# NON-SMOOTH CRITICAL POINT THEORIES WITH APPLICATIONS IN ELLIPTIC PROBLEMS AND THE THEORY OF GEODESICS

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# Contents

# Preface

Many problems in analysis can be reduced into the form of functional equations $A(u) = 0$, the solution $u$ being an admissible function belonging to some Banach space $X$. Generally, these equations are nonlinear. A particular class of functional equations is the class of Euler-Lagrange equations

$$E'(u) = 0 \tag{1}$$

for a functional $E$ on $X$ which is Fréchet differentiable with derivative $E'$. For these kinds of equations an extensive theory has been developed; the collection of various methods to find solutions for the equation (12) is called *Critical Point Theory.* An element $u$ which satisfies the equation (12) is a critical point of $E$. Many problems from mathematical physics, differential geometry, optimal control and numerical analysis can be reduced to the problem of finding critical points of a suitable function. For instance, in the theory of PDEs many elliptic boundary value problems can be solved by this argument, constructing an appropriate function whose critical points will be the (weak) solutions of the considered problem. In geometry, the geodesics are the critical points of the energy functional.

However, in practice, one often encounters problems where the appropriate function for a problem is *non-smooth.*

Chang [**Ch(1981)**] studied the problem

$$-\Delta u = \Psi(u), \tag{2}$$

which is subjected to a standard Dirichlet boundary condition on a bounded domain $\Omega$, and $\Psi$ is only a locally bounded measurable function without any continuity property. The corresponding functional to (13) is of the form

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega \int_0^{u(x)} \Psi(t)dtdx$$

defined on $H_0^1(\Omega)$. Since $\Psi$ is not continuous, $E$ may be not differentiable, only a locally Lipschitz function. Motivated by this kind of problems, in 1981, Chang [**Ch(1981)**] extended the critical point theory to *locally Lipschitz functions*; the generalized gradient of Clarke [**Cl(1983)**] being the basic tool in his investigations.

Later, in 1994, Degiovanni and Marzocchi [**DMa(1994)**] introduced the notion of weak slope for *continuous functions* as a new generalization of the classical critical point theory.

We point out that the set of critical points in the sense of Chang of a given locally Lipschitz function may be strictly larger than the set in the sense of Degiovanni and Marzocchi. Therefore, these two theories are different and the results, obtained from them, in general can not be compared. The collection of the results of the above theories is called *Non-smooth Critical Point Theory.*

Both critical point theories have several applications: in the theory of PDEs, see Chang [**Ch(1981)**], Motreanu [**Mot(1995)**]; in the theory of hemivariational inequalities, see Motreanu and Panagiotopoulos [**MP(1999)**]; in the theory of geodesics, see Canino [**Ca(1988)**], Degiovanni and Morbini [**DMo(1999)**]. We remark that Frigon [**F(1998)**] developed a critical point theory for *set-valued maps* (based mainly on the theory of Degiovanni and Marzocchi), observing that it is more natural to work on the graph of the map than on its domain (see also [**KV(2001)**], [**KV(2002)**]).

The present thesis is based on these theories. In order to establish our results, we will apply a set of results from the non-smooth critical point theory.

In Chapter 1 notions and results of Non-smooth Analysis which are used in the next chapters are presented. In Section 1.1 some elements of the theory of critical points for locally Lipschitz and continuous functions are given. In Section 1.2 we recall the non-smooth versions of the Mountain Pass Theorem of Ambrosetti-Rabinowitz [**AR(1973)**], the Fountain Theorem of Bartsch [**Ba(1993)**], the Principle of Symmetric Criticality of Palais [**Pal(1979)**], and a recent result of Ricceri [**Ric(2000)-1**].

In Chapter 2 we study two different type of hemivariational inequalities on *unbounded domains.* These kind of problems appear in the Mathematical Physics (Schrödinger, Klein-Gordon equations). The "classical" theory of hemivariational inequalities has been developed only for problems which are formulated on *bounded domains,* see [**MP(1999)**], [**NP(1995)**], [**Pan(1985)**]. In Section 2.2 we study the existence of infinitely many radial respective non-radial solutions for a class of hemivariational inequalities in $\mathbb{R}^N$, see [**K(2003)-1**]. Section 2.3 is devoted to the study of the multiplicity of solutions for a class of eigenvalue problems for hemivariational inequalities in *strip-like domains,* i.e. domains of the type $\Omega = \omega \times \mathbb{R}^{N-m}$, where $\omega$ is an open bounded domain in $\mathbb{R}^m$ and $N \geq m+2$, see [**K(2003)-2**].

In Chapter 3 we restrict our attention to the study of quasilinear elliptic *systems* of gradient type on strip-like domains. In Section 3.2 we obtain a new Boccardo-de Figueiredo [**BdF(2002)**] type result by means of Mountain Pass Theorem, see [**K(2003)-3**]. In Section 3.3 we study the corresponding eigenvalue problem to the above one, guaranteeing the existence of an open interval $\Lambda \subset [0, \infty[$ such that for all $\lambda \in \Lambda$ our problem has at least three distinct solutions, see [**K(2003)-4**]. This result is based on the abstract critical point theorem of Ricceri [**Ric(2000)-1**].

Chapter 4 is devoted to study the existence and multiplicity of geodesics in various settings. The purpose of Section 4.2 is to examine the existence and the number of Finsler-geodesics joining orthogonally two submanifolds $M_1$ and $M_2$ when a Finsler metric is given on a complete Riemannian manifold $M$. Under various conditions on $M$ and on the submanifolds $M_1$ and $M_2$, we state several multiplicity results, see [**KKV(2003)**]. In Section 4.3 we study the existence of isometry-invariant geodesics on certain subset $M$ of $\mathbb{R}^n$ which has Lipschitz boundary. Namely, if $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isometry of finite order which leaves invariant $M$, under suitable topological restrictions the existence of a general $A_0$-invariant geodesic on $M$ is proven, see [**KKV(2001)**]. This result is based on the critical point theory of Degiovanni and Marzocchi [**DMa(1994)**] and can be considered as a natural generalization of Grove's results [**Gro(1973)**], [**Gro(1974)**].

The results of Chapters 2, 3 and 4 are presented in the papers [**K(2003)-1**], [**K(2003)-4**], [**K(2003)-3**], [**K(2003)-2**], [**KKV(2001)**] and [**KKV(2003)**].

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CHAPTER 1

# Preliminaries

## 1.1. Critical point notions for non-smooth functions

Let $(X, \|\cdot\|)$ be a real normed space, and let $h : X \to \mathbb{R}$ be a function.

DEFINITION 1.1. *Let* $h \in C^1(X, \mathbb{R})$. *A point* $u \in X$ *is a* critical point *of* $h$ *if* $h'(u) = 0$.

A function $h : X \to \mathbb{R}$ is called *locally Lipschitz* if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|h(u_1) - h(u_2)| \leq L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$. The *generalized directional derivative* of $h$ at the point $u \in X$ in the direction $z \in X$ is

$$h^0(u; z) = \limsup_{w \to u,\, t \to 0^+} \frac{h(w + tz) - h(w)}{t}.$$

Denoting by $X^*$ the topological dual of $X$, the *generalized gradient* of $h$ at $u \in X$ is defined by

$$\partial h(u) = \{x^* \in X^* : \langle x^*, z\rangle_X \leq h^0(u; z) \text{ for all } z \in X\},$$

which is a nonempty, convex and $w^*$-compact subset of $X^*$, where $\langle\cdot,\cdot\rangle_X$ is the duality pairing between $X^*$ and $X$.

We say that $h$ is *regular at* $u \in X$ *in the sense of Clarke* [**Cl(1983)**] (shortly, regular at $u \in X$), if for all $z \in X$ the usual one-sided directional derivative

$$h'(u; z) = \lim_{t \to 0^+} \frac{h(u + tz) - h(u)}{t}$$

exists and $h'(u; z) = h^0(u; z)$. $h$ is *regular on* $X$ *in the sense of Clarke* (shortly, *regular on* $X$) if it is regular at every point $u \in X$.

DEFINITION 1.2. [**Ch(1981)**] *Let* $h : X \to \mathbb{R}$ *be a locally Lipschitz function. A point* $u \in X$ *is a critical point of* $h$ *if* $0 \in \partial h(u)$, *that is,* $h^0(u; w) \geq 0$ *for all* $w \in X$.

This notion was introduced by Chang [**Ch(1981)**] and it is clear that if $h \in C^1(X, \mathbb{R})$, Definition 1.2 reduces to Definition 1.1. We define $m_h(u) = \inf\{\|x^*\|_X : x^* \in \partial h(u)\}$ (we used the notation $\|x^*\|_X$ instead of $\|x^*\|_{X^*}$). Since $\partial h(u)$ is $w^*$-compact, $m_h(u)$ is attained.

In the sequel, we deal with functions which are not necessarily locally Lipschitz. Given a metric space $(X, d)$, we denote

$$B(u; \delta) = \{v \in X : d(u, v) < \delta\}$$

for each $u \in X$ and each real number $\delta > 0$. The set $B(u;\delta)$ is called the *open ball* of radius $\delta$ around $u$.

DEFINITION 1.3. [**DMa(1994)**] *Let $(X,d)$ be a metric space, let $f : X \to \mathbb{R}$ be a continuous function, and let $u \in X$. We denote by $|df|(u)$ the supremum of the numbers $\sigma \in [0,\infty[$ such that there exist a number $\delta > 0$ and a continuous map $\mathcal{H} : B(u;\delta) \times [0,\delta] \to X$ such that for all $v \in B(u;\delta)$ and every $t \in [0,\delta]$ the following inequalities hold:*

$$d(\mathcal{H}(v,t),v) \leq t;$$
$$f(\mathcal{H}(v,t)) \leq f(v) - \sigma t.$$

The extended real number $|df|(u)$ is called the *weak slope* of the function $f$ at $u$. This notion was introduced by Degiovanni and Marzocchi [**DMa(1994)**] and independently by Ioffe and Schwartzmann [**IS(1996)**].

By means of $d$ the set $X \times \mathbb{R}$ can be endowed with the metric $\rho : (X \times \mathbb{R}) \times (X \times \mathbb{R}) \to \mathbb{R}$ defined by

$$\rho((u,b),(v,c)) = \sqrt{d(u,v)^2 + |b-c|^2},$$

whenever $(u,b),\ (v,c) \in X \times \mathbb{R}$.

Given a function $f : X \to \mathbb{R} \cup \{\infty\}$, we set

$$\mathcal{D}(f) = \{u \in X : f(u) < \infty\};$$

$$\mathrm{epi}(f) = \{(u,\alpha) \in X \times \mathbb{R} : f(u) \leq \alpha\}.$$

The set $\mathrm{epi}(f)$ will be endowed with the metric $d_e$ defined by

$$d_e = \rho|_{\mathrm{epi}(f)\times\mathrm{epi}(f)}.$$

Further, we define the function $\mathcal{G}_f : \mathrm{epi}(f) \to \mathbb{R}$ by $\mathcal{G}_f(u,\xi) = \xi$. Since this function is Lipschitz continuous of constant 1, we have

$$|d_e\mathcal{G}_f|(u,\xi) \leq 1 \text{ for all } (u,\xi) \in \mathrm{epi}(f).$$

Moreover, the following proposition holds.

PROPOSITION 1.1. [**DMa(1994)**] *Let $(X,d)$ be a metric space, let $f : X \to \mathbb{R}$ be a continuous function, and let $(u,\xi) \in \mathrm{epi}(f)$. Then*

$$|d_e\mathcal{G}_f|(u,\xi) = \begin{cases} \frac{|df|(u)}{\sqrt{1+|df|(u)^2}}, & \text{if } f(u) = \xi \text{ and } |df|(u) < \infty; \\ 1, & \text{if } f(u) < \xi \text{ or } |df|(u) = \infty. \end{cases}$$

By using $\mathcal{G}_f$ the notion of the weak slope of a continuous function can be extended to lower semicontinuous functions.

DEFINITION 1.4. *Let $(X,d)$ be a metric space, let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function, and let $u \in \mathcal{D}(f)$ be a fixed point. We set*

$$|df|(u) = \begin{cases} \frac{|d_e\mathcal{G}_f|(u,f(u))}{\sqrt{1-|d_e\mathcal{G}_f|^2(u,f(u))}}, & \text{if } |d_e\mathcal{G}_f|(u,f(u)) < 1; \\ \infty, & \text{if } |d_e\mathcal{G}_f|(u,f(u)) = 1. \end{cases}$$

DEFINITION 1.5. [**DMa(1994)**] *Let $(X,d)$ be a metric space, let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function, and let $u \in \mathcal{D}(f)$. We say that $u$ is a* critical point *of $f$ if $|df|(u) = 0$. $K = \{u \in X : |df|(u) = 0\}$ is the set of critical points of $f$. $K_c = K \cap f^{-1}(c)$, $c \in \mathbb{R}$.*

Let us mention here that $|df|(u) = \|f'(u)\|$ when $X$ is a Finsler manifold of class $C^1$ and $f$ is of class $C^1$ (see [**DMa(1994)**]). In particular, Definition 1.5 reduces to Definition 1.1. When $f$ is a locally Lipschitz continuous function, the situation changes dramatically.

PROPOSITION 1.2. [**DMa(1994)**] *Let $X$ be a Banach space and let $f : X \to \mathbb{R}$ be a locally Lipschitz function. Then*

$$|df|(u) \geq \min\{\|x^*\| : x^* \in \partial f(u)\} \text{ for all } u \in X.$$

The above inequality may be strict. To see this, we give a simple example.

EXAMPLE 1.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = a|y - m|x|| - \sigma x$, with $a, m, \sigma > 0$ and $am \geq \sigma$. Then $0 \in \partial f(0)$, but $|df|(0) > 0$.

The above example shows that the set of critical points in the sense of Definition 1.2 of a given locally Lipschitz continuous function may be strictly larger than the set in the sense of Definition 1.5.

## 1.2. Some results in critical point theory

Let $(X, \|\cdot\|)$ be a real normed space, and let $h : X \to \mathbb{R}$ be a locally Lipschitz function.

DEFINITION 1.6. [**Ch(1981)**] *The function $h$* satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ *(shortly $(PS)_c$), if every sequence $\{u_n\} \subset X$ such that $h(u_n) \to c$ and $m_h(u_n) \to 0$ contains a convergent subsequence in the norm of $X$.*

DEFINITION 1.7. [**KP(2000)**] *The function $h$* satisfies the Cerami condition at level $c \in \mathbb{R}$ *(shortly $(C)_c$), if every sequence $\{u_n\} \subset X$ such that $h(u_n) \to c$ and $(1 + \|u_n\|)m_h(u_n) \to 0$ contains a convergent subsequence in the norm of $X$.*

It is clear that $(PS)_c$ implies $(C)_c$. In [**KMV(2003)**], we introduced the following compactness condition.

DEFINITION 1.8. [**KMV(2003)**] *Let $\varphi : X \to \mathbb{R}$ be a globally Lipchitz function such that $\varphi(x) \geq 1$ for all $x \in X$. The function $h$* satisfies the $(\varphi - C)$–condition at level $c \in \mathbb{R}$ *(shortly $(\varphi - C)_c$), if every sequence $\{u_n\} \subset X$ such that $h(u_n) \to c$ and $\varphi(u_n)m_h(u_n) \to 0$ contains a convergent subsequence in the norm of $X$.*

The compactness $(\varphi - C)_c$–condition globalizes Definitions 1.6 and 1.7. Indeed, if $\varphi \equiv 1$ we have the $(PS)_c$-condition while if $\varphi(x) = 1 + \|x\|$ we obtain the $(C)_c$-condition. Moreover, the $(\varphi - C)_c$-condition suffices to give rise to a deformation lemma and the corresponding *Mountain Pass Theorem.*

THEOREM 1.1. [**KMV(2003)**] *Let $X$ be a Banach space, let $h : X \to \mathbb{R}$ be a locally Lipschitz function with $h(0) = 0$ and $\varphi : X \to \mathbb{R}$ be a globally Lipchitz function such that $\varphi(x) \geq 1$ for all $x \in X$. Suppose that there exist an element $e \in X$ and constants $\rho, \eta > 0$ such that*

*i) $h(u) \geq \eta$ for all $u \in X$ with $\|u\| = \rho$;*

*ii) $\|e\| > \rho$ and $h(e) \leq 0$;*

*iii) $h$ satisfies the $(\varphi - C)_c$ condition, where*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} h(\gamma(t)),$$

*with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$.*

*Then the "minimax" value $c$ in iii) is a critical value of $h$, i.e. there exists $u_0 \in X$ such that $0 \in \partial h(u_0)$ and $h(u_0) = c$. In addition, $c \geq \eta$.*

In order to obtain multiplicity results, we recall the non-smooth version of the *Fountain Theorem* of Bartsch (see [**Ba(1993)**, Theorem 2.25]).

THEOREM 1.2. *Let $E$ be a Hilbert space, $\{e_j : j \in \mathbb{N}\}$ an orthonormal basis of $E$ and set $E_k = \operatorname{span}\{e_1, \ldots, e_k\}$. Let $h : E \to \mathbb{R}$ be an even, locally Lipschitz function such that:*

*i) $h$ satisfies $(C)_c$ for all $c > h(0)$;*

*ii) For all $k \geq 1$ there exists $R_k > 0$ such that $h(u) \leq h(0)$, for all $u \in E_k$ with $\|u\| \geq R_k$;*

*iii) There exist $k_0 \geq 1$, $b > h(0)$ and $\rho > 0$ such that $h(u) \geq b$ for all $u \in E_{k_0}^{\perp}$ with $\|u\| = \rho$.*

*Then $h$ possesses a sequence of critical values $\{c_k\}$ such that $c_k \to \infty$ as $k \to \infty$.*

In the study of eigenvalue problems, we will use the non-smooth version of Ricceri's result (see [**Ric(2000)-1**, Theorem 1]), proved by Marano and Motreanu [**MM(2002)**].

THEOREM 1.3. [**MM(2002)**, Theorem B] *Let $X$ be a separable and reflexive Banach space, let $\Psi_1, \Psi_2 : X \to \mathbb{R}$ two locally Lipschitz functions and $\Lambda$ be a real interval. Suppose that:*

*i) $\Psi_1$ is weakly sequentially lower semicontinuous while $\Psi_2$ is weakly sequentially continuous.*

*ii) For every $\lambda \in \Lambda$, the function $\Psi_1 + \lambda\Psi_2$ satisfies $(PS)_c$, $c \in \mathbb{R}$, together with $\lim_{\|u\| \to \infty}(\Psi_1(u) + \lambda\Psi_2(u)) = \infty$.*

*iii) There exists a continuous concave function $h : \Lambda \to \mathbb{R}$ such that*

$$\sup_{\lambda \in \Lambda} \inf_{u \in X}(\Psi_1(u) + \lambda\Psi_2(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \in \Lambda}(\Psi_1(u) + \lambda\Psi_2(u) + h(\lambda)).$$

*Then there is an open interval $\Lambda_0 \subseteq \Lambda$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda_0$ the function $\Psi_1 + \lambda\Psi_2$ has at least three critical points in $X$ having norm less than $\sigma$.*

Let $G$ be a compact Lie group which acts on the real Banach space $(X, \|\cdot\|)$, that is, the action $G \times X \to X : \ (g, u) \mapsto gu$ is continuous. A function $h : X \to \mathbb{R}$

is $G-$*invariant* if $h(gu) = h(u)$ for all $g \in G$ and $u \in X$. Denoting by $X^G \stackrel{\text{not.}}{=} \text{Fix}_G X = \{u \in X : gu = u \text{ for all } g \in G\}$, we have the *Principle of Symmetric Criticality*, proved by Krawcewicz and Marzantowicz [**KM(1990)**, p. 1045].

THEOREM 1.4. *Let $h : X \to \mathbb{R}$ be a $G$-invariant, locally Lipschitz functional. If $h_G$ denotes the restriction of $h$ to $X^G$ and $u \in X^G$ is a critical point of $h_G$ then $u$ is a critical point of $h$.*

The next result will be used in the study of isometry-invariant geodesics.

THEOREM 1.5. [**DMo(1999)**, Proposition 2.3] *Let $(X, d)$ be a metric space, let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function, and let $(u, \lambda) \in \text{epi}(f)$. We assume that there exist $\delta, c, \sigma > 0$ and a continuous map $\mathcal{H} : \{\ v \in B(u; \delta) : f(v) < \lambda + \delta\ \} \times [0, \delta] \to M$ such that for all $v \in B(u; \delta)$ with $f(v) < \lambda + \delta$ and all $t \in [0, \delta]$ we have*

$$d(\mathcal{H}(v, t), v) \le ct,$$
$$f(\mathcal{H}(v, t)) \le f(v) - \sigma t.$$

*Then we have*

$$|d_e \mathcal{G}_f|(u, \lambda) \ge \frac{\sigma}{\sqrt{\sigma^2 + c^2}}.$$

*In particular, if $\lambda = f(u)$, then we have $|df|(u) \ge \sigma/c$.*

CHAPTER 2

# Hemivariational inequalities in unbounded domains

## 2.1. Introduction

The study of *variational inequalities* began in the sixties with the pioneering work of Lions and Stampacchia [**LS(1967)**]. The connection of this theory with the notion of the subdifferential of a convex function was achieved by Moreau [**Mor(1968)**], who introduced the notion of convex superpotentials which permitted the formulation and study in the weak form of a wide ranging class of complicated problems in Mechanics and Engineering (see Duvaut and Lions [**DL(1976)**]). All the inequality problems studied in that period were related to convex energy functions and therefore were linked with the notion of monotonicity. Motivated by some problems from mechanics, Panagiotopoulos introduced in [**Pan(1993), Pan(1985)**] the notion of nonconvex superpotential by using the generalized gradient of Clarke. Due to the lack of convexity, new types of variational expressions were obtained; these are the so-called *Hemivariational Inequalities.* The hemivariational inequalities appears as a generalization of the variational inequalities, but actually they are much more general than these ones, because they are not equivalent to minimum problems. They are no longer connected with monotonicity, but since the main ingredient of their study is based on the notion of Clarke subdifferential of a locally Lipschitz funtion, the theory of hemivariational inequalities appears as a new field of Non-smooth Analysis.

For a comprehensive treatment of the hemivariational inequality problems we refer to the monographs Naniewicz and Panagiotopoulos [**NP(1995)**] (based on pseudomonotonicity), Motreanu and Panagiotopoulos [**MP(1999)**], Motreanu and Rădulescu [**MR(2003)**] (based on compactness arguments).

In the above works (and in references therein) there are studied elliptic problems on *bounded domains.* In this chapter we treat two different kinds of hemivariational inequalities problems on *unbounded domains.* In the unbounded case the problem is more delicate, due to the lack of compactness in the Sobolev embeddings. Variational and/or topological methods are combined with different technics to overcome this difficulty: approximation by bounded sub-domains (see [**E(1983)**]); the use of weighted Sobolev spaces in order to obtain compact embeddings (see [**BHK(1983)**]); the use of Sobolev spaces with symmetric functions (see [**E(1983)**]).

In Section 2.2 we establish the existence of infinitely many radial respective non-radial solutions for a class of hemivariational inequalities in $\mathbb{R}^N$ (Theorems 2.1 and 2.2), applying the Fountain Theorem and the Principle of Symmetric Criticality for locally Lipschitz functions.

In Section 2.3 we study the multiplicity of solutions for a class of eigenvalue problems for hemivariational inequalities in strip-like domains, i.e. domains of the type $\Omega = \omega \times \mathbb{R}^{N-m}$, where $\omega$ is an open bounded domain in $\mathbb{R}^m$ and $N \geq m+2$. The first result gives a new approach to treat eigenvalue problems on strip-like domains. This technic is based on the recent critical-point result of Marano and Motreanu (Theorem 1.3) which will be combined with the Principle of Symmetric Criticality for locally Lipschitz functions, establishing for certain eigenvalues the existence of at least three distinct, axially symmetric solutions for the studied problem (Theorem 2.3). In the case of strip-like domains, the space of axially symmetric functions has been the main tool in the investigations, due to its "good behavior" concerning the compact embeddings (note that $N \geq m+2$, see [**EL(1983)**]); this is the reason why many authors used this space in their works (see [**E(1983), FZ(2001), Gr(1987), T(1998)**]). On the other hand, no attention has been paid in the literature to the existence of axially *non*-symmetric solutions. Therefore, the study of existence of axially non-symmetric solutions for our hemivariation inequality constitutes the second task of this section. The Fountain Theorem provides not only infinitely many axially symmetric solutions but also axially non-symmetric solutions, when $N = m+4$ or $N \geq m+6$ (Theorem 2.4). Some examples are included in the final, illustrating the applicability of our result.

## 2.2. Hemivariational inequalities in $\mathbb{R}^N$

Let $N \geq 2$ and $p \in ]2, 2^*[$, ($2^* = 2N/(N-2)$, if $N \geq 3$ and $2^* = \infty$, if $N = 2$). Let $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (see [**AF(1990)**]) which is locally Lipschitz in the second variable, fulfilling the following condition:

**(F1)** $F(x, 0) = 0$ and there exists $c_1 > 0$ such that

$$|\xi| \leq c_1(|s| + |s|^{p-1})$$

for all $\xi \in \partial F(x, s)$, $s \in \mathbb{R}$, and for a.e. $x \in \mathbb{R}^N$.

The set $\partial F(x, s)$ is the (partial) generalized gradient of $F(x, \cdot)$ at $s \in \mathbb{R}$. Denote by $F^0(x, s; z)$ the (partial) generalized directional derivative of $F(x, \cdot)$ at the point $s \in \mathbb{R}$ in the direction $z \in \mathbb{R}$.

The purpose of this section is to study the following hemivariational inequality problem:

(P) *Find* $u \in H^1(\mathbb{R}^N)$ *such that*

$$\int_{\mathbb{R}^N} (\nabla u \nabla w + uw)dx + \int_{\mathbb{R}^N} F^0(x, u(x); -w(x))dx \geq 0 \quad for\ all\ w \in H^1(\mathbb{R}^N).$$

As usual, $H^1(\mathbb{R}^N)$ is the Sobolev space with the inner product

$$(u, v)_1 = \int_{\mathbb{R}^N} [\nabla u(x) \nabla v(x) + u(x)v(x)]dx$$

and the corresponding norm

$$\|u\|_1 = \left[\int_{\mathbb{R}^N} (|\nabla u(x)|^2 + |u(x)|^2)dx\right]^{1/2}.$$

REMARK 2.1. In particular, let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a measurable (not necessarily continuous) function such that there exists $c > 0$ with

$$|f(x,s)| \leq c(|s| + |s|^{p-1}), \tag{3}$$

for all $s \in \mathbb{R}$ and for a.e. $x \in \mathbb{R}^N$. Let us define $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$F(x,s) = \int_0^s f(x,t)dt \tag{4}$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Then $F$ is a Carathéodory function which is locally Lipschitz in the second variable and $F(x,0) = 0$, a.e. $x \in \mathbb{R}^N$. Moreover, $F$ satisfies the growth condition from (F1). Indeed, since $f(x,\cdot) \in L^\infty_{loc}(\mathbb{R})$ a.e. $x \in \mathbb{R}^N$, by [**MP(1999)**, Proposition 1.7, p.13] we have

$$\partial F(x,s) = [\underline{f}(x,s), \overline{f}(x,s)]$$

for all $s \in \mathbb{R}$ and for a.e. $x \in \mathbb{R}^N$, where

$$\underline{f}(x,s) = \lim_{\delta\to 0^+} \text{essinf}\{f(x,t) : |t-s| < \delta\},$$

and

$$\overline{f}(x,s) = \lim_{\delta\to 0^+} \text{esssup}\{f(x,t) : |t-s| < \delta\}.$$

From the above relation and (3) the desired inequality yields.

Moreover, when $f \in C^0(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, due to (4), the inequality from (P) takes the form

$$\int_{\mathbb{R}^N} (\nabla u \nabla w + uw)dx - \int_{\mathbb{R}^N} f(x,u(x))w(x)dx = 0 \quad \text{for all } w \in H^1(\mathbb{R}^N),$$

i.e. $u \in H^1(\mathbb{R}^N)$ is a weak solution of

$$\begin{cases} -\Delta v + v = f(x,v), \\ v \in H^1(\mathbb{R}^N). \end{cases} \tag{P$'$}$$

Many papers are concerned with the existence and multiplicity of solutions for problems related to (P$'$); see Bartsch and Willem [**BWi(1993)-1, BWi(1993)-2**], Bartsch and Wang [**BWa(1995)**], Strauss [**Stra(1977)**] (in autonomous case), Gidas, Ni and Nirenberg [**GNN(1981)**], Gazzola and Rădulescu [**GR(2000)**], and references therein. The interest in this equation comes from various problems in mathematics and physics (cosmology, constructive field theory, solitary waves, nonlinear Klein-Gordon or Schrödinger equations), see [**AD(1970), CGM(1978), Stra(1977), Rab)1988)**].

Under suitable hypotheses mainly on $f$, Strauss [**Stra(1977)**], Bartsch and Willem [**BWi(1993)-2**], Berestycki and Lions [**BL(1983)**], Struwe [**Stru(1990)**] obtained existence results concerning the radial solutions of problems closely related to (P$'$). Bartsch and Willem [**BWi(1993)-1**] observed that a careful choice of a subgroup of the orthogonal group $O(N)$ in certain dimensions assures the existence of infinitely many non-radial solutions of (P′). In general, a functional of class $C^1$ is constructed which is invariant under a subgroup action of $O(N)$, whose restriction to the appropriate subspace of invariant functions admits critical points. Due to the principle of symmetric criticality of Palais [**Pal(1979)**], these points will be also critical points of the original functional, and they are exactly the radial (resp. non-radial) solutions of (P′), depending on the choice of the subgroup of $O(N)$. We emphasize that in the above works the nonlinear term $f$ is continuous. A good survey for these problems is the book of Willem [**W(1995)**].

In practical problems, in (P′) may appear functions $f$ which are not continuous, see Gazzola and Rădulescu [**GR(2000)**] and the very recent monograph of Motreanu and Rădulescu [**MR(2003)**]. Clearly, in this case the classical framework (described above) is not working. Starting from this point of view, we propose a more general problem, i.e. to study the existence of radial (resp. non-radial) solutions of (P). Our appropriate functionals will be $O(N)$-invariant and only locally Lipschitz; therefore we cannot apply the classical machinery described above. Thanks to Theorem 1.4 we are able to guarantee critical points (in the sense of Chang [**Ch(1981)**]) of the above mentioned functionals, applying a weak version of Theorem 1.2; the corresponding critical points will be radial (resp. non-radial) solutions of (P). These existence theorems improve some results from [**BWi(1993)-1, BWi(1993)-2, Stra(1977), W(1995)**]. On the other hand, we emphasize that our main results can be applied to several concrete cases where the earlier results fail.

In order to obtain existence results, we impose further assumptions on the nonlinear term $F$ :

**(F2)** $F(x,-s) = F(x,s)$ for all $s \in \mathbb{R}$ and for a.e. $x \in \mathbb{R}^N$;

**(F3)** $F(gx,s) = F(x,s)$ for all $s \in \mathbb{R}$, $g \in O(N)$ and for a.e. $x \in \mathbb{R}^N$;

**(F4)** There exist $\alpha > 2$, $\lambda \in [0, (\alpha-2)/2[$ and $c_4 > 0$ such that for a.e. $x \in \mathbb{R}^N$ and all $s \in \mathbb{R}$

$$\alpha F(x,s) + F^0(x,s;-s) - \lambda s^2 \leq 0, \qquad \text{(F4 - a)}$$

and

$$c_4(|s|^\alpha - |s|^2) \leq F(x,s); \qquad \text{(F4 - b)}$$

**(F5)** $\lim_{s\to 0^+} \frac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{|s|} = 0$ uniformly for a.e. $x \in \mathbb{R}^N$.

Let

$$H^1_{O(N)}(\mathbb{R}^N) \overset{\text{not.}}{=} H^1(\mathbb{R}^N)^{O(N)} = \{u \in H^1(\mathbb{R}^N) : gu = u \text{ for all } g \in O(N)\}.$$

The action of $O(N)$ on $H^1(\mathbb{R}^N)$ is defined by $gu(x) = u(g^{-1}x)$ for all $g \in O(N)$, $u \in H^1(\mathbb{R}^N)$ and for a.e. $x \in \mathbb{R}^N$. It is clear that $O(N)$ acts isometrically on $H^1(\mathbb{R}^N)$, i.e. $\|gu\|_1 = \|u\|_1$ for all $g \in O(N)$, $u \in H^1(\mathbb{R}^N)$. The elements of $H^1_{O(N)}(\mathbb{R}^N)$ are the *radial* functions of $H^1(\mathbb{R}^N)$.

The first main result of this section can be formulated as follows.

Theorem 2.1. [**K(2003)-1**] *If the assumptions* (F1)-(F5) *hold, then* (P) *has infinitely many radial solutions.*

The purpose of the following result is to establish the existence of non-radial solutions of (P), which is a non-smooth version of the result of Bartsch and Willem, see [**W(1995)**, Theorem 3.13, p. 63]. A function from $H^1(\mathbb{R}^N)$ is *non-radial,* if does not belong to $H^1_{O(N)}(\mathbb{R}^N)$.

THEOREM 2.2. [**K(2003)-1**] *If the assumptions* (F1)-(F5) *hold and* $N = 4$ *or* $N \geq 6$*, then* (P) *has infinitely many non-radial solutions.*

We denote by $\lfloor u \rfloor$ the nearest integer to $u \in \mathbb{R}$, if $u + \frac{1}{2} \notin \mathbb{Z}$; otherwise we put $\lfloor u \rfloor = u$.

EXAMPLE 2.1. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = \int_0^s \lfloor |t|t \rfloor dt.$$

Then the conclusion of Theorem 2.1 holds for $N \in \{2, 3, 4, 5\}$. Moreover, if $N = 4$, the conclusion of Theorem 2.2 holds too.

### 2.3. Multiple solutions for an eigenvalue problem for hemivariational inequalities in strip-like domains

Throughout this section, $\Omega$ will be a *strip-like domain,* that is $\Omega = \omega \times \mathbb{R}^{N-m}$, where $\omega$ is a bounded open set in $\mathbb{R}^m$ with smooth boundary and $m \geq 1$, $N \geq m+2$. $H^1_0(\Omega)$ is the usual Sobolev space endowed with the inner product $\langle u, v \rangle_0 = \int_\Omega \nabla u \nabla v$ and norm $\| \cdot \|_0 = \sqrt{\langle \cdot, \cdot \rangle_0}$, while the norm of $L^\alpha(\Omega)$ will be denoted by $\| \cdot \|_\alpha$. Since $\Omega$ has the cone property, we have the continuous embedding $H^1_0(\Omega) \hookrightarrow L^\alpha(\Omega)$, $\alpha \in [2, 2^*]$, that is, there exists $k_\alpha > 0$ such that $\|u\|_\alpha \leq k_\alpha \|u\|_0$ for all $u \in H^1_0(\Omega)$. $2^* = 2N(N-2)^{-1}$ is the Sobolev critical exponent.

Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which is locally Lipschitz in the second variable such that

**(F1)** $F(x, 0) = 0$, and there exist $c_1 > 0$ and $p \in ]2, 2^*[$ such that

$$|\xi| \leq c_1(|s| + |s|^{p-1})$$

for all $s \in \mathbb{R}$, $\xi \in \partial F(x, s)$ and a.e. $x \in \Omega$.

In this section we study the following *eigenvalue problem for hemivariational inequalities.* For $\lambda > 0$, denote by (EPHI$_\lambda$):
*Find* $u \in H^1_0(\Omega)$ *such that*

$$\int_\Omega \nabla u \nabla w + \lambda \int_\Omega F^0(x, u(x); -w(x))dx \geq 0 \quad for\ all\ w \in H^1_0(\Omega).$$

REMARK 2.2. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable (not necessarily continuous) function and suppose that there exists $c > 0$ such that $|f(x, s)| \leq c(|s| + |s|^{p-1})$ for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Define $F : \Omega \times \mathbb{R} \to \mathbb{R}$ by $F(x, s) = \int_0^s f(x, t)dt$. Then $F$ is a Carathéodory function which is locally Lipschitz in the second variable which satisfies the growth condition from (F1), see similarly in Remark 2.1.

Moreover, if $f$ is continuous in the second variable, then $\partial F(x,s) = \{f(x,s)\}$ for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Therefore, the inequality from (EPHI$_\lambda$) takes the form

$$\int_\Omega \nabla u \nabla w - \lambda \int_\Omega f(x,u(x))w(x)dx = 0 \quad \text{for all } w \in H_0^1(\Omega),$$

that is, $u \in H_0^1(\Omega)$ is a weak solution of

$$-\Delta u = \lambda f(x,u) \text{ in } \Omega, \quad u \in H_0^1(\Omega), \tag{EP$_\lambda$}$$

in the usual sense.

Under some restrictive conditions on the nonlinear term $f$, (EP$_\lambda$) has been firstly studied by Esteban [**E(1983)**]. Further investigations, closely related to [**E(1983)**] can be found in the papers of Burton [**Bu(1985)**], Fan and Zhao [**FZ(2001)**], Grossinho [**Gr(1987)**], Schindler [**Schi(1992)**], Tersian [**T(1998)**].

We say that a function $h : \Omega \to \mathbb{R}$ is *axially symmetric,* if $h(x,y) = h(x,gy)$ for all $x \in \omega$, $y \in \mathbb{R}^{N-m}$ and $g \in O(N-m)$. In particular, we denote by $H_{0,s}^1(\Omega)$ the closed subspace of axially symmetric functions of $H_0^1(\Omega)$. $u \in H_0^1(\Omega)$ is called *axially non-symmetric,* if it is not axially symmetric.

For the first result, we make the following assumptions on the nonlinearity term $F$.

(**F2**) $$\lim_{s\to 0} \frac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{s} = 0 \quad \text{uniformly for a.e. } x \in \Omega.$$

**(F3)** There exist $q \in ]0,2[$, $\nu \in [2,2^*]$, $\alpha \in L^{\nu/(\nu-q)}(\Omega)$ and $\beta \in L^1(\Omega)$ such that
$$F(x,s) \le \alpha(x)|s|^q + \beta(x)$$
for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

**(F4)** There exists $u_0 \in H_{0,s}^1(\Omega)$ such that $\int_\Omega F(x,u_0(x))dx > 0$.

THEOREM 2.3. [**K(2003)-2**] *Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies* (F1)-(F4) *and $F(\cdot,s)$ is axially symmetric for all $s \in \mathbb{R}$. Then there exist an open interval $\Lambda_0 \subset [0,\infty[$ and a number $\sigma > 0$ such that for all $\lambda \in \Lambda_0$,* (EPHI$_\lambda$) *has at least three distinct solutions which are axially symmetric having $\|\cdot\|_0$-norms less than $\sigma$.*

The following theorem can be considered as an extension of Bartsch and Willem's result (see [**BWi(1993)-1**]) to the case of strip-like domains. We require the following assumption on $F$.

**(F5)** There exist $\nu > 0$ and $\gamma \in L^\infty(\Omega)$ with $\text{essinf}_{x\in\Omega}\gamma(x) = \gamma_0 > 0$ such that
$$2F(x,s) + F^0(x,s;-s) \le -\gamma(x)|s|^\nu$$
for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

THEOREM 2.4. [**K(2003)-2**] *Let* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a function which satisfies* (F1), (F2), *and* (F5) *for some* $\nu > \max\{2, N(p-2)/2\}$. *If* $F$ *is axially symmetric in the first variable and even in the second variable then* (EPHI$_\lambda$) *has infinitely many axially symmetric solutions for all* $\lambda > 0$. *In addition, if* $N = m + 4$ *or* $N \geq m + 6$, (EPHI$_\lambda$) *has infinitely many axially non-symmetric solutions.*

In the final of this section, we give some examples.

EXAMPLE 2.2. Let $p \in ]2, 2^*[$ and $a : \Omega \to \mathbb{R}$ be a continuous, non-negative, not identically zero, axially symmetric function with compact support in $\Omega$. Then there exist an open interval $\Lambda_0 \subset [0, \infty[$ and a number $\sigma > 0$ such that for all $\lambda \in \Lambda_0$, the problem

$$-\Delta u = \lambda a(x)|u|^{p-2}u \cos|u|^p \text{ in } \Omega,\ u \in H_0^1(\Omega)$$

has at least three distinct, axially symmetric solutions which have norms less than $\sigma$.

EXAMPLE 2.3. Let $a : \Omega \to \mathbb{R}$ be as in Example 2.2 and let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by $F(x, s) = a(x) \min\{s^3, |s|^q\}$, where $q \in ]0, 2[$ is a fixed number. The conclusion of Theorem 2.3 holds in dimensions $N \in \{3, 4, 5\}$.

EXAMPLE 2.4. Let $p \in ]2, 2^*[$. Then, for all $\lambda > 0$, the problem

$$-\Delta u = \lambda|u|^{p-2}u \text{ in } \Omega,\ u \in H_0^1(\Omega),$$

has infinitely many axially symmetric solutions. Moreover, if $N = m + 4$ or $N \geq m + 6$, the problem has infinitely many axially non-symmetric solutions.

CHAPTER 3

# Quasilinear elliptic systems of gradient type on strip-like domains

## 3.1. Introduction

In Section 2.3 we treated a class of elliptic problems is strip-like domains in the *scalar* case. In this chapter we are interested in the existence of nontrivial solutions of the *gradient system*

$$\text{(S)} \qquad \begin{cases} -\Delta_p u = F_u(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a strip-like domain in $\mathbb{R}^N$, $1 < p, q < N$, $F_u$ is the partial derivative of $F$ with respect to $u$ (similarly for $F_v$), and $\Delta_\alpha u = \text{div}(|\nabla u|^{\alpha-2}\nabla u)$, $\alpha \in \{p,q\}$.

Systems of the form (S) have been the object of intensive investigations on *bounded domains*; the reader can consult the papers of Boccardo and de Figueiredo [**BdF(2002)**], de Figueiredo [**dF(1998)**], de Nápoli and Mariani [**dNM(2002)**], Felmer, Manásevich and de Thélin [**FMT(1992)**], Vélin and de Thélin [**VT(1993)**].

Recently, Carrião and Miyagaki [**CaMi(2002)**] studied the existence of positive non-trivial solutions of a related problem to (S) (namely, $p = q$) on strip-like domains and on domains which are situated between two infinite cylinders. They assumed that the nonlinear term $F$ has some sort of homogeneity and in addition the right-hand side of $(S_\lambda)$ is perturbed by a gradient type derivative of a $p^*$-homogeneous term. Their approach is based on a suitable version of the concentration compactness principle. Although we do not treat the critical case in the present thesis, we allow the case $p \neq q$ and we do not assume any homogeneity properties on $F$.

In Section 3.2, motivated by the works of Clarke [**Cl(1983)**], P. D. Panagiotopoulos [**Pan(1993), Pan(1985)**], Gazzola and Rădulescu [**GR(2000)**], Motreanu and Papagiotopoulos [**MP(1999)**], and the very recent monograph of Motreanu and Rădulescu [**MR(2003)**], where several non-smooth applications are given, we will *relax* in (S) the *classical regularity condition* of the nonlinear term $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$. More precisely, we will assume only that $F$ is a *locally Lipschitz function* which is regular on $\mathbb{R}^2$ in the sense of Clarke [**Cl(1983)**]. In this setting, problem (S) requires a suitable reformulation which is inspired by the theory of hemivariational inequalities. For simplicity, we consider here only the autonomous case, i.e., $F$ will be supposed to be $x$-independent. Under some growth conditions on $F$ we will guarantee a non-trivial solution of (S) (Theorem 3.1).

In Section 3.3 we study the multiplicity of solutions of the eigenvalue elliptic problem, corresponding to (S):

$$(\mathrm{S}_\lambda)\qquad \begin{array}{ll} -\Delta_p u = \lambda F_u(x,u,v) & \text{in } \Omega \\ -\Delta_q v = \lambda F_v(x,u,v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{array}$$

The main result of this section (Theorem 3.2) states the existence of an open interval $\Lambda \subset [0,\infty[$ such that for all $\lambda \in \Lambda$, the system $(\mathrm{S}_\lambda)$ has at least three distinct solutions.

## 3.2. An existence result for elliptic systems

Let $\Omega$ be a strip-like domain, that is $\Omega = \omega \times \mathbb{R}^{N-m}$, where $\omega$ is a bounded open set in $\mathbb{R}^m$ with smooth boundary and $m \geq 1$, $N \geq m+2$. We assume the following growth conditions on the partial generalized gradients of the locally Lipschitz function $F : \mathbb{R}^2 \to \mathbb{R}$ :

**(F1)** There exist $c_1 > 0$ and $r \in ]p,p^*[$, $s \in ]q,q^*[$ such that

$$|w_u| \leq c_1(|u|^{p-1} + |v|^{(p-1)q/p} + |u|^{r-1}), \tag{5}$$

$$|w_v| \leq c_1(|v|^{q-1} + |u|^{(q-1)p/q} + |v|^{s-1}) \tag{6}$$

for all $(u,v) \in \mathbb{R}^2$, $w_u \in \partial_1 F(u,v)$ and $w_v \in \partial_2 F(u,v)$.

The number $\alpha^* = N\alpha/(N-\alpha)$ $(\alpha \in \{p,q\})$ is the Sobolev critical exponent. Since $\Omega$ has the cone property, we have the Sobolev embeddings $W_0^{1,\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$ $(\beta \in [\alpha,\alpha^*]$, $\alpha \in \{p,q\})$. Therefore, the space $W_0^{1,\alpha}(\Omega)$ can be endowed with the norm

$$\|u\|_{1,\alpha} = (\int_\Omega |\nabla u|^\alpha)^{1/\alpha} \quad (\alpha \in \{p,q\})$$

and there exists $c_{\beta,\alpha} > 0$ such that $\|u\|_\beta \leq c_{\beta,\alpha}\|u\|_{1,\alpha}$ for all $u \in W_0^{1,\alpha}(\Omega)$.

Now, we are in the position to formulate our problem, denoted by $(\mathrm{S}')$:
*Find* $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *such that:*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla w + \int_\Omega F_1^0(u(x),v(x);-w(x))dx \geq 0,$$

$$\int_\Omega |\nabla v|^{q-2}\nabla v \nabla y + \int_\Omega F_2^0(u(x),v(x);-y(x))dx \geq 0$$

*for all* $w \in W_0^{1,p}(\Omega)$ *and for all* $y \in W_0^{1,q}(\Omega)$.

Remark 3.1. When $F \in C^1(\mathbb{R}^2,\mathbb{R})$ then $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ solves $(\mathrm{S}')$ if and only if $(u,v)$ is a weak solution of (S) in the usual sense. Therefore, the formulation of $(\mathrm{S}')$ recovers the classical problem (S).

We require the following further set of assumptions on $F$:
**(F2)** $F$ is regular on $\mathbb{R}^2$, and $F(0,0)=0$.
**(F3)** There exist $c_2,\ \mu,\ \nu>0$ such that

$$-c_2(|u|^\mu+|v|^\nu)\geq F(u,v)+\frac{1}{p}F_1^0(u,v;-u)+\frac{1}{q}F_2^0(u,v;-v) \tag{7}$$

for all $(u,v)\in\mathbb{R}^2$.

$$\lim_{u,v\to 0}\frac{\max\{|w_u|:w_u\in\partial_1F(u,v)\}}{|u|^{p-1}} = \lim_{u,v\to 0}\frac{\max\{|w_v|:w_v\in\partial_2F(u,v)\}}{|v|^{q-1}}=0. \tag{F4}$$

Our main result can be formulated as follows:

THEOREM 3.1. [**K(2003)-3**] *Let* $F:\mathbb{R}^2\to\mathbb{R}$ *be a locally Lipschitz function satisfying* (F1) *with* $ps=qr$*,* (F2), (F4) *and* (F3) *for some*

$$\mu>\max\{p,N(r-p)/p\}\ \text{ and }\nu>\max\{q,N(s-q)/q\}. \tag{8}$$

*Then* (S′) *possesses at least one nontrivial solution whose components are axially symmetric.*

EXAMPLE 3.1. Let $p=\frac{3}{2}$, $q=\frac{9}{4}$, $\Omega=]a,b[\times\mathbb{R}^2$ $(a<b)$ and

$$F(u,v)=u^2+|v|^{\frac{7}{2}}+\frac{1}{4}\max\{|u|^{\frac{5}{2}},|v|^{\frac{5}{2}}\}.$$

Since $F$ has neither homogeneity nor differentiability properties and the domain is not bounded, the earlier well known results (see for instance [**BdF(2002)**], [**CaMi(2002)**]) cannot be applied. $F$ being convex and locally Lipschitz function, (F2) holds (see Clarke [**Cl(1983)**, Proposition 2.3.6]), while (F4) can be verified easily. Choosing $r=\frac{5}{2}$, $s=\frac{15}{4}$ and $\nu=\frac{5}{2}$, $\mu=\frac{7}{2}$, the assumptions (F1) and (F3) hold. Therefore we can apply Theorem 3.1, obtaining at least a nonzero solution for (S′).

### 3.3. An eigenvalue problem for elliptic systems

In this section, we will study the multiplicity of solutions of the eigenvalue elliptic problem

$$(\mathrm{S}_\lambda)\qquad \begin{cases} -\Delta_p u=\lambda F_u(x,u,v) & \text{in }\Omega\\ -\Delta_q v=\lambda F_v(x,u,v) & \text{in }\Omega\\ u=v=0 & \text{on }\partial\Omega,\end{cases}$$

where $\Omega$ is the strip-like domain from Section 3.2, $1<p,q<N$, and $\lambda>0$. We will work under the following hypotheses on the nonlinear term $F$:

**(F0)** $F:\Omega\times\mathbb{R}^2\to\mathbb{R}$ is a Carathéodory function on the first variable, $(s,t)\mapsto F(x,s,t)$ is of class $C^1$ and $F(x,0,0)=0$ for all $x\in\Omega$.
**(F1)** There exist $c_1>0$ and $r\in]p,p^*[$, $s\in]q,q^*[$ such that

$$|F_u(x,u,v)|\leq c_1(|u|^{p-1}+|v|^{(p-1)q/p}+|u|^{r-1}), \tag{9}$$

(10) $$|F_v(x,u,v)| \le c_1(|v|^{q-1} + |u|^{(q-1)p/q} + |v|^{s-1})$$

for all $(u,v) \in \mathbb{R}^2$ and $x \in \Omega$.

**(F2)** $$\lim_{u,v\to 0} \frac{F_u(x,u,v)}{|u|^{p-1}} = \lim_{u,v\to 0} \frac{F_v(x,u,v)}{|v|^{q-1}} = 0 \text{ uniformly for all } x \in \Omega.$$

**(F3)** There exist $p_1 \in ]0,p[$, $q_1 \in ]0,q[$, $\mu \in [p,p^*]$, $\nu \in [q,q^*]$ and $\alpha \in L^{\mu/(\mu-p_1)}(\Omega)$, $\beta \in L^{\nu/(\nu-q_1)}(\Omega)$, $\gamma \in L^1(\Omega)$ such that

$$F(x,u,v) \le \alpha(x)|u|^{p_1} + \beta(x)|v|^{q_1} + \gamma(x)$$

for all $(u,v) \in \mathbb{R}^2$ and $x \in \Omega$.

**(F4)** There exist $u_0 \in W_G^p$ and $v_0 \in W_G^q$ such that

$$\int_\Omega F(x,u_0(x),v_0(x))dx > 0.$$

The main result of this section can be formulated as follows.

THEOREM 3.2. [**K(2003)-4**] *Let $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a function which satisfies* (F0)-(F4). *If $F$ is axially symmetric in the first variable and $ps = qr$ then there exist an open interval $\Lambda \subset [0,\infty[$ and $\sigma > 0$ such that for all $\lambda \in \Lambda$ the system* ($S_\lambda$) *has at least three distinct weak solutions (denote them by $(u_\lambda^i, v_\lambda^i)$, $i \in \{1,2,3\}$), the functions $u_\lambda^i$, $v_\lambda^i$ are axially symmetric, and $\|u_\lambda^i\|_{1,p} < \sigma$, $\|v_\lambda^i\|_{1,q} < \sigma$, $i \in \{1,2,3\}$.*

EXAMPLE 3.2. Let $\Omega = \omega \times \mathbb{R}^2$, where $\omega$ is a bounded open interval in $\mathbb{R}$. Let $a : \Omega \to \mathbb{R}$ be a continuous, non-negative, not identically zero, axially symmetric function with compact support in $\Omega$. Then there exist an open interval $\Lambda \subset [0,\infty[$ and a number $\sigma > 0$ such that for every $\lambda \in \Lambda$, the system

$$\begin{cases} -\Delta_{3/2}u = 5/2\lambda a(x)|u|^{1/2}u\cos(|u|^{5/2}+|v|^3) & \text{in } \Omega \\ -\Delta_{9/4}u = 3\lambda a(x)|v|v\cos(|u|^{5/2}+|v|^3) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least three distinct weak solutions with the properties from Theorem 3.2.

CHAPTER 4

# Applications in the theory of geodesics

## 4.1. Introduction

The study of geodesics on a Riemann manifold $M$ is a classical area in differential geometry and global analysis. Let $M_1$ respectively $M_2$ be two submanifolds of $M$. Many authors studied the problem (see [**Gro(1973)**], [**Ser(1951)**], [**Mot(1995)**], [**Wa(1990)**], [**Schw(1964)**]):

*What is the number of geodesics with endpoints in $M_1$ and $M_2$ and which are orthogonal to $M_1$ and $M_2$ ?*

The purpose of Section 4.2 is to examine the existence and the number of *Finsler-geodesics* joining orthogonally $M_1$ and $M_2$ when a Finsler metric is given on a complete Riemannian manifold $M$. The existence of closed geodesics in the case of Finsler spaces has been studied by Mercuri [**Mer(1977)**]. We consider only such a Finsler metric which dominates the underlying Riemannian structure of the manifold. Following [**Mer(1977)**] we describe the Riemann-Hilbert manifold $\Lambda_N M$ of absolutely continuous maps from the unit interval $I = [0, 1]$ to $M$ with endpoints in $N \subset M \times M$. We show that the energy integral $\tilde{L}$ is of class $C^{2-}$ on $\Lambda_N M$, and the geodesics of the Finsler metric $F$ joining orthogonally $M_1$ and $M_2$ are just the critical points of the energy integral $\tilde{L}\colon \Lambda_{M_1 \times M_2} M \to \mathbb{R}$ (in the sense of Definition 1.1). We prove that the energy functional $\tilde{L}\colon \Lambda_N M \to \mathbb{R}$ of a Finsler metric satisfies the Palais-Smale condition (Theorem 4.3). Applying the results of Grove [**Gro(1973)**], Serre [**Ser(1951)**] and Schwartz [**Schw(1964)**], we deduce some multiplicity results for geodesics of Finsler manifolds joining $M_1$ and $M_2$ (Theorems 4.4, 4.5 and 4.6).

The existence of isometry-invariant geodesics on Riemannian manifold has been studied firstly by Grove in [**Gro(1973)**], [**Gro(1974)**], [**Gro(1975)**]. In Section 4.3 we study the existence of $A_0$-invariant geodesics on a subset $M$ of $\mathbb{R}^n$ (with Lipshitz boundary), where the linear isometry $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ of finite order leaves invariant the set $M$. Under some restrictions on the topology of $M$ and the fixed points of $A_0$, we will establish the existence of at least one non-trivial $A_0$-invariant geodesic on $M$ (see Theorem 4.7).

## 4.2. Geodesics on Finsler manifolds

Let $M$ be an $n$-dimensional Riemannian manifold and $I = [0, 1]$ the unit interval. Now, we consider the manifold $L_1^2(I, M)$ of absolutely continuous maps from $I = [0, 1]$ to $M$ with locally square integrable derivative. The space $L_1^2(I, M)$ has a natural complete Riemannian-Hilbert structure given by

$$\langle X, Y\rangle_c' = \int_I \langle X_c(t), Y_c(t)\rangle_{c(t)} + \langle \nabla_c X_c(t), \nabla_c Y_c(t)\rangle_{c(t)} dt,$$

where $X$ and $Y$ are arbitrary elements of $T_c L_1^2(I, M) = H^1(c^*TM)$, the set of all absolutely continuous vector fields $X$ along $c$ with square integrable covariant derivative $\nabla_c X$.

Let $P : L_1^2(I, M) \to M \times M$ be the projection, defined by $P(c) = (c(0), c(1))$ for all $c \in L_1^2(I, M)$ and let $N \subset M \times M$ be a submanifold of $M \times M$ of codimension $k$. ¿From the expression of local coordinates we have that $P$ is submersion. Then we have that $P^{-1}(N)$ is a submanifold of $L_1^2(I, M)$ of codimension $k$. We denote $P^{-1}(N)$ by $\Lambda_N M$.

Definition 4.1. *A Finsler metric on a manifold $M$ is a continuous function $F : TM \to \mathbb{R}_+$ satisfying the following properties:*

*i) $F$ is $C^\infty$ on $TM \setminus \{0\}$;*

*ii) $F(u) > 0$ for all $u \in TM \setminus \{0\}$;*

*iii) $F(tu) = |t|F(u)$ for all $t \in \mathbb{R}$, $u \in TM$;*

*iv) For all $p \in M$ the indicatrix $I_F(p) = \{u \in T_pM : F(u) < 1\}$ is strongly convex.*

*A manifold $M$ endowed with a Finsler metric is called a Finsler space* [**AP(1994)**], [**BC(1993)**], [**MA(1997)**]. *We say that a Finsler metric $F$ dominates a Riemannian metric $g$ of the manifold if for some $H_0 > 0$:*

$$F(u) \geq H_0\|u\| \quad \textit{for all } u \in TM$$

*where $\|\cdot\|$ denotes the Riemannian norm.*

Remark 4.1. 1. If we consider the function $L = F^2$, then $L$ is of class $\mathcal{C}^1$ and $dL$ is locally Lipschitz on $TM$. The function $L$ is of class $\mathcal{C}^2$ if and only if $F$ is a norm of a Riemannian metric.

2. It is clear that if the manifold $M$ is compact, then any Finsler metric dominates a Riemannian metric on $M$. Namely, considering the Loewner ellipsoid of the indicatrix in each tangent space we get a Riemannian metric on the manifold, dominated by the Finsler metric, for

$$H_0 = \inf\{F(u) : \|u\| = 1,\, u \in TM\}$$

is positive due to the compactness of $M$.

3. It is known from [**BIK(1993)**] that if $F_1$ and $F_2$ are Finsler metrics on a manifold then $\sqrt{F_1^2 + F_2^2}$ is a Finsler metric as well. This means that for any Riemannian metric $g$ on $M$ the Finsler metric

$$\tilde{F}(u) = \sqrt{F^2(u) + g(u, u)}$$

dominates $g$ with the constant $H_0 = 1$.

DEFINITION 4.2. *The function $L = F^2$ induces a map $\tilde{L} : \Lambda_N M \to \mathbb{R}$ defined by*

$$\tilde{L}(c) = \int_I L(\dot{c}(t))dt \qquad for\ all\ c \in \Lambda_N M$$

*and is called the energy integral.*

THEOREM 4.1. *The energy integral $\tilde{L}$ is well-defined and $C^{2-}$ on $\Lambda_N M$, i.e. $\tilde{L}$ is of class $C^1$ and the differential of $\tilde{L}$ is locally Lipschitzian.*

Now we generalize the notion that a curve orthogonally joins two submanifolds of a Finsler manifold. Here we use the machinery of Abate and Patrizio's book [**AP(1994)**].

DEFINITION 4.3. *We say that a curve $c\colon I \to M$ orthogonally joins two submanifolds $M_1$ and $M_2$ if $\langle U^H | T^H \rangle_{\dot{c}(0)} = 0$ and $\langle V^H | T^H \rangle_{\dot{c}(1)} = 0$ hold for all $U \in T_{c(0)} M_1$, and $V \in T_{c(1)} M_2$ respectively, where $T = \dot{c}$.*

THEOREM 4.2. [**KKV(2003)**] *Let $M_1$ and $M_2$ be submanifolds of $M$. Then $c \in \Lambda_{M_1 \times M_2} M$ is a critical point of $\tilde{L}\colon \Lambda_{M_1 \times M_2} M \to \mathbb{R}$ if and only if $c$ is a Finsler-geodesic on $M$ joining orthogonally $M_1$ and $M_2$.*

THEOREM 4.3. [**KKV(2003)**] *Let $F$ be a dominating Finsler metric on a complete Riemannian manifold $M$, and $N \subset M \times M$ be a closed submanifold of $M \times M$ such that $P_1(N) \subset M$ or $P_2(N) \subset M$ is compact. Then $\tilde{L} : \Lambda_N M \to \mathbb{R}_+$ satisfies the Palais-Smale condition, i.e. any sequence $\{c_n\} \subset \Lambda_N M$ with $|\tilde{L}(c_n)| <$ const. and $\|d\tilde{L}(c_n)\| \to 0$ as $n \to \infty$, contains a convergent subsequence.*

In the final of this section we state some multiplicity results, which generalize some of the results of Grove [**Gro(1973)**], Serre [**Ser(1951)**] and Schwartz [**Schw(1964)**], respectively.

THEOREM 4.4. [**KKV(2003)**] *Let $M$ be a smooth, complete, finite dimensional Riemannian manifold with a dominating Finsler metric $F$ and let $M_1$ and $M_2$ be closed submanifolds of $M$ with say $M_1$ compact. Then in any homotopy class of curves from $M_1$ to $M_2$ there exists a Finsler–geodesic joining orthogonally $M_1$ and $M_2$ with length smaller than that of any other curve in this class. Furthermore, there are at least $\mathrm{cat}\,\Lambda_{M_1 \times M_2} M$ geodesics joining orthogonally $M_1$ and $M_2$.*

THEOREM 4.5. [**KKV(2003)**] *Let $M$ be a smooth, compact, connected, finite dimensional Finsler manifold. We suppose that $M$ is simply connected and let $M_1, M_2$ be two closed submanifolds of $M$ such that $M_1 \cap M_2 = \emptyset$, $M_1$ is contractible. Then there are infinitely many Finsler–geodesics joining orthogonally $M_1$ and $M_2$.*

THEOREM 4.6. [**KKV(2003)**] *Let $M$ be a smooth, complete, non contractible, finite dimensional Riemannian manifold endowed with a dominating Finsler metric $F$ and let $M_1$ and $M_2$ be two closed and contractible submanifolds of $M$ such that $M_1$ or $M_2$ is compact. Then there exist infinitely many Finsler-geodesics joining orthogonally $M_1$ and $M_2$.*

### 4.3. Isometry-invariant geodesics with Lipschitz obstacle

Each $\gamma \in W^{1,2}(a,b;\mathbb{R}^n)$ will be identified with its continuous representative $\tilde{\gamma} : [a,b] \to \mathbb{R}^n$. We will denote by $\|\cdot\|_{1,2}$ and $\|\cdot\|_p$ the usual norms in $W^{1,2}(a,b;\mathbb{R}^n)$ and $L^p(a,b;\mathbb{R}^n)$ with $1 \le p \le \infty$. Let $M$ be a closed subset of $\mathbb{R}^n$. We consider

$$W^{1,2}(a,b;M) = \{\ \gamma \in W^{1,2}(a,b;\mathbb{R}^n)\ \ : \gamma(s) \in M \text{ for all } \ s \in [a,b]\ \}$$

and let the functional $E_{a,b} : W^{1,2}(a,b;M) \to \mathbb{R}$ defined by

$$E_{a,b}(\gamma) := \frac{1}{2}\int_a^b |\gamma'(s)|^2 ds.$$

Definition 4.4. [**DMo(1999)**] *Let $a, b \in \mathbb{R}, a < b$. A curve $\gamma \in W^{1,2}(a,b;M)$ is* energy–stationary *if it is not possible to find $\delta, c, \sigma > 0$ and a map*

$$\mathcal{H} : \{\eta \in W^{1,2}(a,b;M) :\ \ \|\eta - \gamma\|_{1,2} < \delta\ \} \times [0,\delta] \longrightarrow W^{1,2}(a,b;M)$$

*such that*

*a) $\mathcal{H}$ is continuous from the topology of $L^2(a,b;\mathbb{R}^n)\times\mathbb{R}$ to that of $L^2(a,b;\mathbb{R}^n)$;*

*b) for every $\eta \in W^{1,2}(a,b;M)$ with $\|\eta-\gamma\|_{1,2} < \delta$ and $t \in [0,\delta]$, we have*

$$\Big(\mathcal{H}(\eta,t) - \eta\Big) \in W_0^{1,2}(a,b;\mathbb{R}^n);$$

$$\|\mathcal{H}(\eta,t) - \eta\|_2 \le ct; \quad E_{a,b}(\mathcal{H}(\eta,t)) \le E_{a,b}(\eta) - \sigma t.$$

Definition 4.5. [**DMo(1999)**] *Let $I$ be an interval in $\mathbb{R}$ with $int(I) \neq \emptyset$. A continuous map $\gamma : I \to M$ is a* geodesic *on $M$, if every $s \in int(I)$ admits a neighborhood $[a,b]$ in $I$, such that $\gamma_{|[a,b]} \in W^{1,2}(a,b;M)$ and $\gamma_{|[a,b]}$ is energy-stationary.*

In the sequel, let $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry, i.e. $A_0$ is a linear map and $\langle A_0x, A_0y\rangle = \langle x,y\rangle$ for all $x,y \in \mathbb{R}^n$, where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. Moreover, let us suppose that $A_0(M) = M$.

Definition 4.6. *An $A_0$-*invariant geodesic *on $M$ is a geodesic $\gamma : \mathbb{R} \to M$ such that $A_0(\gamma(s)) = \gamma(s+1)$ for all $s \in \mathbb{R}$. The above curve is non-trivial if this does not reduce to a point. Otherwise, this point will be a fixed point of $A_0$.*

Let

$$X_{A_0} = \{\gamma \in W^{1,2}(0,1;M)\ \ :\ \ A_0(\gamma(0)) = \gamma(1)\ \}$$

and we define the functional $f_{A_0} : L^2(0,1;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ by

$$f_{A_0}(\gamma) = \begin{cases} E_{0,1}(\gamma), & \text{if } \ \gamma \in X_{A_0} \\ +\infty, & \text{otherwise.} \end{cases}$$

Naturally, the functional $f_{A_0}$ is lower semi-continuous and is $A_0$–invariant, i.e. $f_{A_0}(A_0\gamma) = f_{A_0}(\gamma) = f_{A_0}(A_0^{-1}\gamma)$ for all $\gamma \in L^2(0,1,\mathbb{R}^n)$.

Proposition 4.1. *If $\gamma \in X_{A_0}$ is a critical point of $f_{A_0}$, then $\gamma$ is a geodesic on $M$.*

Proposition 4.2. *For every $\gamma \in X_{A_0}$ we have the relation*

$$|df_{A_0}|(A_0\gamma) = |df_{A_0}|(\gamma).$$

We introduce the following notation

$$A_0^n = \begin{cases} A_0 \circ A_0 \circ \cdots \circ A_0, & n \in \mathbb{Z}_+ \\ id_{\mathbb{R}^n}, & n = 0 \\ A_0^{-1} \circ A_0^{-1} \circ \cdots \circ A_0^{-1}, & n \in \mathbb{Z}_- \end{cases}$$

where $A_0 \circ A_0 \circ \cdots \circ A_0$ and $A_0^{-1} \circ A_0^{-1} \circ \cdots \circ A_0^{-1}$ denote the compositions of $|n|$-times.

REMARK 4.2. If $\gamma \in X_{A_0}$, then $|df_{A_0}|(A_0^n\gamma) = |df_{A_0}|(\gamma)$ for all $n \in \mathbb{Z}$. Indeed, in a similar way as in Proposition 4.2, it is possible to prove that $|df_{A_0}|(A_0^{-1}\gamma) = |df_{A_0}|(\gamma)$. The rest follows by induction.

COROLLARY 4.1. *If $\gamma \in X_{A_0}$ is a critical point of $f_{A_0}$, then $A_0^n\gamma$ is a geodesic on $M$ for every $n \in \mathbb{Z}$.*

PROPOSITION 4.3. *If $\gamma \in X_{A_0}$ is a critical point of $f_{A_0}$, then*

$$\widehat{\gamma}(s) = \begin{cases} \gamma(s + \frac{1}{2}), & 0 \le s \le \frac{1}{2} \\ A_0\gamma(s - \frac{1}{2}), & \frac{1}{2} \le s \le 1 \end{cases} \tag{11}$$

*is a geodesic on $M$.*

REMARK 4.3. Let $\gamma \in X_{A_0}$ be a geodesic on $M$. We define the curve $\tilde{\gamma} : \mathbb{R} \to M$ by $\tilde{\gamma}(t) = A_0^{[t]}(\gamma(\{t\}))$, where $[t]$ is the integer part of $t \in \mathbb{R}$, and $\{t\} = t - [t]$. We observe that for all $s \in \mathbb{R}$

$$A_0(\tilde{\gamma}(s)) = A_0^{1+[s]}(\gamma(\{s\})) = A_0^{[s+1]}(\gamma(\{s+1\})) = \tilde{\gamma}(s+1)$$

i.e. $\tilde{\gamma}$ is $A_0$–invariant geodesic on $M$. Therefore, if we can guarantee that $\gamma \in X_{A_0}$ is a (non–trivial) curve which is a critical point of $f_{A_0}$, the above construction can be applied for constructing $A_0$–invariant geodesics on $M$.

REMARK 4.4. We denote by $\mathrm{Fix}_M A_0$ the fixed points of the isometry $A_0$ on $M$. We observe that $\overline{\mathcal{G}}^0_{f_{A_0}}$ is homeomorphic to $\mathrm{Fix}_M A_0$. Indeed, let $(\gamma, \xi) \in \overline{\mathcal{G}}^0_{f_{A_0}} = \{ (\gamma, \xi) \in \mathrm{epi}(f_{A_0}) \;:\; \xi \le 0 \}$, therefore $f_{A_0}(\gamma) \le \xi \le 0$. From this, we get that $\gamma \in X_{A_0}$ and $|\gamma'(s)| = 0$ a.e., therefore $\gamma(s) = x_0 \in M$. Since $A_0(\gamma(0)) = \gamma(1)$ it follows that $x_0 \in \mathrm{Fix}_M A_0$.

The following topological hypotheses will be assumed:

**(H1)** If the elements of $\mathrm{Fix}_M A_0$ are isolated, then $\mathrm{Fix}_M A_0$ is not homotopically equivalent with $\Lambda_{G(A_0)}(M)$.

**(H2)** The inclusion $i : \mathrm{Fix}_M A_0 \hookrightarrow \Lambda_{G(A_0)}(M)$ does not induce an isomorphism in the Alexander-Spanier cohomology, see [**Sp(1966)**].

The main theorem of this section can be formulated as follows.

THEOREM 4.7. [**KKV(2001)**] *Let $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry of finite order and $g : \mathbb{R}^n \to \mathbb{R}$ a $G$-invariant locally Lipschitz map, where $G = \langle A_0 \rangle$. Suppose that*

$$\forall x \in \mathbb{R}^n : g(x) = 0 \Rightarrow 0 \notin \partial g(x).$$

*and let*

$$M = \{x \in \mathbb{R}^n : g(x) \le 0\}.$$

*If $M$ is compact and* (H1) *or* (H2) *holds, then there exists at least one non-trivial $A_0$–invariant geodesic on $M$.*

REMARK 4.5. In fact, since $A_0$ is of finite order, the above $A_0$-invariant geodesic on $M$ is a closed geodesic on $M$. Therefore, if we want to guarantee the existence of a closed geodesic on $M$, it is enough to establish the existence of an $A_0$-invariant geodesic.

EXAMPLE 4.1. If $A_0 = id_{\mathbb{R}^n}$, then we obtain the notion of the closed geodesics, introduced in [**DMo(1999)**]. Of course, $\mathrm{Fix}_M A_0 = M$ and $\Lambda_{G(A_0)}(M) = \Lambda(M)$ is the free loop space. Moreover, if $M$ is defined as above, and we suppose that it is compact, simply connected and non-contractible in itself, then (according to a result of Vigué-Poirrier and Sullivan [**VPS(1976)**]) the hypothesis (H2) holds. Therefore, there exists a non-trivial closed geodesic on $M$.

EXAMPLE 4.2. If $\mathrm{Fix}_M A_0 = \emptyset$, then both hypotheses hold, therefore there exists at least one $A_0$-invariant geodesic.

EXAMPLE 4.3. Let

$$M = \{x \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i^2 \leq 1 \leq \sum_{i=1}^{3} |x_i|\}.$$

Let $A_0(x_1, x_2, x_3) = (x_1, -x_2, -x_3)$. The fixed points of $A_0$ are $\mathrm{Fix}_M A_0 = \{(1,0,0), (-1,0,0)\}$. Of course, $A_0^2 = id_{\mathbb{R}^3}$ and $\Lambda_{G(A_0)}M$ is 0-connected, therefore the two sets above are not homotopically equivalent, i.e. the (H1) holds. Therefore, there exists an $A_0$–invariant geodesic on $M$.

# Összefoglaló

## NEM-SIMA KRITIKUS PONT ELMÉLETEK ÉS ALKALMAZÁSAIK ELLIPTIKUS FELADATOK, ILLETVE GEODETIKUSOK TANULMÁNYOZÁSÁBAN

Alexandru Kristály

Nagyon sok analízisben felmerülő probléma írható fel $A(u) = 0$ egyenlet alakba, minek megoldása egy olyan függvény, mely egy jól rögzített $X$ Banach tér egy eleme. Általában, ezek az egyenletek nem lineárisak. Egy sajátos osztálya ezeknek az egyenleteknek az Euler-Lagrange típusú egyenlet, melynek a formája

$$E'(u) = 0, \tag{12}$$

ahol $E$ egy olyan funkcionál, mely az $X-$en értelmezett és Fréchet differenciálható, deriváltja $E'$. A fenti egyenletek tanulmányozására egy átfogó elmélet lett kidolgozva; a benne lévő módszerek összességét, melyek az (12) egyenlet megoldásait biztosítják, *kritikus pont elmélet*nek nevezzük. Egy olyan $u$ elem, mely kielégíti az (12) egyenletet, az $E$ funkcionál kritikus pontjának nevezzük. A matematikai fizika, differenciálgeometria, optimalizálás, numerikus analízis sok problémája visszavezethető egy megfelelő funkcionál kritikus pontjának vizsgálatára. Például, a parciális differenciálegyenletek elméletében egy klasszikus Dirichlet típusú feladat esetén egy megfelelően értelmezett funkcionál kritikus pontja az eredeti feladat egy (gyenge) megoldása lesz. A geometriában, a geodetikus vonalak az energia funkcionálok kritikus pontjai.

Ennek ellenére, a gyakorlatban sok olyan problémával találkozunk, ahol a funkcionál nem differenciálható.

Chang [**Ch(1981)**] a

$$-\Delta u = \Psi(u) \tag{13}$$

egyenlet megoldásainak létezését tanulmányozta standard Dirichlet határfeltételek mellett egy $\Omega$ korlátos tartományon, ahol $\Psi$ egy lokálisan korlátos, mérhető függvény, mely nem rendelkezik semmiféle folytonossági tulajdonsággal. A (13) feladathoz társított funkcionál alakja a következő:

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega \int_0^{u(x)} \Psi(t)dtdx,$$

mely a $H_0^1(\Omega)$ téren értelmezett. Mivel $\Psi$ nem folytonos, így $E$ nem lesz differenciálható, csak lokálisan Lipschitz függvény. Ez a típusú feladat motiválta 1981-ben Chang-ot, hogy a [**Ch(1981)**] dolgozatban kiterjessze a kritikus pont elméletet *lokálisan Lipschitz függvény*ekre; a Clarke-féle *általánosított gradiens* képezte az alapeszközt a munkájában, lásd [**Cl(1983)**].

Később, 1994-ben Degiovanni és Marzocchi [**DMa(1994)**] bevezette a *gyenge lejtő* fogalmat *folytonos függvény*ekre. Ez a lépés a klasszikus kritikus pont elmélet egy új általánosításának tekinthető.

Megjegyezzük, hogy egy lokálisan Lipschitz függvény Chang értelemben vett kritikus pontjainak halmaza bővebb lehet, mint a Degiovanni és Marzocchi értelemben vett halmaz. Így hát, ez a két elmélet különbözik egymástól és a segítségükkel kapott eredmények általában nem hasonlíthatóak össze. A fenti két elméletben ismert eredmények összességét *nem-sima kritikus pont elmélet*nek nevezzük.

Mindkét kritikus pont elméletnek több alkalmazása van: parciális differenciál egyenletek elméletében, lásd [**Ch(1981)**]; hemivariációs egyenlőtlenségek elméletében, lásd Motreanu és Panagiotopoulos [**MP(1999)**]; a geodetikusok elméletében, lásd Degiovanni és Morbini [**DMo(1999)**]. Megjegyezzük, hogy Frigon [**F(1998)**] kidolgozott egy *többértékű leképzés*ekre érvényes kritikus pont elméletet (ami főként a Degiovanni és Marzocchi elméleten alapszik), ahol észrevette, hogy természetesebb a leképzés grafikonján dolgozni, mint annak az értelmezési tartományán (lásd továbbá [**KV(2001)**], [**KV(2002)**]). A jelen tézis ezeken az elméleteken alapszik.

Az első fejezetben a nem-sima analízis néhány fogalmát és eredményét mutatjuk be, melyek a dolgozat további részében lesznek használva. Pontosabban, a lokálisan Lipschitz és folytonos függvények kritikus pont elméletéből néhány fontos eredményt adunk meg, mint az Ambrosetti-Rabinowitz [**AR(1973)**] "Mountain Pass" tétel nem-sima formáját, Bartsch [**Ba(1993)**] "Fountain" tételét, a Palais-féle [**Pal(1979)**] *kritikus szimmetria elvét* és Ricceri [**Ric(2000)-1**] egy új eredményét.

A második fejezetben két különböző típusú hemivariációs egyenlőtlenséget tanulmányozunk *nemkorlátos halmaz*okon. Ezek a problémák a matematikai fizikában (Schrödinger, Klein-Gordon egyenletek) jelennek meg. A "klasszikus" hemivariációs egyenlőtlenségek elmélete csak *korlátos halmaz*okon megjelenő feladatokra volt kifejlesztve, lásd [**MP(1999)**], [**NP(1995)**], [**Pan(1985)**]. A nemlineáris tagra kirótt néhány természetes növekedési feltétel mellett, végtelen sok radiális és nemradiális megoldás létezését biztosítjuk, egy, az $\mathbb{R}^N$-en értelmezett, bizonyos típusú hemivariációs egyenlőtlenség osztályra, lásd [**K(2003)-1**]. Továbbá, egy olyan hemivariációs egyenlőtlenséghez rendelt sajátérték feladat megoldásainak multiplicitását vizsgáljuk, mely egy *szalagszerű tartomány*on van megfogalmazva, azaz olyan halmazon, melynek formája $\Omega = \omega \times \mathbb{R}^{N-m}$, ahol $\omega$ egy nyílt, korlátos halmaz $\mathbb{R}^m$-ben és $N \geq m + 2$, lásd [**K(2003)-2**].

A harmadik fejezetben kvázilineáris elliptikus egyenletrendszerek megoldásának létezését tanulmányozzuk szalagszerű tartományokon. A "Mountain Pass" tétel segítségével egy új, Boccardo-de Figueiredo [**BdF(2002)**] típusú eredményt kapunk, lásd [**K(2003)-3**]. A továbbiakban, a fenti rendszerhez társított sajátérték feladatot tanulmányozzuk, biztosítva egy olyan nyílt $\Lambda \subset [0, \infty[$ intervallum létezését, hogy bármely $\lambda \in \Lambda$ esetén a sajátérték feladatnak legalább három különböző megoldása van, lásd [**K(2003)-4**]. Ez az eredmény Ricceri absztrakt kritikus pont tételén alapszik, lásd [**Ric(2000)-1**].

A negyedik fejezetben geodetikus vonalak létezését és multiplicitását tanulmányozzuk különböző keretek között. Először Finsler-geodetikusok létezését és számát vizsgáljuk, melyek összekötnek merőlegesen két, $M_1$ és $M_2$ részsokaságot, mikor a Finsler metrika egy teljes $M$ Riemann sokaságon van értelmezve. Az $M$, $M_1$ és $M_2$ halmazokra kirótt különböző feltételek mellett több multiplicitási tételt kapunk, lásd [**KKV(2003)**]. Továbbá, izometriára invariáns geodetikusok létezését vizsgáljuk olyan $\mathbb{R}^n$-beli $M$ halmazokon, melyek Lipschitz-határral rendelkeznek. Pontosabban, ha $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ egy véges rendű lineáris izometria, mely invariánsan hagyja az $M$ halmazt, bizonyos topológiai megkötések mellett, $A_0$-ra invariáns geodetikusok létezését biztosítjuk $M$-en, lásd [**KKV(2001)**]. Ez az eredmény Degiovanni és Marzocchi [**DMa(1994)**] kritikus pont elméletén alapszik és Grove [**Gro(1974)**] eredményeinek természetes általánosításának tekinthetjük.

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