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# On convexity with respect to Chebyshev systems and Cauchy-Schwarz type inequalities for solutions of Levi–Civita-type functional equations

Thesis for the degree
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Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences of the University of Debrecen in order to obtain a PhD Degree in Natural Sciences from the University of Debrecen.
I declare that the results published in this thesis are not reported in any other PhD theses.

Debrecen, April 15, 2024.

Mahmood Kamil Shihab
signature of the candidate

Hereby I confirm that Mahmood Kamil Shihab candidate conducted his studies with my supervision within the Mathematical Analysis Program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen between 2020 and 2024. The independent studies and research work of the candidate significantly contributed to the results published in this thesis.
I also declare that the results published in the thesis are not reported in any other PhD theses.
I support the acceptance of the dissertation.
Debrecen, April 15, 2024.

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signature of the supervisor

# On convexity with respect to Chebyshev systems and Cauchy-Schwarz type inequalities for solutions of Levi–Civita-type functional equations

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## Notation and Symbols

| | |
|---|---|
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{R}_+$ | the set of nonnegative real numbers |
| $\mathbb{N}$ | the set of natural numbers |
| $\mathbb{Q}$ | the set of rational numbers |
| $\mathbb{C}$ | the set of complex numbers |
| $\Re$ | a real part of a complex number |
| $\Im$ | an imaginary part of a complex number |
| $I, J$ | a nonempty open interval |
| $\mathbb{K}$ | a subfield of $\mathbb{R}$ |

# Introduction

## 1. A summary of basic convexity properties

Let $I$ be a subinterval of $\mathbb{R}$, a function $f : I \to \mathbb{R}$ is said to be a *convex* if it satisfies the inequality

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2) \qquad (x_1, x_2 \in I, t \in [0,1]),$$

$f$ is called *Wright convex* if it satisfies the inequality

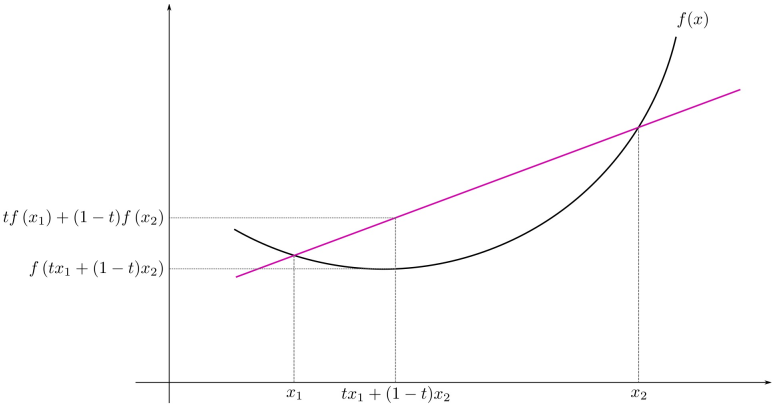
$$f(tx_1 + (1-t)x_2) + f((1-t)x_1 + tx_2) \le f(x_1) + f(x_2) \qquad (x_1, x_2 \in I, t \in [0,1])$$

and $f$ is *Jensen convex* if it satisfies

$$f\Big(\frac{x_1 + x_2}{2}\Big) \le \frac{f(x_1) + f(x_2)}{2} \qquad (x_1, x_2 \in I).$$

One can see that convexity yields Wright convexity and Wright convexity yields Jensen convexity and non of these implications can be reversed. For an overview about the generalizations, stability and regularity properties of Wright convex functions, we refer to the list of references, which may give the impression that this subfield of functional equations and inequalities is still in the focus of research.

The geometric meaning of convexity is that the line segment connecting any two point $(x_1, f(x_1))$ and $(x_2, f(x_2))$ of $I \times \mathbb{R}$ is above the graph of the function $f$ over the interval $[x_1, x_2]$.



In what follows, we describe various settings and generalizations of convexity.

**1.1. Godunova–Levin functions.** We say that a function $f : I \to \mathbb{R}$ is a Godunova–Levin function if, for all $x_1, x_2 \in I$ and $\alpha \in ]0,1[$, we have that the inequality

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \frac{f(x_1)}{\alpha} + \frac{f(x_2)}{1-\alpha}$$

holds. This class of functions was firstly introduced by Godunova and Levin [**13**]. For more characteristics of Godunova–Levin function. see [**27**, **28**].

**1.2. Breckner's $s$-convexity.** Let us fix a real number $s \in ]0,1]$. We say that a function $f : I \to \mathbb{R}$ is $s$-convex if the inequality

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha^s f(x_1) + (1-\alpha)^s f(x_2) \qquad (x_1, x_2 \in I, \alpha \in ]0,1[)$$

holds. This class of functions was firstly introduced by Breckner [**10**]. Clearly $s$-convexity is equivalent to standard convexity with $s = 1$, for more properties of $s$-convexity see [**15**].

**1.3. Varošanec's $h$-convexity.** Let $h : [0,1] \to \mathbb{R}$ be a positive function. A function $f : I \to \mathbb{R}$ is said to be *$h$-convex* if the inequality

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \qquad (x, y \in I, t \in [0,1])$$

holds. It easy to see that if $h(t) = t$, then the notion of $h$-convexity is equivalent to standard convexity. If $h(t) = \frac{1}{t}$ and $h(t) = t^s$, then this concept reduces to the previous notions. Varošanec introduced this concept of convexity, we refer to the paper [**45**] for more details about this class of functions.

**1.4. Toader's $m$-convexity.** Assume that $\inf(I)=0$ and let $m\in]0,1[$ be fixed. A function $f:I\to\mathbb{R}$ is said to be *$m$-convex* if the inequality

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y)\qquad(x,y\in I,t\in[0,1])$$

holds. This notion of $m$-convexity was introduced by Toader [**44**].

Lara, Matkowski, Merentes, Quintero and Wróbel [**23**] generalized the concept of $m$-convexity as follows: let $\mu:[0,1]\to[0,1]$ be a function such that $\mu(t)\le 1-t$ for all $t\in[0,1]$. A function $f:I\to\mathbb{R}$ is said to be *$\mu$-convex* if the inequality

$$f(tx+\mu(t)y)\le tf(x)+\mu(t)f(y)\qquad(x,y\in I,t\in[0,1])$$

holds.

**1.5. Matkowski–Rätz's $(M,N)$-convexity.** Let $I$ be a subinterval of $\mathbb{R}$. A function $M:I^2\to I$ is called mean if the following two inequalities

$$\max(x,y)\le M(x,y)\le\min(x,y)\qquad(x,y\in I)$$

hold. Let $J$ be another subinterval and $N:J^2\to J$ be another mean. A function $f:I\to J$ is called *$(M,N)$-convex* if

$$f(M(x,y))\le N(f(x),f(y))\qquad(x,y\in I)$$

holds. If $M(x,y):=N(x,y):=\frac{x+y}{2}$, then $(M,N)$-convexity coincides with Jensen convexity in the standard sense. This concept of convexity was introduced and investigated by Matkowski and Rätz [**26**].

**1.6. Matkowski's $\alpha$-convexity.** Let $\alpha:I\to\mathbb{R}$ be an arbitrary strictly monotone function. A function $f:I\to\mathbb{R}$ is said to be *convex with respect to* $\alpha$ if the inequality

$$\begin{aligned}f(tx+(1-t)y)\le{}&\frac{\alpha(tx+(1-t)y)-\alpha(y)}{\alpha(x)-\alpha(y)}f(x)\\&+\frac{\alpha(x)-\alpha((1-t)x+ty)}{\alpha(x)-\alpha(y)}f(y)\qquad(x,y\in I,\ t\in]0,1[)\end{aligned}$$

holds. If this inequality ir required only for $t=\frac{1}{2}$, then $f$ is said to be *Jensen convex with respect to* $\alpha$. Matkowski [**25**] introduced the convexity with respect to $\alpha$. For more properties of this class of functions, we refer to [**25**].

**1.7. Beckenbach convexity.** Beckenbach [1] extended the concept of convexity of functions to a more general notion. A family $\mathcal{F}$ of continuous real functions defined on the open interval $I$ is called a two dimensional *Beckenbach* family if it satisfies the following condition:

(C) For all $x_1, x_2 \in I$ with $x_1 < x_2$ and $y_1, y_2 \in \mathbb{R}$, there is a unique member $F$ of $\mathcal{F}$ such that $F(x_1) = y_1$ and $F(x_2) = y_2$.

We will denote the unique member of $\mathcal{F}$ that interpolates the points $(x_1, y_1)$ and $(x_2, y_2)$ by $F_{(x_1,y_1),(x_2,y_2)}$.

Given a two dimensional Beckenbach family $\mathcal{F}$, a function $f : I \to \mathbb{R}$ is called $\mathcal{F}$-*convex*, if

$$f(x) \leq F_{(x_1,f(x_1)),(x_2,f(x_2))}(x)$$

holds for all $x, x_1, x_2 \in I$ with $x_1 < x < x_2$. If $\mathcal{F} = \{f : I \to \mathbb{R} \mid \exists a, b \in \mathbb{R}\colon f(x) = ax + b,\ (x \in I)\}$ then the concept of $\mathcal{F}$-convexity is equivalent to standard convexity.

Beckenback and Bing [2] also generalized the concept of Jensen convexity to this setting. Their definition is as follows: a function $f : I \to \mathbb{R}$ is said to be $\mathcal{F}$-Jensen-convex if it satisfies the inequality

$$f\Big(\frac{x_1 + x_2}{2}\Big) \leq F_{(x_1,f(x_1)),(x_2,f(x_2))}\Big(\frac{x_1 + x_2}{2}\Big)$$

for all $x_1, x_2 \in I$ with $x_1 < x_2$.

## 2. The main goals of this thesis

This dissertation is divided into three chapters. In what follows, we give a brief description for each of them.

Chapter one is devoted to give an elementary proof for the decomposition theorem of Maksa and Páles [24] which is an extension of the Ng theorem [30], (Ng characterized a Wright convex function as a function which is the sum of a convex and an additive function). Maksa and Páles [24] proved that a real function is Wright convex of order $n$ if and only if it is a sum of two functions: a convex function of order $n$ and a generalized polynomial of degree at most $n$. In our proof we adopt the method of Páles [33].

The main purpose of Chapter two is to introduce various convexity notions with respect to a given positive Chebyshev system $\omega$ and give relations among them. In one of our main result, in Theorem 2.6, we generalize the celebrated theorem of Bernstein–Doetsch [6] to the setting of $\omega$-Jensen convexity. From this , we derive that a locally bounded function is $\omega$-Jensen affine if and only if it is the linear combination of the members of the Chebyshev system. In section 3 of this Chapter we extend the notion of Wright convexity to the setting of Chebyshev systems and point out that it is an intermediate convexity property between $\omega$-convexity and $\omega$-Jensen convexity. We also generalize the decomposition theorem of higher-order Wright convex function (obtained by Maksa and Páles [**24**] in 2009) to certain Chebyshev systems.

The main goal of Chapter three is to show that if a real valued function defined on a groupoid satisfies a certain Levi–Civita-type functional equation, then it also fulfills a Cauchy–Schwarz-type functional inequality. In particular, if the groupoid is the multiplicative structure of commutative ring, then we can establish the existence of nontrivial additive functions satisfying inequalities connected to the multiplicative structure.

CHAPTER 1

# Decomposition of higher-order Wright convex functions revisited

## 1. Preliminaries

Throughout this Chapter, let $I$ denote a proper open subinterval of the real line. For a function $f : I \to \mathbb{R}$, one can define the following notions of convexity:
– $f$ is called *Jensen convex* on $I$, if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \qquad (x, y \in I).$$

– $f$ is called *Wright convex* on $I$, if

$$f(tx+(1-t)y)+f((1-t)x+ty) \le f(x)+f(y) \quad (x, y \in I,\, t \in [0,1]).$$

– $f$ is called *convex* on $I$, if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y) \qquad (x, y \in I,\, t \in [0,1]).$$

A celebrated result of C. T. Ng [**30**] established a deeper connection between convexity and Wright convexity. It characterizes Wright convex functions as those functions that are of the form $f = g + a$, where $g$ is convex and $a$ is additive. The original proof of the paper [**30**] applied de Bruijn's theorem [**11**] which is related to functions which have continuous differences. Several subsequent proofs of the result of Ng (c.f., Nikodem [**31**] and Kominek [**19**]) used another approach, which was based on Rode's theorem [**40**]. Basically, all the previously known proofs used transfinite induction for the construction of the additive part $a$ of the decomposition. In a recent paper [**33**], Páles obtained a new proof in which the convex summand $g$ was first constructed in an elementary way. Therefore, there was no transfinite induction involved.

In the paper [**24**], Maksa and Páles extended the decomposition theorem of Ng to the context of higher-order convexity notions. They proved that a real function is Wright convex of order $n$ if and only if it can be decomposed as the sum of a convex function of order $n$ and a polynomial function of order at most $n$. Their proof was again using transfinite tools in the background. The main purpose of this Chapter is to adopt the methods of the paper [**33**] and establish a new and elementary proof for the theorem of Maksa and Páles.

## 2. Higher-order convex and Wright convex functions

In what follows, we are going to define several higher-order convexity concepts in terms of *difference operators* and *divided differences*.

We recall that, for a fixed real number $h$, the operator $\Delta_h$, acting on a real function $f : I \to \mathbb{R}$, is defined by

$$\Delta_h f(x) := f(x+h) - f(x) \qquad (x \in I \cap (I-h)).$$

Obviously, if $|h|$ is small enough, then $I \cap (I-h)$ is a non-void open interval again. The product of these operators can also be defined in the usual way (see e.g. Kuczma [**21**]).

Given a fixed $n \in \mathbb{N}$, a map $f : I \to \mathbb{R}$ is said to be *Jensen convex of order* $n$ (briefly $n$-Jensen convex) if

$$\Delta_h^{n+1} f(x) \geq 0 \qquad \big(h > 0,\, x \in I \cap (I-(n+1)h)\big). \tag{1.1}$$

A map $f : I \to \mathbb{R}$ is said to be *Wright convex of order* $n$ (briefly $n$-Wright convex) if it satisfies the functional inequality

$$\begin{aligned} &\Delta_{h_1} \cdots \Delta_{h_{n+1}} f(x) \geq 0 \\ &\qquad \big(h_1, \ldots, h_{n+1} > 0,\, x \in I \cap (I - (h_1 + \cdots + h_{n+1}))\big). \end{aligned} \tag{1.2}$$

In the investigation of functional inequalities (1.1) and (1.2), those maps that fulfill these inequalities with equality play a fundamental role in the theory of linear functional equations. Therefore, for $n \in \mathbb{N}$, we consider the equation

$$\Delta_h^{n+1} f(x) = 0 \qquad (h > 0,\, x \in I \cap (I-(n+1)h)),$$

which is termed the *Fréchet functional equation* in this theory. It is well-known (see [**21**], [**43**]) that $f : I \to \mathbb{R}$ satisfies this equation if and only if it is a *polynomial function of degree at most* $n$, i.e., it has the representation

$$f(x) = a_0 + a_1(x) + \cdots + a_n(x) \qquad (x \in I),$$

where $a_0 \in \mathbb{R}$ and $a_k$ is the *diagonalization* of some $k$-additive and symmetric function $A_k : \mathbb{R}^k \to \mathbb{R}$, that is, $a_k(x) = A_k(x, \ldots, x)$, $(x \in \mathbb{R},\ k = 1, \ldots, n)$. Standard polynomials are exactly the continuous polynomial functions. On the other hand, using Hamel bases, it is not difficult to construct non-continuous polynomial functions (see [**21**]).

The *divided difference* of the function $f : I \to \mathbb{R}$ with respect to the pairwise distinct points $x_0, \dots, x_n \in I$ is defined by

$$[x_0, \dots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}.$$

Obviously, divided differences are symmetric functions of their variables, furthermore, it is easy to show that they enjoy the following recursive property

$$[x_0, \dots, x_n; f] = \frac{[x_1, \dots, x_n; f] - [x_0, \dots, x_{n-1}; f]}{x_n - x_0}$$

for all $n \in \mathbb{N}$ and pairwise distinct elements $x_0, \dots, x_n \in I$.

Based on the works of T. Popoviciu [**38**,**39**], given $n \in \mathbb{N}$, a map $f : I \to \mathbb{R}$ is said to be *convex of order* $n$ *on* $I$ (shortly $n$-convex on $I$) if the inequality

$$[x_0, x_1, \dots, x_n, x_{n+1}; f] \geq 0 \tag{1.3}$$

holds for all pairwise distinct elements $x_0, x_1, \dots, x_n, x_{n+1} \in I$. Due to the symmetry of divided differences, without loss of generality, we may assume $x_0 < x_1 < \dots < x_n < x_{n+1}$ here.

The following result was obtained in the book [**21**] and in a more general form in the paper [**12**].

LEMMA 1.1. *Let* $n \in \mathbb{N}$. *Then every* $n$*-convex function is* $n$*-Wright convex, and every* $n$*-Wright convex function is* $n$*-Jensen convex.*

One of the main results of the paper [**24**] established the following generalization of Ng's decomposition theorem [**30**].

THEOREM 1.2. *Let* $n \in \mathbb{N}$ *and* $f : I \to \mathbb{R}$ *be an* $n$*-Wright convex function. Then, there exist an* $n$*-convex function* $g : I \to \mathbb{R}$ *and a polynomial function* $P : \mathbb{R} \to \mathbb{R}$ *of degree at most* $n$ *such that*

$$f(x) = g(x) + P(x) \qquad (x \in I).$$

Our aim is to obtain a new proof for this result.

LEMMA 1.3. *Let* $n \in \mathbb{N}$ *and* $f : I \to \mathbb{R}$ *be an* $n$*-Jensen convex function. Then there exists a continuous* $n$*-convex function* $g : I \to \mathbb{R}$ *such that* $g|_{I\cap\mathbb{Q}} = f|_{I\cap\mathbb{Q}}$.

PROOF. Using [**12**, Lemma 5.1], we have the following identity

$$[x, x+h, \dots, x+nh, x+(n+1)h; f] = \frac{\Delta_h^{n+1} f(x)}{(n+1)!\,h^{n+1}}$$
$$(h > 0,\ x \in I \cap (I - (n+1)h)).$$

Therefore, the $n$-Jensen convexity of $f$ implies that

$$[x, x+h, x+2h \dots, x+(n+1)h; f] \geq 0 \qquad (h > 0,\ x \in I \cap (I-(n+1)h)).$$

In the terminology of the paper [**12**], this property says that $f$ is $(t_0, \dots, t_n)$-convex with $t_0 = \cdots = t_n = 1$. According to [**12**, Theorem 3.2], it follows that $f$ is $(r_0, \dots, r_n)$-convex for all positive rational numbers $(r_0, \dots, r_n)$-convex, that is

$$\tag{1.4} \begin{gathered}[x, x+r_0h, x+(r_0+r_1)h, \dots, x+(r_0+\cdots+r_n)h; f] \geq 0 \\ (h > 0,\ x \in I \cap (I-(r_0+\cdots+r_n)h)).\end{gathered}$$

We now deduce that $f$ satisfies the $n$-convexity property with rational arguments, that is,

$$\tag{1.5} \begin{gathered}[x_0, x_1, \dots, x_n, x_{n+1}; f] \geq 0 \\ (x_0, \dots, x_{n+1} \in I \cap \mathbb{Q} \text{ such that } x_i \neq x_j \text{ if } i \neq j).\end{gathered}$$

Indeed, let $x_0, x_1, \dots, x_n, x_{n+1} \in I \cap \mathbb{Q}$ be arbitrary. Without loss of generality, we may assume that $x_0 < x_1 < \cdots < x_n < x_{n+1}$. Applying now the inequality (1.4) with $x := x_0$, $h = 1$ and $r_i := x_{i+1} - x_i$ for $i \in \{0, \dots, n\}$, we can see that (1.5) holds.

Claim: For any compact subinterval $[a, b] \subseteq I$, there exists $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x-y| \qquad (x, y \in [a, b] \cap \mathbb{Q}).$$

To show this claim, let $[a, b] \subseteq I$. Without loss of generality, we can assume that $a, b \in \mathbb{Q}$. Let $x, y \in [a, b] \cap \mathbb{Q}$ with $x < y$ be arbitrary. Then, for all pairwise distinct elements $u_1, \dots, u_n \in (I \setminus [a, b]) \cap \mathbb{Q}$, we get

$$\tag{1.6} [y, x, u_1, \dots, u_n; f] \geq 0,$$

that is,

$$\frac{1}{y-x}\left(\frac{f(y)}{\prod_{j=1}^{n}(y-u_j)} - \frac{f(x)}{\prod_{j=1}^{n}(x-u_j)}\right) \geq -\sum_{i=1}^{n} \frac{f(u_i)}{(u_i-x)(u_i-y)\prod_{j\in\{1,\dots,n\}\setminus\{i\}}(u_i-u_j)}.$$

The mapping

$$[a, b]^2 \ni (x, y) \mapsto -\sum_{i=1}^{n} \frac{f(u_i)}{(u_i-x)(u_i-y)\prod_{j\in\{1,\dots,n\}\setminus\{i\}}(u_i-u_j)}$$

is continuous on $[a,b]^2$ and therefore it is bounded from below by a constant $C(u_1,\dots,u_n)$, therefore, for all $x,y\in[a,b]\cap\mathbb{Q}$, the inequality (1.6) implies

$$\frac{1}{y-x}\left(\frac{f(y)}{\prod_{j=1}^{n}(y-u_j)}-\frac{f(x)}{\prod_{j=1}^{n}(x-u_j)}\right)\geq C(u_1,\dots,u_n). \tag{1.7}$$

Let first $u_1<\dots<u_n<a$ be fixed elements of $I\cap\mathbb{Q}$ and define $U(t):=\prod_{j=1}^{n}(t-u_j)$ for $t\in[a,b]$. Then $U$ is an increasing and positive polynomial on $[a,b]$. Therefore $U\leq U(b)$ and there exists a positive number $M>0$ such that $|U'|\leq M$ on $[a,b]$. Hence, by the Lagrange mean value theorem, $U$ is Lipschitzian over $[a,b]$ with a Lipschitz modulus $M$. From (1.7), for all $x,y\in[a,b]\cap\mathbb{Q}$, it follows that

$$\frac{1}{y-x}\left(\frac{f(y)}{U(y)}-\frac{f(x)}{U(x)}\right)\geq C(u_1,\dots,u_n). \tag{1.8}$$

By putting $x=a$, this inequality yields

$$\begin{aligned} f(y)&\geq\frac{U(y)}{U(a)}f(a)+U(y)C(u_1,\dots,u_n)(y-a)\\ &\geq-\frac{U(y)}{U(a)}|f(a)|-U(y)|C(u_1,\dots,u_n)|(y-a)\\ &\geq-\frac{U(b)}{U(a)}|f(a)|-U(b)|C(u_1,\dots,u_n)|(b-a), \end{aligned}$$

which shows that $f$ is bounded from below on $[a,b]\cap\mathbb{Q}$. On the other hand, putting $y=b$ in (1.8), we can obtain that

$$\begin{aligned} f(x)&\leq\frac{U(x)}{U(b)}f(b)+U(x)C(u_1,\dots,u_n)(x-b)\\ &\leq\frac{U(x)}{U(b)}|f(b)|+U(x)|C(u_1,\dots,u_n)|(b-x)\\ &\leq|f(b)|+U(b)|C(u_1,\dots,u_n)|(b-a), \end{aligned}$$

which shows that $f$ is also bounded from above on $[a,b]\cap\mathbb{Q}$. Thus, there exists a nonnegative number $K$ such that $|f(x)|\leq K$ for $x\in[a,b]\cap\mathbb{Q}$. The inequality (1.8) now yields

$$\begin{aligned} f(y)-f(x) &\geq \frac{U(y)-U(x)}{U(x)}f(x)+U(y)C(u_1,\dots,u_n)(y-x) \\ &\geq -\frac{U(y)-U(x)}{U(x)}|f(x)|-U(y)|C(u_1,\dots,u_n)|(y-x) \\ &\geq -\frac{M(y-x)}{U(a)}K-U(b)|C(u_1,\dots,u_n)|(y-x) \\ &= -\left(\frac{MK}{U(a)}+U(b)|C(u_1,\dots,u_n)|\right)(y-x). \end{aligned}$$

Let, additionally $b<u_n'$. Then $V(t):=(t-u_n')\prod_{j=1}^{n-1}(t-u_j)$ for $t\in[a,b]$. Then $V$ is negative polynomial on $[a,b]$. Therefore there exist positive numbers $M_0, M_1$ and $M_2$ such that $M_0\leq|V|\leq M_1$ and $|V'|\leq M_2$ on $[a,b]$. Hence again by the Lagrange mean value theorem $V$ is Lipschitzian over $[a,b]$ with a Lipschitz modulus $M_2$. From (1.7) for all $x,y\in[a,b]\cap\mathbb{Q}$ we have

$$\frac{1}{y-x}\left(\frac{f(y)}{V(y)}-\frac{f(x)}{V(x)}\right)\geq C(u_1,\dots,u_{n-1},u_n'),$$

which is equivalent to

$$\begin{aligned} f(y)-f(x) &\leq \frac{V(y)-V(x)}{V(x)}f(x)+V(y)C(u_1,\dots,u_{n-1},u_n')(y-x) \\ &\leq \left|\frac{V(y)-V(x)}{V(x)}f(x)\right|+|V(y)C(u_1,\dots,u_{n-1},u_n')|(y-x) \\ &\leq \frac{M_2|y-x|}{M_0}K+M_1|C(u_1,\dots,u_{n-1},u_n')|(y-x) \\ &= \left(\frac{M_2K}{M_0}+M_1|C(u_1.\dots,u_{n-1},u_n')|\right)(y-x) \end{aligned}$$

Define

$$\begin{aligned} L:=\max\Bigg\{&\left(\frac{MK}{U(a)}+U(b)|C(u_1,\dots,u_n)|\right), \\ &\qquad\qquad \left(\frac{M_2K}{M_0}+M_1|C(u_1,\dots,u_{n-1},u_n')|\right)\Bigg\}. \end{aligned}$$

Therefore $f$ is Lipschitz with modulus $L$ on the dense set $[a,b]\cap\mathbb{Q}$. By applying [**33**, Lemma 1] with $D=[a,b]\cap\mathbb{Q}$, we get that there exists a continuous function $g:I\to\mathbb{R}$ such that $g|_{I\cap\mathbb{Q}}=f|_{I\cap\mathbb{Q}}$.

To complete the proof, we have to show that $g$ is $n$-convex. Let $y_0, \dots, y_{n+1}$ be arbitrary pairwise distinct elements of $I$. Then, for all $j \in \{0, \dots, n+1\}$, there exist rational sequences $(x_{k,j})_{k\in\mathbb{N}}$ converging to $y_j$ as $k \to \infty$ with the property that the elements $x_{k,0}, x_{k,1}, \dots, x_{k,n}, x_{k,n+1}$ are pairwise distinct for all $k \in \mathbb{N}$. Then, by applying the $n$-convexity property of $f$ with rational arguments (i.e., inequality (1.5)), for all $k \in \mathbb{N}$, we get

$$[x_{k,0}, x_{k,1}, \dots, x_{k,n}, x_{k,n+1}; g] = [x_{k,0}, x_{k,1}, \dots, x_{k,n}, x_{k,n+1}; f] \geq 0$$

By the continuity of $g$ the $(n+1)$st-order divided difference of $g$ is a continuous function of its arguments. Thus, upon taking the limit as $k \to \infty$, the above inequality yields that

$$[y_0, y_1, \dots, y_n, y_{n+1}; g] \geq 0.$$

Therefore, $g$ is $n$-convex, indeed. □

Now we give a new proof for Theorem 1.2.

PROOF OF THEOREM 1.2. Since $f : I \to \mathbb{R}$ is $n$-Wright convex, therefore by Lemma 1.1, $f$ is $n$-Jensen convex, i.e, $f$ satisfies (1.1). In particular, $f|_{I\cap\mathbb{Q}}$ is $n$-Jensen convex on $I \cap \mathbb{Q}$. Thus, in view of Lemma 1.3, there exists a continuous $n$-convex function $g : I \to \mathbb{R}$ such that $f|_{I\cap\mathbb{Q}} = g|_{I\cap\mathbb{Q}}$.

To complete the proof, we show that $f - g$ is a polynomial function of degree at most $n$. For this we prove that $\Delta_h^{n+1}(f - g)(x) = 0$ for $h > 0$ and $x \in I \cap (I - (n+1)h)$. This equation is equivalent to

$$\Delta_h^{n+1} f(x) = \Delta_h^{n+1} g(x) \qquad (h > 0, x \in I \cap (I - (n+1)h)). \tag{1.9}$$

More generally, we will show that

$$\Delta_{h_0} \cdots \Delta_{h_n} f(x) = \Delta_{h_0} \cdots \Delta_{h_n} g(x) \tag{1.10}$$

holds for all $h_0, \dots, h_n > 0$ and $x \in I \cap (I - (h_0 + \cdots + h_n))$.

By the $n$-Wright convexity of $f$, for all $h_0, \dots, h_n > 0$ and $x \in I \cap (I - (h_0 + \cdots + h_n))$, we have the inequality

$$\Delta_{h_0} \cdots \Delta_{h_n} f(x) \geq 0. \tag{1.11}$$

This implies that $\Delta_{h_1} \cdots \Delta_{h_n} f : I \cap (I - (h_1 + \cdots + h_n)) \to \mathbb{R}$ is nondecreasing for all $h_1, \dots, h_n > 0$. On the other hand,$\Delta_{h_1} \cdots \Delta_{h_n} g : I \cap (I - (h_1 + \cdots + h_n)) \to \mathbb{R}$ is continuous and the equality $f|_{I\cap\mathbb{Q}} = g|_{I\cap\mathbb{Q}}$ gives us $\Delta_{h_1} \cdots \Delta_{h_n} f(x) = \Delta_{h_1} \cdots \Delta_{h_n} g(x)$ for $x \in I \cap (I - (h_1 + \cdots + h_n)) \cap \mathbb{Q}$ whenever $h_1, \dots, h_n \in \mathbb{Q}_+$. Applying [**33**, Lemma 3], it follows that these two functions are equal to each other also at irrational points of $I \cap (I-(h_1+\cdots+h_n))$, that is,

$$\begin{aligned}\Delta_{h_1}\cdots\Delta_{h_n} f(x) =& \Delta_{h_1}\cdots\Delta_{h_n} g(x)\\ &\qquad (h_1,\dots,h_n\in\mathbb{Q}_+, x\in I\cap(I-(h_1+\cdots+h_n))).\end{aligned}$$

Applying this equality at $x+h_0$ and at $x$, and then subtracting the two equalities side by side, we can see that (1.10) is valid if $h_0>0$, $h_1,\dots,h_n\in\mathbb{Q}_+$ and $x\in I\cap(I-(h_0+\cdots+h_n))$.

Let $k\in\{0,\dots,n\}$ and consider the statement $S_k$ which says that (1.10) holds if $h_0,\dots,h_k>0$, $h_{k+1},\dots,h_n\in\mathbb{Q}_+$ and $x\in I\cap(I-(h_0+\cdots+h_n))$. According to the previous argument, we have that $S_0$ is true. Now assume that, for some $k\in\{0,\dots,n-1\}$, the statement $S_k$ holds. We show that $S_{k+1}$ is also valid. To prove this let $h_0,\dots,h_{k+1}>0$, $h_{k+2},\dots,h_n\in\mathbb{Q}_+$ and $x\in I\cap(I-(h_0+\cdots+h_n))$ and let $h'_{k+1}<h_{k+1}$ be an arbitrary rational number. Then, by the $n$-Wright convexity of $f$, we have

$$\Bigg(\prod_{i\in\{0,\dots,n\}\setminus\{k+1\}}\Delta_{h_i}\Bigg)\Delta_{h_{k+1}-h'_{k+1}} f(x+h'_{k+1})\ge 0.$$

Therefore, using the statement $S_k$ in the last step, we get

$$\begin{aligned}&\Delta_{h_0}\cdots\Delta_{h_n} f(x)\\ &=\Bigg(\prod_{i\in\{0,\dots,n\}\setminus\{k+1\}}\Delta_{h_i}\Bigg)\cdot\Delta_{h_{k+1}} f(x)\\ &=\Bigg(\prod_{i\in\{0,\dots,n\}\setminus\{k+1\}}\Delta_{h_i}\Bigg)\big(f(x+h_{k+1})-f(x)\big)\\ &=\Bigg(\prod_{i\in\{0,\dots,n\}\setminus\{k+1\}}\Delta_{h_i}\Bigg)\big(f(x+h_{k+1})-f(x+h'_{k+1})+f(x+h'_{k+1})-f(x)\big)\\ &=\Bigg(\prod_{i\in\{0,\dots,n\}\setminus\{k+1\}}\Delta_{h_i}\Bigg)\big(\Delta_{h_{k+1}-h'_{k+1}} f(x+h'_{k+1})+\Delta_{h'_{k+1}} f(x)\big)\\ &\ge\Bigg(\prod_{i\in\{0,\dots,n\}\setminus\{k+1\}}\Delta_{h_i}\Bigg)\Delta_{h'_{k+1}} f(x)\\ &=\Bigg(\prod_{i\in\{0,\dots,n\}\setminus\{k+1\}}\Delta_{h_i}\Bigg)\Delta_{h'_{k+1}} g(x).\end{aligned}$$

Using that $g$ is continuous, after taking the limit $h'_{k+1}\to h_{k+1}$, we get that

$$\Delta_{h_0}\cdots\Delta_{h_n} f(x)\ge\Bigg(\prod_{i\in\{0,\dots,n\}\setminus\{k+1\}}\Delta_{h_i}\Bigg)\Delta_{h_{k+1}} g(x)=\Delta_{h_0}\cdots\Delta_{h_n} g(x).$$

To prove the other direction let $h''_{k+1} > h_{k+1}$ be an arbitrary rational number, again by the $n$-Wright convexity of $f$ and statement $S_k$, we get

$$\begin{aligned}
&\Delta_{h_0} \cdots \Delta_{h_n} f(x)\\
&\qquad = \Bigg( \prod_{i \in \{0,\dots,n\} \setminus \{k+1\}} \Delta_{h_i} \Bigg) \cdot \Delta_{h_{k+1}} f(x)\\
&\qquad = \Bigg( \prod_{i \in \{0,\dots,n\} \setminus \{k+1\}} \Delta_{h_i} \Bigg) \big( f(x + h_{k+1}) - f(x) \big)\\
&\qquad = \Bigg( \prod_{i \in \{0,\dots,n\} \setminus \{k+1\}} \Delta_{h_i} \Bigg) \big( f(x + h_{k+1}) - f(x + h''_{k+1})\\
&\qquad\qquad\qquad\qquad\qquad\qquad + f(x + h''_{k+1}) - f(x) \big)\\
&\qquad = \Bigg( \prod_{i \in \{0,\dots,n\} \setminus \{k+1\}} \Delta_{h_i} \Bigg) \big( - \Delta_{h''_{k+1} - h_{k+1}} f(x + h_{k+1}) + \Delta_{h''_{k+1}} f(x) \big)\\
&\qquad \le \Bigg( \prod_{i \in \{0,\dots,n\} \setminus \{k+1\}} \Delta_{h_i} \Bigg) \Delta_{h''_{k+1}} f(x)\\
&\qquad = \Bigg( \prod_{i \in \{0,\dots,n\} \setminus \{k+1\}} \Delta_{h_i} \Bigg) \Delta_{h''_{k+1}} g(x).
\end{aligned}$$

Upon taking the limit $h''_{k+1} \to h_{k+1}$ and using that $g$ is continuous, we get that

$$\Delta_{h_0} \cdots \Delta_{h_n} f(x) \le \Bigg( \prod_{i \in \{0,\dots,n\} \setminus \{k+1\}} \Delta_{h_i} \Bigg) \Delta_{h_{k+1}} g(x) = \Delta_{h_0} \cdots \Delta_{h_n} g(x).$$

Combining the above two inequalities, we can see that $S_{k+1}$ is valid and hence, we have proved that $S_k$ is true for all $k \in \{0, \dots, n\}$, in particular, for $k = n$. This means that (1.10) is satisfied for any $h_0, \cdots, h_n > 0$. Consequently, it holds with $h_0 = \cdots = h_n = h > 0$, proving that (1.9) is satisfied. This implies that $f - g$ is a polynomial function of degree at most $n$. $\square$

## CHAPTER 2

# On convexity properties with respect to a Chebyshev system

## 1. Preliminaries

The simplex of strictly ordered $n$-tuples of a set $H \subset \mathbb{R}$, denoted by $\sigma_n(H)$, is defined by

$$\sigma_n(H) := \{(x_1, \dots, x_n) \in H \quad | \quad x_1 < \dots < x_n\}.$$

Obviously, the $\sigma_n(H)$ is a nonempty set if and only if the cardinality $|H|$ of $H$ is bigger than or equal to $n$. We adopt that $|H| \geq n$. Let $\omega = (\omega_1, \dots, \omega_n) : H \to \mathbb{R}^n$ be a vector-valued function, and define the functional operator $\Phi_\omega := \Phi_{(\omega_1,\dots,\omega_n)} : \sigma_n(H) \to \mathbb{R}$ by

$$\Phi_\omega(x_1, \dots, x_n) := \begin{vmatrix} \omega_1(x_1) \dots \omega_1(x_n) \\ \vdots \quad \ddots \quad \vdots \\ \omega_n(x_1) \dots \omega_n(x_n) \end{vmatrix} \qquad ((x_1, \dots, x_n) \in \sigma_n(H)).$$

A continuous function $\omega$ is said to be an *$n$-dimensional positive (respectively negative) Chebyshev system over $H$* if $\Phi_\omega$ is strictly positive (respectively, strictly negative) over $\sigma_n(H)$. The system $\omega$ is called an *$n$-dimensional Chebyshev system over $H$* if it is either a positive or a negative Chebyshev system over $H$. If $\omega : \mathbb{R} \to \mathbb{R}^n$ equals the *$n$-dimensional standard or polynomial system $\pi_n : I \to \mathbb{R}^n$*, which is defined by

$$\pi_n(t) := (1, t, \dots, t^{n-1}) \qquad (t \in \mathbb{R}),$$

then, by computing Vandermonde determinants, one can easily show that it is a positive Chebyshev system. More generally, if $p_1 < \dots < p_n$ are given exponents, then one can show that the system given by

$$\mathbb{R}_+ \ni t \mapsto (t^{p_1}, \dots, t^{p_n})$$

is also a positive Chebyshev system on $\mathbb{R}_+$. Important Chebyshev system arise also related to hyperbolic and trigonometric functions. For instance, for all $n \in \mathbb{N}$, the systems given by

$$I \ni t \mapsto (\cos(t), \sin(t), \dots, \cos(nt), \sin(nt)),$$
$$I \ni t \mapsto (1, \cos(t), \sin(t), \dots, \cos(nt), \sin(nt))$$

are positive $2n$- and $(2n+1)$-dimensional Chebyshev systems over any nonempty open interval $I$ with length less than or equal to $\pi$ and $2\pi$, respectively. (For the proof of this statements, see the introduction of the paper [**34**].) There are analogous Chebyshev systems in terms of hyperbolic functions as well. We give the standard examples of convex functions with respect of Chebyshev system.

(i) Polynomial system: $\omega(x) := (1, x, \dots, x^n)$;
(ii) exponential system: $\omega(x) := (1, \exp(x), \dots, \exp(nx))$;
(iii) hyperbolic system: $\omega(x) := (1, \cosh(x), \sinh(x), \dots, \cosh(nx), \sinh(nx))$ and
(iv) trigonometric system: $\omega(x) := (1, \cos(x), \sin(x), \dots, \cos(nx), \sin(nx))$,

where $\omega : I \to \mathbb{R}^n$ for all items (i), (ii) and (iii), and $\omega : ]-\frac{\pi}{2}, \frac{\pi}{2}[ \to \mathbb{R}^n$ for (iv). For further standard applications of Chebyshev systems, we refer to the monographs [**7**], [**17**] and [**18**].

In what follows, we recall some definitions from the paper [**34**] (see also the paper [**12**] for these definitions in the polynomial setting). Let $I \subset \mathbb{R}$ be a nonvoid interval, $n \in \mathbb{N}$ and let $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an $n$-dimensional positive Chebyshev system over $I$. For a function $f : I \to \mathbb{R}$, the functional operator $\Phi_{(\omega,f)} : \sigma_{n+1}(I) \to \mathbb{R}$ is defined by $\Phi_{(\omega,f)} := \Phi_{(\omega_1,\dots,\omega_n,f)}$.

For a given vector $t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, a function $f : I \to \mathbb{R}$ is said to be $(t,\omega)$-*convex* if

$$\Phi_{(\omega,f)}(x, x + t_1 h, \dots, x + (t_1 + \dots + t_n)h) \geq 0 \tag{2.1}$$

holds for all $h > 0$, $x \in I$ with $x + (t_1 + \dots + t_n)h \in I$. If $T \subseteq \mathbb{R}_+$ and $f$ is $(t,\omega)$-convex for every $t \in T^n$, then $f$ is called $(T,\omega)$-*convex*.

If $t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$ and (2.1) is satisfied with equality, then $f$ is called a $(t,\omega)$-*affine function*. If $T \subseteq \mathbb{R}_+$ and $f$ is $(t,\omega)$-affine for every $t \in T^n$, then $f$ is called $(T,\omega)$-*affine*. In particular, we say that $f$ is $\omega$-*Jensen convex* if it is $(\{1\},\omega)$-convex, i.e., if

$$\Phi_{(\omega,f)}(x, x + h, \dots, x + nh) \geq 0 \tag{2.2}$$

holds for all $h > 0$, $x \in I$ with $x + nh \in I$. If (2.2) is valid with equality instead of inequality, then $f$ is said to be $\omega$-*Jensen affine*.

A function $f$ is termed $\omega$-*convex* if it is $(\mathbb{R}_+,\omega)$-convex. It is easy to see that $f$ is $\omega$-convex on $I$ if and only if

$$\Phi_{(\omega,f)}(x_0, x_1, \dots, x_n) \geq 0 \qquad ((x_0, x_1, \dots, x_n) \in \sigma_{n+1}(I)). \tag{2.3}$$

A function $f$ is called $\omega$-*affine* if (2.3) is satisfied with equality.

We have to mention that in the case when $\omega = \pi_n$, then the concepts of $\omega$-convexity and $\omega$-Jensen convexity, was introduced by Hopf [**14**] and Popoviciu [**39**] (see also the book [**21**] by Kuczma) and these properties were called convexity and Jensen convexity of order $(n-1)$, respectively. If a function $f : I \to \mathbb{R}$ is $n$ times differentiable, then it is convex of order $(n-1)$, i.e., convex with respect to the polynomial system $\pi_n$ if and only if the $n$th derivative of $f$ is nonnegative over $I$. In the particular case when $n = 2$, this is the standard characterization of convexity of twice differentiable functions.

It is a nontrivial question whether or not $\omega$-convex functions form a proper subclass of $\omega$-Jensen convex functions. Depending on the Chebyshev system, the answer could be positive and negative as well. On the other hand, it is well-known that, for all $n \geq 2$, $\pi_n$-convex functions form a proper subset of $\pi_n$-Jensen convex functions. By [**32**, Theorem 2], it follows that, for any additive function $A : \mathbb{R} \to \mathbb{R}$ and $n \in \mathbb{N}$, the function $f := A^{n-1}$ is Jensen affine (and hence it is Jensen convex) of order $n-1$. On the other hand, $f$ is convex of order $n-1$ if and only if $A$ is continuous. Therefore, if $A$ is discontinuous, then $f$ cannot be convex of order $n-1$. For a construction of a Jensen convex function of order $n-1$ which is not Wright convex of order $n-1$, we refer to the paper [**29**].

The following result shows that, for any nonempty set $T \subseteq \mathbb{R}_+$, $(T,\omega)$-convexity implies $\omega$-Jensen convexity.

THEOREM 2.1. *Let $T \subseteq \mathbb{R}_+$ be a nonempty set. If a function $f : I \to \mathbb{R}$ is $(T,\omega)$-convex (resp. $(T,\omega)$-affine), then it is $(\mathbb{Q}_+,\omega)$-convex (resp. $(\mathbb{Q}_+,\omega)$-affine), in particular, it is $\omega$-Jensen convex (resp. $\omega$-Jensen affine).*

PROOF. The result immediately follows from [**34**, Theorem 5]. □

In section 2 we show that to any $\omega$-Jensen convex (resp. $\omega$-Jensen affine) function there exists a continuous $\omega$-convex (resp. $\omega$-affine) function so that these two functions coincide on dense subset of their domains. As a corollary, we obtain that $\omega$-convex functions are automatically continuous. The Bernstein-Doetsch theorem is generalized to the setting of $\omega$-convexity with respect to a positive Chebyshev system, that is, we show that an $\omega$-Jensen convex function which is bounded over some nonempty open subinterval, is also $\omega$-convex. We also establish characterizations of $\omega$-Jensen affine functions in several settings. A sufficient condition on $\omega$ that ensures the existence of discontinuous $\omega$-Jensen affine (resp. $\omega$-Jensen convex) function is given as well. Some of the results of this section extend that of the paper [**25**] by Matkowski.

In Section 3, we generalize the concept of Wright convexity to $\omega$-Wright convexity with respect to a positive Chebyshev system, and establish its relationship with $\omega$-convexity and $\omega$-Jensen convexity by showing that $\omega$-Wright convexity is an intermediate property. The question whether the inclusions are proper or not remains open for the general setting. We also extend Ng's decomposition theorem to the setting of $\omega$-Wright convexity with respect certain positive Chebyshev systems. Finally, in the two dimensional case, we show that $\omega$-Wright convexity could be equivalent to Wright convexity for certain positive Chebyshev systems.

## 2. Results on $\omega$-Jensen functions

In what follows, if $D$ is a subset of $I$ and $f : D \to \mathbb{R}$, then $f$ is said to be *compactly uniformly continuous* if, for all compact subintervals $[a, b]$ of $I$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in [a, b] \cap D$ with $|x - y| < \delta$, we have that $|f(x) - f(y)| < \varepsilon$.

LEMMA 2.2. *Let $D$ be a subset of $I$ and let $f : D \to \mathbb{R}$ be a compactly uniformly continuous function. Then $f$ admits a continuous extension to $I$. Provided that $D$ is dense, the extension is unique.*

The verification of this lemma is standard and straightforward, however, for the convenience of the reader, we provide its proof.

PROOF. Let $x \in I$ be fixed and let $(x_n)$ be an arbitrary sequence in $D$ such that $x_n \to x$ as $n \to \infty$. We point out first that $(f(x_n))$ is a Cauchy sequence. The set $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact, therefore there exists a compact subinterval $[a, b] \subset I$ which contains all the members of the sequence $(x_n)$.

Let $\varepsilon > 0$ be arbitrary. By the compact uniform continuity of $f$, there exists $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon$ for all $u, v \in [a, b] \cap D$ with $|u - v| < \delta$. The sequence $(x_n)$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that, for all $n, m \geq N$, we have $|x_n - x_m| < \delta$. Consequently,

$$|f(x_n) - f(x_m)| < \varepsilon$$

for all $n, m \geq N$. Therefore $(f(x_n))$ is a Cauchy sequence which implies its convergence. Denote its limit by $g(x)$.

We show that $g$ is well defined at $x$. If $(x_n)$ and $(y_n)$ are two sequences in $D$ converging to $x$, then the sequence $(x_1, y_1, x_2, y_2, x_3, y_3 \dots)$ is also converges to $x$. What we have proved above implies that $(f(x_1), f(y_1), f(x_2), f(y_2), f(x_3), f(y_3), \dots)$ is a convergent sequence and hence the subsequences $(f(x_n)$ and $(f(y_n))$ must have the same limit. Therefore the value of $g$ at $x$ does not depend on the choice of the sequence converging to $x$.

Clearly, if $x \in D$, then the sequence $x_n = x$ converges to $x$, hence $g(x) = f(x)$. Therefore, $g$ is an extension of $f$.

In order to prove that $g$ is continuous on $I$, we show that it is compactly uniformly continuous on $I$. To see this, let $[a,b] \subseteq I$. Let $\varepsilon > 0$ be arbitrary and, using the compact uniform continuity of $f$, choose $\delta > 0$ such that, for all $u, v \in [a,b] \cap D$ with $|u-v| < \delta$, we have $|f(u) - f(v)| \leq \varepsilon$.

Let $x, y \in [a,b]$ be arbitrary with $|x-y| < \delta$ and we are going to show that $|g(x) - g(y)| \leq \varepsilon$. Indeed, consider two sequences $(x_n)$ and $(y_n)$ in $[a,b] \cap D$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then $|x_n - y_n| \to |x-y| < \delta$, hence, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $|x_n - y_n| < \delta$. Therefore, according to the choice of $\delta$ (with $u := x_n$ and $v := y_n$), we get that $|f(x_n) - f(y_n)| < \varepsilon$ holds for all $n \geq N$. Upon taking the limit $n \to \infty$, it follows that $|g(x) - g(y)| \leq \varepsilon$. $\square$

THEOREM 2.3. *Let $n \geq 2$ and let $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an $n$-dimensional positive Chebyshev system and let $\mathbb{K}$ be a subfield of $\mathbb{R}$. If a function $f : I \to \mathbb{R}$ is $(\mathbb{K}_+, \omega)$-convex (resp. $(\mathbb{K}_+, \omega)$-affine), then there exists a continuous $\omega$-convex (resp. $\omega$-affine) function $g : I \to \mathbb{R}$ such that $g|_{I \cap \mathbb{K}} = f|_{I \cap \mathbb{K}}$.*

(Here, and in the sequel, $\mathbb{K}_+$ denotes the intersection $\mathbb{K} \cap \mathbb{R}_+$.)

PROOF. If $f$ is $(\mathbb{K}_+, \omega)$-convex, then by definition we have the inequality

$$\Phi_{(\omega,f)}(x_0, x_1, \dots, x_n) \geq 0 \qquad ((x_0, \dots, x_n) \in \sigma_{n+1}(I \cap \mathbb{K})). \tag{2.4}$$

Indeed, apply the inequality (2.1) with $x := x_0$, $h := 1$, and $t_i := x_i - x_{i-1} \in \mathbb{K}_+$. If $f$ is assumed to be $(\mathbb{K}_+, \omega)$-affine, then (2.4) is satisfied with equality.

Using the continuity of the function $\omega$, we are going to show that the restricted function $f|_{I \cap \mathbb{K}}$ is compactly uniformly continuous. To prove this let $[a,b] \subset I$ be arbitrary. Without loss of generality, we can assume that $a, b \in \mathbb{K}$. (In fact, if one of $a$ or $b$ is not in $\mathbb{K}$, then, using the density of $\mathbb{K}$, we can find $a', b' \in \mathbb{K} \cap I$ such that $a' < a$ and $b < b'$ and then we can work on the closed interval $[a', b']$.) We fix some elements $u_0 < u_1 < \cdots < u_{n-2} < a$ and $b < v$ of the set $I \cap \mathbb{K}$. Then, for $x, y \in [a,b] \cap \mathbb{K}$ with $x < y$, we have that $(u_0, \dots, u_{n-2}, x, y), (u_0, \dots, u_{n-3}, x, y, v) \in \sigma_{n+1}(I \cap \mathbb{K})$, therefore (2.4) implies

$$\begin{aligned} &\Phi_{(\omega,f)}(u_0, \dots, u_{n-2}, x, y) \geq 0 \qquad \text{and} \\ &\Phi_{(\omega,f)}(u_0, \dots, u_{n-3}, x, y, v) \geq 0. \end{aligned} \tag{2.5}$$

The first of these inequalities implies that

$$\begin{vmatrix} \omega_1(u_0) & \dots & \omega_1(u_{n-2}) & \omega_1(x) & \omega_1(y) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \omega_n(u_0) & \dots & \omega_n(u_{n-2}) & \omega_n(x) & \omega_n(y) \\ f(u_0) & \dots & f(u_{n-2}) & f(x) & f(y) \end{vmatrix} \geq 0.$$

Developing this determinant by the last row, this inequality is equivalent to

$$f(y)P(x) - f(x)P(y) \geq R(x,y) \qquad ((x,y) \in \sigma_2(I \cap \mathbb{K})), \tag{2.6}$$

where $P : [a,b] \to \mathbb{R}$ and $R : [a,b]^2 \to \mathbb{R}$ are defined by

$$P(z) := \Phi_\omega(u_0, \dots, u_{n-2}, z)$$

and

$$R(x,y) := \sum_{i=0}^{n-2} (-1)^{n-1-i} f(u_i) \Phi_\omega(u_0, \dots, u_{i-1}, u_{i-1}, \dots, u_{n-2}, x, y)$$

For $z \in [a,b]$, we have that $(u_0, \dots, u_{n-2}, z) \in \sigma_n(I)$. Therefore, by the positivity and continuity of the Chebyshev system, it follows that $P$ is positive and continuous over $[a,b]$. We can also see that $R$ is continuous on $[a,b]^2$ and $R(z,z) = 0$ for all $z \in [a,b]$. Substituting $x = a$ into (2.6), we get that

$$f(y) \geq \frac{R(a,y) + f(a)P(y)}{P(a)} \qquad (y \in [a,b] \cap \mathbb{K}).$$

The right hand side of this inequality is a continuous function of $y$ over the compact interval $[a,b]$, therefore it is bounded from below. Hence $f|_{[a,b]\cap\mathbb{K}}$ is also bounded from below. Putting $y = b$ into (2.6), it follows that

$$\frac{f(b)P(x) - R(x,b)}{P(b)} \geq f(x) \qquad (x \in [a,b] \cap \mathbb{K}).$$

Arguing similarly as above, this inequality yields that $f|_{[a,b]\cap\mathbb{K}}$ is bounded from above hence there exists a positive number $K$ such that, for $x \in [a,b] \cap \mathbb{K}$, we have $|f(x)| \leq K$.

Now we consider the second inequality in (2.5). It can be rewritten as

$$\begin{vmatrix} \omega_1(u_0) & \dots & \omega_1(u_{n-3}) & \omega_1(x) & \omega_1(y) & \omega_1(v) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \omega_n(u_0) & \dots & \omega_n(u_{n-3}) & \omega_n(x) & \omega_n(y) & \omega_n(v) \\ f(u_0) & \dots & f(u_{n-3}) & f(x) & f(y) & f(v) \end{vmatrix} \geq 0.$$

Developing this determinant by its last row, this inequality is equivalent to

$$f(x)Q(y) - f(y)Q(x) \geq S(x,y) \qquad ((x,y) \in \sigma_2(I \cap \mathbb{K})), \tag{2.7}$$

where $Q : [a,b] \to \mathbb{R}$ and $S : [a,b]^2 \to \mathbb{R}$ are defined by

$$Q(z) := \Phi_\omega(u_0, \dots, u_{n-3}, z, v)$$

and

$$\begin{aligned}S(x,y) := \sum_{i=0}^{n-3}(-1)^{n-1-i} f(u_i)\Phi_\omega(u_0, \dots, u_{i-1}, u_{i-1}, \dots, u_{n-3}, x, y, v)\\ - f(v)\Phi_\omega(u_0, \dots, u_{n-3}, x, y).\end{aligned}$$

The inclusion $(u_0, \dots, u_{n-3}, z, v) \in \sigma_n(I \cap \mathbb{K})$, the positivity and continuity of the Chebyshev system yield that $Q$ is a positive and continuous function over $[a,b]$. We also have that $S$ is continuous over $[a,b]^2$ and $S(z,z) = 0$ for all $z \in [a,b]$.

For $x, y \in [a,b] \cap \mathbb{K}$ the inequalities (2.6) and (2.7) imply that

$$\begin{aligned} f(y) - f(x) &\geq \frac{P(y) - P(x)}{P(x)} f(x) + \frac{R(x,y)}{P(x)} \\ &\geq -K\frac{|P(y) - P(x)|}{P(x)} + \frac{R(x,y)}{P(x)} =: A(x,y), \\ f(y) - f(x) &\leq \frac{Q(y) - Q(x)}{Q(x)} f(x) + \frac{S(x,y)}{Q(x)} \\ &\leq \frac{|Q(y) - Q(x)|}{Q(x)} K + \frac{S(x,y)}{Q(x)} =: B(x,y). \end{aligned} \tag{2.8}$$

The functions $A$ and $B$ defined on the right hand side of these inequalities are continuous over $[a,b]^2$ and hence they are uniformly continuous over $[a,b]^2$. Therefore, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(x,y), (u,v) \in [a,b]^2$ with $\|(x,y) - (u,v)\| < \delta$, we have

$$\max(|A(x,y) - A(u,v)|, |B(x,y) - B(u,v)|) < \varepsilon.$$

In particular, if $|x - y| < \delta$, then substituting $(u,v) = (x,x)$ and using that $A$ and $B$ vanish at diagonal points of the square $[a,b]^2$, the above inequality implies that

$$\max(|A(x,y)|, |B(x,y)|) < \varepsilon.$$

Therefore, in view of the inequalities in (2.8), $x, y \in [a,b] \cap \mathbb{K}$ with $|x - y| < \delta$, we obtain that

$$|f(y) - f(x)| < \varepsilon.$$

This proves that $f|_{[a,b]\cap\mathbb{K}}$ is uniformly continuous and hence $f|_{I\cap\mathbb{K}}$ is compactly uniformly continuous. If $f$ is $(\mathbb{K}_+, \omega)$-affine, then it is also $(\mathbb{K}_+, \omega)$-convex, therefore we have the same conclusion.

In view of Lemma 2.2 with the dense set $D = I \cap \mathbb{K}$, there exists a continuous function $g : I \to \mathbb{R}$ such that $g(x)|_{I\cap\mathbb{K}} = f(x)|_{I\cap\mathbb{K}}$.

Finally, we show that $g$ is $\omega$-convex (resp. $\omega$-affine). Let $(y_0, \dots, y_n)$ be an arbitrary element of $\sigma_{n+1}(I)$. Then, by the density of $\mathbb{K}$ in $I$, for each $j \in \{0, \dots, n\}$, there exists a sequence $(x_{k,j})_{k\in\mathbb{N}}$ in $\sigma_{n+1}(I \cap \mathbb{K})$ converging to $y_j$ as $k \to \infty$. Then, applying the $(\mathbb{K}_+, \omega)$-convexity (resp. the $(\mathbb{K}_+, \omega)$-affinity) of $f$, we have that (2.4) is valid, we obtain

$$\Phi_{(\omega,g)}(x_{k,0}, x_{k,1}, \dots, x_{k,n}) = \Phi_{(\omega,f)}(x_{k,0}, x_{k,1}, \dots, x_{k,n}) \geq 0$$
$$(\text{resp. } \Phi_{(\omega,g)}(x_{k,0}, x_{k,1}, \dots, x_{k,n}) = \Phi_{(\omega,f)}(x_{k,0}, x_{k,1}, \dots, x_{k,n}) = 0).$$

By the continuity of $g$, the function $\Phi_{(\omega,g)}$ is continuous. Upon taking the limit $k \to \infty$, the above inequality (resp. equality) implies that

$$\Phi_{(\omega,g)}(y_0, \dots, y_n) \geq 0 \qquad (\text{resp. } \Phi_{(\omega,g)}(y_0, \dots, y_n) = 0).$$

Therefore, $g$ is $\omega$-convex (resp. $\omega$-affine). □

COROLLARY 2.4. *Let $n \geq 2$ and $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an $n$-dimensional positive Chebyshev system. If $f : I \to \mathbb{R}$ is $\omega$-convex (resp. $\omega$-affine), then it is continuous on $I$.*

PROOF. The corollary follows by applying Theorem 2.3 with $\mathbb{K} := \mathbb{R}$. □

COROLLARY 2.5. *Let $n \geq 2$ and $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an $n$-dimensional positive Chebyshev system. If $f : I \to \mathbb{R}$ is $\omega$-Jensen convex (resp. $\omega$-Jensen affine), then there exists a continuous $\omega$-convex (resp. $\omega$-affine) function $g : I \to \mathbb{R}$ such that $g|_{I\cap\mathbb{Q}} = f|_{I\cap\mathbb{Q}}$.*

PROOF. In view of Theorem 2.1, the $\omega$-Jensen convexity (resp. $\omega$-Jensen affinity) of $f$ implies that it is $(\mathbb{Q}_+, \omega)$-convex (resp. $(\mathbb{Q}_+, \omega)$-affine). Now the statement of the corollary follows by applying Theorem 2.3 with $\mathbb{K} := \mathbb{Q}$. □

The following statement is the extension of the celebrated Bernstein–Doetsch theorem [**6**] to the setting of $\omega$-Jensen convexity.

THEOREM 2.6. *If $f : I \to \mathbb{R}$ is $\omega$-Jensen convex and bounded on a nonempty open subset of $I$, then it is continuous on $I$.*

PROOF. Let $U$ be a nonvoid open subinterval of $I$ such that $f$ is bounded on $U$ by $K \geq 0$. In the first part of the proof, we are going to show that $f$ is locally bounded on $I$, i.e., for every $v \in I$, there is an open set $V \subseteq I$ containing $v$ such that $f$ is bounded on $V$.

Let $v \in I$ be arbitrary. If $v \in U$, then the statement holds with $V = U$. Therefore, we may assume that $v \notin U$. Choose a closed interval $[a, b] \subset U$. Then either $v < a$ or $b < v$. We consider now the case when $v < a$.

We choose some rational numbers $a - v < r_1 < \cdots < r_n < b - v$. Then we have that $v < a < v + r_1 < \cdots < v + r_n < b$. One can construct a bounded neighborhood $W$ of $v$ such that $\overline{W} \subseteq I$ and, for all $x \in W$, we have $x < a < x + r_1 < \cdots < x + r_n < b$. Now the $\omega$-Jensen convexity of $f$ and Theorem 2.1 yield that $f$ is $(\mathbb{Q}_+, \omega)$-convex, hence we can get that

$$\Phi_{\omega,f}(x, x + r_1, \dots, x + r_n) \geq 0$$

for all $x \in W$. This inequality implies that

$$\begin{vmatrix} \omega_1(x) & \omega_1(x + r_1) & \dots & \omega_1(x + r_n) \\ \vdots & \vdots & \ddots & \\ \omega_n(x) & \omega_n(x + r_1) & \dots & \omega_n(x + r_n) \\ f(x) & f(x + r_1) & \dots & f(x + r_n) \end{vmatrix} \geq 0.$$

Developing the determinant by last row, we obtain

$$\begin{aligned} (-1)^n f(x) \Phi_\omega(x + r_1, \dots, x + r_n) + \sum_{i=1}^{n} (-1)^{n+i} f(x + r_i) \\ \cdot \Phi_\omega(x, x + r_1, \dots, x + r_{i-1}, x + r_{i+1}, \dots, x + r_n) \geq 0. \end{aligned} \tag{2.9}$$

By the boundedness of $f$ on $U$, the inclusion $x + r_i \in [a, b] \subseteq U$ we obtain that $(-1)^{n+i} f(x + r_i) \leq K$. The continuity of $\omega$ and positivity of Chebyshev system yield that the function

$$x \mapsto \sum_{i=1}^{n} \frac{\Phi_\omega(x, x + r_1, \dots, x + r_{i-1}, x + r_{i+1}, \dots, x + r_n)}{\Phi_\omega(x + r_1, \dots, x + r_n)}$$

is bounded from above by a positive number $L$ over the compact set $\overline{W}$. Therefore, the inequality (2.9) implies that, for all $x \in W$,

$$\begin{aligned} &(-1)^{n-1} f(x) \\ &\leq \sum_{i=1}^{n} (-1)^{n+i} f(x + r_i) \frac{\Phi_\omega(x, x + r_1, \dots, x + r_{i-1}, x + r_{i+1}, \dots, x + r_n)}{\Phi_\omega(x + r_1, \dots, x + r_n)} \\ &\leq KL, \end{aligned}$$

which implies that $(-1)^{n-1} f$ is bounded from above over $W$.

To prove that $(-1)^{n-1} f$ is bounded from below over a neighborhood $V \subseteq W$ of $v$, we additionally fix $v_0 \in I$ such that $v_0 < v$. Now choose rational numbers

$$1 < \frac{a - v_0}{v - v_0} < r_2 < r_3 < \cdots < r_n < \frac{b - v_0}{v - v_0}.$$

Then we have

$$a < v_0 + r_2(v - v_0) < v_0 + r_3(v - v_0) < \cdots < v_0 + r_n(v - v_0) < b.$$

Now one can construct a neighborhood $V$ of $v$ such that $V \subseteq W$ and, for all $x \in V$, we have

$$\begin{aligned} v_0 + r_0(x - v_0) < v_0 + r_1(x - v_0) < a < v_0 + r_2(x - v_0) \\ < \cdots < v_0 + r_n(x - v_0) < b, \end{aligned}$$

where $r_0 = 0$ and $r_1 = 1$.

Again applying Theorem 2.1, we conclude that

$$\begin{aligned} \Phi_{(\omega,f)}(v_0 + r_0(x - v_0), v_0 + r_1(x - v_0), v_0 + r_2(x-v_0), \ldots, \\ v_0 + r_n(x - v_0)) \geq 0, \end{aligned}$$

that is,

$$\Phi_{(\omega,f)}(v_0, x, v_0 + r_2(x - v_0), \ldots, v_0 + r_n(x - v_0)) \geq 0,$$

for all $x \in V$. This inequality, for $x \in V$, is equivalent to

$$\begin{vmatrix} \omega_1(v_0) & \omega_1(x) & \omega_1(v_0 + r_2(x - v_0)) & \ldots & \omega_1(v_0 + r_n(x - v_0)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_n(v_0) & \omega_n(x) & \omega_n(v_0 + r_2(x - v_0)) & \ldots & \omega_n(v_0 + r_n(x - v_0)) \\ f(v_0) & f(x) & f(v_0 + r_2(x - v_0)) & \ldots & f(v_0 + r_n(x - v_0)) \end{vmatrix} \geq 0.$$

For brevity, we write $s_i(x) := v_0 + r_i(x - v_0)$, $i \in \{2, \ldots, n\}$. Developing the determinant by its last row, we obtain

$$\begin{aligned} &(-1)^n f(v_0)\Phi_\omega(x, s_2(x), \ldots, s_n(x)) + (-1)^{n-1} f(x)\Phi_\omega(v_0, s_2(x), \ldots, s_n(x)) \\ &+ \sum_{i=2}^{n} (-1)^{n-i} f(s_i)\Phi_\omega(v_0, x, s_2(x) \ldots, s_{i-1}(x), \ldots, s_{i+1}(x), \ldots, s_n(x)) \geq 0, \end{aligned}$$

which yields
(2.10)

$$\begin{aligned} (-1)^{n-1} f(x) \geq (-1)^{n-1} f(v_0) \frac{\Phi_\omega(x, s_2(x), \ldots, s_n(x))}{\Phi_\omega(v_0, s_2(x), \ldots, s_n(x))} \\ + \sum_{i=2}^{n} (-1)^{n-i-1} f(s_i) \frac{\Phi_\omega(v_0, x, s_2(x) \ldots, s_{i-1}(x), \ldots, s_{i+1}(x), \ldots, s_n(x))}{\Phi_\omega(v_0, s_2(x), \ldots, s_n(x))}. \end{aligned}$$

By the boundedness of $f$ on $U$ and the inclusion $s_i(x) \in [a, b] \subseteq U$, we get $(-1)^{n-i-1} f(s_i(x)) \geq -K$. The continuity and the positivity of Chebyshev system $\omega$ yield that the functions

$$x \mapsto \frac{\Phi_\omega(x, s_2(x), \ldots, s_n(x))}{\Phi_\omega(v_0, s_2(x), \ldots, s_n(x))}$$

and

$$x \mapsto \sum_{i=1}^{n} \frac{\Phi_\omega(v_0, x, s_2(x) \dots, s_{i-1}(x), \dots, s_{i+1}(x), \dots, s_n(x))}{\Phi_\omega(v_0, s_2(x), \dots, s_n(x))}$$

are bounded from above by positive numbers $M$ and $N$, respectively, over the compact set $\overline{V} \subseteq \overline{W}$. Therefore, the inequality (2.10) implies, for all $x \in V$, that

$$(-1)^{n-1} f(x) \geq -|f(v_0)|M - KN,$$

which proves that $(-1)^{n-1} f$ is bounded from below over $V$. From the two-sided boundedness, it follows that $(-1)^{n-1} f$ is bounded over $V$, consequently, $f$ is also bounded over $V$.

To complete the proof of the theorem, we have to verify the continuity of $f$ at any point of $I$. Let $v \in I$ be arbitrary. Then, according to what we have proved in the first part, there exists a neighborhood $V \subseteq I$ of $v$ such that $f$ is bounded on $V$ by $K$. We are going to show that $f$ is uniformly continuous on every compact subinterval $[a, b]$ of $V$. This, in particular, implies the continuity of $f$ at $v$.

Let $[a, b] \subseteq V$. We fix additional elements

$$a_1 < b_1 < \cdots < a_{n-1} < b_{n-1} < a < b < a_n < b_n$$

in $V$. Then define the functions

$$\Psi_1 : [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times [a, b]^2 \to \mathbb{R}$$

and

$$\Psi_2 : [a_1, b_1] \times \cdots \times [a_{n-2}, b_{n-2}] \times [a, b]^2 \times [a_n, b_n] \to \mathbb{R}$$

by

$$\Psi_1(v_1, \dots, v_{n-1}, x, y) := \left| 1 - \frac{\Phi_\omega(v_1, \dots, v_{n-1}, y)}{\Phi_\omega(v_1, \dots, v_{n-1}, x)} \right| + \sum_{i=1}^{n-1} \left| \frac{\Phi_\omega(v_1, \dots, v_{i-1}, v_{i+1} \dots, v_{n-1}, x, y)}{\Phi_\omega(v_1, \dots, v_{n-1}, x)} \right|.$$

and

$$\Psi_2(v_1, \dots, v_{n-2}, x, y, v_n) := \left| 1 - \frac{\Phi_\omega(v_1, \dots, v_{n-2}, x, v_n)}{\Phi_\omega(v_1, \dots, v_{n-2}, y, v_n)} \right| + \frac{\Phi_\omega(v_1, \dots, v_{n-2}, x, y)}{\Phi_\omega(v_1, \dots, v_{n-2}, y, v_n)} + \sum_{i=1}^{n-2} \left| \frac{\Phi_\omega(v_1, \dots, v_{i-1}, v_{i+1} \dots, v_{n-2}, x, y, v_n)}{\Phi_\omega(v_1, \dots, v_{n-2}, y, v_n)} \right|.$$

Then the functions $\Psi_1$ and $\Psi_2$ are continuous over a compact rectangle, therefore, they are uniformly continuous. On the other hand,

$\Psi_1(v_1,\dots,v_{n-1},x,x)=\Psi_2(v_1,\dots,v_{n-2},x,x,v_n)=0$ for all $x\in[a,b]$ and $(v_1,\dots,v_{n-1},v_n)\in[a_1,b_1]\times\cdots\times[a_{n-1},b_{n-1}]\times[a_n,b_n]$.

Thus, for every $\varepsilon>0$, there exists $\delta>0$ such that, for all $x,y\in[a,b]$ with $|x-y|<\delta$ and for all $(v_1,\dots,v_{n-1},v_n)\in[a_1,b_1]\times\cdots\times[a_{n-1},b_{n-1}]\times[a_n,b_n]$,

$$\begin{aligned}\Psi_1(v_1,\dots,&v_{n-1},x,y)\\&=\Psi_1(v_1,\dots,v_{n-1},x,y)-\Psi_1(v_1,\dots,v_{n-1},x,x)<\frac{\varepsilon}{K},\\ \Psi_2(v_1,\dots,&v_{n-2},x,y,v_n)\\&=\Psi_2(v_1,\dots,v_{n-2},x,y,v_n)-\Psi_2(v_1,\dots,v_{n-2},x,x,v_n)<\frac{\varepsilon}{K}.\end{aligned}$$

Now choose $x,y\in[a,b]$ with $x<y<x+\delta$. Then choose the rational numbers $\lambda_1,\dots,\lambda_n$ such that

$$\frac{a_i-x}{y-x}<\lambda_i<\frac{b_i-x}{y-x},\qquad i\in\{1,\dots,n\}.$$

Then, with the notation $v_i:=(1-\lambda_i)x+\lambda_i y$, we have that

$$a_i<v_i<b_i,\quad i\in\{1,\dots,n\}.$$

First define

$$r_i:=\frac{v_i-v_1}{y-v_1}\quad(i\in\{1,\dots,n-1\}),\quad r_n:=\frac{x-v_1}{y-v_1},\quad r_{n+1}:=\frac{y-v_1}{y-v_1}.$$

Clearly, due to the chain of inequalities $v_1<v_2\cdots<v_{n-1}<x<y$, we have that $r_1=0<r_2<\cdots<r_n<r_{n+1}=1$. On the other hand, for $i\in\{1,\dots,n-1\}$,

$$\begin{aligned}r_i=\frac{v_i-v_1}{y-v_1}&=\frac{(1-\lambda_i)x+\lambda_i y-((1-\lambda_1)x+\lambda_1 y)}{y-((1-\lambda_1)x+\lambda_1 y))}\\&=\frac{(\lambda_i-\lambda_1)(y-x)}{(1-\lambda_1)(y-x)}=\frac{\lambda_i-\lambda_1}{1-\lambda_1}\in\mathbb{Q},\end{aligned}$$

and

$$r_n=\frac{x-v_1}{y-v_1}=\frac{x-((1-\lambda_1)x+\lambda_1 y)}{y-((1-\lambda_1)x+\lambda_1 y))}=\frac{\lambda_1(x-y)}{(1-\lambda_1)(y-x)}=\frac{-\lambda_1}{1-\lambda_1}\in\mathbb{Q}.$$

Using Theorem 2.1, the $\omega$-Jensen convexity of $f$ implies that

$$\begin{aligned}\Phi_{(\omega,f)}(v_1+r_1(y-v_1),\dots,v_1+r_{n-1}(y-v_1),v_1+&r_n(y-v_1),\\&v_1+r_{n+1}(y-v_1))\geq 0,\end{aligned}$$

which, according to the definition of the numbers $r_0,r_1,\dots,r_{n-1},r_n$, yields that

$$\Phi_{(\omega,f)}(v_1,\dots,v_{n-1},x,y)\geq 0.$$

Now from the above inequality, we get

$$\begin{vmatrix} \omega_1(v_1) & \dots & \omega_1(v_{n-1}) & \omega_1(x) & \omega_1(y) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \omega_n(v_1) & \dots & \omega_n(v_{n-1}) & \omega_n(x) & \omega_n(y) \\ f(v_1) & \dots & f(v_{n-1}) & f(x) & f(y) \end{vmatrix} \geq 0.$$

Developing this determinant by the last row, we obtain

$$\begin{aligned} &f(y)\Phi_\omega(v_1,\dots,v_{n-1},x) - f(x)\Phi_\omega(v_1,\dots,v_{n-1},y) \\ &\quad + \sum_{i=1}^{n-1}(-1)^{n-i-1} f(v_i)\Phi_\omega(v_1,\dots,v_{i-1},v_{i+1}\dots,v_{n-1},x,y) \geq 0. \end{aligned}$$

By the positivity of Chebyshev system and the boundedness of $f$ over $V$, the above inequality yields

$$\begin{aligned} f(x)-f(y) &\leq f(x)\biggl(1-\frac{\Phi_\omega(v_1,\dots,v_{n-1},y)}{\Phi_\omega(v_1,\dots,v_{n-1},x)}\biggr) \\ &\qquad + \sum_{i=1}^{n-1}(-1)^{n-i-1} f(v_i)\frac{\Phi_\omega(v_1,\dots,v_{i-1},v_{i+1}\dots,v_{n-1},x,y)}{\Phi_\omega(v_1,\dots,v_{n-1},x)} \\ &\leq K\Psi_1(v_1,\dots,v_{n-1},x,y) < \varepsilon. \end{aligned} \tag{2.11}$$

Secondly define

$$\begin{gathered} s_i := \frac{v_i - v_1}{v_n - v_1} \quad (i \in \{1,\dots,n-2\}\cup\{n\}), \\ s'_{n-1} := \frac{x - v_1}{v_n - v_1}, \qquad s''_{n-1} := \frac{y - v_1}{v_n - v_1}. \end{gathered}$$

Due to the chain inequalities $v_1 < v_2 \cdots < v_{n-2} < x < y < v_n$, we have that $s_1 = 0 < s_2 < \cdots < s'_{n-1} < s''_{n-1} < s_n = 1$. On the other hand, for $i \in \{1,\dots,n-2\}\cup\{n\}$,

$$\begin{aligned} s_i = \frac{v_i - v_1}{v_n - v_1} &= \frac{(1-\lambda_i)x + \lambda_i y - ((1-\lambda_1)x + \lambda_1 y)}{(1-\lambda_n)x + \lambda_n y - ((1-\lambda_1)x + \lambda_1 y)} \\ &= \frac{(\lambda_i - \lambda_1)(y-x)}{(\lambda_n - \lambda_1)(y-x)} = \frac{\lambda_i - \lambda_1}{\lambda_n - \lambda_1} \in \mathbb{Q}, \end{aligned}$$

$$\begin{aligned} s'_{n-1} = \frac{x - v_1}{v_n - v_1} &= \frac{x - ((1-\lambda_1)x + \lambda_1 y)}{(1-\lambda_n)x + \lambda_n y - ((1-\lambda_1)x + \lambda_1 y)} \\ &= \frac{-\lambda_1(y-x)}{(\lambda_n - \lambda_1)(y-x)} = \frac{-\lambda_1}{\lambda_n - \lambda_1} \in \mathbb{Q} \end{aligned}$$

and

$$s''_{n-1} = \frac{y - v_1}{v_n - v_1} = \frac{y - ((1-\lambda_1)x + \lambda_1 y)}{(1-\lambda_n)x + \lambda_n y - ((1-\lambda_1)x + \lambda_1 y)}$$
$$= \frac{(1-\lambda_1)(y-x)}{(\lambda_n - \lambda_1)(y-x)} = \frac{1-\lambda_1}{\lambda_n - \lambda_1} \in \mathbb{Q}.$$

Now using Theorem 2.1, the $\omega$-Jensen convexity of $f$ implies that

$$\Phi_{(\omega,f)}(v_1 + s_1(v_n - v_1), \dots, v_1 + s'_{n-1}(v_n - v_1), v_1 + s''_{n-1}(v_n - v_1),$$
$$v_1 + s_n(v_n - v_1)) \geq 0,$$

which, according to the definition of the numbers $s_1, s_2, \dots, s'_{n-1}, s''_{n-1}, s_n$, yields that

$$\Phi_{(\omega,f)}(v_1, \dots, v_{n-2}, x, y, v_n) \geq 0.$$

This inequality can be written as

$$\begin{vmatrix} \omega_1(v_1) & \dots & \omega_1(v_{n-2}) & \omega_1(x) & \omega_1(y) & \omega_1(v_n) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \omega_n(v_1) & \dots & \omega_n(v_{n-2}) & \omega_n(x) & \omega_n(y) & \omega_n(v_n) \\ f(v_1) & \dots & f(v_{n-2}) & f(x) & f(y) & f(v_n) \end{vmatrix} \geq 0.$$

Developing this determinant by the last row, we obtain

$$f(v_n)\Phi_\omega(v_1, \dots, v_{n-2}, x, y) - f(y)\Phi_\omega(v_1, \dots, v_{n-2}, x, v_n)$$
$$+ f(x)\Phi_\omega(v_1, \dots, v_{n-2}, y, v_n)$$
$$+ \sum_{i=1}^{n-2}(-1)^{n-i-1} f(v_i)\Phi_\omega(v_1, \dots, v_{i-1}, v_{i+1} \dots, v_{n-2}, x, y, v_n) \geq 0,$$

by the positivity of Chebyshev system, the above inequality is equivalent to (2.12)

$$f(y) - f(x) \leq f(y)\left(1 - \frac{\Phi_\omega(v_1, \dots, v_{n-2}, x, v_n)}{\Phi_\omega(v_1, \dots, v_{n-2}, y, v_n)}\right)$$
$$+ f(v_n)\frac{\Phi_\omega(v_1, \dots, v_{n-2}, x, y)}{\Phi_\omega(v_1, \dots, v_{n-2}, y, v_n)}$$
$$+ \sum_{i=1}^{n-2}(-1)^{n-i-1} f(v_i)\frac{\Phi_\omega(v_1, \dots, v_{i-1}, v_{i+1} \dots, v_{n-2}, x, y, v_n)}{\Phi_\omega(v_1, \dots, v_{n-2}, y, v_n)}$$
$$\leq K\Psi_2(v_1, \dots, v_{n-2}, x, y, v_n) < \epsilon.$$

Hence the inequalities (2.11) and (2.12) imply that, for all $x, y \in [a, b]$ with $|x - y| < \delta$,

$$|f(x) - f(y)| < \varepsilon$$

holds. This proves that $f$ is uniformly continuous on $[a,b]$. The closed interval $[a,b]\subseteq V$ was arbitrary, therefore $f$ is continuous on $V$. $\square$

THEOREM 2.7. *Let $f : I \to \mathbb{R}$ be a function which is bounded on a nonempty open subset of $I$. Then it is $\omega$-Jensen affine if and only if $f = \alpha_1\omega_1+\cdots+\alpha_n\omega_n$ for some $\alpha_1,\dots,\alpha_n \in \mathbb{R}$.*

PROOF. Assume first that $f$ is an $\omega$-Jensen affine function. Then, it is $\omega$-Jensen convex, and using Theorem 2.6, it follows that $f$ is continuous on $I$. We show first that, for all $x_0, x_1,\dots,x_n \in I$,

$$\Phi_{\omega,f}(x_0,x_1,\dots,x_n) = 0. \tag{2.13}$$

Indeed, if two of points $x_0,x_1,\dots,x_n$ coincide, then this equality is obvious. We may assume that these points are pairwise distinct moreover that $x_0 < x_1 < \cdots < x_n$. Let $0 < r_{1,k} < \cdots < r_{n,k}$ be rational sequences converging to $0 < x_1 - x_0 < \cdots < x_n - x_0$, respectively. Then, according to Theorem 2.1, the $\omega$-Jensen affinity of $f$, for all $k \in \mathbb{N}$, yields that

$$\Phi_{\omega,f}(x_0, x_0 + r_{1,k},\dots,x_0 + r_{n,k}) = 0.$$

Using the continuity of $f$ and taking the limit $k \to \infty$, it follows that (2.13) holds.

Let us fix $x_1 < \cdots < x_n$ in $I$ arbitrarily. Then,

$$\Phi_{\omega,f}(x, x_1,\dots,x_n) = 0.$$

holds for all $x \in I$, i.e.,

$$\begin{vmatrix} \omega_1(x) & \omega_1(x_1) & \dots & \omega_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n(x) & \omega_n(x_1) & \dots & \omega_n(x_n) \\ f(x) & f(x_1) & \dots & f(x_n) \end{vmatrix} = 0.$$

Developing this determinant by the first column, we obtain

$$\sum_{i=1}^{n}(-1)^{i-1}\omega_i(x)\Phi_{\omega_1,\dots,\omega_{i-1},\omega_{i+1},\dots,\omega_n,f}(x_1,\dots,x_n) \\ + (-1)^n f(x)\Phi_\omega(x_1,\dots,x_n) = 0.$$

Therefore,

$$f(x) = \sum_{i=1}^{n}(-1)^{n-i}\frac{\Phi_{\omega_1,\dots,\omega_{i-1},\omega_{i+1},\dots,\omega_n,f}(x_1,\dots,x_n)}{\Phi_\omega(x_1,\dots,x_n)}\omega_i(x).$$

Now, with an obvious choice of $\alpha_1,\dots,\alpha_n \in \mathbb{R}$, we can see that $f = \alpha_1\omega_1 + \cdots + \alpha_n\omega_n$ holds.

The reversed statement is obvious, if $f$ is a linear combination of the coordinate functions of $\omega$, then $\Phi_{\omega,f}$ is identically zero by standard properties of determinants. $\square$

The following question seems to be important: What is a necessary and sufficient condition on $\omega$ that ensures the existence of discontinuous $\omega$-Jensen affine or discontinuous $\omega$-Jensen convex functions? The following result provides a sufficient condition. To formulate and prove this condition, we need the following lemma.

LEMMA 2.8. *Let* $\omega : I \to \mathbb{R}^n$ *be a positive Chebyshev system. Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $M = (a_{i,j})_{1\leq i\leq n,\, 0\leq j\leq n-1} \in \mathbb{R}^{n\times n}$ *such that*

$$\omega_i(x) = (a_{i,n-1}x^{n-1} + \cdots + a_{i,1}x + a_{i,0}) \cdot \omega_0(x) \qquad (x \in I,\ i \in \{1,\dots,n\}). \tag{2.14}$$

*Then* $\det(M) > 0$ *and, for all* $(x_1,\dots,x_n) \in \sigma_n(I)$,

$$\Phi_\omega(x_1,\dots,x_n) = \omega_0(x_1)\cdots\omega_0(x_n) \cdot \det(M) \cdot \Phi_{\pi_n}(x_1,\dots,x_n). \tag{2.15}$$

*Additionally, let* $f : I \to \mathbb{R}$*. Then, for all* $(x_0,\dots,x_n) \in \sigma_{n+1}(I)$,

$$\Phi_{\omega,f}(x_0,\dots,x_n) = \omega_0(x_0)\cdots\omega_0(x_n) \cdot \det(M) \cdot \Phi_{\pi_n,f/\omega_0}(x_0,\dots,x_n). \tag{2.16}$$

PROOF. The equality in (2.14) and the product rule for determinants imply, for all $(x_1,\dots,x_n) \in \sigma_n(I)$, that

$$\begin{aligned}
&\Phi_\omega(x_1,\dots,x_n)\\
&\quad= \begin{vmatrix} \big(\sum_{i=0}^{n-1} a_{1,i}x_1^i\big)\omega_0(x_1) & \dots & \big(\sum_{i=0}^{n-1} a_{1,i}x_n^i\big)\omega_0(x_n) \\ \vdots & \ddots & \vdots \\ \big(\sum_{i=0}^{n-1} a_{n,i}x_1^i\big)\omega_0(x_1) & \dots & \big(\sum_{i=0}^{n-1} a_{n,i}x_n^i\big)\omega_0(x_n) \end{vmatrix}\\
&\quad= \omega_0(x_1)\cdots\omega_0(x_n) \begin{vmatrix} \sum_{i=0}^{n-1} a_{1,i}x_1^i & \dots & \sum_{i=0}^{n-1} a_{1,i}x_n^i \\ \vdots & \ddots & \vdots \\ \sum_{i=0}^{n-1} a_{n,i}x_1^i & \dots & \sum_{i=0}^{n-1} a_{n,i}x_n^i \end{vmatrix}\\
&\quad= \omega_0(x_1)\cdots\omega_0(x_n) \begin{vmatrix} a_{1,0} & \dots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n,0} & \dots & a_{n,n-1} \end{vmatrix} \cdot \begin{vmatrix} x_1^0 & \dots & x_n^0 \\ \vdots & \ddots & \vdots \\ x_1^{n-1} & \dots & x_n^{n-1} \end{vmatrix}\\
&\quad= \omega_0(x_1)\cdots\omega_0(x_n) \det(M) \cdot \Phi_{\pi_n}(x_1,\dots,x_n),
\end{aligned}$$

which proves (2.15).

The value of the determinant $\Phi_{\pi_n}(x_1,\dots,x_n)$ equals $\prod_{1\le i<j\le n}(x_j-x_i)>0$ (because it is of Vandermonde-type). Therefore, the positivity of the Chebyshev system $\omega$, the positivity of the function $\omega_0$ and the equality (2.15) yield that $\det(M)>0$.

The equalities in (2.14) and the product rule for determinants again imply, for all $(x_0,\dots,x_n)\in\sigma_{n+1}(I)$, that

$$\begin{aligned}
&\Phi_{\omega,f}(x_0,\dots,x_n)\\
&\quad=\begin{vmatrix}
\big(\sum_{i=0}^{n-1}a_{1,i}x_0^i\big)\omega_0(x_0) & \dots & \big(\sum_{i=0}^{n-1}a_{1,i}x_n^i\big)\omega_0(x_n)\\
\vdots & \ddots & \vdots\\
\big(\sum_{i=0}^{n-1}a_{n,i}x_0^i\big)\omega_0(x_0) & \dots & \big(\sum_{i=0}^{n-1}a_{n,i}x_n^i\big)\omega_0(x_n)\\
f(x_0) & \dots & f(x_n)
\end{vmatrix}\\
&\quad=\omega_0(x_0)\cdots\omega_0(x_n)\begin{vmatrix}
\sum_{i=0}^{n-1}a_{1,i}x_0^i & \dots & \sum_{i=0}^{n-1}a_{1,i}x_n^i\\
\vdots & \ddots & \vdots\\
\sum_{i=0}^{n-1}a_{n,i}x_0^i & \dots & \sum_{i=0}^{n-1}a_{n,i}x_n^i\\
\frac{f}{\omega_0}(x_0) & \dots & \frac{f}{\omega_0}(x_n)
\end{vmatrix}\\
&\quad=\omega_0(x_0)\cdots\omega_0(x_n)\begin{vmatrix}
a_{1,0} & \dots & a_{1,n-1} & 0\\
\vdots & \ddots & \vdots & \\
a_{n,0} & \dots & a_{n,n-1} & 0\\
0 & \dots & 0 & 1
\end{vmatrix}\cdot\begin{vmatrix}
x_0^0 & \dots & x_n^0\\
\vdots & \ddots & \vdots\\
x_0^{n-1} & \dots & x_n^{n-1}\\
\frac{f}{\omega_0}(x_0) & \dots & \frac{f}{\omega_0}(x_n)
\end{vmatrix}\\
&\quad=\omega_0(x_0)\cdots\omega_0(x_n)\cdot\det(M)\cdot\Phi_{\pi_n,f/\omega_0}(x_0,\dots,x_n),
\end{aligned}$$

which shows the validity of (2.16). □

THEOREM 2.9. *Assume that there exist a positive continuous function* $\omega_0 : I\to\mathbb{R}_+$ *and a matrix* $M=(a_{i,j})_{1\le i\le n,\,0\le j\le n-1}\in\mathbb{R}^{n\times n}$ *such that* (2.14) *holds. Then* $f:I\to\mathbb{R}$ *is an* $\omega$*-Jensen convex (resp.* $\omega$*-Jensen affine) function if and only if* $\frac{f}{\omega_0}$ *is a* $\pi_n$*-Jensen convex (resp.* $\pi_n$*-Jensen affine) function.*

PROOF. According to formula (2.16) of Lemma 2.8, for all $h>0$ and $x\in I\cap(I-nh)$, we have that

$$\begin{aligned}
&\Phi_{\omega,f}(x,x+h,\dots,x+nh)\\
&\qquad=\prod_{i=0}^{n}\omega_0(x+ih)\cdot\det(M)\cdot\Phi_{\pi_n,f/\omega_0}(x,x+h,\dots,x+nh).
\end{aligned}$$

Due to the positivity of $\det(M)$ and the positivity of the function $\omega_0$, it follows that the inequality $\Phi_{\omega,f}(x,x+h,\dots,x+nh)\ge 0$ holds if and only if

$\Phi_{\pi_n,f/\omega_0}(x, x+h, \dots, x+nh) \geq 0$ is valid. This shows that $f$ is $\omega$-Jensen convex if and only if $\frac{f}{\omega_0}$ is $\pi_n$-Jensen convex.

Similarly,

$$\Phi_{\omega,f}(x, x+h, \dots, x+nh) = 0$$

if and only if

$$\Phi_{\pi_n,f/\omega_0}(x, x+h, \dots, x+nh) = 0,$$

which proves that $f$ is $\omega$-Jensen affine if and only if $\frac{f}{\omega_0}$ is $\pi_n$-Jensen affine. □

We need to recall the following characterization of $\pi_n$-Jensen affine functions.

THEOREM 2.10. *A function $f : I \to \mathbb{R}$ is $\pi_n$-Jensen affine if and only if there exist a constant $A_0 \in \mathbb{R}$, an additive function $A_1 : \mathbb{R} \to \mathbb{R}$, a symmetric biadditive function $A_2 : \mathbb{R}^2 \to \mathbb{R}$, ..., and a symmetric $(n-1)$-additive function $A_{n-1} : \mathbb{R}^{n-1} \to \mathbb{R}$ such that*

$$f(x) = A_{n-1}(x, \dots, x) + \cdots + A_2(x, x) + A_1(x) + A_0 \quad (x \in I). \tag{2.17}$$

COROLLARY 2.11. *Assume that there exist a positive continuous function $\omega_0 : I \to \mathbb{R}_+$ and a matrix $(a_{i,j})_{1 \le i \le n,\, 0 \le j \le n-1} \in \mathbb{R}^{n \times n}$ such that* (2.14) *holds. Then $f : I \to \mathbb{R}$ is an $\omega$-Jensen affine function if and only if there exist a constant $A_0 \in \mathbb{R}$, an additive function $A_1 : \mathbb{R} \to \mathbb{R}$, a symmetric biadditive function $A_2 : \mathbb{R}^2 \to \mathbb{R}$, ..., and a symmetric $(n-1)$-additive function $A_{n-1} : \mathbb{R}^{n-1} \to \mathbb{R}$ such that*

$$f(x) = \big(A_{n-1}(x, \dots, x) + \ldots + A_2(x, x) + A_1(x) + A_0\big)\omega_0(x) \quad (x \in I). \tag{2.18}$$

PROOF. Assume first that $f$ is $\omega$-Jensen affine. Then, by Theorem 2.9, $\frac{f}{\omega_0}$ is $\pi_n$-Jensen affine. Hence, according to Theorem 2.10, there exist a constant $A_0 \in \mathbb{R}$, and additive function $A_1 : \mathbb{R} \to \mathbb{R}$, a symmetric biadditive function $A_2 : \mathbb{R}^2 \to \mathbb{R}$, ..., a symmetric $(n-1)$-additive function $A_{n-1} : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\frac{f}{\omega_0}(x) = A_{n-1}(x, \dots, x) + \cdots + A_2(x, x) + A_1(x) + A_0 \qquad (x \in I).$$

This proves that $f$ is of the form (2.18).

To prove the reversed implication, assume that there exist a constant $A_0 \in \mathbb{R}$, and additive function $A_1 : \mathbb{R} \to \mathbb{R}$, a symmetric biadditive function $A_2 : \mathbb{R}^2 \to \mathbb{R}$, ..., a symmetric $(n-1)$-additive function $A_{n-1} : \mathbb{R}^{n-1} \to \mathbb{R}$ such that (2.18) holds. Then, according to Theorem 2.10, $\frac{f}{\omega_0}$ is a $\pi_n$-Jensen affine function. In view of Theorem 2.9, this implies that $f$ is $\omega$-Jensen affine. □

## 3. Wright convexity with respect to Chebyshev systems

In 1954, Wright [**46**] introduced a concept of convexity which is stronger than Jensen convexity and weaker than convexity. A function $f : I \to \mathbb{R}$ is called *Wright convex* if

$$f(tx + (1-t)y) + f((1-t)x + ty) \le f(x) + f(y) \quad (x, y \in I,\, t \in [0,1]).$$

One can easily see that convexity implies Wright convexity, and, by putting $t = \frac{1}{2}$ into the above inequality, we can see that Jensen convexity is a consequence of Wright convexity.

A characterization and the ultimate understanding of Wright convexity was established by Ng [**30**], who proved that $f : I \to \mathbb{R}$ is Wright convex if and only if it is of the form $f = g + A|_I$, where $g : I \to \mathbb{R}$ is convex and $A : \mathbb{R} \to \mathbb{R}$ is additive. If $A$ is discontinuous, then $f$ will be discontinuous and hence cannot be convex. On the other hand, if $A$ is a discontinuous additive function, then $|A|$ is Jensen convex but not Wright convex.

The concept of Wright convexity is closely related to Schur convexity, sometimes it is termed ultramodularity and has applications, for instance, in the theory of copulas and $t$-norms (see [**41**] and the references there in).

A higher-order generalization of Wright convexity was introduced by Gilányi and Páles [**12**] as follows. In this paper, a function $f : I \to \mathbb{R}$ was called *Wright convex of order* $(n-1)$, if

$$\Delta_{h_1} \cdots \Delta_{h_n} f(x) \ge 0$$

holds for all $h_1, \dots, h_n > 0$ and $x \in I \cap (I - (h_1 + \cdots + h_n))$. One can easily see that Wright convexity of order $1$ is equivalent to Wright convexity in the standard sense.

In what follows, we extend the notion of higher-order Wright convexity to the setting of positive Chebyshev systems. Let $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be a positive $n$-dimensional Chebyshev system. We say that $\overline{\omega} : I \to \mathbb{R}^{n+1}$ *is an extension of* $\omega$ if there exists a continuous function $\omega_{n+1} : I \to \mathbb{R}$ such that $\overline{\omega} := (\omega_1, \dots, \omega_n, \omega_{n+1})$ and $\overline{\omega}$ is a positive $(n+1)$-dimensional Chebyshev system.

Let $\omega$ be a positive $n$-dimensional Chebyshev system and $\overline{\omega}$ be an arbitrarily fixed extension of $\omega$. We say that a function $f : I \to \mathbb{R}$ *is* $\overline{\omega}$*-Wright convex* if, for all $h_1, \dots, h_n > 0$ and $x \in I \cap (I - (h_1 + \cdots + h_n))$, the inequality

$$\sum_{(i_1,\dots,i_n)} \frac{\Phi_{(\omega,f)}(x, x + h_{i_1}, \dots, x + h_{i_1} + \cdots + h_{i_n})}{\Phi_{\overline{\omega}}(x, x + h_{i_1}, \dots, x + h_{i_1} + \cdots + h_{i_n})} \ge 0 \tag{2.19}$$

holds, where the summation is taken over all permutation $(i_1, \dots, i_n)$ of the elements $\{1, \dots, n\}$.

Our first result establishes the connections between $\omega$-convexity, $\overline{\omega}$-Wright convexity and $\omega$-Jensen convexity.

THEOREM 2.12. *Let $\omega$ be a positive $n$-dimensional Chebyshev system and $\overline{\omega}$ be an extension of $\omega$. Then every $\omega$-convex function is $\overline{\omega}$-Wright convex and every $\overline{\omega}$-Wright convex function is $\omega$-Jensen convex.*

PROOF. Assume first that $f : I \to \mathbb{R}$ is $\omega$-convex. Then, for all $h_1, \dots, h_n > 0$ and $x \in I \cap (I - (h_1 + \dots + h_n))$ all permutation $(i_1, \dots, i_n)$ of the elements $\{1, \dots, n\}$, we have that

$$\Phi_{(\omega,f)}(x, x + h_{i_1}, \dots, x + h_{i_1} + \dots + h_{i_n}) \geq 0.$$

On the other hand, by the positivity of the Chebyshev system $\overline{\omega}$, we also have that

$$\Phi_{\overline{\omega}}(x, x + h_{i_1}, \dots, x + h_{i_1} + \dots + h_{i_n}) > 0.$$

These inequalities yield that (2.19) is valid on the domain indicated and hence $f$ is $\overline{\omega}$-Wright convex.

To verify the second assertion, assume that $f : I \to \mathbb{R}$ is $\overline{\omega}$-Wright convex. Taking $h_1 := \dots = h_n := h > 0$ in inequality (2.19), for all $h > 0$ and $x \in I \cap (I - nh)$, we obtain that

$$\frac{\Phi_{(\omega,f)}(x, x + h, \dots, x + nh)}{\Phi_{\overline{\omega}}(x, x + h, \dots, x + nh)} \geq 0. \tag{2.20}$$

Due to the positivity of the Chebyshev system $\overline{\omega}$, it follows that $\Phi_{\overline{\omega}}(x, x + h, \dots, x + nh)$ is positive, therefore, we can conclude that $\Phi_{(\omega,f)}(x, x + h, \dots, x + nh) \geq 0$, which shows that $f$ is $\omega$-Jensen convex. □

The next theorem describes the connection between $\overline{\omega}$-Wright convexity and Wright convexity of order $(n - 1)$. In what follows, the symbol $[x_0, x_1, \dots, x_n, g]$ denotes the standard $n$th-order divided difference of a function $g : I \to \mathbb{R}$ at the pairwise distinct nodes $x_1, x_1, \dots, x_n \in I$.

THEOREM 2.13. *Assume that there exist a positive continuous function $\omega_0 : I \to \mathbb{R}_+$ and a matrix $M := (a_{i,j})_{1 \leq i \leq n, 0 \leq j \leq n-1} \in \mathbb{R}^{n \times n}$ such that* (2.14) *holds. Define $\omega_{n+1} : I \to \mathbb{R}$ by $\omega_{n+1}(t) := t^n \omega_0(t)$. Then $\overline{\omega} := (\omega, \omega_{n+1})$ is an extension of the Chebyshev system $\omega$. In addition, we have the following assertions:*

*(i) For all $(x_0, x_1, \dots, x_n) \in \sigma_{n+1}(I)$, the equality*

$$\frac{\Phi_{(\omega,f)}(x_0, x_1, \dots, x_n)}{\Phi_{\overline{\omega}}(x_0, x_1, \dots, x_n)} = [x_0, x_1, \dots, x_n; f/\omega_0] \tag{2.21}$$

*holds. Furthermore, a function* $f : I \to \mathbb{R}$ *is* $\omega$*-convex if and only if* $f/\omega_0$ *is convex of order* $(n-1)$*.*

*(ii) For all* $h_1, \dots, h_n > 0$ *and* $x \in I \cap (I - (h_1 + \cdots + h_n))$*, the equality*

$$\begin{aligned} \sum_{(i_1,\dots,i_n)} & \frac{\Phi_{(\omega,f)}(x, x+h_{i_1}, \dots, x+h_{i_1}+\cdots+h_{i_n})}{\Phi_{\overline{\omega}}(x, x+h_{i_1}, \dots, x+h_{i_1}+\cdots+h_{i_n})} \\ & = \frac{\Delta_{h_1}\cdots\Delta_{h_n}(f/\omega_0)(x)}{h_1\cdots h_n} \end{aligned} \tag{2.22}$$

*holds. Furthermore, a function* $f : I \to \mathbb{R}$ *is* $\overline{\omega}$*-Wright convex if and only if* $f/\omega_0$ *is Wright convex of order* $(n-1)$*.*

PROOF. We first verify that $\overline{\omega} = (\omega, \omega_{n+1})$ is a positive Chebyshev system. Let $(x_0, x_1, \dots, x_n) \in \sigma_{n+1}(I)$. Then, applying the equality (2.16) of Lemma 2.8 with $f := \omega_{n+1}$, we get

$$\begin{aligned} &\Phi_{\overline{\omega}}(x_0, x_1, \dots, x_n) \\ &\quad = \omega_0(x_0)\cdots\omega_0(x_n)\cdot\det(M)\cdot\Phi_{(\pi_n,\omega_{n+1}/\omega_0)}(x_0, x_1, \dots, x_n) \\ &\quad = \omega_0(x_0)\cdots\omega_0(x_n)\cdot\det(M)\cdot\Phi_{\pi_{n+1}}(x_0, x_1, \dots, x_n) > 0. \end{aligned} \tag{2.23}$$

The last inequality is due to the fact that $\Phi_{\pi_{n+1}}(x_0, x_1, \dots, x_n)$ is a Vandermonde determinant and $x_0 < x_1 < \cdots < x_n$. This proves that $\overline{\omega}$ is a positive Chebyshev system, indeed.

In the rest of the proof denote $g := f/\omega_0$. To show that assertion (i) holds, let $(x_0, x_1, \dots, x_n) \in \sigma_{n+1}(I)$ be fixed. In view of the Lemma 2.8, we have the equality (2.16). Combining this equality with (2.23), we can obtain

$$\frac{\Phi_{(\omega,f)}(x_0, x_1, \dots, x_n)}{\Phi_{\overline{\omega}}(x_0, x_1, \dots, x_n)} = \frac{\Phi_{(\pi_n,g)}(x_0, x_1, \dots, x_n)}{\Phi_{\pi_{n+1}}(x_0, x_1, \dots, x_n)}.$$

From the theory of divided differences, we have the identity

$$\frac{\Phi_{(\pi_n,g)}(x_0, x_1, \dots, x_n)}{\Phi_{\pi_{n+1}}(x_0, x_1, \dots, x_n)} = [x_0, x_1, \dots, x_n; g],$$

which, together with the previous equality shows that (2.21) holds.

The function $f$ is $\overline{\omega}$-convex if and only if, for all $(x_0, x_1, \dots, x_n) \in \sigma_{n+1}(I)$, the left hand side of (2.21) is nonnegative. According to this equality, this happens if and only if the right hand side is nonnegative, i.e., if $f/\omega_0$ is convex of order $(n-1)$.

To show assertion (ii), let $h_1, \dots, h_n > 0$ and $x \in I \cap (I - (h_1 + \cdots + h_n))$ be fixed. Therefore, with the substitutions $x_j := x + h_1 + \cdots + h_j$, (where

$j \in \{0, \dots, n\}$), (2.21) implies that

$$\frac{\Phi_{(\omega,f)}(x, x+h_1, \dots, x+h_1+\dots+h_n)}{\Phi_{\overline{\omega}}(x, x+h_1, \dots, x+h_1+\dots+h_n)} \\ = [x, x+h_1, \dots, x+h_1+\dots+h_n; g].$$

Applying this equality for $(h_{i_1}, \dots, h_{i_n})$ (instead of $(h_1, \dots, h_n)$), where $(i_1, \dots, i_n)$ is an arbitrary permutation of $(1, \dots, n)$, we can see that

$$\sum_{(i_1,\dots,i_n)} \frac{\Phi_{(\omega,f)}(x, x+h_{i_1}, \dots, x+h_{i_1}+\dots+h_{i_n})}{\Phi_{\overline{\omega}}(x, x+h_{i_1}, \dots, x+h_{i_1}+\dots+h_{i_n})} \\ = \sum_{(i_1,\dots,i_n)} [x, x+h_{i_1}, \dots, x+h_{i_1}+\dots+h_{i_n}; g].$$

On the other hand, from the paper [**12**], we have that

$$\sum_{(i_1,\dots,i_n)} [x, x+h_{i_1}, \dots, x+h_{i_1}+\dots+h_{i_n}; g] = \frac{\Delta_{h_1} \dots \Delta_{h_n} g(x)}{h_1 \cdots h_n}$$

holds, which, together with the previous equality implies (2.22).

The function $f$ is $\overline{\omega}$-Wright convex if and only if, for all $h_1, \dots, h_n > 0$ and $x \in I \cap (I - (h_1 + \dots + h_n))$, the left hand side of (2.22) is nonnegative. According to this equality this happens to be valid if and only if the right hand side is nonnegative, i.e., if $f/\omega_0$ is Wright convex of order $(n-1)$. ☐

In the following result, we establish a characterization theorem for $\overline{\omega}$-Wright convexity provided that the underlying Chebyshev system is strongly related to the polynomial one. This result generalizes the decomposition theorem of Maksa and Páles [**24**] which is related to the polynomial system. An alternative and more elementary proof of that theorem has been recently given by the authors in [**36**].

THEOREM 2.14. *Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $M := (a_{i,j})_{1 \le i \le n, 0 \le j \le n-1} \in \mathbb{R}^{n \times n}$ *such that* (2.14) *holds. Define* $\omega_{n+1} : I \to \mathbb{R}$ *by* $\omega_{n+1}(t) := t^n \omega_0(t)$ *and set* $\overline{\omega} := (\omega, \omega_{n+1})$. *Then a function* $f : I \to \mathbb{R}$ *is* $\overline{\omega}$*-Wright convex if and only if there exist an* $\omega$*-convex function* $F : I \to \mathbb{R}$ *and, for each* $k \in \{1, \dots, n-1\}$, *a symmetric* $k$*-additive mapping* $A_k : \mathbb{R}^k \to \mathbb{R}$ *and a real constant* $A_0$ *such that, for all* $x \in I$,

$$f(x) = F(x) + (A_0 + A_1(x) + \dots + A_{n-1}(x, \dots, x))\omega_0(x). \tag{2.24}$$

PROOF. To prove the necessity, assume that the function $f$ is $\overline{\omega}$-Wright convex. Then the assertion (ii) of Theorem 2.13 implies that $f/\omega_0$ is Wright convex of order $(n-1)$. The decomposition theorem of higher order Wright convex functions [**24**] implies that there exist a function $G : I \to \mathbb{R}$ which is convex of order $(n-1)$, a real constant $A_0$ and, for each $k \in \{1, \dots, n-1\}$, a symmetric $k$-additive mapping $A_k : \mathbb{R}^k \to \mathbb{R}$ such that, for all $x \in I$,

$$\frac{f}{\omega_0}(x) = G(x) + A_0 + A_1(x) + \cdots + A_{n-1}(x, \dots, x). \tag{2.25}$$

This implies that (2.24) holds with $F := G\omega_0$ and $F/\omega_0$ is convex of order $(n-1)$. By assertion (i) of Theorem 2.13, it follows that the function $F$ is $\omega$-convex.

To prove the sufficiency, assume that (2.24) holds, multiplying (2.24) by $1/\omega_0(x)$ implies that

$$\frac{f}{\omega_0}(x) = \frac{F}{\omega_0}(x) + (A_0 + A_1(x) + \cdots + A_{n-1}(x, \dots, x)).$$

Since the function $F$ is $\omega$-convex, therefore the assertion (i) of Theorem 2.13 implies that $F/\omega_0$ is convex of order $(n-1)$. Again, by the decomposition theorem of higher order Wright convex functions [**24**], we can conclude that $f/\omega_0$ is Wright convex of order $(n-1)$. Thus, the assertion (ii) of Theorem 2.13 yields that the function $f$ is $\overline{\omega}$-Wright convex. ☐

In our subsequent result we will prove that if the extension of the two dimensional polynomial system is not a polynomial of at most second degree, then the convexity with respect to the two dimensional polynomial system (i.e., standard convexity) is equivalent to Wright convexity with respect to this extension. For the proof of this result, we will need the following characterization of a polynomial of at most second degree.

LEMMA 2.15. *Let $\rho : I \to \mathbb{R}$ be a continuous function which satisfies the functional equation*

$$\frac{\rho(z)-\rho(y)}{z-y} = \frac{\rho(z+u)-\rho(y-u)}{(z+u)-(y-u)}, \qquad u \geq 0,\ y, z \in (I+u) \cap (I-u),\ z+u \geq y. \tag{2.26}$$

*Then $\rho$ is a polynomial of at most second degree over $I$.*

PROOF. If $u > 0$ and $y \in (I+u) \cap (I-u)$, then the limit of the right hand side exists as $z \to y$ by the continuity of $\rho$, which shows that $\rho$ is differentiable at $y$ and we get

$$\rho'(y) = \frac{\rho(y+u)-\rho(y-u)}{2u}, \qquad u > 0,\ y \in (I+u) \cap (I-u). \tag{2.27}$$

Since $u > 0$ was arbitrary, it follows that $\rho$ is differentiable everywhere on $I$. Now the right hand side of the above equality is differentiable with respect to $y$, which implies that $\rho$ is twice differentiable. Repeating this argument, it follows that $\rho$ is three times differentiable on $I$. We are going to show that $\rho'''$ is identically zero on $I$.

Let $y \in I$ be fixed arbitrarily. Rearranging the equation (2.27), we can obtain that

$$2u\rho'(y) = \rho(y+u) - \rho(y-u), \qquad u \in (y-I) \cap (I-y).$$

Differentiating this equality three times with respect to $u$, we get that

$$0 = \rho'''(y+u) + \rho'''(y-u), \qquad u > 0,\, y \in (I+u) \cap (I-u).$$

With the substitution $u = 0$, we conclude that $\rho'''(y) = 0$. Therefore, $\rho'''$ is identically zero on $I$. This yields that $\rho$ has to be a polynomial of at most second degree. □

THEOREM 2.16. *Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $M := (a_{i,j})_{1\le i\le 2,\, 0\le j\le 1} \in \mathbb{R}^{2\times 2}$ *such that* (2.14) *holds for* $n = 2$. *Assume that* $\omega_3 : I \to \mathbb{R}$ *is a continuous function such that* $\overline{\omega} = (\omega_1, \omega_2, \omega_3)$ *is an extension of* $\omega = (\omega_1, \omega_2)$ *and* $\omega_3/\omega_0$ *is not a polynomial of at most second degree. Then every* $\overline{\omega}$*-Wright convex function is* $\omega$*-convex, i.e.,* $\overline{\omega}$*-Wright convexity is equivalent to* $\omega$*-convexity.*

PROOF. Under the conditions of the theorem, assume that $f : I \to \mathbb{R}$ is an $\overline{\omega}$-Wright convex function. That is, the inequality

$$\frac{\Phi_{(\omega,f)}(x, x+h_1, x+h_1+h_2)}{\Phi_{\overline{\omega}}(x, x+h_1, x+h_1+h_2)} + \frac{\Phi_{(\omega,f)}(x, x+h_2, x+h_1+h_2)}{\Phi_{\overline{\omega}}(x, x+h_2, x+h_1+h_2)} \ge 0$$

holds for all $h_1, h_2 > 0$ and $x \in I \cap (I - (h_1+h_2))$. Using Lemma 2.8, we can see that this inequality is equivalent to

$$\frac{\omega_0(x)\omega_0(x+h_1)\omega_0(x+h_1+h_2)\cdot\det(M)\cdot\Phi_{\pi_2,f/\omega_0}(x,x+h_1,x+h_1+h_2)}{\omega_0(x)\omega_0(x+h_1)\omega_0(x+h_1+h_2)\cdot\det(M)\cdot\Phi_{\pi_2,\omega_3/\omega_0}(x,x+h_1,x+h_1+h_2)}$$
$$+\frac{\omega_0(x)\omega_0(x+h_2)\omega_0(x+h_1+h_2)\cdot\det(M)\cdot\Phi_{\pi_2,f/\omega_0}(x,x+h_2,x+h_1+h_2)}{\omega_0(x)\omega_0(x+h_2)\omega_0(x+h_1+h_2)\cdot\det(M)\cdot\Phi_{\pi_2,\omega_3/\omega_0}(x,x+h_2,x+h_1+h_2)} \ge 0,$$

which simplifies to

$$\frac{\Phi_{\pi_2,f/\omega_0}(x,x+h_1,x+h_1+h_2)}{\Phi_{\pi_2,\omega_3/\omega_0}(x,x+h_1,x+h_1+h_2)}+\frac{\Phi_{\pi_2,f/\omega_0}(x,x+h_2,x+h_1+h_2)}{\Phi_{\pi_2,\omega_3/\omega_0}(x,x+h_2,x+h_1+h_2)} \ge 0. \tag{2.28}$$

This means that $g := f/\omega_0$ is $\overline{\pi}_2$-Wright convex, where $\overline{\pi}_2(t) := (1, t, \rho(t))$ and $\rho := \omega_3/\omega_0$. Observe that, for $\varphi \in \{g, \rho\}$ and $i \in \{1, 2\}$, we have

$$\Phi_{(\pi_2,\varphi)}(x, x + h_i, x + h_1 + h_2) = \begin{vmatrix} 1 & 1 & 1 \\ x & x + h_i & x + h_1 + h_2 \\ \varphi(x) & \varphi(x + h_i) & \varphi(x + h_1 + h_2) \end{vmatrix}$$
$$= h_{3-i}\varphi(x) - (h_1 + h_2)\varphi(x + h_i) + h_i\varphi(x + h_1 + h_1).$$

Using this formula, the inequality (2.28) now states that

$$\begin{aligned} &\frac{h_2 g(x) - (h_1 + h_2)g(x + h_1) + h_1 g(x + h_1 + h_2)}{h_2\rho(x) - (h_1 + h_2)\rho(x + h_1) + h_1\rho(x + h_1 + h_2)} \\ &\qquad + \frac{h_1 g(x) - (h_1 + h_2)g(x + h_2) + h_2 g(x + h_1 + h_2)}{h_1\rho(x) - (h_1 + h_2)\rho(x + h_2) + h_2\rho(x + h_1 + h_2)} \geq 0 \end{aligned} \tag{2.29}$$

holds for all $h_1, h_2 > 0$ and $x \in I \cap (I - (h_1 + h_2))$.

By our assumptions, $(\pi_2, \rho)$ is an extension of $\pi_2$ and $\rho$ is not a polynomial of at most second degree. Using Lemma 2.15, it follows that $\rho$ cannot satisfy the functional equation (2.26), which means that there exist $u \geq 0$, $y, z \in (I + u) \cap (I - u)$ with $z + u \geq y$ such that

$$\frac{\rho(z) - \rho(y)}{z - y} \neq \frac{\rho(z + u) - \rho(y - u)}{(z + u) - (y - u)}.$$

Let $x := y - u$, $h := z - y + 2u$, $t := z - y + u$. Then $z = x + t$, $y = x + h - t$ and $z + u = x + h$, therefore the above relations state that

$$h(\rho(x + t) - \rho(x + h - t)) + (h - 2t)(\rho(x + h) - \rho(x)) \neq 0 \tag{2.30}$$

for some $h > 0$, $x \in I \cap (I - h)$ and $t \in (0, h)$.

Let $h > 0$ and $x \in I \cap (I - h)$ be fixed such that, for some $t \in (0, h)$, (2.30) holds. Define $T \subseteq (0, h)$ to be the set of those values $t$ for which (2.30) is valid. Then the set $T$ is nonempty and, by the continuity of $\rho$, it is also open. Let $T_+$ and $T_-$ denote the (disjoint) subsets of those elements $t \in T$, for which the left hand side of (2.30) is positive and negative, respectively. Then at least one of these subsets is nonempty (and also open).

Since $g$ is $\overline{\pi}_2$-Wright convex, hence Theorem 2.12 implies that $g$ is $\pi_2$-Jensen convex, i.e., it is Jensen convex in the standard sense. According to Rodé's Theorem [**40**], $g$ is the pointwise maximum of Jensen affine functions, i.e., for all $p \in I$, there exists an additive function $A_p : \mathbb{R} \to \mathbb{R}$ such that

$$g(y) \geq A_p(y - p) + g(p) \qquad (p, y \in I). \tag{2.31}$$

Substituting $y := x + t$ and $y := x + h - t$, for $t \in T$ and $p \in I$, we get

$$g(x + t) \geq A_p(x + t - p) + g(p)$$

and

$$g(x+h-t) \geq A_p(x+h-t-p) + g(p).$$

Therefore, with $h_1 := t$ and $h_2 := h-t$, the inequality (2.29) yields that

$$\frac{(h-t)g(x) - h\big(A_p(x+t-p)+g(p)\big) + tg(x+h)}{(h-t)\rho(x) - h\rho(x+t) + t\rho(x+h)} + \frac{tg(x) - h\big(A_p(x+h-t-p)+g(p)\big) + (h-t)g(x+h)}{t\rho(x) - h\rho(x+h-t) + (h-t)\rho(x+h)} \geq 0.$$

Using the additivity of $A_p$ and moving the terms containing $A_p(t)$ to the right hand side, this inequality is equivalent to

$$\begin{aligned} &\frac{(h-t)g(x) - h\big(A_p(x-p)+g(p)\big) + tg(x+h)}{(h-t)\rho(x) - h\rho(x+t) + t\rho(x+h)} + \frac{tg(x) - h\big(A_p(x+h-p)+g(p)\big) + (h-t)g(x+h)}{t\rho(x) - h\rho(x+h-t) + (h-t)\rho(x+h)} \\ &\geq \frac{h(\rho(x+t)-\rho(x+h-t))+(h-2t)(\rho(x+h)-\rho(x))}{((h-t)\rho(x)-h\rho(x+t)+t\rho(x+h))(t\rho(x)-h\rho(x+h-t)+(h-t)\rho(x+h))} hA_p(t). \end{aligned} \tag{2.32}$$

This inequality shows that $A_p$ is bounded from above on $T_+$ and is bounded from below on $T_-$. Therefore, $A_p$ is bounded from above or from below on a nonempty open subset of $T$. In view well-known properties of additive functions, this implies that $A_p$ is continuous, i.e., there exists a real constant $a_p$ such that $A_p(x) = a_p x$ holds for all $x \in \mathbb{R}$. Thus, by (2.31), we can see that $g$ is the pointwise maximum of continuous affine functions, Therefore, $g$ must be a convex function (in the standard sense). From this it follows that $f = g\omega_0$ is $\omega$-convex. □

It seems to be an open problem whether an analogue of the previous theorem is valid for the 3- or higher-dimensional setting.

## CHAPTER 3

# Cauchy–Schwarz-type inequalities for solutions of Levi–Civita-type functional equations

## 1. Preliminaries

In the theory of real and additive functions (see the monograph [**21**] of Kuczma) there are several results which establish the existence of a discontinuous additive function which satisfies further algebraic conditions. One of the first problems of this kind was posed by Szabó [**42**] motivated by a question of Benz [**4**] and solved by Kominek, Reich, and Schwaiger [**20**]. They proved that if $A : \mathbb{R} \to \mathbb{R}$ is an additive function which satisfies the equality $A(x)A(y) = 0$ for all $(x, y) \in C$, where $C$ is a the unit circle, or is a hyperbola, or is an algebraic curve given by polynomials, then $A$ has to be equal to zero identically. Boros and Fechner [**8**] and Boros, Fechner and Kutas [**9**] extended these results to sets defined via generalized polynomials and to quadratic functions instead of additive ones, respectively, and they also examined the stability versions of such problems.

In [**9**], the case when $C$ is the graph of the hyperbola $xy = 1$ was left open. Kannappan [**16**, Chapter 1] proved that if, for some positive constant $a$, an additive function $A$ satisfies the condition $A(x)A(1/x) = a$ for all $x \neq 0$, then $A$ has to be continuous. On the other hand, according to the remarks [**3**] and [**5**], there exist discontinuous additive functions which fulfill the inequality $A(x)A(1/x) > 0$ for all $x \neq 0$. On the other hand, using the theory of valuations of fields, Kutas [**22**, Theorem 24] proved that there exists a nonzero (henceforth discontinuous) additive function which satisfies the equality $A(x)A(1/x) = 0$ for all $x \neq 0$.

The above results motivated us to construct discontinuous additive real functions that enjoy properties that are connected to the multiplicative structure. It turned out that such properties could be possessed if the additive function satisfies Levi–Civita-type functional equations with respect to the multiplicative structure.

More generally, let $(G, *.)$ be a groupoid. (Recall that a pair $(G, *.)$ is said to be a groupoid if $*$ is a binary operation on $G$, i.e., $* : G \times G \to G$.) Let $A : G \to \mathbb{R}$ be a function such that there exist functions $f_1, \dots, f_n, g_1, \dots, g_n :$

$G \to \mathbb{R}$ such that the functional equation

$$A(x * y) = f_1(x)g_1(y) + \cdots + f_n(x)g_n(y) \qquad (x, y \in G)$$

is fulfilled. Under certain assumptions on $n$ and on the functions $f_1, \ldots, f_n, g_1, \ldots, g_n$, we are going to prove that for all $x, y \in G$ the function $A$ will satisfy either the inequality $A(x * y)^2 \leq A(x * x)A(y * y)$ or the reversed one $A(x * x)A(y * y) \leq A(x * y)^2$. In the important particular case when the groupoid is the multiplicative structure of a commutative ring and $A$ is additive, we will establish the existence of nontrivial additive functions which satisfy one of the above mentioned inequalities.

## 2. The inequality $A(x * y)^2 \leq A(x * x)A(y * y)$

In our first result we assume that the function $A$ satisfies a Levi–Civita-type functional equation over a groupoid.

PROPOSITION 3.1. *Let $(G, *)$ be a groupoid and let $A : G \to \mathbb{R}$ be a function. Assume that there exist $n \in \mathbb{N}$ and functions $f_1, \ldots, f_n : G \to \mathbb{R}$ such that $A$ satisfies the Levi–Civita-type functional equation*

$$A(x * y) = f_1(x)f_1(y) + \cdots + f_n(x)f_n(y) \tag{3.1}$$

*for all $x, y \in G$. Then, $A$ fulfills the functional inequality*

$$A(x * y)^2 \leq A(x * x)A(y * y) \tag{3.2}$$

*for all $x, y \in G$.*

PROOF. Let $x, y \in G$. In view of the functional equation (3.1), the inequality (3.2) can be rewritten as

$$\begin{aligned}&\big(f_1(x)f_1(y) + \ldots + f_n(x)f_n(y)\big)^2\\ &\qquad\qquad \leq \big(f_1(x)^2 + \cdots + f_n(x)^2\big)\big(f_1(y)^2 + \cdots + f_n(y)^2\big),\end{aligned}$$

which follows from the Cauchy–Schwarz inequality when we apply it to the $n$-dimensional vectors $(f_1(x), \ldots, f_n(x))$ and $(f_1(y), \ldots, f_n(y))$. □

If the groupoid is the multiplicative semigroup of a commutative ring $(R, +, \cdot)$ and $A$ is additive, then we can establish a characterization of the corresponding inequality. Recall that in a ring, the product $x \cdot y$ of the elements $x, y \in R$ is simply denoted by $xy$, and $x^2$ is defined to be the product $x \cdot x$.

THEOREM 3.2. *Let $(R, +, \cdot)$ be a commutative ring and let $A : R \to \mathbb{R}$ be an additive function. Then $A$ satisfies the inequality*

$$A(xy)^2 \leq A(x^2)A(y^2) \tag{3.3}$$

*for all* $x, y \in R$ *if and only if at least one of the following conditions hold*

(i) $A(x^2) \geq 0$ *for all* $x \in R$,
(ii) $A(x^2) \leq 0$ *for all* $x \in R$.

PROOF. Assume first that $A$ satisfies inequality (3.3) for all $x, y \in R$, but none of the conditions (i) and (ii) is valid. Then, there exist $x, y \in R$ such that

$$A(x^2) < 0 < A(y^2).$$

These inequalities imply that

$$A(x^2)A(y^2) < 0 \leq A(xy)^2,$$

which contradicts the inequality (3.3).

To prove the the reverse implication, assume that $A$ satisfies condition (i) and let $x, y \in R$ be fixed. Then, for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we get that

$$\begin{aligned} 0 \leq A((nx + ky)^2) &= A(n^2x^2 + 2nkxy + k^2y^2) \\ &= n^2A(x^2) + 2nkA(xy) + k^2A(y^2). \end{aligned}$$

Dividing this inequality by $n^2$, we can conclude that

$$0 \leq A(x^2) + 2\frac{k}{n}A(xy) + \frac{k^2}{n^2}A(y^2).$$

Because $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ were arbitrary, we obtain that

$$0 \leq A(x^2) + 2rA(xy) + r^2A(y^2)$$

is valid for all rational number $r$. By the density of rational numbers, it follows that the above inequality is true for all real number $r$. The polynomial on the right hand side cannot have two distinct real roots, therefore, its discriminant has to be non positive, i.e.,

$$(2A(xy))^2 - 4A(x^2)A(y^2) \leq 0.$$

This inequality reduces to (3.3).

In the case when condition (ii) holds, then the additive function $(-A)$ satisfies condition (i) and hence the inequality (3.3) holds with $(-A)$ instead of $A$, which again shows that (3.3) is valid. □

## 3. The inequality $A(x * x)A(y * y) \leq A(x * y)^2$

In the subsequent two propositions, we present two Levi–Civita-type functional equations which imply the inequality in the title of this section.

PROPOSITION 3.3. *Let $(G, *)$ be a groupoid and $A : G \to \mathbb{R}$ be a function. Assume that there exist two functions $f, g : G \to \mathbb{R}$ such that the Levi–Civita-type functional equation*

$$A(x * y) = f(x)f(y) - g(x)g(y) \tag{3.4}$$

*holds for all $x, y \in G$. Then $A$ satisfies the functional inequality*

$$A(x * x)A(y * y) \le A(x * y)^2 \tag{3.5}$$

*for all $x, y \in G$.*

PROOF. Let $x, y \in G$. According to the functional equation (3.4), the inequality (3.5) can be rewritten as

$$(f(x)^2 - g(x)^2)(f(y)^2 - g(y)^2) \le (f(x)f(y) - g(x)g(y))^2.$$

Observe that this inequality is equivalent to

$$0 \le (g(x)f(y) - f(x)g(y))^2,$$

which is obviously valid. □

PROPOSITION 3.4. *Let $(G, *)$ be a groupoid. Let $A : R \to \mathbb{R}$ be a function. Assume that there exist $f, g : R \to \mathbb{R}$ such that the Levi–Civita-type functional equation*

$$A(x * y) = f(x)g(y) + g(x)f(y) \tag{3.6}$$

*holds for all $x, y \in G$. Then, for all $x, y \in G$, $A$ satisfies the functional inequality* (3.5)*.*

PROOF. Let $x, y \in G$. According to the functional equation (3.6), the inequality (3.5) can be rewritten as

$$4f(x)g(x)f(y)g(y) \le (f(x)g(y) + g(x)f(y))^2.$$

Observe that this inequality is equivalent to

$$0 \le (g(x)f(y) - f(x)g(y))^2,$$

which is obviously valid. □

COROLLARY 3.5. *Assume that $A : \mathbb{R} \to \mathbb{R}$ satisfies the Leibniz Rule with respect to multiplication, i.e.,*

$$A(xy) = xA(y) + A(x)y \qquad (x, y \in \mathbb{R}).$$

*Then, for all $x, y \in \mathbb{R}$, the inequality*

$$A(x^2)A(y^2) \le A(xy)^2 \tag{3.7}$$

*holds.*

PROOF. Observe that with groupoid $(G,*) := (\mathbb{R},\cdot)$ and with the notations $g := A$ and $f(x) := x$, $(x \in G)$, the equality (3.6) of Proposition 3.4 holds. Therefore, $A$ satisfies inequality (3.5) for all $x, y \in \mathbb{R}$, hence (3.7) is also satisfied. □

In particular, if $A : \mathbb{R} \to \mathbb{R}$ is a derivation (i.e., $A$ is additive and satisfies the Leibniz Rule with respect to multiplication), then the above corollary implies that it fulfills the inequality (3.7).

If the groupoid is the multiplicative semigroup of a commutative ring $(R, +, \cdot)$ and $A$ is additive, then we can establish a characterization of the inequality (3.5) over a particular subset of the ring.

THEOREM 3.6. *Let $(R, +, \cdot)$ be a commutative ring with a multiplicative unit element $e$ and $A : R \to \mathbb{R}$ be an additive function with $A(e) \neq 0$. Let the subset $R_A \subseteq R$ be defined by*

$$R_A := \{x \in R \mid 0 \leq A(x^2)A(e)\}.$$

*Then $e \in R_A$ and $A$ satisfies functional inequality*

$$A(x^2)A(y^2) \leq A(xy)^2 \tag{3.8}$$

*for all $x, y \in R_A$ if and only if*

$$A(x^2)A(e) \leq A(x)^2 \tag{3.9}$$

*for all $x \in R_A$.*

PROOF. The inclusion $e \in R_A$ is obvious. Now, putting $y := e$, we can see that the inequality (3.8) implies (3.9).

To prove the reversed implication, assume that (3.9) is valid for all $x \in R_A$. Then it is also valid for all $x \in R$, since, for $x \in R \setminus R_A$, the left hand side of the inequality is negative, while the right hand side is nonnegative. Introduce the function $A_0 := A/A(e)$. Then, $A_0$ is additive, $A_0(e) = 1$ and, dividing (3.9) by $A(e)^2 > 0$ side by side, for all $x \in R$, we get that

$$A_0(x^2) \leq A_0(x)^2 \tag{3.10}$$

Let $x, y \in R_A$ be fixed and $n \in \mathbb{N}$, $k \in \mathbb{Z}$ be arbitrary. Then, (3.10) yields

$$A_0((nx + ky)^2) \leq A_0(nx + ky)^2.$$

Using the additivity of $A_0$, we get

$$n^2A_0(x^2) + 2nkA_0(xy) + k^2A_0(y^2)$$
$$\leq n^2A_0(x)^2 + 2nkA_0(x)A_0(y) + k^2A_0(y)^2.$$

Dividing this inequality by $n^2$, we obtain

$$\begin{aligned} A_0(x^2) + 2\tfrac{k}{n}A_0(xy) &+ \left(\tfrac{k}{n}\right)^2 A_0(y^2) \\ &\leq A_0(x)^2 + 2\tfrac{k}{n}A_0(x)A_0(y) + \left(\tfrac{k}{n}\right)^2 A_0(y)^2. \end{aligned}$$

Therefore, for any rational number $r \in \mathbb{Q}$,

$$0 \leq (A_0(x)^2 - A_0(x^2)) + 2r(A_0(x)A_0(y) - A_0(xy)) + r^2(A_0(y)^2 - A_0(y^2)).$$

Using the continuity of both sides as a function of $r$, it follows that the same inequality is valid for all $r \in \mathbb{R}$. Thus, the discriminant of this quadratic polynomial has to be nonpositive, i.e.,

$$(A_0(x)A_0(y) - A_0(xy))^2 \leq (A_0(x)^2 - A_0(x^2))(A_0(y)^2 - A_0(y^2)) \tag{3.11}$$

and hence

$$\begin{aligned} |A_0(x)A_0(y) - A_0(xy)| &\leq \sqrt{(A_0(x)^2 - A_0(x^2))(A_0(y)^2 - A_0(y^2))} \\ &= Q(x)Q(y), \end{aligned}$$

thus

$$\big||A_0(x)A_0(y)| - |A_0(xy)|\big| \leq |A_0(x)A_0(y) - A_0(xy)| \leq Q(x)Q(y),$$

where $Q(u) := \sqrt{A_0(u)^2 - A_0(u^2)} \geq 0$ $(u \in R)$. Then, for all $u \in R$,

$$A_0(u)^2 = Q(u)^2 + A_0(u^2). \tag{3.12}$$

Therefore $|A_0(xy)|$ satisfies the inequality

$$|A_0(x)A_0(y)| - Q(x)Q(y) \leq |A_0(xy)| \leq |A_0(x)A_0(y)| + Q(x)Q(y). \tag{3.13}$$

We are going to show that

$$A_0(x^2)A_0(y^2) \leq A_0(xy)^2. \tag{3.14}$$

To see this inequality, we will prove that

$$Q(x)^2Q(y)^2 \leq A_0(x)^2A_0(y)^2. \tag{3.15}$$

and

$$A_0(x^2)A_0(y^2) \leq \Big(|A_0(x)A_0(y)| - Q(x)Q(y)\Big)^2. \tag{3.16}$$

Since $x$ and $y$ belong to $R_A$, therefore, we have that $A_0(x^2) \geq 0$ and $A_0(y^2) \geq 0$, then, with $u \in \{x, y\}$, the equality (3.12) implies that

$$Q(x)^2 \leq A_0(x)^2 \qquad \text{and} \qquad Q(y)^2 \leq A_0(y)^2.$$

Multiplying these inequalities side by side, we get that (3.15) holds. Therefore, we can conclude that

$$Q(x)Q(y) \leq |A_0(x)A_0(y)| \tag{3.17}$$

which is equivalent to (3.15).

By the obvious inequality

$$\big(|A_0(x)|Q(y) - |A_0(y)|Q(x)\big)^2 \geq 0,$$

we have that

$$2|A_0(x)A_0(y)|Q(x)Q(y) \leq A_0(x)^2Q(y)^2 + A_0(y)^2Q(x)^2. \tag{3.18}$$

Therefore using the equality (3.12) with $u \in \{x, y\}$ and the inequality (3.18), we obtain

$$\begin{aligned} A_0(x^2)A_0(y^2) &= A_0(x)^2A_0(y)^2 - A_0(x)^2Q(y)^2 - Q(x)^2A_0(y)^2 \\ &\qquad + Q(x)^2Q(y)^2 \\ &\leq A_0(x)^2A_0(y)^2 - 2|A_0(x)A_0(y)|Q(x)Q(y) + Q(x)^2Q(y)^2 \\ &= (|A_0(x)A_0(y)| - Q(x)Q(y))^2. \end{aligned}$$

This shows that the inequality (3.16) holds.

In view of (3.17), the first inequality in (3.13) implies that

$$(|A_0(x)A_0(y)| - Q(x)Q(y))^2 \leq A_0(xy)^2.$$

This, combined with the inequality (3.16) yields that (3.14) is valid, indeed. Therefore,

$$A(x^2)A(y^2) = A_0(x^2)A_0(y^2)A(e)^2 \leq A_0^2(xy)A(e)^2 = A(xy)^2,$$

which completes the proof of the inequality (3.8) for $x, y \in R_A$. ☐

In the following example we show that the additivity of the function $A$ in Theorem 3.6 is necessary.

EXAMPLE 3.7. Let $q \in (0, 1)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x & x \neq 1, \\ q & x = 1. \end{cases}$$

Clearly, $f$ is not additive. Therefore, for $x \notin \{1, -1\}$, we have that

$$f(x^2)f(1) = qx^2 \leq x^2 = f(x)^2$$

for $x = \pm 1$,

$$f(1^2)f(1) = q^2 = f(1)^2, \qquad f((-1)^2)f(1) = q^2 < 1 = f(-1)^2,$$

which shows that (3.9) is satisfied for all $x \in \mathbb{R}$. On the other hand, for $x, y \in \mathbb{R} \setminus \{1, -1\}$ with $xy = 1$, we can conclude that

$$f(x^2)f(y^2) = x^2y^2 = 1 > q^2 = f(xy)^2,$$

which shows that (3.8) is not satisfied.

The next example shows that if the function $A$ in Theorem 3.6 is non-additive, continuous and satisfies $A(e) = 0$, then the conclusion of Theorem 3.6 may not be valid.

EXAMPLE 3.8. Let $A : \mathbb{R} \to \mathbb{R}$ be defined as $A(x) = |x - 1|$. Note that $A$ is continuous and not additive. Since $A(1) = 0$ this implies that

$$A(x^2)A(1) = 0 \le (x - 1)^2 = A(x)^2.$$

Thus the inequality (3.9) holds for all $x \in \mathbb{R}$. On the other hand we have that

$$A(x^2)A(y^2) = |x^2 - 1||y^2 - 1| \qquad \text{and} \qquad A(xy)^2 = (xy - 1)^2.$$

Hence for $x = 2$ and $y = \frac{1}{2}$ we have that $A(x^2)A(y^2) = \frac{9}{4}$ but $A(xy)^2 = 0$, therefore the inequality (3.8) does not hold.

## 4. Consequences of systems of Levi–Civita-type equations

THEOREM 3.9. *Let $(G, *)$ be a groupoid and $A, B : G \to \mathbb{R}$ be functions. Assume that there exist $f, g : G \to \mathbb{R}$ such that $A$ and $B$ satisfy the following system of Levi–Civita-type functional equations*

$$\begin{aligned} A(x * y) &= f(x)f(y) - g(x)g(y) \qquad \textit{and} \\ B(x * y) &= f(x)g(y) + g(x)f(y) \end{aligned} \tag{3.19}$$

*for all $x, y \in G$. Then the inequalities*

$$-B(x * y)^2 \le A(x * x)A(y * y) \le A(x * y)^2 \tag{3.20}$$

*and*

$$-A(x * y)^2 \le B(x * x)B(y * y) \le B(x * y)^2 \tag{3.21}$$

*hold for all $x, y \in G$.*

PROOF. In view of two functional equations in (3.19), for $x, y \in G$, we have that

$$\begin{aligned} &B(x * y)^2 + A(x * x)A(y * y) \\ &\qquad = f(x)^2g(y)^2 + 2f(x)g(y)g(x)f(y) + g(x)^2f(y)^2 \\ &\qquad\qquad + (f(x)^2 - g(x)^2)(f(y)^2 - g(y)^2) \\ &\qquad = (f(x)f(y) + g(x)g(y))^2 \ge 0, \end{aligned}$$

which proves the left hand side inequality in (3.20). The right hand side inequality in (3.20) is a direct consequence of Proposition 3.3.

Again, in view of two equations in (3.19), for $x, y \in G$ we have that

$$\begin{aligned}A(x*y)^2 &+ B(x*x)B(y*y)\\ &= f(x)^2f(y)^2 - 2f(x)f(y)g(x)g(y) + g(x)^2g(y)^2\\ &\quad + 4f(x)g(x)f(y)g(y)\\ &= (f(x)f(y)+g(x)g(y))^2 \geq 0.\end{aligned}$$

This implies the left hand side inequality in (3.21). On the other hand, applying Proposition 3.4 for the function $B$ instead of $A$, we obtain that

$$B(x*x)B(y*y) \leq B(x*y)^2.$$

This shows that the second inequality of (3.21) holds for $x, y \in G$. ☐

An interesting consequence of the functional equations in (3.19) is that $A$ and $B$ satisfy the following identity:

$$B(x*y)^2 + A(x*x)A(y*y) = A(x*y)^2 + B(x*x)B(y*y) \quad (x, y \in G).$$

Therefore, the inequalities (3.20) and (3.21) can be expressed as the following chain of inequalities

$$\begin{aligned}0 &\leq A(x*x)A(y*y) + (B(x*y))^2\\ &= B(x*x)B(y*y) + A(x*y)^2 \leq A(x*y)^2 + B(x*y)^2 \qquad (x, y \in G).\end{aligned}$$

The following result is probably well-known, but we could not find an exact reference for it.

COROLLARY 3.10. *For all $x, y \in \mathbb{R}$, we have*

$$\begin{aligned}&-\sin(x+y)^2 \leq \cos(2x)\cos(2y) \leq \cos(x+y)^2 \qquad \textit{and}\\ &-\cos(x+y)^2 \leq \sin(2x)\sin(2y) \leq \sin(x+y)^2.\end{aligned}$$

PROOF. Observe that the trigonometric functions $\cos : \mathbb{R} \to \mathbb{R}$ and $\sin : \mathbb{R} \to \mathbb{R}$ satisfy the functional equations

$$\begin{aligned}\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \qquad \text{and}\\ \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y)\end{aligned}$$

for all $x, y \in \mathbb{R}$. Therefore, (3.19) holds with $A := f := \cos$ and $B := g := \sin$ over the groupoid $(\mathbb{R}, +)$. Consequently, (3.20) and (3.21) are satisfied for all $x, y \in \mathbb{R}$, which imply the assertion. ☐

COROLLARY 3.11. *Let* $(G,*)$ *be a groupoid and let* $\varphi : G \to \mathbb{C}$ *be a homomorphism into the multiplicative semigroup of complex numbers. Define* $A := \Re\varphi$ *and* $B := \Im\varphi$*. Then, for all* $x,y \in G$*, the inequalities* (3.20) *and* (3.21) *hold.*

PROOF. Using the multiplicativity of $\varphi$, for all $x,y \in G$, we get that

$$\begin{aligned} A(x*y) &= \Re(\varphi(x*y)) = \Re(\varphi(x)\varphi(y)) \\ &= \Re((A(x)+iB(x))(A(y)+iB(y))) = A(x)A(y) - B(x)B(y), \\ B(x*y) &= \Im(\varphi(x*y)) = \Im(\varphi(x)\varphi(y)) \\ &= \Im((A(x)+iB(x))(A(y)+iB(y))) = A(x)B(y) + B(x)A(y). \end{aligned}$$

Therefore, the functional equations in (3.19) are satisfied with $f := A$ and $g := B$. Thus, according to Theorem 3.9, we obtain that the inequalities (3.20) and (3.21) hold for all $x,y \in G$, which was to be shown. □

COROLLARY 3.12. *Let* $\varphi : \mathbb{C} \to \mathbb{C}$ *be an automorphism of the field* $\mathbb{C}$*. Define* $A := \Re\varphi$ *and* $B := \Im\varphi$*. Then* $A : \mathbb{C} \to \mathbb{R}$ *and* $B : \mathbb{C} \to \mathbb{R}$ *are additive mappings, furthermore, for all* $x,y \in \mathbb{C}$*,*

$$-B(xy)^2 \le A(x^2)A(y^2) \le A(xy)^2 \quad \text{and} \quad -A(xy)^2 \le B(x^2)B(y^2) \le B(xy)^2. \tag{3.22}$$

PROOF. Using the additivity of $\varphi$, for all $x,y \in \mathbb{C}$, we obtain that

$$\begin{aligned} A(x+y) &= \Re(\varphi(x+y)) = \Re(\varphi(x)+\varphi(y)) \\ &= \Re((A(x)+iB(x))+(A(y)+iB(y))) = A(x)+A(y), \\ B(x+y) &= \Im(\varphi(x+y)) = \Im(\varphi(x)+\varphi(y)) \\ &= \Im((A(x)+iB(x))+(A(y)+iB(y))) = B(x)+B(y). \end{aligned}$$

These equalities show that $A : \mathbb{C} \to \mathbb{R}$ and $B : \mathbb{C} \to \mathbb{R}$ are additive mappings.

By the multiplicativity of $\varphi$, it maps the groupoid $(G,*) := (\mathbb{C},\cdot)$ into itself. Thus, according to Corollary 3.11, we obtain that the inequalities (3.20) and (3.21) hold for all $x,y \in \mathbb{C}$. This yields the assertion. □

The following result is a counterpart of Theorem 3.9.

THEOREM 3.13. *Let* $(G,*)$ *be a groupoid and* $A,B : G \to \mathbb{R}$ *be functions. Assume that there exist* $f,g : G \to \mathbb{R}$ *such that* $A$ *and* $B$ *satisfy the following Levi–Civita-type functional equations*

$$\begin{aligned} A(x*y) &= f(x)f(y) + g(x)g(y) \qquad \text{and} \\ B(x*y) &= f(x)g(y) + g(x)f(y) \end{aligned} \tag{3.23}$$

*for all $x, y \in G$. Then the inequalities*

$$B(x * x)B(y * y) \leq A(x * y)^2 \leq A(x * x)A(y * y) \tag{3.24}$$

*and*

$$B(x * x)B(y * y) \leq B(x * y)^2 \leq A(x * x)A(y * y). \tag{3.25}$$

*hold for all $x, y \in G$.*

PROOF. In view of two functional equations in (3.23), for $x, y \in G$, we have that

$$\begin{aligned}
&A(x * y)^2 - B(x * x)B(y * y) \\
&\qquad = f(x)^2 f(y)^2 + 2f(x)f(y)g(x)g(y) + g(x)^2 g(y)^2 \\
&\qquad\qquad - 4f(x)g(x)f(y)g(y) \\
&\qquad = (f(x)f(y) - g(x)g(y))^2 \geq 0.
\end{aligned}$$

This implies the left hand side inequality in (3.24). On the other hand, applying Proposition 3.1 for the function $A$ and $n = 2$, $f_1 := f$, $f_2 := g$, we obtain that the second inequality of (3.24) holds for $x, y \in G$.

Again, in view of two equations in (3.23), for $x, y \in G$, we have that

$$\begin{aligned}
&A(x * x)A(y * y) - B(x * y)^2 \\
&\qquad = (f(x)^2 + g(x)^2)(f(y)^2 + g(y)^2) - f(x)^2 g(y)^2 \\
&\qquad\qquad - 2f(x)g(y)g(x)f(y) - g(x)^2 f(y)^2 \\
&\qquad = (f(x)f(y) - g(x)g(y))^2 \geq 0,
\end{aligned}$$

which proves the right hand side inequality in (3.25). The left hand side inequality in (3.25) is a direct consequence of Proposition 3.4 (applied to $B$ instead of $A$). ☐

An interesting consequence of the functional equations in (3.19) is that $A$ and $B$ satisfy the following identity:

$$B(x * x)B(y * y) + A(x * x)A(y * y) = A(x * y)^2 + B(x * y)^2 \quad (x, y \in G).$$

The following result is probably also well-known, but we could not find a reference for it.

COROLLARY 3.14. *For all $x, y \in \mathbb{R}$, we have*

$$\sinh(2x)\sinh(2y) \leq \sinh(x + y)^2 < \cosh(x + y)^2 \leq \cosh(2x)\cosh(2y) \tag{3.26}$$

PROOF. Observe that the hyperbolic functions $\cosh : \mathbb{R} \to \mathbb{R}$ and $\sinh : \mathbb{R} \to \mathbb{R}$ satisfy the functional equations

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \qquad \text{and}$$
$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

for all $x, y \in \mathbb{R}$. Therefore, (3.23) holds with $A := f := \cosh$ and $B := g := \sinh$ over the groupoid $(\mathbb{R}, +)$. Consequently, (3.24) and (3.25) are satisfied for all $x, y \in \mathbb{R}$, which imply the first and last inequalities in (3.26). The central inequality follows from the identity $\cosh^2 - \sinh^2 = 1$. $\square$

To formulate the next result, let $p$ be a square free positive integer and let $\mathbb{Q}(\sqrt{p})$ denote the subfield of $\mathbb{R}$ generated by $\sqrt{p}$. Then, one can see that $\mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} : a, b \in \mathbb{Q}\}$. In what follows, we equip $\mathbb{Q}(\sqrt{p})$ with the topology inherited from $\mathbb{R}$.

THEOREM 3.15. *Let $p$ be a square free positive integer. Then there exist two discontinuous additive functions $A : \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ and $B : \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ such that the functions $A$ and $B$ fulfill the inequalities*

$$\begin{aligned} B(x^2)B(y^2) \le A(xy)^2 \le A(x^2)A(y^2), \\ B(x^2)B(y^2) \le B(xy)^2 \le A(x^2)A(y^2) \end{aligned} \tag{3.27}$$

*for all $x, y \in \mathbb{Q}(\sqrt{p})$.*

PROOF. Define the functions $A : \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ and $B : \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ by

$$A(a + b\sqrt{p}) := a \qquad \text{and} \qquad B(a + b\sqrt{p}) := b\sqrt{p} \qquad (a, b \in \mathbb{Q}).$$

We show that $A$ and $B$ are additive. Indeed, let $x = a_1 + b_1\sqrt{p}$ and $y = a_2 + b_2\sqrt{p}$ be two arbitrary points of $\mathbb{Q}(\sqrt{p})$, where $a_1, a_2, b_1, b_2 \in \mathbb{Q}$. According to the definition of $A$ and $B$, we have that

$$A(x+y) = A(a_1 + a_2 + (b_1 + b_2)\sqrt{p}) = a_1 + a_2 = A(x) + A(y)$$

and

$$B(x+y) = B(a_1 + a_2 + (b_1 + b_2)\sqrt{p}) = (b_1 + b_2)\sqrt{p} = B(x) + B(y).$$

This proves that $A$ and $B$ are additive, indeed.

Next we prove that $A$ and $B$ satisfy the functional equations in (3.23) with $f := A$, and $g := B$, where the groupoid $(G, *)$ is equal to $(\mathbb{Q}(\sqrt{p}), \cdot)$. Indeed, let $x = a_1 + b_1\sqrt{p}$ and $y = a_2 + b_2\sqrt{p}$ be two arbitrary points of $\mathbb{Q}(\sqrt{p})$, where $a_1, a_2, b_1, b_2 \in \mathbb{Q}$. Then

$$\begin{aligned} A(xy) &= A((a_1a_2 + b_1b_2p) + (a_1b_2 + a_2b_1)\sqrt{p}) \\ &= a_1a_2 + b_1b_2p = A(x)A(y) + B(x)B(y) \end{aligned}$$

and similarly,

$$\begin{aligned}B(xy) &= B((a_1a_2 + pb_1b_2) + (a_1b_2 + a_2b_1)\sqrt{p})\\ &= (a_1b_2 + a_2b_1)\sqrt{p} = A(x)B(y) + B(x)A(y).\end{aligned}$$

Therefore, according to Theorem 3.13, the inequalities (3.24) and (3.25) holds, which prove that the inequalities in (3.27) are also valid.

Finally, we show that the functions $A$ and $B$ are discontinuous at the point $u := 1 + \sqrt{p}$. By the density of the set $\mathbb{Q}$ in $\mathbb{R}$, there exists a sequence $(x_n)$ of rational numbers converging to $u/\sqrt{p}$. Then the sequence $(x_n\sqrt{p})$ converges to $u$. We have that $A(u) = 1$ but, for all $n \in \mathbb{N}$, $A(x_n\sqrt{p}) = 0$, therefore $A$ is discontinuous at $u$. Furthermore, there exists a sequence $(y_n)$ of rational numbers, which converges to $u$. Since $B(u) = \sqrt{p}$ and $B(y_n) = 0$ for all $n \in \mathbb{N}$, thus we can conclude that the function $B$ is also discontinuous at $u$. □

We note that there does not exist a discontinuous additive function $A : \mathbb{R} \to \mathbb{R}$ such that the inequality $A(xy)^2 \leq A(x^2)A(y^2)$ be valid for all real numbers $x, y$. Indeed, if $A$ is a discontinuous additive function satisfying this inequality, then $A(u)$ is not zero for some $u > 0$. With the substitution $y := \sqrt{u}$, the inequality shows that $A$ is either nonnegative (if $A(u) > 0$) or nonpositive (if $A(u) < 0$) on the set of positive numbers. This, by classical results on additive functions (see [**21**]), implies that $A(x) = ax$ for some $a \in \mathbb{R}$, and hence $A$ is continuous.

Motivated by the above remark, we could formulate the following open problem: Find a description or characterization of those maximal subrings (or subfields) of $\mathbb{R}$ such that system of inequalities in (3.27) holds for a discontinuous pair $(A, B)$ of additive functions which are defined on this subring (or subfield).

# Summary

In this section we summarize the most important results of this PhD dissertation. We mention some of the interesting lemmas, propositions, theorems, and corollaries based on our research which can be found with full details in the papers [**36**], [**37**] and [**35**].

## Decomposition of higher-order Wright convex functions revisited

Chapter one is devoted to give an elementary proof for the decomposition theorem of Maksa and Páles [**24**] which is an extension of the Ng theorem [**30**], (Ng characterized a Wright convex function as the sum of a convex and an additive function). Maksa and Páles [**24**] proved that a real function is Wright convex of order $n$ if and only if it sum of a convex function of order $n$ and a polynomial function of degree at most $n$. In our proof we adopted the method of Páles [**33**].

In what follows, we are going to define several higher-order convexity concepts in terms of *difference operators* and *divided differences*.

We recall that, for a fixed real number $h$, the operator $\Delta_h$, acting on a real function $f : I \to \mathbb{R}$, is defined by

$$\Delta_h f(x) := f(x+h) - f(x) \qquad (x \in I \cap (I-h)).$$

Given a fixed $n \in \mathbb{N}$, a map $f : I \to \mathbb{R}$ is said to be *Jensen convex of order* $n$ (briefly $n$-Jensen convex) if

$$\Delta_h^{n+1} f(x) \geq 0 \qquad \big(h > 0,\, x \in I \cap (I - (n+1)h)\big). \tag{1}$$

A map $f : I \to \mathbb{R}$ is said to be *Wright convex of order* $n$ (briefly $n$-Wright convex) if it satisfies the functional inequality

$$\begin{aligned}&\Delta_{h_1} \cdots \Delta_{h_{n+1}} f(x) \geq 0\\ &\qquad\qquad \big(h_1, \dots, h_{n+1} > 0,\, x \in I \cap (I - (h_1 + \cdots + h_{n+1}))\big).\end{aligned} \tag{2}$$

In the investigation of functional inequalities (1) and (2), those maps that fulfill these inequalities with equality play a fundamental role in the theory of linear functional equations. Therefore, for $n \in \mathbb{N}$, we consider the equation

$$\Delta_h^{n+1} f(x) = 0 \qquad (h > 0,\, x \in I \cap (I - (n+1)h)),$$

which is termed the *Fréchet functional equation* in this theory. It is well-known (see [**21**], [**43**]) that $f : I \to \mathbb{R}$ satisfies this equation if and only if it is a *polynomial function of degree at most* $n$, i.e., it has the representation

$$f(x) = a_0 + a_1(x) + \cdots + a_n(x) \qquad (x \in I),$$

where $a_0 \in \mathbb{R}$ and $a_k$ is the *diagonalization* of some $k$-additive and symmetric function $A_k : \mathbb{R}^k \to \mathbb{R}$, that is, $a_k(x) = A_k(x, \ldots, x)$, $(x \in \mathbb{R},\ k = 1, \ldots, n)$. Standard polynomials are exactly the continuous polynomial functions. On the other hand, using Hamel bases, it is not difficult to construct non-continuous polynomial functions (see [**21**]).

The *divided difference* of the function $f : I \to \mathbb{R}$ with respect to the pairwise distinct points $x_0, \ldots, x_n \in I$ is defined by

$$[x_0, \ldots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}.$$

Obviously, divided differences are symmetric functions of their variables, furthermore, it is easy to show that they enjoy the following recursive property

$$[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}$$

for all $n \in \mathbb{N}$ and pairwise distinct elements $x_0, \ldots, x_n \in I$.

Based on the works of T. Popoviciu [**38**,**39**], given $n \in \mathbb{N}$, a map $f : I \to \mathbb{R}$ is said to be *convex of order* $n$ *on* $I$ (shortly $n$-convex on $I$) if the inequality

$$[x_0, x_1, \ldots, x_n, x_{n+1}; f] \geq 0 \tag{3}$$

holds for all pairwise distinct elements $x_0, x_1, \ldots, x_n, x_{n+1} \in I$. Due to the symmetry of divided differences, without loss of generality, we may assume $x_0 < x_1 < \cdots < x_n < x_{n+1}$ here.

The following result was obtained in the book [**21**] and in a more general form in the paper [**12**].

LEMMA. *Let* $n \in \mathbb{N}$. *Then every* $n$*-convex function is* $n$*-Wright convex, and every* $n$*-Wright convex function is* $n$*-Jensen convex.*

One of the main results of the paper [**24**] established the following generalization of Ng's decomposition theorem [**30**].

THEOREM. *Let* $n \in \mathbb{N}$ *and* $f : I \to \mathbb{R}$ *be an* $n$*-Wright convex function. Then, there exist an* $n$*-convex function* $g : I \to \mathbb{R}$ *and a polynomial function* $P : \mathbb{R} \to \mathbb{R}$ *of degree at most* $n$ *such that*

$$f(x) = g(x) + P(x) \qquad (x \in I).$$

Our aim was to obtain a new and transfinite induction-free proof for this result. To accomplish this goal, we needed the following lemma, which was our most important tool.

LEMMA. *Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be an $n$-Jensen convex function. Then there exists a continuous $n$-convex function $g : I \to \mathbb{R}$ such that $g|_{I\cap\mathbb{Q}} = f|_{I\cap\mathbb{Q}}$.*

### On convexity properties with respect to a Chebyshev system

The main role of Chapter two is to define the Wright convexity with respect to Chebyshev system and generalize the decomposition theorem of Ng [**30**] to a certain Chebyshev system.

The simplex of strictly ordered $n$-tuples of a set $H \subset \mathbb{R}$, denoted by $\sigma_n(H)$, is defined by

$$\sigma_n(H) := \{(x_1, \dots, x_n) \in H^n \quad | \quad x_1 < \dots < x_n\}.$$

We assume that $|H| \geq n$. Let $\omega = (\omega_1, \dots, \omega_n) : H \to \mathbb{R}^n$ be a vector-valued function, and define the functional operator $\Phi_\omega := \Phi_{(\omega_1,\dots,\omega_n)} : \sigma_n(H) \to \mathbb{R}$ by

$$\Phi_\omega(x_1, \dots, x_n) := \begin{vmatrix} \omega_1(x_1) \dots \omega_1(x_n) \\ \vdots \quad \ddots \quad \vdots \\ \omega_n(x_1) \dots \omega_n(x_n) \end{vmatrix} \qquad ((x_1, \dots, x_n) \in \sigma_n(H)).$$

A continuous function $\omega$ is said to be an *$n$-dimensional positive (respectively negative) Chebyshev system over $H$* if $\Phi_\omega$ is strictly positive (respectively, strictly negative) over $\sigma_n(H)$. The system $\omega$ is called an *$n$-dimensional Chebyshev system over $H$* if it is either a positive or a negative Chebyshev system over $H$. If $\omega : \mathbb{R} \to \mathbb{R}^n$ equals the *$n$-dimensional standard or polynomial system $\pi_n : I \to \mathbb{R}^n$*, which is defined by

$$\pi_n(t) := (1, t, \dots, t^{n-1}) \qquad (t \in \mathbb{R}),$$

and turns out to be a positive Chebyshev system. More generally, if $p_1 < \dots < p_n$ are given exponents, then the system given by

$$\mathbb{R}_+ \ni t \mapsto (t^{p_1}, \dots, t^{p_n})$$

is also a positive Chebyshev system on $\mathbb{R}_+$. Important Chebyshev system arise also related to hyperbolic and trigonometric functions. For instance, for all $n \in \mathbb{N}$, the systems given by

$$I \ni t \mapsto (\cos(t), \sin(t), \dots, \cos(nt), \sin(nt)),$$
$$I \ni t \mapsto (1, \cos(t), \sin(t), \dots, \cos(nt), \sin(nt))$$

are positive $2n$- and $(2n+1)$-dimensional Chebyshev systems over any nonempty open interval $I$ with length less than or equal to $\pi$ and $2\pi$, respectively. (For the proof of this statements, see the introduction of the paper [**34**].) There are analogous Chebyshev systems in terms of hyperbolic functions as well. We give the standard examples of convex functions with respect of Chebyshev system.

(i) Polynomial system: $\omega(x) := (1, x, \dots, x^n)$;
(ii) exponential system: $\omega(x) := (1, \exp(x), \dots, \exp(nx))$;
(iii) hyperbolic system: $\omega(x) := (1, \cosh(x), \sinh(x), \dots, \cosh(nx), \sinh(nx))$ and
(iv) trigonometric system: $\omega(x) := (1, \cos(x), \sin(x), \dots, \cos(nx), \sin(nx))$,

where $\omega : I \to \mathbb{R}^n$ for all items (i), (ii) and (iii), and $\omega : ]-\frac{\pi}{2}, \frac{\pi}{2}[ \to \mathbb{R}^n$ for (iv). For further standard applications of Chebyshev systems, we refer to the monographs [**7**], [**17**] and [**18**].

In what follows, we recall some definitions from the paper [**34**] (see also the paper [**12**] for these definitions in the polynomial setting). Let $I \subset \mathbb{R}$ be a nonvoid interval, $n \in \mathbb{N}$ and let $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an $n$-dimensional positive Chebyshev system over $I$. For a function $f : I \to \mathbb{R}$, the functional operator $\Phi_{(\omega,f)} : \sigma_{n+1}(I) \to \mathbb{R}$ is defined by $\Phi_{(\omega,f)} := \Phi_{(\omega_1,\dots,\omega_n,f)}$.

For a given vector $t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, a function $f : I \to \mathbb{R}$ is said to be $(t, \omega)$-*convex* if

$$\Phi_{(\omega,f)}(x, x + t_1 h, \dots, x + (t_1 + \dots + t_n)h) \geq 0 \tag{4}$$

holds for all $h > 0$, $x \in I$ with $x + (t_1 + \dots + t_n)h \in I$. If $T \subseteq \mathbb{R}_+$ and $f$ is $(t, \omega)$-convex for every $t \in T^n$, then $f$ is called $(T, \omega)$-*convex*.

If $t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$ and (4) is satisfied with equality, then $f$ is called a $(t, \omega)$-*affine function*. If $T \subseteq \mathbb{R}_+$ and $f$ is $(t, \omega)$-affine for every $t \in T^n$, then $f$ is called $(T, \omega)$-*affine*. In particular, we say that $f$ is $\omega$-*Jensen convex* if it is $(\{1\}, \omega)$-convex, i.e., if

$$\Phi_{(\omega,f)}(x, x + h, \dots, x + nh) \geq 0 \tag{5}$$

holds for all $h > 0$, $x \in I$ with $x + nh \in I$. If (5) is valid with equality instead of inequality, then $f$ is said to be $\omega$-*Jensen affine*.

A function $f$ is termed $\omega$-*convex* if it is $(\mathbb{R}_+, \omega)$-convex. It is easy to see that $f$ is $\omega$-convex on $I$ if and only if

$$\Phi_{(\omega,f)}(x_0, x_1, \dots, x_n) \geq 0 \qquad ((x_0, x_1, \dots, x_n) \in \sigma_{n+1}(I)). \tag{6}$$

A function $f$ is called $\omega$-*affine* if (6) is satisfied with equality.

We have to mention that in the case when $\omega = \pi_n$, then the concepts of $\omega$-convexity and $\omega$-Jensen convexity, was introduced by Hopf [**14**] and Popoviciu [**39**] (see also the book [**21**] by Kuczma) and these properties were called convexity and Jensen convexity of order $(n-1)$, respectively. If a function $f : I \to \mathbb{R}$ is $n$ times differentiable, then it is convex of order $(n-1)$, i.e., convex with respect to the polynomial system $\pi_n$ if and only if the $n$th derivative of $f$ is nonnegative over $I$. In the particular case when $n = 2$, this is the standard characterization of convexity of twice differentiable functions.

It is a nontrivial statement whether or not $\omega$-convex functions form a proper subclass of $\omega$-Jensen convex functions. Depending on the Chebyshev system, the answer could be positive and negative as well. On the other hand, it is well-known that, for all $n \geq 2$, $\pi_n$-convex functions form a proper subset of $\pi_n$-Jensen convex functions. By [**32**, Theorem 2], it follows that, for any additive function $A : \mathbb{R} \to \mathbb{R}$ and $n \in \mathbb{N}$, the function $f := A^{n-1}$ is Jensen affine (and hence it is Jensen convex) of order $n-1$. On the other hand, $f$ is convex of order $n-1$ if and only if $A$ is continuous. Therefore, if $A$ is discontinuous, then $f$ cannot be convex of order $n-1$. For a construction of a Jensen convex function of order $n-1$ which is not Wright convex of order $n-1$, we refer to the paper [**29**].

In what follows we give a short description of this Chapter.

The following result shows that, for any nonempty set $T \subseteq \mathbb{R}_+$, $(T,\omega)$-convexity implies $\omega$-Jensen convexity.

THEOREM. *Let $T \subseteq \mathbb{R}_+$ be a nonempty set. If a function $f : I \to \mathbb{R}$ is $(T,\omega)$-convex (resp. $(T,\omega)$-affine), then it is $(\mathbb{Q}_+,\omega)$-convex (resp. $(\mathbb{Q}_+,\omega)$-affine), in particular, it is $\omega$-Jensen convex (resp. $\omega$-Jensen affine).*

If $D$ is a subset of $I$ and $f : D \to \mathbb{R}$, then $f$ is said to be *compactly uniformly continuous* if, for all compact subintervals $[a,b]$ of $I$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in [a,b] \cap D$ with $|x-y| < \delta$, we have that $|f(x) - f(y)| < \varepsilon$.

LEMMA. *Let $D$ be a subset of $I$ and let $f : D \to \mathbb{R}$ be a compactly uniformly continuous function. Then $f$ admits a continuous extension to $I$. Provided that $D$ is dense, the extension is unique.*

THEOREM. *Let $n \geq 2$ and let $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an $n$-dimensional positive Chebyshev system and let $\mathbb{K}$ be a subfield of $\mathbb{R}$. If a function $f : I \to \mathbb{R}$ is $(\mathbb{K}_+,\omega)$-convex (resp. $(\mathbb{K}_+,\omega)$-affine), then there exists a continuous $\omega$-convex (resp. $\omega$-affine) function $g : I \to \mathbb{R}$ such that $g|_{I\cap\mathbb{K}} = f|_{I\cap\mathbb{K}}$.*

(Here, and in the sequel, $\mathbb{K}_+$ denotes the intersection $\mathbb{K} \cap \mathbb{R}_+$.)

COROLLARY. *Let $n \geq 2$ and $\omega = (\omega_1, \ldots, \omega_n) : I \to \mathbb{R}^n$ be an $n$-dimensional positive Chebyshev system. If $f : I \to \mathbb{R}$ is $\omega$-convex (resp. $\omega$-affine), then it is continuous on $I$.*

COROLLARY. *Let $n \geq 2$ and $\omega = (\omega_1, \ldots, \omega_n) : I \to \mathbb{R}^n$ be an $n$-dimensional positive Chebyshev system. If $f : I \to \mathbb{R}$ is $\omega$-Jensen convex (resp. $\omega$-Jensen affine), then there exists a continuous $\omega$-convex (resp. $\omega$-affine) function $g : I \to \mathbb{R}$ such that $g|_{I\cap\mathbb{Q}} = f|_{I\cap\mathbb{Q}}$.*

The following statement is the extension of the celebrated Bernstein–Doetsch theorem [**6**] to the setting of $\omega$-Jensen convexity.

THEOREM. *If $f : I \to \mathbb{R}$ is $\omega$-Jensen convex and bounded on a nonempty open subset of $I$, then it is continuous on $I$.*

THEOREM. *Let $f : I \to \mathbb{R}$ be a function which is bounded on a nonempty open subset of $I$. Then it is $\omega$-Jensen affine if and only if $f = \alpha_1\omega_1 + \cdots + \alpha_n\omega_n$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.*

LEMMA. *Let $\omega : I \to \mathbb{R}^n$ be a positive Chebyshev system. Assume that there exist a positive continuous function $\omega_0 : I \to \mathbb{R}_+$ and a matrix $M = (a_{i,j})_{1\leq i\leq n,\, 0\leq j\leq n-1} \in \mathbb{R}^{n\times n}$ such that*

$$\omega_i(x) = (a_{i,n-1}x^{n-1} + \cdots + a_{i,1}x + a_{i,0}) \cdot \omega_0(x) \qquad (x \in I,\ i \in \{1, \ldots, n\}). \tag{7}$$

*Then* $\det(M) > 0$ *and, for all $(x_1, \ldots, x_n) \in \sigma_n(I)$,*

$$\Phi_\omega(x_1, \ldots, x_n) = \omega_0(x_1) \cdots \omega_0(x_n) \cdot \det(M) \cdot \Phi_{\pi_n}(x_1, \ldots, x_n).$$

*Additionally, let $f : I \to \mathbb{R}$. Then, for all $(x_0, \ldots, x_n) \in \sigma_{n+1}(I)$,*

$$\Phi_{\omega,f}(x_0, \ldots, x_n) = \omega_0(x_0) \cdots \omega_0(x_n) \cdot \det(M) \cdot \Phi_{\pi_n, f/\omega_0}(x_0, \ldots, x_n).$$

THEOREM. *Assume that there exist a positive continuous function $\omega_0 : I \to \mathbb{R}_+$ and a matrix $M = (a_{i,j})_{1\leq i\leq n,\, 0\leq j\leq n-1} \in \mathbb{R}^{n\times n}$ such that* (7) *holds. Then $f : I \to \mathbb{R}$ is an $\omega$-Jensen convex (resp. $\omega$-Jensen affine) function if and only if $\frac{f}{\omega_0}$ is a $\pi_n$-Jensen convex (resp. $\pi_n$-Jensen affine) function.*

THEOREM. *A function $f : I \to \mathbb{R}$ is $\pi_n$-Jensen affine if and only if there exist a constant $A_0 \in \mathbb{R}$, an additive function $A_1 : \mathbb{R} \to \mathbb{R}$, a symmetric biadditive function $A_2 : \mathbb{R}^2 \to \mathbb{R}$, …, and a symmetric $(n-1)$-additive function $A_{n-1} : \mathbb{R}^{n-1} \to \mathbb{R}$ such that*

$$f(x) = A_{n-1}(x, \ldots, x) + \cdots + A_2(x, x) + A_1(x) + A_0 \quad (x \in I).$$

COROLLARY. *Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $(a_{i,j})_{1\le i\le n,\,0\le j\le n-1} \in \mathbb{R}^{n\times n}$ *such that* (7) *holds. Then* $f : I \to \mathbb{R}$ *is an* $\omega$*-Jensen affine function if and only if there exist a constant* $A_0 \in \mathbb{R}$*, an additive function* $A_1 : \mathbb{R} \to \mathbb{R}$*, a symmetric biadditive function* $A_2 : \mathbb{R}^2 \to \mathbb{R}, \dots,$ *and a symmetric* $(n-1)$*-additive function* $A_{n-1} : \mathbb{R}^{n-1} \to \mathbb{R}$ *such that*

$$f(x) = \big(A_{n-1}(x,\dots,x) + \cdots + A_2(x,x) + A_1(x) + A_0\big)\omega_0(x) \qquad (x \in I).$$

We establishes the connections between $\omega$-convexity, $\overline{\omega}$-Wright convexity and $\omega$-Jensen convexity.

THEOREM. *Let* $\omega$ *be a positive* $n$*-dimensional Chebyshev system and* $\overline{\omega}$ *be an extension of* $\omega$*. Then every* $\omega$*-convex function is* $\overline{\omega}$*-Wright convex and every* $\overline{\omega}$*-Wright convex function is* $\omega$*-Jensen convex.*

The next theorem describes the connection between $\overline{\omega}$-Wright convexity and Wright convexity of order $(n-1)$. In what follows, the symbol $[x_0, x_1, \dots, x_n, g]$ denotes the standard $n$th-order divided difference of a function $g : I \to \mathbb{R}$ at the pairwise distinct nodes $x_1, x_1, \dots, x_n \in I$.

THEOREM. *Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $M := (a_{i,j})_{1\le i\le n,\,0\le j\le n-1} \in \mathbb{R}^{n\times n}$ *such that* (7) *holds. Define* $\omega_{n+1} : I \to \mathbb{R}$ *by* $\omega_{n+1}(t) := t^n\omega_0(t)$*. Then* $\overline{\omega} := (\omega, \omega_{n+1})$ *is an extension of the Chebyshev system* $\omega$*. In addition, we have the following assertions:*

*(i) For all* $(x_0, x_1, \dots, x_n) \in \sigma_{n+1}(I)$*, the equality*

$$\frac{\Phi_{(\omega,f)}(x_0, x_1, \dots, x_n)}{\Phi_{\overline{\omega}}(x_0, x_1, \dots, x_n)} = [x_0, x_1, \dots, x_n; f/\omega_0]$$

*holds. Furthermore, a function* $f : I \to \mathbb{R}$ *is* $\omega$*-convex if and only if* $f/\omega_0$ *is convex of order* $(n-1)$*.*

*(ii) For all* $h_1, \dots, h_n > 0$ *and* $x \in I \cap (I - (h_1 + \cdots + h_n))$*, the equality*

$$\sum_{(i_1,\dots,i_n)} \frac{\Phi_{(\omega,f)}(x, x+h_{i_1}, \dots, x+h_{i_1}+\cdots+h_{i_n})}{\Phi_{\overline{\omega}}(x, x+h_{i_1}, \dots, x+h_{i_1}+\cdots+h_{i_n})} = \frac{\Delta_{h_1}\cdots\Delta_{h_n}(f/\omega_0)(x)}{h_1\cdots h_n}$$

*holds. Furthermore, a function* $f : I \to \mathbb{R}$ *is* $\overline{\omega}$*-Wright convex if and only if* $f/\omega_0$ *is Wright convex of order* $(n-1)$*.*

In the following result, we establish a characterization theorem for $\overline{\omega}$-Wright convexity provided that the underlying Chebyshev system is strongly related to the polynomial one. This result generalizes the decomposition theorem of Maksa and Páles [**24**] which is related to the polynomial system. An alternative and more elementary proof of that theorem has been recently given by the authors in [**36**].

THEOREM. *Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $M := (a_{i,j})_{1\le i\le n,\, 0\le j\le n-1} \in \mathbb{R}^{n\times n}$ *such that* (7) *holds. Define* $\omega_{n+1} : I \to \mathbb{R}$ *by* $\omega_{n+1}(t) := t^n\omega_0(t)$ *and set* $\overline{\omega} := (\omega, \omega_{n+1})$. *Then a function* $f : I \to \mathbb{R}$ *is* $\overline{\omega}$*-Wright convex if and only if there exist an* $\omega$*-convex function* $F : I \to \mathbb{R}$ *and, for each* $k \in \{1, \dots, n-1\}$, *a symmetric* $k$*-additive mapping* $A_k : \mathbb{R}^k \to \mathbb{R}$ *and a real constant* $A_0$ *such that, for all* $x \in I$,

$$f(x) = F(x) + (A_0 + A_1(x) + \cdots + A_{n-1}(x, \dots, x))\omega_0(x).$$

In our subsequent result we will prove that if the extension of the two dimensional polynomial system is not a polynomial of at most second degree, then the convexity with respect to the two dimensional polynomial system (i.e., standard convexity) is equivalent to Wright convexity with respect to this extension. For the proof of this result, we will need the following characterization of a polynomial of at most second degree.

LEMMA. *Let* $\rho : I \to \mathbb{R}$ *be a continuous function which satisfies the functional equation*

$$\frac{\rho(z) - \rho(y)}{z - y} = \frac{\rho(z+u) - \rho(y-u)}{(z+u) - (y-u)},$$
$$u \ge 0,\ y, z \in (I+u) \cap (I-u),\ z + u \ge y.$$

*Then* $\rho$ *is a polynomial of at most second degree over* $I$.

THEOREM. *Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $M := (a_{i,j})_{1\le i\le 2,\, 0\le j\le 1} \in \mathbb{R}^{2\times 2}$ *such that* (7) *holds for* $n = 2$. *Assume that* $\omega_3 : I \to \mathbb{R}$ *is a continuous function such that* $\overline{\omega} = (\omega_1, \omega_2, \omega_3)$ *is an extension of* $\omega = (\omega_1, \omega_2)$ *and* $\omega_3/\omega_0$ *is not a polynomial of at most second degree. Then every* $\overline{\omega}$*-Wright convex function is* $\omega$*-convex, i.e.,* $\overline{\omega}$*-Wright convexity is equivalent to* $\omega$*-convexity.*

## Cauchy–Schwarz-type inequalities and Levi–Civita-type functional equations

Let $(G, *)$ be a groupoid. (Recall that a pair $(G, *)$ is said to be a groupoid if $\cdot$ is a binary operation on $G$, i.e., $\cdot : G \times G \to G$.) Let $A : G \to \mathbb{R}$ be a function such that there exist functions $f_1, \dots, f_n, g_1, \dots, g_n : G \to \mathbb{R}$ such that the functional equation

$$A(x * y) = f_1(x)g_1(y) + \cdots + f_n(x)g_n(y) \qquad (x, y \in G)$$

is fulfilled. Under certain assumptions on $n$ and on the functions $f_1, \ldots, f_n, g_1, \ldots, g_n$, we are going to prove that for all $x, y \in G$ the function $A$ will satisfy either the inequality $A(x * y)^2 \le A(x * x)A(y * y)$ or the reversed one $A(x * x)A(y * y) \le A(x * y)^2$. In the important particular case when the groupoid is the multiplicative structure of a commutative ring and $A$ is additive, we will establish the existence of nontrivial additive functions which satisfy one of the above mentioned inequalities.

In our first result we assume that the function $A$ satisfies a Levi–Civita-type functional equation over a groupoid.

PROPOSITION. *Let $(G, *)$ be a groupoid and let $A : G \to \mathbb{R}$ be a function. Assume that there exist $n \in \mathbb{N}$ and functions $f_1, \ldots, f_n : G \to \mathbb{R}$ such that $A$ satisfies the Levi–Civita-type functional equation*

$$A(x * y) = f_1(x)f_1(y) + \cdots + f_n(x)f_n(y) \tag{8}$$

*for all $x, y \in G$. Then, $A$ fulfills the functional inequality*

$$A(x * y)^2 \le A(x * x)A(y * y) \tag{9}$$

*for all $x, y \in G$.*

If the groupoid is the multiplicative semigroup of a commutative ring $(R, +, \cdot)$ and $A$ is additive, then we can establish a characterization of the corresponding inequality. Recall that in a ring, the product $x \cdot y$ of the elements $x, y \in R$ is simply denoted by $xy$, and $x^2$ is defined to be the product $x \cdot x$.

THEOREM. *Let $(R, +, \cdot)$ be a commutative ring and let $A : R \to \mathbb{R}$ be an additive function. Then $A$ satisfies the inequality*

$$A(xy)^2 \le A(x^2)A(y^2) \tag{10}$$

*for all $x, y \in R$ if and only if at least one of the following conditions hold*

*(i) $A(x^2) \ge 0$ for all $x \in R$,*
*(ii) $A(x^2) \le 0$ for all $x \in R$.*

In the subsequent two propositions, we present two Levi–Civita-type functional equations which imply the reversed inequality for $A$.

PROPOSITION. *Let $(G, *)$ be a groupoid and $A : G \to \mathbb{R}$ be a function. Assume that there exist two functions $f, g : G \to \mathbb{R}$ such that the Levi–Civita-type functional equation*

$$A(x * y) = f(x)f(y) - g(x)g(y)$$

*holds for all* $x, y \in G$. *Then* $A$ *satisfies the functional inequality*

$$A(x * x)A(y * y) \leq A(x * y)^2 \tag{11}$$

*for all* $x, y \in G$.

PROPOSITION. *Let* $(G, *)$ *be a groupoid. Let* $A : R \to \mathbb{R}$ *be a function. Assume that there exist* $f, g : R \to \mathbb{R}$ *such that the Levi–Civita-type functional equation*

$$A(x * y) = f(x)g(y) + g(x)f(y)$$

*holds for all* $x, y \in G$. *Then, for all* $x, y \in G$, $A$ *satisfies the functional inequality* (11).

COROLLARY. *Assume that* $A : \mathbb{R} \to \mathbb{R}$ *satisfies the Leibniz Rule with respect to multiplication, i.e.,*

$$A(xy) = xA(y) + A(x)y \qquad (x, y \in \mathbb{R}).$$

*Then, for all* $x, y \in \mathbb{R}$, *the inequality*

$$A(x^2)A(y^2) \leq A(xy)^2 \tag{12}$$

*holds.*

In particular, if $A : \mathbb{R} \to \mathbb{R}$ is a derivation (i.e., $A$ is additive and satisfies the Leibniz Rule with respect to multiplication), then the above corollary implies that it fulfills the inequality (12).

If the groupoid is the multiplicative semigroup of a commutative ring $(R, +, \cdot)$ and $A$ is additive, then we can establish a characterization of the inequality (11) over a particular subset of the ring.

THEOREM. *Let* $(R, +, \cdot)$ *be a commutative ring with a multiplicative unit element* $e$ *and* $A : R \to \mathbb{R}$ *be an additive function with* $A(e) \neq 0$. *Let the subset* $R_A \subseteq R$ *be defined by*

$$R_A := \{x \in R \mid 0 \leq A(x^2)A(e)\}.$$

*Then* $e \in R_A$ *and* $A$ *satisfies the following functional inequality*

$$A(x^2)A(y^2) \leq A(xy)^2 \tag{13}$$

*for all* $x, y \in R_A$ *if and only if*

$$A(x^2)A(e) \leq A(x)^2 \tag{14}$$

*for all* $x \in R_A$.

In the following example we show that the additivity of the function $A$ in this theorem is necessary.

EXAMPLE. Let $q \in (0,1)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x & x \neq 1, \\ q & x = 1. \end{cases}$$

Clearly, $f$ is not additive. Therefore, for $x \notin \{1, -1\}$, we have that

$$f(x^2)f(1) = qx^2 \leq x^2 = f(x)^2$$

for $x = \pm 1$,

$$f(1^2)f(1) = q^2 = f(1)^2, \qquad f((-1)^2)f(1) = q^2 < 1 = f(-1)^2,$$

which shows that (14) is satisfied for all $x \in \mathbb{R}$. On the other hand, for $x, y \in \mathbb{R} \setminus \{1, -1\}$ with $xy = 1$, we can conclude that

$$f(x^2)f(y^2) = x^2y^2 = 1 > q^2 = f(xy)^2,$$

which shows that the above theorem is not satisfied.

The next example shows that if the function $A$ in the above theorem is non-additive, continuous and satisfies $A(e) = 0$, then the conclusion of this theorem may not be valid.

EXAMPLE. Let $A : \mathbb{R} \to \mathbb{R}$ be defined as $A(x) = |x - 1|$. Note that $A$ is continuous and not additive. Since $A(1) = 0$ this implies that

$$A(x^2)A(1) = 0 \leq (x-1)^2 = A(x)^2.$$

Thus the inequality (14) holds for all $x \in \mathbb{R}$. On the other hand we have that

$$A(x^2)A(y^2) = |x^2 - 1||y^2 - 1| \qquad \text{and} \qquad A(xy)^2 = (xy - 1)^2.$$

Hence for $x = 2$ and $y = \frac{1}{2}$ we have that $A(x^2)A(y^2) = \frac{9}{4}$ but $A(xy)^2 = 0$, therefore the inequality (13) does not hold.

THEOREM. *Let $(G, *)$ be a groupoid and $A, B : G \to \mathbb{R}$ be functions. Assume that there exist $f, g : G \to \mathbb{R}$ such that $A$ and $B$ satisfy the following system of Levi–Civita-type functional equations*

$$\begin{aligned} A(x * y) &= f(x)f(y) - g(x)g(y) \qquad \textit{and} \\ B(x * y) &= f(x)g(y) + g(x)f(y) \end{aligned} \tag{15}$$

*for all $x, y \in G$. Then the inequalities*

$$-B(x * y)^2 \leq A(x * x)A(y * y) \leq A(x * y)^2 \tag{16}$$

*and*

$$-A(x * y)^2 \leq B(x * x)B(y * y) \leq B(x * y)^2 \tag{17}$$

*hold for all $x, y \in G$.*

An interesting consequence of the functional equations in (15) is that $A$ and $B$ satisfy the following identity:

$$B(x*y)^2 + A(x*x)A(y*y) = A(x*y)^2 + B(x*x)B(y*y), \ (x, y \in G).$$

Therefore, the inequalities (16) and (17) can be expressed as the following chain of inequalities

$$\begin{aligned} 0 &\le A(x*x)A(y*y) + (B(x*y))^2 \\ &= B(x*x)B(y*y) + A(x*y)^2 \le A(x*y)^2 + B(x*y)^2 \qquad (x, y \in G). \end{aligned}$$

The following result is probably well-known, but we could not find an exact reference for it.

COROLLARY. *For all $x, y \in \mathbb{R}$, we have*

$$\begin{aligned} -\sin(x+y)^2 \le \cos(2x)\cos(2y) \le \cos(x+y)^2 \qquad &\textit{and} \\ -\cos(x+y)^2 \le \sin(2x)\sin(2y) \le \sin(x+y)^2. \end{aligned}$$

COROLLARY. *Let $(G, *)$ be a groupoid and let $\varphi : G \to \mathbb{C}$ be a homomorphism into the multiplicative semigroup of complex numbers. Define $A := \Re\varphi$ and $B := \Im\varphi$. Then, for all $x, y \in G$, the inequalities* (16) *and* (17) *hold.*

COROLLARY. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an automorphism of the field $\mathbb{C}$. Define $A := \Re\varphi$ and $B := \Im\varphi$. Then $A : \mathbb{C} \to \mathbb{R}$ and $B : \mathbb{C} \to \mathbb{R}$ are additive mappings, furthermore, for all $x, y \in \mathbb{C}$,*

$$\begin{aligned} -B(xy)^2 \le A(x^2)A(y^2) \le A(xy)^2 \qquad &\textit{and} \\ -A(xy)^2 \le B(x^2)B(y^2) \le B(xy)^2. \end{aligned}$$

THEOREM. *Let $(G, *)$ be a groupoid and $A, B : G \to \mathbb{R}$ be functions. Assume that there exist $f, g : G \to \mathbb{R}$ such that $A$ and $B$ satisfy the following Levi–Civita-type functional equations*

$$\begin{aligned} A(x*y) = f(x)f(y) + g(x)g(y) \qquad &\textit{and} \\ B(x*y) = f(x)g(y) + g(x)f(y) \end{aligned}$$

*for all $x, y \in G$. Then the inequalities*

$$B(x*x)B(y*y) \le A(x*y)^2 \le A(x*x)A(y*y)$$

*and*

$$B(x*x)B(y*y) \le B(x*y)^2 \le A(x*x)A(y*y).$$

*hold for all $x, y \in G$.*

An interesting consequence of the functional equations in (15) is that $A$ and $B$ satisfy the following identity:

$$B(x*x)B(y*y)+A(x*x)A(y*y)=A(x*y)^2+B(x*y)^2 \quad (x,y\in G).$$

The following result is probably also well-known, but we could not find a reference for it.

COROLLARY. *For all $x,y\in\mathbb{R}$, we have*

$$\sinh(2x)\sinh(2y)\le\sinh(x+y)^2<\cosh(x+y)^2\le\cosh(2x)\cosh(2y)$$

To formulate the next result, let $p$ be a square free positive integer and let $\mathbb{Q}(\sqrt{p})$ denote the subfield of $\mathbb{R}$ generated by $\sqrt{p}$. Then, one can see that $\mathbb{Q}(\sqrt{p})=\{a+b\sqrt{p} : a,b\in\mathbb{Q}\}$. In what follows, we equip $\mathbb{Q}(\sqrt{p})$ with the topology inherited from $\mathbb{R}$.

THEOREM. *Let $p$ be a square free positive integer. Then there exist two discontinuous additive functions $A:\mathbb{Q}(\sqrt{p})\to\mathbb{R}$ and $B:\mathbb{Q}(\sqrt{p})\to\mathbb{R}$ such that the functions $A$ and $B$ fulfill the inequalities*

$$\begin{aligned}B(x^2)B(y^2)&\le A(xy)^2\le A(x^2)A(y^2),\\ B(x^2)B(y^2)&\le B(xy)^2\le A(x^2)A(y^2)\end{aligned}\tag{18}$$

*for all $x,y\in\mathbb{Q}(\sqrt{p})$.*

We note that there does not exist a discontinuous additive function $A:\mathbb{R}\to\mathbb{R}$ such that the inequality $A(xy)^2\le A(x^2)A(y^2)$ be valid for all real numbers $x,y$. Indeed, if $A$ is a discontinuous additive function satisfying this inequality, then $A(u)$ is not zero for some $u>0$. With the substitution $y:=\sqrt{u}$, the inequality shows that $A$ is either nonnegative (if $A(u)>0$) or nonpositive (if $A(u)<0$) on the set of positive numbers. This, by classical results on additive functions (see [**21**]), implies that $A(x)=ax$ for some $a\in\mathbb{R}$, and hence $A$ is continuous.

Motivated by the above remark, we could formulate the following open problem: Find a description or characterization of those maximal subrings (or subfields) of $\mathbb{R}$ such that system of inequalities in (18) holds for a discontinuous pair $(A,B)$ of additive functions which are defined on this subring (or subfield).

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