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Dynamic complexity of a discretized predator-prey system with Allee effect and herd behaviour

Yashra Javaid^a, Shireen Jawad^b, Rizwan Ahmed^a, Ali Hasan Ali^{c,d,e} and Badr Rashwani^f

^aDepartment of Mathematics, Air University Multan Campus, Multan, Pakistan; ^bDepartment of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq; ^cInstitute of Mathematics, University of Debrecen, Debrecen, Hungary; ^dDepartment of Business Management, Al-imam University College, Balad, Iraq; ^eTechnical Engineering College, Al-Ayen University, Dhi Qar, Iraq; ^fMathematics Department, Science Faculty, Tishreen University, Latakia, Syria

ABSTRACT

This paper investigates the dynamics of a discrete-time predator-prey system in which the prey population is impacted by the Allee effect. Possible fixed points in the system are studied for their existence and topological categorization. Moreover, the presence and direction of period-doubling and Neimark-Sacker bifurcations at the interior fixed point are examined through the application of bifurcation theory and the centre manifold theorem. A hybrid approach is adopted to control chaotic behaviour and prevent bifurcations. Numerical examples are provided to substantiate our theoretical findings. The Allee effect has been shown to affect the dynamics of the system using numerical simulations. The moderate Allee effect stabilizes predator and prey populations, facilitating ecological cohabitation and persistence.

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1. Introduction

Predator-prey interactions are a common and important phenomenon in the biological realm. This predator-prey interaction is well represented by a popular mathematical model that was separately developed by Lotka [1] and Volterra [2]. However, since it oversimplifies the intricacies of the actual world, academics have made many changes to this system over time to improve its realism and understandability. Numerous variables, including functional response, fear, harvesting, refuge, and the Allee effect, influence the dynamic properties of predator-prey systems [3–11]. The actions and population dynamics of both predators and prey in the ecological system are significantly shaped by these variables. In order to get a clearer understanding of the complex dynamics that control predator-prey interactions in diverse ecological situations, researchers have been actively examining these components. These modifications and comprehensive investigations have transformed predator-prey models into useful tools

CONTACT Badr Rashwani  rashwanibadr@gmail.com  Mathematics Department, Science Faculty, Tishreen University, Latakia P.O. Box 2230, Syria

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for comprehending the dynamics of ecological systems. They considerably contribute to the fields of population ecology and ecosystem dynamics by offering insightful information about the coexistence, population oscillations, and stability of predator and prey species.

In the last few decades, the study of biological systems has focussed more on the complex processes that go beyond stability and bifurcation. An important component of ecological modelling has been the emergence of chaotic behaviour, which is defined by sensitive dependency on initial conditions and the existence of irregular, unexpected patterns. Bifurcation theory, particularly through phenomena such as period-doubling and Neimark-Sacker bifurcations, serves as a pivotal gateway to chaos in predator-prey models. While stability and bifurcation analysis are useful for understanding how biological systems behave over time, chaotic dynamics show how complicated and nonlinear predator-prey relationships are. We refer the readers to [12–17] for a thorough examination of stability, bifurcation, and chaos.

The refuge is crucial for the survival and stability of a species, as shown both theoretically and practically. Refuges may be divided into two categories: those that involve a constant proportion of prey and those that maintain a fixed quantity of prey. Notably, Chen et al. [18] have investigated the significance of prey refuges in stabilizing prey-prey interactions. Their study demonstrates how dynamic behaviours that are strongly driven by the prey refuge parameter may coexist with an ongoing prey refuge. Prey refuges are generally acknowledged to be essential for averting extinction and reducing predator-prey oscillations within ecological systems. Ma et al. [19] looked into the processes behind the stabilizing effects of refuges, which provided more insights into the impacts of prey refuge on predator-prey dynamics. Mukherjee [20] also carried out research on the impact of immigration and refuge in a predator-prey system, offering insightful information on how these elements interact and affect population dynamics. See [21–24] and the references therein for further biological context and findings about the impact of a prey refuge.

A fundamental aspect governing all interactions between predators and prey in population dynamics is the functional response. It establishes a close connection with the prey population and is essential in deciding how much prey each predator eats. In 1965, Holling [25] proposed three functional responses, which were later expanded upon by other researchers such as Crowley-Martin [26] and Beddington-DeAngelis [27,28]. Following that, other studies examined systems established for predator-prey interactions with different types of functional responses [29–34]. Herd behaviour is observed in certain prey populations, wherein interactions between prey and predators occur mostly near the edges of the prey population. The Holling-type functional response is insufficient to accurately explain this type of interaction. The prey population exhibits collective behaviour. Therefore, the rate at which a predator captures its prey varies from typical systems. For example, the rate at which a fish captures zooplankton in the ocean is higher than its rate of capturing phytoplankton. Phytoplankton demonstrates collective behaviour in this particular setting. In order to analyse the herd behaviour of the prey population, Ajraldi et al. [35] utilized the square root of the prey population. This analysis revealed that the predator interacted with the prey in the outer corridor of the herd. The square root functional response of the predator-prey system has been investigated by several authors [36–40].

We consider the following predator-prey system [33]:

$$\begin{cases} \frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) - a(1-c)\sqrt{xy}, \\ \frac{dy}{dt} = b(1-c)\sqrt{xy} - dy, \end{cases} \quad (1)$$

where x is the prey density and y is the predator density satisfying the initial conditions $x(0) > 0, y(0) > 0$. Moreover, r, k, a , and c are the prey's natural growth rate, the carrying capacity of its environment, the rate at which it eats, and the rate at which it hides. The parameters b and d are the prey's conversion rate and the predator's death rate. r, k, a, b, d are all positive numbers and $0 < c < 1$.

The Lotka-Volterra system assumes a direct proportionality between the rate of consumption of a predator and the availability of its prey. This finding suggests that the consumption of prey by predators is limited only by the abundance of available prey resources. While the feasibility of this approach may be evident in situations where prey populations are few, it becomes unfeasible in scenarios characterized by high prey densities, since predators face limitations in terms of time and digestion. Several enhancements to the system have been suggested by researchers, such as the incorporation of functional responses of Holling type and density-dependent prey growth. The use of the Allee effect may serve as a means to modify the Lotka-Volterra system in an alternative manner. The incorporation of the Allee effect into the system is widely acknowledged as enhancing the realism of prey-predator interaction modelling. The Allee effect was first proposed by Allee in 1931 [41]. This study reveals a clear relationship between any measure of the fitness of a species and the size of its population. The primary factors contributing to the Allee effect include challenges associated with mate selection, the negative consequences of inbreeding, impaired social functioning within small communities, the need for predator avoidance, and the exploitation of limited food resources [42–44]. Various natural species have shown indications of the Allee effect, including a range of organisms such as plants, insects, marine invertebrates, birds, and mammals [45]. Recent research has shown that the Allee effect has substantial consequences for the dynamic stability analysis of population systems. This phenomenon has the potential to contribute either destabilizing or stabilizing impacts to the system. The Allee effect has the potential to induce complicated dynamics within predator-prey systems via its capacity to change the system's dynamics in unexpected ways [46–52].

Thus, motivated by the above discussion, a natural question arises: When the Allee effect is subject to the first species of system (1), what will happen to the dynamical properties? Hence, the system (1) is extended by adding the Allee effect in the prey population. Thus, the following modified system is obtained:

$$\begin{cases} \frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) \left(\frac{x}{\beta + x}\right) - a(1-c)\sqrt{xy}, \\ \frac{dy}{dt} = b(1-c)\sqrt{xy} - dy, \end{cases} \quad (2)$$

where $\frac{x}{\beta+x}$ stands for the Allee effect on the prey population and $\beta > 0$ is the Allee effect constant. Equation (2), which represents the variational of the system mentioned above,

exhibits a singularity by the square root term. To better comprehend it, we apply the transformations $x(t) = u2(t)$ and $y(t) = v(t)$. Equivalent to this transformation, the system denoted by (2) converts into the subsequent system:

$$\begin{cases} \frac{du}{dt} = \frac{1}{2} \left(\left(1 - \frac{u^2}{k} \right) \left(\frac{ru^3}{\beta + u^2} \right) - a(1 - c)uv \right), \\ \frac{dv}{dt} = b(1 - c)uv - dv. \end{cases} \quad (3)$$

When representing dynamical systems, one of two approaches is frequently used: i) discrete-time systems, which are governed by difference equations; or ii) continuous-time systems, which are described by differential equations. Over time, academics have undertaken comprehensive inquiries into the nonlinear dynamic characteristics demonstrated by continuous systems. In contemporary times, researchers have exhibited a notable surge in their focus on discrete-time systems [53–56]. This is because discrete systems are considerably more effective than continuous systems at nonoverlapping generation. Ecosystems with non-overlapping generations are especially well-suited for discrete-time models, which provide an organized framework for comprehending their dynamics. Among the various examples, ecosystems reliant on annual plants as a primary food source exemplify this phenomenon. Here, herbivores, such as certain insects or small mammals, follow distinct generational cycles, mirroring the annual plant life cycle. In the same way, populations of marine plankton, which include both phytoplankton and zooplankton, reproduce quickly and have short lifetimes, resulting in non-overlapping generations. Similarly, because of their rapid rates of reproduction and short lifespans, bacteria and the virus that hunts them down, bacteriophages, exhibit non-overlapping generations. All of these examples highlight how important discrete-time models are for representing the complex dynamics of predator-prey interactions in environments with non-overlapping generations. Additionally, the utilization of discrete-time systems facilitates the acquisition of numerical solutions. Furthermore, an abundance of research has demonstrated that discrete-time systems exhibit dynamical behaviours that are more intricate compared to continuous-time systems [57–60]. So, discrete-time systems are more attractive than continuous-time systems. There are several methods for converting continuous differential systems to their discrete counterparts. Among these, the Euler method is the much simpler numerical method, widely employed in the literature to discretize continuous differential systems. The system to be analysed in this work is obtained by using the forward Euler method on the system (3) as follows:

$$\begin{cases} u_{n+1} = u_n + \frac{h}{2} \left((ru_n) \left(\left(1 - \frac{u_n^2}{k} \right) \left(\frac{u_n^2}{\beta + u_n^2} \right) - (1 - c)av_n \right) \right), \\ v_{n+1} = v_n + h \left((1 - c)bu_n - d \right) v_n, \end{cases} \quad (4)$$

where $h > 0$ stands for the step size.

The following are the key contributions to this paper:

- The Allee effect is incorporated into system (1), and subsequently, we derive the discrete system using the forward Euler method.
- The existence and stability of fixed points are investigated.

- We explore Neimark-Sacker (NS) bifurcations and period-doubling (PD) at the positive fixed point.
- The conditions for the existence and direction of both kinds of bifurcations are examined at the positive fixed point.
- A hybrid control approach is employed to control chaos in the system.
- Numerical examples are presented to describe the rich dynamics of the system.

The paper is presented in the following format: The stability section examines the presence and topological categorization of fixed points. An examination of local bifurcation analysis at the positive fixed point is presented in Section 3. Hybrid control is implemented in Section 4 to manage and regulate bifurcations and disorder. To validate and describe the theoretical outcomes, numerical examples are provided in Section 5. Finally, a summary of our analysis can be found in Section 6.

2. Existence and topological classification of fixed points

Solving the following set of equations gives us the fixed points (\bar{u}, \bar{v}) of the system (4):

$$\begin{cases} \bar{u} = \bar{u} + \frac{h}{2} \left(r \left(1 - \frac{(\bar{u})^2}{k} \right) \left(\frac{(\bar{u})^3}{\beta + (\bar{u})^2} \right) - a(1 - c)\bar{v} \right), \\ \bar{v} = \bar{v} + h(b(1 - c)\bar{u} - d)\bar{v}. \end{cases} \quad (5)$$

By simple computations, one can obtain the following fixed points of the system (4):

$$E_0 = (0, 0), \quad E_1 = (\sqrt{k}, 0), \quad E_2 = \left(\frac{d}{b(1 - c)}, \frac{d^3 r(-d^2 + b^2 k(1 - c)^2)}{ab^3 k(1 - c)^4 (d^2 + b^2 \beta(1 - c)^2)} \right).$$

E_0, E_1 are boundary fixed points, and E_2 is the unique interior fixed point of the system (4) iff the carrying capacity of prey meets the condition $k > \frac{d^2}{b^2(1 - c)^2}$.

To classify the fixed points of the system (4), we employ the two following results:

Lemma 2.1 ([61]): *Let $\Lambda(\gamma) = \gamma^2 + S_1\gamma + S_0$ be the characteristic polynomial of $J(\bar{u}, \bar{v})$ and γ_1, γ_2 are solutions of $\Lambda(\gamma) = 0$, then (\bar{u}, \bar{v}) is a*

- sink if $|\gamma_1| < 1$ and $|\gamma_2| < 1$,*
- source if $|\gamma_1| > 1$ and $|\gamma_2| > 1$,*
- saddle point if $|\gamma_1| < 1$ and $|\gamma_2| > 1$ (or $|\gamma_1| > 1$ and $|\gamma_2| < 1$),*
- non-hyperbolic point if either $|\gamma_1| = 1$ or $|\gamma_2| = 1$.*

Lemma 2.2 ([61]): *Let $\Lambda(\gamma) = \gamma^2 + S_1\gamma + S_0$. Assume that $\Lambda(1) > 0$. If γ_1, γ_2 are two roots of $\Lambda(\gamma) = 0$, then*

- $|\gamma_1| < 1$ and $|\gamma_2| < 1$ if $\Lambda(-1) > 0$ and $S_0 < 1$,*
- $|\gamma_1| < 1$ and $|\gamma_2| > 1$ (or $|\gamma_1| > 1$ and $|\gamma_2| < 1$) if $\Lambda(-1) < 0$,*
- $|\gamma_1| > 1$ and $|\gamma_2| > 1$ if $\Lambda(-1) > 0$ and $S_0 > 1$,*
- $\gamma_1 = -1$ and $|\gamma_2| \neq 1$ if $\Lambda(-1) = 0$ and $S_1 \neq 0, 2$,*
- γ_1 and γ_2 are complex and $|\gamma_{1,2}| = 1$ if $S_1^2 - 4S_0 < 0$ and $S_0 = 1$.*

The variational matrix of the following system:

$$\begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}$$

is the matrix provided by

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$

Thus, the variational matrix of the system (4) determined at any point (u, v) is provided by

$$J(u, v) = \begin{bmatrix} -hru^4(3u^2 + 5\beta) & \frac{1}{2}a(-1 + c)h \\ \frac{+k((2 + hr)u^4 + (4 + 3hr)u^2\beta + 2\beta^2)}{2k(u^2 + \beta)^2} & 1 - dh - b(-1 + c)hu \\ -b(-1 + c)hv & \end{bmatrix}.$$

2.1. Topological classification of $E_0(0, 0)$

We take into account the following set:

$$\Gamma_0 = \{(a, b, c, d, r, k, \beta, h) \in \mathbb{R}_+^8 \mid 0 < c < 1\},$$

where \mathbb{R}_+^8 is the set of all points in the Euclidean space \mathbb{R}^8 where all coordinates are positive.

We get the following result by applying Lemma 2.1:

Proposition 2.3: *Within the domain Γ_0 , the fixed point $E_0(0, 0)$ of the system (4) is non-hyperbolic.*

Proof: The variational matrix is provided at $E_0(0, 0)$ by

$$J(0, 0) = \begin{bmatrix} 1 & -\frac{1}{2}a(1 - c)h \\ 0 & 1 - dh \end{bmatrix}.$$

The eigenvalues of $J(0, 0)$ are $\gamma_1 = 1$ and $\gamma_2 = 1 - dh$. Since $|\gamma_1| = 1$, therefore $(0, 0)$ is non-hyperbolic according to Lemma 2.1. ■

2.2. Topological classification of $E_1(\sqrt{k}, 0)$

By applying Lemma 2.1, the following conclusion is reached:

Proposition 2.4: *Within the domain Γ_0 , the fixed point $E_1(\sqrt{k}, 0)$ of the system (4) is a*

- (i) *sink if $r < \frac{2(k+\beta)}{hk}$ and $b(1 - c)\sqrt{k} < d < \frac{2}{h} + b(1 - c)\sqrt{k}$,*
- (ii) *saddle point if any one of the following is true:*

- (a) $r > \frac{2(k+\beta)}{hk}$ and $(1-c)b\sqrt{k} < d < \frac{2}{h} + (1-c)b\sqrt{k}$,
 (b) $rhk < 2(k+\beta)$ and either $d < b(1-c)\sqrt{k}$ or $d > \frac{2}{h} + b(1-c)\sqrt{k}$,
 (iii) source if $rhk > 2(k+\beta)$ and any one of the following is true:
 (a) $d < (1-c)b\sqrt{k}$,
 (b) $d > \frac{2}{h} + (1-c)b\sqrt{k}$,
 (iv) non-hyperbolic point if any one of the following is true:
 (a) $rhk = 2(k+\beta)$,
 (b) $d = (1-c)b\sqrt{k}$,
 (c) $d = \frac{2}{h} + (1-c)b\sqrt{k}$.

Proof: The variational matrix is provided at $E_1(\sqrt{k}, 0)$ by

$$J(\sqrt{k}, 0) = \begin{bmatrix} 1 - \frac{hkr}{k+\beta} & \frac{1}{2}a(-1+c)h \\ 0 & 1 - dh - (-1+c)bh\sqrt{k} \end{bmatrix}.$$

The eigenvalues of $J(\sqrt{k}, 0)$ are $\gamma_1 = 1 - \frac{hkr}{k+\beta}$ and $\gamma_2 = 1 - dh - b(-1+c)h\sqrt{k}$. ■

2.3. Topological classification of $E_2 \left(\frac{d}{b(1-c)}, \frac{d^3r(-d^2+b^2k(1-c)^2)}{ab^3k(1-c)^4(d^2+b^2\beta(1-c)^2)} \right)$

We consider the following set:

$$\Gamma_2 = \left\{ (a, b, c, d, r, k, \beta, h) \in \mathbb{R}_+^8 \mid 0 < c < 1, k > \frac{d^2}{b^2(1-c)^2} \right\}.$$

The variational matrix is provided at E_2 by

$$J(E_2) = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & 1 \end{bmatrix}, \quad (6)$$

where

$$j_{11} = 1 + \frac{d^6hr}{(b-bc)^2k(d^2+b^2(-1+c)^2\beta)^2} + \frac{3d^2hr}{2(d^2+b^2(-1+c)^2\beta)} - \frac{d^4hr(5d^2+b^2(-1+c)^2(2k+5\beta))}{2b^2(-1+c)^2k(d^2+b^2(-1+c)^2\beta)^2},$$

$$j_{12} = \frac{1}{2}a(-1+c)h, \quad j_{21} = \frac{d^3h(d^2-b^2(-1+c)^2k)r}{ab^2(-1+c)^3k(d^2+b^2(-1+c)^2\beta)}.$$

then, The formula for the characteristic polynomial of $J(E_2)$ is

$$\Lambda(\gamma) = \gamma^2 + S_1\gamma + S_0, \quad (7)$$

where

$$S_1 = -\frac{-3d^6hr + b^4(-1+c)^4d^2k(8+3hr)\beta + 4b^6(-1+c)^6k\beta^2 + b^2(-1+c)^2d^4(k(4+hr) - 5hr\beta)}{2b^2(-1+c)^2k(d^2 + b^2(-1+c)^2\beta)^2},$$

$$S_0 = (-d^6h(3+dh)r + b^4(-1+c)^4d^2k(4+3hr+dh^2r)\beta + 2b^6(-1+c)^6k\beta^2 + b^2(-1+c)^2d^4(k(2+hr+dh^2r) - h(5+dh)r\beta)) / (2b^2(-1+c)^2k(d^2 + b^2(-1+c)^2\beta)^2).$$

By simple computations, we obtain

$$\Lambda(1) = \frac{d^3h^2(-d^2 + b^2(-1+c)^2k)r}{2b^2(-1+c)^2k(d^2 + b^2(-1+c)^2\beta)}, \quad \Lambda(0) = S_0,$$

$$\Lambda(-1) = (-d^6h(6+dh)r + b^4(-1+c)^4d^2k(16+6hr+dh^2r)\beta + 8b^6(-1+c)^6k\beta^2 + b^2(-1+c)^2d^4(k(8+2hr+dh^2r) - h(10+dh)r\beta)) / (2b^2(-1+c)^2k(d^2 + b^2(-1+c)^2\beta)^2).$$

It is obvious that $\Lambda(1) > 0$ in Γ_2 . Assume that

$$k_1 = \frac{d^2}{b^2(-1+c)^2}, \quad k_2 = \frac{d^2(3d^2 + 5b^2(-1+c)^2\beta)}{b^2(-1+c)^2(d^2 + 3b^2(-1+c)^2\beta)},$$

$$h_1 = \frac{-3d^4 + b^2(-1+c)^2d^2(k-5\beta) + 3b^4(-1+c)^4k\beta}{d(d^2 - b^2(-1+c)^2k)(d^2 + b^2(-1+c)^2\beta)},$$

$$r_1 = (8b^2(-1+c)^2k(d^2 + b^2(-1+c)^2\beta)^2) / (d^2h(6d^4 + d^5h - 2b^2(-1+c)^2d^2(k-5\beta) - b^2(-1+c)^2d^3h(k-\beta) - 6b^4(-1+c)^4k\beta - b^4(-1+c)^4dhk\beta)).$$

We get the following result by using Lemmas 2.2 and 2.1:

Proposition 2.5: *Within the domain Γ_2 , the fixed point E_2 is a*

- (i) *sink if $k_1 < k < k_2$, $h < h_1$, and $r < r_1$,*
- (ii) *saddle point if $k_1 < k < k_2$, $h < 2h_1$, and $r > r_1$,*
- (iii) *source if any one of the following is true:*
 - (a) *$k_1 < k < k_2$, $h_1 < h < 2h_1$, and $r < r_1$,*
 - (b) *$k_1 < k < k_2$ and $h \geq 2h_1$,*
 - (c) *$k \geq k_2$,*
- (iv) *non-hyperbolic point if any one of the following is true:*
 - (a) *$r > \frac{2(d^2+b^2(-1+c)^2\beta)}{d^2h}$ and*

$$k = k_\xi := (d^4hr(6d^2 + d^3h + 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta)) / (b^2(-1+c)^2(d^5h^2r + 2d^4(4+hr) + b^2(-1+c)^2d^3h^2r\beta + 2b^2(-1+c)^2d^2(8+3hr)\beta + 8b^4(-1+c)^4\beta^2)),$$

(b)

$$k = k_\eta := \frac{d^2 (3d^2 + d^3h + 5b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta)}{b^2(-1+c)^2 (d^2 + d^3h + 3b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta)}.$$

If condition (4.a) of Proposition 2.5 is satisfied, it can be concluded that one of the eigenvalues of $J(E_2)$ is -1 , while the other eigenvalue is distinct from both 1 and -1 . When (4.b) of Proposition 2.5 is true, the eigenvalues of $J(E_2)$ consist of a pair of complex conjugates with unit modulus.

Let

$$\Omega_{PDB} = \left\{ (a, b, c, d, r, k, \beta, h) \in \mathbb{R}_+^8 \mid 0 < c < 1, k > \frac{d^2}{b^2(1-c)^2}, \right. \\ \left. r > \frac{2(d^2 + b^2(-1+c)^2\beta)}{d^2h}, k = k_\xi \right\}.$$

Then, if parameters vary in a close neighbourhood of Ω_{PDB} , E_2 may go through the PD bifurcation.

Let

$$\Omega_{NSB} = \left\{ (a, b, c, d, r, k, \beta, h) \in \mathbb{R}_+^8 \mid 0 < c < 1, k > \frac{d^2}{b^2(1-c)^2}, k = k_\eta \right\}.$$

Then E_2 may go through the NS bifurcation when parameters vary in a close neighbourhood of Ω_{NSB} .

3. Local bifurcation analysis

Using the aforementioned information, we examine the NS and PD bifurcations of the interior fixed point E_2 . To accomplish this, we apply the centre manifold theorem and bifurcation theory. As a bifurcation parameter, the carrying capacity of the environment for the prey, k , is chosen. For a more in-depth discussion of bifurcation theory, readers are directed to [62–66]. These bifurcations indicate significant alterations in the system's dynamics, offering insights into scenarios where even slight modifications to parameters lead to substantial transformations in the dynamics of predator-prey interactions. Furthermore, understanding the functions of NS and PD bifurcations facilitates the development of effective conservation and management strategies to ensure the sustained coexistence of prey and predator populations, thereby contributing to our comprehension of ecosystem dynamics.

3.1. PD bifurcation

In ecology, PD bifurcations are critical points at which an ecological system's behaviour changes qualitatively. Consider a scenario where the numbers of predators and prey fluctuate predictably throughout time, resembling a cyclic pattern of rapid growth and sudden decline. During a PD bifurcation, the intensity of this rhythmic activity increases, and the period between the highest and lowest points in population numbers begins to double. This

may have significant ecological impacts. If the environment suddenly increases resources, predator numbers may rise, causing prey populations to drop rapidly. Following this reduction in prey numbers, predator populations may decrease, enabling prey populations to rebound. However, predator numbers may rise again, causing predation to rise and prey populations to fall. In this subsection, we examine PD bifurcation for a fixed point E_2 of system (4) by taking k as the bifurcation parameter. If $k = k_\xi$, then system (4) becomes:

$$\begin{cases} u_{n+1} = u_n + \frac{h}{2} \left(r \left(1 - \frac{u_n^2}{k_\xi} \right) \left(\frac{u_n^3}{\beta + u_n^2} \right) - a(1-c)v_n \right), \\ v_{n+1} = v_n + h((1-c)bu_n - d)v_n. \end{cases} \quad (8)$$

Next, we take into account the following perturbation of system (8):

$$\begin{cases} u_{n+1} = u_n + \frac{h}{2} \left(r \left(1 - \frac{u_n^2}{k_\xi + \delta} \right) \left(\frac{u_n^3}{\beta + u_n^2} \right) - a(1-c)v_n \right), \\ v_{n+1} = v_n + h((1-c)bu_n - d)v_n, \end{cases} \quad (9)$$

where δ is a perturbation in bifurcation parameter k_ξ and $|\delta| \ll 1$. Now, it is noted that by using transformation $w_n = u_n - \frac{d}{b(1-c)}$, $z_n = v_n - \frac{d^3 r(-d^2 + b^2(k_\xi + \delta)(1-c)^2)}{ab^3(k_\xi + \delta)(1-c)^4(d^2 + b^2\beta(1-c)^2)}$, one can translate the fixed point E_2 to $(0, 0)$. With this translation, the system (9) is changed to

$$\begin{bmatrix} w_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} w_n \\ z_n \end{bmatrix} + \begin{bmatrix} F(w_n, z_n, \delta) \\ G(w_n, z_n, \delta) \end{bmatrix}, \quad (10)$$

where

$$l_{11} = \frac{-6d^2 + d^3(h - hr) - 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta}{6d^2 + d^3h + 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta}, \quad l_{12} = \frac{1}{2}a(-1+c)h,$$

$$l_{21} = \frac{d^3(8 - 4hr) + 8b^2(-1+c)^2d\beta}{a(-1+c)(6d^2 + d^3h + 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta)}, \quad l_{22} = 1,$$

$$F(w_n, z_n, \delta) = a_1 w_n^2 + a_2 w_n^3 + a_3 w_n \delta + a_4 w_n^2 \delta + a_5 w_n \delta^2 + O((|w_n| + |z_n| + |\delta|)^4),$$

$$G(w_n, z_n, \delta) = b_1 w_n z_n + b_2 w_n \delta + b_3 w_n \delta^2 + O((|w_n| + |z_n| + |\delta|)^4),$$

$$a_1 = (b(-1+c)(3d^5 h^2 r + 6d^4(4+hr) + 7b^2(-1+c)^2 d^3 h^2 r \beta + 6b^2(-1+c)^2 d^2(12+5hr)\beta + 80b^4(-1+c)^4 \beta^2)) / (2d(d^2 + b^2(-1+c)^2\beta)(6d^2 + d^3h + 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta)),$$

$$a_2 = -(b^2(-1+c)^2(d^7 h^2 r + 2d^6(4+hr) + 2b^2(-1+c)^2 d^5 h^2 r \beta + 4b^2(-1+c)^2 d^4(8+hr)\beta + 9b^4(-1+c)^4 d^3 h^2 r \beta^2 + 10b^4(-1+c)^4 d^2(4+5hr)\beta^2 + 80b^6(-1+c)^6 \beta^3)) / (2d^2(d^2 + b^2(-1+c)^2\beta)^2(6d^2 + d^3h + 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta)),$$

$$a_3 = (b^2(-1+c)^2(3d^2 + 5b^2(-1+c)^2\beta)(d^5 h^2 r + 2d^4(4+hr) + b^2(-1+c)^2 d^3 h^2 r \beta + 2b^2(-1+c)^2 d^2(8+3hr)\beta + 8b^4(-1+c)^4 \beta^2)) / (2d^4 hr(d^2 + b^2(-1+c)^2\beta)^2(6d^2 + d^3h + 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta)^2),$$

$$\begin{aligned}
 a_4 &= - (b^3(-1+c)^3(d^5h^2r + 2d^4(4+hr) + b^2(-1+c)^2d^3h^2r\beta + 2b^2(-1+c)^2d^2(8+3hr)\beta \\
 &\quad + 8b^4(-1+c)^4\beta^2)(3d^4 + 9b^2(-1+c)^2d^2\beta + 10b^4(-1+c)^4\beta^2) / \\
 &\quad (2d^5hr(d^2 + b^2(-1+c)^2\beta)^3 (6d^2 + d^3h + 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta)^2), \\
 a_5 &= - (b^4(-1+c)^4(3d^2 + 5b^2(-1+c)^2\beta)(d^5h^2r + 2d^4(4+hr) \\
 &\quad + b^2(-1+c)^2d^3h^2r\beta + 2b^2(-1+c)^2d^2(8+3hr)\beta + 8b^4(-1+c)^4\beta^2)^3) \\
 &\quad / (2d^8h^2r^2(d^2 + b^2(-1+c)^2\beta)^2(6d^2 + d^3h + 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta)^3), \\
 b_1 &= -b(-1+c)h, \\
 b_2 &= - (b^2(-1+c)(d^5h^2r + 2d^4(4+hr) + b^2(-1+c)^2d^3h^2r\beta + 2b^2(-1+c)^2d^2(8+3hr)\beta \\
 &\quad + 8b^4(-1+c)^4\beta^2)^2) / (ad^3hr(d^2 + b^2(-1+c)^2\beta)(6d^2 + d^3h + 10b^2(-1+c)^2\beta \\
 &\quad + b^2(-1+c)^2dh\beta)^2), \\
 b_3 &= (b^4(-1+c)^3(d^5h^2r + 2d^4(4+hr) + b^2(-1+c)^2d^3h^2r\beta + 2b^2(-1+c)^2d^2(8+3hr)\beta \\
 &\quad + 8b^4(-1+c)^4\beta^2)^3) / (ad^7h^2r^2(d^2 + b^2(-1+c)^2\beta)(6d^2 + d^3h + 10b^2(-1+c)^2\beta \\
 &\quad + b^2(-1+c)^2dh\beta)^3).
 \end{aligned}$$

The eigenvalues of $J(E_2)$ for system (4) are $\gamma_1 = -1$ and

$$\gamma_2 = \frac{6d^2 + d^3h(3 - hr) + 10b^2(-1+c)^2\beta + 3b^2(-1+c)^2dh\beta}{6d^2 + d^3h + 10b^2(-1+c)^2\beta + b^2(-1+c)^2dh\beta}.$$

When the parameters change in a small area near Ω_{PDB} . After that, we use the following transformation:

$$\begin{bmatrix} w_n \\ z_n \end{bmatrix} = \begin{bmatrix} l_{12} & l_{12} \\ -1 - l_{11} & \gamma_2 - a_{11} \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix}. \quad (11)$$

As a result, the system (10) becomes

$$\begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} \phi(e_n, f_n, \delta) \\ \psi(e_n, f_n, \delta) \end{bmatrix}, \quad (12)$$

where

$$\begin{aligned}
 \phi(e_n, f_n, \delta) &= c_1e_n^2\delta + c_2e_nf_n\delta + c_3f_n^2\delta + c_4f_n^2 + c_5e_n^3 + c_6e_n^2f_n + c_7e_nf_n^2 + c_8f_n^3 + c_9e_n^2 \\
 &\quad + c_{10}e_nf_n + c_{11}e_n\delta + c_{12}f_n\delta + c_{13}e_n\delta^2 + c_{14}f_n\delta^2 + O((|e_n| + |f_n| + |\delta|)^4),
 \end{aligned}$$

$$\begin{aligned}
 \psi(e_n, f_n, \delta) &= d_1e_n^2\delta + d_2e_nf_n\delta + d_3f_n^2\delta + d_4e_n^2 + d_5e_n^3 + d_6e_n^2f_n + d_7e_nf_n^2 + d_8f_n^3 + d_9e_n\delta \\
 &\quad + d_{10}f_n\delta + d_{11}e_n\delta^2 + d_{12}f_n\delta^2 + d_{13}e_nf_n + d_{14}f_n^2 + O((|e_n| + |f_n| + |\delta|)^4),
 \end{aligned}$$

$$c_1 = -\frac{a_4l_{12}(l_{11} - \gamma_2)}{1 + \gamma_2}, \quad c_2 = -\frac{2a_4l_{12}(l_{11} - \gamma_2)}{1 + \gamma_2}, \quad c_3 = -\frac{a_4l_{12}(l_{11} - \gamma_2)}{1 + \gamma_2},$$

$$c_4 = -\frac{(a_1 - b_1)l_{12}(l_{11} - \gamma_2)}{1 + \gamma_2}, \quad c_5 = -\frac{a_2l_{12}^2(l_{11} - \gamma_2)}{1 + \gamma_2}, \quad c_6 = -\frac{3a_2l_{12}^2(l_{11} - \gamma_2)}{1 + \gamma_2},$$

$$\begin{aligned}
c_7 &= -\frac{3a_2 l_{12}^2 (l_{11} - \gamma_2)}{1 + \gamma_2}, & c_8 &= -\frac{a_2 l_{12}^2 (l_{11} - \gamma_2)}{1 + \gamma_2}, \\
c_9 &= \frac{l_{12} (b_1 (1 + l_{11}) + a_1 (-l_{11} + \gamma_2))}{1 + \gamma_2}, \\
c_{10} &= \frac{l_{12} (b_1 (1 + 2l_{11} - \gamma_2) + 2a_1 (-l_{11} + \gamma_2))}{1 + \gamma_2}, \\
c_{11} &= \frac{-b_2 l_{12} + a_3 (-l_{11} + \gamma_2)}{1 + \gamma_2}, & c_{12} &= \frac{-b_2 l_{12} + a_3 (-l_{11} + \gamma_2)}{1 + \gamma_2}, \\
c_{13} &= \frac{-b_3 l_{12} + a_5 (-l_{11} + \gamma_2)}{1 + \gamma_2}, & c_{14} &= \frac{-b_3 l_{12} + a_5 (-l_{11} + \gamma_2)}{1 + \gamma_2}, \\
d_1 &= \frac{a_4 (1 + l_{11}) l_{12}}{1 + \gamma_2}, & d_2 &= \frac{2a_4 (1 + l_{11}) l_{12}}{1 + \gamma_2}, & d_3 &= \frac{a_4 (1 + l_{11}) l_{12}}{1 + \gamma_2}, \\
d_4 &= \frac{(a_1 - b_1) (1 + l_{11}) l_{12}}{1 + \gamma_2}, & d_5 &= \frac{a_2 (1 + l_{11}) l_{12}^2}{1 + \gamma_2}, & d_6 &= \frac{3 a_2 (1 + l_{11}) l_{12}^2}{1 + \gamma_2}, \\
d_7 &= \frac{3a_2 (1 + l_{11}) l_{12}^2}{1 + \gamma_2}, & d_8 &= \frac{a_2 (1 + l_{11}) l_{12}^2}{1 + \gamma_2}, & d_9 &= \frac{a_3 (1 + l_{11}) + b_2 l_{12}}{1 + \gamma_2}, \\
d_{10} &= \frac{a_3 (1 + l_{11}) + b_2 l_{12}}{1 + \gamma_2}, & d_{11} &= \frac{a_5 (1 + l_{11}) + b_3 l_{12}}{1 + \gamma_2}, & d_{12} &= \frac{a_5 (1 + l_{11}) + b_3 l_{12}}{1 + \gamma_2}, \\
d_{13} &= \frac{l_{12} (2a_1 (1 + l_{11}) + b_1 (-1 - 2l_{11} + \gamma_2))}{1 + \gamma_2}, \\
d_{14} &= \frac{l_{12} (a_1 (1 + l_{11}) + b_1 (-l_{11} + \gamma_2))}{1 + \gamma_2}.
\end{aligned}$$

The centre manifold at origin in a close neighbourhood of $\delta = 0$ for the system (12) is then computed. It may be expressed in the following way:

$$Q^C(0, 0, 0) = \{(e_n, f_n, \delta) \in \mathbb{R}^3 \mid f_n = M_0 \delta + M_1 e_n^2 + M_2 e_n \delta + M_3 \delta^2 + O((|e_n| + |\delta|)^3)\},$$

where

$$M_0 = 0, \quad M_1 = -\frac{d_4}{-1 + \gamma_2}, \quad M_2 = -\frac{d_9}{1 + \gamma_2}, \quad M_3 = 0.$$

The system (12) that is restricted to $Q^C(0, 0, 0)$ is provided by

$$\begin{aligned}
\Psi(e_n, \delta) &:= e_{n+1} = -e_n + e_n^2 \delta c_1 + e_n^3 c_5 + e_n^2 c_9 + e_n \delta c_{11} + e_n \delta^2 c_{13} \\
&+ e_n \delta c_2 \left(-\frac{e_n^2 d_4}{-1 + \gamma_2} - \frac{e_n \delta d_9}{1 + \gamma_2} \right) + e_n^2 c_6 \left(-\frac{e_n^2 d_4}{-1 + \gamma_2} - \frac{e_n \delta d_9}{1 + \gamma_2} \right) \\
&+ e_n c_{10} \left(-\frac{e_n^2 d_4}{-1 + \gamma_2} - \frac{e_n \delta d_9}{1 + \gamma_2} \right) + \delta c_{12} \left(-\frac{e_n^2 d_4}{-1 + \gamma_2} - \frac{e_n \delta d_9}{1 + \gamma_2} \right)
\end{aligned}$$

$$\begin{aligned}
 & + \delta^2 c_{14} \left(-\frac{e_n^2 d_4}{-1 + \gamma_2} - \frac{e_n \delta d_9}{1 + \gamma_2} \right) + \delta c_3 \left(-\frac{e_n^2 d_4}{-1 + \gamma_2} - \frac{e_n \delta d_9}{1 + \gamma_2} \right)^2 \\
 & + c_4 \left(-\frac{e_n^2 d_4}{-1 + \gamma_2} - \frac{e_n \delta d_9}{1 + \gamma_2} \right)^2 + e_n c_7 \left(-\frac{e_n^2 d_4}{-1 + \gamma_2} - \frac{e_n \delta d_9}{1 + \gamma_2} \right)^2 \\
 & + c_8 \left(-\frac{e_n^2 d_4}{-1 + \gamma_2} - \frac{e_n \delta d_9}{1 + \gamma_2} \right)^3.
 \end{aligned}$$

We need the following two values, π_1 and π_2 , to be non-zero for PD bifurcation to occur.

$$\pi_1 = 2c_{11} = - \frac{(b(-1+c)d^4(8+h(2+dh)r) + b^3(-1+c)^3d^2(16+h(6+dh)r)\beta + 8b^5(-1+c)^5\beta^2)}{d^4hr(d^2 + b^2(-1+c)^2\beta)^2(d^2(-12+dh(-4+hr)) - 4b^2(-1+c)^2(5+dh)\beta)}, \quad (13)$$

$$\begin{aligned}
 \pi_2 & = 2 \left(c_5 + c_9^2 - \frac{c_{10}d_4}{-1 + \gamma_2} \right) \\
 & = - \left((a^2b^2(-1+c)^4h^2(-d^6(16+2dh(36+dh(6-dh))) + h(4+dh(20+dh(9+2dh))))r \right. \\
 & \quad - 2b^2(-1+c)^2d^4(32+4hr+dh(144+56hr+dh(22+15hr+dh(-3+2hr))))\beta \\
 & \quad - b^4(-1+c)^4d^2(80+100hr+dh(456+108hr+dh(52+21hr+2dh(-3+hr))))\beta^2 \\
 & \quad \left. + 2b^6(-1+c)^6(-80+dh(-120+dh(-10+dh)))\beta^3 \right) \\
 & \quad / (4(d^3 + b^2(-1+c)^2d\beta)^2(d^2(-12+hd(-4+hr)) - 4b^2(-1+c)^2(5+dh)\beta)). \quad (14)
 \end{aligned}$$

The analysis mentioned above has led to the following result:

Theorem 3.1: *Let $(a, b, c, d, r, k, \beta, h) \in \Omega_{PDB}$. If π_1, π_2 specified in (13) and (14) are non-zero and k varies in a close neighbourhood of k_ξ , the system (4) undergoes PD bifurcation at E_2 . Furthermore, if $\pi_2 > 0$ (respectively $\pi_2 < 0$), the period-2 orbits that bifurcate from E_2 are stable (respectively, unstable).*

3.2. NS bifurcation

NS bifurcations are the emergence of more complicated dynamics in a biological system, which are often marked by unexpected changes and chaotic behaviour. Consider a scenario where the changes in the numbers of predators and prey are unpredictable. Due to NS bifurcation, even small changes in the environment or predator-prey interactions can cause big changes in population numbers that are hard to predict. This lack of stability can affect the way ecosystems work, making it hard for ecologists to guess how populations will respond to changes in the environment. For instance, rapid changes in the environment, such as habitat loss or the introduction of invasive species, may trigger chaotic fluctuations in predator and prey populations, leading to disruptions in trophic interactions and ecosystem functioning. In this subsection, we examine NS bifurcation at E_2 when parameters vary in a close neighbourhood of Ω_{NSB} .

If $k = k_\eta$, then system (4) becomes:

$$\begin{cases} u_{n+1} = u_n + \frac{h}{2} \left(r \left(1 - \frac{u_n^2}{k_\eta} \right) \left(\frac{u_n^3}{\beta + u_n^2} \right) - a(1-c)v_n \right), \\ v_{n+1} = v_n + h((1-c)bu_n - d)v_n. \end{cases} \quad (15)$$

Next, we take into account the following perturbation of system (15):

$$\begin{cases} u_{n+1} = u_n + \frac{h}{2} \left(r \left(1 - \frac{u_n^2}{k_\eta + \delta} \right) \left(\frac{u_n^3}{\beta + u_n^2} \right) - a(1-c)v_n \right), \\ v_{n+1} = v_n + h((1-c)bu_n - d)v_n. \end{cases} \quad (16)$$

where δ is a perturbation in bifurcation parameter k_η and $|\delta| \ll 1$. Next, it is observed that by using the transformation $w_n = u_n - \frac{d}{b(1-c)}$, $z_n = v_n - \frac{d^3 r(-d^2 + b^2(k_\eta + \delta)(1-c)^2)}{ab^3(k_\eta + \delta)(1-c)^4(d^2 + b^2\beta(1-c)^2)}$, one can translate the fixed point E_2 to $(0, 0)$. With this translation, the system (16) is changed to

$$\begin{bmatrix} w_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} w_n \\ z_n \end{bmatrix} + \begin{bmatrix} F(w_n, z_n, \delta) \\ G(w_n, z_n, \delta) \end{bmatrix}, \quad (17)$$

where

$$m_{11} = \frac{1}{2} \left(2 + \frac{3d^2 hr}{d^2 + b^2(-1+c)^2\beta} - \frac{3d^6 hr}{(b-bc)^2(d^2 + b^2(-1+c)^2\beta)^2(k+\delta)} - \frac{d^4 hr(2k+5\beta+2\delta)}{(d^2 + b^2(-1+c)^2\beta)^2(k+\delta)} \right),$$

$$m_{12} = \frac{1}{2}(-1+c)ah, \quad m_{21} = \frac{d^3 hr(d^2 - b^2(-1+c)^2(k+\delta))}{ab^2(-1+c)^3(d^2 + b^2(-1+c)^2\beta)(k+\delta)}, \quad m_{22} = 1,$$

$$F(w_n, z_n) = a_1 w_n^3 + a_2 w_n^2 + O((|w_n| + |z_n|)^4),$$

$$G(w_n, z_n) = b_1 w_n z_n + O((|w_n| + |z_n|)^4),$$

$$a_1 = - (hr(d^8 + 4(b-bc)^2 d^6 \beta + 5(b-bc)^4 d^4 \beta^2 + 10(b-bc)^6 d^2 \beta^3 - (b-bc)^4 d^4 \beta(k+\delta) + 6(b-bc)^6 d^2 \beta^2(k+\delta) - (b-bc)^8 \beta^3(k+\delta)) / \left(2(b-bc)^8 \left(\frac{d^2}{(b-bc)^2} + \beta \right)^4 (k+\delta) \right),$$

$$a_2 = \frac{hr(3d^7 + 9b^2(-1+c)^2 d^5 \beta - 3b^6(-1+c)^6 d \beta^2(k+\delta) + b^4(-1+c)^4 d^3 \beta(k+10\beta+\delta))}{2b(-1+c)(d^2 + b^2(-1+c)^2\beta)^3(k+\delta)},$$

$$b_1 = -b(-1+c)h.$$

Consider the characteristic equation of the variational matrix for system (17) evaluated at origin, given by

$$\gamma^2 - R(\delta)\gamma + T(\delta) = 0, \quad (18)$$

where

$$R(\delta) = (12d^8 - 2d^9 h(-2 + hr) + 12b^8(-1+c)^8 \beta^3 \delta + 4b^8(-1+c)^8 dh\beta^3 \delta - 2b^4(-1+c)^4 d^5 h\beta((-6 + hr)\beta - 2(3 + hr)\delta) - b^2(-1+c)^2 d^7 h(4(-3 + hr)\beta - (4 + hr)\delta) + b^2(-1+c)^2 d^6(44\beta + (4 + hr)\delta) + b^6(-1+c)^6 d^3 h\beta^2(4\beta + 3(4 + hr)\delta) + b^4(-1+c)^4 d^4 \beta(26\beta + (10 + 3hr)\delta) + b^6(-1+c)^6 d^2 \beta^2$$

$$\begin{aligned}
 & \times (20\beta + (28 + 9hr)\delta) / (2(d^2 + b^2(-1 + c)^2\beta)^2(3d^4 + d^5h \\
 & + 3b^4(-1 + c)^4\beta\delta + b^4(-1 + c)^4dh\beta\delta + b^2(-1 + c)^2d^3h(\beta + \delta) \\
 & + b^2(-1 + c)^2d^2(5\beta + \delta))), \\
 T(\delta) = & (2d^9h + 6b^8(-1 + c)^8\beta^3\delta + 2b^8(-1 + c)^8dh\beta^3\delta + 2b^2(-1 + c)^2d^7h \\
 & \times (3\beta + \delta + hr\delta) + d^8(6 + b^2(-1 + c)^2h^3r\delta) + 2b^4(-1 + c)^4d^5h\beta(3\beta + (3 + 4hr)\delta) \\
 & + b^6(-1 + c)^6d^2\beta^2(10\beta + (14 + 9hr)\delta) + 2b^6(-1 + c)^6d^3h\beta^2(\beta + 3(\delta + hr\delta)) \\
 & + b^2(-1 + c)^2d^6((2 + hr)\delta + 2\beta(11 + b^2(-1 + c)^2h^3r\delta)) + b^4(-1 + c)^4d^4\beta(2(5 + 3hr)\delta \\
 & + \beta(26 + b^2(-1 + c)^2h^3r\delta))) / (2(d^2 + b^2(-1 + c)^2\beta)^2(3d^4 + d^5h + 3b^4(-1 + c)^4\beta\delta \\
 & + b^4(-1 + c)^4dh\beta\delta + b^2(-1 + c)^2d^3h(\beta + \delta) + b^2(-1 + c)^2d^2(5\beta + \delta))).
 \end{aligned}$$

The solutions of (18) are

$$\gamma_{1,2} = \frac{R(\delta)}{2} \pm \frac{i}{2} \sqrt{4T(\delta) - R^2(\delta)}. \quad (19)$$

Since $(a, b, c, d, r, k, \beta, h) \in \Omega_{NSB}$, we have, $|\gamma_{1,2}| = T(\delta)$. Moreover, $(|\gamma_{1,2}|)_{\delta=0} = T(0) = 1$ and

$$\left(\frac{d|\gamma_{1,2}|}{d\delta} \right)_{\delta=0} = \frac{b^2(-1 + c)^2hr(d^2 + d^3h + 3b^2(-1 + c)^2\beta + b^2(-1 + c)^2dh\beta)^2}{4(d^2 + b^2(-1 + c)^2\beta)^2(3d^2 + d^3h + 5b^2(-1 + c)^2\beta + b^2(-1 + c)^2dh\beta)} \neq 0.$$

Moreover, it is also needed that $\gamma_{1,2}^k \neq 1$ for $k \in \{1, 2, 3, 4\}$ at $\delta = 0$, that is identical to $R(0) \neq \pm 2, -1, 0$. Since $(a, b, c, d, r, k, \beta, h) \in \Omega_{NSB}$, thus it follows that

$$R(0) = \frac{6d^2 + d^3h(2 - hr) + 10b^2(-1 + c)^2\beta + 2b^2(-1 + c)^2dh\beta}{3d^2 + d^3h + 5b^2(-1 + c)^2\beta + b^2(-1 + c)^2dh\beta} < 2.$$

Moreover, $R(0) \neq -2, -1, 0$ implies that $r \neq 4H, 3H, 2H$, respectively, where

$$H = \frac{3d^2 + d^3h + 5b^2(-1 + c)^2\beta + b^2(-1 + c)^2dh\beta}{d^3h^2}.$$

We utilize the following mapping to convert the linearized part of (17) into canonical form at $\delta = 0$:

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} m_{12} & 0 \\ \alpha_1 - m_{11} & -\alpha_2 \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix}, \quad (20)$$

where

$$\begin{aligned}
 \alpha_1 &= \frac{R(0)}{2} = \frac{6d^2 + d^3h(2 - hr) + 10b^2(-1 + c)^2\beta + 2b^2(-1 + c)^2dh\beta}{2(3d^2 + d^3h + 5b^2(-1 + c)^2\beta + b^2(-1 + c)^2dh\beta)}, \\
 \alpha_2 &= \frac{\sqrt{4T(0) - R^2(0)}}{2} = \frac{-\sqrt{d^3h^2r(-12d^2 + d^3h(-4 + hr) - 20b^2(-1 + c)^2\beta - 4b^2(-1 + c)^2dh\beta)}}{2(3d^2 + d^3h + 5b^2(-1 + c)^2\beta + b^2(-1 + c)^2dh\beta)}.
 \end{aligned}$$

Upon using the transformation (20), the system (17) takes the following form:

$$\begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} \Upsilon(e_n, f_n) \\ \chi(e_n, f_n) \end{bmatrix}, \quad (21)$$

where

$$\Upsilon(e_n, f_n) = C_1 e_n^2 + C_2 e_n^3 + O((|e_n| + |f_n|)^4),$$

$$\chi(e_n, f_n) = D_1 e_n f_n + D_2 e_n^3 + D_3 e_n^2 + O((|e_n| + |f_n|)^4),$$

$$C_1 = \frac{ab(-1+c)^2 dh^2 r(3d^2 + 3d^3 h + 15b^2(-1+c)^2 \beta + 7b^2(-1+c)^2 dh\beta)}{4(d^2 + b^2(-1+c)^2 \beta)(3d^2 + d^3 h + 5b^2(-1+c)^2 \beta + b^2(-1+c)^2 dh\beta)},$$

$$C_2 = -\frac{(a^2 b^2(-1+c)^4 h^3 r(d^4 + d^5 h + 2b^2(-1+c)^2 d^2 \beta + 2b^2(-1+c)^2 d^3 h\beta + 25b^4(-1+c)^4 \beta^2 + 9b^4(-1+c)^4 dh\beta^2))}{(8(d^2 + b^2(-1+c)^2 \beta)^2 \times (3d^2 + d^3 h + 5b^2(-1+c)^2 \beta + b^2(-1+c)^2 dh\beta))},$$

$$D_1 = -\frac{1}{2} ab(-1+c)^2 h^2,$$

$$D_2 = -\frac{(a^2 b^2(-1+c)^4 d^3 h^5 r^2(d^4 + d^5 h + 2b^2(-1+c)^2 d^2 \beta + 2b^2(-1+c)^2 d^3 h\beta + 25b^4(-1+c)^4 \beta^2 + 9b^4(-1+c)^4 dh\beta^2))}{(16v(d^2 + b^2(-1+c)^2 \beta)^2 \times (3d^2 + d^3 h + 5b^2(-1+c)^2 \beta + b^2(-1+c)^2 dh\beta)^2)},$$

$$D_3 = \frac{(ab(-1+c)^2 d^3 h^4 r(2d^5 h + 3d^4(2+hr) + b^2(-1+c)^2 d^2(16+7hr)\beta + 10b^4(-1+c)^4 \beta^2 + b^2(-1+c)^2 d\beta(15r + 2b^2(-1+c)^2 h\beta) + d^3(3r + 4b^2(-1+c)^2 h\beta))}{(8\alpha_2(d^2 + b^2(-1+c)^2 \beta)(3d^2 + d^3 h + 5b^2(-1+c)^2 \beta + b^2(-1+c)^2 dh\beta)^2)}.$$

We take into consideration the first Lyapunov exponent, which was determined as follows to investigate the NS bifurcation's direction:

$$L = \left(\left[-\operatorname{Re} \left(\frac{(1-2\gamma_1)\gamma_2^2}{1-\gamma_1} I_{20} I_{11} \right) - \frac{1}{2} |I_{11}|^2 - |I_{02}|^2 + \operatorname{Re}(\gamma_2 I_{21}) \right] \right)_{\delta=0}, \quad (22)$$

where

$$I_{20} = \frac{1}{8} [\Upsilon_{e_n e_n} - \Upsilon_{f_n f_n} + 2\chi_{e_n f_n} + i(\chi_{e_n e_n} - \chi_{f_n f_n} - 2\Upsilon_{e_n f_n})],$$

$$I_{11} = \frac{1}{4} [\Upsilon_{e_n e_n} + \Upsilon_{f_n f_n} + i(\chi_{e_n e_n} + \chi_{f_n f_n})],$$

$$I_{02} = \frac{1}{8} [\Upsilon_{e_n e_n} - \Upsilon_{f_n f_n} - 2\chi_{e_n f_n} + i(\chi_{e_n e_n} - \chi_{f_n f_n} + 2\Upsilon_{e_n f_n})],$$

$$I_{21} = \frac{1}{16} [\Upsilon_{e_n e_n e_n} + \Upsilon_{e_n f_n f_n} + \chi_{e_n e_n f_n} + \chi_{f_n f_n f_n} + i(\chi_{e_n e_n e_n} + \chi_{e_n f_n f_n} - \Upsilon_{e_n e_n f_n} - \Upsilon_{f_n f_n f_n})].$$

We get the following theorem as a result of the calculations described above:

Theorem 3.2: Let $(a, b, c, d, r, k, \beta, h) \in \Omega_{\text{NSB}}$. If $r \neq 4H, 3H, 2H$, where

$$H = \frac{3d^2 + d^3 h + 5b^2(-1+c)^2 \beta + b^2(-1+c)^2 dh\beta}{d^3 h^2}.$$

If $L \neq 0$, then fixed point E_2 of system (4) goes through NS bifurcation when the parameter k differs in a close neighbourhood of k_η . Moreover, if $L > 0$ (or $L < 0$), then a repelling (or attracting) invariant closed curve bifurcates from E_2 for $k < k_\eta$ (or $k > k_\eta$).

4. Chaos control

The objective of minimizing disorder and optimizing dynamical systems by particular performance requirements is desirable. Chaos control techniques find extensive application in a multitude of engineering and applied research fields. Throughout history, unstable oscillations and distances have been regarded unfavourably. Within the realm of mathematical biology, these factors are significant because they harm the ability of biological populations to reproduce. It is possible to devise a controller capable of altering the bifurcation properties of a specified nonlinear dynamical system to achieve specific desirable dynamical properties and regulate disorder in the presence of PD and NS bifurcations. In the context of a discrete-time system, a variety of chaos control methods exist. This section focuses on the hybrid control method [67], which efficiently controls chaos by addressing both bifurcation effects. The method of hybrid control combines state feedback and parameter perturbation. This combination is used to stabilize the unstable periodic orbits in the chaotic attractor of the system (4). As a result, the controlled system is stable across a wide range of parameters. By deploying the hybrid control method, the following controlled system is obtained:

$$\begin{cases} u_{n+1} = \rho \left(u_n + \frac{h}{2} \left(\left(1 - \frac{u_n^2}{k} \right) \left(\frac{ru_n^3}{\beta + u_n^2} \right) - a(1 - c)v_n \right) \right) + (1 - \rho)u_n, \\ v_{n+1} = \rho v_n (1 + h(b(1 - c)u_n - d)) + (1 - \rho)v_n, \end{cases} \quad (23)$$

where $\rho \in (0, 1)$ is the control parameter. The fixed points of both the original system (4) and the controlled system (23) are same. By selecting the suitable value of ρ , the bifurcation for E_2 of system (4) can be advanced, postponed, or even annihilated. The variational matrix at E_2 for system (23) may be computed as follows:

$$J(E_2) = \begin{bmatrix} 1 + \frac{d^2 hr(-3d^4 + b^2(-1 + c)^2 d^2(k - 5\beta) + 3b^4(-1 + c)^4 k\beta)\rho}{2k(b(-1 + c)d^2 + b^3(-1 + c)^3\beta)^2} & \frac{1}{2} a(-1 + c)h\rho \\ \frac{d^3 h(d^2 - b^2(-1 + c)^2 k)r\rho}{ab^2(-1 + c)^3 k(d^2 + b^2(-1 + c)^2\beta)} & 1 \end{bmatrix}.$$

Let

$$F(\theta) = \theta^2 + A_1\theta + A_0 \quad (24)$$

be the characteristic polynomial of $J(E_2)$, where

$$A_1 = - \frac{4b^6(-1 + c)^6 k\beta^2 - 3d^6 hr\rho + b^4(-1 + c)^4 d^2 k\beta(8 + 3hr\rho) + b^2(-1 + c)^2 d^4(4k + hr(k - 5\beta)\rho)}{2b^2(-1 + c)^2 k(d^2 + b^2(-1 + c)^2\beta)^2},$$

$$A_0 = (2b^6(-1+c)^6k\beta^2 - d^6hr\rho(3+dh\rho) + b^4(-1+c)^4d^2k\beta(4+hr\rho(3+dh\rho)) \\ + b^2(-1+c)^2d^4(-hr\beta\rho(5+dh\rho) + k(2+hr\rho(1+dh\rho)))) \\ / (2b^2(-1+c)^2k(d^2 + b^2(-1+c)^2\beta^2)).$$

According to Lemma 2.2, the positive fixed of system (23) is a sink if $F(1) > 0$, $F(-1) > 0$ and $F(0) < 1$. Thus, we obtain the following result:

Theorem 4.1: *The fixed point E_2 of the system (23) is a sink if following conditions are satisfied:*

(i)

$$\frac{d^2}{b^2(-1+c)^2} < k < \frac{3d^4 + 5b^2(-1+c)^2d^2\beta}{b^2(-1+c)^2d^2 + 3b^4(-1+c)^4\beta},$$

(ii)

$$0 < h < \frac{-3d^4 + b^2(-1+c)^2d^2(k-5\beta) + 3b^4(-1+c)^4k\beta}{d(d^2 - b^2(-1+c)^2k)(d^2 + b^2(-1+c)^2\beta)\rho},$$

(iii)

$$0 < r < (8b^2(-1+c)^2k(d^2 + b^2(-1+c)^2\beta^2) / (d^2h\rho(6d^4 - 2b^2(-1+c)^2d^2(k-5\beta) \\ - 6b^4(-1+c)^4k\beta + d^5h\rho - b^2(-1+c)^2d^3h(k-\beta)\rho - b^4(-1+c)^4dhk\beta\rho)).$$

5. Numerical examples

In this section, a numerical investigation is conducted to validate the theoretical findings. Our primary objective is to analyse the dynamics of model (4) at the positive fixed point E_2 . The objective of this analysis is to provide empirical evidence and computational support for the implications derived from our theoretical framework. For this purpose, the parametric values are not derived from any existing experimental data or model, since this is a new model without any known experimental data. In this numerical investigation, the parametric values are selected randomly from the allowable domain of parameters. MATHEMATICA is used for computations, whereas MATLAB is utilized to draw graphs.

5.1. Bifurcation analysis by using k as bifurcation parameter

Let us take the parameter values as $a = 1.2$, $b = 0.6$, $c = 0.8$, $d = 0.1$, $r = 1.5$, $h = 4$, $\beta = 0.9$ with initial conditions $u_0 = 0.85$, $v_0 = 0.25$ and varying $k \in [0.55, 1.30]$. For these values, the PD bifurcation value is $k_\xi = 0.77889$ and the NS bifurcation value is $k_\eta = 1.24365$. The positive fixed point E_2 is a sink if $0.77889 < k < 1.24365$.

The positive fixed point is obtained as $E_2 = (0.833333, 0.24594)$ for $k \approx 0.77889$. The eigenvalues of $J(E_2)$ are $\gamma_1 = -1$ and $\gamma_2 = 0.971668$. Through simple computations, we obtain $\pi_1 = 13.1353 \neq 0$ and $\pi_2 = -5.32322 < 0$. This proves the correctness of Theorem 3.1. Moreover, the positive fixed point is obtained as $E_2 = (0.833333, 1.00176)$ for

$k \approx 1.24365$. The eigenvalues of $J(E_2)$ are $\gamma_{1,2} = 0.884598 \pm 0.466355i$. Through simple computations, we obtain

$$\begin{aligned} I_{20} &= 0.43814 + 0.136352i, & I_{11} &= 0.986828 + 0.272703i, \\ I_{02} &= 0.551014 + 0.136352i, & I_{21} &= -0.361328 - 0.089413i. \end{aligned}$$

Thus, it is obtained that $L = -1.48855 < 0$. This proves the correctness of Theorem 3.2.

Figure 1(a,b) show prey and predator bifurcation diagrams for $k \in [0.55, 1.30]$. These confirm that the positive fixed losses its stability via PD bifurcation for $k \leq 0.77889$. Moreover, it losses its stability via NS bifurcation for $k \geq 1.24365$. Higher or lower carrying capacity destabilizes the system, resulting in more complicated dynamical behaviours. The maximum Lyapunov exponents corresponding to Figure 1(a,b) are presented in Figure 1(c).

Following that, Figure 2(a–g) illustrate phase portraits of the system (4) for different values of k . It can be observed that the fixed point E_2 is locally asymptotically stable for values of k between 0.77889 and 1.24365. However, it loses stability at about $k_\eta = 1.24365$, which leads to an NS bifurcation in the model (4). For $k \geq 1.24365$, an invariant curve emerges

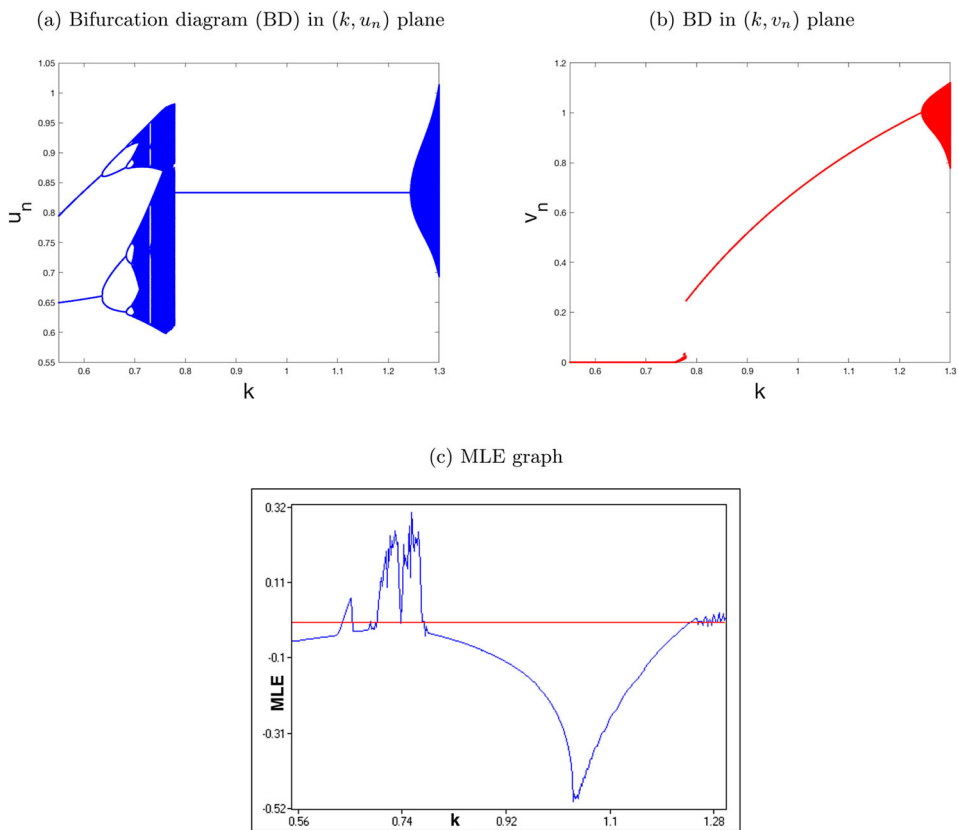


Figure 1. Bifurcation diagrams and MLE graph of model (4) for $a = 1.2, r = 1.5, b = 0.6, h = 4, c = 0.8, d = 0.1, \beta = 0.9, u_0 = 0.85, v_0 = 0.25, k \in [0.55, 1.30]$. (a) Bifurcation diagram (BD) in (k, u_n) plane. (b) BD in (k, v_n) plane and (c) MLE graph.

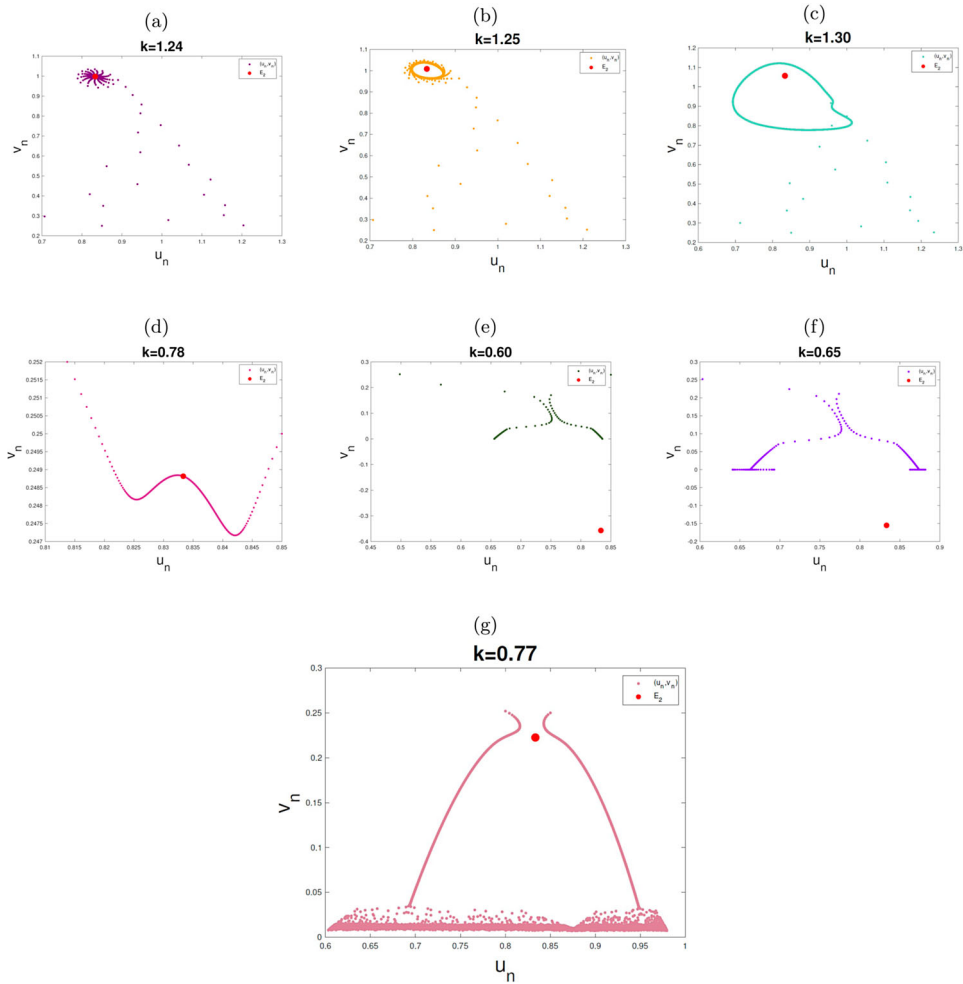


Figure 2. Phase portraits of (4) with $a = 1.2, r = 1.5, b = 0.6, h = 4, c = 0.8, d = 0.1, \beta = 0.9, u_0 = 0.85, v_0 = 0.25, k \in \{0.60, 0.65, 0.77, 0.78, 1.24, 1.25, 1.30\}$.

from the fixed point E_2 , the radius of which grows as k grows. Moreover, the fixed point becomes unstable at $k_\xi = 0.77889$ due to PD bifurcation. These illustrate that a moderate level of Allee effect is beneficial for both prey and predators. The moderate Allee effect stabilizes predator and prey populations, facilitating ecological cohabitation and persistence. The discretized system (4) demonstrates a PD path to the chaotic phenomena. A positive Lyapunov exponent is evidence that chaos exists in the system. A chaotic, strange attractor is depicted in Figure 2(g). Subsequently, we want to evaluate the efficacy of the hybrid control approach. For the controlled system (23), we consider $\rho = 0.95, a = 1.2, r = 1.5, b = 0.6, h = 4, c = 0.8, d = 0.1, \beta = 0.9, \rho = 0.95, u_0 = 0.85, v_0 = 0.25$ and vary $k \in [0.55, 1.30]$. The interior fixed point E_2 is a sink if $0.763139 < k < 1.24802$. The bifurcation diagrams for the controlled system Figure 3(a,b) depict that the interior fixed point E_2 is stable for a larger interval than the original system (4). By employing small values of the control parameter ρ , it is possible to extend the stability interval.

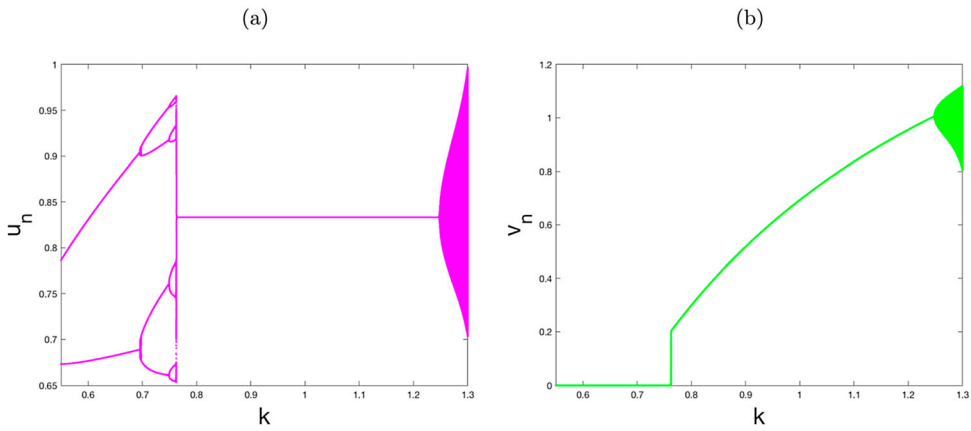


Figure 3. Bifurcation diagrams of (23) for $\rho = 0.95, a = 1.2, b = 0.6, c = 0.8, d = 0.1, r = 1.5, h = 4, \beta = 0.9, \rho = 0.95, u_0 = 0.85, v_0 = 0.25, k \in [0.55, 1.30]$.

5.2. Bifurcation analysis by using β as bifurcation parameter

We assume the following set of parameters' values and initial conditions:

$$\begin{aligned} a = 1.2, \quad k = 1.2, \quad b = 0.6, \quad h = 4, \quad c = 0.8, \quad d = 0.1, \\ r = 1.5, \quad u_0 = 0.8, \quad v_0 = 0.7. \end{aligned}$$

For these values, the NS bifurcation value is $\beta_\eta = 1.43331$. The interior fixed point E_2 is a sink if $0 < \beta < 1.43331$. The interior fixed point E_2 loses its stability for $\beta \geq 1.43331$ due to NS bifurcation. The positive fixed point is obtained as $E_2 = (0.833333, 0.716146)$. The eigenvalues of $J(E_2)$ are $\gamma_{1,2} = 0.9175 \pm 0.397736i$. Through simple computations, we obtain

$$\begin{aligned} I_{20} = 0.339231 + 0.09426i, \quad I_{11} = 0.793662 + 0.18852i, \\ I_{02} = 0.454431 + 0.09426i, \quad I_{21} = -0.352646 - 0.073147i. \end{aligned}$$

As a result, $L = -1.09679 < 0$ is found. This demonstrates that Theorem 3.2 is accurate. Figure 1(a,b) show prey and predator bifurcation diagrams for $\beta \in [1.15, 1.65]$. The maximum Lyapunov exponents corresponding to Figure 4(a,b) are depicted in Figure 4(c).

We consider the parameter values of the controlled system (23) to be the same as those in the uncontrolled system, with $\rho = 0.95$. The interior fixed point E_2 is a sink if $0 < k < 1.50057$. The bifurcation diagrams for the controlled system Figure 5(a,b) depict that the NS bifurcation is delayed. The controlled system goes through NS bifurcation when the Allee effect parameter β goes through $\beta_\eta = 1.50057$. The delay of the NS bifurcation across a wide range of β values can be achieved through the use of small values of the control parameter ρ .

5.3. Influence of Allee effect

The aim of this section is to investigate the influence of the Allee effect on the dynamics of system (4). Incorporating the Allee effect into a predator-prey system yields insightful

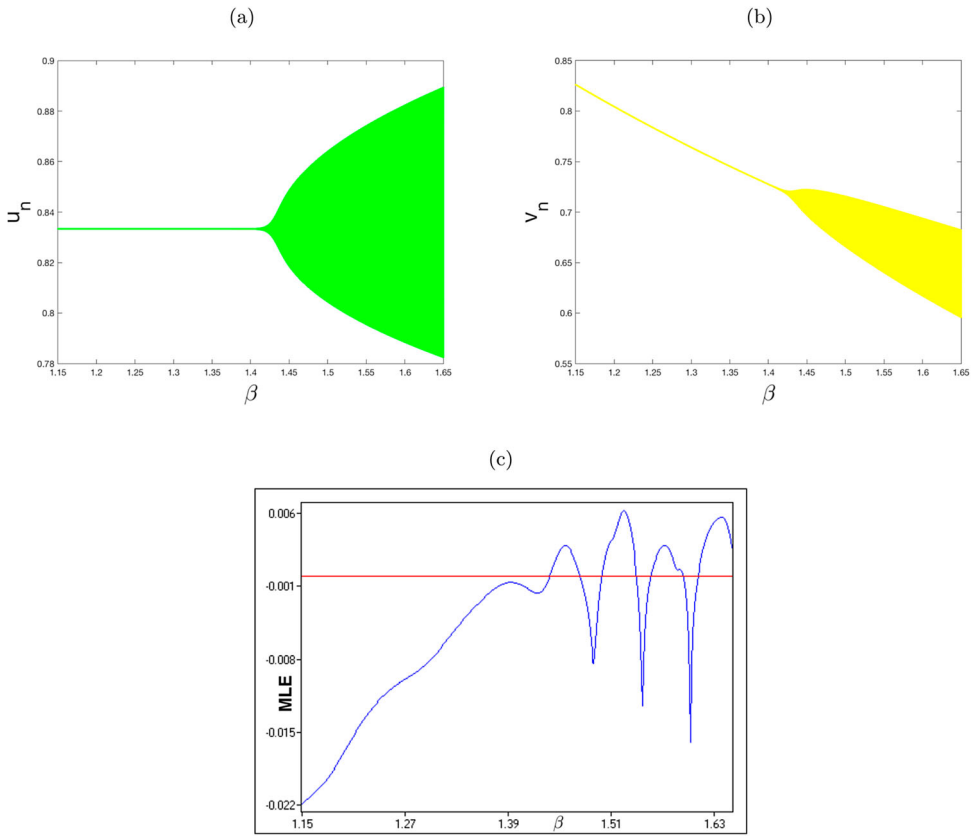


Figure 4. Bifurcation diagrams of (4) with $a = 1.2, k = 1.2, b = 0.6, h = 4, c = 0.8, d = 0.1, r = 1.5, u_0 = 0.8, v_0 = 0.7, \beta \in [1.15, 1.65]$.

findings that underscore the intricate dynamics of species interactions in ecological systems. While the Allee effect is directly applied to the prey population, its influence extends beyond that, indirectly impacting predator dynamics through the interaction between predator and prey populations. One can observe in Figure 1 that system (4) goes through NS and PD bifurcations at the positive fixed point E_2 by varying the Allee effect parameter β . These show that the Allee effect can stabilize or destabilize the system. Notably, a moderate level of the Allee effect appears to be advantageous for both prey and predator populations, as evidenced by the occurrence of these bifurcations. Furthermore, the positive fixed point $E_2 = (x_2, y_2)$ is dependent on the Allee effect parameter β . Through simple calculations, one can obtain that $\frac{dx_2}{d\beta} = 0$ and

$$\frac{dy_2}{d\beta} = -\frac{d^3 r(-d^2 + b^2 k(1 - c)^2)}{abk(1 - c)^2(d^2 + b^2 \beta(1 - c)^2)^2} < 0.$$

Ecologically, it represents that when the Allee effect strengthens in the prey population, it might become slightly more difficult for predators to locate and capture prey individuals. This could be due to various factors, such as decreased aggregation behaviour or reduced mating success among prey individuals. A decrease in prey availability could lead

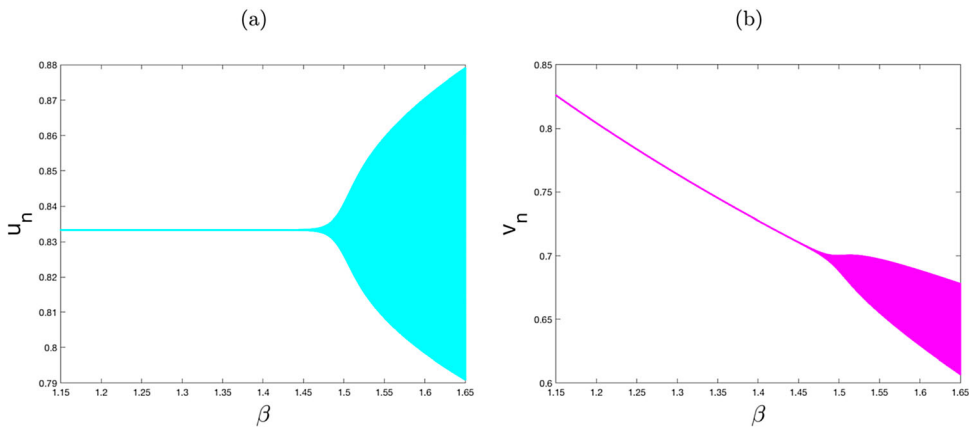


Figure 5. Bifurcation diagrams of (23) with $\rho = 0.95, a = 1.2, k = 1.2, b = 0.6, h = 4, c = 0.8, d = 0.1, r = 1.5, u_0 = 0.8, v_0 = 0.7, \beta \in [1.15, 1.65]$.

to reduced reproduction rates or increased mortality among predators, ultimately resulting in a decrease in predator population growth rate.

6. Conclusion

In this work, we have considered the complex dynamical behaviours of a predator-prey system with the Allee effect on the prey population. We conduct a local stability analysis of the fixed points and provide a detailed investigation of local bifurcations at the positive fixed point. Our findings demonstrate that system (4) undergoes PD and NS bifurcations. Moreover, a hybrid control technique is employed to control bifurcation and chaos in the system (4). As a result, it is possible to control both bifurcation forms using a broad range of control parameters. To confirm our theoretical results, numerical simulations are conducted. When the carrying capacity k is used as the bifurcation parameter, numerical simulations demonstrate that the system (4) shows exciting dynamics. According to a biological perspective, the environmental carrying capacity may significantly impact how the system (4) evolves. The proper carrying capacity may aid in the stabilization of the dynamical system (4). Instead, a larger or lower carrying capacity might cause the system to become unstable and exhibit more complicated dynamical behaviours.

Moreover, varying the Allee effect parameter β leads to both PD and NS bifurcations, serving as a crucial bifurcation parameter that may either stabilize or destabilize the predator-prey system. Interestingly, a moderate level of the Allee effect seems to be advantageous for both prey and predator populations. The stability of the positive fixed point shows intricate long-term dynamics, with the Allee effect affecting predator populations. Despite not directly affecting predators, the Allee effect affects prey availability and hunting success, thus altering predator dynamics.

We can learn more about ecosystem stability in the future by investigating the global dynamics of the predator-prey model with the Allee effect. By examining the behaviour of the system by varying parameters, codimension-2 bifurcation analysis may shed light on new bifurcation phenomena and their implications for population dynamics. Due to the

Allee effect's large influence on predator populations, future studies might concentrate on minimizing its effects. Introducing additional food sources or alternate prey species may provide extra resources to offset prey shortages.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Availability of data and materials

Data sharing not applicable to this article as no data-sets were generated or analysed during this study.

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