Béla Brindza
(1958–2003)

Béla Brindza was born in 1958 in Csongrád, where he completed both primary and secondary school. His gift for mathematics became manifest in his high school years. He participated with great success in national and international competitions; in particular, he was a successful problem solver of the "Középiskolai Matematikai Lapok". His achievement at the National Secondary School Mathematics Competition ensured his automatic admission to József Attila University (JATE) in Szeged. After his first year in Szeged he continued his studies at Kossuth Lajos University (KLTE) in Debrecen, from where he graduated in mathematics in 1983. He completed his PhD thesis [2] in that same year, while still a student. After graduation, with Kálmán Győry as his supervisor he finished and defended his candidate's thesis [9]. For his thesis [36] he was awarded the academic doctoral degree in 1999.

He started teaching in his university years as a demonstrator. From 1983 he was an instructor at KLTE until 1987, when he won a three year postdoctoral scholarship at Macquarie University in Sydney. In 1990 he became an associate professor at KLTE, and then from 1992 to 1996 he taught at Kuwait University. After his return in 1996 he was granted a scholarship of the Hungarian National Scientific Foundation; and from 1997 he was associate professor at Kossuth University again. In 2000 he was appointed full professor. He was also holder of the Széchenyi Professorial Scholarship from 1998 to 2001. He reviewed articles regularly for the Mathematical Reviews and for the Zentralblatt für Mathematik. From 1992 he was member of our periodical's editorial board.

His scientific achievement was rewarded by the Kató Rényi Prize of the János Bolyai Mathematical Association in 1982, the Géza Grünwald Prize in 1985, and the Academic Prize of the Hungarian Academy of Sciences in 1992. He published his results in 48 papers, exposing them at a number of universities and international conferences in Cambridge, Durham, Ann Arbor, Oberwolfach, Leiden, Sydney, Tokyo and Kyoto. A whole chapter is devoted to Béla Brindza's results in Shorey and Tijdeman's book Exponential Diophantine Equations, a classical work on this subject.

Béla Brindza had an all-round talent for mathematics. Besides his research work, he took an active part in mathematical life also at the level of personal contacts. In particular, he paid a great deal of attention to gifted young people. He conducted "student-olympics" study groups on mathematics for high school students, while also guiding the professional development of younger colleagues (Lajos Hajdu, Kálmán Liptai, László Szalay, Attila Bérczes, Ákos Pintér). He was an outstanding lecturer, liked and respected by his students. His special courses were lively and very popular as he scintillated with wit.

Béla Brindza was an erudite, cheerful, sociable person, warm-hearted and helpful. He had a passion for music. He was still very young when his health betrayed him. A grave illness struck him down and he is no longer with us. With his untimely death Hungarian Mathematics lost an outstanding scientist, and the University of Debrecen an eminent professor.

The scientific research work of Béla Brindza was almost entirely devoted to diophantine number theory. He made significant contributions to the theory of diophantine equations. His attention was mainly focused on deriving bounds for the solutions and the number of solutions of important classical equations which play a central rôle in modern diophantine analysis and have a great number of applications. In Brindza's papers modern, powerful methods, including the theory of linear forms in logarithms, were successfully combined with new, original ideas and techniques elaborated by him. In what follows, we give a brief survey of his outstanding scientific achievements.

1) Perhaps the most important results of Brindza are related to *superelliptic equations*. Let $f(X)$ be a polynomial with integer coefficients, and let $r_1, \dots, r_n$ denote the multiplicities of its zeros. For given $m \geq 2$, the *superelliptic equation*

$$f(x) = y^m \quad \text{in integers } x, y \tag{1}$$

has only finitely many solutions, unless $\{m/(m, r_1), \dots, m/(m, r_n)\}$ is a permutation of one of the n-tuples $\{t, 1, \dots, 1\}$, $t \geq 1$, and $\{2, 2, 1, \dots, 1\}$. This theorem, due to LeVeque (1964), was ineffective. Baker (1969) was the first to derive bounds for the solutions of (1), at last in the particular case when $m \geq 3$ and f possesses at least 2 simple zeros, or $m = 2$ and f possesses at least 3 simple zeros. As a graduate student, Brindza [3] made effective LeVeque's theorem in full generality. In fact he proved this in the more general situation when the ground ring is the ring of S-integers of a number field. In the last two decades Brindza's result has often been quoted, treated and applied in the literature.

Schinzel and Tijdeman (1976) showed that if in (1) m is also unknown, then m can be effectively bounded above in terms of f, provided that f has at least 2 distinct zeros. In his joint paper [41] with Bérczes and Hajdu, Brindza gave an explicit upper bound for m in terms of the degree and height of f. Further, together with Evertse and Győry he proved in [22] that, for irreducible f, m can be estimated from above only in terms of the degree and discriminant of f. These exerted considerable influence on latter investigations of Haristoy, Győry, Pink, Pintér and others.

To extend the applicability of his results concerning (1), he showed in

[12] that if

$$f(X) = f_1^{k_1}(X) + \ldots + f_N^{k_N}(X)$$

with coprime non-constant polynomials $f_1, \ldots, f_N \in \mathbb{Z}[X]$ such that $\mu = \min_i k_i > N(N-1)$, then $f(X)$ has at least $\mu/(N-1)$ distinct zeros. This implies that for such a polynomial $f(X)$ with $N \geq 2$, m must be bounded in (1).

2) Brindza's interest concerning (1) was greately motivated by *Schäffer's equation*

$$S_k(x) = y^m \quad \text{in integers } x, y > 1, \tag{2}$$

where k and $m > 1$ are integers and $S_k(x) = 1^k + 2^k + \ldots + x^k$. Since $S_k(X)$ can be written as $(B_{k+1}(X+1) - B_{k+1}(0))/(k+1)$ where $B_{k+1}(X)$ denotes the $(k+1)$th Bernoulli polynomial, equation (2) is in fact a special superelliptic equation. Schäffer (1956) characterized those pairs (k, m) for which (2) has infinitely many solutions. This was extended by Győry, Tijdeman and Voorhoeve (1979) to the more general equation when in (2) m is also unknown and $S_k(x)$ is replaced by $S_k(x) + R(x)$, where $R(x)$ is any polynomial with coefficients in $\mathbb{Z}$. The above-mentioned results were ineffective. By means of his general effective theorem on equation (1) Brindza [4] succeeded to make effective these results. Moreover, he generalized them in an effective form for even more general equation of the shape $F(S_k(x)) = y^m$, where $F \in (\mathbb{Z}[X])[Y]$ satisfies some natural conditions.

He derived also upper bounds for the number of solutions of (2). In [19] he proved that for any given $m \geq 3$ with $m \neq 4$, equation (2) has at most e^{7k} solutions. Further, when in (2) the exponent m is also unknown he proved with Pintér [44] that apart from the case $(k, m) = (3, 4)$, equation (2) possesses at most $\max\{c, e^{3k}\}$ solutions, where c is an effectively computable absolute constant.

In his nice joint article [47] with Bilu, Kirschenhofer, Pintér and Tichy it is showed that for given k, $l \geq 2$ and for sufficiently large integer x, $S_k(x)$ is not a product of l consecutive integers. Further, if $k \neq l$ then $S_k(x) \neq S_l(y)$ holds, provided x, y are sufficiently large. To prove these, the authors had to completely characterize the decomposable Bernoulli polynomials, which is a result of independent interest.

3) A famous theorem of Tijdeman (1976) states that the *Catalan equation*

$$x^m - y^n = 1 \quad \text{in integers } x, y, m, n > 1 \tag{3}$$

has only finitely many solutions, and all of them can be, at least in principle, effectively determined. In [8], Brindza, Győry and Tijdeman extended this result to the number field case when x, y are integers in an arbitrary but fixed number field K. They showed that if $m, n > 1$, $mn > 4$ and x, y are not roots of unity, then m, n and the heights of x, y can be estimated from above by an effective constant depending only on K. In contrast with polynomial diophantine equations, the proof was not merely a straightforward generalization of the rational case. In the general situation new arguments were needed to prove that m and n are bounded and that x or y has a bounded norm. Later, Brindza [13] further generalized this result for the cases of S-integers x, y of K.

Following a different approach, recently Mihailescu proved that the only solution of (3) is $3^2 - 2^3 = 1$. It would be interesting to obtain similar results in the more general cases considered by Brindza.

4) Many number theoretical problems can be reduced to *unit equations* of the form

$$au + bv = c \quad \text{in } u, v \in U_K, \tag{4}$$

where a, b, c are given non-zero elements in a number field K, and U_K denotes the unit group of K. By a theorem of Evertse, Győry, Stewart and Tijdeman (1988) (4) has at most 2 solutions for "almost all" triples of a, b, c, up to a proportional factor. In his joint paper [18] with Győry, Brindza considerably improved this result in the important special case when a, b, c, are rational numbers. Then one may assume that a, b, c are coprime positive integers and $c \geq a, b$. In [18] it is proved that for all but finitely many triples a, b, c of coprime positive integers, (4) has at most one solution u, v (up to conjugacy), and u, v must belong to $\mathbb{Q}$ or a real quadratic subfield of K. It is also showed in an effective form that if c is sufficiently large and $ab > 1$ or $[K : \mathbb{Q}]$ is odd, then no solution exists. This surprising result was the first application of the simultaneous Baker's method to diophantine equations.

Effective versions of *Dirichlet's unit theorem* play an important rôle in applications of unit equations. In [24] Brindza employed a simple argument from the geometry of numbers to obtain a set of generators of small height for the non-torsion subgroup of the S-unit group in a number field. His ingenious proof has exerted an influence on latter investigations of Hajdu, Bugeaud, Győry and others.

5) Several remarkable results were obtained by Brindza on the Thue equation

$$F(x,y) = m \quad \text{in } x, y \in \mathbb{Z}, \tag{5}$$

where F is an irreducible binary form of degree $n \geq 3$ with integer coefficients and m is a non-zero integer. By Thue's theorem (1909) (5) has only finitely many solutions. The first explicit bound for $\max\{|x|, |y|\}$ was given by Baker (1968). This was later improved by several people. A surprisingly good upper bound was derived by Brindza in terms of the height $H(F)$ of F. Namely, he derived in [22] with Evertse and Győry a bound of the form $c_1 H(F)^{3/n} |D(F) \cdot m|^{c_2}$, where $D(F)$ denotes the discriminant of F and c_1, c_2 are effectively computable numbers depending only on the splitting field of F.

There exist uniform upper bounds for the number of solutions of (5) which are independent of the coefficients of F. Bombieri and Schmidt (1987) established e.g. the bound cn^{s+1}, where c is an absolute constant and s denotes the number of distinct prime factors of m. Brindza [32] proved that if m is large enough with respect to $|D(F)|$ then the number of solutions does not exceed

$$n(6^s + 6).$$

It is a remarkable fact that this bound is already best possible in terms of n. In his joint paper [42] with Pintér, van der Poorten and Waldschmidt he obtained the bound $2n^2(s+1) + 13n$ for the number of those solutions x, y for which $\max\{|x|, |y|\}$ is larger than a relatively small and explicitly given bound which depends only on F and m. In other words, only a "few" solutions of (5) can be "large".

6) In [42], the following *new gap principle of Brindza* was the main tool. Let $t \in \mathbb{N}$ and $\lambda, \eta_0, \ldots, \eta_{t-1}, \mu, \psi_0, \ldots, \psi_{t-1} \in \mathbb{C}^*$. Suppose that the

equation

$$\lambda \boldsymbol{\eta}^{\boldsymbol{k}} + \mu \boldsymbol{\psi}^{\boldsymbol{k}} = 1,$$

where $\boldsymbol{\eta}^{\boldsymbol{k}} = \eta_0^{k_0} \dots \eta_{t-1}^{k_{t-1}}$, $\boldsymbol{\psi}^{\boldsymbol{k}} = \psi_0^{k_0} \dots \psi_{t-1}^{k_{t-1}}$, has at least $t+2$ solutions $\boldsymbol{k_j} = (k_{0,j}, \dots, k_{t-1,j}) \in \mathbb{Z}^t,\ j = 1, 2, \dots, t+2$. Let

$$K = \max_{0 \le i \le t-1,\, 1 \le j \le t+1} \{2, |k_{i,j+1} - k_{i,j}|\}.$$

If $|\lambda \boldsymbol{\eta}^{\boldsymbol{k_1}}| \ge 6$ and $|\boldsymbol{\eta}^{\boldsymbol{k_{j+1}}} / \boldsymbol{\eta}^{\boldsymbol{k_j}}| \ge 9(t+1)^{t/2} K^t$ for $j = 1, \dots, t+1$, then

$$(t+1)^{t/2} K^t / 4 \ge |\lambda \boldsymbol{\eta}^{\boldsymbol{k_1}}|.$$

In case of Thue equations this ensures exponential gaps between the solutions, provided that the equation has sufficiently many solutions. We note that the earlier gap principles yielded only polynomial gaps.

In [40] Brindza applied his gap principle to the *generalized Ramanujan–Nagell equation*

$$f(x) = b^z \quad \text{in } x, z \in \mathbb{Z},\ z > 1, \tag{6}$$

where $b \in \mathbb{Z}$ with $|b| > 1$ and $f \in \mathbb{Z}[X]$ is of degree n with at least two distinct zeros, one of which is not rational. He showed that the number of solutions of (6) is at most $6n^2(ns+n+3)(s+1)$, subject to the condition that $|b|$ is sufficiently large. Here s denotes the number of distinct prime factors of b. It is likely that Brindza's gap principle will lead to further applications as well.

7) Brindza was also interested in *diophantine equations over function fields* and *finitely generated domains*. Joining the investigations of Schmidt, Győry, Mason and others, he established several nice results in this direction.

Let K be a function field in one variable over an algebraically closed field $\Bbbk$ of characteristic 0, S a finite set of places of K, and $f(X)$ a polynomial with coefficients in K and with zero multiplicities $r_1, \dots, r_n$, respectively. As an analogue of his remarkable result over number fields, Brindza [10] derived together with Mason an explicit upper bound for the heights of S-integral solutions x, y of (1), subject again to the condition that $\{m/(m, r_1), \dots, m/(m, r_n)\}$ is not a permutation of either $\{t, 1, \dots, 1\}$ or

$\{2, 2, 1, \ldots, 1\}$. Further, with Pintér and Végső he gave in [28] an explicit upper bound also for m, under the condition that in (1) $n \geq 2$, $x, y \in K$ and $y \notin \Bbbk$.

As a common generalization of Thue equations and superelliptic equations Brindza considered in [11] the equation

$$F(x, y) = cz^m \quad \text{in relatively prime } S\text{-integers } x, y, z, \tag{7}$$

where F is a binary form with coefficients in K and $c \in K^*$. Genaralizing and improving a theorem of W. Schmidt (1980), he obtained an explicit bound for the heights of the solutions of (7) in the case that F has at least three distinct simple linear factors over the splitting field of F. Moreover, he gave also an explicit upper bound for m when $z \notin \Bbbk$.

Lang (1962) extended several classical diophantine finiteness results to that general situation when the ground ring R is any given finitely generated integral domain over $\mathbb{Z}$. Lang's results were, however, ineffective. Győry (1983) worked out a method to establish effective finiteness theorems for Thue equations and more generally for decomposable form equations over R. Combining this method with his own results obtained over number fields and function fields Brindza [17], [27] generalized his effective finiteness theorems on equations (1) and (3) for the finitely generated case.

8) Finally, let us mention an interesting result of Brindza in *algebraic coding theory*. Brindza's attention was drawn to the following problem by K. Buzási. Let K be a finite field of characteristic 2, and KG the group algebra of G over K where $G = (a) \times (b)$ and $a^p = b^p = 1$ for an odd prime p. Then $KG = I_0 \oplus \ldots \oplus I_{p+1}$, where each I_n is a minimal ideal generated by e_n, $n = 0, 1, \ldots, p+1$,

$$e_0 = \sum_{g \in G} g, \qquad e_{p+1} = \sum_{i=0}^{p-1} b^i \big(a + \ldots + a^{p-1}\big)$$

and

$$e_n = \sum_{i=0}^{p-1} \big(ab^{n-1}\big)^i \big(b + \ldots + b^{p-1}\big), \quad n = 1, \ldots, p.$$

Brindza [6] as a graduate student disproved an old conjecture of S. D. Berman (1967) by proving that if $p \equiv -1 \pmod 4$ and if $I \subset KG$ is an ideal

which is the sum of three minimal ideals different from I_0 and I_{p+1}, then the code distance $d(I)$ is equal to $2(p-1)$.

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