



Analysis of a regeneration period of the diesel particulate filter using queueing model with correlated arrivals and phase-type service[☆]

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ARTICLE INFO

Keywords:

MAP/PH/1/K queue

N-policy

Random environment

Regeneration period

Diesel particulate filter

ABSTRACT

This paper develops a queueing model for analyzing the regeneration period of a Diesel Particulate Filter (DPF) with the objective of providing a quantitative tool for optimal regeneration control. The system is modeled as a finite-capacity queue *MAP/PH/1/K* with Markovian arrival process (*MAP*) and phase-type (*PH*) service, incorporating a *N*-policy, setup time, service interruptions, reset mechanism and a random environment representing urban, mixed, and highway driving modes. The soot accumulation corresponds to arrivals, the soot burning to service, and regeneration triggers to the *N*-policy. The main contributions include the derivation of the Laplace–Stieltjes transform of the regeneration period, *z*-transform of the number of service interruptions, recursive moment relations, parametric approximation of the regeneration period distribution by a lognormal-gamma mixture, and the maximum queue length distribution during regeneration. The numerical results show that the optimal threshold decreases with increasing correlation and varies between driving modes, ranging from $N^* = 40$ for urban driving with low correlation to $N^* = 10$ for driving on the highway with high correlation. All analytical results are validated through discrete-event simulation. The proposed framework provides a quantitative basis for adaptive regeneration strategies that balance fuel consumption, filter lifetime, and emissions compliance under real-time driving conditions.

1. Introduction

Queueing systems with Markovian arrival processes (*MAP*) and phase-type (*PH*) service distributions have emerged as powerful analytical tools for modeling complex queueing systems where correlation and non-exponential patterns play critical roles [1]. Unlike traditional Poisson-based models, the *MAP/PH* frameworks capture the pulsed nature of real-world arrival processes and the structured variability of service times, making them particularly suitable for applications ranging from telecommunications to manufacturing systems [2]. The concept of the *N*-policy originated in the seminal work [3], where the authors introduced the policy for a single-server queueing system as a form of operating rule to manage server activation and deactivation. Under the classical *N*-policy, the server remains idle until the queue length reaches a threshold *N*. This mechanism was designed to balance the trade-off between maintaining an idle server and incurring the costs associated with frequent server startups and shutdowns.

[☆] The research is funded by the Federal Ministry for Women, Science and Research, as well as by the Hungarian Ministry of Culture and Innovation, the project N1210u2 of the Austro-Hungarian Campaign.

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<https://doi.org/10.1016/j.rico.2026.100792>

Received 14 April 2026; Received in revised form 1 July 2026; Accepted 2 August 2026

Available online 10 August 2026

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The past decade has witnessed significant advances in queueing models that incorporate various operational policies and system interruptions. In paper [4], the authors analyzed a $MAP/PH/1$ queueing system with catastrophic delay action and a standby server under N -policy, demonstrating the importance of considering catastrophic events that can wipe out all customers. Their work highlighted how such disruptions fundamentally alter the dynamics of the system and require specialized analytical treatment. Similarly, the paper [5] extended this line of research considering differentiated vacations, vacation interruption under N -policy, optional services, and breakdown–repair mechanisms in a queue $MAP/PH/1$. The comprehensive analysis introduced included stability conditions, analysis of busy periods, and cost optimization, establishing a methodological foundation for complex queueing systems with multiple operational phases. The $MAP/PH/1$ queue with working vacations, vacation interruptions, and N -policy has been studied in Ref. [6]. Service interruptions represent another crucial dimension of queueing system realism. These interruptions may arise from server breakdowns, maintenance activities, or external interruptions, and their duration often follows complex patterns. The phase-type representation of the interruption times allows for modeling various interruption duration distributions while maintaining analytical tractability. Recent work [7] on waiting times in queueing systems with variable cross-correlated arrival rates emphasized that service interruptions contribute significantly to total system variability, and this variability combines additively with arrival variability when determining queue performance. The regeneration mechanism in queueing systems, where the system periodically returns to an empty state, has received renewed attention due to applications in inventory management, manufacturing, and environmental systems. In paper [8], the authors demonstrated that correlation in arrival processes fundamentally affects blocking probabilities and waiting time distributions, and these effects can be quantified through systematic analysis of regeneration cycles. Their work provided closed-form expressions for performance distributions, establishing that correlation effects are attributable to correlation alone rather than other traffic features.

The motivation for this research stems from an industry-academia collaboration with BMW within the joint project “Statistical analysis of data to improve the regeneration efficiency of the diesel particulate filter” (2025–2026). The DPF is a critical component in modern diesel vehicles, responsible for trapping harmful particulate matter from exhaust emissions to meet stringent environmental regulations. Over time, accumulated soot increases back-pressure, degrading engine performance and fuel economy, necessitating periodic regeneration – a high-temperature process that oxidizes trapped soot and restores filter function. The paper [9] developed dynamic models of DOC (Diesel Oxidation Catalyst) - DPF after-treatment systems for regeneration control, demonstrating that regeneration events can be triggered passively through catalysts or actively through fuel injection strategies. In Ref. [10], the authors further advanced the control-oriented modeling of catalytic DPFs, emphasizing the need for real-time management of regeneration processes considering back-pressure and thermal state constraints. The practical challenge facing our industry partner is optimizing the timing and frequency of these regeneration events. Premature regeneration wastes fuel and reduces filter longevity, while delayed regeneration risks excessive back-pressure, potential filter damage, and emissions non-compliance. Current practice relies on threshold-based triggers using differential pressure sensors, but these approaches do not fully exploit the stochastic nature of soot accumulation under varying driving conditions.

The DPF regeneration problem presents remarkable parallels with queueing systems under the N -policy. The soot accumulation process corresponds to customer arrivals, the filter’s capacity represents the finite buffer K , the regeneration trigger threshold N parallels the N -policy activation level, the heating and oxidation process resembles setup time, and the return to clean filter state mirrors system reset. Service interruptions naturally correspond to changes in engine operating conditions that temporarily suspend effective oxidation of soot. Moreover, the random environment representing different driving conditions (city, highway, mixed) directly influences soot accumulation rates, analogous to queueing models with Markov arrival processes operating in a Markovian random environment [11]. Despite these clear parallels, no existing study has developed a unified queueing framework that simultaneously incorporates all these features while specifically targeting DPF regeneration applications. The $MAP/PH/1/K$ model with the N -policy, setup time, service interruption, reset mechanism, and random environment fills this gap by providing a comprehensive mathematical framework. The analysis of the regeneration period includes the following contributions:

- The Laplace–Stieltjes transform of the regeneration period and the probability generating function of the number of interruptions occurring during this period are derived in vector–matrix form. These transforms provide a complete probabilistic characterization of the regeneration dynamics and serve as the foundation for subsequent moment analysis.
- Recursive relations are established for the computation of arbitrary-order moments of both the regeneration period and the number of interruptions. These relations enable efficient numerical evaluation of the moments without the need for explicit inversion of the transforms.
- Based on the computed moments, the continuous distribution of the regeneration period is approximated by a two-component mixture of lognormal and gamma distributions, while the discrete distribution of the number of interruptions is approximated by a negative binomial distribution. The parameters of both approximations are estimated using a hybrid procedure combining the method of moments and maximum likelihood estimation, with the latter performed via the Expectation–Maximization (EM) algorithm.
- A maximal queue length analysis during the regeneration period is conducted, and the corresponding distribution function is derived. This analysis provides insights into the worst-case soot accumulation behavior and is of practical importance for filter design and control.
- The influence of correlation in the arrival process and environmental states (driving modes) on the distribution functions is systematically investigated. The analysis reveals that both factors significantly affect the tail behavior of the regeneration period distribution, with urban driving and high correlation leading to the heaviest tails.

- All analytical results are validated through discrete-event simulation. The comparison between the theoretical distributions and the empirical distributions obtained from simulation confirms the accuracy of the proposed queueing model and the correctness of the analytical derivations.

By analyzing how correlation structures and environmental transitions affect regeneration timing and effectiveness, this research provides quantitative tools for optimizing DPF regeneration strategies to balance fuel consumption, regeneration frequency, and filter longevity. The results have a broader applicability to any system that exhibits threshold-based activation, setup delays, service interruptions, and periodic resetting, including batch manufacturing processors, communication networks with sleep modes, and inventory systems with emergency ordering policies. Diesel vehicle manufacturers have the opportunity using the proposed model to adapt the DPF regeneration procedure to drivers, taking into account the specific characteristics of vehicle operation, and this adaptation can be carried out in dynamic mode.

The remainder of this paper is organized as follows. Section 2 describes the mathematical model and briefly presents the stationary distribution of the system state. The specific values of the system parameters used for numerical experiments throughout the paper are discussed in Section 3. Section 4 presents algorithmic solutions for the characteristics of the regeneration period, with illustrative examples on correlation and driving pattern analysis. Section 5 provides the distribution of the number of interruptions during the regeneration period. The maximum queue length analysis during a regeneration period is performed in Section 6. Finally, Section 7 concludes the paper with remarks on implementation and future research directions

2. Mathematical model

Consider the finite-capacity single server queueing system of type $MAP/PH/1/K$ with Markovian arrival process (MAP), phase-type (PH) service times, threshold-based N -policy startup policy, random service interruptions and a reset mechanism when the system becomes full. Based on the application of this system to describe the DPF regeneration process, it is assumed that the incoming flow of one gram of soot particles can be described as MAP . Moreover, since the accumulation of particles depends on the speed of the vehicle, the behavior of the system depends on the state of a random environment. For example, the low soot accumulation process with average accumulation rates of approximately 0.05 – 0.3 grams per hour (g/h) of engine operation corresponds to a highway driving at average speeds exceeding 90 km/h. The middle accumulation of soot is characterized by rates at 0.3–0.5 g/h and corresponds to speeds between 50 and 90 km/h, and the high accumulation process with average accumulation rates of approximately 0.5–1.5 g/h corresponds to driving through the city at average speeds not exceeding 50 km/h. The transition parameters in a set of states of a random environment may vary depending on driving style and can be adjusted by the vehicle's on-board computer based on available statistical information. Throughout this paper, let $e(m)$, $I(m)$ denote, respectively, the column vector of dimension m consisting of 1's and an identity matrix of dimension m . The vector $e(\cdot)$ will denote a column vector of 1's of appropriate dimension. The zero matrix block with dimension $n \times m$ will be denoted by $0_{n \times m}$.

Let $\{\Phi_e(t)\}_{t \geq 0}$ denote the stochastic process for the states of random environment which is assumed to be an irreducible continuous-time Markov chain with the infinitesimal generator H , where $\Phi_e(t) = 1, \dots, m_e$ and by π_e the corresponding stationary vector satisfying $\pi_e H = 0$, $\pi_e e(m_e) = 1$. For the fixed state ϕ_e of the process $\{\Phi_e(t)\}_{t \geq 0}$, we define MAP which is governed by the $m_a \times m_a$ matrices $D_0^{(\phi_e)}$ and $D_1^{(\phi_e)}$, respectively, for transitions without arrivals and transitions accompanied by arrivals under the fixed $\phi_e = 1, \dots, m_e$. We note that the transitions of the process $\{\Phi_e(t)\}_{t \geq 0}$ cannot change the states of the arrival process; therefore, only the transition rates of MAP can be changed. Arrivals occur only when the server is idle. The infinitesimal generator of the underlying Markov chain is $D^{(\phi_e)} = D_0^{(\phi_e)} + D_1^{(\phi_e)}$. The stationary vector of MAP is defined as $\pi_a^{(\phi_e)}$, satisfying the system $\pi_a^{(\phi_e)} D^{(\phi_e)} = 0$, $\pi_a^{(\phi_e)} e(m_a) = 1$. By arrivals, the number of customers (the whole number of grams of soot) in the system increases by one (by one gram). The average arrival rate $\lambda^{(\phi_e)}$, the second moment of the inter-arrival times $\lambda_2^{(\phi_e)}$, and the correlation coefficient for lag 1 $\rho^{(\phi_e)}$ are calculated as follows:

$$\begin{aligned} \lambda^{(\phi_e)} &= \pi_a^{(\phi_e)} D_1^{(\phi_e)} e(m_a), \quad \lambda_2^{(\phi_e)} = 2\pi_a^{(\phi_e)} (-\lambda^{(\phi_e)} D_0^{(\phi_e)})^{-1} e(m_a), \\ \rho^{(\phi_e)} &= \frac{\lambda^{(\phi_e)} \pi_a^{(\phi_e)} (-D_0^{(\phi_e)})^{-1} D_1^{(\phi_e)} (-D_0^{(\phi_e)})^{-1} e(m_a) - 1}{2\lambda^{(\phi_e)} \pi_a^{(\phi_e)} (-D_0^{(\phi_e)})^{-1} e(m_a) - 1}. \end{aligned}$$

The service process is phase-type with a number of phases m_s . It has a representation (μ, M) , where μ is the initial probability vector of dimension m_s and M is a subgenerator matrix of dimension $m_s \times m_s$. Phase-dependent service completion rates are defined by the vector $M^0 = -M e(m_s)$ and the average service rate is $\mu_s = -(\mu M^{-1} e(m_s))^{-1}$. By completion of the service, the number of customers in the system is reduced by one. The phase-type service interruption time has representation (α, A) , with an initial m_i -dimensional vector α and a $m_i \times m_i$ subgenerator A . Interruption occurrence rates are defined by $A^0 = -A e(m_i)$ and the average rate is $\alpha_i = -(\alpha A^{-1} e(m_i))^{-1}$. The effects of service interruption are as follows. When an interruption occurs, the service stops, the server becomes idle, and the customer remains in the system. To reduce the state space, we do not track the states of the random environment and MAP phases during busy periods of server, that is, the environment state information and the MAP phase is lost. Interruption of service when the number of remaining customers is less than or equal to the threshold level L will mean successful regeneration. Otherwise, regeneration is called incomplete. We will assume that successful filter regeneration is achieved when the filter is burned to such a level that no more than 10% of the threshold level N of soot remains in the filter. This condition may vary according to the specified physical characteristics of the DPF and the soot burning process itself. The environment state and the MAP phase must be recovered probabilistically after a service interruption when the server becomes

idle again. It is done using the stationary distributions $\pi_e = (\pi_{e,1}, \dots, \pi_{e,m_e})$ and $\pi_a^{(\phi_e)} = (\pi_{a,1}^{(\phi_e)}, \dots, \pi_{a,m_a}^{(\phi_e)})$. The startup process is of the phase-type. It starts automatically if the server is idle when the number of customers in the system reaches the threshold level N . The process has a representation (γ, Γ) with dimension m_u . The completion rates for the startup are $\Gamma^0 = -\Gamma e(m_u)$ and the average rate is $\gamma_u = -(\gamma \Gamma^{-1} e(m_u))^{-1}$. After startup completion, the service begins with a new customer. The phase-type reset process begins when the system has K customers and the server is still idle. The *MAP* phases cannot be tracked in this state. The PH process has a representation (β, B) with dimension m_r . Reset completion rates are defined by $B^0 = -B e(m_r)$ and the average rate is $\beta_r = -(\beta B^{-1} e(m_r))^{-1}$. After reset time, the system becomes empty and returns to level 0. In this case, the re-initialization distribution π_a is used for the arrival process.

The following notations are used to describe other components of the corresponding stochastic process. Denote by

- $N(t)$ the number of customers in the system at time t , $N(t) = 0, 1, \dots, K$,
- $D(t)$ the status of the server at time t , where

$$D(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is busy,} \end{cases}$$

- $\Phi_a(t)$ the phase of the MAP process, $\Phi_a(t) = 1, \dots, m_a$,
- $\Phi_s(t)$ the phase of the PH-type service process, $\Phi_s(t) = 1, \dots, m_s$,
- $\Phi_i(t)$ the phase of the PH-type service interruption process, $\Phi_i(t) = 1, \dots, m_i$,
- $\Phi_u(t)$ the phase of the PH-type startup process, $\Phi_u(t) = 1, \dots, m_u$,
- $\Phi_r(t)$ the phase of the PH-type reset process, $\Phi_r(t) = 1, \dots, m_r$.

The queueing process is defined on the state space:

$$E = \bigcup_{n=0}^K l(n),$$

where level $l(0)$ corresponds to the empty system,

$$l(0) = \bigcup_{\phi_e=1}^{m_e} \bigcup_{\phi_a=1}^{m_a} \{(0, 0, \phi_e, \phi_a)\}.$$

with cardinality $|l(0)| = l_0 = m_e m_a$. Levels $l(n)$, $1 \leq n \leq N-1$, below the threshold,

$$l(n) = l(n, 0) \cup l(n, 1) = \bigcup_{\phi_e=1}^{m_e} \bigcup_{\phi_a=1}^{m_a} \{(n, 0, \phi_e, \phi_a)\} \bigcup \bigcup_{\phi_s=1}^{m_s} \bigcup_{\phi_i=1}^{m_i} \{(n, 1, \phi_s, \phi_i)\},$$

have cardinality $|l(n)| = |l(n, 0)| + |l(n, 1)| = l_0 + l_1 = m_e m_a + m_s m_i$ and include states $(n, 0, \phi_e, \phi_a)$ with n customers, idle server, random environment state $\phi_e = 1, \dots, m_e$, *MAP* phase $\phi_a = 1, \dots, m_a$ and $(n, 1, \phi_s, \phi_i)$ with n customers, busy server, service phase $\phi_s = 1, \dots, m_s$, interruption phase $\phi_i = 1, \dots, m_i$.

Levels $l(n)$, $N \leq n \leq K-1$, at and above threshold,

$$l(n) = l(n, 0) \cup l(n, 1) = \bigcup_{\phi_e=1}^{m_e} \bigcup_{\phi_a=1}^{m_a} \bigcup_{\phi_u=1}^{m_u} \{(n, 0, \phi_e, \phi_a, \phi_u)\} \bigcup \bigcup_{\phi_s=1}^{m_s} \bigcup_{\phi_i=1}^{m_i} \{(n, 1, \phi_s, \phi_i)\},$$

have cardinality $|l(n)| = |l(n, 0)| + |l(n, 1)| = l_2 + l_1 = m_e m_a m_u + m_s m_i$ and consist of states $(n, 0, \phi_e, \phi_a, \phi_u)$ with n customers, idle server, random environment state $\phi_e = 1, \dots, m_e$, *MAP* phase $\phi_a = 1, \dots, m_a$, startup phase $\phi_u = 1, \dots, m_u$, and states $(n, 1, \phi_s, \phi_i)$ with n customers, busy server, service phase $\phi_s = 1, \dots, m_s$ and service interruption phase $\phi_i = 1, \dots, m_i$.

The reset level $l(K)$ of the full system

$$l(K) = \bigcup_{\phi_r=1}^{m_r} \{(K, 0, \phi_r)\},$$

where $(K, 0, \phi_r)$ has K customers, idle server, and reset process at phase $\phi_r = 1, \dots, m_r$. The states in each level $l(n)$ are enumerated in lexicographic order. The total number of states in the state space E is equal to

$$|E| = l_0(K-1) + l_1(N-1) + l_2(K-N) + m_r.$$

Note that if the server is idle, i.e., $D(t) = 0$ and $0 \leq N(t) \leq N-1$, the values of the components $\Phi_s(t)$, $\Phi_i(t)$, $\Phi_u(t)$, and $\Phi_r(t)$ are not defined. If $D(t) = 0$ and $N \leq N(t) \leq K-1$, then the components $\Phi_s(t)$, $\Phi_i(t)$, and $\Phi_r(t)$ are not defined. In case of a busy server, where $D(t) = 1$ with $1 \leq N(t) \leq K-1$, the components $\Phi_e(t)$, $\Phi_a(t)$, $\Phi_u(t)$ and $\Phi_r(t)$ are not defined. In states of a reset level, the components $\Phi_a(t)$, $\Phi_u(t)$, $\Phi_s(t)$, and $\Phi_i(t)$ are not defined. Taking into account this comment, the multi-dimensional process

$$\{X(t)\}_{t \geq 0} = \{N(t), D(t), \Phi_e(t), \Phi_a(t), \Phi_u(t), \Phi_s(t), \Phi_i(t), \Phi_r(t)\}_{t \geq 0} \quad (1)$$

is an irreducible regular continuous-time Markov chain (CTMC) defined in state space E .

Then, the infinitesimal generator of the Markov chain $\{X(t)\}_{t \geq 0}$ for a fixed level N has the form

$$Q^{(N)} = \begin{pmatrix} l(0) & l(1) & l(2) & \dots & l(N-1) & l(N) & l(N+1) & \dots & l(K) \\ \begin{pmatrix} Q_{1,0} & Q_{0,1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ Q_{2,0} & Q_{1,1} & Q_{0,2} & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & Q_{2,1} & Q_{1,1} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \dots & 0 \\ 0 & \dots & 0 & Q_{2,1} & Q_{1,1} & Q_{0,3} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & Q_{2,2} & Q_{1,2} & Q_{0,4} & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & Q_{2,3} & Q_{1,2} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & Q_{2,3} & Q_{1,2} & Q_{0,5} \\ Q_{3,0} & 0 & \dots & \dots & 0 & 0 & 0 & 0 & B \end{pmatrix} & \begin{pmatrix} l(0) \\ l(1) \\ l(2) \\ \vdots \\ l(N-1) \\ l(N) \\ l(N+1) \\ \vdots \\ l(K-1) \\ l(K) \end{pmatrix} \end{pmatrix}, \quad (2)$$

where non-zero blocks $Q_{1,\cdot}$, $Q_{0,\cdot}$ and $Q_{2,\cdot}$ specify the transitions respectively to stay in a certain level $l(n)$, to go from level $l(n-1)$ to level $l(n)$ according to arrivals and to go from level $l(n+1)$ to $l(n)$ due to service completion. The block $Q_{3,0}$ is responsible for the reset process, i.e., the transition from $l(K)$ to $l(0)$. The matrices are defined as follows:

$$\begin{aligned} Q_{1,0} &= D_0 + H \otimes I(m_a), \quad Q_{1,1} = \begin{pmatrix} D_0 + H \otimes I(m_a) & 0_{l_0 \times l_1} \\ A^0 \otimes e(m_s) \otimes (\pi_a \text{diag}(\pi_e) \otimes I(m_a)) & A \oplus M \end{pmatrix}, \\ Q_{1,2} &= \begin{pmatrix} D_0 \oplus \Gamma + H \otimes I(m_a m_u) & e(l_0) \otimes (\Gamma^0 \otimes \mu^\top \otimes \alpha^\top) \\ A^0 \otimes e(m_s) \otimes (\pi_a \text{diag}(\pi_e) \otimes I(m_a)) \otimes \gamma^\top & A \oplus M \end{pmatrix}, \\ Q_{0,1} &= \begin{pmatrix} D_1 & 0_{l_0 \times l_1} \\ 0_{l_1 \times l_0} & 0_{l_1 \times l_1} \end{pmatrix}, \quad Q_{0,2} = \begin{pmatrix} D_1 & 0_{l_0 \times l_1} \\ 0_{l_1 \times l_0} & 0_{l_1 \times l_1} \end{pmatrix}, \\ Q_{0,3} &= \begin{pmatrix} D_1 \otimes \gamma^\top & 0_{l_0 \times l_1} \\ 0_{l_1 \times l_2} & 0_{l_1 \times l_1} \end{pmatrix}, \quad Q_{0,4} = \begin{pmatrix} D_1 \otimes I(m_u) & 0_{l_2 \times l_1} \\ 0_{l_1 \times l_2} & 0_{l_1 \times l_1} \end{pmatrix}, \\ Q_{0,5} &= \begin{pmatrix} (\beta \otimes (D_1 e(l_0)) \otimes e(m_u))^\top \\ 0_{l_1 \times m_r} \end{pmatrix}, \quad Q_{2,0} = \begin{pmatrix} 0_{l_0 \times l_0} \\ e(m_i) \otimes M^0 \otimes (\pi_a \text{diag}(\pi_e) \otimes I(m_a)) \end{pmatrix}, \\ Q_{2,1} &= \begin{pmatrix} 0_{l_0 \times l_0} & 0_{l_0 \times l_1} \\ 0_{l_1 \times l_0} & I(m_i) \otimes (M^0 \otimes \mu^\top) \end{pmatrix}, \quad Q_{2,2} = \begin{pmatrix} 0_{l_2 \times l_0} & 0_{l_2 \times l_1} \\ 0_{l_1 \times l_0} & I(m_i) \otimes (M^0 \otimes \mu^\top) \end{pmatrix}, \\ Q_{2,3} &= \begin{pmatrix} 0_{l_2 \times l_2} & 0_{l_2 \times l_1} \\ 0_{l_1 \times l_2} & I(m_i) \otimes (M^0 \otimes \mu^\top) \end{pmatrix}, \quad Q_{3,0} = B^0 \otimes (\pi_a \text{diag}(\pi_e) \otimes I(m_a)), \\ D_0 &= \text{diag}(D_0^{(1)}, \dots, D_0^{(m_e)}), \quad D_1 = \text{diag}(D_1^{(1)}, \dots, D_1^{(m_e)}). \end{aligned}$$

The structure of the generator $Q^{(N)}$ confirms that the process $\{X(t)\}_{t \geq 0}$ is a type of level-dependent quasi-birth-and-death process with special boundary conditions. Since CTMC (1) is irreducible, regular, and has finite state space E , the following limits exist for stationary probabilities:

$$\pi_x = \lim_{t \rightarrow \infty} \mathbb{P}[X(t) = x], \quad x \in E.$$

Let $\pi = (\pi(0), \pi(1), \dots, \pi(K))$, with sub vectors $\pi(n)$ partitioned with respect to levels $l(n)$, $0 \leq n \leq K$, denote the stationary probability vector of the infinitesimal matrix $Q^{(N)}$ defined in (2). This vector satisfies the system

$$\pi Q^{(N)} = 0, \quad \pi e(|E|) = 1.$$

The stationary distribution was obtained by solving a finite block almost three-diagonal system in paper [12]. We also refer the reader to Refs. [11,13], where the details for computing the vector π for the finite block tridiagonal system are presented. The computation for our model can be carried out in a similar manner using, for example, a block forward-elimination-backward-substitution technique.

The probability distribution for the number of customers in the system is computed by

$$\pi_n := \lim_{t \rightarrow \infty} \mathbb{P}[N(t) = n] = \pi(n) e(\cdot) = \begin{cases} \pi(n) e(l_0), & n = 0, \\ \pi(n) e(l_0 + l_1), & 1 \leq n \leq N-1, \\ \pi(n) e(l_2 + l_1), & N \leq n \leq K-1, \\ \pi(n) e(m_r), & n = K. \end{cases}$$

The average number of customers in the system $\bar{N}$ is calculated by

$$\bar{N} = \sum_{n=1}^K n \pi_n.$$

The arrival rate of arrivals to a reset level $l(K)$

$$A_r = \pi(K-1) Q_{0,5} e(l_2 + l_1).$$

The probability that the server is in the startup state

$$P_u = \sum_{n=N}^{K-1} \pi(n) \binom{e(l_2)}{0_{l_1 \times 1}}.$$

The service interruption rate in levels $l(n)$, $L+1 \leq n \leq K-1$ for incomplete regenerations

$$\begin{aligned} A_i &= \sum_{n=L+1}^{N-1} \pi(n) \left(A^0 \otimes e(m_s) \otimes (\pi_a \text{diag}(\pi_e) \otimes I(m_a)) \right) e(l_0 + l_1) \\ &+ \sum_{n=N}^{K-1} \pi(n) \left(A^0 \otimes e(m_s) \otimes (\pi_a \text{diag}(\pi_e) \otimes I(m_a)) \otimes \gamma^\top \right) e(l_2 + l_1). \end{aligned}$$

Let c_h be the holding cost per unit of time for each customer present in the system, c_r the fixed cost incurred for each visit to the states in $l(K)$ due to the system blocking in the reset states, c_u be the start-up cost per unit of time for the server performing the pre-service work, and c_i be the fixed cost incurred for the service activation plus service interruption in levels $l(n)$, $L+1 \leq n \leq K-1$ leading to incomplete regeneration. Combining the aforementioned cost elements yields the following total expected cost function per unit time

$$\overline{C}^{(N)} = c_h \overline{N} + c_r A_r + c_u P_u + c_i A_i \quad (3)$$

that can be minimized with respect to the threshold N .

3. The parameter values for numerical experiments

For numerical experiments, we consider the $MAP/PH/1/80$ queueing system throughout the remainder of this paper, where the finite capacity 80 corresponds to the maximum soot level threshold in grams. We now specify the parameter values for the phase-type service, startup, service interruption, and reset processes. The service process is assumed to be of Erlang type with $m_s = 5$ phases, modeling the nearly deterministic soot oxidation (burning) process,

$$M = \begin{pmatrix} -200 & 200 & 0 & 0 & 0 \\ 0 & -200 & 200 & 0 & 0 \\ 0 & 0 & -200 & 200 & 0 \\ 0 & 0 & 0 & -200 & 200 \\ 0 & 0 & 0 & 0 & -200 \end{pmatrix}, M^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 200 \end{pmatrix}, \mu = (1, 0, 0, 0, 0), \mu_s = 40,$$

which guaranties a small variance $\sigma_s^2 = 0.000125$. The reset process is described by following parameters,

$$B = \begin{pmatrix} -2.50 & 1.09 \\ 0.50 & -3.50 \end{pmatrix}, \beta = (0.7 \quad 0.3), B^0 = (1.41, 3.00)^\top, \beta_r = 1.994.$$

The parameters of the MAP with $m_e = 4$ environmental phases are generated according to the matching procedure described in paper [14]. For each state $\phi_e = 1, \dots, m_e$, we specify three distinct values of the correlation function $\rho^{(\phi_e)}$, representing the autocorrelation of the inter-arrival times. Additionally, for each correlation value, three possible average arrival rates $\lambda^{(\phi_e)}$ are defined, which depend on the current environmental state ϕ_e . These environmental states reflect the vehicle speed level, with higher speeds corresponding to lower soot accumulation rates, as the engine operates more efficiently under steady highway driving conditions.

- For $\rho^{(\phi_e)} = 0$,

- (a) $\phi_e = 1$ for slow speed driving, $\lambda^{(1)} = 1.50$, $\lambda_2^{(1)} = 1$,

$$\begin{aligned} D_0^{(1)} &= \begin{pmatrix} -1.26062 & 1.26062 & 0. & 0. \\ 0. & -3.003 & 0. & 0. \\ 0. & 0. & -1.1255 & 1.1255 \\ 0. & 0. & 0. & -3.003 \end{pmatrix}, \\ D_1^{(1)} &= \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0.98104 & 1.35596 & 0.249612 & 0.416388 \\ 0. & 0. & 0. & 0. \\ 0.350732 & 0.48477 & 0.812364 & 1.35514 \end{pmatrix} \end{aligned}$$

- (b) $\phi_e = 2$ for middle speed driving, $\lambda^{(2)} = 0.503$, $\lambda_2^{(2)} = 7$,

$$D_0^{(2)} = \begin{pmatrix} -0.608759 & 0.608759 & 0. & 0. \\ 0. & -1.00503 & 0. & 0. \\ 0. & 0. & -0.694814 & 0.694814 \\ 0. & 0. & 0. & -1.00503 \end{pmatrix},$$

$$D_1^{(2)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0.436563 & 0.284177 & 0.196538 & 0.0877476 \\ 0. & 0. & 0. & 0. \\ 0.132183 & 0.0860436 & 0.543945 & 0.242853 \end{pmatrix}$$

(c) $\phi_e = 3$ for high speed driving, $\lambda^{(3)} = 0.300$, $\lambda_2^{(3)} = 18$,

$$D_0^{(3)} = \begin{pmatrix} -0.419338 & 0.419338 & 0. & 0. \\ 0. & -0.600601 & 0. & 0. \\ 0. & 0. & -0.531171 & 0.531171 \\ 0. & 0. & 0. & -0.600601 \end{pmatrix},$$

$$D_1^{(3)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0.290171 & 0.125429 & 0.163614 & 0.0213859 \\ 0. & 0. & 0. & 0. \\ 0.0805024 & 0.0347979 & 0.4292 & 0.0561005 \end{pmatrix}.$$

• For $\rho^{(\phi_e)} = 0.2$,

(a) $\phi_e = 1$ for slow speed driving, $\lambda^{(1)} = 1.50$, $\lambda_2^{(1)} = 2$,

$$D_0^{(1)} = \begin{pmatrix} -1.01798 & 1.01798 & 0. & 0. \\ 0. & -7.10518 & 0. & 0. \\ 0. & 0. & -0.543434 & 0.543434 \\ 0. & 0. & 0. & -0.992423 \end{pmatrix},$$

$$D_1^{(1)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0.927674 & 5.5472 & 0.345147 & 0.285163 \\ 0. & 0. & 0. & 0. \\ 0.0442613 & 0.264669 & 0.374269 & 0.309224 \end{pmatrix}$$

(b) $\phi_e = 2$ for the middle speed driving, $\lambda^{(2)} = 0.503$, $\lambda_2^{(2)} = 15$,

$$D_0^{(2)} = \begin{pmatrix} -0.395125 & 0.395125 & 0. & 0. \\ 0. & -2.37791 & 0. & 0. \\ 0. & 0. & -0.236649 & 0.236649 \\ 0. & 0. & 0. & -0.385137 \end{pmatrix},$$

$$D_1^{(2)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0.353399 & 1.7734 & 0.154298 & 0.0968158 \\ 0. & 0. & 0. & 0. \\ 0.0188403 & 0.094543 & 0.16698 & 0.104773 \end{pmatrix}$$

(c) $\phi_e = 3$ for the high speed driving, $\lambda^{(3)} = 0.300$, $\lambda_2^{(3)} = 26$,

$$D_0^{(3)} = \begin{pmatrix} -0.375492 & 0.375492 & 0. & 0. \\ 0. & -1.42104 & 0. & 0. \\ 0. & 0. & -0.323809 & 0.323809 \\ 0. & 0. & 0. & -0.338108 \end{pmatrix},$$

$$D_1^{(3)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0.311433 & 0.867176 & 0.232174 & 0.0102524 \\ 0. & 0. & 0. & 0. \\ 0.020495 & 0.0570676 & 0.249527 & 0.0110187 \end{pmatrix}.$$

• For $\rho^{(\phi_e)} = 0.4$,

(a) $\phi_e = 1$ for slow speed driving, $\lambda^{(1)} = 1.50$, $\lambda_2^{(1)} = 3$,

$$D_0^{(1)} = \begin{pmatrix} -1.28225 & 1.28225 & 0. & 0. \\ 0. & -12.3038 & 0. & 0. \\ 0. & 0. & -0.534109 & 0.534109 \\ 0. & 0. & 0. & -0.560595 \end{pmatrix},$$

$$D_1^{(1)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 1.22538 & 10.5327 & 0.519959 & 0.0257841 \\ 0. & 0. & 0. & 0. \\ 0.0149355 & 0.128378 & 0.397567 & 0.0197149 \end{pmatrix}$$

(b) $\phi_e = 2$ for middle speed driving, $\lambda^{(2)} = 0.503$, $\lambda_2^{(2)} = 24$,

$$D_0^{(2)} = \begin{pmatrix} -0.462225 & 0.462225 & 0. & 0. \\ 0. & -4.11776 & 0. & 0. \\ 0. & 0. & -0.205475 & 0.205475 \\ 0. & 0. & 0. & -0.208281 \end{pmatrix},$$

$$D_1^{(2)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0.439344 & 3.47458 & 0.201088 & 0.00274587 \\ 0. & 0. & 0. & 0. \\ 0.00585662 & 0.0463174 & 0.154004 & 0.00210293 \end{pmatrix}$$

(c) $\phi_e = 3$ for high speed driving, $\lambda^{(3)} = 0.300$, $\lambda_2^{(3)} = 65$,

$$D_0^{(3)} = \begin{pmatrix} -0.282633 & 0.282633 & 0. & 0. \\ 0. & -2.46076 & 0. & 0. \\ 0. & 0. & -0.128182 & 0.128182 \\ 0. & 0. & 0. & -0.128468 \end{pmatrix},$$

$$D_1^{(3)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0.268168 & 2.06665 & 0.125661 & 0.000279606 \\ 0. & 0. & 0. & 0. \\ 0.00367142 & 0.028294 & 0.096288 & 0.000214249 \end{pmatrix}.$$

For numerical experiments, we study three main cases corresponding to different structures of the infinitesimal generator matrix H , which governs transitions between the environmental states $\phi_e = 1, \dots, m_e$.

- Case 1: The vehicle is operated mostly in the city,

$$H = \begin{pmatrix} -0.02 & 0.01 & 0.01 \\ 10 & -10.01 & 0.01 \\ 10 & 0.01 & -10.01 \end{pmatrix}, \pi_e = (0.998004, 0.000998, 0.000998).$$

- Case 2: Equal probability of using slow, middle and fast speed driving,

$$H = \begin{pmatrix} -1 & 0.5 & 0.5 \\ 0.5 & -1 & 0.5 \\ 0.5 & 0.5 & -1 \end{pmatrix}, \pi_e = (0.333333, 0.333333, 0.333334).$$

- Case 3: The vehicle is mainly driven on highways,

$$H = \begin{pmatrix} -10.01 & 0.01 & 10 \\ 0.01 & -10.01 & 10 \\ 0.01 & 0.01 & -0.02 \end{pmatrix}, \pi_e = (0.000998, 0.000998, 0.998004).$$

Next, we distinguish between two subcases for the startup and service interruption processes, which influence the length of the regeneration period. The first subcase (a) corresponds to a short regeneration period, characterized by a high startup intensity and a low interruption intensity,

$$(a) \quad \Gamma = \begin{pmatrix} -1.50 & 0.01 \\ 0.05 & -2.80 \end{pmatrix}, \gamma = (0.6 \quad 0.4), \Gamma^0 = (1.49, 2.75)^\top, \gamma_u = 1.821$$

$$A = \begin{pmatrix} -0.75 & 0.70 \\ 0.45 & -1.00 \end{pmatrix}, \alpha = (0.6 \quad 0.4), A^0 = (0.05, 0.55)^\top, \alpha_i = 0.290$$

and the second subcase (b), corresponding to a long regeneration period when the startup intensity is low and the interruption intensity is high, resulting in delayed and frequently interrupted regeneration,

$$(b) \quad \Gamma = \begin{pmatrix} -1.05 & 0.70 \\ 0.35 & -1.40 \end{pmatrix}, \gamma = (0.6 \quad 0.4), \Gamma^0 = (0.35, 1.05)^\top, \gamma_u = 0.673$$

$$A = \begin{pmatrix} -0.75 & 0.50 \\ 0.25 & -1.00 \end{pmatrix}, \alpha = (0.6 \quad 0.4), A^0 = (0.25, 0.75)^\top, \alpha_i = 0.481.$$

As threshold levels, we use optimal thresholds N^* that minimize the cost function $\bar{C}^{(N)}$ in (3) for the cost values $c_h = 0.001$, $c_r = 1500$, $c_u = 1.0$ and $c_i = 5.0$, $L = [0.1N^*]$.

It should be noted that the parameter values employed in the present study are predominantly hypothetical, as comprehensive empirical data on DPF regeneration dynamics are generally not publicly available. In practical applications, the soot accumulation rate depends on a multitude of factors, including engine type and power output, filter design, driving conditions, and ambient operating parameters. Nevertheless, open-source literature provides estimates of soot accumulation rates ranging from 0.0024 g/kWh to 0.2 g/kWh as described in papers [15,16]. To facilitate direct comparison with the proposed model, these values are converted to units of grams per hour of engine operation. Assuming a typical passenger car power output in the range of 15–25 kW, the corresponding soot accumulation rates lie between approximately 0.036 g/h and 3 g/h. The parameter ranges adopted in this study are consistent with these reference values.

The mean startup time, corresponding to the period required for the engine to reach the temperature necessary for regeneration initiation (time to start the highway trip) is assumed to be approximately 1 to 1.5 h. This interval reflects the typical warm-up phase of a diesel engine under normal operating conditions. The time to service interruption, representing the duration during which the regeneration process may be temporarily suspended (e.g., due to driving mode changes or unfavorable exhaust conditions), is set between 2 and 3.4 h. These values seem also to be consistent with observed regeneration patterns in automotive DPF systems. The service process itself is characterized by a soot oxidation rate of 1.5 grams per minute. Consequently, the complete combustion of 20 grams of accumulated soot requires approximately 0.5 h (30 min). This duration is in good agreement with typical active regeneration intervals reported in the automotive literature [17,18]. From a practical standpoint, all parameter values employed in the numerical analysis are physically reasonable and lie within the ranges observed in real-world DPF operation, thereby ensuring that the model captures the essential features of the regeneration process while maintaining tractability for analytical investigation.

The *MAP* provides a flexible modeling framework capable of capturing a wide range of arrival behavior. It is particularly well-suited to the DPF context, where the soot accumulation process is influenced by correlated factors such as engine speed, load, and driving conditions. The Markovian structure of the *MAP* enables the preservation of the correlation structure inherent in the exhaust emission process, while maintaining tractability for analytical solution. The parameters of the process can be estimated from statistical data using the matching procedure discussed in paper [14]. The delays occurring in the system including the startup, service and time to service interruption are modeled using *PH* distributions which are known to form a dense subset in the space of all non-negative distributions as stated in Refs. [19,20]. Consequently, for any non-negative distribution, there exists a sequence of *PH* distributions that converges to it in distribution. The canonical representation of *PH* distributions proposed in paper [21] provides a systematic approach for moment matching. A distinctive feature of the proposed queueing framework is its inherent universality, which stems from the flexibility of the *MAP* and *PH* distributions employed in the model. Hence, we believe that the proposed model can be readily fitted to empirical data using established statistical techniques, enabling its application to specific DPF systems with minimal customization.

Example 1. To validate the queueing model and the matrix-analytic solution, we performed an independent discrete-event simulation of the DPF soot accumulation and regeneration process. The simulation duration is 20000 time units after some warm-up period to eliminate initialization bias. Fig. 1 presents two complementary views of the system behavior: 1(a) shows a representative time trace of the soot level dynamics generated using *MAP/PH/1/80* queueing model, while 1(b) compares the stationary soot level distributions obtained from the matrix-analytic solution (MA) and the simulation (SIM).

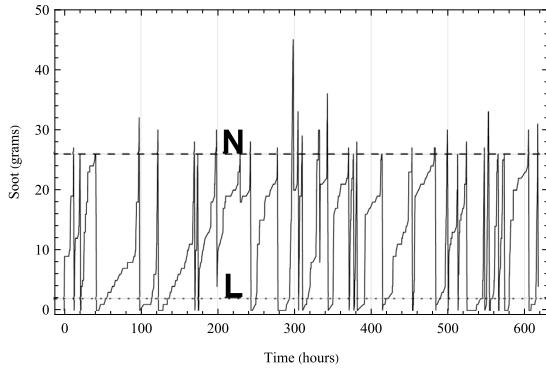
In Fig. 1(a), the soot level increases according to *MAP* process depending on the driving mode monotonically at a varying rate, representing the continuous capture of particulate matter from the exhaust stream, until reaching the regeneration threshold N (horizontal dashed line). Upon reaching this threshold, a regeneration cycle is initiated by generating a *PH* type distributed startup time, during which further soot accumulation remains possible. Once the system reaches the required thermal state for soot oxidation, the *PH*-type (Erlang) distributed service (burning) process commences, resulting in a rapid decrease of the soot level. A successful regeneration resets the soot level to a value below the success threshold L (horizontal dotted line), creating the characteristic sawtooth pattern. The figure also illustrates a service interruption: during the burning process, an interruption occurs in *PH*-type distributed time before the soot level drops below the success threshold L . After the interruption, soot accumulation resumes while the regeneration period continues, demonstrating that the regeneration cycle is not complete until the soot level falls below L . The visit of the reset state K is a rare event therefore it is not included in this realization. In Fig. 1(b), the two stationary state distributions exhibit excellent agreement across the entire support $0 \leq n \leq K$. The results were obtained for the parameters of *MAP* with correlation coefficient equal to $\rho^{(\phi_e)} = 0.4$ and Cases 1(a)–3(a) of infinitesimal generator for environment process. The comparison of stationary state distributions was performed using the Total Variation Distance (TVD),

$$TVD = \frac{1}{2} \sum_{n=0}^K |\pi_n^{(MA)} - \pi_n^{(SIM)}|$$

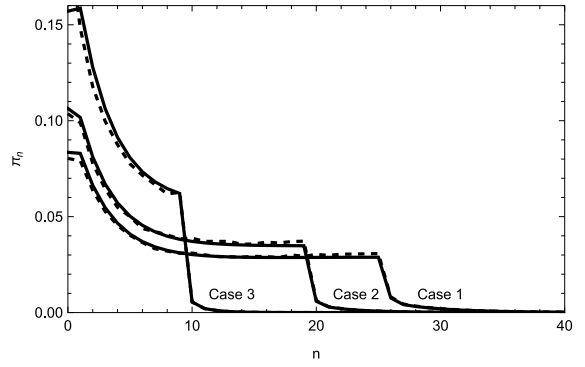
which measures the fraction of differing probability mass. The TVD in all our experiments was below 0.05 which confirm that the matrix-analytic solution is correctly implemented and that the simulation provides a consistent estimate of the stationary behavior.

4. The length of a regeneration period

The regeneration period R is defined as the time interval that begins when a customer enters the system in one of the levels $l(N-1)$ where the server is idle. This interval ends when the system visits again a state with an idle server in level $l(0)$ due to service completion or reset service completion or in levels $l(n)$, $1 \leq n \leq L$, due to service interruption.



(a) Simulated soot accumulation and regeneration process in the DPF with regeneration level N and success level L .



(b) Comparison of stationary soot level distributions obtained by the matrix-analytic method (MA) and discrete-event simulation (SIM).

Fig. 1. Results of the discrete-event simulation.

Denote by R_x the first-passage time to states $y \in l(0) \cup \bigcup_{n=1}^L l(n, 0)$ given that the initial state is $x \in \bigcup_{n=1}^L l(n, 1) \cup \bigcup_{n=L+1}^{K-1} l(n)$ and by

$$r_x(s) = \mathbb{E}[e^{-sR_x}], \text{Re}(s) \geq 0.$$

The following column vectors contain the Laplace–Stieltjes transforms, partitioned according to levels $l(0)$, $l(n, 0)$, $l(n, 1)$, $1 \leq n \leq K-1$ and $l(K)$,

$$\begin{aligned} r_0(s) &= (r_{(0,0,1,1)}(s), \dots, r_{(0,0,m_e,m_a)}(s))^T, \\ r_n(s) &= (r_{n,0}(s), r_{n,1}(s))^T \\ &= \begin{cases} (r_{(n,0,1,1)}(s), \dots, r_{(n,0,m_e,m_a)}(s), r_{(n,1,1,1)}(s), \dots, r_{(n,1,m_s,m_i)}(s))^T, & 1 \leq n \leq N-1 \\ (r_{(n,0,1,1)}(s), \dots, r_{(n,0,m_e,m_a)}(s), r_{(n,1,1,1)}(s), \dots, r_{(n,1,m_s,m_i)}(s))^T, & N \leq n \leq K-1, \end{cases} \\ r_K(s) &= (r_{(K,0,1)}(s), \dots, r_{(K,0,m_r)}(s))^T. \end{aligned}$$

Moreover, since $r_y(s) = 1$, here $r_0(s) = e(l_0)$ and $r_{n,0}(s) = e(l_0)$ for $1 \leq n \leq L$.

Remark 1. According to our definition, the regeneration period R includes visits to states at level $l(K)$, where the DPF is repaired at the auto repair station. If this time must be excluded from the period, it is necessary to set $r_K(s) = 0_{m_r \times 1}$.

Theorem 1. The Laplace–Stieltjes transforms

$$r^{(N)}(s) = (r_{1,1}(s), r_{2,1}(s), \dots, r_{L,1}(s), r_{L+1}(s), \dots, r_{K-1}(s), r_K(s))^T$$

satisfy the following system

$$T_R^{(N)}(s)r^{(N)}(s) = f^{(N)}, \quad (4)$$

where

$$\begin{aligned} T_R^{(N)}(s) &= \bar{Q}^{(N)} - sI(l_0(N-L-1) + l_1(K-1) + l_2(K-N) + m_r), \\ f^{(N)} &= -(A^0 \otimes e(m_s) \otimes e(L), 0_{(l_0(N-L-1)+l_1(K-L-1)+l_2(K-N)) \times 1}, B^0)^T \\ &\quad - (e(m_i) \otimes M^0, 0_{(l_0(N-2)+l_1(K-2)+l_2(K-N)+m_r) \times 1})^T. \end{aligned}$$

The matrix $\bar{Q}^{(N)}$ is obtained from $Q^{(N)}$ by removing the first (block) column, the first (block) row, and by replacing super-diagonal matrices up to the level $l(L)$ by blocks 0. In modified matrix $\bar{Q}_{1,1} = A \oplus M$, $\bar{Q}_{2,1} = I(m_i) \otimes (M^0 \otimes \mu^\top)$, and $\hat{Q}_{2,1} = \begin{pmatrix} 0_{l_0 \times l_1} \\ I(m_i) \otimes (M^0 \otimes \mu^\top) \end{pmatrix}$,

$$\bar{Q}^{(N)} = \begin{pmatrix} l(1) & l(2) & l(3) & \cdots & l(L) & l(L+1) & l(L+2) & \cdots & l(N-1) & l(N) & l(N+1) & \cdots & l(K) \\ \bar{Q}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \bar{Q}_{2,1} & \bar{Q}_{1,1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \bar{Q}_{2,1} & \bar{Q}_{1,1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & \bar{Q}_{2,1} & \bar{Q}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \bar{Q}_{2,1} & \bar{Q}_{1,1} & \bar{Q}_{0,2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \bar{Q}_{2,1} & \bar{Q}_{1,1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \bar{Q}_{2,1} & \bar{Q}_{1,1} & \bar{Q}_{0,3} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \bar{Q}_{2,2} & \bar{Q}_{1,2} & \bar{Q}_{0,4} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \bar{Q}_{2,3} & \bar{Q}_{1,2} & \bar{Q}_{0,4} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{Q}_{2,3} & \bar{Q}_{1,2} & \bar{Q}_{0,5} \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B \end{pmatrix} \begin{matrix} l(1) \\ l(2) \\ l(3) \\ \vdots \\ l(L) \\ l(L+1) \\ l(L+2) \\ \vdots \\ l(N-1) \\ l(N) \\ l(N+1) \\ \vdots \\ l(K-1) \\ l(K) \end{matrix}, \quad (5)$$

Proof. By employing the first step analysis, we get

$$\begin{aligned} r_{(1,1,\phi_s,\phi_i)}(s)(s - M_{\phi_s,\phi_s} - A_{\phi_i,\phi_i}) &= \sum_{j \neq \phi_s} M_{\phi_s,j} r_{(n,1,j,\phi_i)}(s) + \sum_{j \neq \phi_i} A_{\phi_i,j} r_{(n,1,\phi_s,k)}(s) + M_{\phi_s}^0 + A_{\phi_i}^0, \\ r_{(n,1,\phi_s,\phi_i)}(s)(s - M_{\phi_s,\phi_s} - A_{\phi_i,\phi_i}) &= \sum_{j \neq \phi_s} M_{\phi_s,j} r_{(n,1,j,\phi_i)}(s) + \sum_{j \neq \phi_i} A_{\phi_i,j} r_{(n,1,\phi_s,k)}(s) \\ &+ M_{\phi_s}^0 \sum_{j=1}^{m_s} \mu_j r_{(n-1,1,j,\phi_i)}(s) + A_{\phi_i}^0, \quad 2 \leq n \leq L, \\ r_{(n,0,\phi_e,\phi_a)}(s)(s - H_{\phi_e,\phi_e} - D_{0,\phi_a,\phi_a}^{(\phi_e)}) &= \sum_{j \neq \phi_e} H_{\phi_e,j} r_{(n,0,j,\phi_a)}(s) + \sum_{j \neq \phi_a} D_{0,\phi_a,j}^{(\phi_e)} r_{(n,0,\phi_e,j)}(s) \\ &+ \sum_j D_{1,\phi_a,j}^{(\phi_e)} r_{(n+1,0,\phi_e,j)}(s), \quad L+1 \leq n \leq N-1, \\ r_{(n,0,\phi_e,\phi_a,\phi_u)}(s)(s - H_{\phi_e,\phi_e} - D_{0,\phi_a,\phi_a}^{(\phi_e)} - \Gamma_{\phi_u,\phi_u}) &= \sum_{j \neq \phi_e} H_{\phi_e,j} r_{(n,0,j,\phi_a,\phi_u)}(s) + \sum_{j \neq \phi_a} D_{0,\phi_a,j}^{(\phi_e)} r_{(n,0,\phi_e,j,\phi_u)}(s) \\ &+ \sum_{j \neq \phi_u} \Gamma_{\phi_u,j} r_{(n,0,\phi_e,\phi_a,j)}(s) + \sum_j D_{1,\phi_a,j}^{(\phi_e)} r_{(n+1,0,\phi_e,j,\phi_u)}(s) + \Gamma_{\phi_u}^0 \sum_j \sum_k \mu_j \alpha_k r_{(n,1,j,k)}(s), \quad N \leq n \leq K-2, \\ r_{(n,1,\phi_s,\phi_i)}(s)(s - M_{\phi_s,\phi_s} - A_{\phi_i,\phi_i}) &= \sum_{j \neq \phi_s} M_{\phi_s,j} r_{(n,1,j,\phi_i)}(s) + \sum_{j \neq \phi_i} A_{\phi_i,j} r_{(n,1,\phi_s,j)}(s) \\ &+ M_{\phi_s}^0 \sum_j \mu_j r_{(n-1,1,j,\phi_i)}(s) + A_{\phi_i}^0 \sum_u \sum_j \pi_{a,j}^{(u)} \pi_{e,u} r_{(n,0,u,j)}(s), \quad L+1 \leq n \leq N-1, \\ r_{(n,1,\phi_s,\phi_i)}(s)(s - M_{\phi_s,\phi_s} - A_{\phi_i,\phi_i}) &= \sum_{j \neq \phi_s} M_{\phi_s,j} r_{(n,1,j,\phi_i)}(s) + \sum_{j \neq \phi_i} A_{\phi_i,j} r_{(n,1,\phi_s,j)}(s) \\ &+ M_{\phi_s}^0 \sum_j \mu_j r_{(n-1,1,j,\phi_i)}(s) + A_{\phi_i}^0 \sum_u \sum_j \sum_k \pi_{a,j}^{(u)} \pi_{e,u} \gamma_k r_{(n,0,u,j,k)}(s), \quad N \leq n \leq K-1, \\ r_{(K-1,0,\phi_e,\phi_a,\phi_u)}(s)(s - H_{\phi_e,\phi_e} - D_{0,\phi_a,\phi_a}^{(\phi_e)} - \Gamma_{\phi_u,\phi_u}) &= \sum_{j \neq \phi_e} H_{\phi_e,j} r_{(K-1,0,j,\phi_a,\phi_u)}(s) \\ &+ \sum_{j \neq \phi_a} D_{0,\phi_a,j}^{(\phi_e)} r_{(K-1,0,\phi_e,j,\phi_u)}(s) + \sum_{j \neq \phi_u} \Gamma_{\phi_u,j} r_{(K-1,0,\phi_e,\phi_a,j)}(s) + \sum_j \sum_k D_{1,\phi_a,j}^{(\phi_e)} \beta_k r_{(K,0,k)}(s) \\ &+ \Gamma_{\phi_u}^0 \sum_j \sum_k \mu_j \alpha_k r_{(K-1,1,j,k)}(s), \\ r_{(K,0,\phi_r)}(s)(s - B_{\phi_r,\phi_r}) &= \sum_{j \neq \phi_r} B_{\phi_r,j} r_{(K,0,j)}(s) + B_{\phi_r}^0. \end{aligned}$$

By expressing the components of (6) in matrix-vector form, we get the system

$$\begin{aligned} 1. (A \oplus M - sI(l_1))r_{1,1}(s) + A^0 \otimes e(m_s) + e(m_i) \otimes M^0 &= 0, \\ 2. (A \oplus M - sI(l_1))r_{n,1}(s) + I(m_i) \otimes (M^0 \otimes \mu^\top) r_{n-1,1}(s) + A^0 \otimes e(m_s) &= 0, \quad 2 \leq n \leq L, \\ 3. (Q_{1,1} - sI(l_0 + l_1))r_{L+1}(s) + \begin{pmatrix} 0_{l_0 \times l_1} \\ I(m_i) \otimes (M^0 \otimes \mu^\top) \end{pmatrix} r_{L,1}(s) + Q_{0,2} r_{L+2}(s) &= 0, \end{aligned} \quad (7)$$

$$\begin{aligned}
4. & (\mathcal{Q}_{1,1} - sI(l_0 + l_1))r_n(s) + \mathcal{Q}_{2,1}r_{n-1}(s) + \mathcal{Q}_{0,2}r_{n+1}(s) = 0, \quad L+2 \leq n \leq N-2, \\
5. & (\mathcal{Q}_{1,1} - sI(l_0 + l_1))r_{N-1}(s) + \mathcal{Q}_{2,1}r_{N-2}(s) + \mathcal{Q}_{0,3}r_N(s) = 0, \\
6. & (\mathcal{Q}_{1,2} - sI(l_0 + l_2))r_N(s) + \mathcal{Q}_{2,2}r_{N-1}(s) + \mathcal{Q}_{0,4}r_{N+1}(s) = 0, \\
7. & (\mathcal{Q}_{1,2} - sI(l_0 + l_2))r_n(s) + \mathcal{Q}_{2,3}r_{n-1}(s) + \mathcal{Q}_{0,4}r_{n+1}(s) = 0, \quad N+1 \leq n \leq K-2, \\
8. & (\mathcal{Q}_{1,2} - sI(l_0 + l_2))r_{K-1}(s) + \mathcal{Q}_{2,3}r_{K-2}(s) + \mathcal{Q}_{0,5}r_K(s) = 0, \\
9. & (B - sI(m_r))r_K(s) + B^0 = 0,
\end{aligned}$$

If we denote by $f_1 = -A^0 \otimes e(m_s) - e(m_i) \otimes M^0$, $f_n = -A^0 \otimes e(m_s)$, $2 \leq n \leq L$, $f_n = 0_{l_0+l_1 \times 1}$, $L+1 \leq n \leq N-1$, $f_n = 0_{l_0+l_2 \times 1}$, $N \leq n \leq K-1$ and $f_K = -B^0$, then we get the column vector $f^{(N)} = (f_1, f_2, \dots, f_K)^T$ defined in the statement. Then the system (7) obviously implies the required expression (4). $\square$

Corollary 1. The Laplace transforms $r_{n,1}(s)$, $1 \leq n \leq L$, of the vector $r^{(N)}(s)$ satisfy the relations

$$r_{1,1}(s) = -(\bar{\mathcal{Q}}_{1,1} - sI(l_1))^{-1} f_1, \quad (8)$$

$$r_{n,1}(s) = C_1^{n-1}(s)r_{1,1}(s) + I(l_1) - C_1^{n-1}(s)(I(l_1) - C_1(s))^{-1}v(s), \quad 2 \leq n \leq L, \quad (9)$$

where $C_1(s) = -(\bar{\mathcal{Q}}_{1,1} - sI(l_1))^{-1}\bar{\mathcal{Q}}_{2,1}$, $v(s) = (\bar{\mathcal{Q}}_{1,1} - sI(l_1))^{-1}f_2$. The Laplace transforms $r_n(s)$, $L+1 \leq n \leq K$, of the vector $r^{(N)}(s)$ satisfy the relations

$$\begin{aligned}
r_n(s) &= \prod_{j=n}^{K-1} P_j(s)r_K(s) + \sum_{i=n}^{K-1} \prod_{j=i}^{K-1} P_j(s)q_i(s), \quad L+1 \leq n \leq K-1, \\
r_K(s) &= -(B - sI(m_r))^{-1}B^0,
\end{aligned} \quad (10)$$

where

$$\begin{aligned}
P_{L+1}(s) &= -(\mathcal{Q}_{1,1} - sI(l_0 + l_1))^{-1}\mathcal{Q}_{0,2}, \quad q_{L+1}(s) = -(\mathcal{Q}_{1,1} - sI(l_0 + l_1))^{-1}\hat{\mathcal{Q}}_{2,1}r_{L,1}(s), \\
P_n(s) &= -(\mathcal{Q}_{1,1} - sI(l_0 + l_1) + \mathcal{Q}_{2,1}P_{n-1}(s))^{-1}\mathcal{Q}_{0,2}, \\
q_n(s) &= -(\mathcal{Q}_{1,1} - sI(l_0 + l_1) + \mathcal{Q}_{2,1}P_{n-1}(s))^{-1}\mathcal{Q}_{2,1}q_{n-1}(s), \quad L+2 \leq n \leq N-2, \\
P_{N-1}(s) &= -(\mathcal{Q}_{1,1} - sI(l_0 + l_1) + \mathcal{Q}_{2,1}P_{N-2}(s))^{-1}\mathcal{Q}_{0,3}, \\
q_{N-1}(s) &= -(\mathcal{Q}_{1,1} - sI(l_0 + l_1) + \mathcal{Q}_{2,1}P_{N-2}(s))^{-1}\mathcal{Q}_{2,1}q_{N-2}(s), \\
P_N(s) &= -(\mathcal{Q}_{1,2} - sI(l_0 + l_2) + \mathcal{Q}_{2,2}P_{N-1}(s))^{-1}\mathcal{Q}_{0,4}, \\
q_N(s) &= -(\mathcal{Q}_{1,2} - sI(l_0 + l_2) + \mathcal{Q}_{2,2}P_{N-1}(s))^{-1}\mathcal{Q}_{2,2}q_{N-1}(s), \\
P_n(s) &= -(\mathcal{Q}_{1,2} - sI(l_0 + l_2) + \mathcal{Q}_{2,3}P_{n-1}(s))^{-1}\mathcal{Q}_{0,4}, \\
q_n(s) &= -(\mathcal{Q}_{1,2} - sI(l_0 + l_2) + \mathcal{Q}_{2,3}P_{n-1}(s))^{-1}\mathcal{Q}_{2,3}q_{n-1}(s), \quad N+1 \leq n \leq K-2, \\
P_{K-1}(s) &= -(\mathcal{Q}_{1,2} - sI(l_0 + l_2) + \mathcal{Q}_{2,3}P_{K-2}(s))^{-1}\mathcal{Q}_{0,5}, \\
q_{K-1}(s) &= -(\mathcal{Q}_{1,2} - sI(l_0 + l_2) + \mathcal{Q}_{2,3}P_{K-2}(s))^{-1}\mathcal{Q}_{2,3}q_{K-2}(s).
\end{aligned} \quad (11)$$

Proof. The relations (8) and (9) can be derived directly from equations 1 and 2 of the system (7). Indeed,

$$\begin{aligned}
r_{1,1}(s) &= -(\bar{\mathcal{Q}}_{1,1} - sI_1)^{-1}f_1, \\
r_{n,1}(s) &= C_1(s)r_{n-1,1}(s) + v(s), \quad 2 \leq n \leq L.
\end{aligned}$$

Then we get

$$\begin{aligned}
r_{n,1}(s) &= C_1^{n-1}(s)r_{1,1}(s) + \sum_{j=0}^{n-2} C_1^j(s)v(s) \\
&= C_1^{n-1}(s)r_{1,1}(s) + (I_1 - C_1^{n-1}(s))(I_1 - C_1(s))^{-1}v(s).
\end{aligned}$$

Consider now equation 3 of the system (7). It can be rewritten in the form

$$\begin{aligned}
r_{L+1}(s) &= -(\mathcal{Q}_{1,1} - sI(l_0 + l_1))^{-1}\mathcal{Q}_{0,2}r_{L+2}(s) - (\mathcal{Q}_{1,1} - sI(l_0 + l_1))^{-1}\hat{\mathcal{Q}}_{2,1}r_{L,1}(s) \\
&= P_{L+1}(s)r_{L+2}(s) + q_L(s).
\end{aligned}$$

where the boundary condition for $r_{L,1}(s)$ can be calculated by (9). The further usage of back substitution for $L+2 \leq n \leq K-1$ leads to recursive relations of the form

$$r_n(s) = P_n(s)r_{n+1}(s) + q_n(s), \quad (12)$$

where matrices $P_n(s)$ and vectors $q_n(s)$ are defined by (11) dependent on the threshold policy. For $n = K-1$ the boundary condition $r_K(s)$ can be calculated by equation 9 of the system (7),

$$r_K(s) = -(B - sI(m_r))^{-1}B^0.$$

Finally, by means of recursive substitution in (12), we obtain the expression (10). $\square$

We now examine its unconditional version, which has the Laplace–Stieltjes transform

$$R^{(N)}(s) = p^{(N)} \cdot r^{(N)}(s) = \frac{\pi(N-1)Q_{0,3}}{\pi(N-1)Q_{0,3}e^{(l_2+l_1)}} r_N(s). \quad (13)$$

Here $p^{(N)}$ is the row vector of dimension $m_e m_a m_u (K-N) + m_s m_i (K-1) + m_r$ given by

$$p^{(N)} = \frac{(0_{1 \times m_s m_i (N-1)}, \pi(N-1)Q_{0,3}, 0_{1 \times (l_2+l_1)(K-N-1)+m_r})}{\pi(N-1)Q_{0,3}e^{(l_2+l_1)}}.$$

This vector is required, since PASTA does not hold, and we have taken into account the probability that an arriving customer according to the MAP sees the system in state at the level $l(N-1)$.

Remark 2. The calculation of the Laplace–Stieltjes transform $r_N(s)$ in symbolic form is quite difficult if dimensions of the MAP and PH processes are high; therefore, we turn our attention to the first k conditional raw moments $m_x(k) = \mathbb{E}[R_x^k]$ for $k \geq 0$ which can be used to match the suitable parametric probability density function. Using the formula of Leibnitz for the derivative of a product, after differentiation of the equality (4) we obtain

$$T_R^{(N)}(s) \frac{d^k}{ds^k} r^{(N)}(s) - k \frac{d^{k-1}}{ds^{k-1}} r^{(N)}(s) = 0.$$

Denote now by

$$m^{(N)}(k) = (m_{1,1}(k), m_{2,1}(k), \dots, m_{L,1}(k), m_{L+1}(k), \dots, m_{K-1}(k), m_K(k))^T$$

the macro-vector that contains the k th conditional moments partitioned with respect to the levels $l(n, 1)$, $1 \leq n \leq L$, $l(n, 0)$, $l(n, 1)$, $L+1 \leq n \leq K-1$ and $l(K)$,

$$\begin{aligned} m_{n,1}(k) &= (m_{(n,1,1,1)}(k), \dots, m_{(n,1,m_s,m_i)}(k))^T, \quad 1 \leq n \leq L, \\ m_n(k) &= (m_{n,0}(k), m_{n,1}(k))^T \\ &= (m_{(n,0,1,1)}(k), \dots, m_{(n,0,m_e,m_a)}(k), m_{(n,1,1,1)}(k), \dots, m_{(n,1,m_s,m_i)}(k))^T, \quad L+1 \leq n \leq N-1, \\ m_n(k) &= (m_{n,0}(k), m_{n,1}(k))^T \\ &= (m_{(n,0,1,1,1)}(k), \dots, m_{(n,0,m_e,m_a,m_u)}(k), m_{(n,1,1,1)}(k), \dots, m_{(n,1,m_s,m_i)}(k))^T, \quad N \leq n \leq K-1, \\ m_K(k) &= (m_{(K,0,1)}(k), \dots, m_{(K,0,m_r)}(k))^T. \end{aligned}$$

Since $m^{(N)}(k) = (-1)^k \frac{d^k}{ds^k} r^{(N)}(s) \Big|_{s=0}$ and $T_R^{(N)}(0) = \bar{Q}^{(N)}$ we see that

$$\begin{aligned} \bar{Q}^{(N)} m^{(N)}(k) &= -k m^{(N)}(k-1), \quad k \geq 1, \\ m^{(N)}(0) &= e^{(l_0(N-L-1) + l_1(K-1) + l_2(K-N) + m_r)}. \end{aligned} \quad (14)$$

To calculate moments, it is convenient to use the following recursive relations obtained by analogy with the corresponding Laplace–Stieltjes transforms.

Corollary 2. The components $m_{n,1}(k)$, $1 \leq n \leq L$ of the vector $m^{(N)}(k)$ for the conditional moments satisfy the relations

$$m_{1,1}(k) = k!(-\bar{Q}_{1,1}^{-1})^k m_{1,1}(0), \quad (15)$$

$$m_{n,1}(k) = \sum_{j=0}^{k-1} \frac{k!}{j!} (-\bar{Q}_{1,1}^{-1})^{k-j+1} \bar{Q}_{2,1} m_{n-1,1}(j) + m_{1,1}(k), \quad 2 \leq n \leq L. \quad (16)$$

The sub-vectors $m_n(k)$, $L+1 \leq n \leq K-1$ of the vector $m^{(N)}(k)$ satisfy the relations

$$\begin{aligned} m_n(k) &= \prod_{j=n}^{K-1} P_j(0) m_K(k) + \sum_{i=n}^{K-1} \prod_{j=n}^{i-1} P_j(0) v_i, \\ m_K(k) &= k!(B^{-1})^k m_K(0), \end{aligned} \quad (17)$$

where $P_j(0)$, $L+1 \leq j \leq K-1$ and $q_i(0)$, $L+2 \leq i \leq K-1$ are defined by (11) for $s=0$, and

$$\begin{aligned} v_{L+1} &= -\bar{Q}_{1,1}^{-1}(\bar{Q}_{2,1} m_{L,1}(0) + k m_{L+1}(0)), \\ v_n &= -(Q_{1,1} + Q_{2,1} P_{n-1}(0))^{-1} (Q_{2,1} v_{n-1} + k m_n(0)), \quad L+2 \leq n \leq N-1, \\ v_N &= -(Q_{1,2} + Q_{2,2} P_{N-1}(0))^{-1} (Q_{2,2} v_{N-1} + k m_N(0)), \\ v_n &= -(Q_{1,2} + Q_{2,3} P_{n-1}(0))^{-1} (Q_{2,3} v_{n-1} + k m_n(0)), \quad N+1 \leq n \leq K-1. \end{aligned} \quad (18)$$

It can be noticed that the unconditional moments of the regeneration period are defined by

$$M_R(k) = \mathbb{E}[R^k] = p^{(N)} \cdot m^{(N)}(k) = \frac{\pi(N-1)Q_{0.3}}{\pi(N-1)Q_{0.3}e(l_2+l_1)} m_N(k), \quad k \geq 0. \quad (19)$$

Having derived closed-form expressions for the first five moments of the regeneration period using the matrix-analytic framework in accordance with Remark 2, we next address the problem of estimating the parameters of the proposed lognormal-gamma mixture distribution. The probability density function of the mixture is given by:

$$f(t; \Theta) = w \cdot f_{\text{LN}}(t; \mu, \sigma) + (1-w) \cdot f_{\text{Gamma}}(t; a, b), \quad t > 0,$$

where $w \in [0, 1]$ is the mixing weight, f_{LN} denotes the lognormal density with log-mean μ and log-standard deviation σ , and f_{Gamma} denotes the gamma density with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$. The full parameter vector is $\Theta = \{w, \mu, \sigma, a, b\}$. The moments of the mixture are given by:

$$m_k = w \cdot \exp\left(k\mu + \frac{k^2\sigma^2}{2}\right) + (1-w) \cdot \frac{\Gamma(a+k)}{\Gamma(a)b^k}, \quad k = 1, \dots, 5.$$

A natural approach to parameter estimation is the method of moments, where the first five theoretical moments are equated to their empirical counterparts obtained from the matrix-analytic solution. This yields a system of five nonlinear equations:

$$m_k(\Theta) = M_R(k), \quad k = 1, \dots, 5,$$

which can be solved numerically using constrained optimization algorithms (e.g., the Levenberg–Marquardt method). However, we observed that the method of moments does not yield reliable parameter estimates in all parameter regimes. This limitation arises from several factors. First, the system of moment equations may be ill-conditioned when the higher-order moments assume extremely large values, leading to numerical instability. Second, the mapping from the moment space to the parameter space of the mixture is not one-to-one; different parameter combinations can produce similar moment sequences, resulting in non-unique solutions. Third, the moment equations are highly nonlinear, and the optimization algorithm may converge to local minima rather than the global optimum, particularly when the mixing weight w is close to 0 or 1.

To address these limitations, we complement the method of moments with maximum likelihood estimation (MLE) using simulated samples. Specifically, we generate a large synthetic sample of regeneration periods from the theoretical distribution by sampling from the queueing model using discrete-event simulation. For each sample of size n , the log-likelihood function is:

$$\ell(\Theta) = \sum_{i=1}^n \log f(t_i; \Theta),$$

where $f(t_i; \Theta)$ is the mixture density evaluated at the i th simulated observation. The maximum likelihood estimates are obtained by maximizing $\ell(\Theta)$ with respect to Θ :

$$\hat{\Theta}_{\text{MLE}} = \arg \max_{\Theta} \sum_{i=1}^n \log f(t_i; \Theta).$$

The maximization is performed using the Expectation–Maximization (EM) algorithm, which is particularly well-suited for mixture models. The EM algorithm is initialized using the method of moments estimates and iterated until convergence. This hybrid approach combines the method of moments with the statistical efficiency of maximum likelihood estimation. The method of moments provides a robust initialization, while the EM algorithm refines the estimates to achieve the maximum likelihood solution. In cases where the method of moments fails to produce physically plausible parameter values (e.g., $\sigma > 0$, $a > 0$, $0 \leq w \leq 1$), we initialize the EM algorithm using multiple random starting points and select the solution that maximizes the likelihood. This strategy ensures that the final parameter estimates are both statistically efficient and numerically stable.

The validity of the fitted mixture distribution is subsequently assessed using the Cramér–von Mises goodness-of-fit test, comparing the theoretical CDF with the empirical CDF obtained from the discrete-event simulation. The results presented in Example 2 confirm that the lognormal-gamma mixture provides an accurate and statistically consistent representation of the regeneration period distribution in a wide range of parameter configurations.

Example 2. To demonstrate the accuracy of the lognormal-gamma mixture fitting, we compare the theoretical CDFs obtained from the mixture with the empirical CDFs from discrete-event simulation for four representative parameter configurations: Cases 1(a), 1(b), 3(a), and 3(b). The mixture parameters are estimated by matching the first five moments derived from the matrix-analytic solution and by maximum likelihood method. Fig. 2 displays the overlaid CDFs for the four cases with the following fitted parameters:

- Case 1(a) $\Theta = \{0.829, 0.127, 0.402, 1.121, 14.269\}$
- Case 1(b) $\Theta = \{0.809, 0.645, 0.655, 1.765, 13.664\}$
- Case 3(a) $\Theta = \{0.735, 0.089, 0.255, 1.076, 13.308\}$
- Case 3(b) $\Theta = \{0.927, 0.229, 0.816, 3.219, 18.837\}$

In all cases, the fitted mixture distributions exhibit excellent agreement with the empirical CDFs. The p -values of Cramér–von Mises test statistic range from 0.091 to 0.158 are above the 5% significance level, confirming that the null hypothesis is not rejected.

These results validate the lognormal-gamma mixture as a parsimonious and accurate representation of the regeneration period distribution. The mixture successfully captures the key distributional features, including the central mass and the heavy tail, enabling the use of analytical moment-based methods while maintaining fidelity to the underlying stochastic process.

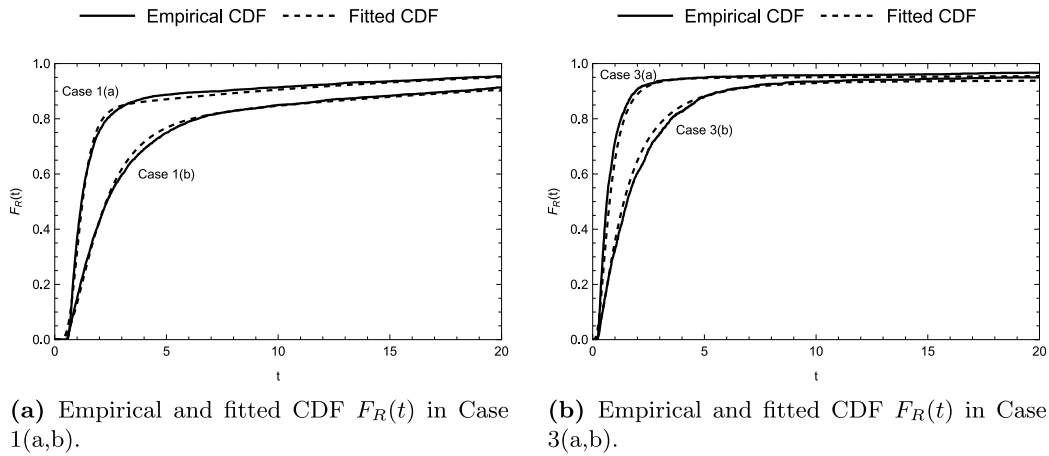
Fig. 2. Empirical and fitted cumulative distribution functions $F_R(t)$.

Table 1

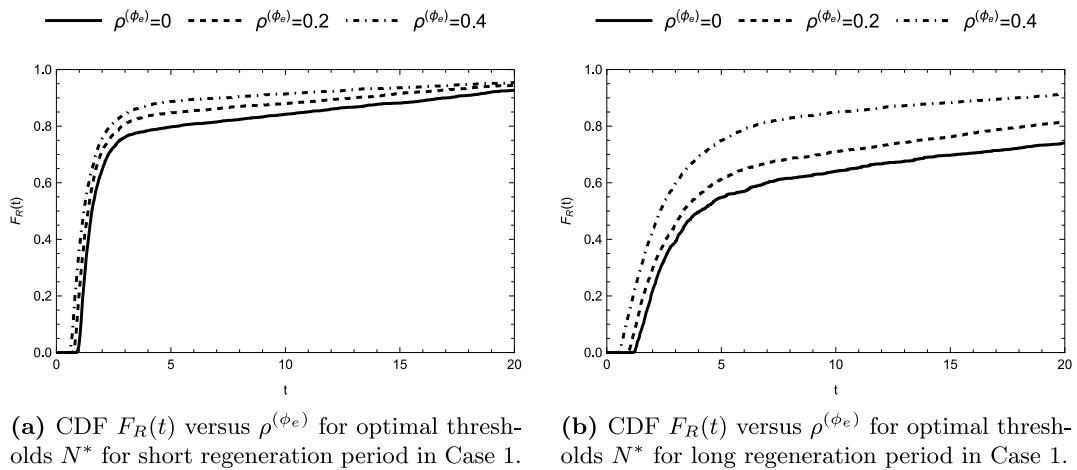
The real and estimated moments versus correlation coefficient $\rho^{(\phi_e)}$ in Case 1.

	$\rho^{(\phi_e)}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$M_R(k)$	0.0	5.593	102.844	3255.058	$1.327 \cdot 10^5$	$6.589 \cdot 10^6$
$\bar{M}_R(k)$		6.141	158.131	7700.32	$5.984 \cdot 10^5$	$73.435 \cdot 10^6$
$M_R(k)$	0.2	4.167	75.288	2695.837	$1.280 \cdot 10^5$	$7.401 \cdot 10^6$
$\bar{M}_R(k)$		4.556	112.307	5054.735	$2.976 \cdot 10^5$	$21.473 \cdot 10^6$
$M_R(k)$	0.4	3.402	72.595	3442.465	$2.171 \cdot 10^5$	$1.655 \cdot 10^7$
$\bar{M}_R(k)$		3.755	84.244	3688.605	$2.168 \cdot 10^5$	$1.584 \cdot 10^7$

Table 2

Estimated coefficients Θ of the mixture distribution for optimal thresholds in Case 1 (a) and (b).

$\rho^{(\phi_e)}$	N^* (a)	Θ (a)	N^* (b)	Θ (b)
0.0	40	{0.735,0.089,0.235,1.076,13.308}	50	{0.576,0.909,0.440,1.752,18.091}
0.2	33	{0.824,0.277,0.358,1.465,13.199}	40	{0.612,0.756,0.452,1.502,16.497}
0.4	26	{0.829,0.127,0.402,1.121,14.269}	23	{0.809,0.645,0.655,1.765,13.664}

Fig. 3. Cumulative distribution functions $F_R(t)$ versus correlation coefficient in Case 1.

Example 3. To investigate the impact of correlation and driving mode on the regeneration period, we analyze the behavior of the proposed queueing model under different correlation structures and operating conditions. The correlation in the soot accumulation process is governed by the underlying Markov chain of the *MAP*, which modulates the arrival rate based on the current phase.

Table 3The real and estimated moments versus correlation coefficient $\rho^{(\phi_e)}$ in Case 2.

	$\rho^{(\phi_e)}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$M_R(k)$	0.0	4.788	103.707	4156.006	$2.104 \cdot 10^5$	$1.273 \cdot 10^7$
$\bar{M}_R(k)$		4.643	122.429	5523.704	$3.206 \cdot 10^5$	$2.265 \cdot 10^7$
$M_R(k)$	0.2	3.958	98.205	5106.597	$3.455 \cdot 10^5$	$2.814 \cdot 10^7$
$\bar{M}_R(k)$		4.151	148.300	10497.214	$10.029 \cdot 10^5$	$11.998 \cdot 10^7$
$M_R(k)$	0.4	3.386	117.030	9208.004	$9.417 \cdot 10^5$	$11.557 \cdot 10^7$
$\bar{M}_R(k)$		3.736	183.056	12594.519	$11.421 \cdot 10^5$	$12.806 \cdot 10^7$

Table 4Estimated coefficients Θ of the mixture distribution for optimal thresholds in Case 2 (a) and (b).

$\rho^{(\phi_e)}$	N^* (a)	Θ (a)	N^* (b)	Θ (b)
0.0	27	{0.787,0.117,0.386,1.828,11.566}	41	{0.646,0.755,0.521,1.711,24.608}
0.2	24	{0.868,0.102,0.443,0.974,24.049}	40	{0.659,0.808,0.470,1.498,27.078}
0.4	20	{0.856,-0.033,0.490,1.234,21.421}	30	{0.749,0.642,0.564,1.178,39.419}

Table 5The real and estimated moments versus correlation coefficient $\rho^{(\phi_e)}$ in Case 3.

	$\rho^{(\phi_e)}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$M_R(k)$	0.0	2.777	98.515	6327.747	$4.945 \cdot 10^5$	$4.523 \cdot 10^7$
$\bar{M}_R(k)$		3.070	108.710	7089.713	$6.045 \cdot 10^5$	$6.322 \cdot 10^7$
$M_R(k)$	0.2	2.407	74.317	4505.295	$3.285 \cdot 10^5$	$2.781 \cdot 10^7$
$\bar{M}_R(k)$		2.407	74.317	4409.362	$3.871 \cdot 10^5$	$4.500 \cdot 10^7$
$M_R(k)$	0.4	1.530	51.478	4668.828	$3.285 \cdot 10^5$	$2.781 \cdot 10^7$
$\bar{M}_R(k)$		1.530	51.478	3384.816	$3.312 \cdot 10^5$	$4.305 \cdot 10^7$

Table 6Estimated coefficients Θ of the mixture distribution for optimal thresholds in Case 3 (a) and (b).

$\rho^{(\phi_e)}$	N^* (a)	Θ (a)	N^* (b)	Θ (b)
0.0	20	{0.916,-0.059,0.482,1.417,19.305}	28	{0.747,0.653,0.575,2.245,23.726}
0.2	17	{0.898,-0.152,0.504,2.052,14.544}	27	{0.797,0.571,0.604,1.524,41.395}
0.4	10	{0.938,0.263,0.865,1.075,13.308}	20	{0.926,0.229,0.816,3.219,18.837}

The higher correlation corresponds to longer sojourn times in the modulating states, leading to persistent periods of high or low accumulation rates. The driving mode reflects the typical operating conditions of the vehicle, including urban, highway, and mixed driving patterns, each characterized by distinct soot emission profiles and accumulation dynamics.

We calculate the first five unconditional moments of the regeneration period R using (19) for Cases 1–3.

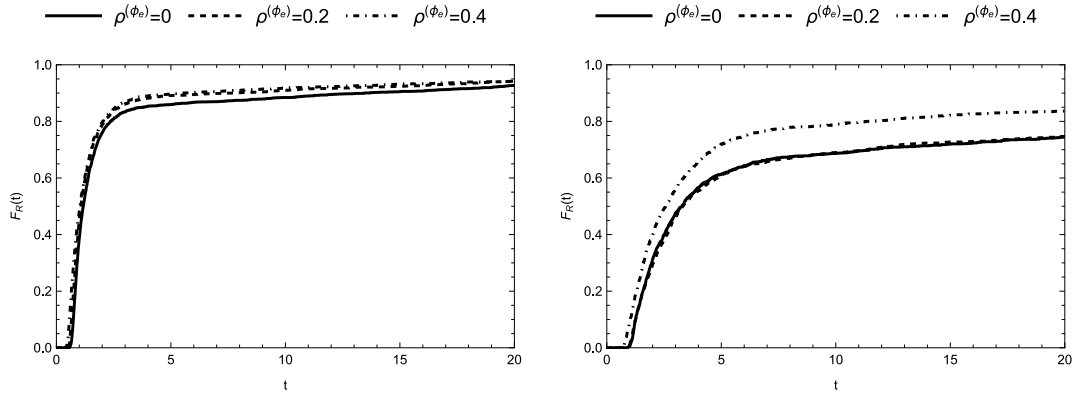
- Case 1: The vehicle is mostly operated in the city.

The results of Case 1 are summarized in Tables 1, 2 and in Fig. 3(a,b).

- Case 2: Equal probability of using slow, middle and fast speed driving. The results of this case are summarized in Tables 3, 4 and in Fig. 4(a,b).
- Case 3: The vehicle is mainly driven on highways. The results of this case are summarized in Tables 5, 6 and in Fig. 5(a,b).

Tables 1, 3, and 5 present a comparative analysis of the first five theoretical moments $M_R(k)$ of the regeneration period, calculated for the optimal thresholds N^* , along with the corresponding moment estimates $\bar{M}_R(k)$ obtained from the fitted lognormal-gamma mixture distribution. The mixture provides a good fit for the duration of the regeneration period in all driving patterns and correlation levels. The mean regeneration period $M_R(1)$ decreases with increasing correlation in all cases. This behavior is expected, as a higher correlation $\rho^{(\phi_e)}$ leads to an increased system load, which in turn shifts the optimal threshold N^* that minimizes the average cost function towards lower values. Consequently, regeneration is triggered more frequently, resulting in a shorter average regeneration period.

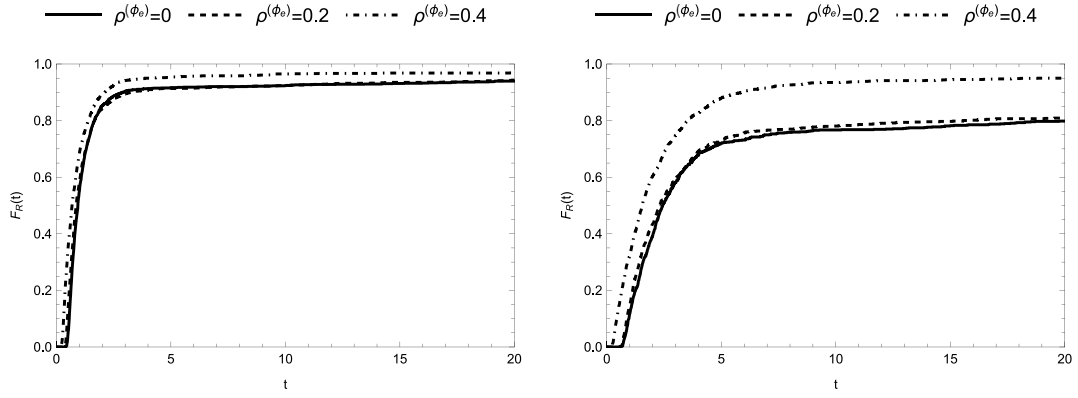
Tables 2, 4 and 6 summarize the estimated parameters Θ of the mixture distribution for the three cases considered. The parameter values of the mixture distributions indicate the presence of a heavy tail in the regeneration period distribution. Furthermore, the distributions exhibit increasingly heavy tails as the correlation coefficient and the soot accumulation rate decrease, as well as when the startup and service interruption parameters are such that the mean regeneration period lengthens. These findings are corroborated by the comparison of cumulative distribution functions $F_R(t)$ for different cases, shown in Figs. 3–5.



(a) CDF $F_R(t)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* for short regeneration period in Case 2.

(b) CDF $F_R(t)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* for long regeneration period in Case 2.

Fig. 4. Cumulative distribution functions $F_R(t)$ versus correlation coefficient in Case 2.



(a) CDF $F_R(t)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* for short regeneration period in Case 3.

(b) CDF $F_R(t)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* for long regeneration period in Case 3.

Fig. 5. Cumulative distribution functions $F_R(t)$ versus correlation coefficient in Case 3.

5. The number of interruptions during the regenerative period

This section analyzes the number of interruptions in the regeneration process until the successful completion of this period. Analysis of this characteristic complements research on the regeneration period, providing a discrete counterpart of the duration of the regeneration period. Next, we introduce some definitions and notations. Denote by I_x the number of interruptions during R_x and by $\xi_x(z) = \mathbb{E}[z^{I_x}]$, $|z| \leq 1$, $x \in E$ the conditional z -transform, where

$$\begin{aligned} \xi_0(z) &= (\xi_{(0,0,1,1)}(z), \dots, \xi_{(0,0,m_e,m_a)}(z))^T, \\ \xi_n(z) &= (\xi_{n,0}(z), \xi_{n,1}(z))^T \\ &= \begin{cases} (\xi_{(n,0,1,1)}(z), \dots, \xi_{(n,0,m_e,m_a)}(z), \xi_{(n,1,1,1)}(z), \dots, \xi_{(n,1,m_s,m_i)}(z))^T, & 1 \leq n \leq N-1 \\ (\xi_{(n,0,1,1)}(z), \dots, \xi_{(n,0,m_e,m_a,m_u)}(z), \xi_{(n,1,1,1)}(z), \dots, \xi_{(n,1,m_s,m_i)}(z))^T, & N \leq n \leq K-1, \end{cases} \\ \xi_K(z) &= (\xi_{K,0,1}(z), \dots, \xi_{K,0,m_r}(z))^T. \end{aligned}$$

Obviously, for the states of levels $l(0)$, $l(n,0)$ for $1 \leq n \leq L$ and $l(K)$ hold $\xi_0(z) = \xi_{n,1}(z) = e(l_0)$, $1 \leq n \leq L$ and $\xi_K(z) = e(m_r)$. Factorial moments are defined as $\tilde{m}_x(k) = \mathbb{E}[I_x \dots (I_x - k + 1)]$, $k \geq 0$, $\tilde{m}_x(0) = 1$. Denote by

$$\tilde{m}^{(N)}(k) = (\tilde{m}_{L+1}(k), \dots, \tilde{m}_{K-1}(k))^T$$

with

$$\tilde{m}_n(k) = (\tilde{m}_{n,0}(k), \tilde{m}_{n,1}(k))^T$$

$$\begin{aligned}
&= (\tilde{m}_{(n,0,1,1)}(k), \dots, \tilde{m}_{(n,0,m_e,m_a)}(k), \tilde{m}_{(n,1,1,1)}(k), \dots, \tilde{m}_{(n,1,m_s,m_i)}(k))^T, \quad L+1 \leq n \leq N-1, \\
\tilde{m}_n(k) &= (\tilde{m}_{n,0}(k), \tilde{m}_{n,1}(k))^T \\
&= (\tilde{m}_{(n,0,1,1)}(k), \dots, \tilde{m}_{(n,0,m_e,m_a,m_u)}(k), \tilde{m}_{(n,1,1,1)}(k), \dots, \tilde{m}_{(n,1,m_s,m_i)}(k))^T, \quad N \leq n \leq K-1.
\end{aligned}$$

The first-step analysis can be used here again to derive the system of equations for the generation functions $\xi_x(z)$, $x \in E$. The system has the same form as in (7) for $s = 0$ where the service interruption rates A^0 are replaced by zA^0 . The resulting system can be represented in the following matrix-vector form.

Theorem 2. *The generating functions*

$$\xi^{(N)}(z) = (\xi_{L+1}(z), \dots, \xi_{K-1}(z))^T$$

satisfy the following system

$$T_I^{(N)}(z)\xi^{(N)}(z) = g^{(N)}(K), \quad (20)$$

where $T_I^{(N)}(z) = \hat{Q}^{(N)} - A^{(N)}(z)$, $g^{(N)}(K) = (\hat{Q}_{2,1}e(l_1), 0_{(l_0(N-L-2)+l_1(K-L-2)+l_2(K-N)) \times 1})^T$. The matrix $\hat{Q}^{(N)}$ is obtained from $\bar{Q}^{(N)}$ by removing the first L block columns and rows, $A^{(N)}(z) = (1-z)\text{diag}(\underbrace{A_1, \dots, A_1}_{N-L-1}, \underbrace{A_2, \dots, A_2}_{K-N})$,

$$\begin{aligned}
A_1 &= \begin{pmatrix} 0_{l_0 \times l_0} & 0_{l_0 \times l_1} \\ A^0 \otimes e(m_s) \otimes (\pi_a \text{diag}(\pi_e) \otimes I(m_a)) & 0_{l_1 \times l_1} \end{pmatrix}, \\
A_2 &= \begin{pmatrix} 0_{l_2 \times l_2} & 0_{l_2 \times l_1} \\ A^0 \otimes e(m_s) \otimes (\pi_a \text{diag}(\pi_e) \otimes I(m_a)) \otimes \gamma^T & 0_{l_1 \times l_1} \end{pmatrix}.
\end{aligned}$$

Corollary 3. *The generating functions $\xi_n(z)$, $L+1 \leq n \leq K$, of the vector $\xi^{(N)}(z)$ satisfy the relations*

$$\xi_n(z) = \prod_{j=n}^{K-1} \tilde{P}_j(z)e(m_r) + \sum_{i=n}^{K-1} \prod_{j=n}^{i-1} \tilde{P}_j(z)\tilde{q}_i(z), \quad L+1 \leq n \leq K-1, \quad (21)$$

where

$$\begin{aligned}
\tilde{P}_{L+1}(z) &= -(Q_{1,1} - (1-z)A_1)^{-1}Q_{0,2}, \quad \tilde{q}_{L+1}(z) = -(Q_{1,1} - (1-z)A_1)^{-1}\hat{Q}_{2,1}e(l_1), \\
\tilde{P}_n(z) &= -(Q_{1,1} - (1-z)A_1 + Q_{2,1}\tilde{P}_{n-1}(z))^{-1}Q_{0,2}, \\
\tilde{q}_n(z) &= -(Q_{1,1} - (1-z)A_1 + Q_{2,1}\tilde{P}_{n-1}(z))^{-1}Q_{2,1}\tilde{q}_{n-1}(z), \quad L+2 \leq n \leq N-2, \\
\tilde{P}_{N-1}(z) &= -(Q_{1,1} - (1-z)A_1 + Q_{2,1}\tilde{P}_{N-2}(z))^{-1}Q_{0,3}, \\
\tilde{q}_{N-1}(z) &= -(Q_{1,1} - (1-z)A_1 + Q_{2,1}\tilde{P}_{N-2}(z))^{-1}Q_{2,1}\tilde{q}_{N-2}(z), \\
\tilde{P}_N(z) &= -(Q_{1,2} - (1-z)A_2 + Q_{2,2}\tilde{P}_{N-1}(z))^{-1}Q_{0,4}, \\
\tilde{q}_N(z) &= -(Q_{1,2} - (1-z)A_2 + Q_{2,2}\tilde{P}_{N-1}(z))^{-1}Q_{2,2}\tilde{q}_{N-1}(z), \\
\tilde{P}_n(z) &= -(Q_{1,2} - (1-z)A_2 + Q_{2,3}\tilde{P}_{n-1}(z))^{-1}Q_{0,4}, \\
\tilde{q}_n(z) &= -(Q_{1,2} - (1-z)A_2 + Q_{2,3}\tilde{P}_{n-1}(z))^{-1}Q_{2,3}\tilde{q}_{n-1}(z), \quad N+1 \leq n \leq K-2, \\
\tilde{P}_{K-1}(z) &= -(Q_{1,2} - (1-z)A_2 + Q_{2,3}\tilde{P}_{K-2}(z))^{-1}Q_{0,5}, \\
\tilde{q}_{K-1}(z) &= -(Q_{1,2} - (1-z)A_2 + Q_{2,3}\tilde{P}_{K-2}(z))^{-1}Q_{2,3}\tilde{q}_{K-2}(z).
\end{aligned} \quad (22)$$

Next, we differentiate the expression in (20) from Theorem 2, and obtain the following relation

$$T_I^{(N)}(z) \frac{d^k}{dz^k} \xi^{(N)}(z) + kA^{(N)}(0) \frac{d^{k-1}}{dz^{k-1}} \xi^{(N)}(z) = 0_{(l_0(N-L-1)+l_1(K-1)+l_2(K-N)+m_r) \times 1}.$$

We note that $\tilde{m}^{(N)}(k) = \left. \frac{d^k}{dz^k} \xi^{(N)}(z) \right|_{z=1}$ and $T_I^{(N)}(1) = \bar{Q}^{(N)}$, we get a recursive relation to compute the vector $\tilde{m}^{(N)}(k)$ of the k th factorial moments,

$$\begin{aligned}
\bar{Q}^{(N)} \tilde{m}^{(N)}(k) &= -kA^{(N)}(0)\tilde{m}^{(N)}(k-1), \quad k \geq 1, \\
\tilde{m}^{(N)}(0) &= e(l_0(N-L-1) + l_1(K-1) + l_2(K-N) + m_r).
\end{aligned} \quad (23)$$

Corollary 4. *The components $\tilde{m}_n(k)$, $L+1 \leq n \leq K-1$ of the vector $\tilde{m}^{(N)}(k)$ of the conditional factorial moments satisfy the relation*

$$\tilde{m}_n(k) = \sum_{i=n}^{K-1} \prod_{j=n}^{i-1} \tilde{P}_j(1)\tilde{v}_i, \quad L+1 \leq n \leq K-1, \quad (24)$$

where

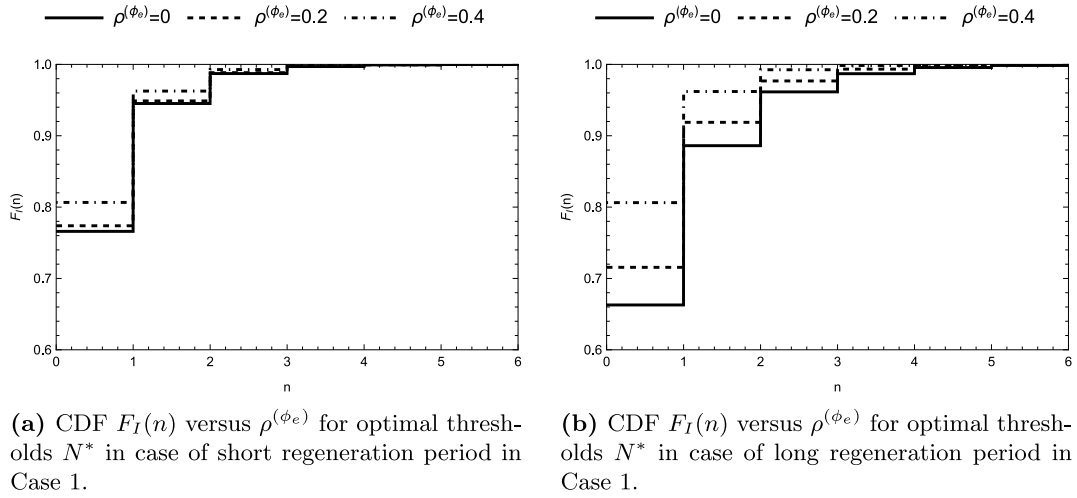
$$\tilde{v}_{L+1} = -kQ_{1,1}^{-1}A_1\tilde{m}_{L+1}(k-1), \quad (25)$$

Table 7The real and estimated moments versus correlation coefficient $\rho^{(\phi_e)}$ in Case 1.

	$\rho^{(\phi_e)}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$M_I(k)$	0.0	0.306	0.187	0.172	0.210	0.321
$\widehat{M}_I(k)$	0.0	0.306	0.187	0.172	0.210	0.322
$M_I(k)$	0.2	0.292	0.171	0.150	0.176	0.257
$\widehat{M}_I(k)$	0.2	0.292	0.171	0.150	0.176	0.257
$M_I(k)$	0.4	0.240	0.115	0.083	0.079	0.095
$\widehat{M}_I(k)$	0.4	0.240	0.115	0.083	0.079	0.095

Table 8Estimated coefficients (r, p) of the negative binomial distribution for optimal thresholds in Case 1 (a) and (b).

$\rho^{(\phi_e)}$	N^* (a)	$(r, p)^*$ (a)	N^* (b)	$(r, p)^*$ (b)
0.0	35	(0.998, 0.766)	50	(0.994, 0.661)
0.2	33	(0.999, 0.774)	40	(0.991, 0.713)
0.4	26	(1.002, 0.807)	23	(0.973, 0.801)

**Fig. 6.** Cumulative distribution functions $F_I(n)$ versus correlation coefficient in Case 1.

$$\begin{aligned}\tilde{v}_n &= -(Q_{1,1} + Q_{2,1}\tilde{P}_{L+1}(1))^{-1}Q_{2,1}(\tilde{v}_{n-1} + kA_1\tilde{m}_n(k-1)), \quad L+2 \leq n \leq N-1, \\ \tilde{v}_N &= -(Q_{1,2} + Q_{2,2}\tilde{P}_{L+1}(1))^{-1}Q_{2,2}(\tilde{v}_{N-1} + kA_2\tilde{m}_N(k-1)), \\ \tilde{v}_n &= -(Q_{1,2} + Q_{2,3}\tilde{P}_{L+1}(1))^{-1}Q_{2,3}(\tilde{v}_{n-1} + kA_2\tilde{m}_n(k-1)), \quad N+1 \leq n \leq K-1.\end{aligned}$$

For the unconditional factorial moments of the number of interruptions, we have

$$M_I(k) = \mathbb{E}[I \dots (I - k + 1)] = \frac{\pi(N-1)Q_{0,3}}{\pi(N-1)Q_{0,3}e(l_2 + l_1)}\tilde{m}_N(k), \quad k \geq 0. \quad (26)$$

Example 4. Here we repeat the calculations from [Example 3](#) for the first five unconditional factorial moments (26). The values of the calculated moments allow us to assume that the negative binomial distribution with a real-valued parameter r for the number of successes, the probability of a success p and the CDF

$$F_I(n) = \sum_{i=0}^n \frac{\Gamma(i+r)}{i!\Gamma(r)} p^r (1-p)^i, \quad n \geq 0$$

is a suitable distribution for moment fitting. Using the first two factorial moments $M_I(1) = \frac{r(1-p)}{p}$ and $M_I(2) = r(r+1)\left(\frac{1-p}{p}\right)^2$ we estimate the unknown parameters r and p . The other three factorial moments can be used for verification.

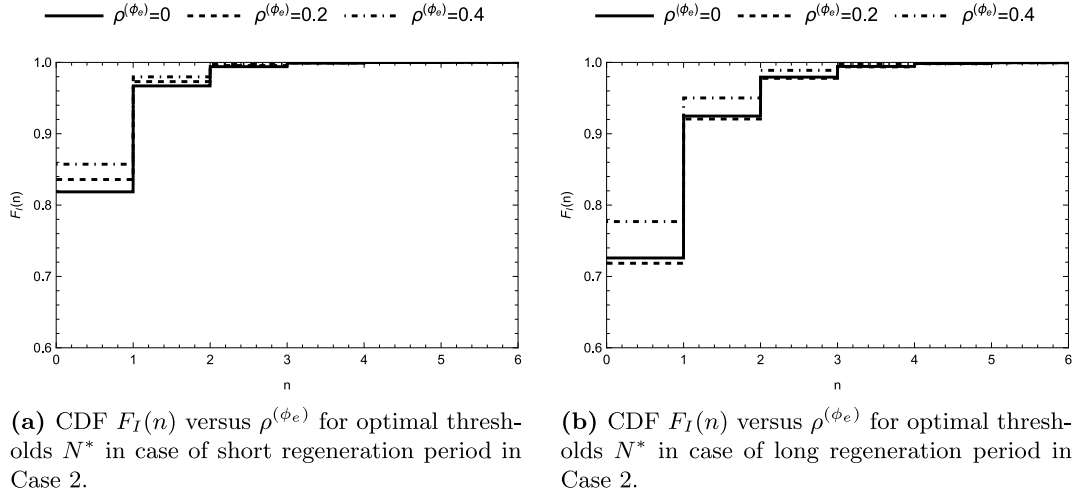
- Case 1: The vehicle is mostly operated in the city. The numerical results for this case are summarized in [Tables 7, 8](#) and in [Fig. 6\(a,b\)](#).

Table 9The real and estimated factorial moments versus correlation coefficient $\rho^{(\phi_e)}$ in Case 2.

	$\rho^{(\phi_e)}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$M_I(k)$	0.0	0.222	0.098	0.065	0.058	0.064
$\widehat{M}_I(k)$	0.0	0.222	0.098	0.065	0.058	0.064
$M_I(k)$	0.2	0.196	0.077	0.045	0.036	0.035
$\widehat{M}_I(k)$	0.2	0.196	0.077	0.045	0.036	0.035
$M_I(k)$	0.4	0.166	0.055	0.027	0.018	0.015
$\widehat{M}_I(k)$	0.4	0.166	0.055	0.027	0.018	0.015

Table 10Estimated coefficients (r, p) of the negative binomial distribution for optimal thresholds in Case 2 (a) and (b).

$\rho^{(\phi_e)}$	N^* (a)	$(r, p)^*$ (a)	N^* (b)	$(r, p)^*$ (b)
0.0	27	(0.999, 0.818)	41	(0.998, 0.725)
0.2	24	(1.002, 0.836)	40	(0.998, 0.718)
0.4	20	(1.011, 0.859)	30	(0.998, 0.776)

**Fig. 7.** Cumulative distribution functions $F_I(n)$ versus correlation coefficient in Case 2.

- Case 2: Equal probability of using slow, middle and fast speed driving. The numerical results for this case are summarized in [Tables 9, 10](#) and in [Fig. 7\(a,b\)](#).
- Case 3: The vehicle is driven mainly on highways. The numerical results for this case are summarized in [Tables 11, 12](#) and in [Fig. 8\(a,b\)](#).

In this example, we studied the discrete counterpart to the regeneration period in the form of the number of service interruptions. [Tables 7–11](#) show that the negative binomial distribution with parameters provides an excellent fit. Moreover, since $r \approx 1$ in all cases, indicating the distribution effectively reduces to a geometric distribution. Higher correlation leads to fewer interruptions because the optimal response to burstiness is to lower the regeneration threshold, resulting in shorter cycles with less exposure time to potential interruptions. The differences in values of the estimated parameters for short and long regeneration periods are illustrated in [Tables 8–12](#). How such a difference influences the form of the CDFs $F_I(n)$ can be identified in [Figs. 6–8](#). We observe that subcase (b) for a longer regeneration period shows greater sensitivity to correlation, suggesting higher interruption rates or costs that drive more aggressive threshold adjustment.

6. Maximum queue length in a regeneration period

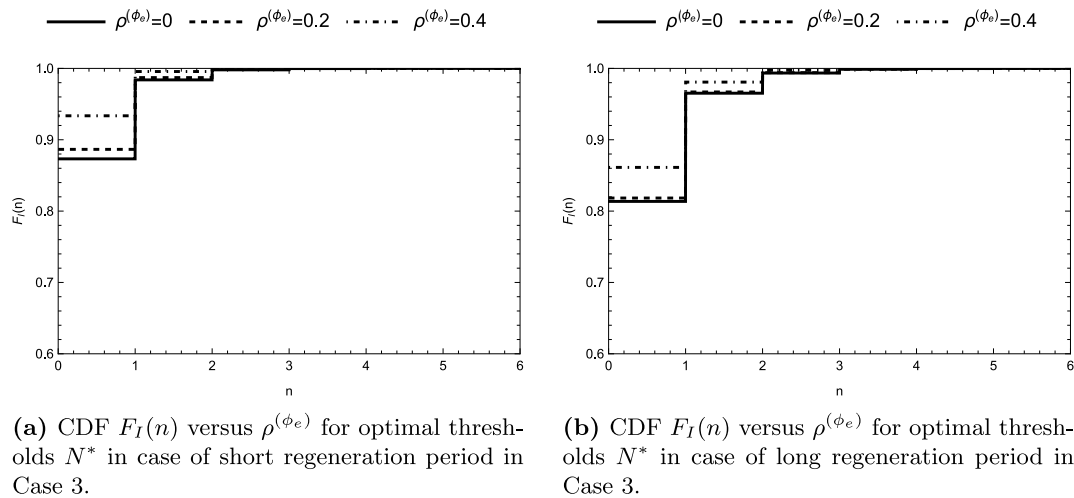
Denote by $N_{\max}$ the maximum queue length or maximum level of soot observed during the regeneration period. Note that for each value $k \geq 1$, the event $N_{\max} < k$ is equivalent to the event that the random process starting in the state of level $l(N)$ hits one of the absorbing states at levels $l(0)$ and $l(n, 0)$, $0 \leq n \leq L$ before hitting the states of level $l(k)$. Thus, calculating the probability $\mathbb{P}[N_{\max} < k]$ boils down to finding the probability of absorption at levels $l(0)$, $l(n, 0)$, $0 \leq n \leq L$ of the finite Markov chain where

Table 11The real and estimated moments versus correlation coefficient $\rho^{(\phi_e)}$ in Case 3.

	$\rho^{(\phi_e)}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$M_I(k)$	0.0	0.145	0.042	0.018	0.011	0.008
$\widehat{M_I}(k)$		0.145	0.042	0.018	0.011	0.008
$M_I(k)$	0.2	0.128	0.033	0.013	0.006	0.004
$\widehat{M_I}(k)$		0.128	0.033	0.013	0.006	0.004
$M_I(k)$	0.4	0.071	0.010	0.002	0.001	$0.2 \cdot 10^{-3}$
$\widehat{M_I}(k)$		0.071	0.010	0.002	0.001	$0.2 \cdot 10^{-3}$

Table 12Estimated coefficients (r, p) of the negative binomial distribution for optimal thresholds in Case 3 (a) and (b).

$\rho^{(\phi_e)}$	N^* (a)	$(r, p)^*$ (a)	N^* (b)	$(r, p)^*$ (b)
0.0	20	(0.999, 0.873)	28	(0.998, 0.813)
0.2	17	(1.000, 0.887)	27	(0.999, 0.818)
0.4	10	(1.019, 0.935)	20	(1.006, 0.862)

**Fig. 8.** Cumulative distribution functions $F_I(n)$ versus correlation coefficient in Case 3.

the level $l(k)$ is also absorbing. Define for fixed k the probability $h_x(k)$ of absorption in the above specified levels starting from $x \in \bigcup_{n=1}^L l(n, 1) \bigcup \bigcup_{n=L+1}^k l(n)$, $N+1 \leq k \leq K$. Obviously, $h_x(k) = 1$ for $x \in \bigcup_{n=1}^L l(n, 1)$ and $h_x(k) = 0$, $x \in l(k)$. Denote by

$$\begin{aligned}
 h_n(k) &= (h_{n,0}(k), h_{n,1}(k))^T \\
 &= (h_{(n,0,1,1)}(k), \dots, h_{(n,0,m_e,m_a)}(k), h_{(n,1,1,1)}(k), \dots, h_{(n,1,m_s,m_i)}(k))^T, \quad L+1 \leq n \leq N-1, \\
 h_n(k) &= (h_{n,0}(k), h_{n,1}(k))^T \\
 &= (h_{(n,0,1,1,1)}(k), \dots, h_{(n,0,m_e,m_a,m_u)}(k), h_{(n,1,1,1)}(k), \dots, h_{(n,1,m_s,m_i)}(k))^T, \quad N \leq n \leq K-1.
 \end{aligned}$$

By conditioning on the next state visited, we obtain the following system.

Theorem 3. *The probabilities*

$$h^{(N)}(k) = (h_{L+1}(k), \dots, h_{K-1}(k))^T, \quad N+1 \leq k \leq K$$

satisfy the following system

$$T_{N_{\max}}^{(N)}(k) h^{(N)}(k) = g^{(N)}(k), \quad (27)$$

where the matrix $T_{N_{\max}}^{(N)}(k)$ is obtained from $\bar{Q}^{(N)}$ by removing the last $K-k+1$ (block) columns and (block) rows, the column vector $g^{(N)}(k)$ is defined as in Theorem 2.

Table 13PDF $f_{N_{\max}}(N^*)$ for the optimal thresholds in Case 1.

$\rho(\phi_s)$	N^* (a)	$f_{N_{\max}}(N^*)$ (a)	N^* (b)	$f_{N_{\max}}(N^*)$ (b)
0.0	35	0.508	50	0.221
0.2	33	0.377	40	0.187
0.4	26	0.289	23	0.173

Table 14PDF $f_{N_{\max}}(N^*)$ for the optimal thresholds in Case 2.

$\rho(\phi_s)$	N^* (a)	$f_{N_{\max}}(N^*)$ (a)	N^* (b)	$f_{N_{\max}}(N^*)$ (b)
0.0	27	0.648	41	0.366
0.2	24	0.503	40	0.283
0.4	20	0.375	30	0.222

Proof. Setting $s = 0$ in the set of Eqs. (7) starting from $n \geq L + 1$, taking into account that $h_{L,1}(k) = e(l_1)$ and $h_k(k) = 0$, we get the system (27). $\square$

As a result, the algorithmic solution of (27) is parallel to Corollary 1.

Corollary 5. The probabilities $h_n(k)$, $L + 1 \leq n \leq m - 1$, of the vector $h^{(N)}(k)$ satisfy the relations

$$h_n(k) = \sum_{i=n}^{k-1} \prod_{j=n}^{i-1} P_j(0)q_i(0), \quad L + 1 \leq n \leq k - 1. \quad (28)$$

The expression (28) for $k = N$ is used to calculate the unconditional CDF $F_{N_{\max}}(k) = \mathbb{P}[N_{\max} < k]$. We get

$$F_{N_{\max}}(k) = \frac{\pi(N-1)Q_{0,3}}{\pi(N-1)Q_{0,3}e(l_2 + l_1)} h_N(k). \quad (29)$$

PDF $f_{N_{\max}}(k)$ is then of the form

$$\begin{aligned} f_{N_{\max}}(k) &= \mathbb{P}[N_{\max} = k] = F_{N_{\max}}(k+1) - F_{N_{\max}}(k) \\ &= \frac{\pi(N-1)Q_{0,3}}{\pi(N-1)Q_{0,3}e(l_2 + l_1)} \prod_{j=N}^{k-1} P_j(0)q_k(0). \end{aligned} \quad (30)$$

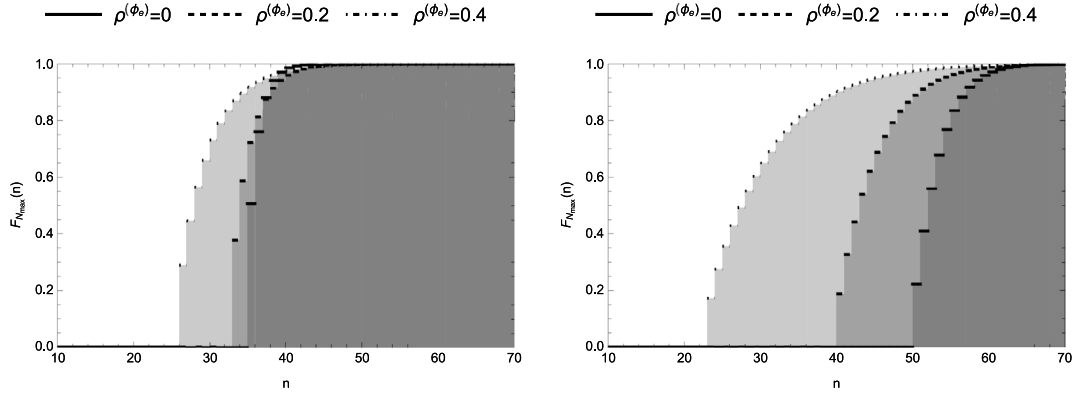
Example 5. Here we present numerical results involving the CDF $F_{N_{\max}}(n)$ and the PDF $f_{N_{\max}}(n)$ defined in (29) and (30). This function is calculated under the same conditions as in previous examples for three main cases and two subcases (a) and (b) for short and long regeneration periods due to different values of the startup and the service interruption rates.

- Case 1: The vehicle is mostly operated in the city. The numerical results for this case are summarized in Table 13 and Fig. 9(a,b).
- Case 2: Equal probability of using slow, middle and fast speed driving. The numerical results for this case are summarized in Table 14 and Fig. 10(a,b).
- Case 3: The vehicle is driven mainly on highways. The numerical results for this case are summarized in Table 15 and Fig. 11(a,b).

In Tables 13–15, we display the probabilities that the maximum accumulated soot level in the filter will not exceed the optimal threshold for the average cost. We remind the reader that reaching the level N signals the start of the regeneration period. As expected, the highest probability $f_{N_{\max}}(20) = 0.905$ was obtained for Case 3(a) with low soot arrival intensity, zero correlation in arrival flow, and short regeneration period. The lowest probability $f_{N_{\max}}(23) = 0.173$ we observe in Case 1(b) for a high arrival intensity, highly correlated arrival flow, and long regeneration period. The intermediate results illustrate how sensitive this probability is to changes in system parameters. We can also observe that as the system load described in Cases 1-3 decreases, the intensity of soot accumulation decreases. As a result, the probability increases that during the regeneration period, the amount of soot will not increase significantly above the specified threshold level. Finally, the influence of the correlation, as well as of the startup and service interruption rates, for three cases of the environment process is analyzed in Figs. 9–11. Here we can see the results presented in the previous tables graphically. Again, as in the previous examples, the CDF $F_{N_{\max}}(n)$ of the random variable $N_{\max}$ has heavier tails as the system load increases, in our case due to the higher soot accumulation intensity, e.g., as in Case 1 and longer preparation times and/or shorter intervals before service interruption as in subcases (b).

7. Conclusions

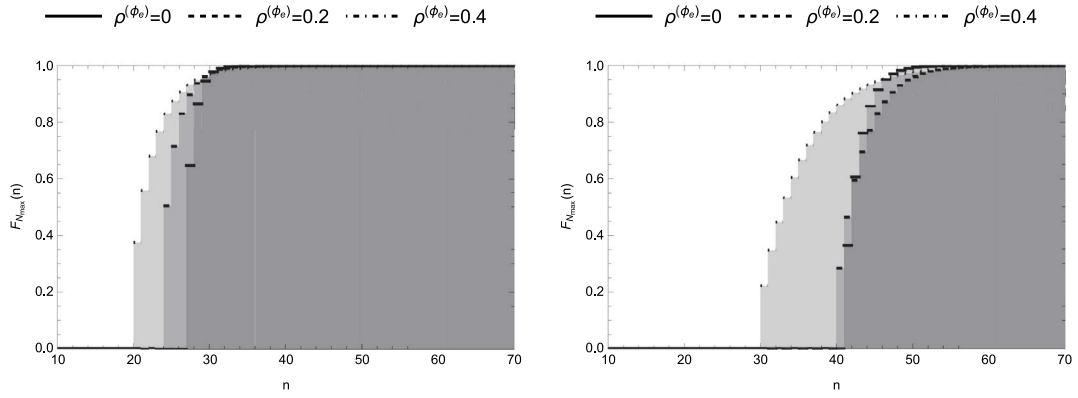
This paper presented a comprehensive analysis of a $MAP/PH/1/K$ queueing system with the N -policy, setup time, service interruptions, reset mechanism, and random environment. The core contribution lies in the algorithmic analysis of the regenerative



(a) CDF $F_{N_{\max}}(n)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* in case of short regeneration period in Case 1.

(b) CDF $F_{N_{\max}}(n)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* in case of long regeneration period in Case 1.

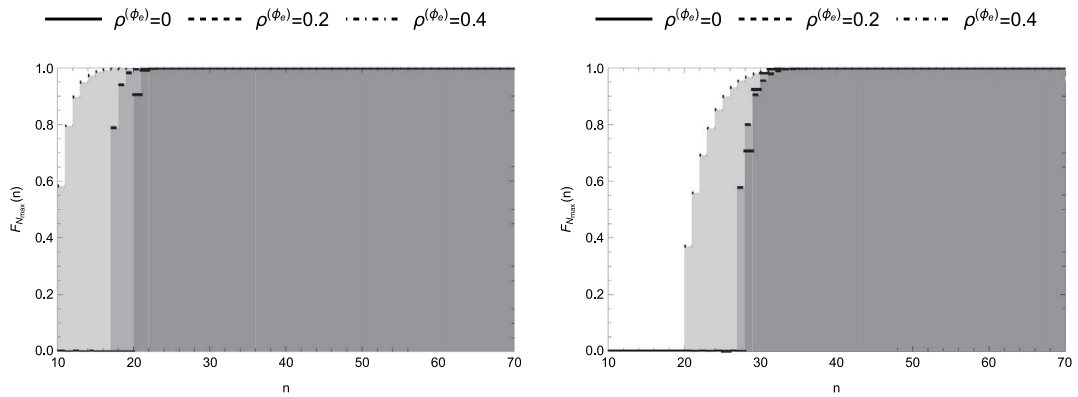
Fig. 9. Cumulative distribution functions $F_{N_{\max}}(n)$ versus correlation coefficient in Case 1.



(a) CDF $F_{N_{\max}}(n)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* in case of short regeneration period in Case 2.

(b) CDF $F_{N_{\max}}(n)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* in case of long regeneration period in Case 2.

Fig. 10. Cumulative distribution functions $F_{N_{\max}}(n)$ versus correlation coefficient in Case 2.



(a) CDF $F_{N_{\max}}(n)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* in case of short regeneration period in Case 3.

(b) CDF $F_{N_{\max}}(n)$ versus $\rho^{(\phi_e)}$ for optimal thresholds N^* in case of long regeneration period in Case 3.

Fig. 11. Cumulative distribution functions $F_{N_{\max}}(n)$ versus correlation coefficient in Case 3.

Table 15PDF $f_{N_{\max}}(N^*)$ for the optimal thresholds in Case 3.

$\rho(\phi_0)$	N^* (a)	$f_{N_{\max}}(N^*)$ (a)	N^* (b)	$f_{N_{\max}}(N^*)$ (b)
0.0	20	0.905	28	0.706
0.2	17	0.787	27	0.578
0.4	10	0.584	20	0.371

period, providing Laplace–Stieltjes transforms for its length, interruption count distributions, and maximal queue length during regeneration. Several extensions merit investigation: (i) incorporating multiple server states with different service rates during partial operation, (ii) considering batch arrivals representing sudden soot accumulation during high-load events, (iii) modeling age-dependent deterioration where filter efficiency degrades over time, and (iv) introducing correlated interruptions where engine conditions exhibit memory beyond phase-type representation. Future work could explore dynamic programming approaches for state-dependent threshold policies, multi-objective optimization balancing fuel consumption, filter longevity, and emissions compliance, and reinforcement learning methods for adaptive policy optimization in unknown environments. The model requires validation against real DPF regeneration data from instrumented vehicles under controlled driving cycles. Collaboration with automotive manufacturers would enable the parameter estimation from production vehicles and refinement of phase-type representations based on empirical oxidation kinetics.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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