn applies and we get that there exists either a unitary or an antiunitary operator U on H such that $\phi(P) = UPU^*$ $(P \in P_1(H))$. Define the map ψ by $\psi(A) = U^*\phi(A)U$ $(A \in S(H))$. It is clear that $\psi\colon S(H) \to S(H)$ satisfies the conditions in Theorem 2.5 and acts as the identity on $P_1(H)$.

It follows that for any density operator $A \in B(H)$ and rank-one projection P on H we have that P commutes with A if and only if P commutes with $\psi(A)$. Just as previously, it implies that for each $A \in S(H)$ and unit vector $x \in H$ we have that x is an eigenvector of A if and only if x is an eigenvector of $\psi(A)$. By this observation and the spectral theorem we have that there exists an orthonormal basis $(e_i)_{i\in I}$ in H whose each member is a common eigenvector of A and $\psi(A)$. For every $i \in I$ the operators A and $e_i \otimes e_i$ commute, so we get that $F(A, e_i \otimes e_i) = F(\psi(A), e_i \otimes e_i)$. But easy calculation yields that for any $B \in S(H)$ and unit vector $x \in H$ we have $F(B, x \otimes x) = \sqrt{\langle Bx, x\rangle}$. Consequently, we deduce

$$\langle Ae_i, e_i\rangle = \langle\psi(A)e_i, e_i\rangle$$

implying that the eigenvalues of A and $\psi(A)$ corresponding to the eigenvectors e_i are the same. This gives us that $\psi(A) = A$ and transforming back to ϕ we obtain the statement of Theorem 2.5. $\square$

2.3. Remarks

We remark that there is interest also in the space of non-normalized density operators, i.e., $C_1(H)^+$ in the place of $S(H)$. Observe that the quantity on the right-hand side of (2.4) is well-defined for any pair A, B of positive trace-class operators on H. One can consider the transformations appearing in Theorem 2.3, Theorem 2.4 and – due to the latter observation – Theorem 2.5 also on $C_1(H)^+$. The arguments we have employed here (with suitable modifications) apply to obtain analogous results for that case as well. We omit the details.

In the theorems of Chapter 2 it is assumed that $\dim H \geq 3$. However, it is a natural question to ask whether the conclusions of these results remain valid for lower dimensional Hilbert spaces. Since in the case where $\dim H < 2$ the answer is trivial, the only interesting case is that where $\dim H = 2$. It is conjectured that the conclusion of Theorem 2.4 is valid in the 2-dimensional case. As for Theorem 2.1, Theorem 2.2 and Theorem 2.5 counterexamples can be given which show that those statements do not hold if $\dim H = 2$.

In [**28**, Remark 1], using a certain construction it was proved that the conclusion of Theorem 2.1 is not valid in the 2-dimensional case. In what follows, we will use essentially the same construction in order to obtain a counterexample for the statement of Theorem 2.2 and Theorem 2.5. Now let $\dim H = 2$. In [**43**, section 5.2] it was proved that there is a continuous orthogonality preserving bijection ψ on $P_1(H)$ which does not leave the transition probability invariant. Using this map define a transformation $\phi\colon S(H) \to S(H)$ in the following way. Let $\phi(1/2I) = 1/2I$. If $1/2I \neq A \in S(H)$, then it is clear that there is a unique set $\{P, Q\} \subset P_1(H)$ of mutually orthogonal projections and a unique collection $\{\lambda, \mu\} \subset \mathbb{R}_+$ such that $\lambda + \mu = 1$ and $A = \lambda P + \mu Q$. Now define

$$\phi(A) = \lambda\psi(P) + \mu\psi(Q).$$

It is easy to check that ϕ leaves the fidelity between commuting operators invariant. Using an argument similar to the one applied in [**28**, Remark 1] we get that ϕ is a commutativity preserving bijection which cannot be written in the form (2.2). Now we conclude that ϕ satisfies the conditions in Theorem 2.2 and Theorem 2.5, however the conclusions of these results do not hold for ϕ.

3

Relative entropy preserving maps on density operators

3.1. Introduction and statement of the results

In this chapter we consider certain kinds of relative entropies on sets of density operators acting on a finite dimensional complex Hilbert space and we study mappings preserving these entropic quantities. We determine the structure of 'a priori' non-surjective maps on the set of all density operators which leave a certain measure of relative entropy invariant and also characterize the surjective maps on the set of all invertible density operators with similar invariance properties. The results of the present section have been published in the work [**27**].

Throughout this chapter, it will be assumed that $n = \dim H < \infty$. The notion of relative entropy is a fundamental concept in quantum information theory. It is used to measure distinguishability between quantum states. In fact, several notions of relative entropy are studied in the literature the most common one being the (standard) relative entropy introduced by Umegaki. In this chapter we consider other concepts of relative entropy such as the Belavkin-Staszewski, the Tsallis and the quadratic relative entropy and the Jensen-Shannon divergence (see, e.g. [**14, 36, 1, 18**]). For any $A, B \in S(H)$ and given $0 < q < 1$ these quantities are defined as follows.

(i) Umegaki relative entropy: $S(A\|B) = \operatorname{tr} A(\log A - \log B)$ if $\operatorname{supp} A \subset \operatorname{supp} B$, and $S(A\|B) = \infty$ otherwise

(ii) Belavkin-Staszewski relative entropy:
$S_B(A\|B) = \operatorname{tr} \sqrt{A} \log \sqrt{A} B^{-1} \sqrt{A}$ if $\operatorname{supp} A \subset \operatorname{supp} B$, and $S_B(A\|B) = \infty$ otherwise

(iii) Tsallis relative entropy: $S_T(A\|B) = 1/(1-q)(1 - \operatorname{tr} A^q B^{1-q})$

(iv) Quadratic relative entropy: $S_Q(A\|B) = \operatorname{tr} A^{-1}(A-B)^2$ if $\operatorname{supp} B \subset \operatorname{supp} A$, and $S_Q(A\|B) = \infty$ otherwise

(v) Jensen-Shannon divergence:

$$D_J(A\|B) = \frac{S\left(A \left\| \frac{1}{2}(A+B)\right.\right) + S\left(B \left\| \frac{1}{2}(A+B)\right.\right)}{2}$$

As for these definitions, in this section supp denotes the support of a density operator (which is the orthogonal complement of its kernel) and log stands for the logarithm with base 2. Moreover, in this chapter the -1$^{\text{th}}$ power and the logarithm of a positive operator on H are only taken on its range. It is an important property of the quantities (i)–(v) that they are nonnegative and equal to 0 if and only if $A = B$. Moreover, the Jensen-Shannon divergence has been proved to be the square of a metric on $S(H)$ if $\dim H = 2$ and it is conjectured that the same is true in any finite dimension, too (see [**6**]).

In the paper [**23**] Molnár showed that any surjective map on $S(H)$ which preserves the Umegaki relative entropy is implemented by a unitary or an antiunitary operator. Recently, significantly refining and modifying the arguments in the proof, in [**29**] the authors managed to prove that the same conclusion holds also in the case where the surjectivity of transformations is not assumed. As non-surjective versions of classical theorems like Wigner's theorem or the fundamental theorem of projective geometry are much more useful and applicable than their bijective versions, we were motivated to investigate transformations preserving a given relative entropy without the surjectivity assumption. It turns out below that the method of the proof presented in [**29**] can also be applied for maps leaving any of the quantities (ii),(iv) invariant. Concerning the relative entropy (iii) we show that its non-surjective preservers are also implemented by unitary or antiunitary operators and as for the relative entropy (v), we prove that the result in [**31**] on the form of surjective transformations on $S(H)$ preserving that quantity can be extended to the case of non-surjective maps as well. Thus we obtain a complete description of 'a priori' non-surjective transformations on $S(H)$ preserving any of the relative entropies (ii)–(v). It is not difficult to check that every map of the form $A \mapsto UAU^*$ ($A \in S(H)$) with a unitary or an antiunitary operator U on H preserves all of the relative entropies (ii)–(v). The following result tells that the reverse statement is also true: each transformation which leaves one of the quantities (ii)–(v) invariant is necessarily of that form.

Theorem 3.1. (Molnár, Nagy [**27**])
Let $X(.\|.)$ denote any of the relative entropies (ii)–(v). *Suppose that ϕ is a*

selfmap of $S(H)$ which satisfies that

$$X(\phi(A)\|\phi(B)) = X(A\|B) \tag{3.1}$$

holds for all $A, B \in S(H)$. Then we have either a unitary or an antiunitary operator U on H such that ϕ is of the form

$$\phi(A) = UAU^* \quad (A \in S(H)).$$

Relating to problems in quantum information theory and quantum statistics where differential geometric considerations and corresponding strong analytical tools are applied, instead of the whole set of density operators one usually considers only the collection $M(H)$ of invertible elements of $S(H)$. The reason is that from differential geometric point of view $M(H)$ is a much more appropriate set, namely, a manifold. This gives rise to study our transformations also on this restricted domain. In [**25**] Molnár described the form of surjective maps on $M(H)$ which preserve one of the relative entropies (i),(ii). Adopting the method developed there, we prove that a similar conclusion holds also for surjective maps preserving one of the quantities (iii),(iv). The corresponding statement reads as follows.

Theorem 3.2. (Molnár, Nagy [**27**])
Let $X(.\|.)$ denote any of the quantities (iii),(iv). *Suppose that $\phi\colon M(H) \to M(H)$ is a surjective transformation such that* (3.1) *holds for all $A, B \in M(H)$. Then we have either a unitary or an antiunitary operator U on H such that ϕ is of the form*

$$\phi(A) = UAU^* \quad (A \in M(H)).$$

3.2. Proofs

First we remark that clearly $\operatorname{rng} A = \operatorname{supp} A$ $(A \in S(H))$. We shall need the fact that the relative entropies (i)–(iv) are definite. By this property we mean that for each $A, B \in S(H)$ we have

$$A = B \iff X(A\|B) = 0,$$

where $X(.\|.)$ denotes any of the quantities (i)–(iv). Concerning the Umegaki relative entropy, this is very well known (see [**35**, Theorem 11.7]). Now let $A, B \in S(H)$. We learn from [**14**, Corollary 2.6.] that one has $S_B(A\|B) \geq S(A\|B)$, moreover by [**41**, Proposition 6] the inequality $S_Q(A\|B) \geq S(A\|B)$ holds. Using the definiteness and the nonnegativity of the Umegaki relative entropy it follows that the relative entropies S_B, S_Q are definite. As for the Tsallis relative entropy, we can use the criterion of equality in Hölder's inequality concerning p-norms as follows. By [**19**, Theorem 2.3] we infer that

$$\begin{aligned}\operatorname{tr} A^q B^{1-q} &= |\operatorname{tr} A^q B^{1-q}| \leq \operatorname{tr} |A^q B^{1-q}| \\ &= \|A^q B^{1-q}\|_1 \leq \|A^q\|_{\frac{1}{q}} \|B^{1-q}\|_{\frac{1}{1-q}} = 1\end{aligned}$$

and in the second inequality equality occurs if and only if A^2 is a scalar multiple of B^2. Since A and B are density operators, this is equivalent to $A = B$. The definiteness of the Tsallis relative entropy follows.

In our arguments we will use the following simple observation.

(*) A density operator $A \in S(H)$ belongs to $P_1(H)$ exactly when it is contained in a set of n pairwise orthogonal elements of $S(H)$.

We now present the proof of the first result of this chapter.

Proof of Theorem 3.1. According to the considered relative entropy, we divide the proof into four cases. In the case of the relative entropies (ii),(iv),(v), we use the approaches developed by Molnár and his coauthors in the papers [**29**] and [**31**]. Therefore, below we present only the main steps of the proofs pointing out the necessary alterations. The reader is supposed to have the mentioned papers at hand.

CASE 1. Suppose now that $X(.\|.)$ is the Jensen-Shannon divergence. Following the proof in [**31**] one can check that the arguments applied there in the course of verifying the orthogonality preserving property of ϕ are all valid for non-surjective transformations as well. Thus for any $A, B \in S(H)$ we have that A and B are orthogonal if and only if the same holds for $\phi(A)$ and $\phi(B)$. By (*), it follows that ϕ leaves the set $P_1(H)$ invariant. Next, we can apply the method given in the corresponding part of [**31**] to prove that

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ$$

holds for arbitrary elements P, Q of $P_1(H)$. Referring to Theorem 1.3, we infer that there is a unitary or an antiunitary operator U on H such that

$$\phi(P) = UPU^* \quad (P \in P_1(H)).$$

One can check that from this point the argument presented in [**31**] can be applied again and hence we obtain the statement of Theorem 3.1 in the case of the Jensen-Shannon divergence.

CASE 2. In this part of the proof $X(.\|.)$ is the Belavkin-Staszewski relative entropy. Here we can adopt the approach developed in [**29**]. As above, we recall the main steps of the argument in [**29**] and elaborate in more details only those points which must be handled differently. For technical reasons we define the quantity

$$\tilde{S}_B(A\|B) = \begin{cases} 2^{S_B(A\|B)}, & \text{if } \operatorname{supp} A \subset \operatorname{supp} B \\ \infty, & \text{otherwise,} \end{cases}$$

where $A, B \in S(H)$. It is clear that we have

$$\tilde{S}_B(\phi(A)\|\phi(B)) = \tilde{S}_B(A\|B) \quad (A, B \in S(H)).$$

Now we deduce a formula which we shall use in the proof. Let $A \in S(H)$ and $P \in P_1(H)$ be such that $\operatorname{supp} P \subset \operatorname{supp} A$. Easy computation shows that we have $\tilde{S}_B(P\|A) = \langle A^{-1}x, x\rangle$, where x is a unit vector in $\operatorname{rng} P$. This yields that

$$\tilde{S}_B(P\|A) = \operatorname{tr} A^{-1}P. \tag{3.2}$$

We show that ϕ preserves the rank of operators. This is a consequence of the following observations. On the one hand, we see that ϕ preserves the strict inclusion between the supports of density operators on H. On the other hand, an element of $S(H)$ is of rank k if and only if there is a strictly increasing sequence (with respect to inclusion) consisting of the supports of n density operators on H such that its k-th element is $\operatorname{supp} A$.

Next we assume that H is 2-dimensional. We first assert that ϕ can only enlarge the convex hull of the spectrum of the elements of $S(H)$. To see this, we recall that ϕ preserves the rank-one projections (i.e. rank-one density operators), so the claim holds for rank-one elements. Any rank-two density operator on H can be written in the form $\lambda P + \mu Q$ with certain mutually orthogonal projections $P, Q \in P_1(H)$ and positive numbers λ, μ which satisfy $\lambda + \mu = 1$. By (3.2) it is easy to see that for any $R \in P_1(H)$ we have

$$\tilde{S}_B(R\|\lambda P + \mu Q) = \frac{\operatorname{tr} PR}{\lambda} + \frac{\operatorname{tr} QR}{\mu}. \tag{3.3}$$

It follows that $\tilde{S}_B(R\|\lambda P + \mu Q)$ is a convex combination of $1/\lambda$ and $1/\mu$. Clearly, similar formula holds for $\tilde{S}_B(\phi(R)\|\phi(\lambda P + \mu Q))$, too. Letting R run through $P_1(H)$, one can deduce the assertion as in the corresponding part of the proof in [**29**].

Next we claim that ϕ is injective. Indeed, this follows immediately from the fact that $S_B(.\|.)$ is definite.

In the key step of the proof we show that $\phi\left(\frac{1}{2}I\right) = \frac{1}{2}I$. To verify this, assume on the contrary that $\phi\left(\frac{1}{2}I\right) = \lambda_1 P_1 + \mu_1 Q_1$, where $\lambda_1, \mu_1 > 0$ and $P_1, Q_1 \in P_1(H)$ are fixed elements such that $\lambda_1 > \mu_1$, $\lambda_1 + \mu_1 = 1$, and P_1

and Q_1 are mutually orthogonal. Observe that $\tilde{S}_B\left(R \middle\| \frac{1}{2}I\right) = 2$ holds for all $R \in P_1(H)$. By (3.3) it follows from the properties of ϕ that

$$2 = \tilde{S}_B\left(\phi(R) \middle\| \phi\left(\frac{1}{2}I\right)\right) = \frac{\operatorname{tr} P_1\phi(R)}{\lambda_1} + \frac{\operatorname{tr} Q_1\phi(R)}{\mu_1}. \tag{3.4}$$

Fix an orthonormal basis in H whose members belong to the ranges of P_1 and Q_1, respectively. It is easy to see that the matrix of any element of $P_1(H)$ with respect to this basis is of the form

$$\begin{pmatrix} a & \sqrt{a(1-a)}\varepsilon \\ \sqrt{a(1-a)}\overline{\varepsilon} & 1-a \end{pmatrix},$$

where $0 \leq a \leq 1$ and $\varepsilon \in \mathbb{C}$ is such that $|\varepsilon| = 1$. Therefore, the matrix of $\phi(R)$ is also of that form. Moreover, as in (3.4) the quantities λ_1, μ_1 are constant, the number $a = \operatorname{tr} P_1\phi(R)$ in the matrix of $\phi(R)$ must also remain constant (and different from 0,1) as R varies. We can rewrite the equation (3.4) in the form

$$2 = \frac{a}{\lambda_1} + \frac{1-a}{1-\lambda_1},$$

where a is the common entry in the upper left corner of the matrices of $\phi(R)$'s ($R \in P_1(H)$).

Consider now the iterated transformation $\phi \circ \phi$ and let λ_2, μ_2 be the eigenvalues of $\phi\left(\phi\left(\frac{1}{2}I\right)\right)$. Moreover, suppose that $\lambda_2 > \mu_2$. Then we clearly have $\lambda_2 \geq \lambda_1$. Just as in [**29**] we deduce that $\phi\left(\phi\left(\frac{1}{2}I\right)\right) = \lambda_2 P_1 + \mu_2 Q_1$ and

$$2 = \frac{a}{\lambda_2} + \frac{1-a}{1-\lambda_2}.$$

It is obvious that the solutions of the equation

$$2 = \frac{a}{\lambda} + \frac{1-a}{1-\lambda} \tag{3.5}$$

are $\lambda = 1/2$ and $\lambda = a$. Now suppose that $\lambda_2 = \lambda_1$. Then we deduce that $\phi\left(\phi\left(\frac{1}{2}I\right)\right) = \phi\left(\frac{1}{2}I\right)$ and by the injectivity of ϕ this implies $\phi\left(\frac{1}{2}I\right) = \frac{1}{2}I$, a contradiction. Therefore we must have $\lambda_2 > \lambda_1$. It is clear that $\lambda_1 > 1/2$ and λ_1, λ_2 satisfy (3.5). It follows that $\lambda_2 = 1/2$ which is untenable. Thus $\phi\left(\frac{1}{2}I\right) = \frac{1}{2}I$ must hold.

Now we proceed in the same way as in [**29**]. Namely, using the fact that ϕ sends $\frac{1}{2}I$ to itself and examining the solutions of some quite simple equations one can infer that ϕ in fact preserves the spectrum of density operators. We deduce that ϕ sends mutually orthogonal rank-one projections to mutually orthogonal rank-one projections. Moreover, the equality

$$\phi(\lambda P + \mu Q) = \lambda\phi(P) + \mu\phi(Q)$$

holds for any $P, Q \in P_1(H)$ and $\lambda, \mu > 0$ which satisfy $PQ = 0$ and $\lambda+\mu = 1$. From this we conclude that ϕ preserves the transition probability between rank-one projections on H.

In the rest of CASE 2 we consider an arbitrary finite dimensional space H. Just as in the corresponding part of the argument in [**29**] we infer that the conclusion deduced in the 2-dimensional case holds also in any finite dimension. Namely, we conclude that

$$\operatorname{tr}\phi(P)\phi(Q) = \operatorname{tr} PQ \quad (P, Q \in P_1(H)).$$

Then applying Theorem 1.3 we obtain that

$$\phi(P) = UPU^* \quad (P \in P_1(H))$$

holds for some unitary or antiunitary operator U on H.

To complete the proof, we define the transformation $\psi\colon S(H) \to S(H)$ by $\psi(A) = U^*\phi(A)U$ $(A \in S(H))$. This map preserves $\tilde{S}_B(.\|.)$ and acts as the identity on $P_1(H)$. Let $A \in S(H)$. Just as in [**29**], we deduce that $\operatorname{rng}\psi(A) = \operatorname{rng} A$. Moreover, we obviously have

$$\tilde{S}_B(P\|A) = \tilde{S}_B(P\|\psi(A))$$

for any $P \in P_1(H)$ which projects into $\operatorname{rng} A = \operatorname{rng}\psi(A)$. Then, by (3.2) it follows that $\operatorname{tr} A^{-1}P = \operatorname{tr}\psi(A)^{-1}P$. Since P was arbitrary, we obtain that $\psi(A)^{-1} = A^{-1}$, which means that $\psi(A) = A$. Now the statement of Theorem 3.1 follows in CASE 2.

CASE 3. Suppose now that $X(.\|.)$ is the quadratic relative entropy. Easy calculation shows that the equality

$$S_Q(A\|B) = \operatorname{tr} A^{-1}B^2 - 1 \tag{3.6}$$

holds for any $A, B \in S(H)$ with the property that $\operatorname{supp} B \subset \operatorname{supp} A$. For each pair $A, B \in S(H)$ we define the quantity

$$\tilde{S}_Q(A\|B) = \begin{cases} S_Q(B\|A) + 1, & \text{if } \operatorname{supp} A \subset \operatorname{supp} B \\ \infty, & \text{otherwise.} \end{cases}$$

Observe that by (3.2) we have $\tilde{S}_Q(P\|A) = \tilde{S}_B(P\|A)$ for any $A \in S(H)$ and $P \in P_1(H)$ which projects into $\operatorname{supp} A$ ($\tilde{S}_B(.\|.)$ is defined in CASE 2). Using this observation, it is easy to check that replacing $\tilde{S}_B$ by $\tilde{S}_Q$ in the argument in CASE 2, the method there applies again and thus we obtain the statement of Theorem 3.1 in CASE 3.

CASE 4. In what follows we assume that $X(.\|.)$ is the Tsallis relative entropy. Let $A, B \in S(H)$. Observe that we have

$$\operatorname{tr} \phi(A)^q \phi(B)^{1-q} = \operatorname{tr} A^q B^{1-q}. \tag{3.7}$$

We assert that one has $AB = 0$ if and only if $\operatorname{tr} A^q B^{1-q} = 0$. In fact, it is easy to see that $\operatorname{rng} A^q = \operatorname{rng} A$ and $\operatorname{rng} B^{1-q} = \operatorname{rng} B$. We infer that

$$AB = 0 \iff \operatorname{rng} A \perp \operatorname{rng} B \iff \operatorname{rng} A^q \perp \operatorname{rng} B^{1-q} \iff A^q B^{1-q} = 0 \\ \iff \operatorname{tr} A^q B^{1-q} = 0$$

and thus the required equivalence follows. Now we deduce that ϕ preserves orthogonality, and by (*) this implies that ϕ leaves the set $P_1(H)$ invariant. Using (3.7), it follows that for any $P, Q \in P_1(H)$ we have

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ.$$

Thus Theorem 1.3 applies and we obtain that there is a unitary or an antiunitary operator U on H such that

$$\phi(P) = UPU^* \quad (P \in P_1(H)).$$

Consider the transformation $\psi\colon S(H) \to S(H)$ defined by

$$\psi(A) = U^* \phi(A) U \ (A \in S(H)).$$

It is clear that ψ preserves $S_T(.\|.)$ and thus it satisfies (3.7). Moreover, ψ acts as the identity on $P_1(H)$. Let $A \in S(H)$ and $P \in P_1(H)$. Then, referring to (3.7) we obtain that

$$\operatorname{tr} \psi(A)^q P^{1-q} = \operatorname{tr} A^q P^{1-q},$$

where the left-hand side equals $\operatorname{tr} \psi(A)^q P$ and the right-hand side equals $\operatorname{tr} A^q P$. Since P was arbitrary, we infer that $\psi(A)^q = A^q$, i.e., $\psi(A) = A$ and this completes the proof in CASE 4. □

Now we prove the second theorem of this chapter.

Proof of Theorem 3.2. We can use arguments similar to the ones followed in the proofs of [**25**, Theorems 3,5]. Therefore, like above we present only the main steps. First observe that since the relative entropies (iii) and (iv) are definite, ϕ is injective. We extend ϕ onto $B(H)^+_{-1}$ by the formula

$$\psi(A) = (\operatorname{tr} A)\phi\left(\frac{1}{\operatorname{tr} A} A\right) \quad (A \in B(H)^+_{-1}).$$

One can check that the transformation $\psi\colon B(H)^+_{-1} \to B(H)^+_{-1}$ is bijective. Concerning the case of the Tsallis relative entropy, we see that

$$\operatorname{tr}\phi(A)^q\phi(B)^{1-q} = \operatorname{tr}A^qB^{1-q} \quad (A, B \in M(H)).$$

Easy computation shows that then one has

$$\operatorname{tr}\psi(A)^q\psi(B)^{1-q} = \operatorname{tr}A^qB^{1-q} \quad (A, B \in B(H)^+_{-1}). \tag{3.8}$$

As for the quadratic relative entropy, using (3.6) we see that

$$\operatorname{tr}\phi(A)^{-1}\phi(B)^2 = \operatorname{tr}A^{-1}B^2$$

for any $A, B \in M(H)$. By the definition of ψ, we deduce

$$\operatorname{tr}\psi(A)^{-1}\psi(B)^2 = \operatorname{tr}A^{-1}B^2 \quad (A, B \in B(H)^+_{-1}). \tag{3.9}$$

We proceed as follows. In the case of the Tsallis relative entropy, one can infer from (3.8) (using the bijectivity of ψ) that

$$A^q \leq B^q \iff \psi(A)^q \leq \psi(B)^q$$

holds for any $A, B \in B(H)^+_{-1}$. Next, defining the selfmap Ψ of $B(H)^+_{-1}$ by

$$\Psi(A) = \left(\psi\left(A^{\frac{1}{q}}\right)\right)^q \quad (A \in B(H)^+_{-1})$$

we see that Ψ is an order automorphism of $B(H)^+_{-1}$ meaning that it is bijective and

$$A \leq B \iff \Psi(A) \leq \Psi(B)$$

holds for any $A, B \in B(H)^+_{-1}$. We can do something similar in the case of the quadratic relative entropy. Indeed, from (3.9) one can deduce that

$$A^{-1} \leq B^{-1} \iff \psi(A)^{-1} \leq \psi(B)^{-1} \quad (A, B \in B(H)^+_{-1}).$$

This apparently gives us that ψ is an order automorphism of $B(H)^+_{-1}$. To sum up, just as in the proof of [**25**, Theorem 5], we end up with an order automorphism of $B(H)^+_{-1}$ regardless of which of the relative entropies (iii),(iv) is considered. The order automorphisms of the set of all invertible positive operators acting on a complex Hilbert space is characterized in [**25**, Theorem 1] as the transformations implemented by bijective bounded linear or conjugate-linear operators on the underlying Hilbert space. Namely, in the case of the Tsallis relative entropy, it gives us that Ψ is of the form

$$\Psi(A) = TAT^* \quad (A \in B(H)^+_{-1}),$$

where T is a bijective linear or conjugate-linear operator on H. Concerning ψ, this implies that

$$\psi(A) = (TA^qT^*)^{\frac{1}{q}} \quad (A \in B(H)^+_{-1}).$$

As for the quadratic relative entropy, we obtain that ψ is of the form

$$\psi(A) = TAT^* \quad (A \in B(H)^+_{-1}),$$

where T is just as above.

Now substitute these forms of ψ into the equations (3.8) and (3.9), respectively. We obtain in the first case that

$$\operatorname{tr} TA^qT^*(TB^qT^*)^{\frac{1}{q}-1} = \operatorname{tr} A^qB^{1-q}, \tag{3.10}$$

while in the second case we have

$$\operatorname{tr}(TAT^*)^{-1}(TBT^*)^2 = \operatorname{tr} A^{-1}B^2 \tag{3.11}$$

for all $A, B \in B(H)^+_{-1}$. Considering the equation (3.10), observe that the expressions on both sides are defined for all positive operators A, B on H and they are continuous functions of those variables. As for the equation (3.11), it is clear that the expressions on both sides are defined for any positive operator B on H and they are also continuous in B. Consequently, (3.10) holds also for all positive operators A, B on H and (3.11) is necessarily satisfied by any positive operators A, B on H, A being invertible. Pick an arbitrary $P \in P_1(H)$ and insert $A = B = P$ and $A = I, B = P$ in (3.10) and (3.11), respectively. In that way, using some elementary computations, one can conclude that T is an isometry and hence a unitary or an antiunitary operator on H. In the case of the quadratic relative entropy we are then done, while concerning the Tsallis relative entropy we have to argue a bit more. Namely, we see that

$$\psi(A) = (TA^qT^*)^{\frac{1}{q}} = T(A^q)^{\frac{1}{q}}T^* = TAT^*$$

holds for all $A \in B(H)^+_{-1}$. The statement of Theorem 3.2 follows. □

3.3. An open problem

We conclude this chapter with an unsolved problem. This is the general question concerning the characterization of non-surjective transformations on $M(H)$ preserving different kinds of relative entropies. We guess that their structure can be described (we suspect that they are implemented by unitary or antiunitary operators). But the arguments we developed in [**29**] and used also here (in the proof of Theorem 3.1) to handle non-surjective transformations on $S(H)$ can not be applied and hence a different approach should be worked out. We propose this question as an open problem for further research.

4

Isometries of positive operators and identification lemmas

4.1. Introduction and statement of the results

In the present chapter we consider certain kinds of metrics on spaces of positive operators. These distances are in an intimate connection with the p-norms. We describe the general form of the surjective isometries of the corresponding spaces. The key of the proof of certain theorems of this chapter is the application of so-called identification lemmas, the present section contains two of them. Some of the results of the chapter have been appeared in [**27, 34**], while the others are unpublished statements of the author.

The problem of describing the morphisms of a given structure appears in many parts of mathematics and plays a crucial role in most of the cases. The natural morphisms of metric spaces are the isometries, they have a vast literature. Concerning results on linear isometries of normed spaces we refer to the two volume set [**11, 12**]. In the case where a linear structure is not present, the problem of isometries becomes much more difficult. Such spaces are investigated in numerous areas of mathematics, e.g. in the Hilbert space formalism of quantum mechanics. Among the nonlinear structures appearing in quantum theory one of the most important is the set of density operators. On this set several metrics are studied, e.g. the Bures metric and those coming from the trace norm or the Hilbert-Schmidt norm which are special cases of the p-norms.

In what follows, we fix the notation used in this chapter. Let $1 \leq p \leq \infty$. We denote by d_p the metric induced by the p-norm. Let $C_p(H)_1^+$ signify the set of those elements of $C_p(H)^+$ whose p-norm is 1. The set $C_p(H)_1^+$ can be regarded as the positive part of the unit sphere in $C_p(H)$. Observe that $C_1(H)_1^+ = S(H)$. As usual, the symbol $\oplus$ stands for the direct sum of bounded linear operators on Hilbert spaces.

The theorem of Wigner on quantum mechanical symmetry transformations motivated the study of different sorts of "symmetries" on certain quantum structures. For example, in the paper [**30**] Molnár and Timmermann described the structure of surjective isometries of the spaces $S(H)$ and $C_1(H)^+$ with respect to d_1. In this section, motivated by these descriptions we present several results related to the ones in [**30**].

The key of the proof of some of these results is the application of particular identification lemmas. In the thesis, this term concerns the following type of assertions. Suppose we are given a metric space X and a subset E of X. Determine whether the below property holds. If $a, b \in X$ are such that for all $e \in E$ the distances from a, respectively b to e are the same, then $a = b$. This feature means that the elements of X are uniquely determined by their distances to the members of E, i.e., using the metric on X we can recover X from E. We call such assertions identification lemmas.

In our first identification result a parameter $\gamma \geq 1$ appears only for technical reasons (however, it makes its statement more general). It reads as follows.

Lemma 4.1. (Nagy [**34**])
Suppose that $\dim H < \infty$ *and let* $1 \leq p, \gamma < \infty$ *be fixed numbers. If* $A, B \in C_p(H)_1^+$ *are such that the equality*

$$d_p(A, \gamma P) = d_p(B, \gamma P)$$

holds for any $P \in P_1(H)$, *then* $A = B$.

We remark that in the case where $p = \gamma = 1$, Lemma 4.1 has a nice geometrical content. To see this, we recall the well-known fact that the extreme points of the set of the density operators acting on H are the elements of $P_1(H)$. Having this assertion in mind, the mentioned case of Lemma 4.1 can be interpreted as follows. Considering $S(H)$ as a metric space equipped with d_1, each density operator on H can be recovered from its distances to the extreme points of $S(H)$. The following lemma shows that this holds also for the metric which comes from the operator norm. These observations show that the identification results in this chapter are interesting on their own right.

Lemma 4.2. (Nagy)
Suppose that $n = \dim H < \infty$ *and that* $A, B \in S(H)$ *are such that*

$$d_\infty(A, P) = d_\infty(B, P)$$

for any $P \in P_1(H)$. *Then* $A = B$.

In what follows, we present those results of the present chapter which concern isometries on spaces of positive operators. As we have already mentioned, in [**30**] the authors investigated the isometries of $C_p(H)^+$ and $C_p(H)^+_1$ with respect to d_p only in the case $p = 1$. We can extend that investigation in two possible directions. One of them is the problem of isometries of $C_1(H)^+$ and $C_1(H)^+_1$ with respect to d_p for some $1 < p \le \infty$. So this extension is obtained from the original problem by changing only the metric. Changing also the underlying sets gives rise to the other one. Namely, it is a natural problem to investigate the isometries of $C_p(H)^+$ and $C_p(H)^+_1$ endowed with d_p $(1 < p \le \infty)$.

The following discussion is related to the first of the above extensions. In the next part of this chapter let $1 < p < \infty$. We remark that as it is well-known $C_1(H)^+ \subset C_1(H) \subset C_p(H)$, thus d_p is defined on $C_1(H)^+$. Now we formulate two results on the isometries of $(C_1(H)^+, d_p)$. The first one is a straightforward corollary of Theorem 4.5 below, therefore we omit its proof. It reads as follows. Any surjective isometry of $C_1(H)^+$ with respect to d_p is of the form $A \mapsto UAU^*$ $(A \in C_1(H)^+)$ with some unitary-antiunitary operator U on H. Observe that if $\dim H < \infty$, then $C_1(H) = C_p(H)$, therefore we see that Theorem 4.6 describes the structure of the isometries of $(C_1(H)^+, d_p)$ in the case where H is finite dimensional.

As for $C_1(H)^+_1 = S(H)$, we recall that the result in [**30**] on the isometries of this space endowed with d_1 concerns surjective transformations. In our related theorem we consider the non-surjective isometries of $S(H)$ with respect to d_p. We remark that each transformation of the form $A \mapsto VAV^*$ $(A \in S(H))$ with some linear or conjugate-linear isometry V of H is an isometry of $S(H)$ with respect to d_p. The assertion below shows that the reverse statement is also true, all distance preserving maps of $(S(H), d_p)$ must be of that form. The case $\dim H < \infty$ of this result appeared as part of Theorem 1 in the author's paper [**27**] joint with Molnár.

Theorem 4.3. (Molnár, Nagy)
Let $1 < p < \infty$ and suppose that $\phi\colon S(H) \to S(H)$ is an isometry with respect to d_p. Then there exists either a linear or a conjugate-linear isometry V of H such that ϕ is of the form

$$\phi(A) = VAV^* \quad (A \in S(H)).$$

In the finite dimensional case Theorem 4.3 extends to the isometries of $S(H)$ with respect to d_∞, too. Before presenting the corresponding result we mention an important feature of the p-norms. Let $1 \le p \le \infty$. It is known that for any $A \in C_p(H)$ and (anti)unitary operators U, V on

H one has $UAV \in C_p(H)$ and $\|UAV\|_p = \|A\|_p$. We deduce that the p-norms are unitarily invariant and that the transformations of the form $A \mapsto UAU^*$ ($A \in S(H)$) with a unitary or an antiunitary operator U on H are isometries of $S(H)$ with respect to d_∞. The following theorem tells us that there are no other kinds of distance preserving maps of $(S(H), d_\infty)$ provided that $\dim H < \infty$.

Theorem 4.4. (Nagy [**34**])
Assume that $\dim H < \infty$ *and that* $\phi\colon S(H) \to S(H)$ *is an isometry with respect to* d_∞*. Then there exists either a unitary or an antiunitary operator* U *on* H *such that* ϕ *is of the form*

$$\phi(A) = UAU^* \quad (A \in S(H)). \tag{4.1}$$

Above we have mentioned two extensions of an investigation and the next part of this section is devoted to the second one. Namely, the following discussion is related to the isometries of $C_p(H)^+$ and $C_p(H)_1^+$ with respect to d_p ($1 < p < \infty$). We start with the corresponding discussion about $C_p(H)^+$. Let $1 < p < \infty$. It is well-known that in this case $(C_p(H), \|.\|_p)$ is strictly convex (see, e.g. [**19**, Theorem 2.4.]). Therefore, as we shall see, the convexity of $C_p(H)^+$ will be a great advantage in examining the isometries of $C_p(H)^+$. Turning to the corresponding results, we present the general form of the surjective isometries of the set of all positive operators in $C_p(H)$ endowed with d_p in the statements below.

Theorem 4.5. (Nagy)
Let $1 < p < \infty$ *and suppose that* $\phi\colon C_p(H)^+ \to C_p(H)^+$ *is a surjective isometry with respect to* d_p*. Then we have either a unitary or an antiunitary operator* U *on* H *such that* ϕ *is of the form*

$$\phi(A) = UAU^* \quad (A \in C_p(H)^+).$$

As for the finite dimensional case, the following assertion describes the structure of the isometries of the space of complex positive semidefinite matrices equipped with d_p.

Theorem 4.6. (Nagy)
Suppose that $\dim H < \infty$ *and let* $1 < p < \infty$*. Moreover, assume that* $\phi\colon C_p(H)^+ \to C_p(H)^+$ *is an isometry with respect to* d_p*. Then we have a positive operator* $X \in B(H)$ *and either a unitary or an antiunitary operator* U *on* H *such that* ϕ *is of the form*

$$\phi(A) = UAU^* + X \quad (A \in C_p(H)^+).$$

As for the case $p = 2$ of Theorem 4.5, it is well-known that $C_2(H)$ is a complex Hilbert space endowed with the Hilbert-Schmidt inner product $\langle .,. \rangle_{\mathrm{HS}}$ which is defined by

$$\langle A, B\rangle_{\mathrm{HS}} = \mathrm{tr}\, AB^* \quad (A, B \in C_2(H)).$$

Moreover, the norm induced by $\langle .,. \rangle_{\mathrm{HS}}$ is clearly $\|.\|_2$. By the previous facts, there are a bulk of surjective isometries of $C_2(H)$. However, Theorem 4.5 tells us that all such transformations of the smaller space $C_2(H)^+$ have a very special form.

We proceed with the description of the isometries of $(C_p(H)_1^+, d_p)$ (in this paragraph $1 < p < \infty$). In the case where the domain of isometries is $C_p(H)^+$ the corresponding problem has been examined previously. As for the present case, the fact that – unlike the latter set – $C_p(H)_1^+$ is not convex makes the problem much more complicated. In our below result we determine the structure of the surjective isometries of $(C_p(H)_1^+, d_p)$.

Theorem 4.7. (Nagy [**34**])
Let $1 < p < \infty$ and suppose that $\phi\colon C_p(H)_1^+ \to C_p(H)_1^+$ is a surjective isometry with respect to d_p. Then we have either a unitary or an antiunitary operator U on H such that ϕ is of the form

$$\phi(A) = UAU^* \quad (A \in C_p(H)_1^+). \tag{4.2}$$

This result together with the corresponding one in [**30**] yields that for any $1 \le p < \infty$ the surjective isometries of $(C_p(H)_1^+, d_p)$ can be written in the form (4.2). As for the finite dimensional case of the previous theorem, it follows from the last part of the proof of Theorem 4.6 that in this case any isometry of $(C_p(H)_1^+, d_p)$ $(1 < p < \infty)$ is of the form (4.2).

We conclude this section with an application of Theorem 4.7. In [**17**, Definition 2.] Ma, Zhang and Chen introduced certain metrics ρ_p by slightly modifying the functions d_p $(1 \le p < \infty)$. The previous distances are given by the formula

$$\rho_p(A, B) = d_p\left(A^{\frac{1}{p}}, B^{\frac{1}{p}}\right) \quad (1 \le p < \infty,\ A, B \in S(H)).$$

Observe that $\rho_1 = d_1$. Theorem 4.7 has the following consequence concerning the isometries of $(S(H), \rho_p)$ $(1 < p)$.

Corollary 4.8. (Nagy)
Let $1 < p < \infty$ and suppose that $\phi\colon S(H) \to S(H)$ is a surjective isometry with respect to ρ_p. Then we have either a unitary or an antiunitary operator U on H such that ϕ is of the form (4.1).

4.2. Proofs

Throughout this section we shall use the following notation and assertions. In the case where H is finite dimensional denote by n the dimension of H. Suppose that $\dim H < \infty$ and let $T\colon H \to H$ be a self-adjoint operator (or equivalently an $n \times n$ complex Hermitian matrix). The symbol $(\lambda_i(T))_{i=1}^n$ signifies the increasing sequence of the eigenvalues of T (counted according to their multiplicities) and $\mathcal{M}_T$ stands for the eigensubspace of T corresponding to $\lambda_n(T)$.

The simple observation below can be proved by the Cauchy-Schwarz inequality.

(*) Assume that H is finite dimensional. Then for any unit vector $x \in H$ we have $\langle Ax, x\rangle \leq \lambda_n(A)$ and equality holds if and only if x is an eigenvector of A corresponding to $\lambda_n(A)$.

The following assertion is similar to (*) in Section 3.2.

(**) Suppose that $\dim H < \infty$ and $\mathcal{M} \subset B_s(H)$ is a set of nonzero operators such that its rank-one elements are exactly the members of $P_1(H)$. Then for any $A \in \mathcal{M}$ we have that A is a rank-one projection if and only if there are n pairwise orthogonal elements of $\mathcal{M}$ such that one of them is A.

We remark that it is a straightforward consequence of the definition of p-classes that any finite rank operator in $B(H)$ belongs to $C_p(H)$ $(1 \leq p \leq \infty)$.

In this section, we will use the following lemma several times.

Lemma 4.9. *If $1 \leq p \leq \infty$ and ϕ is an isometry of $(P_1(H), d_p)$, then*

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \quad (P, Q \in P_1(H)).$$

Proof. First, we define a function $f_p\colon [0,1] \to \mathbb{R}$ in the following way. For any $p < \infty$ let $f_p(x) = 2^{1/p}\sqrt{1-x}$ and let $f_\infty(x) = \sqrt{1-x}$ $(x \in [0,1])$. We assert that $d_p(P, Q) = f_p(\operatorname{tr} PQ)$ $(P, Q \in P_1(H))$. Since this claim clearly holds in the case where $P = Q$, we assume that P and Q are different. Let $x \in \operatorname{rng} P$ and $y \in \operatorname{rng} Q$ be unit vectors. It is apparent that x and y generate $\operatorname{rng}(P - Q)$ and they form a basis in it. It is easy to check that, with respect to the basis $\{x, y\}$, the matrix of the restriction of $P - Q$ onto its range is

$$\begin{pmatrix} 1 & \langle y, x\rangle \\ -\langle x, y\rangle & -1 \end{pmatrix}.$$

Elementary computation gives that the eigenvalues of this matrix are $-\sqrt{1 - \operatorname{tr} PQ}$ and $\sqrt{1 - \operatorname{tr} PQ}$ (observe that $\operatorname{tr} PQ = |\langle x, y\rangle|^2$). This implies that the nonzero elements of the spectrum of $P - Q$ are exactly these values and now the desired assertion can be deduced from the equality (1.3).

Let $P, Q \in P_1(H)$. On the one hand, by the previous paragraph it is clear that $f_p(\operatorname{tr} \phi(P)\phi(Q)) = f_p(\operatorname{tr} PQ)$. On the other hand, observe that f_p is injective and then the statement of Lemma 4.9 follows. □

We now turn to the proof of our first identification result.

Proof of Lemma 4.1. Apparently, in the following argument we may and do suppose that $n > 1$. Let $T \in C_p(H)_1^+$ and define the function $f_T : P_1(H) \to \mathbb{R}$ by

$$f_T(P) = d_p(T, \gamma P)^p \quad (P \in P_1(H)).$$

We are going to show that f_T uniquely determines $\lambda_n(T)$ and $\mathcal{M}_T$. To do this, we assert that

$$\min f_T(P_1(H)) = (\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p$$

and f_T attains its minimum exactly for those rank-one projections on H which project into $\mathcal{M}_T$. For the proof, first observe that since $\|T\|_p^p = \sum_{i=1}^n \lambda_i(T)^p = 1$, we have

$$f_T(P) = (\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p \quad (P \in P_1(H),\ \operatorname{rng} P \subset \mathcal{M}_T). \tag{4.3}$$

We have to prove that $(\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p$ is a lower bound of the range of f_T. To this end, let $P \in P_1(H)$. We learn from [**4**, Theorem 9.7] that if $\|.\|$ is a unitarily invariant norm on the space of $n \times n$ complex matrices and R, S are Hermitian matrices, then

$$\| \operatorname{diag}(\lambda_n(R), \dots, \lambda_1(R)) - \operatorname{diag}(\lambda_n(S), \dots, \lambda_1(S))\| \leq \|R - S\|,$$

where diag(.) denotes the diagonal matrix whose diagonal is the given sequence. We have seen in Section 4.1. that the p-norm is unitarily invariant, thus the previous inequality yields

$$(\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p = \sum_{i=1}^n |\lambda_i(T) - \lambda_i(\gamma P)|^p \leq \|T - \gamma P\|_p^p = f_T(P).$$

This together with equality (4.3) implies that

$$\min f_T(P_1(H)) = (\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p \tag{4.4}$$

and f_T attains its minimum for any element of $P_1(H)$ which projects into $\mathcal{M}_T$.

In what follows let P be a rank-one projection on H which minimizes f_T. We have to show that P projects into $\mathcal{M}_T$. To do this, first we give a lower bound for $f_T(P)$. According to [**19**, Lemma 2.2.], for any linear operator K on H and orthonormal basis $\{\varphi_i\}_{i=1}^n$ in H we have $\sum_{i=1}^n |\langle K\varphi_i, \varphi_i\rangle|^p \le \|K\|_p^p$. Let x be a unit vector in rng P and choose an orthonormal basis $\{\varphi_i\}_{i=1}^n$ in H such that $\varphi_1 = x$. Apply the preceding assertion for this basis and for the operator $K = T - \gamma P$ in order to get

$$(\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p = f_T(P) = \|T - \gamma P\|_p^p \ge |\langle Tx, x\rangle - \gamma|^p + \sum_{i=2}^n \langle T\varphi_i, \varphi_i\rangle^p. \tag{4.5}$$

Since T is positive, by (*) one has $\langle Tx, x\rangle \le \lambda_n(T) \le 1 \le \gamma$. Referring to (4.5) we thus get

$$\sum_{i=2}^n \langle T\varphi_i, \varphi_i\rangle^p \le 1 - \lambda_n(T)^p. \tag{4.6}$$

We emphasize that this relation is valid for all orthonormal bases $\{\varphi_i\}_{i=2}^n$ in $(\text{rng}\, P)^\perp$.

Now define $Q = P^\perp$ and $\mathcal{L} = \text{rng}\, Q$. It is clear that

$$\sum_{i=2}^n \langle T\varphi_i, \varphi_i\rangle^p = \sum_{i=2}^n \langle QTQ\varphi_i, \varphi_i\rangle^p$$

for all orthonormal bases $\{\varphi_i\}_{i=2}^n \subset \mathcal{L}$. It is obvious that there is an orthonormal basis in $\mathcal{L}$ consisting of eigenvectors of $(QTQ)|_{\mathcal{L}}$. Inserting such a basis in (4.6) and using the last displayed equality we obtain

$$1 - \lambda_n(T)^p \ge \text{tr}\,((QTQ)|_{\mathcal{L}})^p\,. \tag{4.7}$$

It is clear that $(QTQ)^p$ is 0 on $\mathcal{L}^\perp$. Therefore, using the fact that the p-norm is invariant under taking adjoints (see [**19**, Theorem 1.3.]), we get that

$$\begin{aligned} \text{tr}\,((QTQ)|_{\mathcal{L}})^p &= \text{tr}(QTQ)^p|_{\mathcal{L}} = \text{tr}(QTQ)^p \\ &= \left\| \sqrt{T}Q \right\|_{2p}^{2p} = \left\| Q\sqrt{T} \right\|_{2p}^{2p} = \text{tr}\left(\sqrt{T}Q\sqrt{T}\right)^p \\ &= \text{tr}\left(T - \sqrt{T}P\sqrt{T}\right)^p = \|T - \sqrt{T}P\sqrt{T}\|_p^p. \end{aligned} \tag{4.8}$$

Let $N = \sqrt{T}P\sqrt{T}$. By [**4**, Theorem 9.7], (4.8) and (4.7), we infer

$$\sum_{i=1}^n |\lambda_i(T) - \lambda_i(N)|^p \le \|T - N\|_p^p \le 1 - \lambda_n(T)^p. \tag{4.9}$$

Observe that N is an element of $P_1(H)$ multiplied by a scalar which is easily seen to be $\operatorname{tr} N$. Hence

$$\lambda_n(N) = \operatorname{tr} N = \langle Tx, x\rangle$$

and by (4.9) it follows that

$$|\lambda_n(T) - \langle Tx, x\rangle|^p + 1 - \lambda_n(T)^p \le 1 - \lambda_n(T)^p.$$

Now we deduce $\langle Tx, x\rangle = \lambda_n(T)$. Using (*), we conclude that x is an eigenvector of T corresponding to $\lambda_n(T)$, which means that P projects into $\mathcal{M}_T$.

To sum up, we obtain that f_T attains its minimum exactly for those elements of $P_1(H)$ which project into $\mathcal{M}_T$. Observe that any subspace of H is completely determined by the set of the rank-one projections on H with range contained in it. The previous observations yield that f_T uniquely determines $\mathcal{M}_T$. We assert that from f_T the number $\lambda_n(T)$ can also be recovered. Indeed, it is clear that the function $\lambda \mapsto (\gamma-\lambda)^p + 1 - \lambda^p$ $(\lambda \in]0,1])$ is strictly decreasing and thus injective. Hence, from the value $(\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p$ we can recover $\lambda_n(T)$ and then by (4.4) it follows that f_T uniquely determines also $\lambda_n(T)$.

To complete the proof we use induction on $\dim H$. It is obvious that Lemma 4.1 holds for 1-dimensional complex Hilbert spaces. Suppose now that it is valid for any space of dimension at most $n-1$ and that A, B are operators satisfying the conditions of Lemma 4.1. By what we have proved so far, we see that the maximal eigenvalues of A and B are the same and this holds also for the corresponding eigensubspaces $\mathcal{M}_A, \mathcal{M}_B$. This yields that $A|_{\mathcal{M}_A} = B|_{\mathcal{M}_A}$. If $\mathcal{M}_A = \mathcal{M}_B = H$ or $\lambda_n(A) = \lambda_n(B) = 1$, then both of A and B is 0 on $\tilde{\mathcal{M}} = \mathcal{M}_A^{\perp}$, therefore $A = B$. Otherwise, we easily infer

$$d_p\left(A|_{\tilde{\mathcal{M}}}, \gamma P|_{\tilde{\mathcal{M}}}\right) = d_p\left(B|_{\tilde{\mathcal{M}}}, \gamma P|_{\tilde{\mathcal{M}}}\right) \quad (P \in P_1(H),\ \operatorname{rng} P \subset \tilde{\mathcal{M}}).$$

It is obvious that $0 < \left\|A|_{\tilde{\mathcal{M}}}\right\|_p = \left\|B|_{\tilde{\mathcal{M}}}\right\|_p \le 1$, and the last displayed equality implies that for every $P \in P_1(H)$ projecting into $\tilde{\mathcal{M}}$ we have

$$d_p\left(\frac{1}{\left\|A|_{\tilde{\mathcal{M}}}\right\|_p} A|_{\tilde{\mathcal{M}}}, \frac{\gamma}{\left\|A|_{\tilde{\mathcal{M}}}\right\|_p} P|_{\tilde{\mathcal{M}}}\right) = d_p\left(\frac{1}{\left\|A|_{\tilde{\mathcal{M}}}\right\|_p} B|_{\tilde{\mathcal{M}}}, \frac{\gamma}{\left\|A|_{\tilde{\mathcal{M}}}\right\|_p} P|_{\tilde{\mathcal{M}}}\right).$$

Clearly, $\gamma / \left\|A|_{\tilde{\mathcal{M}}}\right\|_p \ge 1$, and

$$\frac{1}{\left\|A|_{\tilde{\mathcal{M}}}\right\|_p} A|_{\tilde{\mathcal{M}}}, \frac{1}{\left\|A|_{\tilde{\mathcal{M}}}\right\|_p} B|_{\tilde{\mathcal{M}}} \in C_p\left(\tilde{\mathcal{M}}\right)_1^+ .$$

Moreover, it is trivial that when P runs through the set of all rank-one projections on H projecting into $\tilde{\mathcal{M}}$, then $P|_{\tilde{\mathcal{M}}}$ runs through the set $P_1\left(\tilde{\mathcal{M}}\right)$. Thus, by the inductive hypothesis it follows that

$$\left(1/\left\|A|_{\tilde{\mathcal{M}}}\right\|_p\right)A|_{\tilde{\mathcal{M}}} = \left(1/\left\|A|_{\tilde{\mathcal{M}}}\right\|_p\right)B|_{\tilde{\mathcal{M}}}$$

and this together with the previous argument yields $A = B$. The proof of Lemma 4.1 is complete. □

We proceed with the proof of the second lemma of the present chapter.

Proof of Lemma 4.2. It is clear that we may and do assume that $n > 1$. Let T be a positive operator on H such that $\|T\|_1 \leq 1$ and define

$$f_T(P) = d_\infty(T, P) \quad (P \in P_1(H)).$$

It is clear that to prove Lemma 4.2, it is enough to show that f_T uniquely determines T. To do this, first we claim that $\lambda_n(T)$ and $\mathcal{M}_T$ can be recovered from f_T. This will be verified once we have proved that

$$\min f_T(P_1(H)) = 1 - \lambda_n(T) \tag{4.10}$$

and f_T assumes this value exactly for the projections P projecting into $\mathcal{M}_T$. To show this, observe that since $\sum_{i=1}^n \lambda_i(T) \leq 1$, we have

$$f_T(P) = 1 - \lambda_n(T) \quad (P \in P_1(H),\ \operatorname{rng} P \subset \mathcal{M}_T). \tag{4.11}$$

Pick an element $P \in P_1(H)$. Since $\|.\|_\infty$ is unitarily invariant, referring to [**4**, Theorem 9.7], we deduce that

$$1 - \lambda_n(T) \leq \|T - P\|_\infty = f_T(P)$$

and using (4.11) this gives us that $\min f_T(P_1(H)) = 1 - \lambda_n(T)$ and f_T attains this value for any rank-one projection on H whose range is in $\mathcal{M}_T$. We proceed with choosing an element P of $P_1(H)$ which minimizes f_T. Let $x \in \operatorname{rng} P$ be a unit vector. Since the operator norm of a normal operator coincides with its numerical radius we infer

$$1 - \lambda_n(T) = \|T - P\|_\infty \geq |\langle (T - P)x, x\rangle| = 1 - \langle Tx, x\rangle,$$

which implies $\langle Tx, x\rangle \geq \lambda_n(T)$. By (*), it follows that $\langle Tx, x\rangle = \lambda_n(T)$, and hence x is an eigenvector of T corresponding to $\lambda_n(T)$. Finally, we can conclude that P projects into $\mathcal{M}_T$ and thus we have proved the assertion concerning the minimum of f_T.

Now there are two cases. If $\mathcal{M}_T = H$, then it is obvious that $T = \lambda_n(T)I$ and thus we have that f_T uniquely determines T. Otherwise we proceed as follows. Define $\mathcal{L}_T = \mathcal{M}_T^\perp$ and denote by $\tilde{T}$ the restriction of T onto its invariant subspace $\mathcal{L}_T$. Let $P \in P_1(H)$ be such that $\operatorname{rng} P \subset \mathcal{L}_T$. We compute

$$f_T(P) = \max\left\{\lambda_n(T), \|\tilde{T} - P|_{\mathcal{L}_T}\|_\infty\right\}.$$

Let $k = \dim \mathcal{M}_T$. It is clear that $\lambda_i(\tilde{T}) = \lambda_i(T)$ $(i = 1, ..., n-k)$ and the eigensubspaces of $\tilde{T}$ and T corresponding to this common eigenvalue are the same. Therefore applying (4.10) to $\tilde{T}$ gives us that

$$\|\tilde{T} - P|_{\mathcal{L}_T}\|_\infty \geq 1 - \lambda_{n-k}(T) \geq \lambda_n(T),$$

thus

$$f_T(P) = \|\tilde{T} - P|_{\mathcal{L}_T}\|_\infty.$$

This implies that

$$f_{\tilde{T}}(P) = f_T(0 \oplus P) \quad (P \in P_1(\mathcal{L}_T)),$$

where 0 is the zero operator on $\mathcal{M}_T$. Denote by $\mathcal{N}_T$ the eigensubspace of T corresponding to $\lambda_{n-k}(T)$. By the first paragraph, we easily get that from $f_{\tilde{T}}$ we can recover $\lambda_{n-k}(T)$ and $\mathcal{N}_T$. We infer that f_T uniquely determines $\lambda_{n-k}(T)$ and $\mathcal{N}_T$. Continuing this procedure, we deduce that from f_T we can recover $\lambda_i(T)$ $(i = 1, ..., n)$ and the corresponding eigensubspace of T. This yields that f_T uniquely determines T and now the proof of Lemma 4.2 is complete. □

In what follows, we present the proof of the first theorem of the chapter.

Proof of Theorem 4.3. As an initial step, we show that ϕ is affine. To do this, we need a metric characterization of the convex combinations of density operators. It reads as follows. Let $\lambda \in]0,1[$ and $A, B \in S(H)$ be such that $A \neq B$. Then for any $T \in S(H)$ we have $T = \lambda A + (1-\lambda)B$ if and only if $d_p(A,T) = (1-\lambda)d_p(A,B)$ and $d_p(B,T) = \lambda d_p(A,B)$. To see this, suppose that T has the latter properties. Then

$$\|A - T\|_p + \|T - B\|_p = \|A - B\|_p = \|(A - T) + (T - B)\|_p,$$

and, since $C_p(H)$ is strictly convex, this implies that there is a positive real number α such that $A - T = \alpha(T - B)$. Easy calculation shows that

$$T = \frac{1}{\alpha + 1}A + \frac{\alpha}{\alpha + 1}B \text{ and } \alpha = \frac{1}{\lambda} - 1$$

and thus it follows that $T = \lambda A + (1-\lambda)B$. The reverse implication is trivial, therefore we conclude that the desired characterization holds. By the isometry property of ϕ and the previous assertion we deduce that ϕ is affine, i.e., for any $\lambda \in [0,1]$ and for any pair $A, B \in S(H)$ one has

$$\phi(\lambda A + (1-\lambda)B) = \lambda\phi(A) + (1-\lambda)\phi(B).$$

(We remark that if ϕ were assumed to be surjective, already from the affinity of ϕ we could infer that it is of the desired form, see Theorem 1.4. Since here we consider non-surjective maps, we have to work more.)

In the next step we show that ϕ preserves the rank-one projections and the transition probability between them. To verify this, we assert that for any $A, B \in S(H)$ one has

$$\|A - B\|_p = 2^{\frac{1}{p}} \iff A, B \in P_1(H) \text{ and } AB = 0.$$

The implication $\Longleftarrow$ is easy to see. To prove the reverse implication, let $\{e_i\}_{i\in I}$ be an orthonormal basis in H consisting of eigenvectors of $A - B$. We learn from [**19**, Lemma 2.1] that for any positive operator T on H and unit vector $x \in H$ we have $\langle Tx, x\rangle^p \le \langle T^p x, x\rangle$ and equality holds if and only if x is an eigenvector of T. Using this assertion we obtain

$$\begin{aligned} &|\langle Ae_i, e_i\rangle - \langle Be_i, e_i\rangle|^p \le \max\{\langle Ae_i, e_i\rangle^p, \langle Be_i, e_i\rangle^p\} \\ &\le \langle Ae_i, e_i\rangle^p + \langle Be_i, e_i\rangle^p \le \langle A^p e_i, e_i\rangle + \langle B^p e_i, e_i\rangle \quad (i \in I). \end{aligned} \tag{4.12}$$

Then we compute

$$\begin{aligned} 2 = \|A - B\|_p^p &= \sum_{i\in I} |\langle (A - B)e_i, e_i\rangle|^p \\ \le \sum_{i\in I}\langle A^p e_i, e_i\rangle + \sum_{i\in I}\langle B^p e_i, e_i\rangle &= \operatorname{tr} A^p + \operatorname{tr} B^p \le \operatorname{tr} A + \operatorname{tr} B = 2. \end{aligned} \tag{4.13}$$

This yields that in (4.13) and thus in (4.12) we have equalities everywhere. Referring to [**19**, Lemma 2.1] it follows that the e_i's ($i \in I$) are common eigenvectors of A and B. Let λ_i, resp. μ_i be the eigenvalue of A, resp. B corresponding to e_i ($i \in I$). We deduce from (4.13) that $\operatorname{tr} A^p = \operatorname{tr} A = 1$ and $\operatorname{tr} B^p = \operatorname{tr} B = 1$. Hence, there are indices $i_1, i_2 \in I$ such that $\lambda_{i_1} = \mu_{i_2} = 1$ and $\lambda_i = \mu_j = 0$ ($i \in I \setminus \{i_1\},\ j \in I \setminus \{i_2\}$). It implies that $A, B \in P_1(H)$. We clearly have $A \ne B$ and then it follows that for any $i \in I$ at least one of the numbers λ_i and μ_i is 0. We infer that $AB = 0$ and this completes the proof of the reverse implication.

The above verified equivalence yields that ϕ preserves the rank-one projections on H, i.e., $\phi(P) \in P_1(H)$ holds for any $P \in P_1(H)$. Since ϕ is an isometry, by Lemma 4.9 it follows that ϕ preserves the transition probability between the elements of $P_1(H)$. Thus applying Theorem 1.3 to $\phi|_{P_1(H)}$, we obtain that there is either a linear or a conjugate-linear isometry V on H such that

$$\phi(P) = VPV^* \quad (P \in P_1(H)).$$

Denote by $S_f(H)$ the set of the finite rank elements of $S(H)$. Since the operators in $S_f(H)$ are convex combinations of elements of $P_1(H)$ and ϕ is affine, it follows that $\phi(A) = VAV^*$ $(A \in S_f(H))$. According to [**19**, Lemma 5.2.], the collection of all finite rank operators in $B(H)$ is dense in $C_1(H)$ and this implies that the closure of $S_f(H)$ with respect to the topology induced by d_1 is $S(H)$. It is easy to check that $d_p(A, B) \leq d_1(A, B)$ $(A, B \in S(H))$, and it then follows that $S_f(H)$ is dense in $S(H)$ with respect to d_p. By the continuity of ϕ and the previous assertions, we deduce that it is of the form appearing in Theorem 4.3 and this completes the proof. □

The verification of the next result of Section 4.1. reads as follows.

Proof of Theorem 4.4. Pick operators $A, B \in S(H)$. We assert that $d_\infty(A, B) = 1$ if and only if $AB = 0$ and at least one of A and B belongs to $P_1(H)$. In fact, if A and B are orthogonal such that one of them is a rank-one projection, then it is easy to see that their distance with respect to d_∞ is 1. As for the reverse, suppose that the distance from A to B is 1. It is clear that there is a unit vector $x_0 \in H$ such that

$$d_\infty(A, B) = \sup\{|\langle (A - B)x, x\rangle| : x \in H,\ \|x\| = 1\} = |\langle (A - B)x_0, x_0\rangle| = |\langle Ax_0, x_0\rangle - \langle Bx_0, x_0\rangle|.$$

Since the numerical range of density operators lies in $[0, 1]$, it follows that

$$1 = d_\infty(A, B) = |\langle Ax_0, x_0\rangle - \langle Bx_0, x_0\rangle| \leq 1.$$

This yields $|\langle Ax_0, x_0\rangle - \langle Bx_0, x_0\rangle| = 1$, which clearly implies that one of the latter inner products, say $\langle Ax_0, x_0\rangle$ is 1 and the other is 0. This assertion has consequences for both of A and B. Concerning B it implies that $Bx_0 = 0$ and that, denoting by $\mathcal{M}$ the subspace generated by x_0, the space $\mathcal{M}^\perp$ is invariant under B. Concerning A, by (*), it follows that $1 = \langle Ax_0, x_0\rangle \leq \lambda_n(A) \leq 1$, therefore $\lambda_n(A) = 1$ and x_0 is an eigenvector of A corresponding to $\lambda_n(A)$. These facts imply $\lambda_i(A) = 0$ $(i = 1, ..., n - 1)$ which means that $A \in P_1(H)$. Observe that on the one hand B is 0 on $\mathcal{M}$ which yields that the same holds for AB, and on the other hand A is 0 on $\mathcal{M}^\perp$, hence $(AB)|_{\mathcal{M}^\perp} = 0$. Now we conclude that $AB = 0$ and thus we have proved the desired equivalence.

We proceed with choosing an element $P \in P_1(H)$. It is clear that there are n pairwise orthogonal operators in $P_1(H)$ such that one of them is P. Using the above characterization, we obtain that $\phi(P)$ can be included in a set of n pairwise orthogonal density operators. By (**) it follows that $\phi(P) \in P_1(H)$. Hence we deduce that ϕ leaves the set $P_1(H)$ invariant. Lemma 4.9 yields that $\phi|_{P_1(H)} : P_1(H) \to P_1(H)$ preserves the transition probability and then by Theorem 1.3 we get that

$$\phi(P) = UPU^* \quad (P \in P_1(H))$$

with some unitary-antiunitary operator U on H. Let $A \in S(H)$. We infer that

$$d_\infty(U^*\phi(A)U, P) = d_\infty(\phi(A), UPU^*) = d_\infty(A, P)$$

holds for any $P \in P_1(H)$. By Lemma 4.2, we obtain that $U^*\phi(A)U = A$ and this completes the proof of Theorem 4.4. □

In what follows, we prove the fifth theorem of Chapter 4.

Proof of Theorem 4.7. First we present a characterization of orthogonality of operators in terms of their distance. Namely, let $A, B \in C_p(H)_1^+$. Then we have

$$AB = 0 \iff d_p(A, B) = 2^{\frac{1}{p}}. \tag{4.14}$$

In the proof of Theorem 4.3 a similar equivalence has been verified. It is easy to see that the majority of the corresponding argument is valid in the present case as well. Namely, if $d_p(A, B) = 2^{1/p}$, then there is an orthonormal basis $\{e_i\}_{i \in I}$ in H whose members are common eigenvectors of A and B, and $|\lambda_i - \mu_i|^p = \max\{\lambda_i^p, \mu_i^p\}$, where $\lambda_i = \langle Ae_i, e_i\rangle$, respectively $\mu_i = \langle Be_i, e_i\rangle$ is the eigenvalue of A, respectively B corresponding to e_i ($i \in I$). It follows that for any $i \in I$ at least one of the numbers λ_i and μ_i is 0 and hence $AB = 0$. Conversely, if the latter equality holds, then it is easy to check that $d_p(A, B) = 2^{1/p}$. Now we conclude that ϕ preserves orthogonality.

It is clear that ϕ is bijective. Similarly to the second paragraph of the proof of Theorem 2.5 we infer that ϕ preserves the elements of $P_1(H)$ in both directions and then it follows that $\phi|_{P_1(H)} \colon P_1(H) \to P_1(H)$ is bijective. Referring to Lemma 4.9, we obtain that the latter transformation leaves the transition probability invariant. Applying Theorem 1.2, we conclude that there exists either a unitary or an antiunitary operator U on H such that

$$\phi(P) = UPU^* \quad (P \in P_1(H)).$$

Let $A \in C_p(H)_1^+$ be an operator of finite rank. Referring to the preceding paragraph, we deduce that for any $P \in P_1(H)$ the equality $d_p(\phi(A), UPU^*) = d_p(A, P)$ holds true. Since the p-norm is unitarily invariant, it follows that one has $d_p(U^*\phi(A)U, P) = d_p(A, P)$ for each $P \in P_1(H)$. Using the equivalence (4.14), this yields that a rank-one projection on H is orthogonal to $\psi(A) = U^*\phi(A)U$ if and only if it is orthogonal to A. We infer that $\operatorname{rng}\psi(A) = \operatorname{rng} A$. It is clear that when P runs through the set of those elements in $P_1(H)$ which project into $\operatorname{rng} A$, the operator $P|_{\operatorname{rng} A}$ runs through

$P_1(\operatorname{rng} A)$. The previous observations imply that for any $P \in P_1(\operatorname{rng} A)$ the equality $d_p\left(\psi(A)|_{\operatorname{rng} A}, P\right) = d_p\left(A|_{\operatorname{rng} A}, P\right)$ holds. Then Lemma 4.1 applies and we obtain that $\psi(A)|_{\operatorname{rng} A} = A|_{\operatorname{rng} A}$ and hence $\phi(A) = UAU^*$.

To complete the proof let $A \in C_p(H)_1^+$ be arbitrary. The assertion in [**19**, Lemma 5.2.] yields that the collection of the finite rank elements in $C_p(H)_1^+$ is a dense subset of $C_p(H)_1^+$. Since ϕ is continuous, it then follows by the preceding paragraph that $\phi(A) = UAU^*$ and thus the proof of Theorem 4.7 is complete. □

Now we are in a position to verify the corollary in the present chapter.

Proof of Corollary 4.8. Define the map $\psi\colon C_p(H)_1^+ \to B(H)$ by

$$\psi(A) = \phi(A^p)^{\frac{1}{p}} \quad (A \in C_p(H)_1^+).$$

It is easy to check that ψ is a well-defined surjective isometry of $(C_p(H)_1^+, d_p)$. Therefore we can apply Theorem 4.7 to ψ which gives us that $\psi(A) = UAU^*$ $(A \in C_p(H)_1^+)$, where U is a unitary-antiunitary operator on H. Hence, we get that

$$\phi(A) = \left(UA^{\frac{1}{p}}U^*\right)^p = UAU^* \quad (A \in S(H))$$

and this completes the proof of the corollary in this chapter. □

Using Theorem 4.7 we can prove the third theorem of this chapter very easily.

Proof of Theorem 4.5. In the same way as in the first paragraph of the proof of Theorem 4.3 we obtain that ϕ is affine. It is easy to see that the only extreme point of $C_p(H)^+$ is 0 and by the affinity and the surjectivity of ϕ we infer that $\phi(0) = 0$. This assertion has two consequences. On the one hand, by the affinity of ϕ, we deduce that it is positively homogeneous. On the other hand, since ϕ is an isometry, it follows that $\|\phi(A)\|_p = \|A\|_p$ $(A \in C_p(H)^+)$, in particular ϕ preserves the operators of unit norm. Now it is clear that the restriction of ϕ to $C_p(H)_1^+$ is a surjective isometry of this space. Therefore, by Theorem 4.7 there exists either a unitary or an antiunitary operator U on H such that $\phi|_{C_p(H)_1^+}$ is of the form

$$\phi(A) = UAU^* \quad (A \in C_p(H)_1^+).$$

Then it follows by the positive homogeneity of ϕ that it is of the desired form and this completes the proof of Theorem 4.5. □

We close this section with the verification of the fourth theorem of the present chapter.

Proof of Theorem 4.6. Denote by $B(H)^+$ the set of all positive operators on H. Observe that since H is finite dimensional one has $C_p(H)^+ = B(H)^+$. Turning to the verification of Theorem 4.6, just as in the first paragraph of the proof of Theorem 4.3 we get that ϕ is affine. Let $X = \phi(0)$ and define the map $\psi\colon B(H)^+ \to B_s(H)$ by

$$\psi(A) = \phi(A) - X \quad (A \in B(H)^+).$$

It is clear that ψ is affine and since it sends 0 to 0, it follows that

$$\psi(\lambda A) = \lambda\psi(A) \tag{4.15}$$

and $\|\psi(A)\|_p = \|A\|_p$ holds for all $A \in B(H)^+$ and $\lambda \in \mathbb{R}_+$. We assert that ψ maps $B(H)^+$ into itself. To see this, let $A \in B(H)^+$. By (4.15) the range of ψ is closed with respect to multiplication by nonnegative scalars. Choose an arbitrary positive integer n. On the one hand, referring to the latter fact, we obtain that $n\psi(A) \in \psi(B(H)^+)$. On the other hand, the elements of the range of ψ are apparently greater than or equal to $-X$. Hence, we deduce that $n\psi(A) \geq -X$, i.e. $\psi(A) \geq -(1/n)X$, and then letting n tend to ∞ and using the fact that $B(H)^+$ is closed, we infer that $\psi(A) \geq 0$. Thus, $\psi(B(H)^+) \subset B(H)^+$ as required.

By what we have proved so far, we see that ψ is an isometry of $C_p(H)^+$ which preserves the elements of unit norm. Therefore the restriction of ψ to $C_p(H)_1^+$ is an isometry of this space. On the other hand, since $\dim H < \infty$ the space $C_p(H)_1^+$ is compact. However, by [**7**, Excercise 2.4.1] any isometry of a compact metric space is surjective. We infer that $\psi|_{C_p(H)_1^+}$ is a surjective isometry. Now Theorem 4.7 applies and we obtain that there exists either a unitary or an antiunitary operator U on H such that $\psi|_{C_p(H)_1^+}$ is of the form

$$\psi(A) = UAU^* \quad (A \in C_p(H)_1^+).$$

By the positive homogeneity of ψ this implies that $\psi(A) = UAU^*$, i.e. $\phi(A) = UAU^* + X$ $(A \in C_p(H)^+)$ and this completes the proof of Theorem 4.6. □

4.3. Remarks

Let $1 < p < \infty$ and assume that $X \in C_p(H)^+$ is a nonzero operator. Observe that the map $A \mapsto A + X$ is an isometry of $C_p(H)^+$ which cannot be written in the form $A \mapsto UAU^*$, where U is a unitary or an antiunitary operator on H. This shows that the surjectivity condition in Theorem 4.5 cannot be omitted.

We have seen that in the finite dimensional case the conclusion of Theorem 4.7 holds for any isometry. As for infinite dimensional spaces, concerning arbitrary isometries we cannot expect such a regular form like (4.2). To see this, consider the following example. Assume that $\dim H = \infty$ and let H_1 be a finite dimensional nontrivial subspace of H. For a given $1 \leq p < \infty$ define the transformation $\phi\colon C_p(H)_1^+ \to C_p(H)_1^+$ by

$$\phi(A) = 0 \oplus A \quad (A \in C_p(H)_1^+),$$

where 0 is the zero operator on H_1. It is trivial that ϕ is an isometry with respect to d_p. Observe that the range of ϕ does not contain those rank-one projections P on H which project into H_1. This shows that ϕ is not surjective and hence it cannot be written in the form (4.2).

It is a natural question to ask whether Theorem 4.7 remains true in the case $p = \infty$. Our conjecture is that the answer to this question is affirmative. However, we could not prove nor disprove it. As for possible proofs, one cannot apply the analogue of Lemma 4.1 in the case $p = \infty$, since it does not hold. In fact, suppose that $2 < \dim H < \infty$ and let $P \neq I$ be a projection on H of rank at least 2. It is easy to see that

$$d_\infty(P, R) = d_\infty(I, R) = 1 \quad (R \in P_1(H))$$

and this shows that Lemma 4.1 is not valid in the case $p = \infty$.

5

Thompson isometries of positive operators

5.1. Introduction and statement of the results

In the paper [**24**] Molnár described the structure of all surjective isometries of $B(H)^+_{-1}$ equipped with the Thompson metric or the Hilbert projective metric under the condition $\dim H \geq 3$. Apparently, it is a natural question to ask what happens in the case where $\dim H = 2$. Indeed, the referee of [**24**] has posed this question which Molnár could not answer that time; he only conjectured that the same conclusion should hold as in higher dimensions. The aim of Chapter 5 is to present a proof of that conjecture. The results of this section have been published in the work [**26**].

We begin with introducing some notions and notation. In this chapter log denotes the natural logarithm. The general definitions of the Thompson metric and the Hilbert projective metric are as follows. Let X be a real normed space and $K \neq \emptyset$ be a closed convex cone in X which satisfies $K \cap (-K) = \{0\}$. For any $x, y \in X$ we write $x \leq y$ if $y - x \in K$. Clearly, $\leq$ is a partial order on X. There is an equivalence relation $\sim$ on $K \setminus \{0\}$ which is defined by $x \sim y$ if and only if we have positive real numbers t and s such that $sx \leq y \leq tx$. The equivalence classes induced by $\sim$ are called components. Let C be a component and for any $x, y \in C$ denote by $M(x/y)$ the quantity $\inf\{t > 0 \mid x \leq ty\}$. The Thompson metric d_T is defined by

$$d_T(x, y) = \log \max\{M(x/y), M(y/x)\} \quad (x, y \in C)$$

on C (for basic properties of d_T we refer to the original source [**42**]). Thompson introduced this metric as a modification of the Hilbert projective metric d_H which is defined by the formula

$$d_H(x, y) = \log M(x/y)M(y/x) \quad (x, y \in C)$$

on the same component. Actually, d_H is not a true metric, only a pseudo-metric. Indeed, for any $x, y \in C$ we have $d_H(x, y) = 0$ if and only if there is a positive number α such that $x = \alpha y$.

It is well-known that in any complex unital C^*-algebra $\mathcal{A}$, the set $\mathcal{A}^+$ of all positive elements of $\mathcal{A}$ is a nonempty closed convex cone in $\mathcal{A}$ considered as a real normed space, and $\mathcal{A}^+ \cap (-\mathcal{A}^+) = \{0\}$. Moreover, the invertible elements in $\mathcal{A}^+$ form a component of $\mathcal{A}^+ \setminus \{0\}$ which we denote by $\mathcal{A}^+_{-1}$. The Thompson metric on this component is in an intimate connection with the natural Finsler geometry of $\mathcal{A}^+_{-1}$ which has many applications in several fields of mathematics (see the introduction of [**24**] and the references therein).

As already mentioned, in [**24**] Molnár described the structure of bijective isometries of the space of all invertible positive operators acting on a complex Hilbert space H endowed with the Thompson metric or the Hilbert projective metric under the condition $\dim H \geq 3$. The aim of this chapter is to extend those results for the remaining case $\dim H = 2$. As in [**24**], it is easy to see that for any bijective bounded linear or conjugate-linear operator S on H and for any function $\tau\colon B(H)^+_{-1} \to]0, \infty[$ the transformations

$$A \mapsto SAS^*, \quad A \mapsto SA^{-1}S^* \quad (A \in B(H)^+_{-1})$$

and

$$A \mapsto \tau(A)SAS^*, \quad A \mapsto \tau(A)SA^{-1}S^* \quad (A \in B(H)^+_{-1})$$

are isometries corresponding to d_T and d_H, respectively. The results in [**24**] tell us that if $\dim H \geq 3$, then there are no other kinds of bijective isometries corresponding to those distances. The content of the following statements is that the same conclusions hold also in the 2-dimensional case.

Theorem 5.1. (Molnár, Nagy [**26**])
Suppose that $\dim H = 2$ *and that* $\Phi\colon B(H)^+_{-1} \to B(H)^+_{-1}$ *is a surjective isometry with respect to* d_T*. Then there exists a bijective linear or conjugate-linear operator* S *on* H *such that* Φ *is either of the form*

$$\Phi(A) = SAS^* \quad (A \in B(H)^+_{-1})$$

or of the form

$$\Phi(A) = SA^{-1}S^* \quad (A \in B(H)^+_{-1}).$$

The result concerning the Hilbert projective metric reads as follows.

Theorem 5.2. (Molnár, Nagy [**26**])
Suppose that $\dim H = 2$ *and that* $\Phi\colon B(H)^+_{-1} \to B(H)^+_{-1}$ *is a surjective isometry with respect to* d_H*. Then there exists a bijective linear or*

conjugate-linear operator S on H and a function $\tau\colon B(H)^+_{-1} \to]0,\infty[$ such that Φ is of the form

$$\Phi(A) = \tau(A)SAS^* \quad (A \in B(H)^+_{-1}).$$

5.2. Proofs

In this section the diameter of any subset K of $\mathbb{R}$ will be denoted by $\operatorname{diam} K$. We now present the proof of the result concerning the Thompson metric.

Proof of Theorem 5.1. We first recall the following formula relating to the Thompson metric:

$$d_T(A,B) = \|\log A^{-1/2}BA^{-1/2}\|_\infty \quad (A,B \in B(H)^+_{-1}). \tag{5.1}$$

A proof can be found, e.g. in [**24**].

By an observation made in the paragraph preceding the formulation of Theorem 5.1 we deduce that the transformation

$$A \longmapsto \Phi(I)^{-1/2}\Phi(A)\Phi(I)^{-1/2} \quad (A \in B(H)^+_{-1})$$

is a surjective isometry with respect to the Thompson metric which has the additional property that it sends I to I. Hence, regarding the statement of Theorem 5.1 we may and do assume that already Φ has this property, i.e. $\Phi(I) = I$.

In the proof of Theorem 1 in [**24**] it has been shown that any surjective Thompson isometry preserves the Jordan triple product, i.e. satisfies

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A) \quad (A,B \in B(H)^+_{-1}). \tag{5.2}$$

Moreover, it was proved at the beginning of the proof of Theorem 1 in [**21**] that if $\dim H \geq 3$, then (5.2) implies that

$$AB = BA \iff \Phi(A)\Phi(B) = \Phi(B)\Phi(A) \quad (A,B \in B(H)^+_{-1}) \tag{5.3}$$

and

$$\Phi(AB) = \Phi(A)\Phi(B) \quad (A,B \in B(H)^+_{-1}, AB = BA). \tag{5.4}$$

One can easily check that those arguments in [**21, 24**] do not require that $\dim H \geq 3$, and hence the above observations remain valid also in the present case $\dim H = 2$.

As the positive scalar multiples of I are exactly the elements of $B(H)^+_{-1}$ which commute with all elements, it follows from (5.3) that there is a function $f\colon\ ]0,\infty[\to]0,\infty[$ such that

$$\Phi(\lambda I) = f(\lambda)I \quad (\lambda > 0).$$

It is clear that $f(1) = 1$. Moreover, it follows from (5.1) that

$$|\log f(\lambda) - \log f(\mu)| = |\log\lambda - \log\mu| \quad (\lambda,\mu > 0). \tag{5.5}$$

Let $F = \log\circ f\circ\exp$. Then (5.5) implies that $F\colon \mathbb{R}\to\mathbb{R}$ is an isometry. It is well-known and in fact very easy to see that there exists a real number a such that either we have

$$F(x) = a + x \quad (x\in\mathbb{R})$$

or we have

$$F(x) = a - x \quad (x\in\mathbb{R}).$$

As $f(1) = 1$, we deduce that either

$$f(\lambda) = \lambda \quad (\lambda > 0)$$

or

$$f(\lambda) = \frac{1}{\lambda} \quad (\lambda > 0).$$

In the first case let $\phi = \Phi$. While in the second case we define $\phi\colon B(H)^+_{-1}\to B(H)^+_{-1}$ by

$$\phi(A) = \Phi(A)^{-1} \quad (A\in B(H)^+_{-1}).$$

Since the inverse operation is a surjective Thompson isometry (see the paragraph before the formulation of Theorem 5.1), the same holds for ϕ. It is clear that

$$\phi(\lambda I) = \lambda I \quad (\lambda > 0)$$

and ϕ satisfies (5.2), (5.3) and (5.4).

The elements of $B(H)^+_{-1}$ are exactly the operators of the form $\lambda P + \mu P^\perp$ where $P\in P_1(H)$ and λ,μ are positive real numbers. Let $P\in P_1(H)$ be fixed. It is easy to see that given two different positive real numbers λ and μ, an invertible positive operator commutes with $\lambda P+\mu P^\perp$ if and only if it is of the form $\lambda' P + \mu' P^\perp$ where $\lambda',\mu' > 0$. This, together with (5.3), yields that there exists a rank-one projection $\widetilde{P}$ such that for any positive real numbers λ,μ we have scalars $\widetilde{\lambda},\widetilde{\mu} > 0$ for which

$$\phi(\lambda P + \mu P^\perp) = \widetilde{\lambda}\cdot\widetilde{P} + \widetilde{\mu}\cdot\widetilde{P}^\perp.$$

Therefore, there are functions $g,h\colon\ ]0,\infty[\to]0,\infty[$ such that

$$\phi(\lambda P + P^\perp) = g(\lambda)\widetilde{P} + h(\lambda)\widetilde{P}^\perp \quad (\lambda > 0). \tag{5.6}$$

It is clear from (5.4) that g and h are multiplicative.

Since ϕ is multiplicative on the commuting elements of $B(H)^+_{-1}$ and leaves the scalar operators fixed, we infer that

$$\phi(P + \mu P^\perp) = \frac{\mu}{g(\mu)}\widetilde{P} + \frac{\mu}{h(\mu)}\widetilde{P}^\perp \quad (\mu > 0).$$

Multiplying this equality with the one in (5.6), we obtain

$$\phi(\lambda P + \mu P^\perp) = \mu g\left(\frac{\lambda}{\mu}\right)\widetilde{P} + \mu h\left(\frac{\lambda}{\mu}\right)\widetilde{P}^\perp \quad (\lambda, \mu > 0). \tag{5.7}$$

Let $\lambda, \mu, \delta, \varepsilon > 0$. Substituting $A = \lambda P + \mu P^\perp$ and $B = \delta P + \varepsilon P^\perp$ into the equality (5.1) and using (5.7) and the isometry property of ϕ we obtain

$$\max\left\{\left|\log\frac{\delta}{\lambda}\right|, \left|\log\frac{\varepsilon}{\mu}\right|\right\} = \max\left\{\left|\log\frac{\varepsilon}{\mu}g\left(\frac{\delta\mu}{\lambda\varepsilon}\right)\right|, \left|\log\frac{\varepsilon}{\mu}h\left(\frac{\delta\mu}{\lambda\varepsilon}\right)\right|\right\}.$$

This yields

$$\max\{|\log t|, |\log s|\} = \max\left\{\left|\log sg\left(\frac{t}{s}\right)\right|, \left|\log sh\left(\frac{t}{s}\right)\right|\right\} \quad (s, t > 0). \tag{5.8}$$

Substituting $s = 1$ into this equation, we see

$$|\log g(t)| \leq |\log t|, \ |\log h(t)| \leq |\log t| \quad (t > 0).$$

Consequently, for the additive functions $G = \log \circ g \circ \exp$ and $H = \log \circ h \circ \exp$ we have

$$|G(t) - G(s)| = |G(t - s)| \leq |t - s| \text{ and } |H(t) - H(s)| \leq |t - s|$$

for every $s, t \in \mathbb{R}$. This gives us that G and H are continuous and hence they are scalar multiples of the identity on $\mathbb{R}$. This implies that g and h are power functions which means that we have real numbers a and b such that

$$g(x) = x^a \quad (x > 0)$$

and

$$h(x) = x^b \quad (x > 0).$$

Substituting $s = 1, t = e$ respectively $s = e, t = 1$ into (5.8) we obtain that

$$\max\{|a|, |b|\} = \max\{|a - 1|, |b - 1|\} = 1.$$

From this we infer that either $a = 1$ and $b = 0$, or $a = 0$ and $b = 1$. The equation (5.7) implies that in the first case we have

$$\phi(\lambda P + \mu P^\perp) = \lambda\widetilde{P} + \mu\widetilde{P}^\perp \quad (\lambda, \mu > 0)$$

while in the second case one has

$$\phi(\lambda P + \mu P^\perp) = \mu\widetilde{P} + \lambda\widetilde{P}^\perp \quad (\lambda, \mu > 0).$$

In both cases we can conclude

$$\sigma_p(\phi(A)) = \sigma_p(A) \quad (A \in B(H)^+_{-1}). \tag{5.9}$$

This implies that ϕ preserves the trace of operators

$$\operatorname{tr}\phi(A) = \operatorname{tr} A \quad (A \in B(H)^+_{-1}). \tag{5.10}$$

Define the transformation $\psi\colon P_1(H) \to B(H)$ by

$$\psi(P) = \phi(I + P) - I \quad (P \in P_1(H)).$$

The elements of $B(H)^+_{-1}$ with spectrum $\{1, 2\}$ are exactly the operators of the form $I + P$ where $P \in P_1(H)$. By (5.9) this yields that the range of ψ is in $P_1(H)$. Clearly, $\psi\colon P_1(H) \to P_1(H)$ is bijective. Let $P, Q \in P_1(H)$. Easy computation shows that

$$\operatorname{tr}(I + P)(I + Q)(I + P) = 6 + 3 \operatorname{tr} PQ$$

and

$$\operatorname{tr}(I + \psi(P))(I + \psi(Q))(I + \psi(P)) = 6 + 3 \operatorname{tr}\psi(P)\psi(Q).$$

Referring to the equality (5.2) and to the trace preserving property of ϕ we obtain

$$\operatorname{tr} PQ = \operatorname{tr}\psi(P)\psi(Q).$$

To sum up, ψ is a bijection which preserves the transition probability. Hence, by Theorem 1.2, ψ is of the form

$$\psi(P) = UPU^* \quad (P \in P_1(H)) \tag{5.11}$$

with some unitary or antiunitary operator U on H. From (5.11) it follows that

$$\phi(I + P) = U(I + P)U^* \quad (P \in P_1(H)).$$

Define the map $\widetilde{\phi}$ by

$$\widetilde{\phi}(A) = U^*\phi(A)U \quad (A \in B(H)^+_{-1}).$$

It is clear that $\widetilde{\phi}\colon B(H)^+_{-1} \to B(H)^+_{-1}$ is a surjective Thompson isometry with the additional property that $\widetilde{\phi}(I + P) = I + P$ $(P \in P_1(H))$. Therefore $\widetilde{\phi}$ satisfies (5.2), (5.3), (5.9) and (5.10).

Pick any $P \in P_1(H)$ and let $\lambda, \mu > 0$. By the commutativity preserving property of $\widetilde{\phi}$ and the equality

$$\widetilde{\phi}(2P + P^\perp) = 2P + P^\perp$$

we obtain that there are positive scalars α, β such that

$$\widetilde{\phi}(\lambda P + \mu P^\perp) = \alpha P + \beta P^\perp.$$

Let $T = \lambda P + \mu P^\perp$. We compute on the one hand

$$\operatorname{tr}(I+P)T(I+P) = \operatorname{tr}\widetilde{\phi}((I+P)T(I+P)) = \operatorname{tr}\widetilde{\phi}(I+P)\widetilde{\phi}(T)\widetilde{\phi}(I+P) =$$
$$\operatorname{tr}(I+P)\widetilde{\phi}(T)(I+P) = \operatorname{tr}\widetilde{\phi}(T) + 3\operatorname{tr}\widetilde{\phi}(T)P = \operatorname{tr}T + 3\alpha,$$

and on the other hand

$$\operatorname{tr}(I+P)T(I+P) = \operatorname{tr}T + 3\operatorname{tr}TP = \operatorname{tr}T + 3\lambda.$$

This gives us that $\alpha = \lambda$ and next that $\beta = \mu$. Consequently, we obtain that $\widetilde{\phi}(A) = A$ $(A \in B(H)^+_{-1})$. Transforming back first to ϕ and then to Φ and having in mind how we have reached the assumption $\Phi(I) = I$, we deduce that our original transformation Φ is necessarily of one of the forms appearing in the statement of Theorem 5.1. □

Turning to the proof of the second statement in this chapter, we denote by $\overline{A}$ the set of all positive scalar multiples of the operator $A \in B(H)^+_{-1}$. As noted in Section 5.1., we have

$$\overline{A} = \{B \in B(H)^+_{-1} \ : \ d_H(A,B) = 0\} \quad (A \in B(H)^+_{-1}).$$

Proof of Theorem 5.2. We know from the introduction of [**24**] that the Hilbert projective metric can be calculated in the following way:

$$d_H(A,B) = \operatorname{diam}\log\left(\sigma_p\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right) \quad (A,B \in B(H)^+_{-1}). \tag{5.12}$$

Similarly to the proof of Theorem 5.1 we see that the map

$$A \mapsto \Phi(I)^{-1/2}\Phi(A)\Phi(I)^{-1/2} \quad (A \in B(H)^+_{-1})$$

is a surjective isometry of $B(H)^+_{-1}$ with respect to the Hilbert projective metric which has the additional property that it sends I to I. Therefore we may and do assume that already Φ has this property, i.e. $\Phi(I) = I$. Then in the same way as in the proof of [**24**, Theorem 2] one can deduce that

$$\overline{\Phi(ABA)} = \overline{\Phi(A)\Phi(B)\Phi(A)} \quad (A,B \in B(H)^+_{-1}). \tag{5.13}$$

Again, as in the proof of [**24**, Theorem 2] this yields that Φ preserves commutativity.

Let now $P \in P_1(H)$ be fixed. In a similar way as in the proof of Theorem 5.1 we infer that there is a projection $\widetilde{P} \in P_1(H)$ such that for any numbers $\lambda, \mu > 0$ we have positive scalars $\widetilde{\lambda}, \widetilde{\mu}$ for which

$$\Phi(\lambda P + \mu P^\perp) = \widetilde{\lambda}\cdot\widetilde{P} + \widetilde{\mu}\cdot\widetilde{P}^\perp.$$

Consequently, there exist functions $f, g\colon]0,\infty[\to]0,\infty[$ such that

$$\Phi(\lambda P + P^\perp) = f(\lambda)\widetilde{P} + g(\lambda)\widetilde{P}^\perp \quad (\lambda > 0). \tag{5.14}$$

Let $\lambda, \mu > 0$. Using the isometry property of Φ for $A = \lambda P + P^\perp$ and $B = \mu P + P^\perp$, and applying (5.12), (5.14) we obtain

$$\left|\log \frac{\mu}{\lambda}\right| = \left|\log \frac{f(\mu)}{f(\lambda)} - \log \frac{g(\mu)}{g(\lambda)}\right|.$$

Define $h = f/g$. We infer that

$$|\log \mu - \log \lambda| = |\log h(\mu) - \log h(\lambda)| \quad (\lambda, \mu > 0).$$

Since $h(1) = 1$, just as in the proof of Theorem 5.1 we deduce that either

$$h(\lambda) = \lambda \quad (\lambda > 0)$$

or

$$h(\lambda) = \frac{1}{\lambda} \quad (\lambda > 0).$$

In the first case we compute

$$\overline{\Phi(\lambda P + \mu P^\perp)} = \Phi(\overline{\lambda P + \mu P^\perp}) = \Phi(\overline{(\lambda/\mu)P + P^\perp}) =$$
$$\overline{\Phi((\lambda/\mu)P + P^\perp)} = \overline{(\lambda/\mu)\widetilde{P} + \widetilde{P}^\perp} = \overline{\lambda\widetilde{P} + \mu\widetilde{P}^\perp}$$

while in the second case we have

$$\overline{\Phi(\lambda P + \mu P^\perp)} = \Phi(\overline{\lambda P + \mu P^\perp}) = \Phi(\overline{(\lambda/\mu)P + P^\perp}) =$$
$$\overline{\Phi((\lambda/\mu)P + P^\perp)} = \overline{(\mu/\lambda)\widetilde{P} + \widetilde{P}^\perp} = \overline{\mu\widetilde{P} + \lambda\widetilde{P}^\perp}$$

for any scalars $\lambda, \mu > 0$. Therefore in either case there is a projection $\widehat{P} \in P_1(H)$ such that

$$\overline{\Phi(\lambda P + \mu P^\perp)} = \overline{\lambda\widehat{P} + \mu\widehat{P}^\perp} \tag{5.15}$$

holds for all $\lambda, \mu > 0$. It now easily follows that

$$\frac{\max \sigma_p(\Phi(A))}{\min \sigma_p(\Phi(A))} = \frac{\max \sigma_p(A)}{\min \sigma_p(A)} \tag{5.16}$$

holds for every $A \in B(H)^+_{-1}$. We also infer that to the given rank-one projection P there corresponds a projection $\psi(P) \in P_1(H)$ such that

$$\overline{\Phi(I + P)} = \overline{I + \psi(P)}.$$

It is clear that $\psi(P)$ is uniquely determined by this equality.

Now pick two arbitrary rank-one projections $P, Q \in P_1(H)$. Define

$$S = (I + P)(I + Q)(I + P), \; T = (I + \psi(P))(I + \psi(Q))(I + \psi(P)).$$

It follows from (5.13) that $\overline{\Phi(S)} = \overline{T}$. By (5.16) this implies that

$$\frac{\max \sigma_p(S)}{\min \sigma_p(S)} = \frac{\max \sigma_p(T)}{\min \sigma_p(T)}. \tag{5.17}$$

It is apparent that

$$\operatorname{tr} S = 6 + 3 \operatorname{tr} PQ, \ \operatorname{tr} T = 6 + 3 \operatorname{tr} \psi(P)\psi(Q) \tag{5.18}$$

and

$$\det S = \det T = 8. \tag{5.19}$$

By (5.19) we have

$$\frac{\max \sigma_p(S)}{\min \sigma_p(S)} = \frac{(\max \sigma_p(S))^2}{8}, \ \frac{\max \sigma_p(T)}{\min \sigma_p(T)} = \frac{(\max \sigma_p(T))^2}{8}$$

and these together with (5.17) yield that $\max \sigma_p(S) = \max \sigma_p(T)$. We next obtain that $\min \sigma_p(S) = \min \sigma_p(T)$, thus $\operatorname{tr} S = \operatorname{tr} T$. By (5.18) this implies that

$$\operatorname{tr} \psi(P)\psi(Q) = \operatorname{tr} PQ.$$

Since P and Q were arbitrary, this means that the map $\psi \colon P_1(H) \to P_1(H)$ preserves the transition probability. Therefore applying Theorem 1.3, we obtain that there is either a unitary or an antiunitary operator U on H such that

$$\psi(P) = UPU^* \quad (P \in P_1(H)).$$

We then infer

$$\overline{\Phi(2P + P^\perp)} = \overline{I + \psi(P)} = \overline{2UPU^* + (UPU^*)^\perp} \quad (P \in P_1(H)).$$

Taking (5.15) into consideration, it is easy to verify that $\widehat{P} = UPU^*$ and hence

$$\overline{\Phi(\lambda P + \mu P^\perp)} = \overline{\lambda UPU^* + \mu(UPU^*)^\perp}$$

holds for all scalars $\lambda, \mu > 0$ ($P \in P_1(H)$). This clearly yields

$$\overline{\Phi(A)} = \overline{UAU^*} \quad (A \in B(H)^+_{-1}).$$

Having in mind the definition of the classes $\overline{A}$ ($A \in B(H)^+_{-1}$) and the reduction $\Phi(I) = I$ what we have employed in the proof, we obtain the statement of Theorem 5.2. □

5.3. Remarks

One may wonder why the inverse operation does not show up in the formulation of Theorem 5.2. The fact is that it does in some hidden way. To see this, let $\{e_1, e_2\}$ be an orthonormal basis of the 2-dimensional space H. Define the map $U\colon H \to H$ by

$$Ux = \langle e_2, x\rangle e_1 - \langle e_1, x\rangle e_2 \quad (x \in H).$$

It is easy to see that U is an antiunitary operator with the property that

$$P^{\perp} = UPU^* \quad (P \in P_1(H)).$$

One can deduce that

$$A^{-1} = \frac{1}{\det A} UAU^* \quad (A \in B(H)^+_{-1})$$

implying that every transformation on $B(H)^+_{-1}$ of the form

$$A \longmapsto \tau(A) S A^{-1} S^*$$

with bijective linear or conjugate-linear operator $S : H \to H$ and function $\tau\colon B(H)^+_{-1} \to]0, \infty[$ is also of the form that appears in Theorem 5.2.

Summary

This dissertation contains results about preserver problems on positive operators. It consists of five chapters, a summary (both in English and in Hungarian) and a bibliography. In Chapter 1, we give a general overview about preserver problems on algebraic structures of operators and then in Section 1.1, we collect those preliminaries and terminology which are necessary for the material of the thesis. The results the dissertation is built upon are included in Chapters 2-5. In what follows we summarize them.

Chapter 2 is devoted to problems concerning bijective transformations on density operators (i.e., positive operators with unit trace) with invariance properties related to commutativity. The corresponding results describe the general forms of these maps. Before we formulate our theorem on the structure of commutativity preserving bijections of the set $S(H)$ of density operators acting on H, we introduce some notation which will be used in it. If $A \in B(H)$ and $\lambda \in \sigma_p(A)$ is an eigenvalue of A, then denote by $P_A(\lambda)$ the projection on H whose range is the eigensubspace of A corresponding to λ. A more restrictive version of the following result is the statement [**33**, Theorem 1.] which contains a separability condition, too.

Theorem. (Nagy)
Assume that $\dim H \geq 3$. *Moreover, suppose that* $\phi \colon S(H) \to S(H)$ *is a bijection which preserves commutativity, i.e. satisfies*

$$\phi(A)\phi(B) = \phi(B)\phi(A) \iff AB = BA$$

for any $A, B \in S(H)$. *Then there is a unitary or an antiunitary operator* U *on H, and for any* $A \in S(H)$ *there exists an injective function* $f_A \colon \sigma_p(A) \to [0, 1]$ *such that*

$$\phi(A) = U\left(\sum_{\lambda\in\sigma_p(A)} f_A(\lambda)P_A(\lambda)\right)U^*.$$

The above statement shows that the form of the commutativity preserving bijections of $S(H)$ is slightly irregular. To obtain a more regular form we have to assume a bit more requirements concerning the transformations under consideration. In the results of Chapter 2, we determine the structure of those bijections on $S(H)$ which preserve not only commutativity but also a certain measure of it. We define this quantity as follows. For a fixed unitarily invariant norm $\|.\|\colon C_1(H) \to \mathbb{R}$ and any $A, B \in S(H)$ the measure of commutativity between them is

$$\|AB - BA\|.$$

The below assertions describe the structure of the bijections on $S(H)$ which preserve the last displayed quantity. Those special cases of these statements when $\|.\|$ is a p-norm ($1 \leq p \leq \infty$) have been appeared in the author's paper [**33**] as Theorems 2 and 3.

Theorem. (Nagy)
Suppose that $\dim H = \infty$ *and that* $\phi\colon S(H) \to S(H)$ *is a bijection which satisfies*

$$\|\phi(A)\phi(B) - \phi(B)\phi(A)\| = \|AB - BA\| \tag{6.1}$$

for each pair $A, B \in S(H)$*. Then we have a unitary or an antiunitary operator* U *on* H *such that* ϕ *can be written in the form*

$$\phi(A) = UAU^* \quad (A \in S(H)).$$

As for finite dimensional spaces we have the following statement.

Theorem. (Nagy)
Suppose that $3 \leq \dim H < \infty$ *and that* $\phi\colon S(H) \to S(H)$ *is a bijective mapping with the property that* (6.1) *holds for every* $A, B \in S(H)$*. Then there is a unitary or an antiunitary operator* U *on* H *such that for each* $A \in S(H)$ *one has*

$$\phi(A) = UAU^*$$

or

$$\phi(A) = \frac{2}{\dim H} I - UAU^*.$$

In the last result of Chapter 2, we give the form of those bijective commutativity preserving transformations on $S(H)$ which leave the fidelity between commuting elements invariant. We recall that this quantity which has a fundamental role in quantum information theory is defined in the following way. For any $A, B \in S(H)$ the fidelity $F(A, B)$ between them is

$$F(A, B) = \operatorname{tr} \sqrt{\sqrt{A} B \sqrt{A}}.$$

The last theorem in the second chapter reads as follows.

Theorem. (Nagy [**33**])
Assume that $\dim H \geq 3$. *Furthermore, suppose that* $\phi\colon S(H) \to S(H)$ *is a bijection which preserves commutativity and has the property that*

$$F(\phi(A), \phi(B)) = F(A, B)$$

holds for each pair $A, B \in S(H)$ *which satisfies* $AB = BA$. *Then there exists a unitary or an antiunitary operator* U *on* H *such that* ϕ *can be written in the form*

$$\phi(A) = UAU^* \quad (A \in S(H)).$$

We proceed with a summary of the results in Chapter 3 in which the finite dimensionality of H is assumed. This chapter contains theorems on relative entropy preserving maps of subsets of density operators. Relative entropy is a fundamental notion in quantum information theory which has several versions. We recall the definitions of those which are considered in the third chapter.

(i) Umegaki relative entropy: $S(A\|B) = \operatorname{tr} A(\log A - \log B)$ if $\operatorname{supp} A \subset \operatorname{supp} B$, and $S(A\|B) = \infty$ otherwise
(ii) Belavkin-Staszewski relative entropy:
$S_B(A\|B) = \operatorname{tr} \sqrt{A} \log \sqrt{A} B^{-1} \sqrt{A}$ if $\operatorname{supp} A \subset \operatorname{supp} B$, and $S_B(A\|B) = \infty$ otherwise
(iii) Tsallis relative entropy: $S_T(A\|B) = 1/(1-q)(1 - \operatorname{tr} A^q B^{1-q})$
(iv) Quadratic relative entropy: $S_Q(A\|B) = \operatorname{tr} A^{-1}(A-B)^2$ if $\operatorname{supp} B \subset \operatorname{supp} A$, and $S_Q(A\|B) = \infty$ otherwise
(v) Jensen-Shannon divergence:
$$D_J(A\|B) = \frac{S\left(A \middle\| \frac{1}{2}(A+B)\right) + S\left(B \middle\| \frac{1}{2}(A+B)\right)}{2}$$

In these definitions, $A, B \in S(H)$, the number $0 < q < 1$ is fixed, supp signifies the support of a density operator (which is the orthogonal complement of its kernel) and log denotes the logarithm with base 2. Furthermore, in this dissertation the -1$^{\text{th}}$ power and the logarithm of a positive operator on H are only taken on its range.

In [**29**], the authors have determined the structure of those transformations on $S(H)$ which preserve the Umegaki relative entropy. Motivated by their result, we have described the general form of all mappings of the set of density operators on H which leave one of the quantities (ii)–(v) invariant. We emphasize that in the corresponding statement there are no assumptions on the transformation in question only that it should preserve the relative entropy under consideration. However the conclusion is that such maps have a very simple form, namely they are implemented by unitary-antiunitary operators. This is formulated in the first theorem of Chapter 3.

Theorem. (Molnár, Nagy [**27**])
Let $X(.\|.)$ *signify any of the quantities* (ii)–(v). *Assume that* $\phi\colon S(H) \to S(H)$ *is a mapping with the property that*

$$X(\phi(A)\|\phi(B)) = X(A\|B) \tag{6.2}$$

holds for all $A, B \in S(H)$. *Then there exists a unitary or an antiunitary operator* U *on* H *such that* ϕ *can be written in the form*

$$\phi(A) = UAU^* \quad (A \in S(H)).$$

In the differential geometric aspects of quantum theory, instead of $S(H)$ they usually consider the set $M(H)$ of invertible density operators on H. The reason is that from differential geometric point of view $M(H)$ is a much richer structure, it is a manifold. In [**25**] Molnár has determined the general form of those surjective transformations on $M(H)$ which preserve the relative entropy (i) or (ii). We have extended the corresponding results to the case of the quantities (iii),(iv) and obtained the second theorem in Chapter 3.

Theorem. (Molnár, Nagy [**27**])
Let $X(.\|.)$ *signify one of the relative entropies* (iii),(iv). *Assume that* ϕ *is a surjective selfmap of* $M(H)$ *with the property that* (6.2) *holds for each pair* $A, B \in M(H)$. *Then there exists a unitary or an antiunitary operator* U *on* H *such that* ϕ *can be written in the form*

$$\phi(A) = UAU^* \quad (A \in M(H)).$$

Chapter 4 is devoted to the investigation of isometries on spaces of positive operators and it also deals with some identification lemmas. Before summarizing the results of this chapter, we recall the main notation used in it. Let $1 \le p < \infty$. The metric induced by the p-norm is denoted by d_p (this concerns also the case $p = \infty$), the symbol $C_p(H)^+$ stands for the set of all positive operators in the von Neumann - Schatten p-class and $C_p(H)^+_1$ signifies the set of those elements in $C_p(H)^+$ which have unit p-norm. After this reminder, we can formulate the identification lemmas appearing in the fourth chapter. It is an immediate consequence of them that, roughly speaking, in finite dimension the elements of $C_p(H)_1^+$, respectively $S(H)$ can be identified by their distances to the members of $P_1(H)$ in the metric d_p, respectively d_∞.

Lemma. (Nagy [**34**])
Assume that H is finite dimensional and let $1 \leq p, \gamma < \infty$ be fixed numbers. If $A, B \in C_p(H)_1^+$ have the property that

$$d_p(A, \gamma P) = d_p(B, \gamma P)$$

is satisfied by each $P \in P_1(H)$, then $A = B$.

Lemma. (Nagy)
Assume that H is finite dimensional and let $A, B \in S(H)$ be such that

$$d_\infty(A, P) = d_\infty(B, P)$$

for each $P \in P_1(H)$. Then $A = B$.

In the fourth chapter, there are several results concerning the isometries of $C_p(H)_1^+$ and those of $C_p(H)^+$ $(1 \leq p < \infty)$. In the first theorem of Chapter 4, we determine the structure of the isometries of $(S(H), d_p)$ $(1 < p < \infty)$. This result appeared as part of [**27**, Theorem 1] under the additional condition $\dim H < \infty$. It reads as follows.

Theorem. (Molnár, Nagy)
Let $1 < p < \infty$ be a fixed number and assume that ϕ is an isometry of $(S(H), d_p)$. Then we have a linear or a conjugate-linear isometry V on H with the property that ϕ can be written in the form

$$\phi(A) = VAV^* \quad (A \in S(H)).$$

The next result of Chapter 4 describes the structure of the isometries of $(S(H), d_\infty)$ under the condition $\dim H < \infty$.

Theorem. (Nagy [**34**])
Suppose that H is finite dimensional and that ϕ is an isometry of $(S(H), d_\infty)$. Then we have a unitary or an antiunitary operator U on H with the property that ϕ can be written in the form

$$\phi(A) = UAU^* \quad (A \in S(H)). \tag{6.3}$$

In what follows we recall the third theorem of the fourth chapter which describes the general form of the surjective isometries of $C_p(H)^+$ with respect to d_p $(1 < p < \infty)$.

Theorem. (Nagy)
Let $1 < p < \infty$ and assume that ϕ is a surjective isometry of $(C_p(H)^+, d_p)$. Then there exists a unitary or an antiunitary operator U on H which has the property that ϕ can be written in the form

$$\phi(A) = UAU^* \quad (A \in C_p(H)^+).$$

As for finite dimensional spaces, we present the structure of the isometries of the cone of complex positive semidefinite matrices with respect to d_p in the statement below ($1 < p < \infty$).

Theorem. (Nagy)
Assume that $\dim H < \infty$ and let $1 < p < \infty$. Furthermore, suppose that $\phi\colon C_p(H)^+ \to C_p(H)^+$ is an isometry with respect to d_p. Then there is a positive operator $X \in B(H)$ and there exists either a unitary or an antiunitary operator U on H such that ϕ can be written in the form

$$\phi(A) = UAU^* + X \quad (A \in C_p(H)^+).$$

There is a statement in Chapter 4 concerning the description of the surjective isometries of $(C_p(H)_1^+, d_p)$ ($1 < p < \infty$). Now we remind the reader of it.

Theorem. (Nagy [**34**])
Let $1 < p < \infty$ be a number and assume that ϕ is a surjective isometry of $(C_p(H)_1^+, d_p)$. Then there exists a unitary or an antiunitary operator U on H which has the property that ϕ can be written in the form

$$\phi(A) = UAU^* \quad (A \in C_p(H)_1^+).$$

There is a corollary of this statement in Chapter 4. It concerns the isometries of $(S(H), \rho_p)$, where $1 \leq p < \infty$ and ρ_p is the function defined by

$$\rho_p(A, B) = d_p\left(A^{\frac{1}{p}}, B^{\frac{1}{p}}\right) \quad (A, B \in S(H)).$$

These metrics are introduced in quantum information theory to measure the distance between quantum states the quantum mechanical objects represented by density operators. Turning to the mentioned corollary, we recall that it describes the general form of the surjective isometries of $(S(H), \rho_p)$. It can be read below.

Corollary. (Nagy)
Let $1 < p < \infty$ and assume that ϕ is a surjective isometry of $(S(H), \rho_p)$. Then there exists a unitary or an antiunitary operator U on H which has the property that ϕ can be written in the form (6.3).

We close the summary with a review of the results in Chapter 5. These theorems concern the structure of the surjective isometries of the space of invertible positive operators on H equipped with the Thompson metric or the Hilbert projective metric under the condition $\dim H = 2$. In what follows, we recall the definitions of these metrics. Let X be a real normed space and $K \neq \emptyset$ be a closed convex cone in X with the property that $K \cap (-K) = \{0\}$. We define a relation $\leq$ on the space X in the following way. For a given pair $x, y \in X$ of vectors $x \leq y$ if and only if $y - x \in K$. An equivalence relation $\sim$ is defined on $K \setminus \{0\}$ by $x \sim y$ if and only if there exist numbers $s, t > 0$ with the property that $sx \leq y \leq tx$. The equivalence classes of $K \setminus \{0\}$ induced by $\sim$ are called components. Let C be a component and for each pair $x, y \in C$ define $M(x/y) = \inf\{t > 0 \mid x \leq ty\}$. The Thompson metric d_T is defined by

$$d_T(x, y) = \log \max\{M(x/y), M(y/x)\} \quad (x, y \in C)$$

on C, where log signifies the natural logarithm. One defines the Hilbert projective metric d_H by

$$d_H(x, y) = \log M(x/y)M(y/x) \quad (x, y \in C)$$

on C. We remark that d_H is not in fact a metric, it is just a pseudo-metric.

It is well-known that the set $B(H)^+$ of all positive operators on H is a nonempty closed convex cone in $B(H)$ considered as a real normed space, and $B(H)^+ \cap (-B(H)^+) = \{0\}$. Moreover, the set $B(H)^+_{-1}$ of the invertible elements in $B(H)^+$ is a component of $B(H)^+ \setminus \{0\}$. In [**24**] Molnár has determined the general forms of the surjective isometries of $B(H)^+_{-1}$ with respect to d_T or d_H under the condition $\dim H \geq 3$. The results of the fifth chapter extend the theorems in the cited paper to the 2-dimensional case. They read as follows.

Theorem. (Molnár, Nagy [**26**])
Assume that H is 2-dimensional and that ϕ is a surjective isometry of $(B(H)^+_{-1}$ with respect to d_T. Then we have a bijective linear or conjugate-linear operator S on H with the property that ϕ can be written in the form

$$\phi(A) = SAS^* \quad (A \in B(H)^+_{-1})$$

or in the form

$$\phi(A) = SA^{-1}S^* \quad (A \in B(H)^+_{-1}).$$

Theorem. (Molnár, Nagy [**26**])
Assume that H is 2-dimensional and that $\phi\colon B(H)^+_{-1} \to B(H)^+_{-1}$ is a surjective isometry with respect to d_H. Then we have a bijective linear or

conjugate-linear operator S on H and a function $\tau\colon B(H)^+_{-1} \to]0,\infty[$ with the property that ϕ can be written in the form

$$\phi(A) = \tau(A) S A S^* \quad (A \in B(H)^+_{-1}).$$

Összefoglalás

Jelen disszertáció pozitív operátorok struktúrái megőrzési problémáival kapcsolatos eredményeket tartalmaz. Öt fejezetből, egy angol és egy magyar nyelvű összefoglalóból és egy irodalomjegyzékből áll. Az első fejezet áttekintést ad operátorok algebrai struktúrái megőrzési problémáiról, ezt követően pedig az 1.1 szakaszban összefoglaljuk azon alapismereteket és azt a terminológiát, melyeket a dolgozatban használunk. A 2-5 fejezetek tartalmazzák a disszertáció alapjául szolgáló eredményeket. Az alábbiakban összegezzük ezeket.

A második fejezet olyan problémák vizsgálatát tartalmazza, melyekben a sűrűségoperátorok (H azon A pozitív operátorai, melyekre $\operatorname{tr} A = 1$) halmazán értelmezett, kommutativitással kapcsolatos invariancia tulajdonságokkal bíró, bijektív transzformációk szerepelnek. A kapcsolódó eredmények leírják ezen leképezések általános alakját. Mielőtt megfogalmazzuk a H-n ható sűrűségoperátorok $S(H)$ halmazán értelmezett, kommutálástartó bijekciók struktúrájáról szóló tételünket, bevezetünk egy benne szereplő jelölést. Ha $A \in B(H)$ egy operátor, $\lambda \in \sigma_p(A)$ pedig egy sajátértéke A-nak, akkor jelölje $P_A(\lambda)$ azon H-n ható projekciót, mely értékkészlete az A operátor λ-hoz tartozó sajátaltere. Az alábbi eredmény általánosítása a [**33**, Theorem 1.] állításnak, mely egy szeparábilitási feltételt is tartalmaz.

TÉTEL. (Nagy)
Tegyük fel, hogy $\dim H \geq 3$. *Továbbá legyen* $\phi\colon S(H) \to S(H)$ *egy olyan bijekció, mely tartja a kommutálást, azaz teljesül rá, hogy tetszőleges* $A, B \in S(H)$ *esetén* $\phi(A)\phi(B) = \phi(B)\phi(A)$ *akkor és csak akkor áll fenn, ha* $AB = BA$. *Ekkor van olyan* U *unitér vagy antiunitér operátor* H*-n, és bármely* $A \in S(H)$ *esetén létezik olyan* $f_A\colon \sigma_p(A) \to [0, 1]$ *injektív függvény, melyekre*

$$\phi(A) = U\left(\sum_{\lambda\in\sigma_p(A)} f_A(\lambda)P_A(\lambda)\right)U^*.$$

A fenti állítás mutatja, hogy $S(H)$ kommutálástartó bijekciói alakja kevéssé reguláris. Ahhoz, hogy regulárisabb alakot kapjunk, valamivel több feltételt kell szabnunk a szóban forgó transzformációkra. A második fejezetben levő eredmények megadják $S(H)$ azon bijekciói struktúráját, melyek nemcsak a kommutálást, hanem annak egyfajta mértékét is megőrzik. Ezen mennyiséget a következőképpen definiáljuk. A H-n ható trace-operátorok vektorterén adott tetszőlegesen rögzített $\|.\|$ unitér invariáns norma és bármely $A, B \in S(H)$ esetén a kommutálásuk mértékét az

$$\|AB - BA\|$$

számként értelmezzük. Az alábbi állítások megadják $S(H)$ azon bijekcióinak struktúráját, melyek megőrzik az utóbbi mennyiséget. Ezen eredmények azon speciális esetei, melyekben $\|.\|$ egy p-norma ($1 \leq p \leq \infty$) a szerző [**33**] publikációjában jelentek meg (lásd Theorem 2 és Theorem 3).

TÉTEL. (Nagy)
Tegyük fel, hogy $\dim H = \infty$ *és, hogy* $\phi\colon S(H) \to S(H)$ *egy bijekció, mely kielégíti a*

$$\|\phi(A)\phi(B) - \phi(B)\phi(A)\| = \|AB - BA\| \tag{7.1}$$

egyenletet bármely $A, B \in S(H)$ *pár esetén. Ekkor létezik egy* U *unitér vagy antiunitér operátor* H*-n úgy, hogy* ϕ *a*

$$\phi(A) = UAU^* \quad (A \in S(H))$$

alakba írható.

Véges dimenziós alaptér esetén a következő eredmény érvényes.

TÉTEL. (Nagy)
Tegyük fel, hogy $3 \leq \dim H < \infty$ *és, hogy* $\phi\colon S(H) \to S(H)$ *egy olyan bijekció, mely kielégíti* (7.1)*-t bármely* $A, B \in S(H)$ *esetén. Ekkor van olyan* U *unitér vagy antiunitér operátor* H*-n, melyre minden egyes* $A \in S(H)$ *esetén fennáll a*

$$\phi(A) = UAU^*,$$

vagy a

$$\phi(A) = \frac{2}{\dim H} I - UAU^*$$

egyenlőségek valamelyike.

A második fejezet utolsó eredményében megadjuk $S(H)$ azon kommutálástartó bijekciói alakját, melyek invariánsan hagyják a felcserélhető elemek közötti fidelitást. Emlékeztetünk arra, hogy ezen mennyiségnek alapvető szerepe van a kvantum-információelméletben. Tetszőleges $A, B \in S(H)$ esetén az $F(A, B)$ fidelitásuk az

$$F(A, B) = \operatorname{tr} \sqrt{\sqrt{A} B \sqrt{A}}$$

egyenlőséggel van definiálva. A második fejezet utolsó tétele a következőképpen szól.

Tétel. (Nagy [**33**])
Tegyük fel, hogy $\dim H \geq 3$. *Továbbá legyen* $\phi \colon S(H) \to S(H)$ *egy olyan bijekció, mely megőrzi a kommutálást és kielégíti az*

$$F(\phi(A), \phi(B)) = F(A, B)$$

egyenletet minden olyan $A, B \in S(H)$ *esetén, melyekre* $AB = BA$. *Ekkor van olyan* U *unitér vagy antiunitér operátor* H*-n, mellyel* ϕ *a*

$$\phi(A) = UAU^* \quad (A \in S(H))$$

alakba írható.

Az alábbiakban a harmadik fejezetben levő eredményeket foglaljuk össze, s ennek során feltesszük, hogy H véges dimenziós. Ezen fejezet sűrűségoperátorok halmazainak relatív entrópiát megőrző transzformációival kapcsolatos eredményeket tartalmaz. A relatív entrópia egy alapvető mennyiség a kvantum-információelméletben, melynek több változata van. Ezek közül a harmadik fejezetben tárgyalt relatív entrópiák definíciója:

(i) Umegaki relatív entrópia: $S(A\|B) = \operatorname{tr} A(\log A - \log B)$, ha $\operatorname{supp} A \subset \operatorname{supp} B$ és $S(A\|B) = \infty$ egyébként
(ii) Belavkin-Staszewski relatív entrópia:
$S_B(A\|B) = \operatorname{tr} \sqrt{A} \log \sqrt{A} B^{-1} \sqrt{A}$, ha $\operatorname{supp} A \subset \operatorname{supp} B$ és $S_B(A\|B) = \infty$ egyébként
(iii) Tsallis relatív entrópia: $S_T(A\|B) = 1/(1-q)(1 - \operatorname{tr} A^q B^{1-q})$
(iv) Kvadratikus relatív entrópia: $S_Q(A\|B) = \operatorname{tr} A^{-1}(A - B)^2$, ha $\operatorname{supp} B \subset \operatorname{supp} A$ és $S_Q(A\|B) = \infty$ egyébként
(v) Jensen-Shannon divergencia:
$$D_J(A\|B) = \frac{S\left(A \middle\| \frac{1}{2}(A+B)\right) + S\left(B \middle\| \frac{1}{2}(A+B)\right)}{2}$$

Ezen definíciókban $A, B \in S(H)$ operátorok, $0 < q < 1$ rögzített szám és supp jelöli a sűrűségoperátorok tartóját (mely a magjuk ortogonális komplementere), illetve log a 2-es alapú logaritmusfüggvényt. Továbbá megállapodunk abban, hogy egy H-n ható pozitív operátor -1 kitevőjű hatványát illetve logaritmusát csak az operátor értékkészletén képezzük.

A [**29**] cikkben a szerzők megadták $S(H)$ azon transzformációi struktúráját, melyek megőrzik az Umegaki relatív entrópiát. A kapcsolódó eredmény motivált minket arra, hogy meghatározzuk $S(H)$ azon leképezései általános alakját, melyek invariánsan hagyják a (ii)–(v) mennyiségek valamelyikét. Hangsúlyozzuk, hogy a kérdéses transzformációkkal kapcsolatban csak azzal a feltételezéssel élünk, hogy megőrzik a szóban forgó relatív entrópiát. Mindazonáltal a kapcsolódó eredményünkben azt állítjuk, hogy az ilyen transzformációk nagyon egyszerű alakúak. Erről szól a harmadik fejezet első tétele.

TÉTEL. (Molnár, Nagy [**27**])
Jelölje $X(.\|.)$ *a* (ii)–(v) *mennyiségek valamelyikét. Tegyük fel, hogy* ϕ *az* $S(H)$ *halmaz egy olyan transzformációja, mely kielégíti az*

$$X(\phi(A)\|\phi(B)) = X(A\|B) \tag{7.2}$$

egyenletet bármely $A, B \in S(H)$ *esetén. Ekkor van olyan* U *unitér vagy antiunitér operátor* H*-n, mellyel* ϕ *a*

$$\phi(A) = UAU^* \quad (A \in S(H))$$

alakba írható.

A kvantumelmélet differenciálgeometriai vonatkozásaiban $S(H)$ helyett általában a H-n ható invertálható sűrűségoperátorok $M(H)$ halmazát vizsgálják. Ennek az az oka, hogy differenciálgeometriai szempontból $M(H)$ egy sokkal gazdagabb struktúra, egy sokaság. A [**25**] dolgozatban Molnár meghatározta $M(H)$ azon szürjektív transzformációi általános alakját, melyek megőrzik az (i) vagy a (ii) relatív entrópiát. A vonatkozó eredményeket kiterjesztettük a (iii),(iv) mennyiségek esetére is, erről szól a harmadik fejezet második tétele.

TÉTEL. (Molnár, Nagy [**27**])
Jelölje $X(.\|.)$ *a* (iii),(iv) *relatív entrópiák egyikét. Tegyük fel, hogy* ϕ *az* $M(H)$ *halmaz egy olyan szürjektív transzformációja, mely kielégíti a* (7.2) *egyenletet minden* $A, B \in M(H)$ *esetén. Ekkor van olyan* U *unitér vagy antiunitér operátor* H*-n, mellyel* ϕ *a*

$$\phi(A) = UAU^* \quad (A \in M(H))$$

alakba írható.

A negyedik fejezet célja pozitív operátorok terei izometriáinak vizsgálata, továbbá úgynevezett azonossági lemmákat is tanulmányozunk benne.

Mielőtt összegeznénk a fejezet eredményeit, emlékeztetünk az abban használt főbb jelölésekre. Legyen $1 \leq p < \infty$ egy szám. A p-normából származó metrikát d_p jelöli (ez a $p = \infty$ esetre is vonatkozik), a Neumann - Schatten p-osztályban levő pozitív operátorok halmazának jele pedig $C_p(H)^+$. Továbbá használjuk a

$$C_p(H)_1^+ = \{A \in C_p(H)^+ \ : \ \|A\|_p = 1\}$$

definíciót. Ezen emlékeztető után az alábbiakban idézzük fel a negyedik fejezetben található azonossági lemmákat. Ezek azonnali következménye az, hogy – szemléletesen – $C_p(H)_1^+$, illetve $S(H)$ elemei beazonosíthatók a $P_1(H)$-beli operátoroktól a d_p, illetve a d_∞ metrikában mért távolságaik segítségével.

LEMMA. (Nagy [**34**])
Tegyük fel, hogy H véges dimenziós, és legyenek $1 \leq p, \gamma < \infty$ rögzített számok. Ha $A, B \in C_p(H)_1^+$ operátorok úgy, hogy minden $P \in P_1(H)$ esetén teljesül a

$$d_p(A, \gamma P) = d_p(B, \gamma P)$$

egyenlőség, akkor $A = B$.

LEMMA. (Nagy)
Tegyük fel, hogy H véges dimenziós és legyenek $A, B \in S(H)$ operátorok úgy, hogy minden $P \in P_1(H)$ esetén

$$d_\infty(A, P) = d_\infty(B, P).$$

Ekkor $A = B$.

A negyedik fejezetben több eredmény található $C_p(H)_1^+$ és $C_p(H)^+$ izometriáival kapcsolatban ($1 \leq p < \infty$). A fejezet első tételében meghatározzuk az $(S(H), d_p)$ ($1 < p < \infty$) metrikus tér távolságtartó transzformációi struktúráját. Eme eredmény véges dimenziós változatát tartalmazza a [**27**, Theorem 1] állítás is. Az alábbiakban megfogalmazzuk az említett tételt.

TÉTEL. (Molnár, Nagy)
Legyen $1 < p < \infty$ egy rögzített szám és tegyük fel, hogy ϕ egy izometriája $(S(H), d_p)$-nek. Ekkor van olyan V lineáris vagy konjugált-lineáris izometriája H-nak, mellyel ϕ a

$$\phi(A) = VAV^* \quad (A \in S(H))$$

alakba írható.

A negyedik fejezet következő eredménye megadja $(S(H), d_\infty)$ izometriái struktúráját a $\dim H < \infty$ feltétel mellett.

TÉTEL. (Nagy [**34**])
Tegyük fel, hogy H véges dimenziós és, hogy ϕ egy izometriája $(S(H), d_\infty)$-nek. Ekkor van olyan U unitér vagy antiunitér operátor H-n, mellyel ϕ a

$$\phi(A) = UAU^* \quad (A \in S(H)) \tag{7.3}$$

alakba írható.

A következőkben emlékeztetünk a negyedik fejezet harmadik tételére, mely megadja a $C_p(H)^+$ tér d_p-re vonatkozó szürjektív izometriái általános alakját $(1 < p < \infty)$.

TÉTEL. (Nagy)
Legyen $1 < p < \infty$ és tegyük fel, hogy ϕ egy d_p-re vonatkozó szürjektív izometriája $C_p(H)^+$-nak. Ekkor van olyan U unitér vagy antiunitér operátor H-n, mellyel ϕ az alábbi alakba írható:

$$\phi(A) = UAU^* \quad (A \in C_p(H)^+).$$

Ami a véges dimenziós tereket illeti, az alábbi állításban megadjuk a pozitív szemidefinit komplex mátrixok kúpja d_p-re vonatkozó izometriái struktúráját $(1 < p < \infty)$.

TÉTEL. (Nagy)
Legyen $1 < p < \infty$ és tegyük fel, hogy $\dim H < \infty$. *Ha ϕ egy izometriája $(C_p(H)^+, d_p)$-nek, akkor van olyan $X \in B(H)$ pozitív operátor és olyan U unitér vagy antiunitér operátor H-n, melyekkel ϕ az alábbi alakba írható:*

$$\phi(A) = UAU^* + X \quad (A \in C_p(H)^+).$$

A negyedik fejezetben egy $(C_p(H)_1^+, d_p)$ szürjektív izometriái jellemzésével kapcsolatos állítás is található $(1 < p < \infty)$. A következőkben erre emlékeztetjük az olvasót.

TÉTEL. (Nagy [**34**])
Legyen $1 < p < \infty$ egy szám, és tegyük fel, hogy ϕ egy szürjektív izometriája $(C_p(H)_1^+, d_p)$-nek. Ekkor van olyan U unitér vagy antiunitér operátor H-n, mellyel ϕ az alábbi alakba írható:

$$\phi(A) = UAU^* \quad (A \in C_p(H)_1^+).$$

A negyedik fejezetben ezen állítás egy következménye is megtalálható. Ez az $(S(H), \rho_p)$ metrikus tér izometriáira vonatkozik, ahol $1 \leq p < \infty$ egy szám, ρ_p pedig azon függvény, mely a

$$\rho_p(A, B) = d_p\left(A^{\frac{1}{p}}, B^{\frac{1}{p}}\right) \quad (A, B \in S(H))$$

formulával van definiálva. Ezen metrikákat a sűrűségoperátorok által reprezentált kvantummechanikai objektumok, a kevert állapotok távolságának mérésére vezették be a kvantum-információelméletben. Ami az említett következményt illeti, abban megadjuk $(S(H), \rho_p)$ szürjektív izometriái általános alakját. Ezen állítás alább található.

Következmény. (Nagy)
Legyen $1 < p < \infty$ *és tegyük fel, hogy* ϕ *egy szürjektív izometriája* $(S(H), \rho_p)$*-nek. Ekkor megadható olyan* U *unitér-antiunitér operátor* H*-n, mellyel* ϕ *a* (7.3) *alakba írható.*

Az összefoglalást az ötödik fejezetben található eredmények áttekintésével tesszük teljessé. Ezen tételek a $\dim H = 2$ esetben a H-n ható pozitív invertálható operátorok tere Thompson-metrikára illetve Hilbert projektív metrikára vonatkozó szürjektív izometriái struktúrájával kapcsolatosak. Az alábbiakban emlékeztetünk ezen mennyiségek definíciójára. Legyen X egy valós normált tér, $K \neq \emptyset$ pedig egy zárt konvex kúp X-ben, melyre teljesül a $K \cap (-K) = \{0\}$ egyenlőség. Definiáljunk egy $\leq$ relációt X-n a következő módon. Adott $x, y \in X$ elempár esetén $x \leq y$ akkor és csak akkor, ha $y - x \in K$. Ezek után a $\sim$ ekvivalencia relációt $K \setminus \{0\}$-n a következőképpen értelmezzük. Adott $x, y \in K \setminus \{0\}$ esetén $x \sim y$ pontosan akkor, ha léteznek olyan s, t pozitív valós számok, melyekre $sx \leq y \leq tx$. A $K \setminus \{0\}$ halmaz $\sim$ által indukált ekvivalencia osztályait komponenseknek nevezik. Legyen C egy komponens, és tetszőleges $x, y \in C$ elempár esetén definiáljuk az $M(x/y)$ mennyiséget az $M(x/y) = \inf\{t > 0 \mid x \leq ty\}$ formulával. A d_T Thompson-metrika C-n a

$$d_T(x, y) = \ln \max\{M(x/y), M(y/x)\} \quad (x, y \in C)$$

módon van értelmezve. A d_H Hilbert projektív metrikát C-n a

$$d_H(x, y) = \ln M(x/y)M(y/x) \quad (x, y \in C)$$

módon definiáljuk. Megjegyezzük, hogy d_H valójában nem metrika, hanem szemimetrika.

Jól ismert tény, hogy a H-n ható pozitív operátorok $B(H)^+$ halmaza egy nemüres, zárt konvex kúp $B(H)$-ban, mint valós vektortérben, és $B(H)^+ \cap (-B(H)^+) = \{0\}$. Továbbá a $B(H)^+$-beli invertálható elemek $B(H)^+_{-1}$ halmaza komponense $B(H)^+ \setminus \{0\}$-nak. A [**24**] dolgozatban Molnár a $\dim H \geq 3$ feltétel mellett meghatározta a $B(H)^+_{-1}$ tér d_T-re, illetve d_H-ra vonatkozó szürjektív izometriái általános alakjait. Az ötödik fejezetben és az alábbiakban is megtalálható eredmények az idézett cikkben szereplő tételeknek a kétdimenziós esetre való kiterjesztései.

TÉTEL. (Molnár, Nagy [**26**])
Tegyük fel, hogy H *kétdimenziós és* ϕ *szürjektív izometriája* $(B(H)^+_{-1}, d_T)$*-nek. Ekkor van olyan* S *bijektív, lineáris vagy konjugált-lineáris operátor* H*-n, mellyel* ϕ *az alábbi alakok valamelyikébe írható:*

$$\phi(A) = SAS^* \quad (A \in B(H)^+_{-1}), \qquad \phi(A) = SA^{-1}S^* \quad (A \in B(H)^+_{-1}).$$

TÉTEL. (Molnár, Nagy [**26**])
Tegyük fel, hogy H *kétdimenziós és* $\phi\colon B(H)^+_{-1} \to B(H)^+_{-1}$ *egy* d_H*-ra vonatkozó szürjektív izometria. Ekkor van olyan* S *bijektív, lineáris vagy konjugált-lineáris operátor* H*-n és olyan* $\tau\colon B(H)^+_{-1} \to]0, \infty[$ *függvény, melyekkel* ϕ *az alábbi alakba írható:*

$$\phi(A) = \tau(A) SAS^* \quad (A \in B(H)^+_{-1}).$$

Talks held by the author

(1) Some preserver problems on quantum structures, *9th Katowice-Debrecen Winter Seminar on Functional Equations and Inequalities,* Bedlewo, Poland, 2009 February 4–7.

(2) Some preserver problems on quantum states, *The 5th International Students' Conference on Analysis,* Szare, Poland, 2009 February 7–10.

(3) Some preserver problems on quantum states, *Quantum structure 2009*, Kocovce, Slovakia, 2009 February 24–27.

(4) Preservers on sets of positive operators, *The 6th International Students' Conference on Analysis,* Noszvaj, Hungary, 2010 January 31–February 3.

(5) Isometries of spaces of positive operators, *The Sixth Conference on Function Spaces,* Edwardsville, USA, 2010 May 18–22.

(6) Transformations on density operators, *Quantum Structures 2010 Boston,* Boston, USA, 2010 June 21–26 (2010 Best Paper Award of the International Quantum Structures Association).

(7) Transformations on spaces of positive operators, *János Bolyai Memorial Conference,* Budapest, Hungary, 2010 August 30.

(8) Entropy preserving maps and isometries on quantum structures, *The 7th International Students' Conference on Analysis,* Wisla, Poland, 2011 February 5–8.

(9) Sűrűségoperátorokra vonatkozó azonossági lemmák és következményeik, *Az Analízis Tanszék Síkfőkúti Szemináriuma,* Noszvaj, Hungary, 2011 May 27–29.

(10) Relative entropy preserving maps and isometries on density operators, *22nd International Workshop on Operator Theory and its Applications,* Seville, Spain, 2011 July 3–9.

(11) Isometries on positive operators via identification lemmas, *12th Debrecen-Katowice Winter Seminar on Functional Equations and Inequalities,* Hajdúszoboszló, Hungary, 2012 January 25–28.

(12) Isometries of positive operators and identification lemmata, *The 8^{th} International Students' Conference on Analysis,* Noszvaj, Hungary, 2012 January 28–31.

(13) Isometries of positive operators with unit norm, *50^{th} International Symposium on Functional Equations,* Hajdúszoboszló, Hungary, 2012 June 17–24 (ISFE-Medal for outstanding contribution to the 50^{th} International Symposium on Functional Equations).

(14) On some isometries of density operators, *Quantum Structures 2012,* Cagliari, Italy, 2012 July 23–27.

(15) Maps on sets of density operators preserving the Holevo quantity, *13^{th} Katowice-Debrecen Winter Seminar on Functional Equations and Inequalities,* Zakopane, Poland, 2013 January 30–February 2.

(16) Maps on sets of density operators preserving the Holevo quantity, *The 9^{th} International Students' Conference on Analysis,* Ustroń, Poland, 2013 February 2–5.

Publications of the author

1. G. Nagy, *Commutativity preserving maps on quantum states*, Rep. Math. Phys. **63** (2009), 447–464.
2. L. Molnár and G. Nagy, *Thompson isometries on positive operators: The 2-dimensional case,* Electron. J. Linear Algebra **20** (2010), 79–89.
3. L. Molnár and G. Nagy, *Isometries and relative entropy preserving maps on density operators*, Linear Multilinear Algebra **60** (2012), 93–108.
4. G. Nagy, *Isometries on positive operators of unit norm,* Publ. Math. Debrecen, **82** (2013), 183–192.
5. L. Molnár, G. Nagy and P. Szokol, *Maps on density operators preserving quantum f-divergences*, Quantum Inf. Process., to appear, DOI 10.1007/s11128-013-0528-6.
6. L. Molnár and G. Nagy, *Transformations on density operators that leave the Holevo bound invariant*, submitted.

Bibliography

[1] M. Aizenman, H. Araki and E. H. Lieb, *The State of Matter: a volume dedicated to E. H. Lieb,* World Scientific, Singapore, 1994.

[2] V. Bargmann, *Note on Wigner's theorem on symmetry operations,* J. Math. Phys. **5** (1964), 862-–868.

[3] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa and B. Schumacher, *Noncommuting mixed states cannot be broadcast*, Physical Review Letters **76** (1996), 2818–2821.

[4] R. Bhatia, *Perturbation bounds for matrix eigenvalues*, Longman, Essex and Wiley, New York, 1987.

[5] M. Brešar and P. Šemrl, *Linear preservers on B(X),* Banach Cent. Publ. **38** (1997), 49-–58.

[6] J. Briët and P. Harremoës, *Properties of classical and quantum Jensen-Shannon divergence,* Phys. Rev. A, **79** (2009), 052311.

[7] M. Brin and G. Stuck, *Introduction to dynamical systems,* Cambridge University Press, Cambridge, 2002.

[8] P. Busch, *Stochastic Isometries in Quantum Mechanics,* Math. Phys., Analysis and Geometry, **2** (1999), 83–106.

[9] G. Cassinelli, E. De Vito, P. Lahti and A. Levrero, *Symmetry groups in quantum mechanics and the theorem of Wigner on the symmetry transformations,* Rev. Math. Phys. **8** (1997), 921—941.

[10] J. B. Conway, *A Course in Functional Analysis,* Springer-Verlag, New York, 1985.

[11] R. J. Fleming and J. E. Jamison, *Isometries on Banach Spaces: Function Spaces,* CRC Press, Boca Raton, FL, 2003.

[12] R. J. Fleming and J. E. Jamison, *Isometries on Banach Spaces: Vector-valued Function Spaces and Operator Spaces*, CRC Press, Boca Raton, FL, 2007.

[13] G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen,* Sitzungsber. Königl. Preuss. Akad. Wiss. Berlin (1897), 994— 1015.

[14] F. Hiai and D. Petz, *The proper formula for relative entropy and its asymptotics in quantum probability,* Comm. Math. Phys. **143** (1991), 99–114.

[15] C. K. Li and S. Pierce, *Linear preserver problems,* Amer. Math. Monthly **108** (2001), 591—605.

[16] C. K. Li and N. K. Tsing, *Linear preserver problems: A brief introduction and some special techniques,* Linear Algebra Appl. **162-164** (1992), 217-–235.

[17] Z. Ma, F. Zhang and J. Chen, *Metrics Of Quantum States* (2009), arXiv:0812.3016v4.

[18] A. P. Majtey, P. W. Lamberti and D. P. Prato, *Jensen-Shannon divergence as a measure of distinguishability between mixed quantum states,* Phys. Rev. A **72** (2005), 052310.

[19] C. A. McCarthy, c_p, Israel J. Math **5** (1967), 249–271.

[20] L. Molnár, *Fidelity preserving maps on density operators*, Rep. Math. Phys. **48** (2001), 299-303.

[21] L. Molnár, *Non-linear Jordan triple automorphisms of sets of self-adjoint matrices and operators,* Studia Math. **173** (2006), 39–48.

[22] L. Molnár, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces,* Springer, Berlin Heidelberg, 2007.

[23] L. Molnár, *Maps on states preserving the relative entropy,* J. Math. Phys. **49** (2008), 032114.

[24] L. Molnár, *Thompson isometries of the space of invertible positive operators,* Proc. Amer. Math. Soc. **137** (2009), 3849–3859.

[25] L. Molnár, *Order automorphisms on positive definite operators and a few applications*, Linear Algebra Appl., **434** (2011), 2158–2169.

[26] L. Molnár and G. Nagy, *Thompson isometries on positive operators: The 2-dimensional case,* Electron. J. Linear Algebra **20** (2010), 79–89.

[27] L. Molnár and G. Nagy, *Isometries and relative entropy preserving maps on density operators*, Linear Multilinear Algebra **60** (2012), 93–108.

[28] L. Molnár and P. Šemrl, *Nonlinear commutativity preserving maps on self-adjoint operators*, Quart. J. Math. **56** (2005), 589-595.

[29] L. Molnár and P. Szokol, *Maps on states preserving the relative entropy II*, Linear Algebra Appl., **432** (2010), 3343–3350.

[30] L. Molnár and W. Timmermann, *Isometries of quantum states*, J. Phys. A: Math. Gen. **36** (2003), 267–273.

[31] L. Molnár and W. Timmermann, *Maps on quantum states preserving the Jensen-Shannon divergence*, J. Phys. A: Math. Theor. **42** (2009), 015301.

[32] L. Molnár and W. Timmermann, *Transformations on bounded observables preserving measure of compatibility*, Int. J. Theor. Phys. **50** (2011), 3857-3863.

[33] G. Nagy, *Commutativity preserving maps on quantum states*, Rep. Math. Phys. **63** (2009), 447–464.

[34] G. Nagy, *Isometries on positive operators of unit norm,* Publ. Math. Debrecen, **82** (2013), 183–192.

[35] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information,* Cambridge University Press, Cambridge, 2000.

[36] M. Ohya and D. Petz, *Quantum Entropy and its Use,* Springer-Verlag, Heidelberg, 1993.

[37] S. Pierce et al., *A survey of linear preserver problems,* Linear and Multilinear Algebra **33** (1992), 1–129.

[38] J. R. Ringrose, *Compact Non-self-adjoint Operators,* Van Nostrand Reinhold, London, 1971.

[39] C. S. Sharma and D. F. Almeida, *A direct proof of Wigner's theorem on maps which preserve transition probabilities between pure states of quantum systems,* Ann. Phys. **197** (1990), 300-–309.

[40] B. Simon, *Quantum dynamics: from automorphism to hamiltonian,* Princeton University Press, Princeton, 1976, pp. 327—349.

[41] K. Temme, M. J. Kastoryano, M. B. Ruskai, M. M. Wolf and F. Verstraete, *The* χ^2 *– divergence and mixing times of quantum Markov processes,* J. Math. Phys. **51** (2010), 122201.

[42] A. C. Thompson, *On certain contraction mappings in a partially ordered vector space,* Proc. Amer. Math. Soc. **14** (1963), 438–443.

[43] U. Uhlhorn, *Representation of symmetry transformations in quantum mechanics*, Ark. Fysik **23** (1963), 307-340.

[44] A. Uhlmann, *On "partial" fidelities*, Rep. Math. Phys. **45** (2000), 407-418.