THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (PHD)

Sub-Finsler Geometry and Non-positive Curvature in Finsler Geometry

Layth Muhsin Habeeb Alabdulsada

Supervisor: Dr. László Kozma

UNIVERSITY OF DEBRECEN
DOCTORAL SCHOOL OF
MATHEMATICAL AND COMPUTATIONAL SCIENCES

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Introduction

This dissertation consists of two main parts. The first part has intended to clarify what is the relationships among the non-positive curvatures, especially, that in the case of Hilbert metric of a convex domain. The second part and the most important, in terms of the effort, the time spent and the results obtained, is devoted to solve questions of sub-Finslerian geometry. This dissertation is divided into three Chapters, organized as follows:

The first Chapter of the dissertation presents the basic concepts of Differentiable structure (manifolds, tangent spaces, cotangent spaces, vector fields and 1-forms, distributions). We present the distributions of the tangent and cotangent bundles with some examples. Furthermore, we give a brief overview of Finsler geometry, which is a fundamental and natural generalization of Riemannian geometry. Then we summarize the basic facts on Berwald spaces. We give some well known examples of the above concepts.

In the second Chapter, first of all, we explain what the non-positive curvature means by showing their definitions, more precisely, we give the basic definitions of four types of non-positive curvature in geodesic metric spaces, some of their properties and their general relationships, furthermore, list some results important for the next steps. After defining the Hilbert metric of a convex domain, we show that in the case of Hilbert metric, the first two concepts of non-positivity are equivalent if and only if the domain is an ellipsoid. Stepping to the analytical considerations, we restrict ourselves to the to the special case of Berwald space in Finsler geometry. Then we present those relationships and known facts about these concepts which are used to prove our result. Finally, we show that if the Finsler metric induced by the Hilbert metric of a convex domain is Berwald, then the domain should be an ellipsoid.

In third Chapter, we define the sub-Finsler metrics and introduce some of its notions. We provide some examples that satisfy the sub-Finsler property, further, we introduce basic properties of sub-Finsler manifolds. Also, we take a look at the horizontal path-connectedness of the manifold, especially for the Chow's theorem [18]. In the next, we give some basics of generalized connections in sub-Riemannian geometry, with which the normal and abnormal extremal can be characterized in the sub-Riemannian case (see the local version in [39], and the coordinate free version in [29]. Afterwards, we give a brief description of the Legendre transformation, we show the relationship between the sub-Finsler geometry with Lagrange spaces as well as the Hamiltonian spaces.

We introduce the symmetric bracket associated to a sub-Finsler metric, and the symmetric product of an $\mathcal{L}$-connection it is shown, they are coincident if and only if the $\mathcal{L}$-connection is normal. With the help of the generalization of the Bott connection for involutive distribution we can characterize $\mathcal{D}$-adapted and normal connections, resp. Last we turn to the problem of Hopf-Rinow theorem. It is well known that in the Riemannian and Finslerian geometry, there are two concepts of completeness. The first is the completeness in the sense of metric spaces, using the Riemannian metric. Furthermore, a Riemannian or Finsler manifold M is called geodesically complete if any geodesic $\gamma(t)$ starting from $x \in M$ is defined for all values of $t \in \mathbb{R}$. On the other hand, the completeness in the Finsler geometry is divided according to forward and backward distance metric, and forward and backward geodesically completeness.

Hopf-Rinow theorem is a basic theorem of complete Riemannian manifolds, which connects the completeness properties with compactness, and the exponential map. Its consequence says that any two points of the complete manifold can be connected by a length minimizing geodesic. To prove the statements of Hopf-Rinow theorem in sub-Finsler manifolds: we define the sub-Finsler metric on the cotangent space $\mathcal{D}^*$ with help of the Legendre transformation, where we look more closely at a sub-Hamiltonian H defined on the cotangent bundle, induced by the sub-Finslerian metric on $\mathcal{D}^*$.

Afterwards, we construct a sub-Finsler bundle, which plays a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry. Moreover, the sub-Finsler bundle allows an orthonormal frame for the sub-Finsler structure. Also, we introduce the notion of an exponential map in sub-Finsler geometry. At the end our main theorem, namely Hopf-Rinow theorem is stated and proved.

Non-positive curvature in Finsler geometry

Alexandrov Non-positive Curvature

A locally geodesic space (M, d) is said to be an *Alexandrov non-positive curvature space* if for every $p \in M$ there exists $\delta_p > 0$ such that for every $x, y, z \in B(p, \delta_p)$ and any shortest geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = x, \gamma(1) = z$, we have for $0 \leq t \leq 1$

$$d^2(y, \gamma(t)) \leq (1 - t)\ d^2(y, x) + t\ d^2(y, z) - t(1 - t)\ d^2(x, z).$$

Busemann Non-positive Curvature

A locally geodesic space (M, d) is said to be a *Busemann non-positive curvature space* if for every $p \in M$ there exists $\delta_p > 0$ such that for all $x, y, z \in B(p, \delta_p)$ we have

$$d(m(x, y), m(x, z)) \leq \frac{1}{2} d(y, z).$$

Pedersen Non-positive Curvature

Now, define the distance of a curve γ and a point $q \in M$ as

$$\text{dist}(\gamma, q) = \inf\{d(\gamma(t), q)\ :\ 0 \leq t \leq 1\}.$$

A locally geodesic space is said to be a *Pedersen non-positive curvature space* if for every $p \in M$ there exists $\delta_p > 0$ such that for any two shortest geodesics $\gamma_1, \gamma_2 : [0, 1] \to B(p, \delta_p)$ the function $f : [0, 1] \to \mathbb{R}$, defined by

$$f(t) = \text{dist}(\gamma_1, \gamma_2(t))$$

is quasiconvex, i.e., for every $t \in [0, 1]$, $f(t) \leq \max\{f(0), f(1)\}$.

Convex Capsules

Let $\gamma : [a, b] \to M$ be a shortest geodesic and $\alpha > 0$. Attached to γ and α, we define the *capsule* as

$$\mathcal{C}_\gamma(\alpha) = \{q \in M : \text{dist}(\gamma, q) \leq \alpha\}.$$

Let M_0 be a non-empty subset of M. The pair (γ, α) is said to be M_0-admissible if $\mathcal{C}_\gamma(\alpha) \subset M_0$.

We say that a locally geodesic space (M,d) has *convex capsules* if for every $p \in M$ there exists $\delta_p > 0$ such that for every $B(p,\delta_p)$-admissible pair (γ,α), the capsule $\mathcal{C}_\gamma(\alpha)$ is convex.

Remark : In general, an Alexandrov non-positive curvature space is a Busemann non-positive curvature space (see [25], Chapter 2). Moreover, Busemann non-positive curvature property meets the of Pedersen non-positive curvature space and accordingly has a convex capsule. Nevertheless, on Riemannian structures, all the above non-positive curvature notions coincide, and they characterize the non-positivity of the sectional curvature. For a deeper discussion of the non-positive curvature spaces, we indicate the reader to the classical work of Bridson and Haefliger [11] and Busemann [12].

Average angle

Given $\epsilon > 0$, let $\gamma_1 : [0,\epsilon) \longrightarrow X$ and $\gamma_2 : [0;\epsilon) \longrightarrow X$ be two geodesics in a length space X emanating from the same point $p = \gamma_1(0) = \gamma_2(0)$. We define the *angle* $\angle(\gamma_1, p, \gamma_2)$ between γ_1 and γ_2 as

$$\angle(\gamma_1, p, \gamma_2) = \lim_{s,t\to 0} \widetilde{\angle}(\gamma_1(s), p, \gamma_2(t)),$$

if the limit exists, where

$$\widetilde{\angle}(\gamma_1(s), p, \gamma_2(t)) := \arccos \frac{s^2 + t^2 - d(\gamma_1(s),\gamma_2(t))^2}{2st}.$$

Let $\gamma_1 : [0,a] \to M$ and $\gamma_2 : [0,b] \to M$ be two shortest geodesics with $p = \gamma_1(0) = \gamma_2(0)$. *The average angle* between γ_1 and γ_2 at p is defined by

$$\measuredangle(\gamma_1, p, \gamma_2) = \lim_{n\to\infty} A_{\gamma_1,\gamma_2}(\frac{a}{2^n}, \frac{b}{2^n}),$$

if the limit of the sequence exists, where the comparison angle is given by

$$A_{\gamma_1,\gamma_2}(a,b) := \arccos \frac{a^2 + b^2 - d(\gamma_1(a),\gamma_2(b))^2}{2ab}.$$

Let q be an inner point of a shortest geodesic pr, and qs be a shortest geodesic. It is clear that for an Alexandrov non-positive curvature space the sum of adjacent average angles is at least π, i.e., $\measuredangle(p,q,s) + \measuredangle(s,q,r) \geq \pi$.

Remark : In a locally geodesic space the Alexandrov and Busemann non-positive curvature properties are equivalent if and only if the sum of adjacent average angles $\geq \pi$ [19].

The Hilbert metric of a convex domain and its curvature

Let K be a bounded convex open set in $\mathbb{R}^n$ $(n \geq 2)$. The Hilbert metric d_K on K is defined as follows. For any $x \in K$, let $d_K(x,x) = 0$. For distinct points $x, y \in K$, assume that the straight line passing through x, y intersects the boundary ∂K at two points a, b such that the order of these four points on the line is a, x, y, b.

Denote the cross-ratio of the points by

$$[a, x, y, b] = \frac{\|b - x\|}{\|b - y\|} \frac{\|a - y\|}{\|a - x\|},$$

where $\|.\|$ is the Euclidean norm of $\mathbb{R}^n$. Then the *Hilbert metric* is

$$d_K(x, y) = \frac{1}{2} \ln[a, x, y, b],$$

and the metric space (K, d_K) is called a *Hilbert geometry* of the domain K.

Concerning the curvature properties of the Hilbert metric, Busemann ([12], page 108) showed that for any strictly convex domain K, the Hilbert metric d_K satisfies the Pedersen non-positivity curvature property, and, consequently, has a convex capsules. Kelly and Straus published two papers on the curvature of Hilbert geometry in 1958 and 1968. They used the concept of Busemann non-positive curvature, which is a pleasant geometric approach and weaker than Alexandrov's one. Nevertheless in the case of Hilbert metric of a convex domain it has a strong consequence, namely it implies the reduction of the domain to an ellipsoid. In details, it was proved in [26]:

Proposition : If Hilbert metric (K, d_K) has Busemann non-positive curvature, then the domain K is an ellipsoid and the Hilbert metric d_K is hyperbolic, i.e., Riemannian.

Proposition : Let (K, d_K) be the Hilbert metric of a convex domain K. Then the Busemann non-positive curvature is equivalent to the Alexandrov non-positive curvature.

Corollary : In Hilbert geometry, the sum of adjacent average angles is $\geq \pi$.

Finsler structure of the Hilbert metric

Finsler geometry

Let M be an n-dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle. If the continuous function $\hat{F} : TM \longrightarrow [0,\infty)$ satisfies the following conditions:

(i) (**Regularity**) $\hat{F}$ is continuous (C^0) on TM and C^∞ on $TM \setminus \{0\}$.

(ii) (**Positive homogeneity**) $\hat{F}(tu) = t\hat{F}(u)$ for all $t > 0$ and $u \in TM$, i.e., $\hat{F}$ is positively homogeneous of degree one.

(iii) (**Strong convexity**) The matrix $\dfrac{\partial^2 \hat{F}^2}{\partial y^i \partial y^j}(u)$ is positive definite for all $u \in TM \setminus \{0\}$.

Then we say that $\hat{F}$ is a Finsler fundamental function and $(M, \hat{F})$ is a *Finsler manifold.*

Berwald spaces

A Finsler manifold is of *Berwald type* if the Chern connection coefficients Γ^i_{jk} in natural coordinates depend only on the base point (see [10, p. 258]).

Kristály et al. proved in [28] that all mentioned non-positivity properties are equivalent to the analytical condition $\kappa \leq 0$ in the case of Berwald space:

Proposition :[28] Let $(M, \hat{F})$ be a Berwald space where $\hat{F}$ is positively (but perhaps not absolutely) homogeneous of degree one.The following assertions are equivalent:

a) The flag curvature κ of $(M, \hat{F})$ is non-positive.

b) $(M, d_{\hat{F}})$ is a Busemann non-positive curvature space.

c) $(M, d_{\hat{F}})$ is a forward Pedersen non-positive curvature space.

d) $(M, d_{\hat{F}})$ is a backward Pedersen non-positive curvature space.

e) $(M, d_{\hat{F}})$ has convex forward capsules.

f) $(M, d_{\hat{F}})$ has convex backward capsules.

The Hilbert metric d_K of the convex open domain K naturally determines its Hilbert Finsler fundamental function $\hat{F}_K$ as follows ([41]: First the asymmetric Finsler metric, called Funk metric $\widetilde{F}_K$ is defined by

$$p + \frac{1}{\widetilde{F}_K(u)} u \in \partial K \qquad \text{for any } u \in T_pK, \text{and } p \in K,$$

and then $\hat{F}_K$ is obtained by symmetrization:

$$\hat{F}_K(u) = \frac{1}{2}(\widetilde{F}_K(u) + \widetilde{F}_K(-u)).$$

Naturally, $\hat{F}_K$ is a reversible Finsler metric, therefore the forward and backward concepts coincide. It is easy to check that the induced distance of $\hat{F}_K$ is just the Hilbert distance d_K defined above in Definition .

The flag curvature of the Hilbert metrics was computed in 1929 by Funk in dimension 2 and by Berwald in all dimensions. Later T. Okada proposed a more direct computation:

Proposition : [35] The Hilbert geometry (K, d_K) is projectively flat Finsler space of negative constant curvature -1.

Theorem : If the Hilbert metric d_K of a convex domain K is a Berwald metric, then it reduces to a Riemannian metric, and the domain is an ellipsoid.

Corollary : (see [41, Theorem 11.6])The Hilbert metric d_K of a bounded convex domain $K \subset \mathbb{R}^n$ with smooth strongly convex boundary is Riemannian if and only if K is an ellipsoid.

Remark : Conversely, if the domain is not an ellipsoid, then d_K is a non-Berwaldian projectively flat metric.

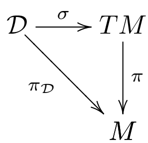
Sub-Finslerian geometry

A setting of sub-Finslerian geometry

Let M be an n–dimensional connected manifold. A sub-Finslerian structure on M is a triple $(\mathcal{D}, \sigma, F)$ where:

(1) $(\mathcal{D}, \pi_{\mathcal{D}})$ is a vector bundle on M.

(2) $\sigma : \mathcal{D} \longrightarrow TM$ is a morphism of vector bundles. In particular, the following diagram is commutative



such that $\pi_{\mathcal{D}} : \mathcal{D} \longrightarrow M$ and $\pi : TM \longrightarrow M$ are the canonical projection.

(3) A function $F : \widetilde{\mathcal{D}} \to \mathbb{R}$, where $\widetilde{\mathcal{D}} = \mathcal{D} \backslash \{0\}$, called a *sub-Finsler metric*, which satisfies the following properties:
- $F_x(v) > 0$ for all $v \in \widetilde{\mathcal{D}}$, $x \in M$.
- F is C^∞ on $\widetilde{\mathcal{D}}$.
- $F_x(\lambda v) = \lambda F_x(v)$ for all $v \in \widetilde{\mathcal{D}}_x$ and $\lambda \in \mathbb{R}_+$.
- The Hessian of F^2 with respect to the vector variables is positive definite.

A *sub-Finsler manifold* $(M, \mathcal{D}, F)$ is a smooth manifold M endowed with a sub-Finslerian structure i.e. the triple $(\mathcal{D}, \sigma, F)$.

Let $\mathcal{D}_x$ be the fiber over $x \in M$. The last condition means that the matrix $\frac{\partial^2 F^2}{\partial v^i \partial v^j}(x, v)$ is positive definite for all $v = (v^1, \dots, v^k) \in \mathcal{D}_x$. Equivalently, the corresponding indicatrix

$$I_x = \{v \,|\, v \in \mathcal{D}_x,\ F_x(v) = 1\}$$

is strictly convex.

The following technique describes the association between the sub-Finsler manifold $(\mathcal{D}, \sigma, F)$ and a Finsler metric $\hat{F}$ on $\mathrm{Im}(\sigma) \subset TM$:

For each $u \in \mathrm{Im}(\sigma)_x \subset T_x M$ and $x \in M$, we have

$$\hat{F}_x(u) = \inf_v \{F_x(v) |\ v \in D_x,\ \sigma(v) = u\}.$$

From now on we suppose that $\mathcal{D} \subset TM$, $\sigma : \mathcal{D} \longrightarrow TM$ is the inclusion $i : \mathcal{D} \longrightarrow TM$ and F is a sub-Finsler metric on $\mathcal{D}$.

As in the sub-Riemannian case, we call $\mathcal{D}$ the *horizontal distribution*. A curve $\gamma : [0,T] \to M$ is called *horizontal*, or *admissible* if $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in [0,T]$, that is $\gamma(t)$ is tangent to $\mathcal{D}$. The length of a smooth horizontal curve γ is defined as usual by

$$\ell(\gamma) = \int_0^T F(\dot{\gamma}(t))dt.$$

Equivalently and as in Finslerian case, we observe that it suffices to minimize the energy

$$E(\gamma) = \frac{1}{2}\int_0^T F^2(\dot{\gamma}(t))dt.$$

instead of length $\ell(\gamma)$.

The length induces a sub-Finslerian distance $d(x_0, x_1)$ between two points x_0 and x_1 as in Finsler geometry:

$$d(x_0, x_1) = \inf \ell(\gamma),$$

where we consider the infimum over all smooth horizontal curves joining x_0 and x_1. The distance is infinite if there is no such a horizontal curve between x_0 and x_1.

Definition : An absolutely continuous horizontal curve between x_0 and x_1 which realizes the distance $d(x_0, x_1)$ is called a *minimizing* geodesic between x_0 and x_1.

Definition : [33] A distribution $\mathcal{D}$ is said to be *bracket generating* if any local frame X_i of $\mathcal{D}$, together with all of its iterated Lie brackets spans the whole tangent bundle TM.

Theorem :[33](Chow's theorem). If $\mathcal{D}$ is a bracket generating distribution on a connected manifold M then any two points of $\mathcal{D}$ can be joined by a horizontal path.

The problem of minimizing the length of joining two given points x_0 and x_1 is equivalent to a time optimal problem: where the control bundle is $(\mathcal{D}, \pi, M)$ and we are searching for such a curve $\gamma(t)$ and a control curve $v(t) \in \mathcal{D}_{\gamma(t)}$ minimizing the time T needed to connect x_0 and x_1.

Connection

Legendre transformation of a sub-Finslerian manifold

The *sub-Lagrange function*

$$L : \mathcal{D} \longrightarrow \mathbb{R},$$

determined by F is given in the following way:

$$L = \frac{1}{2}F^2.$$

The fiber derivative of L defines the map

$$\mathcal{L}_L : \mathcal{D} \longrightarrow \mathcal{D}^*,$$

$$\mathcal{L}_L(v)(w) = \frac{d}{dt}L_x(v + tw), \text{ such that } v, w \in \mathcal{D}_x,$$

which is known in the literature as the *Legendre transformation* (see [1], [10]). We use the Legendre transformation to carry over the geometrical objects of a sub-Lagrange space from $\mathcal{D}$ onto $\mathcal{D}^*$. Then the relation of the distribution $\mathcal{D}$ of tangent bundle and the distribution $\mathcal{D}^*$ of cotangent bundle is given by Legendre transformation in local coordinates as follows

$$\mathcal{L}_L(x^i, v^a) = (x^i, \frac{\partial L}{\partial v^a}).$$

Then sub-Hamiltonian is given by

$$H : \mathcal{D}^* \longrightarrow \mathbb{R},$$

$$H = \iota_{\mathcal{L}_L^{-1}} - L \circ \mathcal{L}_L^{-1}, \text{ or locally, } H = v^a p_a - L.$$

Secondly, for fiber derivative of H, we define the Legendre transformation of the sub-Hamiltonian H in the following way

$$\mathcal{L}_H : \mathcal{D}^* \longrightarrow \mathcal{D},$$

for any $\alpha, \beta \in \mathcal{D}_x^*$, it holds

$$\beta(\mathcal{L}_H(\alpha)) = \frac{d}{dt}H_x(\alpha + t\beta).$$

This relates the distribution $\mathcal{D}^*$ of cotangent bundle and the distribution $\mathcal{D}$ of tangent bundle locally according to the next expression

$$\mathcal{L}_H(x^i, p_a) = (x^i, \frac{\partial H}{\partial p_a}),$$

we say H is a Hamiltonian if and only if $\mathcal{L}_H$ is local diffeormorphism ([1]).

In other hand, for every $p \in \mathcal{D}^*$, one can define the sub-Finsler metric on the cotangent space $\mathcal{D}^*$ with help of the indicatrix I as follows:

$$F^*(p) := \sup_{w \in I} p(w) = \sup_{0 \neq v \in \mathcal{D}} p[\frac{v}{F(v)}].$$

Moreover, a function F^* defined in the following way

$$F^* : \widetilde{\mathcal{D}}^* \longrightarrow \mathbb{R}.$$

$$F^*(p) = F(v), \text{ where } p = \mathcal{L}_L(v),$$

and

$$H := \frac{1}{2}(F^*)^2,$$

see details in [10].

We recall here the basic relations of non-linear connections of Lagrange and Hamiltonian spaces. For a Lagrange space, there exists a canonical non-linear connection given by (see [21])

$$N^i_j = \frac{1}{2}\frac{\partial G^i}{\partial v^j}; \quad G^i = g^{ij}\left(\frac{\partial^2 L}{\partial v^j \partial x^k} v^k - \frac{\partial L}{\partial x^i}\right). \tag{0.1}$$

In the homogeneous case, i.e. for Finsler manifolds this connection is also called as the Barthel non-linear connection. For Hamiltonian spaces, the non-linear connection is the image of the non-linear connection of the sub-Finsler spaces (Lagrangian spaces) by Legendre transformation.

Definition : A linear transformation $P : TM \longrightarrow \mathcal{D}$ is called a *projection operator* in $\mathcal{D}$, if $P^2 = P$ and $\mathcal{D} = \mathrm{Im}(P)$. If P is a projection operator in $\mathcal{D}$ then

$$TM = \mathrm{Ker}(P) \oplus \mathrm{Im}(P), \quad \text{and} \quad P = 0_{\mathrm{Ker}(P)} \oplus \mathrm{I}_{\mathrm{Im}(P)},$$

such that $\mathcal{D}^{\perp} := \mathrm{Ker}(P)$. Its complement projection is $P^c = id - P$.

For the sake of notation, we shall use the symbols P^* to denote the projection on T^*M and $(P^*)^c$ to denote its complement, respectively, namely

$$P^* : T^*M \longrightarrow (\mathcal{D}^{\perp})^0.$$

where $P^*(\alpha)(u) = \alpha(P(u))$ for all $\alpha \in T^*M$, $P(u) \in \mathcal{D}$ and u is a vector in TM.
Next the projection complement which is given by

$$(P^*)^c : T^*M \longrightarrow \mathcal{D}^0,$$

satisfies the condition $(P^*)^c = id - P^*$, such that for all $(P^*)^c(\alpha) \in \mathcal{D}^0$ we have

$$(P^*)^c(\alpha)(u) = \alpha(u - P(u)) = 0, \text{ if } u \in \mathcal{D}.$$

Furthermore,

$$P^*(\alpha) + (P^*)^c(\alpha) = \alpha, \; P^* + (P^*)^c = id_{T^*M}.$$

Now T^*M can be written as the direct sum of $(\mathcal{D}^{\perp})^0$ and $\mathcal{D}^0$.

After all, one can imagine a picture from above which through it one has a complete conception of generating a Finsler metric from sub-Finsler one by using the upcoming technique. Starting with a sub-Finsler metric F in the subbundle $\mathcal{D}$, we choose an arbitrary $\widetilde{F}$ which is defined in the complement $\mathcal{D}^{\perp}$ associated to P. Now if we take the sum of both metrics we will obtain a *Finsler metric* $\hat{F}$ in TM, specifically

$$\hat{F}^2(u) = F^2(P(u)) + \widetilde{F}^2(P^c(u)) \text{ for all } u \in TM.$$

Comparing the Legendre transformations of the sub-Finsler metric F and the extended Finsler metric $\hat{F}$ one can easily see that the following relations hold:

$$\begin{aligned} \mathcal{L}_L = \mathcal{L}_{\hat{L}}|_{\mathcal{D}}, &\quad P^* \circ \mathcal{L}_L = \mathcal{L}_{\hat{L}} \circ i, \\ \mathcal{L}_H = \mathcal{L}_{\hat{H}}|_{\mathcal{D}^*}, &\quad \mathcal{L}_H \circ i = P^* \circ \mathcal{L}_{\hat{H}}. \end{aligned}$$

Generalized non-linear connection for a sub-Finslerian manifold

Suppose that $(M, \mathcal{D}, F)$ is a sub-Finsler geometry such that M is a smooth manifold of dimension n equipped with distribution $\mathcal{D} \subset TM$ and F is a sub-Finsler metric on $\mathcal{D}$. The natural inclusion $i : \mathcal{D} \longrightarrow TM$ is then a linear bundle mapping fibered over the identity of M. A distribution is also completely characterized by its annihilator, i.e., giving $\mathcal{D}$ is equivalent to specifying the subbundle $\mathcal{D}^0$ of the cotangent bundle T^*M whose fibre over $x \in M$ consists of all covector at x which annihilates all vectors in the subspace $\mathcal{D}_x$ of T_xM. With a sub-Finsler structure one can associate a smooth mapping, defined by

$$E : T^*M \longrightarrow TM, \quad E(\alpha_x) = i(\mathcal{L}_H(i^*(\alpha_x))) \in TM, \tag{0.2}$$

where $i^* : T^*M \longrightarrow \mathcal{D}^*$ is the adjoint mapping of i, i.e., for any $\alpha_x \in T^*_xM$, $i^*(\alpha_x)$ is determined by

$$\langle X_x, i^*(\alpha_x)\rangle = \langle i(X_x), \alpha_x\rangle \text{ for all } X_x \in \mathcal{D}_x,$$

such that $\langle v, \alpha\rangle := \alpha(v)$ for all $v \in \mathcal{D}, \alpha \in \mathcal{D}^*$. Clearly, E is a bundle mapping whose image set is precisely the subbundle $\mathcal{D}$ of TM and whose kernel is the annihilator $\mathcal{D}^0$ of $\mathcal{D}$. To simplify notations we shall often identify an arbitrary vector in $\mathcal{D}$ with its image in TM under i and smooth section of $\mathcal{D}$ (i.e., element of $\Gamma(\mathcal{D})$) will often be regarded as a vector field on M. To E we can further associate a section $\bar{E}$ of $TM \otimes TM \longrightarrow M$ according to

$$\bar{E}(x)(\alpha_x, \beta_x) = \langle E(\alpha_x), \beta_x\rangle = \langle \mathcal{L}_H(i^*(\alpha_x)), i^*(\beta_x)\rangle,$$

for all $x \in M$ and $\alpha_x, \beta_x \in T^*_xM$. One can check the invariance of $\bar{E}$ under the projection for all $\alpha \in \mathfrak{X}^*(M)$, more explicitly

$$(P^* \ \bar{E})(\alpha, \alpha) = \bar{E}(\alpha, \alpha), \qquad \text{for all } (P^*)^c(\alpha) \in \Gamma(\mathcal{H}^0). \tag{0.3}$$

Definition : The map $\nabla : Sec\,\nu \times Sec\,\pi \longrightarrow Sec\,\pi, \quad (s, \sigma) \mapsto \nabla_s\sigma$ is called *a generalized non-linear connection over the anchor map* ϱ if

- $\mathbb{R}$-linear in s and σ;
- additive in s;
- for any $f \in \mathcal{F}(M), \nabla_s(f\sigma) = f\nabla_s\sigma + \varrho(s)(f)\sigma.$

This will applied to sub-Finslerian geometry with the following choices: $V = A = T^*M, \varrho = E : T^*M \longrightarrow TM.$

Definition : An $\mathcal{L}$-*connection* ∇ on a sub-Finsler manifold is a generalized non-linear connection over the induced mapping $E : T^*M \longrightarrow TM$ constructed by Legendre transformation $\mathcal{L}_H : \mathcal{D}^* \longrightarrow \mathcal{D}$ by (0.2).

Definition : The *symmetric bracket* associated to sub-Finsler geometry is mapping

$$\{.,.\} : \mathfrak{X}^*(M) \times \mathfrak{X}^*(M) \longrightarrow \mathfrak{X}^*(M),$$

$$\{\alpha, \beta\} = \bar{\mathcal{L}}_{E(\alpha)}\beta + \bar{\mathcal{L}}_{E(\beta)}\alpha - d(\bar{E}(\alpha, \beta)) - d(\bar{E}(\beta, \alpha)),$$

where $\bar{\mathcal{L}}_X$ is the *Lie derivative* with respect to $X \in \mathfrak{X}^*(M)$.

In the following proposition we list some properties of the above bracket, the first of which justifies the denomination "symmetric bracket". The proofs are straightforward.

Proposition : [29] For any symmetric bracket the following properties are satisfied for any $\alpha, \beta \in \mathfrak{X}^*(M)$:

(1) $\{\alpha, \beta\} = \{\beta, \alpha\}$;

(2) the bracket is $\mathbb{R}$-linear;

(3) $\{f\alpha, \beta\} = E(\beta)(f)\alpha + f\{\alpha, \beta\}$;

(4) $\{\alpha, \gamma\} = \bar{\mathcal{L}}_{E(\alpha)}\gamma$, for any $\gamma \in \Gamma(\mathcal{D}^0)$ and $\{\alpha, \gamma\} = 0$ if both α and γ belong to $\Gamma(\mathcal{D}^0)$.

The first three properties justify the next definition.

Definition : An $\mathcal{L}$-connection ∇ on sub-Finsler manifold $(M, \mathcal{D}, F)$ is said to be a *normal* if the associated symmetric product equals to the symmetric bracket, i.e. if $\langle \alpha, \beta \rangle_\nabla = \{\alpha, \beta\}$ holds for all $\alpha, \beta \in \mathfrak{X}^*(M)$, where the symmetric product of ∇ is given by

$$\langle \alpha, \beta \rangle_\nabla = \nabla_\alpha \beta + \nabla_\beta \alpha, \text{ for all } \alpha, \beta \in \mathfrak{X}^*(M).$$

One can associate a mapping δ to any sub-Finsler manifold $(M, \mathcal{D}, F)$ according to

$$\delta : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}^0) \longrightarrow \mathfrak{X}^*(M), \quad (X, \gamma) \mapsto \delta_X \gamma = i_X d\gamma.$$

It is clear that δ generalizes the Bott connection of involutive distribution to our general case of non-involutive distribution ([13]).

Definition : An $\mathcal{L}$-connection ∇ on sub-Finsler space $(M, \mathcal{D}, F)$ is said to be an *adapted* to the bundle $\mathcal{D}$ (shortly $\mathcal{D}$-adapted) if $\nabla_\alpha \gamma = \delta_\alpha \gamma$ for all $\alpha \in \mathfrak{X}^*(M)$ and $\gamma \in \Gamma(\mathcal{D}^0)$.

We define the *Barthel non-linear connection* $\overline{\nabla}^B$ of the cotangent bundle as follows

$$\overline{\nabla}^B_X \alpha(Y) = X(\alpha(Y)) - \alpha(\nabla^B_X Y),$$

where the Barthel non-linear connection ∇^B on the tangent bundle was locally given in (0.1). Recall that in some literature, the Barthel non-linear connection plays the same role instead of the Levi-Civita connection in case of positively homogeneous.

Theorem : Let ∇ be an $\mathcal{L}$-connection, then the following assertions are equivalent:

(i) ∇ is a normal $\mathcal{L}$-connection;

(ii) For any $\alpha \in \mathfrak{X}^*(M)$, ∇ satisfies:

$$\nabla_\alpha \alpha = \overline{\nabla}^B_{E(\alpha)} P^*(\alpha) + \delta_{E(\alpha)} (P^*)^c(\alpha).$$

The above theorem clarifies, in particular that

$$\nabla_\alpha \beta = \overline{\nabla}^B_{E(\alpha)} P^*(\beta) + \delta_{E(\alpha)} (P^*)^c(\beta),$$

is a non-linear $\mathcal{L}$-connection and it is normal. Moreover, for all $\beta \in \Gamma(\mathcal{D}^0)$ one can verify $\nabla_\alpha \beta = \delta_{E(\alpha)}(\beta)$, i.e., the connection under consideration is also $\mathcal{D}$-adapted. Consequently, we have

Theorem : Given a sub-Finsler structure $(M, \mathcal{D}, F)$, one can always construct a normal and a $\mathcal{D}$-adapted $\mathcal{L}$-connection.

Definition : A non-linear $\mathcal{L}$-connection is called *partial* if for any $\alpha \in \mathfrak{X}^*(M)$ and $\beta \in \Gamma(\mathcal{D}^0)$ we have $\nabla_\beta \alpha = 0$.

Proposition : Let ∇ be a normal $\mathcal{L}$-connection. Then ∇ is partial if and only if ∇ is $\mathcal{D}$-adapted.

Proposition : An $\mathcal{D}$-adapted $\mathcal{L}$-connection is not metrical.

Sub-Finsler bundle

We define in this section a sub-Finsler vector bundle which will play a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry. Let us consider first the covector subbundle $(\mathcal{D}^*, \tau, M)$ with the projection $\tau : \mathcal{D}^* \longrightarrow M$, which is a covector subbundle of rank k $(= \dim \mathcal{D}^*)$ in the cotangent bundle of T^*M. The illustrious role in our consideration will play by the pullback bundle $\tau^*(\tau)$ of τ by τ as follows:

$$\mathrm{pr}_1 : \mathcal{D}^* \times_M \mathcal{D}^* \longrightarrow \mathcal{D}^*, \ (p, q) \mapsto p.$$

Throughout, we shall call the above pullback bundle as the *sub-Finsler bundle* over $\mathcal{D}^*$. The following is summarized for the pullback construction in our case:

$$\mathcal{D}^* \times_M \mathcal{D}^* = \{(p, q) \in \mathcal{D}^* \times \mathcal{D}^* | \ \ \tau(p) = \tau(q)\}.$$

Now, if p is fixed, then

$$\begin{aligned}(\mathrm{pr}_1)^{-1}(p) &= \{(p, q) \in \mathcal{D}^* \times \mathcal{D}^* | \ \ q \in \mathcal{D}^*_{\tau(q)}\} \\ &= \{p\} \times \mathcal{D}^*_{\tau(p)},\end{aligned}$$

is a fiber of the sub-Finsler bundle over $p \in \mathcal{D}^*$.

We can introduce a Riemannian metric g^* in the sub-Finsler vector bundle induced by the Hamiltonian H as follows:

$$g^*_p(q, r) = \frac{\partial^2 H(p + tq + sr)}{\partial t \partial s}|_{t,s=0} \qquad \text{for all } q, r \in \mathcal{D}^*_{\tau(p)},$$

which locally means

$$g^{*ij} = \frac{\partial^2 H}{\partial p_i \partial p_j}.$$

Now the sub-Finsler bundle $\tau^*(\tau) = (\mathcal{D}^* \times \mathcal{D}^*, \mathrm{pr}_1), \mathcal{D}^*$ allows k covector fields $\alpha_1, \alpha_2, \ldots, \alpha_k$ which form an orthonormal frame with respect to the induced Riemannian metric g^*.

Notice that $\alpha_i(p)$ is a covector field that depends on the position $x \in M$ and the direction $p \in \mathcal{D}^*$. Moreover, one can choose in a way that $\alpha_i(p)$ is a homogeneous of degree zero in p i.e. $\alpha_i(tp) = t^0\alpha_i(p) = \alpha_i(p)$. According to the above metric g^{*ij} on M which is a homogeneous of degree zero, we could generate a sub-Hamiltonian formalism in the components p_i (induces naturally by the inner product) as (see [14])

$$H_x(p) = \frac{1}{2} \sum_{i,j=1}^{n} g^{*ij} p_i p_j, \tag{0.4}$$

such that this metric defined in the extended Finsler metric which was shown in [5].

Since the sub-Hamiltonian $H_x(p)$ is homogeneous of degree 2 in the variable p. Consequently, we shall write the sub-Hamiltonian formalism (0.4) in a more useful way using the orthonormality of α_i as follows

$$H_x(p) = \frac{1}{2} \sum_{i=1}^{k} \langle p, \alpha_i(p) \rangle^2, \qquad p \in \mathcal{D}^*. \tag{0.5}$$

One can check the homogeneity of the sub-Hamiltonian formalism $H_x(p)$ easily. The importance of $H_x(p)$ lies on to define sub-Finslerian geodesics.

Definition : A *normal geodesic* between the points A and B is a solution $x(t)$ of the sub-Hamiltonian system

$$\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial p_i}(x,p), \\ \dot{p}_i &= -\frac{\partial H}{\partial x^i}(x,p), \quad i = 1, \dots, n \end{aligned}$$

with the boundary conditions $x(0) = A$ and $x(T) = B$.

Remark : Let $\xi(t) = (x(t), p(t))$ be a solution of the sub-Hamiltonian system and let $x(t)$ be its projection to M. Then every sufficient short sub-arc of the normal geodesic $x(t)$ is a minimizer sub-Finslerian geodesic. This subarc is the unique minimizer joining its end points. (see [22, Theorem 1]).

Remark : In the sub-Finslerian geometry, not all the sub-Finslerian geodesics are normal (contrary to the Finsler geometry). This is due to the fact that the sub-Finslerian geodesics which admits a minimizing geodesic might not be solved the sub-Hamiltonian system. Those minimizer that are not normal geodesics called *singular* or *abnormal* geodesics (see [22], [33]).

Moreover, we call the extremal pair $\xi(t) := (x(t), p(t))$ a *normal extremal* if it is a solution for the sub-Hamiltonian system, otherwise it is called an *abnormal extremal.* After all, by using (0.5) one can generate the following form of the system of differential equations in term of canonical coordinates (x, p):

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = \sum_{j=1}^{k} \langle p, \alpha_j(p) \rangle \ (\delta_i(\alpha_j(p)) + \langle p, D_{p_i} \alpha_j(p) \rangle), \tag{0.6}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = -\sum_{j=1}^{k} \langle p, \alpha_j(p) \rangle \langle p, D_{x^i} \alpha_j(p) \rangle. \tag{0.7}$$

Exponential map in sub-Finsler geometry

Definition : (Lipschitz continuous) A sequence $\gamma_k : [0,T] \longrightarrow M$ is *Lipschitz continuous* if there exists a real $C \geq 0$ that for every $t_1, t_2 \in [0,T]$ we have

$$d(\gamma_k(t_1) - \gamma_k(t_2)) \leq C|t_1 - t_2|.$$

Moreover, C is called a *Lipschitz constant* of γ_k if it is satisfying the above definition.

Proposition : Let M be a sub-Finsler manifold and $x \in M$. Assume that $\bar{B}_x(r)$ is compact, for some $r > 0$. Then for every $y \in B_x(r)$ there is a minimizing geodesic between x and y, i.e.,

$$d(x,y) = \min\{\ell(\gamma) \ |\gamma : [0,T] \longrightarrow M \text{ horizontal}, \ \gamma(0) = x, \gamma(T) = y\}.$$

The exponential map is an essential object in sub-Finslerian geometry, that is the map that parametrizes normal extremals through their initial covectors. We are going to define the exponential map in both of the distribution $\mathcal{D}, \mathcal{D}^*$ of tangent and cotangent bundle, respectively.

Definition : Let $\Omega_x \subset \mathcal{D}_x$ be the domain of the exponential map over $x \in M$ such that Ω_x given by

$$\Omega_x = \{v \in \Omega_x | \ \xi \text{ is defined on the interval } [0,1]\},$$

where $v = \mathcal{L}_H(p)$ by the Legendre transformation of sub-Hamiltonian H. Then the *sub-Finsler exponential map* is defined as follows

$$\exp_x : \Omega_x \subset \mathcal{D}_x \ \longrightarrow \ M, \ v \mapsto \pi_{\mathcal{D}}(\mathcal{L}_H(\xi(1))),$$

here $\xi(t)$ is the normal extremal for every $t \in [0,1]$.

We can do the same in the dual distribution $\mathcal{D}_x^*$, consider $\Omega_x^* \subset \mathcal{D}_x^*$ be the domain of the exponential map over $x \in M$ such that Ω_x^* given by

$$\Omega_x^* = \{p \in \Omega_x^* |\ \xi \text{ is defined on the interval } [0,1]\}\,.$$

Consequently, the *sub-Hamiltonian exponential map* is given by

$$\exp_x^* : \Omega_x^* \subset \mathcal{D}_x^* \longrightarrow M,\ p \mapsto \tau(\xi(1)),$$

where $\xi(t)$ is the same normal extremal as above. The set Ω_x^* containing the origin and star-shaped with respect to 0. Moreover, with the help of Legendre transformation it is fairly easy to see that

$$\exp_x(v) = \exp_x^*(p).$$

Remark : It is clear that in the case of sub-Finsler exponential map the following expressions holds:

$$\exp_x^*[\mathcal{B}_x^*(r)] = B_x(r),$$

$$\exp_x^*[\mathcal{S}_x^*(r)] = S_x(r),$$

which are analogous to the Finslerian context, see Bao et al. [10] for more details. There is also shown that for general Finsler manifolds, the exponential map is C^∞ away from the origin of TM and only C^1 at the origin such that for any $x \in M$,

$$d(\exp_x)_{|0} : T_xM \longrightarrow T_xM$$

is the identity map at the origin $0 \in T_xM$. Moreover, Akbar-Zadeh [3] proved that the exponential map is C^2 map near the origin if and only if the Finsler metric is Berwald type. Furthermore, $\exp_x$ map is indeed C^∞ over TM.

Hopf-Rinow Theorem in sub-Finslerian geometry

Theorem : Let $(M, \mathcal{D}, F)$ be any connected sub-Finsler manifold, where $\mathcal{D}$ is bracket generating distribution. The following conditions are equivalent:

(i) The metric space (M, d) is forward complete.

(ii) The sub-Finsler manifold $(M, \mathcal{D}, F)$ is forward geodesically complete.

(iii) $\Omega_x^* = \mathcal{D}_x^*$, additionally, the exponential map is onto if there are no strictly abnormal minimizer.

(iv) Every closed and forward bounded subset of (M, d) is compact.

Furthermore, for any $x, y \in M$ there exists a minimizing geodesic γ joining x to y, i.e. the length of this geodesic is equal to the distance between these points.

Summary in Hungarian

A disszertáció két fő részből áll. Az első rész célja, hogy tisztázzuk, mi a kapcsolat a nem-pozitív görbületi fogalmak között, különösen a Hilbert metrika esetében. A második rész – a ráfordított kutatási idő, és az elért eredmények szempontjából talán jelentősebb rész, - a szub-Finsler geometria kérdéseinek vizsgálatára irányul. A disszertáció három fejezete a következőképpen tagolódik: Az első fejezet bemutatja a differenciálgeometria alapfogalmait struktúra (sokaságok, érintőterek, kotangens terek, vektormezők és 1-formák, disztribúciók), továbbá példákat mutat be a tangens és kotangens nyalábokra. Ezt követően rövid áttekintést adunk a Finsler geometriáról, amely alapvető és természetes általánosítása a Riemann geometriának. A Finsler struktúra mindkét koordinátától függ, a ponttól és az érintővektortól (sebességtől) is, tehát a sokaság érintőterén megadott $\hat{F} : TM \longrightarrow [0, \infty)$ függvénnyel van meghatározva. Ezt követően összefoglaljuk a Berwald-terek alapvető tényeit. A fenti fogalmakra néhány jól ismert példát is bemutatunk. A második fejezetben először tisztázzuk, hogy mit jelent a nem-pozitív a görbület: bemutatjuk a definícióikat, pontosabban, a nem-pozitív görbület négy típusának definícióját a geodetikus metrikus terekben, azok egyes tulajdonságait és kapcsolatukat, továbbá a következő lépések szempontjából fontos, ismert eredményeket. Egy konvex tartomány Hilbert metrikájának definiálása után, azt mutatjuk meg, hogy a Hilbert metrika esetében az első két nem-pozitivitási fogalom pontosan akkor egyenértékű, ha a tartomány egy ellipszoid. Az analitikai megfontolásokra térve, a Finsler geometriában a Berwald tér speciális esetére szorítkozunk. Majd azokat az összefüggéseket és ismert tényeket mutatjuk be, melyek szükségesek a fő eredmény igazolásához. Végül megmutatjuk, hogy amennyiben a konvex tartomány Hilbert metrikája a Berwald típusú Finsler metrika, akkor a tartomány szükségképpen egy ellipszoid. A harmadik fejezetben definiáljuk a szub-Finsler metrikákat és néhány alapfogalmát. Példákat mutatunk be, amelyek teljesítik a szub-Finsler tulajdonságot, továbbá a szub-Finsler sokaságok alapvető tulajdonságait tárgyaljuk. Áttekintjük a pontpárok horizontális görbékkel való összeköthetőségének kérdését (Chow-tétele [18]). A következőkben bemutatjuk az általánosított konnexiók szerepét a szub-Riemann-geometriában, amellyel a normál és az abnormális extrémumgörbéket jellemezhetjük a szub-Riemann esetben (lásd a lokális verziót [39]-ben, és a koordináta-mentes változatát a [29]-ban.) Röviden ismertetjük a Legendre transzformációt a szub-Finsler geometriai esetben, mind a Lagrange, mind a Hamilton metrikából levezetve. Definiáljuk a szub-Finsler

metrikához tartozó szimmetrikus zárójelet és egy L-konnexió szimmetrikus szorzatát, Megmutatjuk, hogy azok akkor és csak akkor esnek egybe, ha az L-konnexió normális. Az involutív disztribúciók Bott konnexiójának általánosításával jellemezhetjük a D-adaptált és a normál konnexiókat. Végül a Hopf-Rinow tételre térünk. Jól ismert, hogy a Riemann és a Finsler geometriában két teljességi fogalom van. Az első a metrikus terek teljessége. Másrészt egy Riemann vagy Finsler sokaságot geodetikusan teljesnek nevezzük, ha bármely $x \in M$-től kezdődő $\gamma(t)$ geodetikus a $t \in \mathbb{R}$ minden értékére van meghatározva. A Finsler geometriában ezen túlmenően beszélhetünk az előre, illetve visszafelé mért távolságról, és előre, illetve visszafelé tekintett geodetikus teljességről. A Hopf-Rinow tétel a teljes Riemann-sokaságok alaptétele, amely a teljesség tulajdonságot a kompaktsággal és az exponenciális leképezéssel kapcsolja össze. Ennek következménye szerint egy teljes sokaság bármely két pontja összeköthető minimális geodetikus görbével. 1931-ben H. Hopf és W. Rinow csak felületekre mutatta meg a tételt, de a bizonyítás magasabb dimenzióban sem különbözik jelentősen. A Hopf-Rinow-tételt részletesen tanulmányozták mind a Riemann, mind a Finsler geometriákban (ld. [10], [16], [34]). A Finsler esetben elegendő csak előre-teljességet feltételezni. A Hopf-Rinow-tétel állításainak bizonyítása érdekében a szub-Finsler-sokaságok esetén először egy szub-Finsler metrikát definiálunk a $\mathcal{D}^*$ kotangens térben a Legendre transzformáció segítségével, mely egy szub-Hamilton függvényt határoz meg. Ezután egy szub-Finsler nyalábot konstruálunk, amely jelentős szerepet játszik kap a szub-Hamilton függvény előállításában. Ugyanis, a szub-Finsler nyaláb lehetővé teszi egy ortonormált bázismező alkalmazását. Ezután áttekintjük a szub-Finsler geometriai exponenciális leképezés fogalmát és alapvető tulajdonságait. Végül a fő tételt, Hopf-Rinow tételét mondjuk ki és bizonyítjuk a szub-Finsler geometria esetében.

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Neptun ID: KUDQ0K
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List of other publications

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   IF: 0.526 (2017)

Foreign language scientific articles in international journals (2)

3. Vincze, C., Oláh, M., **Alabdulsada, L. M. H.**: On the divergence representation of the Gauss curvature of Riemannian surfaces and its applications.
   Rend. Circ. mat. Palermo. Ser. 2. [Epub], 1-13, 2018. ISSN: 0009-725X.
   DOI: http://dx.doi.org/10.1007/s12215-018-0382-6
4. Jaber, H., **Alabdulsada, L. M. H.**: On Almost contra T*-continuous functions.
   J. Kufa Math. Comput. 1 (6), 1-6, 2012. ISSN: 2076-1171.

Total IF of journals (all publications): 1,522
Total IF of journals (publications related to the dissertation): 0,996

The Candidate's publication data submitted to the iDEa Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

26 February, 2019

DEBRECENI EGYETEM
EGYETEMI ÉS NEMZETI KÖNYVTÁR
H-4002 Debrecen, Egyetem tér 1, Pf.: 400
Tel.: 52/410-443, e-mail: publikaciok@lib.unideb.hu

Nyilvántartási szám: DEENK/42/2019.PL
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Jelölt: Alabdulsada, Layth Muhsin Habeeb
Neptun kód: KUDQ0K
Doktori Iskola: Matematika- és Számítástudományok Doktori Iskola
MTMT azonosító: 10066157

A PhD értekezés alapjául szolgáló közlemények

Idegen nyelvű tudományos közlemények külföldi folyóiratban (1)

1. **Alabdulsada, L. M. H.**, Kozma, L.: On non-positive curvature properties of the Hilbert metric.
   J. Geom. Anal. [Epub], 1-8, 2018. ISSN: 1050-6926.
   DOI: http://dx.doi.org/10.1007/s12220-018-0011-9
   IF: 0.996 (2017)

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H-4002 Debrecen, Egyetem tér 1, Pf.: 400
Tel.: 52/410-443, e-mail: publikaciok@lib.unideb.hu

További közlemények

<u>Idegen nyelvű tudományos közlemények hazai folyóiratban</u> (1)

2. **Alabdulsada, L. M. H.**: On the class of weakly almost contra-T*-continuous functions.
   Publ. Math. Debr. "Accepted by Publisher", 1-9, 2019. ISSN: 0033-3883.
   IF: 0.526 (2017)

<u>Idegen nyelvű tudományos közlemények külföldi folyóiratban</u> (2)

3. Vincze, C., Oláh, M., **Alabdulsada, L. M. H.**: On the divergence representation of the Gauss curvature of Riemannian surfaces and its applications.
   Rend. Circ. mat. Palermo. Ser. 2. [Epub], 1-13, 2018. ISSN: 0009-725X.
   DOI: http://dx.doi.org/10.1007/s12215-018-0382-6

4. Jaber, H., **Alabdulsada, L. M. H.**: On Almost contra T*-continuous functions.
   J. Kufa Math. Comput. 1 (6), 1-6, 2012. ISSN: 2076-1171.

A közlő folyóiratok összesített impakt faktora: 1,522
A közlő folyóiratok összesített impakt faktora (az értekezés alapjául szolgáló közleményekre): 0,996

A DEENK a Jelölt által az iDEa Tudóstérbe feltöltött adatok bibliográfiai és tudománymetriai ellenőrzését a tudományos adatbázisok és a Journal Citation Reports Impact Factor lista alapján elvégezte.

Debrecen, 2019.02.26.