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Generalizations and stability of convexity

Thesis for the Degree of Doctor of Philosophy (PhD)

written by Gábor Marcell Molnár
supervised by Prof. Dr. Zsolt Páles

UNIVERSITY OF DEBRECEN
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Doctoral School of Mathematical and Computational Sciences
Debrecen, 2024.

Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at Debrecen University.
The results published in the thesis are not reported in any other PhD theses.

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Hereby I confirm that Gábor Marcell Molnár candidate conducted his studies with my supervision within the Mathematical analysis, functional equations and inequalities Doctoral Program of the Doctoral School of Doctoral School of Mathematical and Computational Sciences between 2020 and 2024. The independent studies and research work of the candidate significantly contributed to the results published in the thesis.
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GENERALIZATIONS AND STABILITY OF CONVEXITY

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Dissertation advisor: Dr. Zsolt Páles

The official opponents of the dissertation:

	Dr.	....
	Dr.	....

The evaluation committee:

chairperson:	Dr.	....
members:	Dr.	....
	Dr.	....
	Dr.	....
	Dr.	....

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Contents

Notations

The following list outlines various symbols that will be utilized later in the document.

Number sets

$\mathbb{R}$	The set of real numbers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{Z}$	The set of integer numbers.
$\mathbb{N}$	The set of positive integers.
$\mathbb{S}_+$	The positive elements of the set $\mathbb{S}$.

Notations for chapter one

C_x	The set of incides for which an element is n-convex.
C^n	The set of n-convex elements.
$\mathrm{conv}_n(x)$	The n-convex hull of an element x.
K^n	The collection of those elements which have an n-convex hull.
M^n	The set of elements that have an n-convex majorant.
$P_W(S)$	The collection of all nonempty W-invariant subsets of S.
χ_W	The characteristic function of a set W.

$X_{\preceq}$	The set of nonnegative elements of X.
$X_{\prec}$	The set of Archimedean elements of X.
$\mathrm{cl}_{\mathcal{A}}(x)$	The $\mathcal{A}$-closure of the element x.
$\mathrm{Cl}_{\mathcal{A}}$	The set of all elements of X which possess an $\mathcal{A}$-closure.

Notations for chapter two

$T_k(x)$	Chebyshev polynomials of the first kind of order k.
$U_k(x)$	Chebyshev polynomials of the second kind of order k.
$\mathcal{S}(n\|m)$	The set of all real sequences indexed from the nth term to the mth one.
$\mathcal{C}_q^{\cup}(n\|m)$	The set of q-convex sequences indexed from the nth term to the mth one.
$\mathcal{C}_q^{\cap}(n\|m)$	The set of q-concave sequences indexed from the nth term to the mth one.
$\mathcal{A}_q(n\|m)$	The set of q-affine sequences indexed from the nth term to the mth one.

Introduction

This dissertation consists of two distinct topics, separated into two main chapters. The first and the second chapters are based on the papers [MP21] and [MP22], respectively.

In chapter 1 we introduce and investigate the algebraic structures called cornets. Our primary focus is on extending the well-regarded Rådström cancellation principle to these specific objects. This chapter aims to delve into the adaptation and application of the cancellation principle within the context of cornets. Additionally, we establish the fundamental properties of cornets, covering essential aspects such as convexity properties, topological notions and, boundedness.

In chapter 2, we broaden the concepts of convex, concave, and affine sequences by introducing the novel notions of q-convex, q-concave, and q-affine sequences. This chapter not only unveils foundational results but also highlights an unexpected correlation between these sequences and Chebyshev polynomials. Furthermore, we present two practical applications of q-convex, q-concave, and q-affine sequences. The first involves addressing a minimax-type problem, while the second is an application in fixed point theory.

Lastly, unnumbered chapters within the dissertation encompass summaries in both English and Hungarian, distinct bibliographies for each chapter, collection of citations referencing the papers [MP21] and [MP22], along with a compilation of publications validated by DEENK.

Chapter 1

An extension of the Rådström cancellation theorem to cornets

1.1 Introduction to the first chapter

In the theory of convex sets, a basic Cancellation Principle was discovered by Rådström [R52b] in 1952. The Lemma 2 of his paper states that the inclusion

$$A + B \subseteq C + B$$

implies $A \subseteq C$ provided that A, B, C are nonempty subsets of a normed space X, C is closed and convex and B is bounded.

This lemma turned out to be a basic tool in various fields and hundreds of papers have used it by now. For instance, in nonsmooth analysis [BF10, BFR12a, BFR12b, CGT10, GGM16, GM18, HN17, D15], optimization theory [CKR14, JS20, K09], theory of convex sets and functions [BS16, CZ14, DMM11, dBT11, GKKU14, GKKU15, GPPU12, GPPU18, GPU10a, GPU16, GPU19, GP15, GP17, GPU13, GU14, VN12, VN15, I15, K14], set-valued analysis [ANR15, CT13, KPR15, LMNS14, MM14, P09, K19, O17, P13], set-valued differential equations [AB15a, AB15b, BG09, GS08, PS14, M15], set-valued functional equations [BMP18, S09, S09, M12, S13, S15–S17, S19], and iteration theory [AN16, AGMM10, SS12, XNZ11, G12, P11], etc.

These applications motivated us to extend the above Cancellation Principle to a more general setting in the paper [MP21] which, possibly, could allow one to apply it to a broader class of problems. It turns out that the natural setting of the Cancellation Principle is a commutative ordered semigroup which is equipped with a multiplication by natural numbers. These structures will be termed *cornets*. The most important examples for cornets are the families of the nonempty subsets and the nonempty fuzzy subsets of a vector space. In a cornet, one can naturally define the convexity, nonnegativity, Archimedean property, boundedness, closedness of an element. In Sections 2 and 3, we establish the basic properties related to these notions and, finally, in Section 4, we state an abstract form of the Cancellation Principle and also its consequences.

1.2 Cornets and convexity properties in cornets

In the next two definitions, we describe the main structure, the notion of a cornet, that we shall investigate in the sequel.

Definition 1.2.1. [MP21]. An ordered triplet $(X, +, \preceq)$ is called an *ordered commutative semigroup* if

(i) $(X, +)$ is a commutative unital semigroup with a unit element 0;

(ii) $(X, \preceq)$ is a partially ordered set, that is, $\preceq$ is a reflexive, antisymmetric and transitive binary relation on X;

(iii) For all $x, y, z \in X$ with $x \preceq y$, the inequality $x + z \preceq y + z$ holds.

If the partially ordered set $(X, \preceq)$ is complete, i.e., every nonempty lower bounded subset of X has a greatest lower bound, then $(X, +, \preceq)$ is called a *complete ordered commutative semigroup*.

A unital subsemigroup $(S, +)$ of an ordered commutative semigroup $(X, +, \preceq)$ is obviously an ordered commutative subsemigroup with the ordering restricted to S.

In a semigroup $(X, +)$, we naturally have the multiplication by natural numbers which is defined recursively by

$$1 \cdot x := x, \qquad (n + 1) \cdot x := n \cdot x + x \qquad (n \in \mathbb{N}).$$

If the semigroup is unital, then we also define $0 \cdot x := 0$. Using induction, one can easily prove that this multiplication obeys the following rules in an ordered commutative semigroup $(X, +, \preceq)$:

(i) For all $n, m \in \mathbb{N}$ and $x \in X$, $(nm) \cdot x = n \cdot (m \cdot x)$;

(ii) For all $n \in \mathbb{N}$ and $x, y \in X$, $n \cdot (x + y) = n \cdot x + n \cdot y$;

(iii) For all $n, m \in \mathbb{N}$ and $x \in X$, $(n + m) \cdot x = n \cdot x + m \cdot x$;

(iv) For all $n \in \mathbb{N}$ and $x, y \in X$, if $x \preceq y$, then $n \cdot x \preceq n \cdot y$.

In the next definition we present the central concept of our paper.

Definition 1.2.2. [MP21]. An ordered quadruple $(X, +, *, \preceq)$ is called a *cornet* if $(X, +, \preceq)$ is an ordered commutative semigroup and "$*$" is a multiplication of the elements of X by positive integers such that the following conditions hold:

(i) For all $n, m \in \mathbb{N}$ and $x \in X$, $(nm) * x = n * (m * x)$;

(ii) For all $n \in \mathbb{N}$ and $x, y \in X$, $n * (x + y) = n * x + n * y$;

(iii) For all $n, m \in \mathbb{N}$ and $x \in X$, $(n + m) * x \preceq n * x + m * x$;

(iv) For all $n \in \mathbb{N}$ and $x, y \in X$, the inequality $x \preceq y$ holds if and only if $n * x \preceq n * y$;

(v) $1 * x = x$;

(vi) $n * 0 = 0$.

If the partially ordered set $(X, \preceq)$ is complete, then $(X, +, *, \preceq)$ is called a *complete cornet*. A unital subsemigroup $(S, +)$ of a cornet $(X, +, *, \preceq)$ which is also closed with respect to the multiplication $*$ is called a *subcornet* of $(X, +, *, \preceq)$ with the ordering restricted to S.

It is obvious that if $(X,+)$ is a commutative unital semigroup such that, for all $n \in \mathbb{N}$, the mapping $n \mapsto n \cdot x$ is injective, then $(X,+,\cdot,=)$ is a cornet. The following lemma summarizes the basic properties and connection between the two multiplication operations "$\cdot$" and "$*$".

Lemma 1.2.3. [MP21] *Let $(X,+,*,\preceq)$ be a cornet. Then the following two assertions hold.*

(i) For all $n,k \in \mathbb{N}$, $x_1,\dots,x_k \in X$,

$$n \cdot (x_1 + \cdots + x_k) = n \cdot x_1 + \cdots + n \cdot x_k \quad \textit{and}$$
$$n * (x_1 + \cdots + x_k) = n * x_1 + \cdots + n * x_k.$$

In particular, for all $n,m \in \mathbb{N}$ and $x \in X$,

$$n * (m \cdot x) = m \cdot (n * x). \tag{1.1}$$

(ii) For all $n,k_1,\dots,k_n \in \mathbb{N}$ and $x \in X$,

$$(k_1 + \cdots + k_n) \cdot x = k_1 \cdot x + \cdots + k_n \cdot x \quad \textit{and}$$
$$(k_1 + \cdots + k_n) * x \preceq k_1 * x + \cdots + k_n * x.$$

In particular, for all $n,m \in \mathbb{N}$ and $x \in X$,

$$(mn) * x \preceq n \cdot (m * x). \tag{1.2}$$

Proof. We prove (i) by induction on k. If $k=1$, then the equalities hold trivially. The $k=2$ case follows from property (ii) of the two operations "$\cdot$" and "$*$". Assume that (i) holds for some $k \in \mathbb{N}$ and let $n \in \mathbb{N}$ and $x_1,\dots,x_{k+1} \in X$ be arbitrary. Then, by property (ii) of the operation "$*$" and the inductive hypothesis, we get

$$\begin{aligned} n * (x_1 + \cdots + x_k + x_{k+1}) &= n * (x_1 + \cdots + x_k) + n * x_{k+1} \\ &= n * x_1 + \cdots + n * x_k + n * x_{k+1}. \end{aligned}$$

For the operation "$\cdot$", the proof is completely similar.

By taking $k := m$ and $x_1 := \cdots = x_k := x$, the second equality in (i) yields the equality (1.1).

The relations in (ii) will be proved by induction on n. For $n = 1$ both of them hold with equality. For $n = 2$, they are consequences of property (iii) of the two operations "$\cdot$" and "$*$". Assume that (ii) holds for some $n \in \mathbb{N}$ and let $k_1, \dots, k_{n+1} \in \mathbb{N}$ and $x \in C$ be arbitrary. Then, by property (iii) of the operation "$*$" and the inductive hypothesis, we get

$$\begin{aligned}(k_1 + \cdots + k_n + k_{n+1}) * x &\preceq (k_1 + \cdots + k_n) * x + k_{n+1} * x \\ &\preceq k_1 * x + \cdots + k_n * x + k_{n+1} * x.\end{aligned}$$

For the operation "$\cdot$", the proof is completely similar.

By taking $k_1 := \cdots = k_n := m$, the second inequality in (ii) yields property (1.2). □

For a given element $x \in X$, the set of those numbers n for which (1.2) holds with equality if $m = 1$ play a crucial role among the properties of x.

Definition 1.2.4. [MP21]. Let $(X, +, *, \preceq)$ be a cornet and $n \in \mathbb{N}$. An element $x \in X$ will be called *n-convex* if it fulfills the equality $n * x = n \cdot x$. For fixed elements $x \in X$ and $n \in \mathbb{N}$, we introduce the notations

$$C_x := \{n \in \mathbb{N} \mid x \text{ is } n\text{-convex}\} \quad \text{and} \quad C^n := \{x \in X \mid x \text{ is } n\text{-convex}\},$$

respectively. If $C_x = \mathbb{N}$, i.e., if x is n-convex for all $n \in \mathbb{N}$, then we say that x is *convex*.

Lemma 1.2.5. [MP21] *Let $(X, +, *, \preceq)$ be a cornet. Then the following assertions hold:*

(i) For all $x \in X$, the set C_x is a unital multiplicative subsemigroup of $\mathbb{N}$.

(ii) The quadruple $(C^n, +, *, \preceq)$ is a subcornet of $(X, +, *, \preceq)$ for all $n \in \mathbb{N}$.

Proof. Let $x \in X$ be fixed. It is clear that $1 \in C_x$. Let $n, m \in C_x$ be arbitrary. Then, by property (i) of the two multiplication operations, by the n- and m-convexity of x and by (1.2), we have that

$$\begin{aligned}(mn) * x = m * (n * x) = m * (n \cdot x)\\ = n \cdot (m * x) = n \cdot (m \cdot x) = (nm) \cdot x.\end{aligned}$$

This shows that x is also (mn)-convex, i.e., $mn \in C_x$.

For the second assertion, let $n \in \mathbb{N}$ be fixed and $x, y \in C^n$. Using property (ii) of the two multiplication operations and the n-convexity of x and y, we have

$$n * (x + y) = n * x + n * y = n \cdot x + n \cdot y = n \cdot (x + y),$$

therefore, $x + y$ is also n-convex.

If $x \in C^n$ and $m \in \mathbb{N}$, then

$$\begin{aligned}n * (m * x) = (nm) * x = (mn) * x\\ = m * (n * x) = m * (n \cdot x) = n \cdot (m * x),\end{aligned}$$

which proves that $m * x$ is n-convex. □

In what follows, we define the n-convex hull of elements in a cornet $(X, +, *, \preceq)$.

Definition 1.2.6. [MP21]. Let $n \in \mathbb{N}$, $(X, +, *, \preceq)$ and $x \in X$. The *n-convex hull* of x, denoted as $\mathrm{conv}_n(x)$, is an element $y \in C^n$ such that $x \preceq y$ and, whenever $x \preceq z \in C^n$, then $y \preceq z$.

In general, the n-convex hull of an element may not exist, but if it exists, then it is unique. In order to formulate conditions which are sufficient for the existence, we say that the $*$-multiplication in a complete cornet $(X, +, *, \preceq)$ is *n-continuous* (with respect to the ordering "$\preceq$") if, for all nonempty lower bounded subsets $H \subseteq X$, we have

$$\inf(n * H) = n * \inf(H).$$

Proposition 1.2.7. [MP21] *Let $n \in \mathbb{N}$ and let $(X, +, *, \preceq)$ be a complete cornet in which the $*$-multiplication is n-continuous. Then $(C^n, +, *, \preceq)$ is a complete subcornet of the cornet $(X, +, *, \preceq)$. Furthermore, for every element $x \in X$, x admits an n-convex hull if and only if it has an n-convex majorant.*

Proof. Let $H \subseteq C^n$ be a lower bounded subset and denote $x := \inf(H)$. Then, for all $h \in H$,

$$n \cdot x \preceq n \cdot h = n * h,$$

hence

$$n \cdot x \preceq \inf(n * H) = n * \inf(H) = n * x.$$

The reversed inequality is a consequence of (1.2) with $m = 1$, hence $n \cdot x = n * x$ holds, which shows that x is also n-convex. This proves that $(C^n, \preceq)$ is a complete partially ordered set.

To prove the last assertion, let $x \in X$ be arbitrary. If x has an n-convex hull, then it also has an n-convex majorant. Conversely, if x admits an n-convex majorant, then the set

$$H := \{z \in C^n \mid x \preceq z\}$$

is nonempty and lower bounded. According to the first part, the infimum u of H belongs to C^n, that is, u is n-convex. It is clear that u is the n-convex hull of x. □

In a cornet $(X, +, *, \preceq)$, let K^n denote the collection of those elements which have an n-convex hull and let M^n denote the set of those elements that have an n-convex majorant. Obviously, we have $C^n \subseteq K^n \subseteq M^n$. Using this terminology, the previous proposition asserts that if $(X, +, *, \preceq)$ is a complete cornet in which the $*$-multiplication is n-continuous, then $K^n = M^n$.

Proposition 1.2.8. [MP21] *Let $(X, +, *, \preceq)$ be a cornet and let $n \in \mathbb{N}$. Then we have the following assertions.*

(i) If $x \in K^n$*, then* $\mathrm{conv}_n(x) \in C^n$ *and* $x \preceq \mathrm{conv}_n(x)$*. Furthermore,* $\mathrm{conv}_n : K^n \to C^n$ *is a monotone mapping whose set of fixed points is equal to* C^n*. In addition,*

$$\begin{aligned} &\mathrm{conv}_n(x+y) \preceq \mathrm{conv}_n(x) + \mathrm{conv}_n(y) &&\text{if } x, y, x+y \in K^n, \\ &\mathrm{conv}_n(m * x) \preceq m * \mathrm{conv}_n(x) &&\text{if } x, m * x \in K^n. \end{aligned} \tag{1.3}$$

(ii) $(M^n, +, *, \preceq)$ *is a subcornet of* $(X, +, *, \preceq)$.

Proof.

(i) For an arbitrary $x \in K^n$, the inclusion $\mathrm{conv}_n(x) \in C^n$ and the inequality $x \preceq \mathrm{conv}_n(x)$ are consequences of the definition of the n-convex hull. If $x \in C^n$, then the smallest n-convex element which is nonsmaller than x is equal to x, that is, $x = \mathrm{conv}_n(x)$. Conversely, if $x = \mathrm{conv}_n(x)$, then $\mathrm{conv}_n(x) \in C^n$ implies that x must be in C^n. To see that conv_n is monotone, let $x, y \in K^n$ with $x \preceq y$. Then $x \preceq \mathrm{conv}_n(y)$, which yields that $\mathrm{conv}_n(x) \preceq \mathrm{conv}_n(y)$. If $x, y, x+y \in K^n$, then the inequalities $x \preceq \mathrm{conv}_n(x)$ and $y \preceq \mathrm{conv}_n(y)$ imply that $x + y \preceq \mathrm{conv}_n(x) + \mathrm{conv}_n(y) \in C^n$. This proves the first inequality in (1.3). If $x, m*x \in K^n$, then the inequality $x \preceq \mathrm{conv}_n(x)$ yields that $m*x \preceq m*\mathrm{conv}_n(x) \in C^n$. This shows the second inequality in (1.3).

(ii) Let $x, y \in M^n$. Then there exist $u, v \in C^n$ such that $x \preceq u$ and $y \preceq v$. Thus yields that $x + y \preceq u + v \in C^n$, which proves that $x + y \in M^n$. If $x \in M^n$ and $m \in \mathbb{N}$. Then there exist $u \in C^n$ such that $x \preceq u$. This implies $m * x \preceq m * u \in C^n$, which shows that $m * x \in M^n$. □

To illustrate the rich applicability of the above concepts, we provide the most basic examples for cornets in the subsequent three propositions. For these definitions, we introduce the notion of wedge in abelian group setting.

Definition 1.2.9. [MP21]. If $(G, +)$ is an abelian semigroup and $n \in \mathbb{N}$, then for a subset $S \subseteq G$, define

$$n^{-1}(S) := \{x \in G \mid n \cdot x \in S\}.$$

A subsemigroup S of the group $(G, +)$ is said to be n-*divisible* if, for all $x \in S$, the set $n^{-1}(\{x\}) \cap S$ is nonempty. If this set is a singleton, then S is called *uniquely* n-*divisible* and its unique element will be denoted by x/n.

In a unital abelian semigroup G, a subset $W \subseteq G$ is called a *wedge* if the following properties are satisfied:

(i) W is a unital subsemigroup of G.

(ii) If $u, v \in W$ such that $u + v = 0$, then $u = v = 0$.

(iii) For all $n \in \mathbb{N}$, the inverse image $n^{-1}(W)$ is contained in W.

In terms of a wedge $W \subseteq G$, we can define a partial order $\preceq_W$ in the following way: For $x, y \in G$, we say that $x \preceq_W y$ if $y \in x + W$. It immediately follows that $\preceq_W$ is a reflexive, and transitive relation on G. If, in addition, G is cancellative (which is always the case if G is group), then $\preceq_W$ is antisymmetric and hence it is a partial order on G.

Proposition 1.2.10. [MP21] *Let $(G, +)$ be an abelian group and let $W \subseteq G$ be a wedge. Then, for a subsemigroup S of G containing W, the quadruple $(S, +, \cdot, \preceq_W)$ is a cornet in which every element is n-convex for all $n \in \mathbb{N}$. In particular, by taking $W := \{0\}$, it follows that $(G, +, \cdot, =)$ is a cornet.*

Proof. The properties (i), (ii), (iii) of Definition 1.2.2 can easily be verified by induction, moreover, (iii) holds with equality. Thus, it suffices to show that property (iv) is also valid.

Let $n \in \mathbb{N}$ and $x, y \in S$ be arbitrary. Assume first that $x \preceq_W y$ holds. Then $y \in x + W$. The set W is a subsemigroup, therefore, $y \in x + W$ implies that $n \cdot y \in n \cdot x + n \cdot W \subseteq n \cdot x + W$, which yields that $n \cdot x \preceq_W n \cdot y$. On the other hand, if $n \cdot x \preceq_W n \cdot y$ holds, then $n \cdot (y - x) \in W$, consequently $y - x \in n^{-1}(W)$. By condition (iii) of Definition 1.2.9, it follows that $y - x \in W$ must be valid and hence $x \preceq_W y$.

The operation $\cdot$ being the cornet-multiplication implies that every element of S is n-convex for all $n \in \mathbb{N}$. □

Proposition 1.2.11. [MP21] *Let $(G,+)$ be an abelian group, W be a wedge and let S be a subsemigroup of G containing W. Let $P_W(S)$ denote the collection of all nonempty W-invariant subsets A of S, which means that $A+W \subseteq A$ holds. Define the operations $+$ and $*$ by:*

$$\begin{aligned} A+B &:= \{a+b \mid a \in A,\, b \in B\} && (A,B \in P_W(S)), \\ n * A &:= \{n \cdot a + w \mid a \in A,\, w \in W\} && (A \in P_W(S),\, n \in \mathbb{N}). \end{aligned} \tag{1.4}$$

Then $(P_W(S),+,*,\subseteq)$ is a complete cornet with the unit element W. Furthermore, the mapping

$$\varphi(x) := x + W \qquad (x \in S) \tag{1.5}$$

is an injective order reversing homomorphic mapping of $(S,+,\cdot,\preceq_W)$ into the cornet $(P_W(S),+,*,\subseteq)$. In addition, if $n \in \mathbb{N}$ and W is n-divisible, then $A \in P_W(S)$ is n-convex if and only if, for all $x_1,\dots,x_n \in A$, we have

$$n^{-1}(\{x_1+\dots+x_n\}) \cap A \neq \emptyset. \tag{1.6}$$

Proof. If $A,B \in P_W(S)$, then $A+B \subseteq S+S \subseteq S$ and $(A+B)+W = A+(B+W) \subseteq A+B$, which show that $A+B \in P_W(S)$. Therefore, $P_W(S)$ is an abelian semigroup with the addition defined in (1.4). Clearly, for $A \in P_W(S)$, the property $0 \in W$ implies $A \subseteq A+W \subseteq A$, which proves that W is the unit element of the semigroup $(P_W(S),+)$.

The inclusion of sets is trivially a partial order on $P_W(S)$ and the implication $A \subseteq B \Rightarrow A+C \subseteq B+C$ is also obvious for $A,B,C \in P_W(S)$. Therefore, $(P_W(S),+,\subseteq)$ is an ordered abelian semigroup.

First observe that the definition of the multiplication operation $*$ is correct, i.e., $n * A \in P_W(S)$ for all $n \in \mathbb{N}$ and $A \in P_W(S)$.

To see that property (i) holds, let $n,m \in \mathbb{N}$ and $A \in P_W(S)$. First, let $u \in (nm) * A$. Then there exist elements $a \in A$ and $w \in W$ such that $u = (nm)\cdot a + w$. We have that $m \cdot a \in m * A$. Therefore,

$$u = n \cdot (m \cdot a) + w \subseteq n * (m * A).$$

This proves that $(nm) * A \subseteq n * (m * A)$. To verify the reversed inclusion, let $u \in n * (m * A)$. Then there exist $b \in m * A$ and $w \in W$ such that

$u = n \cdot b + w$. Similarly, there exist $a \in A$ and $z \in W$ such that $b = m \cdot a + z$. Combining these equalities, we get that

$$u = n \cdot (m \cdot a + z) + w = (nm) \cdot a + (n \cdot z + w) \in (nm) * A,$$

which completes the proof of the reversed inclusion $n*(m*A) \subseteq (nm)*A$ and property (i) of Definition 1.2.2.

The verification of property (ii) of Definition 1.2.2 is similar, and therefore it is left to the reader.

To show that (iii) of Definition 1.2.2 holds, let $n, m \in \mathbb{N}$ and $A \in P_W(S)$ and $u \in (n+m) * A$. Then there exist $a \in A$ and $w \in W$ such that $u = (n+m) \cdot a + w = n \cdot a + (m \cdot a + w) \in n * A + m * A$. Therefore, $(n+m) * A \subseteq n * A + m * A$.

For the proof of property (iv) of Definition 1.2.2, let $n \in \mathbb{N}$, $A, B \in P_W(S)$. If $A \subseteq B$ holds, then the inclusion $n * A \subseteq n * B$ is obvious. Conversely, assume that $n * A \subseteq n * B$ holds. Then, for an arbitrary $a \in A$, we get that $n \cdot a \in n * A \subseteq n * B$, therefore, there exist $b \in B$ and $w \in W$ such that $n \cdot a = n \cdot b + w$. This yields that $n \cdot (a - b) \in W$, i.e., $a - b \in n^{-1}(W)$. Now the condition (iii) of Definition 1.2.9 gives that $a - b \in W$, which proves that $a = b + w \in B$.

The properties (v) and (vi) of Definition 1.2.2 can easily be seen.

We verify now the completeness of $(P_W(S), +, *, \subseteq)$. Let $\mathcal{A} := \{A_\gamma \mid \gamma \in \Gamma\}$ be a nonempty and lower bounded family of elements of $P_W(S)$. Define $A \subseteq S$ by $A := \bigcap_{\gamma\in\Gamma} A_\gamma$. Then A is nonempty by the existence of a lower bound for the family $\mathcal{A}$. We show that A also belongs to $P_W(S)$. Indeed, if $u \in A + W$, then there exists $a \in A$ and $w \in W$ such that $u = a + w$. We have that $a \in A_\gamma$ for all $\gamma \in \Gamma$, therefore, $u = a + w \in A_\gamma + W \subseteq A_\gamma$ for all $\gamma \in \Gamma$. This proves $u + w \in A$ showing that $A + W \subseteq A$ holds. Thus, $A \in P_W(S)$ is valid. Clearly, A is the greatest lower bound for $\mathcal{A}$ in $P_W(S)$, and hence $(P_W(S), \subseteq)$ is a complete partially ordered set.

Now consider the mapping φ defined by (1.5). For $x \in S$, we have that $\varphi(x) = x + W \subseteq S$ and $\varphi(x) + W = x + W + W \subseteq x + W$, which show that $\varphi(x)$ is in $P_W(S)$. If, for some $x, y \in S$, the equality $\varphi(x) = \varphi(y)$ holds, then $x + W = y + W$, which yields that $x \in y + W$

and $y \in x + W$. Therefore, $x - y \in W \cap (-W) = \{0\}$, showing that $x = y$, which proves the injectivity of φ.

The structure preserving properties are easily seen from the following identities.

$$\varphi(x + y) = x + y + W = (x + W) + (y + W) = \varphi(x) + \varphi(y),$$
$$\varphi(n \cdot x) = n \cdot x + W = n \cdot (x + W) + W = n \cdot \varphi(x) + W = n * \varphi(x).$$

To see that φ is also order reversing, observe that the following inequalities of inclusions are pairwise equivalent:

$$x \preceq_W y \Leftrightarrow y \in x + W \Leftrightarrow y + W \subseteq x + W \Leftrightarrow \varphi(y) \subseteq \varphi(x).$$

Therefore, φ is an order reversing and homomorphic embedding of the quadruple $(S, +, \cdot, \preceq_W)$ into $(P_W(S), +, *, \subseteq)$.

Finally, let $n \in \mathbb{N}$, $A \in P_W(S)$ and assume that W is n-divisible. First suppose that A is n-convex. Let $x_1, \dots, x_n \in A$ be arbitrary. Then

$$x_1 + \cdots + x_n \in n \cdot A \subseteq n * A = \{n \cdot a + w \mid a \in A,\, w \in W\}.$$

Therefore, there exist $a \in A$ and $w \in W$ such that $x_1 + \cdots + x_n = n \cdot a + w$. By the n-divisibility of W, $n^{-1}(\{w\}) \cap W$ is nonempty, therefore, $w = n \cdot v$ for some $v \in W$. On the other hand, $a + v \in A + W \subseteq A$, thus $a + v \in n^{-1}(\{x_1 + \cdots + x_n\}) \cap A$.

To prove the converse, assume that (1.6) is valid for all $x_1, \dots, x_n \in A$. To prove that A is n-convex, it is sufficient to show that $n \cdot A \subseteq n * A$. Let $x \in n \cdot A$. This means that there exist $x_1, \dots, x_n \in A$ such that $x = x_1 + \cdots + x_n$. By (1.6), for some $a \in A$, we have that $x_1 + \cdots + x_n = n \cdot a \in n * A$. Thus $x \in n * A$, which was to be proved. □

Proposition 1.2.12. [MP21] *Let $(G, +)$ be an abelian group, W be a wedge and let S be a uniquely divisible subsemigroup of G which contains W. Let, for $p \in]0, 1]$,*

$$F_W^p(S) := \{f \colon S \to [0, 1] \mid \sup f \geq p \text{ and } f \text{ is } W\text{-nondecreasing}\} \tag{1.7}$$

and define the addition and the scalar multiplication in $F_W^p(S)$ *by*

$$(f \oplus g)(x) := \sup_{\substack{u,v \in S \\ u+v=x}} \min\big(f(u), g(v)\big), \qquad (n \odot f)(x) := f\left(\frac{x}{n}\right) \tag{1.8}$$
$$(f, g \in F_W^p(S),\ x \in S,\ n \in \mathbb{N}).$$

Finally, let $\leq$ *denote the pointwise ordering in* $F_W^p(S)$*. Then the quadruple* $(F_W^p(S), \oplus, \odot, \leq)$ *is a complete cornet whose unit element is the characteristic function of the wedge* W*. Furthermore, the mapping*

$$\Phi(A) := \chi_A \qquad (A \in P_W(S)) \tag{1.9}$$

is an injective cornet-preserving mapping of $(P_W(S), +, *, \subseteq)$ *into the quadruple* $(F_W^1(S), \oplus, \odot, \leq)$*. In addition, a function* $f \in F_W^p(S)$ *is* n*-convex if and only if it is* n*-quasiconcave, i.e., for all* $x_1, \dots, x_n \in S$*,*

$$\min(f(x_1), \dots, f(x_n)) \leq f\Big(\frac{x_1 + \cdots + x_n}{n}\Big). \tag{1.10}$$

Proof. Let $p \in]0, 1]$. First we show that $F_W^p(S)$ is closed under the operation $\oplus$. To see this, let $f, g : S \to [0, 1]$ be W-nondecreasing functions with $\sup f, \sup g \geq p$. If $x \in S$ and $w \in W$, then

$$\begin{aligned}(f \oplus g)(x + w) &= \sup_{\substack{u,v \in S \\ u+v=x+w}} \min\big(f(u), g(v)\big) \\ &\geq \sup_{\substack{u,v' \in S \\ u+v'=x}} \min\big(f(u), g(v' + w)\big) \\ &\geq \sup_{\substack{u,v' \in S \\ u+v'=x}} \min\big(f(u), g(v')\big) = (f \oplus g)(x),\end{aligned}$$

which shows that $f \oplus g$ is also W-nondecreasing. Let $\eta < p$ be arbitrary. Then there exist $u_0, v_0 \in S$ such that $f(u_0) > \eta$ and $g(v_0) > \eta$ hold. Then we have $(f \oplus g)(u_0 + v_0) \geq \min\big(f(u_0), g(v_0)\big) > \eta$, proving that $\sup(f \oplus g) > \eta$. Taking the limit $\eta \to p$, this implies that $f \oplus g \in F_W^p(S)$.

The commutativity of the operation $\oplus$ is a consequence of the commutativity of the group operation $+$ in G. To verify the associativity, let $f, g, h \in F_W^p(S)$. Then, for all $x \in S$,

$$\begin{aligned}((f \oplus g) \oplus h)(x) &= \sup_{\substack{u,v\in S\\ u+v=x}} \min\big((f\oplus g)(u), h(v)\big)\\ &= \sup_{\substack{u,v\in S\\ u+v=x}} \min\left(\sup_{\substack{s,t\in S\\ s+t=u}} \min\big(f(s), g(t)\big), h(v)\big)\right)\\ &= \sup_{\substack{u,v\in S\\ u+v=x}} \sup_{\substack{s,t\in S\\ s+t=u}} \min\big(\min\big(f(s), g(t)\big), h(v)\big)\big)\\ &= \sup_{\substack{s,t,v\in S\\ s+t+v=x}} \min\big(f(s), g(t), h(v)\big).\end{aligned}$$

A similar argument shows that

$$(f \oplus (g \oplus h))(x) = \sup_{\substack{u,s,t\in S\\ u+s+t=x}} \min\big(f(u), g(s), h(t)\big),$$

which results the desired equality $((f \oplus g) \oplus h)(x) = (f \oplus (g \oplus h))(x)$.

To see that the characteristic function χ_W of W is a unital element of the semigroup $F_W^p(S)$, observe that χ_W is a W-nondecreasing function and, for all $x \in S$,

$$\begin{aligned}(f \oplus \chi_W)(x) &= \sup_{\substack{u,v\in S\\ u+v=x}} \min\big(f(u), \chi_W(v)\big)\\ &= \sup_{\substack{u\in S,\, v\in W\\ u+v=x}} f(u) \le \sup_{\substack{u\in S,\, v\in W\\ u+v=x}} f(u+v) = f(x).\end{aligned}$$

On the other hand, by taking $v = 0$, we can see that the inequality

$$f(x) \le \sup_{\substack{u\in S,\, v\in W\\ u+v=x}} f(u)$$

holds, which finally implies the equality $(f \oplus \chi_W)(x) = f(x)$.

It is obvious that $(F_W^p(S), \leq)$ is a partially ordered set. We prove that the operation $\oplus$ is monotone with respect to the ordering $\leq$. Indeed, if $f, g, h \in F_W^p(S)$ and $g \leq h$ on S, then, for all $x \in S$,

$$\begin{aligned}(f \oplus g)(x) &= \sup_{\substack{u,v \in S \\ u+v=x}} \min\big(f(u), g(v)\big) \\ &\leq \sup_{\substack{u,v \in S \\ u+v=x}} \min\big(f(u), h(v)\big) = (f \oplus h)(x).\end{aligned}$$

So far we have shown that $(F_W^p(S), \oplus, \leq)$ is an ordered commutative semigroup. In the rest of the proof, we prove that, this structure with the operation $\odot$ forms a cornet.

First we show that $n \odot f \in F_W^p(S)$ whenever $n \in \mathbb{N}$ and $f \in F_W^p(S)$. Indeed,

$$\sup_{x \in S}(n \odot f)(x) = \sup_{x \in S} f\Big(\frac{x}{n}\Big) \geq \sup_{y \in S} f\Big(\frac{n \cdot y}{n}\Big) = \sup_{y \in S} f(y) \geq p,$$

which proves that $\sup(n \odot f) \geq p$. On the other hand, if $x \leq_W y$, then $\frac{x}{n} \leq_W \frac{y}{n}$. By the W-nondecreasingness of f, this implies $f(\frac{x}{n}) \leq f(\frac{y}{n})$, that is, $(n \odot f)(x) \leq n \odot f)(y)$. Therefore, $(n \odot f)$ is also W-nondecreasing.

For property (i) of Definition 1.2.2, let $n, m \in \mathbb{N}$ and $f \in F_W^p(S)$. Then, for all $x \in S$,

$$\begin{aligned}((nm) \odot f)(x) &= f\Big(\frac{x}{nm}\Big) = f\Big(\frac{x/n}{m}\Big) \\ &= (m \odot f)\Big(\frac{x}{n}\Big) = (n \odot (m \odot f))(x),\end{aligned}$$

which shows the expected identity $(nm) \odot f = n \odot (m \odot f)$.

For property (ii) of Definition 1.2.2, let $n \in \mathbb{N}$ and $f, g \in F_W^p(S)$.

Then, for all $x \in S$,

$$\begin{aligned}(n \odot (f \oplus g))(x) &= (f \oplus g)\Big(\frac{x}{n}\Big) \\ &= \sup_{\substack{u,v \in S \\ u+v=\frac{x}{n}}} \min\big(f(u), g(v)\big) \\ &= \sup_{\substack{u',v' \in S \\ u'+v'=x}} \min\Big(f\Big(\frac{u'}{n}\Big), g\Big(\frac{v'}{n}\Big)\Big) \\ &= \sup_{\substack{u',v' \in S \\ u'+v'=x}} \min\big((n \odot f)(u'), (n \odot g)(v')\big) \\ &= ((n \odot f) \oplus (n \odot g))(x),\end{aligned}$$

which proves the equality $n \odot (f \oplus g) = (n \odot f) \oplus (n \odot g)$.

For property (iii) of Definition 1.2.2, let $n, m \in \mathbb{N}$ and $f \in F^p_W(S)$. Then, for all $x \in S$,

$$\begin{aligned}((n+m) \odot f)(x) &= f\Big(\frac{x}{n+m}\Big) \\ &= \min\Big(f\Big(\frac{1}{n} \cdot \frac{nx}{n+m}\Big), f\Big(\frac{1}{m} \cdot \frac{mx}{n+m}\Big)\Big) \\ &\leq \sup_{\substack{u,v \in S \\ u+v=x}} \min\Big(f\Big(\frac{u}{n}\Big), f\Big(\frac{v}{m}\Big)\Big) \\ &= ((n \odot f) \oplus (m \odot f))(x),\end{aligned}$$

which shows the desired inequality $(n+m) \odot f \leq (n \odot f) \oplus (m \odot f)$.

For property (iv) of Definition 1.2.2, let $n \in \mathbb{N}$ and $f, g \in F^p_W(S)$ with $f \leq g$. Then, for all $x \in S$,

$$(n \odot f)(x) = f\Big(\frac{x}{n}\Big) \leq g\Big(\frac{x}{n}\Big) = (n \odot g)(x),$$

which yields that $n \odot f \leq n \odot g$.

The property (v), which is the equality $1 \odot f = f$, is obvious. The equality $n \odot \chi_W = \chi_W$ easily follows from the equivalence of the inclusions $\frac{x}{n} \in W$ and $x \in W$. Thus property (vi) of Definition 1.2.2 is also satisfied.

We now show that $(F^p_W(S), \leq)$ is a complete partially ordered set. Let $\mathcal{F} := \{f_\gamma \mid \gamma \in \Gamma\}$ be a family of elements in $F^p_W(S)$ bounded from below by $g \in F^p_W(S)$. Define $f\colon S \to [0,1]$ by $f\colon = \inf_{\gamma\in\Gamma} f_\gamma$. We prove that f is also a member of $F^p_W(S)$. By the inequality $g \leq f_\gamma$, it follows that $g \leq f$ and hence $p \leq \sup g \leq f$. Let $x, y \in S$ with $x \preceq_W y$, that is, with $y - x \in W$. Then, for all $\gamma \in \Gamma$ the W-nondecreasingness of f_γ gives $f_\gamma(x) \leq f_\gamma(y)$. Taking the infimum with respect to $\gamma \in \Gamma$ side by side, it follows that $f(x) \leq f(y)$, which proves that f is also W-nondecreasing and hence $f \in F^p_W(S)$. Clearly, f is the infimum of the family $\mathcal{F}$ and this shows that $(F^p_W(S), \leq)$ is a complete partially ordered set.

We verify that the map Φ defined by (1.9) is an injective cornet-preserving mapping of $(P_W(S), +, *, \subseteq)$ into $(F^1_W(S), \oplus, \odot, \leq)$. Clearly, if $A \in P_W(S)$, then $\Phi(a) = \chi_A$ is W-nondecreasing and $\sup \Phi(A) = \sup \chi_A = 1$, which shows that $\Phi(A) \in F^1_W(S)$. The injectivity of Φ is obvious. To prove that Φ preserves the addition, let $A, B \in P_W(S)$. Then, for $x \in S$, it is easy to see that

$$\sup_{\substack{u,v\in S\\ u+v=x}} \min(\chi_A(u), \chi_B(v)) = 1$$

if and only if there exist $u \in A$, $v \in B$ such that $x = u + v$, that is, if $x \in A + B$. This proves that, for all $x \in S$,

$$(\chi_A \oplus \chi_B)(x) = \sup_{\substack{u,v\in S\\ u+v=x}} \min(\chi_A(u), \chi_B(v)) = \chi_{A+B}(x).$$

As a consequence of this equality, it follows that $\Phi(A + B) = \Phi(A) \oplus \Phi(B)$.

Let $A, B \in P_W(S)$ and $n \in \mathbb{N}$. It is clear that

$$\{n \cdot a \mid a \in A\} \subseteq n * A = \{n \cdot a + w \mid a \in A,\ w \in W\}. \tag{1.11}$$

In fact, this inclusion is an equality. To see this, let $x \in S$ be of the form $x = n \cdot a + w$ for some $a \in A$ and $w \in W$. Then, by the divisibility of W, we have that $w/n \in W$. Thus, the W-invariance of A yields that $a' = a + (w/n) \in A$ and hence x is of the form $n \cdot a'$ for some element $a' \in A$, which shows that it belongs to the left hand side set in (1.11).

Using the equality (1.11), for $x \in S$, we have

$$\chi_{n*A}(x) = \chi_A\Big(\frac{x}{n}\Big) = n \odot \chi_A(x),$$

which proves the equality $\Phi(n * A) = n \odot \Phi(A)$.

If $A, B \in P_W(S)$ with $A \subseteq B$, then $\chi_A(x) \leq \chi_B(x)$ holds for all $x \in S$, which shows that $\Phi(A) \leq \Phi(B)$, that is, Φ preserves the ordering as well.

To prove the last assertion of the proposition, assume that $f \in F_W(S)$ is an n-convex element. Let $x_1, \dots, x_n \in S$. By the n-convexity of f, we have that $n \cdot f \leq n \odot f$, that is, for all $x \in S$,

$$\begin{aligned}\sup_{\substack{u_1,\dots,u_n \in S\\ u_1+\dots+u_n=x}} \min\big(f(u_1), \dots, f(u_n)\big) &= (f \oplus \dots \oplus f)(x)\\ &\leq (n \odot f)(x) = f\Big(\frac{x}{n}\Big).\end{aligned}$$

By taking $x := x_1 + \dots + x_n$, with $u_1 := x_1, \dots, u_n := x_n$, it follows that (1.10) holds. The proof of the reversed implication is analogous. □

1.3 Topological notions and boundedness in cornets

In a natural way, we can introduce the notions of nonnegative and Archimedean elements in a cornet with the following definition.

Definition 1.3.1. [MP21]. In a cornet $(X, +, *, \preceq)$ an element $x \in X$ is said to be *nonnegative* if $0 \preceq x$ holds. The element x is called *Archimedean*, denoted by $0 \prec x$, if, for all $u \in X$, there exists $n_0 \in \mathbb{N}$ such that $0 \preceq u + n * x$ for all $n_0 \leq n$. The set of all nonnegative and Archimedean elements in X will be denoted by $X_{\preceq}$ and $X_{\prec}$, respectively.

The properties of nonnegative and Archimedean elements are established in the following assertion.

Proposition 1.3.2. [MP21] *Let $(X, +, *, \preceq)$ be a cornet. Then $X_{\prec}$ is contained in $X_{\preceq}$ and*

$$X_{\prec} + X_{\preceq} \subseteq X_{\prec}. \tag{1.12}$$

In addition, $X_{\prec}$ and $X_{\preceq}$ are subcornets of $(X, +, *, \preceq)$.

Proof. To show that $X_\prec \subseteq X_\preceq$, let $x \in X_\prec$. Then, there exists $n_0 \in \mathbb{N}$ such that $0 \preceq n_0 * x$. Therefore, $n_0 * 0 \preceq n_0 * x$, which implies that $0 \preceq x$, i.e., $x \in X_\preceq$.

Let $x \in X_\prec$ and $y \in X_\preceq$. Then, for any $u \in X$, there exists $n_0 \in \mathbb{N}$ such that $0 \preceq u + n * x$ for all $n_0 \leq n$. On the other hand, $0 = n * 0 \preceq n * y$, therefore $0 \preceq u + n * (x + y)$ holds for all $n_0 \leq n$, which implies that $x + y$ is Archimedean and proves the inclusion (1.12).

If x and y are nonnegative elements, then by the axioms of cornets, $0 \preceq y = 0 + y \preceq x + y$, which shows that $X_\preceq$ is closed under addition. Similarly, if $x \in X_\preceq$ and $m \in \mathbb{N}$, then $0 = m * 0 \preceq m * x$ proving that $X_\preceq$ is closed under $*$-multiplication and hence $(X_\preceq, +, *, \preceq)$ is a subcornet of $(X, +, *, \preceq)$.

By (1.12) we have that $X_\prec + X_\prec \subseteq X_\prec + X_\preceq \subseteq X_\prec$, which shows that $X_\prec$ is closed under addition. To prove that $X_\prec$ is also closed under $*$-multiplication, let $x \in X_\prec$ and $m \in \mathbb{N}$ and let $u \in X$ be arbitrary. Then there exists $n_0 \in \mathbb{N}$ such that $0 \preceq u + n * x$ for all $n \geq n_0$. In particular, we have that $0 \preceq u + (km) * x = u + k * (m * x)$ for all $k \geq \frac{n_0}{m}$, which shows that $m * x$ is also Archimedean. Hence, $(X_\prec, +, *, \preceq)$ is also a subcornet of $(X, +, *, \preceq)$ □

In what follows, we introduce the notions of continuity of the addition, boundedness and closedness with respect to a subsemigroup of Archimedean elements. For comparison, we recall first the standard topological concepts for abelian groups.

Definition 1.3.3. [MP21]. If $(G, +)$ is an abelian group and $\mathcal{T}$ is a Hausdorff topology on G, then we say that $(G, \mathcal{T}, +)$ is a *topological group* if the $(x, y) \mapsto x - y$ is a continuous map of $G \times G$ into G. A subset $U \subseteq G$ is said to be *convex* if, for all $n \in \mathbb{N}$,

$$\{u_1 + \cdots + u_n \mid u_1, \ldots, u_n \in U\} = \{n \cdot u \mid u \in U\}.$$

We say that G is *locally convex* if every neighborhood of 0 contains a convex neighborhood of 0. A subset $H \subseteq G$ is said to be *topologically bounded* if, for all neighborhood U of 0, there exists $n \in \mathbb{N}$ such that $H \subseteq \{u_1 + \cdots + u_n \mid u_1, \ldots, u_n \in U\}$.

Definition 1.3.4. [MP21]. Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$. We say that the *addition is $\mathcal{A}$-continuous* if, for all $a \in \mathcal{A}$, there exists $b \in \mathcal{A}$ such that $b + b \preceq a$ holds.

Lemma 1.3.5. [MP21] *Let $(X, +, *, \preceq)$ be a cornet, let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$ and assume that the addition is $\mathcal{A}$-continuous. Then the "$\cdot$"-multiplication and the $*$-multiplication are $\mathcal{A}$-continuous, that is, for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$, there exists $b \in \mathcal{A}$ such that $n \cdot b \preceq a$ and $n * b \preceq a$, respectively.*

Proof. Let $a \in \mathcal{A}$. By the continuity of addition there exists $a_1 \in \mathcal{A}$ such that $a_1 + a_1 \preceq a$, or in other words $2 \cdot a_1 \preceq a$. Applying the continuity of addition again for $a_1 \in \mathcal{A}$, we get that there exists $a_2 \in \mathcal{A}$ such that $2 \cdot a_2 \preceq a_1$, which implies that $4 \cdot a_2 \preceq a$. Continuing this process, we can construct a sequence $a_k \in \mathcal{A}$ such that $2^k \cdot a_k \preceq a$ holds for all $k \in \mathbb{N}$. Now let $n \in \mathbb{N}$ and choose $k \in \mathbb{N}$ so that $n \leq 2^k$. Then $n \cdot a_k = n \cdot a_k + (2^k - n) \cdot 0 \preceq 2^k \cdot a_k \preceq a$. Thus, $n \cdot b \preceq a$ holds with $b := a_k$. The inequality $n * b \preceq n \cdot b$ implies that $n * b \preceq a$ is also valid. □

Definition 1.3.6. [MP21]. Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$. We say that an element $x \in X$ is *$\mathcal{A}$-bounded* if, for all $a \in \mathcal{A}$, there exists $n_0 \in \mathbb{N}$ such that $x \preceq n * a$ for all $n \geq n_0$.

Proposition 1.3.7. [MP21] *Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$ such that the addition is $\mathcal{A}$-continuous. Then the $\mathcal{A}$-bounded elements form a subcornet of $(X, +, *, \preceq)$.*

Proof. Let x, y be $\mathcal{A}$-bounded elements of X and let $a \in \mathcal{A}$ be arbitrary. Since X is $\mathcal{A}$-topological therefore there exists $b \in \mathcal{A}$ such that $b + b \preceq a$. Using the $\mathcal{A}$-boundedness property of x, y, there exist $k_0, m_0 \in \mathbb{N}$ such that

$$\begin{aligned} x &\preceq k * b \qquad \text{if } k \geq k_0, \\ y &\preceq m * b \qquad \text{if } m \geq m_0. \end{aligned}$$

Thus,

$$x \preceq n * b \qquad \text{if } n \geq \max(k_0, m_0),$$
$$y \preceq n * b \qquad \text{if } n \geq \max(k_0, m_0).$$

Adding the above inequalities side by side, we get

$$x + y \preceq n * (b + b) \preceq n * a \qquad \text{if } n \geq \max(k_0, m_0),$$

which proves that the set of $\mathcal{A}$-bounded elements is closed under addition.

To prove that the set of $\mathcal{A}$-bounded elements is closed under the $*$-multiplication, let $x \in X$ be $\mathcal{A}$-bounded, $m \in \mathbb{N}$ and $a \in \mathcal{A}$. Then, there exist $b \in \mathcal{A}$ such that $m * b \preceq a$. Then there exists $n_0 \in \mathbb{N}$ such that $x \preceq n * b$ holds for all $n \geq n_0$. Using the monotonicity property of $*$-multiplication, for all $n \geq n_0$, it follows that $m * x \preceq m * (n * b) = (mn) * n = n * (m * b) \preceq n * a$. This shows that the set of $\mathcal{A}$-bounded elements is closed under the $*$-multiplication and proves that $\mathcal{A}$-bounded elements form a subcornet of $(X, +, *, \preceq)$. □

Definition 1.3.8. [MP21]. Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$. Given an element $x \in X$, we say that $y \in X$ is the $\mathcal{A}$-*closure* of x if $y \preceq x + a$ holds for all $a \in A$ and, y is the largest element of X with is property, i.e., if $z \preceq x + a$ holds for all $a \in A$, then $z \preceq y$. It is clear that the $\mathcal{A}$-closure of an element, if exists, is unique and is denoted by $\mathrm{cl}_{\mathcal{A}}(x)$. An element x is called $\mathcal{A}$-*closed* if $x = \mathrm{cl}_{\mathcal{A}}(x)$. The set of all elements of X which possess an $\mathcal{A}$-closure will be denoted by $\mathrm{Cl}_{\mathcal{A}}$.

Proposition 1.3.9. [MP21] *Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$ such that X is $\mathcal{A}$-topological. Then we have the following assertions.*

(i) If $x \in \mathrm{Cl}_{\mathcal{A}}$, then $x \preceq \mathrm{cl}_{\mathcal{A}}(x)$.

(ii) If $x, y \in \mathrm{Cl}_{\mathcal{A}}$ and $x \preceq y$, then $\mathrm{cl}_{\mathcal{A}}(x) \preceq \mathrm{cl}_{\mathcal{A}}(y)$.

(iii) If $x \in \mathrm{Cl}_{\mathcal{A}}$, then $\mathrm{cl}_{\mathcal{A}}(x) \in \mathrm{Cl}_{\mathcal{A}}$ and $\mathrm{cl}_{\mathcal{A}}(x) = \mathrm{cl}_{\mathcal{A}}(\mathrm{cl}_{\mathcal{A}}(x))$.

(iv) *If* $x \in \mathrm{Cl}_{\mathcal{A}}$ *and* $x \preceq y \preceq \mathrm{cl}_{\mathcal{A}}(x)$*, then* $y \in \mathrm{Cl}_{\mathcal{A}}$ *and* $\mathrm{cl}_{\mathcal{A}}(y) = \mathrm{cl}_{\mathcal{A}}(x)$*.*

(v) *If* $x, y \in \mathrm{Cl}_{\mathcal{A}}$ *with* $x + y \in \mathrm{Cl}_{\mathcal{A}}$*, then* $\mathrm{cl}_{\mathcal{A}}(x) + \mathrm{cl}_{\mathcal{A}}(y) \in \mathrm{Cl}_{\mathcal{A}}$ *and* $\mathrm{cl}_{\mathcal{A}}(\mathrm{cl}_{\mathcal{A}}(x) + \mathrm{cl}_{\mathcal{A}}(y)) = \mathrm{cl}_{\mathcal{A}}(x + y)$*.*

(vi) *If* $x \in \mathrm{Cl}_{\mathcal{A}}$*,* $n \in \mathbb{N}$ *and* $n \cdot x \in \mathrm{Cl}_{\mathcal{A}}$*, then* $n \cdot \mathrm{cl}_{\mathcal{A}}(x) \in \mathrm{Cl}_{\mathcal{A}}$ *and* $\mathrm{cl}_{\mathcal{A}}(n \cdot \mathrm{cl}_{\mathcal{A}}(x)) = \mathrm{cl}_{\mathcal{A}}(n \cdot x)$*.*

(vii) *If* $x \in \mathrm{Cl}_{\mathcal{A}}$*,* $n \in \mathbb{N}$ *and* $n * x \in \mathrm{Cl}_{\mathcal{A}}$*, then* $n * \mathrm{cl}_{\mathcal{A}}(x) \in \mathrm{Cl}_{\mathcal{A}}$ *and* $\mathrm{cl}_{\mathcal{A}}(n * \mathrm{cl}_{\mathcal{A}}(x)) = \mathrm{cl}_{\mathcal{A}}(n * x)$*.*

(viii) *If* $x \in \mathrm{Cl}_{\mathcal{A}}$ *is* $\mathcal{A}$*-bounded, then* $\mathrm{cl}_{\mathcal{A}}(x)$ *is also* $\mathcal{A}$*-bounded.*

(ix) *If* $n \in \mathbb{N}$*,* $x \in \mathrm{Cl}_{\mathcal{A}}$ *is* n*-convex,* $n * x \in \mathrm{Cl}_{\mathcal{A}}$ *and* $n * \mathrm{cl}_{\mathcal{A}}(x)$ *is* $\mathcal{A}$*-closed, then* $\mathrm{cl}_{\mathcal{A}}(x)$ *is also* n*-convex.*

Proof.

(i) Let $x \in \mathrm{Cl}_{\mathcal{A}}$. Clearly $x \preceq x + a$ holds for all $a \in \mathcal{A}$. This implies that $x \preceq \mathrm{cl}_{\mathcal{A}}(x)$.

(ii) Let $x, y \in \mathrm{Cl}_{\mathcal{A}}$ such that $x \preceq y$. By using the definition of $\mathcal{A}$-closure, for all $a \in \mathcal{A}$, we have that

$$\mathrm{cl}_{\mathcal{A}}(x) \preceq x + a \preceq y + a,$$

which implies that $\mathrm{cl}_{\mathcal{A}}(x) \preceq \mathrm{cl}_{\mathcal{A}}(y)$.

(iii) Let $x \in \mathrm{Cl}_{\mathcal{A}}$. We need to show that $y = \mathrm{cl}_{\mathcal{A}}(x)$ is $\mathcal{A}$-closed. The inequality $y \preceq y + a$ is trivial for all $a \in \mathcal{A}$. Assume that $z \preceq y + a$ holds for all $a \in \mathcal{A}$. Let $b \in \mathcal{A}$ be arbitrary. By the $\mathcal{A}$-continuity of the addition, there exist $a, c \in \mathcal{A}$ such that $a + c \preceq b$. We have that $y \preceq x + c$, hence,

$$z \preceq y + a \preceq (x + c) + a \preceq x + b.$$

Since b was arbitrary, this implies that $z \preceq \mathrm{cl}_{\mathcal{A}}(x) = y$, which shows that y is the $\mathcal{A}$-closure of itself.

(iv) Let $x \in \mathrm{Cl}_{\mathcal{A}}$ and $x \preceq y \preceq \mathrm{cl}_{\mathcal{A}}(x)$. We need to show that $\mathrm{cl}_{\mathcal{A}}(x)$ is the $\mathcal{A}$-closure of y.

On one hand, for all $a \in \mathcal{A}$, we have $\mathrm{cl}_{\mathcal{A}}(x) \preceq x + a \preceq y + a$. On the other hand, assume that $z \in X$ satisfies $z \preceq y + a$ for all $a \in \mathcal{A}$.

Then $z \preceq \mathrm{cl}_{\mathcal{A}}(x) + a$ for all $a \in \mathcal{A}$, which, by property (iii) implies that $z \preceq \mathrm{cl}_{\mathcal{A}}(x)$. This shows that $\mathrm{cl}_{\mathcal{A}}(x)$ is the $\mathcal{A}$-closure of y.

(v) Let $x, y \in X$ be arbitrary. First we show that $\mathrm{cl}_{\mathcal{A}}(x) + \mathrm{cl}_{\mathcal{A}}(y) \preceq x + y + a$ holds for all $a \in \mathcal{A}$. Indeed, if $a \in \mathcal{A}$, then there exist $b, c \in \mathcal{A}$ such that $b + c \preceq a$ and we get

$$\mathrm{cl}_{\mathcal{A}}(x) + \mathrm{cl}_{\mathcal{A}}(y) \preceq (x + b) + (y + c) \preceq x + y + a.$$

This inequality and property (i) imply that $x + y \preceq \mathrm{cl}_{\mathcal{A}}(x) + \mathrm{cl}_{\mathcal{A}}(y) \preceq \mathrm{cl}_{\mathcal{A}}(x + y)$. Applying properties (iii) and (iv), it follows that $\mathrm{cl}_{\mathcal{A}}(x) + \mathrm{cl}_{\mathcal{A}}(y) \in \mathrm{Cl}_{\mathcal{A}}$ and $\mathrm{cl}_{\mathcal{A}}(\mathrm{cl}_{\mathcal{A}}(x) + \mathrm{cl}_{\mathcal{A}}(y)) = \mathrm{cl}_{\mathcal{A}}(x + y)$, which was to be proved.

(vi) Let $x \in \mathrm{Cl}_{\mathcal{A}}$, $n \in \mathbb{N}$ with $n \cdot x \in \mathrm{Cl}_{\mathcal{A}}$. First we show that $n \cdot \mathrm{cl}_{\mathcal{A}}(x) \preceq n \cdot x + a$ holds for all $a \in \mathcal{A}$. Indeed, if $a \in \mathcal{A}$, then there exist $b \in \mathcal{A}$ such that $n \cdot b \preceq a$. Therefore, the inequality $\mathrm{cl}_{\mathcal{A}}(x) \preceq x + b$ implies that

$$n \cdot \mathrm{cl}_{\mathcal{A}}(x) \preceq n \cdot (x + b) = n \cdot x + n \cdot b \preceq n \cdot x + a.$$

In view of property (i) and this inequality, we get that $n \cdot x \preceq n \cdot \mathrm{cl}_{\mathcal{A}}(x) \preceq \mathrm{cl}_{\mathcal{A}}(n \cdot x)$. Applying properties (iii) and (iv), it follows that $n \cdot \mathrm{cl}_{\mathcal{A}}(x) \in \mathrm{Cl}_{\mathcal{A}}$ and $\mathrm{cl}_{\mathcal{A}}(n \cdot \mathrm{cl}_{\mathcal{A}}(x)) = \mathrm{cl}_{\mathcal{A}}(n \cdot x)$, which yields the statement.

(vii) Let $x \in \mathrm{Cl}_{\mathcal{A}}$, $n \in \mathbb{N}$ with $n * x \in \mathrm{Cl}_{\mathcal{A}}$. First we show that $n * \mathrm{cl}_{\mathcal{A}}(x) \preceq n * x + a$ holds for all $a \in \mathcal{A}$. Indeed, if $a \in \mathcal{A}$, then there exist $b \in \mathcal{A}$ such that $n * b \preceq a$. Therefore, the inequality $\mathrm{cl}_{\mathcal{A}}(x) \preceq x + b$ implies that

$$n * \mathrm{cl}_{\mathcal{A}}(x) \preceq n * (x + b) = n * x + n * b \preceq n * x + a.$$

In view of property (i) and this inequality, we get that $n * x \preceq n * \mathrm{cl}_{\mathcal{A}}(x) \preceq \mathrm{cl}_{\mathcal{A}}(n * x)$. Applying properties (iii) and (iv), it follows that $n * \mathrm{cl}_{\mathcal{A}}(x) \in \mathrm{Cl}_{\mathcal{A}}$ and $\mathrm{cl}_{\mathcal{A}}(n * \mathrm{cl}_{\mathcal{A}}(x)) = \mathrm{cl}_{\mathcal{A}}(n * x)$, which yields the statement.

(viii) Let $x \in X$ be an $\mathcal{A}$-bounded element and let $a \in \mathcal{A}$ be arbitrary. Then, for some $b, c \in \mathcal{A}$, we have that $b + c \preceq a$. By the $\mathcal{A}$-boundedness of x, there exists $n_0 \in \mathbb{N}$ such that $x \preceq n * b$ holds for all $n \geq n_0$. On the other hand, $c \in X_{\prec}$, hence $n * c \in X_{\prec}$ for all $n \in \mathbb{N}$. Consequently, for all $n \geq n_0$, we have that

$$\mathrm{cl}_{\mathcal{A}}(x) \preceq x + n * c \preceq n * b + n * c = n * (b + c) \preceq n * a,$$

which shows that $\mathrm{cl}_{\mathcal{A}}(x)$ is also an $\mathcal{A}$-bounded element of X.

(ix) Let $x \in X$ be n-convex such that $n * x \in \mathrm{Cl}_{\mathcal{A}}$ and $n * \mathrm{cl}_{\mathcal{A}}(x)$ is $\mathcal{A}$-closed. Then, the equality $n \cdot x = n * x$ yields that $n \cdot x \in \mathrm{Cl}_{\mathcal{A}}$ and properties (i), (ii), (v) and (vi), (vii) imply that

$$n \cdot \mathrm{cl}_{\mathcal{A}}(x) \preceq \mathrm{cl}_{\mathcal{A}}(n \cdot \mathrm{cl}_{\mathcal{A}}(x)) = \mathrm{cl}_{\mathcal{A}}(n \cdot x) = \mathrm{cl}_{\mathcal{A}}(n * x) = \mathrm{cl}_{\mathcal{A}}(n * \mathrm{cl}_{\mathcal{A}}(x)).$$

Using that $n * \mathrm{cl}_{\mathcal{A}}(x)$ is $\mathcal{A}$-closed, this inequality implies $n \cdot \mathrm{cl}_{\mathcal{A}}(x) \preceq n * \mathrm{cl}_{\mathcal{A}}(x)$. The reversed inequality holds automatically, hence $n \cdot \mathrm{cl}_{\mathcal{A}}(x) = n * \mathrm{cl}_{\mathcal{A}}(x)$, which proves the n-convexity of $\mathrm{cl}_{\mathcal{A}}(x)$. □

In what follows, we investigate the connection among the notions of boundedness, closedness and convexity.

In the subsequent propositions, we consider the cornets that were introduced in Proposition 1.2.10, 1.2.11, 2.3.4 and we determine all bounded, closed and Archimedean elements in these structures.

Proposition 1.3.10. [MP21] *Let $(G, +)$ be a topological abelian group such that there is no proper open subgroup of G. Let $W \subseteq G$ be a wedge with $W^\circ \neq \emptyset$ and let S be a subsemigroup of G containing W. Then we have the following claims:*

(i) In the cornet $(S, +, \cdot, \preceq_W)$ the set of nonnegative elements equals W.

(ii) The set W° is a subsemigroup of the Archimedean elements.

(iii) Every element of S is W°-bounded.

(iv) If, in addition, G is locally convex and W is topologically closed and $W^\circ = W^\circ + W^\circ$, then every element of S is also W°-closed.

Proof. (i) The nonnegativity of an element $x \in S$ with respect to the ordering $\preceq_W$, by definition, means that $x = x - 0 \in W$. This proves that W equals the set of nonnegative elements.

(ii) We have that $W^\circ + W^\circ \subseteq W + W \subseteq W$ and $W^\circ + W^\circ$ is also open, therefore, $W^\circ + W^\circ \subseteq W^\circ$ showing that W° is a subsemigroup of G.

To show that every element of W° is Archimedean, let $x \in W^\circ$ be arbitrary and define

$$T_x := \bigcup_{n \in \mathbb{N}} (W^\circ - n \cdot x) \cap (n \cdot x - W^\circ). \tag{1.13}$$

We show that T_x is an open subgroup of G. The opennes of T_x is obvious. Let $y, z \in T_x$. Then there exist $n, m \in \mathbb{N}$ and $v_1, v_2, w_1, w_2 \in W^\circ$ such that

$$y = v_1 - n \cdot x = n \cdot x - v_2, \qquad z = w_1 - m \cdot x = m \cdot x - w_2.$$

Therefore,

$$y - z = (v_1 + w_2) - (n + m) \cdot x = (n + m) \cdot x - (v_2 + w_1),$$

and hence

$$y - z \in (W^\circ - (n + m) \cdot x) \cap ((n + m) \cdot x - W^\circ) \subseteq T_x,$$

which proves that T_x is an open subgroup of G. Thus, T_x cannot be proper, in other words, $T_x = G$.

Let $u \in S$ be arbitrary. Then $S \subseteq T_x$ implies that there exists $n_0 \in \mathbb{N}$ such that $u \in W^\circ - n_0 \cdot x$. If $n > n_0$, then $(n - n_0) \cdot x \in (n - n_0) \cdot W^\circ \subseteq W^\circ$. Therefore,

$$\begin{aligned} u + n \cdot x = u + n_0 \cdot x + (n - n_0) \cdot x \\ \in (W^\circ - n_0 \cdot x) + n_0 \cdot x + W^\circ \subseteq W^\circ \subseteq W, \end{aligned}$$

i.e., $0 \preceq_W u + n \cdot x$, which proves that any element of W° is Archimedean.

(iii) Now we are going to show that every element of S is W°-bounded. Let $u \in S$ be fixed and $x \in W^\circ$ be arbitrary. As we have seen above, the set T_x defined by (1.13), covers S, therefore, there exists $n_0 \in \mathbb{N}$ such that $u \in n_0 \cdot x - W^\circ$. If $n > n_0$, then $(n - n_0) \cdot x \in (n - n_0) \cdot W^\circ \subseteq W^\circ$. Thus,

$$n \cdot x - u = (n - n_0) \cdot x + n_0 \cdot x - u \in W^\circ + W^\circ \subseteq W^\circ \subseteq W,$$

which shows that $u \preceq_W n \cdot x$ holds for all $n > n_0$. This proves that u is W°-bounded.

(iv) Finally, assume that W is topologically closed. We are going to verify that every element u of S is W°-closed, that is, u is the greatest lower bound of the set $\{u + x \mid x \in W^\circ\}$. It is clear that u is a lower bound. Assume that $v \in S$ is a lower bound for $\{u + x \mid x \in W^\circ\}$, that is, $u - v + x \in W$ holds for all $x \in W^\circ$. Let $x \in W^\circ$ be fixed and $n \in \mathbb{N}$.

Assume that $W^\circ = W^\circ + W^\circ$. Then

$$W^\circ = \{x_1 + \cdots + x_n \mid x_1, \dots, x_n \in W^\circ\}.$$

Therefore, there exist $x_1, \dots, x_n \in W^\circ$ such that $x = x_1 + \cdots + x_n$. This implies that

$$n \cdot (u - v) + x = (u - v + x_1) + \cdots + (u - v + x_n) \in W.$$

If $u - v \notin W$, then $u - v \in G \setminus W$. Then there exists an open convex and symmetric neighborhood of 0 such that $u - v + U \subseteq G \setminus W$. Consider the set S defined by

$$S := \bigcup_{n=1}^{\infty} \{y_1 + \cdots + y_n \mid y_1, \dots, y_n \in U\}.$$

Then S is an open subgroup of G, hence $S = G$, which yields that $x \in S$. This implies that, for some $n \in \mathbb{N}$ and $y_1, \dots, y_n \in U$ the equality $x = y_1 + \cdots + y_n$ holds. By the convexity of U, it follows that there exists $y \in U$, such that $x = n \cdot y$. Thus

$$n \cdot (u - v + y) = n \cdot (u - v) + x \in W,$$

which yields that $u - v + y \in W$ contradicting $u - v + y \in u - v + U \subseteq G \setminus W$. This contradiction shows that $u - v \in W$ must be valid, i.e., $v \preceq_W u$ holds. □

Proposition 1.3.11. [MP21] *Let $(G, +)$ be a topological abelian group such that there is no proper open subgroup of G. Let W be a wedge and let S be a subsemigroup of G containing W. Let $P_W(S)$ denote the collection of all nonempty W-invariant subsets of S. Define the operations $+$ and $*$ by (1.4). Then the following statements hold:*

(i) The set of nonnegative elements of the cornet $(P_W(S), +, *, \subseteq)$ consists of those W-invariant subsets of S that contain 0 (which denotes the neutral element of G).

(ii) The collection $\mathcal{A}$ of those W-invariant subsets which contain an open convex neighborhood $C \in P_W(S)$ of 0 is a subsemigroup of the Archimedean elements.

(iii) An element of $P_W(S)$ is $\mathcal{A}$-bounded if it is the sum of a topologically bounded subset of S and W.

(iv) If, in addition, G is locally convex, then any topologically closed element of $P_W(S)$ is also $\mathcal{A}$-closed and the addition is $\mathcal{A}$-continuous.

Proof. (i) The unit element of $(P_W(S), +, *, \subseteq)$ is the set W. Now, an element $A \in P_W(S)$ is nonnegative if $W \subseteq A$. This inclusion is equivalent to the condition $0 \in A$ because A is W-invariant.

(ii) Let $A, B \in \mathcal{A}$. Then there exist open convex sets $C, D \in P_W(S)$ such that $0 \in C \subseteq A$ and $0 \in D \subseteq B$, Then, by Lemma 1.2.5, the set $C + D$ is convex and also open, furthermore $0 \in C + D \subseteq A + B$. Therefore, $A + B \in \mathcal{A}$, which proves that $(\mathcal{A}, +)$ is a semigroup.

To prove that the elements of $\mathcal{A}$ are Archimedean, let $A \in \mathcal{A}$. Then there exists an open and convex $C \in P_W(S)$ such that $0 \in C \subseteq A$. Define

$$T_C := \bigcup_{n \in \mathbb{N}} (n * C) \cap (-n * C). \tag{1.14}$$

We prove that T_C is a subgroup of G. Let $x, y \in T_C$. Then there there exist $n, m \in \mathbb{N}$ such that $x \in (n*C) \cap (-n*C)$ and $y \in (m*C) \cap (-m*C)$. Using the definition of the $*$ multiplication, it follows that there exist $a_1, a_2, b_1, b_2 \in C$ and $v_1, v_2, w_1, w_2 \in W$ such that

$$x = n \cdot a_1 + v_1 = -n \cdot a_2 - v_2, \qquad y = m \cdot b_1 + w_1 = -m \cdot b_2 - w_2.$$

By the $(n+m)$-convexity of C, it follows that here exist $c_1, c_2 \in C$ and $u_1, u_2 \in W$ such that

$$\begin{aligned} n \cdot a_1 + m \cdot b_2 &= (n+m) \cdot c_1 + u_1, \\ n \cdot a_2 + m \cdot b_1 &= (n+m) \cdot c_2 + u_2. \end{aligned}$$

Then

$$\begin{aligned} x - y &= (n \cdot a_1 + m \cdot b_2) + (v_1 + w_2) \\ &\qquad \subseteq (n+m) \cdot c_1 + u_1 + W \subseteq (n+m) * C, \\ y - x &= (n \cdot a_2 + m \cdot b_1) + (v_2 + w_1) \\ &\qquad \subseteq (n+m) \cdot c_2 + u_2 + W \subseteq (n+m) * C. \end{aligned}$$

Therefore, $x - y \in T_C$, which proves that T_C is a subgroup of G. The openness of C implies that $n \cdot C$ is open for every $n \in \mathbb{N}$. Thus, by the convexity of C, we have that $n*C$ is open for every $n \in \mathbb{N}$. Consequently, T_C is open and hence, by our assumption, T_C is equal to G.

Before proving the further assertions, we show that $n * C \subseteq m * C$ if $n \leq m$. Indeed, let $u \in n * C$. Then, for some $c \in C$ and $w \in W$, we have $u = n \cdot c + w$. If $n \leq m$, then, the m-convexity of C and $0 \in C$ yield that there exist $d \in C$ and $v \in W$ such that $n \cdot c + (m-n) \cdot 0 = n \cdot d + v$. Thus

$$u = n \cdot c + (m-n) \cdot 0 + w = m \cdot d + v + w = m * C,$$

which verifies $n * C \subseteq m * C$.

Let $U \in P_W(S)$ be arbitrary and choose a fixed element $u \in U$. The inclusion $u \in T_C$ yields, for some $n_0 \in \mathbb{N}$, that $-u \in n_0 * C$. For $n \geq n_0$, this implies that $-u \in n * C \subseteq n * A$. Hence $0 \in U + n * A$ holds for all

$n \geq n_0$ which, according to the the first assertion, means that $U + n * A$ is nonnegative for all $n \geq n_0$. This proves that A is Archimedean.

(iii) Let $B \in P_W(S)$ be the sum of a topologically bounded set $D \subseteq S$ and W. Let $A \in \mathcal{A}$ be fixed. Then there exists an open convex set $C \in P_W(S)$ such that $0 \in C \subseteq A$.

By the topological boundedness of D, we can find a number n_0 such that $D \subseteq n_0 \cdot C$. Since $0 \in C$, this implies that $D \subseteq n \cdot C$ for all $n \geq n_0$. By the convexity of C, this yields that $D \subseteq n * C$. Consequently

$$B = D + W \subseteq n * C + W = n * C \subseteq n * A,$$

which proves that B is $\mathcal{A}$-bounded.

(iv) In this part of the proof, we assume that G is a locally convex topological group. Let $D \in P_W(S)$ be a topologically closed set. We need to show that D is the $\mathcal{A}$-closure of itself. The inclusion $D \subseteq D + A$ trivially holds for all $A \in \mathcal{A}$ because $0 \in A$. Assume now that, for some $E \in P_W(S)$, the inclusion $E \subseteq D + A$ holds for all $A \in \mathcal{A}$. We need to show that $E \subseteq D$.

Let $e \in E$ be arbitray and assume that $e \notin D$, i.e., $0 \notin e - D$. Using that D is topologically closed, we have that $e - D$ is closed. Hence 0 is an interior point of its complement. Thus there exists an open convex neighborhood C_0 of zero such that $C_0 \cap (e - D) = \emptyset$. We show that then $(C_0 + W) \cap (e - D) = \emptyset$. Indeed, if this not true, then there exist $c \in C_0, w \in W$ and $d \in D$ such that $c + w = e - d$. Then, by the W-invariance of D, we get $c = e - (d + w) \in e - D$ which contradicts $C_0 \cap (e - D) = \emptyset$. Then $A := C_0 + W$ is an open convex W-invariant neighborhood of zero which is disjoint from $e - D$. This implies that $e \notin A + D$ which contradicts that $E \subseteq D + A$ holds for all $A \in \mathcal{A}$

Finally, we prove that the addition is $\mathcal{A}$-continuous. Let $A \in \mathcal{A}$. We need to show that there exists $B \in \mathcal{A}$ such that $B + B \subseteq A$.

By $A \in \mathcal{A}$, there exists a convex neighborhood $C \in P_W(S)$ of 0 such that $C \subseteq A$. By the continuity of the addition in G, there exists a neighborhood D of 0 such that $D + D \subseteq C$. Using the local convexity of the topology of G, we may assume that D convex. Define B as $D + W$.

Then B is a W-invariant convex neighborhood of 0, hence $B \in \mathcal{A}$, and $B + B = (D + W) + (D + W) \subseteq C + W \subseteq C \subseteq A$. □

Proposition 1.3.12. [MP21] *Let $(G, +)$ be a topological abelian group such that there is no proper open subgroup of G. Let W be a wedge and let S be a uniquely divisible subsemigroup of G containing W. Let, for $p \in]0, 1]$, the set $F^p_W(S)$ be defined by (1.7) and define the operations $\oplus$ and $\odot$ by (1.8). Then, the following statements hold.*

(i) The set of nonnegative elements of the cornets $(F^p_W(S), \oplus, \odot, \leq)$ consists of those W-invariant functions f such that $f(0) = 1$ (here 0 denotes the neutral element of G).

(ii) The cornet $(F^p_W(S), \oplus, \odot, \leq)$ has no Archimedean elements for $p \in]0, 1[$. On the other hand, the collection $\mathcal{A}$ of those $a \in F^1_W(S)$ or which there exists an open convex neighborhood C of 0 such that $a|_C = 1$ is a subsemigroup of the Archimedean elements of $(F^1_W(S), \oplus, \odot, \leq)$.

(iii) An element f of $F^1_W(S)$ is $\mathcal{A}$-bounded if $\operatorname{supp}(f) := \{u \in S \mid f(u) > 0\}$ is covered by the sum of a topologically bounded subset of S and W.

(iv) If, in addition, G is locally convex, then any upper semicontinuous element of $F^1_W(S)$ is also $\mathcal{A}$-closed and the addition $\oplus$ is $\mathcal{A}$-continuous.

Proof. (i) Let $p \in]0, 1]$. The unit element of $(F^p_W(S), \oplus, \odot, \leq)$ is the characteristic function of W. Therefore, by definition, an element $f \in F^p_W(S)$ is nonnegative if $\chi_W \leq f$ which implies that $1 = \chi_W(0) \leq f(0)$, whence $f(0) = 1$ follows. On the other hand, if $f(0) = 1$, then W-nondecreasingness implies of f implies that $1 = f(0) \leq f(w)$. Thus $\chi_W \leq f$ holds.

(ii) Let $p \in]0, 1[$. Assume that $g \in F^p_W(S)$ is an Archimedean element of $F^p_W(S)$. Let $f\colon S \to [0, 1]$ be a constant function with a value

$p \leq f(0) < 1$. Then $f \in F_W^p(S)$, on the other hand, for all $n \in \mathbb{N}$,

$$(f \oplus (n \odot g))(0) = \sup_{\substack{u,v \in S \\ u+v=0}} \min\big(f(u), (n \odot g)(v)\big) \leq f(0) < 1$$

According to the first assertion of this theorem, $f \oplus (n \odot g)$ is not non-negative for all $n \in \mathbb{N}$, which shows that g cannot be Archimedean.

Let $f, g \in \mathcal{A}$. Then there exist open convex neighborhoods C, D of 0 such that $f|_C = 1$ and $g|_D = 1$. Then, for $x \in C + D$, we get

$$(f \oplus g)(x) = \sup_{\substack{u,v \in S \\ u+v=x}} \min\big(f(u), g(v)\big) \geq \sup_{\substack{u \in C, v \in D \\ u+v=x}} \min\big(f(u), g(v)\big) = 1.$$

On the other hand, $C + D$ is an open convex neighborhood of 0 on which $f \oplus g = 1$. Therefore, $f \oplus g \in \mathcal{A}$.

In what follows we show that every element of $\mathcal{A}$ are Archimedean in $(F_W^1(S), \oplus, \odot, \leq)$. Let $f \in \mathcal{A}$ and $g \in F_W^1(S)$ be arbitrary. We need to show that there exists $n_0 \in \mathbb{N}$ such that $(g \oplus (n \odot f))(0) = 1$ holds for all $n \in \mathbb{N}$. Let $0 < \eta < 1$ be arbitrary and choose $u_0 \in S$ such that $g(u_0) \geq \eta$. Then

$$\begin{aligned}(g \oplus (n \odot f))(0) &= \sup_{\substack{u,v \in S \\ u+v=0}} \min\big(g(u), (n \odot f)(v)\big) \\ &\geq \min\big(g(u_0), (n \odot f)(-u_0)\big) \geq \min\big(\eta, f(-\tfrac{u_0}{n})\big).\end{aligned} \tag{1.15}$$

There exists an open convex neighborhood C of 0 such that $f|_C = 1$. Define the set T_C by (1.14), where the operation $*$ is given by (1.4). Then, as we have seen it in the proof of Proposition 1.3.11, $T_C = G$ and $n * C \subseteq m * C$ holds for $n \leq m$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $-u_0 \in n_0 * C \subseteq n * C$ for all $n \geq n_0$. This means that for all $n \geq n_0$ there exists $c \in C$ and $w \in W$ such that $-u_0 = n \cdot c + w$. Thus $-\frac{u_0}{n} = c + \frac{w}{n}$, which implies that $f(-\frac{u_0}{n}) = f(c + \frac{w}{n}) \geq f(c) = 1$. Therefore, $f(-\frac{u_0}{n}) = 1$ for $n \geq n_0$. Combining this with the inequality (1.15), we obtain

$$(g \oplus (n \odot f))(0) \geq \min\big(\eta, f(-\tfrac{u_0}{n})\big) = \eta \qquad (n \geq n_0).$$

Since η was arbitrary, this inequality implies that $(g \oplus (n \odot f))(0) = 1$, for all $n \geq n_0$. According to the first assertion of this theorem, this yields that $g \oplus (n \odot f)$ is a nonnegative element of the cornet $(F_W^1(S), \oplus, \odot, \leq)$. This proves that f is Archimedean in $(F_W^1(S), \oplus, \odot, \leq)$.

(iii) Let f be in $F_W^1(S)$ such that the support of f is covered by the sum of a topologically bounded subset $D \subseteq S$ and W. Let $a \in \mathcal{A}$ be fixed. We need to show that there exists $n_0 \in \mathbb{N}$ such that $f \leq n \odot a$ if $n \geq n_0$.

By $a \in \mathcal{A}$, there exists an open convex neighborhood C of 0 such that $a|_C = 1$. In view of the topological boundedness of D, we can find a number n_0 such that $D \subseteq n_0 \cdot C$. By the convexity of C and $0 \in C$, this implies that $D \subseteq n \cdot C$. Consequently,

$$\operatorname{supp}(f) \subseteq D + W \subseteq n \cdot C + W.$$

Therefore, if $u \in \operatorname{supp}(f)$, then $\frac{u}{n} \in C + W$ for all $n \geq n_0$. The W-nondecreasingness of a and $a|_C = 1$ now yield that $a\left(\frac{u}{n}\right) = 1$ for all $n \geq n_0$. This implies that $f(u) \leq a\left(\frac{u}{n}\right) = (n \odot a)(u)$ holds for all $u \in \operatorname{supp}(f)$ and $n \geq n_0$. This inequality is obvious for $u \in S \setminus \operatorname{supp}(f)$, i.e., if $f(u) = 0$. Thus, we have proved that $f \leq n \odot a$ holds for all $n \geq n_0$, which shows that f is $\mathcal{A}$-bounded.

(iv) in this part of the proof, we assume that G is a locally convex topological group. Let $f \in F_W^1(S)$ be an upper semicontinuous element. To prove that f is $\mathcal{A}$-closed, we need to show that if $g \in F_W^1(S)$ satisfies $g \leq f \oplus a$ for all $a \in \mathcal{A}$, then $g \leq f$.

Let $x \in S$ be fixed and $\varepsilon > 0$ be arbitrary. Then, by the upper semicontinuity of f at x, there exists a neighborhood U of x such that $f(u) \leq f(x) + \varepsilon$ for all $u \in S \cap U$. Observing that $x - U$ is a neighborhood of 0, the local convexity of G implies that there exists a convex neighborhood C of 0 such that $C \subseteq x - U$, i.e., $x - C \subseteq U$. Let $a := \chi_{C+W}$. Then a is W-nondecreasing and $a|_C = 1$, hence $a \in \mathcal{A}$. Therefore, we have $g \leq f \oplus a$, which implies

$$\begin{aligned} g(x) &\leq (f \oplus a)(x) \\ &= \sup_{\substack{u,v \in S \\ u+v=x}} \min\big(f(u), a(v)\big) = \sup_{\substack{u \in S,\, v \in C+W \\ u+v=x}} f(u) \\ &= \sup_{u \in S \cap (x-C-W)} f(u) \leq \sup_{u \in S \cap (U-W)} f(u) \leq \sup_{u \in S \cap U} f(u) \leq f(x) + \varepsilon. \end{aligned}$$

Upon taking the limit $\varepsilon \to 0$, the above inequality yields $f(x) \leq g(x)$, which was to be proved.

Finally, we prove the $\mathcal{A}$-continuity of the operation $\oplus$. Let $a \in \mathcal{A}$. Then there exists a W-invariant convex neighborhood C of 0 such that $a|_C = 1$. Therefore, $\chi_C \leq a$. As we have seen it in the proof of Proposition 1.3.11, there exists a W-invariant convex neighborhood B of 0 such that $B + B \subseteq C$. Then

$$\chi_B \oplus \chi_B = \Phi(B) \oplus \Phi(B) = \Phi(B+B) = \chi_{B+B} \leq \chi_C \leq a.$$

On the other hand $\chi_B \in \mathcal{A}$. This completes the proof of the $\mathcal{A}$-continuity of $\oplus$. □

1.4 Main results

The following result is the extension of the Rådström Cancellation Theorem to the setting of cornets.

Theorem 1.4.1. [MP21] *Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_\prec$ such that the addition is $\mathcal{A}$-continuous. Let $x, y, z \in X$ such that z is $\mathcal{A}$-bounded and y is $\mathcal{A}$-closed and m-convex for some $m \geq 2$. If*

$$x + z \preceq y + z \tag{1.16}$$

holds, then we have $x \preceq y$.

Proof. Let $x, y, z \in X$ such that (1.16) holds. First, for all $n \in \mathbb{N}$, we show that

$$n \cdot x + z \preceq n \cdot y + z. \tag{1.17}$$

For $n = 1$, the inequality is equivalent to (1.16). Assume that (1.17) holds for $n = k \in \mathbb{N}$. Then

$$\begin{aligned}(k+1)\cdot x + z &= k\cdot x + x + z \\ &\preceq k\cdot y + x + z \preceq k\cdot y + y + z = (k+1)\cdot y + z,\end{aligned}$$

which proves that the inequality (1.17) holds for $n = k+1$. This, by the principle of mathematical induction, completes the proof (1.17) for all $n \in \mathbb{N}$.

The m-convexity of the element y implies that $m \in C_y$, which is closed under multiplication by Lemma 1.2.5. Thus, for all $k \in \mathbb{N}$, $m^k \in C_y$. Using this and the inequality (1.2), we obtain that

$$m^k * x \preceq m^k \cdot x \quad \text{and} \quad m^k \cdot y = m^k * y. \tag{1.18}$$

Combining inequalities (1.17) and (1.18), it follows that for all $k \in \mathbb{N}$,

$$m^k * x + z \preceq m^k * y + z. \tag{1.19}$$

In the final step, assuming the $\mathcal{A}$-boundedness of z, we show that (1.19) implies $x \preceq y + a$ for all $a \in \mathcal{A}$.

Let $a \in \mathcal{A}$. Then, using that the addition is $\mathcal{A}$-continuous, we can find $b, c \in \mathcal{A}$ such that $b + c \preceq a$. Then there exists $n_1 \in \mathbb{N}$ such that $0 \preceq z + n * b$ holds for all $n_1 \leq n$. The element z is $\mathcal{A}$-bounded, thus we can find $n_2 \in \mathbb{N}$ such that $z \preceq n * c$ holds for all $n_2 \leq n$. By choosing k so that $\max(n_1, n_2) \leq m^k$ is satisfied, it follows that

$$0 \preceq z + m^k * b \quad \text{and} \quad z \preceq m^k * c.$$

then we have

$$\begin{aligned}m^k * x &\preceq m^k * x + z + m^k * b \preceq m^k * y + z + m^k * b \\ &\preceq m^k * y + m^k * c + m^k * b = m^k * (y + c + b) \preceq m^k * (y + a).\end{aligned}$$

This inequality implies that

$$x \preceq y + a$$

for all $a \in \mathcal{A}$. Now, using that y is $\mathcal{A}$-closed, we can conclude that $x \preceq y$, which is what we wanted to prove. □

In what follows, we present several applications of the above theorem in the particular cornets described in Proposition 1.2.10, Proposition 1.2.11, and Proposition 2.3.4.

Corollary 1.4.2. [MP21] *Let $(G, +)$ be a locally convex topological abelian group such that there is no proper open subgroup of G. Let W be a wedge and let S be a subsemigroup of G containing W. Let $P_W(S)$ denote the collection of W-invariant subsets of S and define the operations $+$ and $*$ by (1.4). Let $A, B, C \in P_W(S)$ such that B is covered by the sum of a topologically bounded subset of S and W and C is a topologically closed m-convex subset of S for some $m \geq 2$. Assume that*

$$A + B \subseteq C + B \tag{1.20}$$

holds. Then $A \subseteq C$.

Proof. In view of Proposition 2.3.4, $(P_W(S), +, *, \subseteq)$ is a cornet and the m-convexity of C implies that C is an m-convex element of this cornet.

Let $\mathcal{A}$ denote the collection of those W-invariant subsets which contain an open convex neighborhood of 0. Then, by assertion (ii) of Proposition 1.3.11, $\mathcal{A}$ is a subsemigroup of the Archimedean elements of the quadruple $(P_W(S), +, *, \subseteq)$. By assertion (iii) of this proposition, we have that B is $\mathcal{A}$-bounded and by the assertion (iv), we obtain that the element C is $\mathcal{A}$-closed and the addition is $\mathcal{A}$-continuous.

Therefore, we can apply Theorem 2.5.1, which shows that the inclusion (1.20) implies $A \subseteq C$. □

Corollary 1.4.3. [MP21] *Let $(G, +)$ be a locally convex topological abelian group such that there is no proper open subgroup of G. Let W be a wedge and let S be a uniquely divisible subsemigroup of G containing W. Let the set $F^1_W(S)$ be defined by (1.7) and define the operations $\oplus$ and $\odot$ by (1.8). Let $f, g, h \in F^1_W(S)$ such that $\operatorname{supp}(h)$ is covered by the sum of a topologically bounded subset of S and W and g is upper semicontinuous and m-quasiconcave for some $m \geq 2$. Assume that*

$$f \oplus h \leq g \oplus h \tag{1.21}$$

holds. Then $f \leq g$.

Proof. In view of Proposition 2.3.4, $(F^1_W(S), \oplus, \odot, \leq)$ is a cornet and the m-quasiconcavity of g implies that g is an m-convex element of this cornet.

Let $\mathcal{A}$ denote the collection of those $a \in F^1_W(S)$ for which there exists an open convex neighborhood C of 0 such that $a|_C = 1$. Then, by assertion (ii) of Proposition 1.3.12, $\mathcal{A}$ is a subsemigroup of the Archimedean elements of $(F^1_W(S), \oplus, \odot, \leq)$. By assertion (iii) of this proposition, we have that h is $\mathcal{A}$-bounded and by the assertion (iv), we obtain that the element g is $\mathcal{A}$-closed and the addition $\oplus$ is $\mathcal{A}$-continuous.

Therefore, we can apply Theorem 2.5.1, which shows that the inequality (1.21) implies $f \leq g$. □

Chapter 2

On convex and concave sequences and their applications

2.1 Introduction to the second chapter

In the theory of convexity, the investigation of convex functions play a fundamental role. We refer to the following monographs for the details: Hardy–Littlewood–Pólya [HLP34], Kuczma [K85], Mitrinović [M70], Mitrinović–Pečarić–Fink [MPF91, MPF93], Niculescu–Persson [NP06], Popoviciu [P44], and Roberts–Varberg [RV73]. The investigation of convex sequences probably started in the book Mitrinović [M70]. This subfield is still very active, some recent results and applications have been obtained by Krasniqi [K16], Niezgoda [N11,N17a,N17b], Sofonoea–Ţincu–Acu [STA18], Tabor–Tabor–Żoldak [TTZ12], Wu–Debnath [WD07], Yıldız [Y18]. In this chapter we introduce the notions of q-convex, q-affine and q-concave sequences and some basic results on them are also presented. Then we establish their surprising connection to Chebyshev polynomials of the first and of the second kind. Finally, an application of them to fixed point theory is presented.

Given $n, m \in \mathbb{Z}$ with $2 \leq m - n$, let $\mathcal{S}(n|m)$ denote the linear space $\mathbb{R}^{\{n,\dots,m\}}$ of all real sequences, i.e., the collection of all functions $p\colon \{n, \dots, m\} \to \mathbb{R}$. It is natural to define the notions of concavity, convexity and affinity for the elements of $\mathcal{S}(n|m)$. A sequence $p =$

$(p_n, \dots, p_m) \in \mathcal{S}(n|m)$ is called *convex* if, for all $i \in \{n+1, \dots, m-1\}$,

$$2p_i \leq p_{i-1} + p_{i+1}. \tag{2.1}$$

If, for all $i \in \{n+1, \dots, m-1\}$, the reversed inequality holds in (2.1), then the sequence is termed *concave*. Finally, if a sequence is simultaneously convex and concave, then it is said to be *affine*. If the inequality (2.1) holds with strict inequality sign, then we speak about strict convexity and concavity, respectively.

In what follows, we extend the above definitions and introduce the notions of q-convex, q-concave, and q-affine sequences with respect to a positive number q. A sequence $p = (p_n, \dots, p_m) \in \mathcal{S}(n|m)$ is called *q-convex* if, for all $i \in \{n+1, \dots, m-1\}$,

$$2qp_i \leq p_{i-1} + p_{i+1}. \tag{2.2}$$

If, for all $i \in \{n+1, \dots, m-1\}$, the reversed inequality holds in (2.2), then the sequence is termed *q-concave*. If a sequence is simultaneously q-convex and q-concave, then it is said to be *q-affine*.

We can easily see that the strict convexity of a positive (or negative) sequence implies its q-convexity for some q. Indeed, if $p \in \mathcal{S}(n|m)$ is a positive strictly convex sequence then, for all $i \in \{n+1, \dots, m-1\}$,

$$1 < \frac{p_{i-1} + p_{i+1}}{2p_i}.$$

Therefore,

$$1 < q := \min_{i \in \{n+1, \dots, m-1\}} \frac{p_{i-1} + p_{i+1}}{2p_i},$$

which implies that p is q-convex with a number $q > 1$. Analogously, if $p \in \mathcal{S}(n|m)$ is a negative strictly convex sequence, then it is q-convex with a number $0 < q < 1$.

The subclasses of q-convex and q-concave sequences in $\mathcal{S}(n|m)$ will be denoted by $\mathcal{C}_q^{\cup}(n|m)$ and $\mathcal{C}_q^{\cap}(n|m)$, respectively. Finally, $\mathcal{A}_q(n|m)$ will stand for the subclass of q-affine sequences, that is,

$$\mathcal{A}_q(n|m) := \mathcal{C}_q^{\cup}(n|m) \cap \mathcal{C}_q^{\cap}(n|m).$$

It is easy to see that $\mathcal{A}_q(n|m)$ is a linear subspace of $\mathcal{S}(n|m)$, and both $\mathcal{C}_q^{\cup}(n|m)$ and $\mathcal{C}_q^{\cap}(n|m)$ are convex cones in $\mathcal{S}(n|m)$, i.e., they are closed with respect to linear combinations with nonnegative coefficients.

2.2 Auxiliary results for Chebyshev polynomials

For $k \in \mathbb{Z}$, let $T_k\colon \mathbb{R} \to \mathbb{R}$ and $U_k\colon \mathbb{R} \to \mathbb{R}$ denote the Chebyshev polynomials of the first and of the second kind of order k, which are defined by the system of equations for $k \in \mathbb{Z}$

$$\begin{aligned} &T_0(x) := 1, \quad T_1(x) := x, \quad\ \ T_{k-1}(x) + T_{k+1}(x) = 2xT_k(x), \\ &U_0(x) := 1, \quad U_1(x) := 2x, \quad U_{k-1}(x) + U_{k+1}(x) = 2xU_k(x), \end{aligned} \tag{2.3}$$

respectively. The last equalities in (2.3) rewritten as

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x),$$

can be used to compute T_k and U_k for $k \geq 2$ recursively. If we rewrite them as

$$T_{k-1}(x) = 2xT_k(x) - T_{k+1}(x), \quad U_{k-1}(x) = 2xU_k(x) - U_{k+1}(x),$$

then T_k and U_k can be determined for $k \leq -1$. One can easily prove that, for $k \in \mathbb{Z}$,

$$T_{-k} = T_k \quad \text{and} \quad U_{-k} = -U_{k-2}.$$

In particular, $U_{-1} = 0$. It is clear that, for $k \geq 0$, the degree of T_k and U_k equals to k. It is well-known that these polynomials satisfy, for all $u \in \mathbb{R}$ and $k \in \mathbb{Z}$, the equalities

$$T_k(\cos(u)) = \cos(ku) \quad \text{and} \quad T_k(\cosh(u)) = \cosh(ku) \tag{2.4}$$

and

$$U_k(\cos(u)) = \frac{\sin((k+1)u)}{\sin(u)} \quad \text{and} \quad U_k(\cosh(u)) = \frac{\sinh((k+1)u)}{\sinh(u)}. \tag{2.5}$$

From these representations it easily follows that the roots of T_k (for $k \neq 0$) and U_{k-1} (for $k \notin \{-1, 0, 1\}$) are given by

$$\left\{ \cos\left(\frac{2i-1}{2k}\pi\right) \mid i \in \{1, \ldots, |k|\} \right\} \quad \text{and}$$
$$\left\{ \cos\left(\frac{i}{k}\pi\right) \mid i \in \{1, \ldots, |k|-1\} \right\},$$

respectively. Therefore, the largest root of T_k (for $k \neq 0$) and U_{k-1} (for $k \notin \{-1, 0, 1\}$) are given by

$$\cos\left(\frac{\pi}{2k}\right) \quad \text{and} \quad \cos\left(\frac{\pi}{k}\right),$$

respectively.

Lemma 2.2.1. [MP22] *For $0 \leq x < 1$, the sequence $(T_k(x))_{k=1}^{\tau(x)}$ is strictly decreasing, where $\tau(x) := \left\lfloor \frac{\pi}{\arccos(x)} \right\rfloor$. For $x > 1$, the sequence $(T_k(x))_{k=0}^{\infty}$ is strictly increasing.*

Proof. If $x > 1$, then there exists $u > 0$ such that $x = \cosh(u)$. Thus, in view of the second formula in (2.4), we have

$$T_k(x) = T_k(\cosh(u)) = \cosh(ku) \qquad (k \in \mathbb{N} \cup \{0\}),$$

which by the strict monotonicity of the $\cosh$ function implies that the right hand side is a strictly increasing function of k.

If $0 \leq x < 1$, then there exists $u \in]0, \frac{\pi}{2}]$ such that $x = \cos(u)$. In view of the first formula in (2.4), we have

$$T_k(x) = T_k(\cos(u)) = \cos(ku) \qquad (k \in \mathbb{N} \cup \{0\}),$$

which, using that $\cos$ is strictly decreasing on $[0, \pi]$, implies that $T_k(x)$ is strictly decreasing for $k \in \{0, \ldots, \left\lfloor \frac{\pi}{u} \right\rfloor\}$. □

Lemma 2.2.2. [MP22] *Let* $n \geq 3$ *be an odd number. Then, for all* $x_1, \dots, x_n \in \mathbb{R}$ *with the notation* $x_{i+n} := x_i$ *(*$i \in \{1, \dots, n\}$*), we have*

$$\begin{aligned} &\sum_{i=1}^{n} \sin\left(\sum_{j=1}^{n-1}(-1)^j x_{i+j}\right)\sin(x_i) = 0 \quad \textit{and} \\ &\sum_{i=1}^{n} \sin\left(\sum_{j=1}^{n-1}(-1)^j x_{i+j}\right)\cos(x_i) = 0. \end{aligned} \tag{2.6}$$

Proof. Let $x_1, \dots, x_n \in \mathbb{R}$ and denote

$$y_i := \sum_{j=1}^{n-1}(-1)^j x_{i+j} \qquad (i \in \{1, \dots, n-1\}).$$

Then, by the well-known product-to-sum identities

$$\begin{aligned} 2\sin\left(\sum_{j=1}^{n-1}(-1)^j x_{i+j}\right)\sin(x_i) &= 2\sin(x_i)\sin(y_i) \\ &= \cos(x_i - y_i) - \cos(x_i + y_i), \\ 2\sin\left(\sum_{j=1}^{n-1}(-1)^j x_{i+j}\right)\cos(x_i) &= 2\cos(x_i)\sin(y_i) \\ &= \sin(x_i + y_i) - \sin(x_i - y_i). \end{aligned}$$

Observe that, by the equality $x_i = x_{i+n}$ and by the oddness of n, we have

$$\begin{aligned} x_i - y_i = x_i - \sum_{j=1}^{n-1}(-1)^j x_{i+j} &= x_i + x_{i+1} + \sum_{j=2}^{n-1}(-1)^{j-1} x_{i+j} \\ &= x_{i+1} + (-1)^{n-1} x_{i+n} + \sum_{j=1}^{n-2}(-1)^j x_{i+1+j} = x_{i+1} + y_{i+1}. \end{aligned}$$

Therefore

$$2\sin\left(\sum_{j=1}^{n-1}(-1)^j x_{i+j}\right)\sin(x_i) = \cos(x_{i+1}+y_{i+1}) - \cos(x_i+y_i),$$
$$2\sin\left(\sum_{j=1}^{n-1}(-1)^j x_{i+j}\right)\cos(x_i) = \sin(x_i+y_i) - \sin(x_{i+1}+y_{i+1}).$$

Summing up these equalities side by side for $i \in \{1,\dots,n\}$, respectively, we can see that the right hand sides are telescopic sums which are equal to zero, hence both equalities in (2.6) hold true. □

Lemma 2.2.3. [MP22] *For all $i, j, k \in \mathbb{Z}$, we have*

$$\begin{aligned} U_{k-j-1}U_i + U_{j-i-1}U_k &= U_{k-i-1}U_j \quad \textit{and} \\ U_{k-j-1}T_i + U_{j-i-1}T_k &= U_{k-i-1}T_j. \end{aligned} \tag{2.7}$$

Furthermore, for $i, j \in \mathbb{Z}$, we also have

$$U_{i-j} + U_{i+j} = 2T_jU_i \quad \textit{and} \quad T_{i-j} + T_{i+j} = 2T_jT_i. \tag{2.8}$$

Proof. In the particular case $n = 3$, with $x_1 := x$, $x_2 := y$ and $x_3 := z$, the identities in (2.6) yield

$$\begin{aligned} \sin(z-y)\sin(x) + \sin(y-x)\sin(z) &= \sin(z-x)\sin(y), \\ \sin(z-y)\cos(x) + \sin(y-x)\cos(z) &= \sin(z-x)\cos(y). \end{aligned} \tag{2.9}$$

Let $q \in]-1,1[$ be arbitrary, let $u := \arccos(q)$ and let $i, j, k \in \mathbb{Z}$. With the substitutions $(x,y,z) := ((i+1)u, (j+1)u, (k+1)u)$ and $(x,y,z) := (iu, ju, ku)$, the first and second identities in (2.9) imply

$$\frac{\sin((k-j)u)}{\sin(u)}\frac{\sin((i+1)u)}{\sin(u)} + \frac{\sin((j-i)u)}{\sin(u)}\frac{\sin((k+1)u)}{\sin(u)} = \frac{\sin((k-i)u)}{\sin(u)}\frac{\sin((j+1)u)}{\sin(u)},$$

$$\frac{\sin((k-j)u)}{\sin(u)}\cos(iu)+\frac{\sin((j-i)u)}{\sin(u)}\cos(ku) \\ =\frac{\sin((k-i)u)}{\sin(u)}\cos(ju).$$

In view of (2.4), from these equalities we can easily obtain that

$$U_{k-j-1}(q)U_i(q)+U_{j-i-1}(q)U_k(q)=U_{k-i-1}(q)U_j(q), \\ U_{k-j-1}(q)T_i(q)+U_{j-i-1}(q)T_k(q)=U_{k-i-1}(q)T_j(q)$$

hold for all $q\in\,]-1,1[$ and hence for all $q\in\mathbb{R}$. This completes the proof of the equalities in (2.7).

To prove (2.8), let $q\in\,]-1,1[$ be arbitrary, let $u:=\arccos(q)$ and $i,j\in\mathbb{Z}$. Using (2.4) and the addition formula for the sine and cosine functions, we obtain

$$\begin{aligned} U_{i-j}(q)+U_{i+j}(q) &= U_{i-j}(\cos(u))+U_{i+j}(\cos(u)) \\ &= \frac{\sin((i-j+1)u)}{\sin(u)}+\frac{\sin((i+j+1)u)}{\sin(u)} \\ &= 2\frac{\sin((i+1)u)}{\sin(u)}\cos(ju) \\ &= 2U_i(\cos(u))T_j(\cos(u))=2U_i(q)T_j(q) \end{aligned}$$

and

$$\begin{aligned} T_{i-j}(q)+T_{i+j}(q) &= T_{i-j}(\cos(u))+T_{i+j}(\cos(u)) \\ &= \cos((i-j)u)+\cos((i+j)u) \\ &= 2\cos(iu)\cos(ju) \\ &= 2T_i(\cos(u))T_j(\cos(u))=2T_i(q)T_j(q). \end{aligned}$$

This completes the proof of (2.8). □

Observe that, in the particular case $j=1$, the equalities in (2.8) reduce to the recursive formulas in (2.3)

Remark 2.2.4. For the difference of two Chebyshev polynomials of the second kind, using the equality $-U_k = U_{-k-2}$, we can deduce the following identity:

$$\begin{aligned} U_{i+j} - U_{i-j} &= U_{i+j} + U_{-i+j-2} \\ &= U_{(j-1)+(i+1)} + U_{(j-1)-(i+1)} = 2T_{i+1}U_{j-1}. \end{aligned} \tag{2.10}$$

On the other hand, to compute the difference of two Chebyshev polynomials of the first kind, the following equality can be established:

$$T_{j-i}(q) - T_{j+i}(q) = 2(1-q^2)U_{j-1}(q)U_{i-1}(q). \tag{2.11}$$

To prove this, let $q \in]-1,1[$ be arbitrary, let $u := \arccos(q)$ and $i, j \in \mathbb{Z}$. Using (2.4) and the addition formula for the cosine function, we get

$$\begin{aligned} 2U_{j-1}(q)U_{i-1}(q) = 2U_{j-1}(\cos(u))U_{i-1}(\cos(u)) &= 2 \cdot \frac{\sin(ju)}{\sin(u)} \frac{\sin(iu)}{\sin(u)} \\ &= \frac{\cos((j-i)u) - \cos((j+i)u)}{\sin^2(u)} \\ &= \frac{T_{j-i}(q) - T_{j+i}(q)}{1-\cos^2(u)} = \frac{T_{j-i}(q) - T_{j+i}(q)}{1-q^2}. \end{aligned}$$

From here, (2.11) directly follows.

2.3 q-concave, q-convex and q-affine sequences

The next proposition shows that $\mathcal{A}_q(n|m)$ is a two dimensional subspace of $\mathcal{S}(n|m)$.

Proposition 2.3.1. [MP22] *A sequence $p \in \mathcal{S}(n|m)$ is q-affine if and only if there exist $a, b \in \mathbb{R}$ such that*

$$p_i := aU_{i-n}(q) + bT_{i-n}(q) \qquad (i \in \{n, \dots, m\}). \tag{2.12}$$

In addition, if $p \in \mathcal{A}_q(n|m)$, then, for all $i, j, k \in \{n, \dots, m\}$,

$$U_{k-j-1}(q)p_i + U_{j-i-1}(q)p_k = U_{k-i-1}(q)p_j. \tag{2.13}$$

In particular, for $i \in \{n, \dots, m\}$ *and* $j \in \{1, \dots, \min(i-n, m-i)\}$,

$$p_{i-j} + p_{i+j} = 2T_j(q)p_i. \tag{2.14}$$

Proof. First assume that $p = (p_n, \dots, p_m)$ is q-affine. Define

$$a := \frac{p_{n+1}}{q} - p_n, \qquad b := 2p_n - \frac{p_{n+1}}{q}.$$

We prove the equality (2.12) by induction with respect to i. Observe that $p_n = a + b = aU_0(q) + bT_0(q)$ and $p_{n+1} = a(2q) + bq = aU_1(q) + bT_1(q)$, which show that (2.12) holds for $i = n$ and $i = n+1$. Assume that we have proved (2.12) for $i \le \ell$, where $n+1 \le \ell \le m-1$. Then, applying the q-affinity of the sequence, the inductive hypothesis and finally the recursive property of Chebyshev polynomials, we obtain

$$\begin{aligned} p_{\ell+1} &= 2qp_\ell - p_{\ell-1} \\ &= 2q(aU_{\ell-n}(q) + bT_{\ell-n}(q)) - (aU_{\ell-1-n}(q) + bT_{\ell-1-n}(q)) \\ &= a(2qU_{\ell-n}(q) - U_{\ell-1-n}(q)) + b(2qT_{\ell-n}(q) - T_{\ell-1-n}(q)) \\ &= aU_{\ell+1-n}(q) + bT_{\ell+1-n}(q). \end{aligned}$$

This shows the validity of (2.12) for $i = \ell + 1$.

For the sufficiency part of our assertion, assume that (2.12) holds for some $a, b \in \mathbb{R}$. Then, by the recursive property of Chebyshev polynomials, for $i \in \{n+1, \dots, m-1\}$, we have that

$$\begin{aligned} p_{i+1} &= aU_{i+1-n}(q) + bT_{i+1-n}(q) \\ &= a(2qU_{i-n}(q) - U_{i-1-n}(q)) + b(2qT_{i-n}(q) - T_{i-1-n}(q)) \\ &= 2q(aU_{i-n}(q) + bT_{i-n}(q)) - (aU_{i-1-n}(q) + bT_{i-1-n}(q)) \\ &= 2qp_i - p_{i-1}, \end{aligned}$$

which proves that p is a q-affine sequence.

To verify the last two assertions let $p \in \mathcal{A}_q(n|m)$. Then, as we have seen it, (2.12) holds for some $a, b \in \mathbb{R}$.

Let first $i, j, k \in \{n, \dots, m\}$ be arbitrary. Then, applying Lemma 2.2.3, we get

$$U_{k-j-1}(q)U_{i-n}(q) + U_{j-i-1}(q)U_{k-n}(q) = U_{k-i-1}(q)U_{j-n}(q) \quad \text{and}$$
$$U_{k-j-1}(q)T_{i-n}(q) + U_{j-i-1}(q)T_{k-n}(q) = U_{k-i-1}T_{j-n}(q).$$

Multiplying the first and second equalities by a and b, respectively, and then adding them up side by side, we obtain

$$U_{k-j-1}(q)(aU_{i-n}(q) + bT_{i-n}(q)) + U_{j-i-1}(q)(aU_{k-n}(q) + bT_{k-n}(q)) = U_{k-i-1}(q)(aU_{j-n}(q) + bT_{j-n}(q)),$$

which, in view of (2.12), shows that (2.13) holds.

Finally, let $i \in \{n, \dots, m\}$ and $j \in \{1, \dots, \min(i-n, m-i)\}$. In view of (2.8), we have that

$$U_{i-j-n}(q) + U_{i+j-n}(q) = 2T_j(q)U_{i-n}(q),$$
$$T_{i-j-n}(q) + T_{i+j-n}(q) = 2T_j(q)T_{i-n}(q).$$

Multiplying the first and second equalities by a and b, respectively, and then adding them up side by side, we obtain

$$p_{i-j} + p_{i+j} = (aU_{i-j-n}(q) + bT_{i-j-n}(q)) + (aU_{i+j-n}(q) + bT_{i+j-n}(q)) = 2T_j(q)(aU_{i-n}(q) + bT_{i-n}(q)) = 2T_j(q)p_i.$$

This completes the proof of (2.14). □

In the following statement, we establish some properties of the class of q-concave (and hence of q-convex) sequences.

Proposition 2.3.2. [MP22] *The cone $\mathcal{C}_q^{\cap}(n|m)$ is closed with respect to the pointwise minimum and the cone $\mathcal{C}_q^{\cup}(n|m)$ is closed with respect to the pointwise maximum.*

Proof. To prove the statement for $\mathcal{C}_q^{\cap}(n|m)$, let $p, r \in \mathcal{C}_q^{\cap}(n|m)$ be arbitrary and denote $s := \min(p, r)$ (i.e., $s_i := \min(p_i, r_i)$ for all $i \in$

$\{n, \ldots, m\}$). Let $i \in \{n+1, \ldots, m-1\}$. Then, by the q-concavity of p and r, we have

$$s_{i-1} + s_{i+1} \le p_{i-1} + p_{i+1} \le qp_i \quad \text{and} \quad s_{i-1} + s_{i+1} \le r_{i-1} + r_{i+1} \le qr_i.$$

Therefore,

$$s_{i-1} + s_{i+1} \le \min(qp_i, qr_i) = q \min(p_i, r_i) = qs_i,$$

which shows that s is also q-concave. The proof of the statement for $\mathcal{C}_q^{\cup}(n|m)$ is analogous. □

As q-affine sequences are q-concave and also q-convex, we obtain that the pointwise minimum and maximum of a finite family of q-affine sequences are q-concave and also q-convex, respectively.

Proposition 2.3.3. [MP22] *Let $i, j, k \in \{n, \ldots, m\}$ with $i < j < k$. Assume that*

$$q \ge \cos\left(\frac{\pi}{\max(j-i, k-j)}\right). \tag{2.15}$$

Then, for all $p \in \mathcal{C}_q^{\cap}(n|m)$,

$$U_{k-j-1}(q)p_i + U_{j-i-1}(q)p_k \le U_{k-i-1}(q)p_j. \tag{2.16}$$

In particular, if $i \in \{n+1, \ldots, m-1\}$ and $j \in \{1, \ldots, \min(i-n, m-i)\}$ and

$$q > \cos\left(\frac{\pi}{j}\right), \tag{2.17}$$

then

$$p_{i-j} + p_{i+j} \le 2T_j(q)p_i. \tag{2.18}$$

Proof. We shall verify (2.16) by induction on $\ell := k - i$. If $\ell = 2$, that is, $j - i = k - j = 1$, then (2.16) is equivalent to the q-concavity of p, because $U_0(q) = 1$ and $U_1(q) = 2q$.

Assume that we have verified (2.16) for all $i < j < k$ with $k - i \le \ell$, where $\ell \ge 2$. Suppose that $k - i = \ell + 1 \ge 3$ and (2.15) holds. Then $\max(j - i, k - j) \ge 2$. We now distinguish two cases.

The first the case is when $j-i \geq 2$. Then $k-(i+1) = \ell$ and $j-i \leq k-i-1 \leq \ell$ and, using (2.15), it follows that

$$q \geq \cos\left(\frac{\pi}{\max(j-(i+1), k-j)}\right) \quad \text{and}$$
$$q \geq \cos\left(\frac{\pi}{\max((i+1)-i, j-(i+1))}\right).$$

Thus, applying the inductive hypotheses for the triplets $i+1 < j < k$ and for $i < i+1 < j$, we obtain

$$U_{k-j-1}(q)p_{i+1} + U_{j-i-2}(q)p_k \leq U_{k-i-2}(q)p_j,$$
$$U_{j-i-2}(q)p_i + U_0(q)p_j \leq U_{j-i-1}(q)p_{i+1}.$$

The inequality (2.15) shows that q is nonsmaller than the largest roots of U_{j-i-1} and U_{k-j-1}, hence $U_{j-i-1}(q) \geq 0$ and $U_{k-j-1}(q) \geq 0$. Multiplying the first inequality by $U_{j-i-1}(q)$, the second one by $U_{k-j-1}(q)$, and adding up the inequalities so obtained side by side, we get

$$U_{k-j-1}(q)U_{j-i-2}(q)p_i + U_{j-i-1}(q)U_{j-i-2}(q)p_k \leq \big(U_{j-i-1}(q)U_{k-i-2}(q) - U_{k-j-1}(q)U_0(q)\big)p_j.$$

On the other hand, applying Lemma 2.2.3 for the numbers $k-j-1 < k-i-2 < k-i-1$, we have that

$$U_{j-i-1}(q)U_{k-i-2}(q) = U_0(q)U_{k-j-1}(q) + U_{j-i-2}(q)U_{k-i-1}(q).$$

Therefore, the above inequality can be rewritten as

$$U_{k-j-1}(q)U_{j-i-2}(q)p_i + U_{j-i-1}(q)U_{j-i-2}(q)p_k \leq U_{j-i-2}(q)U_{k-i-1}(q)p_j.$$

By (2.15), q is strictly bigger than $\cos\left(\frac{\pi}{j-i-1}\right)$, which is the largest root of U_{j-i-2} if $j-i > 2$, therefore $U_{j-i-2}(q) > 0$. If $i-j = 2$, then $U_{j-i-2}(q) = U_0(q) = 1 > 0$. Now dividing the last inequality by this positive value side by side, we arrive at the desired inequality (2.16).

The proof in the second case when $k-j \geq 2$ is completely analogous, therefore it is omitted.

Finally, let $i \in \{n+1, \dots, m-1\}$ and $j \in \{1, \dots, \min(i-n, m-i)\}$ and assume that (2.17) is satisfied. We apply the previous statement to the triplet $(i-j, i, i+j)$. Then, also using identity (2.8), we get

$$U_{j-1}(q)p_{i-j} + U_{j-1}(q)p_{i+j} \leq U_{2j-1}(q)p_i = 2U_{j-1}(q)T_j(q)p_i. \quad (2.19)$$

In view of (2.17), we have that q is bigger than the largest root of U_{j-1} if $j \geq 2$, hence $U_{j-1}(q) > 0$. This inequality is obviously true if $j = 1$. Thus, after dividing (2.19) by $U_{j-1}(q)$ side by side, this inequality implies (2.18). □

Proposition 2.3.4. [MP22] *Let $j, k \in \{n, \dots, m\}$ with $j < k$. In addition, assume that*

$$q > \cos\left(\frac{\pi}{k-j}\right). \quad (2.20)$$

Let $p \in \mathcal{C}_q^{\cap}(n|m)$ and define

$$r_i := p_k \frac{U_{i-j-1}(q)}{U_{k-j-1}(q)} + p_j \frac{U_{k-i-1}(q)}{U_{k-j-1}(q)} \qquad (i \in \{n, \dots, m\}).$$

Then, $r = (r_n, \dots, r_m)$ is a q-affine sequence and, for $i \in \{n, \dots, m\}$,

$$r_i \begin{cases} \geq p_i & \text{if } i < j \text{ or } k < i. \\ = p_i & \text{if } i \in \{j, k\}. \\ \leq p_i & \text{if } j < i < k. \end{cases}$$

Proof. If $k - j = 1$, then $U_{k-j-1}(q) = U_0(q) = 1 > 0$. If $k - j \geq 2$, then q is bigger than the largest root of U_{k-j-1}. Therefore $U_{k-j-1}(q) > 0$ and hence the sequence (r_i) is well-defined. From the recursive formula (2.3) of Chebyshev polynomials of the second kind, for $i \in \{n+1, \dots, m-1\}$, it follows that

$$U_{(i-1)-j-1}(q) + U_{(i+1)-j-1}(q) = 2qU_{i-j-1}(q),$$
$$U_{k-(i-1)-1}(q) + U_{k-(i+1)-1}(q) = 2qU_{k-i-1}(q).$$

Multiplying theses equalities by $\frac{p_k}{U_{k-j-1}(q)}$ and by $\frac{p_j}{U_{k-j-1}(q)}$, respectively, and then adding them up side by side, we obtain that $r_{i-1} + r_{i+1} = 2qr_i$, which shows that (r_i) is a q-affine sequence.

If $i = j$, or $i = k$, then, by $U_{-1} = 0$, we can see that $r_j = p_j$ and $r_k = p_k$. Suppose first that $j < i < k$. From the equality (2.13) of the second assertion of Proposition 2.3.1 applied to the q-affine sequence (r_i), we get

$$U_{k-j-1}(q)r_i = U_{k-i-1}(q)r_j + U_{i-j-1}(q)r_k.$$

On the other hand, applying inequality (2.16) of Proposition 2.3.3 for the q-concave sequence (p_i), we get

$$U_{k-i-1}(q)p_j + U_{i-j-1}(q)p_k \leq U_{k-j-1}(q)p_i$$

Using that $r_j = p_j$ and $r_k = p_k$, it follows that

$$\begin{aligned} U_{k-j-1}(q)r_i &= U_{k-i-1}(q)r_j + U_{i-j-1}(q)r_k \\ &= U_{k-i-1}(q)p_j + U_{i-j-1}(q)p_k \leq U_{k-j-1}(q)p_i, \end{aligned}$$

which, by $U_{k-j-1}(q) > 0$ simplifies to the inequality $r_i \leq p_i$.

For the remaining inequalities, suppose first that $i < j$. By the q affinity of (r_i), the second assertion of Proposition 2.3.1 implies

$$U_{k-i-1}(q)r_j = U_{k-j-1}(q)r_i + U_{j-i-1}(q)r_k$$

and hence

$$U_{k-j-1}(q)r_i = U_{k-i-1}(q)r_j - U_{j-i-1}(q)r_k.$$

On the other hand, applying inequality (2.16) of Proposition 2.3.3 for the q-concave sequence (p_i), we get

$$U_{k-j-1}(q)p_i + U_{j-i-1}(q)p_k \leq U_{k-i-1}(q)p_j$$

and hence

$$U_{k-i-1}(q)p_j - U_{j-i-1}(q)p_k \geq U_{k-j-1}(q)p_i.$$

Combining these inequalities and using $r_j = p_j$ and $r_k = p_k$, we can conclude that

$$\begin{aligned} U_{k-j-1}(q)r_i &= U_{k-i-1}(q)r_j - U_{j-i-1}(q)r_k \\ &= U_{k-i-1}(q)p_j - U_{j-i-1}(q)p_k \geq U_{k-j-1}(q)p_i. \end{aligned}$$

This inequality, by $U_{k-j-1}(q) > 0$, is equivalent to $r_i \geq p_i$ as desired.

The proof of $r_i \geq p_i$ in the case $k < i$ is completely similar and therefore omitted. □

In the following proposition, we establish a characterization of q-concave sequences.

Proposition 2.3.5. [MP22] *Let $p \in \mathcal{S}(n|m)$. Then p is q-concave if and only if, for all $j \in \{n, \dots, m-1\}$, there exists $r \in \mathcal{A}_q(n|m)$ such that*

$$p_j = r_j, \quad p_{j+1} = r_{j+1}, \quad \textit{and} \quad p_i \leq r_i \quad \textit{for} \quad i \in \{n, \dots, m\}. \tag{2.21}$$

Proof. Assume first that p is q-concave and let $j \in \{n, \dots, m-1\}$. Then, with $k = j+1$, we can see that (2.20) holds, therefore applying Proposition 2.3.4, the sequence $r \in \mathcal{S}(n|m)$ defined by

$$r_i := p_{j+1} U_{i-j-1}(q) + p_j U_{j-i}(q)$$

is q-affine and satisfies all the conditions in (2.21).

To prove the sufficiency part of the assertion, let us assume that for $j \in \{n, \dots, m-1\}$, there exists q-affine sequence $r^j \in \mathcal{A}_q(n|m)$ such that

$$p_j = r^j_j, \quad p_{j+1} = r^j_{j+1}, \quad \text{and} \quad p_i \leq r^j_i \quad \text{for} \quad i \in \{n, \dots, m\}.$$

Then, it follows that

$$p_i = \min_{n \leq j \leq m-1} r^j_i,$$

which shows that p is the pointwise minimum of finitely many (in fact, $m-n$) q-affine sequences. Thus, by Proposition 2.3.2, it follows that p is q-concave. □

2.4 A minimax-type problem

Throughout this section, n, m are integers with $2 \leq m-n$ and we consider the following minimum problem: Let $M : \mathbb{R}_+^{m-n-1} \to \mathbb{R}_+$ be an

$(m-n-1)$-variable mean. Our aim is to find the largest nonnegative constant C_M such that, for all $p \in \mathcal{S}(n|m)$ with $p_n, p_m \geq 0$ and $p_{n+1}, \ldots, p_{m-1} > 0$,

$$C_M \leq M\left(\frac{p_n + p_{n+2}}{2p_{n+1}}, \ldots, \frac{p_{i-1} + p_{i+1}}{2p_i}, \ldots, \frac{p_{m-2} + p_m}{2p_{m-1}}\right).$$

By taking p as a constant sequence, one can see that the right hand side of this inequality then equals 1, hence it follows that $C_M \leq 1$. As we shall see below, this estimate can be essentially improved for several concrete means.

In the case when M is the $(m-n-1)$-variable arithmetic mean A_{m-n-1}, we can obtain the following result.

Proposition 2.4.1. [MP22] $C_A = \frac{m-n-2}{m-n-1}$, *that is, for all* $p \in \mathcal{S}(n|m)$ *with* $p_n, p_m \geq 0$ *and* $p_{n+1}, \ldots, p_{m-1} > 0$,

$$\frac{m-n-2}{m-n-1} \leq \frac{1}{m-n-1} \sum_{i=n+1}^{m-1} \frac{p_{i-1} + p_{i+1}}{2p_i} \tag{2.22}$$

and the constant on the left hand side is the best possible.

Proof. If $m - n = 2$, that is, $m = n + 2$, then the left hand side of (2.22) equals zero, thus, the inequality is trivial. On the other hand, for $(p_n, p_{n+1}, p_{n+2}) = (0, 1, 0)$ equality holds in (2.22). Thus, in the rest of the proof, we may assume that $m - n > 2$.

To prove (2.22), let $p \in \mathcal{S}(n|m)$ with $p_n, p_m \geq 0$ and $p_{n+1}, \ldots, p_{m-1} > 0$. Then (using the arithmetic-geometric mean inequality in the last step), we obtain

$$\begin{aligned}\sum_{i=n+1}^{m-1}\frac{p_{i-1}+p_{i+1}}{2p_i}&=\frac{p_n+p_{n+2}}{2p_{n+1}}+\sum_{i=n+2}^{m-2}\frac{p_{i-1}+p_{i+1}}{2p_i}+\frac{p_{m-2}+p_m}{2p_{m-1}}\\&\geq\frac{p_{n+2}}{2p_{n+1}}+\sum_{i=n+2}^{m-2}\left(\frac{p_{i-1}}{2p_i}+\frac{p_{i+1}}{2p_i}\right)+\frac{p_{m-2}}{2p_{m-1}}\\&=\sum_{i=n+1}^{m-2}\frac{1}{2}\left(\frac{p_{i+1}}{p_i}+\frac{p_i}{p_{i+1}}\right)\geq\sum_{i=n+1}^{m-2}\sqrt{\frac{p_{i+1}}{p_i}\cdot\frac{p_i}{p_{i+1}}}\\&=m-n-2.\end{aligned}$$

Dividing the above obtained inequality by $m-n-1$ side by side, we can see that (2.22) holds. On the other hand, for $(p_n,p_{n+1},\dots,p_{m-1},p_m)=(0,1,\dots,1,0)$ equality holds in (2.22), therefore, the left hand side of (2.22) is the largest possible, indeed. □

In order to reach a higher level of generality, for $r\in[-\infty,\infty]$ and $k\in\mathbb{N}$, we define the *k-variable rth power mean (or Hölder mean)* of the variables $u_1,\dots,u_k\in\mathbb{R}_+$ by

$$H_{r,k}(u_1,\dots,u_k):=\begin{cases}\min(x_1,\dots,x_k) & \text{if } r=-\infty,\\ \left(\dfrac{u_1^r+\dots+u_k^r}{k}\right)^{\frac{1}{r}} & \text{if } r\in\mathbb{R}\setminus\{0\},\\ \sqrt[k]{u_1\cdots u_k} & \text{if } r=0,\\ \max(x_1,\dots,x_k) & \text{if } r=\infty.\end{cases}$$

Obviously, the mean $H_{1,k}$ equals the k-variable arithmetic mean A_k and $H_{0,k}$ equals the k-variable geometric mean G_k. It is well known that, for all $k\in\mathbb{N}$ and $-\infty\leq r\leq s\leq\infty$, the comparison inequality $H_{r,k}\leq H_{s,k}$ holds. In particular, $G_k\leq A_k$, which is the celebrated inequality between the geometric and arithmetic means.

For the investigation of the more general problem in terms of power means, for $r\in\mathbb{R}$ and $k\in\mathbb{N}$, we introduce the function $F_{r,k}\colon\mathbb{R}_+^k\to\mathbb{R}_+$

by

$$F_{r,k}(u_1,\ldots,u_k) := u_1^r + \sum_{i=1}^{k-1}\Big(\frac{1}{u_i}+u_{i+1}\Big)^r + \frac{1}{u_k^r}.$$

Lemma 2.4.2. [MP22] *Let $r>0$ and $k\in\mathbb{N}$. Then*

$$F_{r,k} \geq \begin{cases} 2 & \text{if } k=1,\\ 2^{\frac{r+1}{2}}+(k-2)2^r+2^{\frac{r+1}{2}} & \text{if } k\geq 2,\ r\leq 1,\\ 2^r k^{1-r}\big(2^{\frac{1-r}{2r}}+(k-2)+2^{\frac{1-r}{2r}}\big)^r & \text{if } k\geq 2,\ r\geq 1, \end{cases} \tag{2.23}$$

and the estimates are sharp if $k\in\{1,2\}$ or $r=1$. Furthermore, for all $k\in\mathbb{N}$

$$F_{r,k} \geq k2^{r+\frac{1-r}{k}}, \tag{2.24}$$

which is also sharp if $k\in\{1,2\}$ or $r=1$. In the particular case when k is odd, we also have that

$$F_{r,k} \geq k+1, \tag{2.25}$$

which is sharp if $k=1$ and which is sharper than (2.23) and (2.24) if r is a sufficiently small positive number.

Proof. If $k=1$ then, by the arithmetic-geometric mean inequality, for all $u_1\in\mathbb{R}_+$, we easily get

$$F_{r,1}(u_1) = u_1^r+\frac{1}{u_1^r} = 2A_2\Big(u_1^r,\frac{1}{u_1^r}\Big) \geq 2G_2\Big(u_1^r,\frac{1}{u_1^r}\Big) = 2\sqrt{u_1^r\cdot\frac{1}{u_1^r}} = 2.$$

Observe that $F_{r,1}(1)=2$, hence the lower estimate 2 is best possible in this case.

Now assume that $r\leq 1$ and $k\geq 2$, and let $u_1,\ldots,u_k\in\mathbb{R}_+$ be arbitrary. Then, by the comparison inequality $H_{r,2}\leq H_{1,2}=A_2$, for all $i\in\{1,\ldots,k-1\}$, we get

$$\begin{aligned}\frac{1}{2}\Big(\frac{1}{u_i}+u_{i+1}\Big) &= A_2\Big(\frac{1}{u_i},u_{i+1}\Big)\\ &\geq H_{r,2}\Big(\frac{1}{u_i},u_{i+1}\Big) = \bigg(\frac{1}{2}\Big(\frac{1}{u_i^r}+(u_{i+1})^r\Big)\bigg)^{\frac{1}{r}}.\end{aligned}$$

Using this inequality and the arithmetic-geometric mean inequality at the last step, we obtain

$$\begin{aligned} F_{r,k}(u_1,\dots,u_k) &:= u_1^r + \sum_{i=1}^{k-1} 2^r \Big(\frac{1}{2}\Big(\frac{1}{u_i} + u_{i+1}\Big)\Big)^r + \frac{1}{u_k^r} \\ &\geq u_1^r + \sum_{i=1}^{k-1} 2^r \frac{1}{2}\Big(\frac{1}{u_i^r} + (u_{i+1})^r\Big) + \frac{1}{u_k^r} \\ &= u_1^r + 2^{r-1}\frac{1}{u_1^r} + \sum_{i=2}^{k-1} 2^{r-1}\Big(\frac{1}{u_i^r} + u_i^r\Big) + 2^{r-1}u_k + \frac{1}{u_k} \\ &= 2A_2\Big(u_1^r, 2^{r-1}\frac{1}{u_1^r}\Big) + \sum_{i=2}^{k-1} 2^r A_2\Big(\frac{1}{u_i^r}, u_i^r\Big) + 2A_2\big(2^{r-1}u_k, \frac{1}{u_k}\big) \\ &\geq 2G_2\Big(u_1^r, 2^{r-1}\frac{1}{u_1^r}\Big) + \sum_{i=2}^{k-1} 2^r G_2\Big(\frac{1}{u_i^r}, u_i^r\Big) + 2G_2\big(2^{r-1}u_k, \frac{1}{u_k}\big) \\ &= 2^{\frac{r+1}{2}} + (k-2)2^r + 2^{\frac{r+1}{2}}. \end{aligned}$$

This proves the assertion when $r \leq 1$ and $k \geq 2$. Finally, by arithmetic-geometric mean inequality again, we get

$$\begin{aligned} F_{r,k}(u_1,\dots,u_k) &\geq 2^{\frac{r+1}{2}} + (k-2)2^r + 2^{\frac{r+1}{2}} \\ &= kA_k\big(2^{\frac{r+1}{2}}, 2^r, \dots, 2^r, 2^{\frac{r+1}{2}}\big) \\ &\geq kG_k\big(2^{\frac{r+1}{2}}, 2^r, \dots, 2^r, 2^{\frac{r+1}{2}}\big) \\ &= k\Big(2^{(k-2)r+r+1}\Big)^{\frac{1}{k}} = k2^{r+\frac{1-r}{k}}, \end{aligned}$$

which shows that (2.24) is also valid.

In the case $r \geq 1$ and $k \geq 2$, using the comparison inequality $A_{2k} = H_{1,2k} \geq H_{\frac{1}{r},2k}$ and the 2-variable arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
F_{r,k}(u_1,\dots,u_k) &= u_1^r + \sum_{i=1}^{k-1} 2\cdot\frac{1}{2}\Big(\frac{1}{u_i}+u_{i+1}\Big)^r + \frac{1}{u_k^r}\\
&= 2kA_{2k}\bigg(u_1^r,\dots,\frac{1}{2}\Big(\frac{1}{u_i}+u_{i+1}\Big)^r,\frac{1}{2}\Big(\frac{1}{u_i}+u_{i+1}\Big)^r,\dots\frac{1}{u_k^r}\bigg)\\
&\geq 2kH_{\frac{1}{r},2k}\bigg(u_1^r,\dots,\frac{1}{2}\Big(\frac{1}{u_i}+u_{i+1}\Big)^r,\frac{1}{2}\Big(\frac{1}{u_i}+u_{i+1}\Big)^r,\dots\frac{1}{u_k^r}\bigg)\\
&= 2k\bigg(\frac{1}{2k}\bigg(u_1+\sum_{i=1}^{k-1} 2\cdot 2^{-\frac{1}{r}}\Big(\frac{1}{u_i}+u_{i+1}\Big)+\frac{1}{u_k}\bigg)\bigg)^r\\
&= (2k)^{1-r}\bigg(u_1+2^{1-\frac{1}{r}}\frac{1}{u_1}+\sum_{i=2}^{k-1}2^{1-\frac{1}{r}}\Big(\frac{1}{u_i}+u_i\Big)+2^{1-\frac{1}{r}}u_k+\frac{1}{u_k}\bigg)^r\\
&\geq (2k)^{1-r}\big(2\cdot 2^{\frac{r-1}{2r}}+2(k-2)2^{1-\frac{1}{r}}+2\cdot 2^{\frac{r-1}{2r}}\big)^r\\
&= 2^r k^{1-r}\big(2^{\frac{1-r}{2r}}+(k-2)+2^{\frac{1-r}{2r}}\big)^r.
\end{aligned}$$

This proves the assertion when $r\geq 1$ and $k\geq 2$. Finally, by arithmetic-geometric mean inequality again, we get

$$\begin{aligned}
F_{r,k}(u_1,\dots,u_k) &\geq 2^r k^{1-r}\big(2^{\frac{1-r}{2r}}+(k-2)+2^{\frac{1-r}{2r}}\big)^r\\
&= 2^r kA_k\big(2^{\frac{1-r}{2r}},1,\dots,1,2^{\frac{1-r}{2r}}\big)^r\\
&\geq 2^r kG_k\big(2^{\frac{1-r}{2r}},1,\dots,1,2^{\frac{1-r}{2r}}\big)^r\\
&= 2^r k\sqrt[k]{2^{\frac{1-r}{r}}} = k2^{r+\frac{1-r}{k}},
\end{aligned}$$

which shows that (2.24) is also valid.

If $k=2$, then the lower estimates (2.23) and (2.24) simplify to the inequality

$$F_{r,2}\geq 2^{\frac{3+r}{2}}.$$

On the other hand, with $u_1 := 2^{\frac{r-1}{2r}}$ and $u_2 := 2^{\frac{1-r}{2r}}$, one can see that

$$F_{r,2}(u_1,u_2) = u_1^r+\Big(\frac{1}{u_1}+u_2\Big)^r+\frac{1}{u_2^r} = 2^{\frac{r+1}{2}}+2^{\frac{1+r}{2}} = 2^{\frac{3+r}{2}},$$

which proves that the lower estimate $2^{\frac{3+r}{2}}$ is sharp.

If $r = 1$, then all the lower estimates simplify to the inequality

$$F_{1,k} \geq 2k,$$

which is attained at $u_1 = \cdots = u_k = 1$. This proves that the lower estimate $2k$ is sharp in this case.

Finally, we prove that (2.25) holds. This inequality is a consequence of (2.23) in the case $k = 1$. Thus, we may assume that $k \geq 3$ is odd. Then, for $u_1, \ldots, u_k \in \mathbb{R}_+$, we get

$$\begin{aligned}
&F_{r,k}(u_1, \ldots, u_k) \\
&= u_1^r + \left(\frac{1}{u_1} + u_2\right)^r + \sum_{i=2}^{k-2} \left(\frac{1}{u_i} + u_{i+1}\right)^r + \left(\frac{1}{u_{k-1}} + u_k\right)^r + \frac{1}{u_k^r} \\
&\geq u_1^r + \frac{1}{u_1^r} + \sum_{j=1}^{\frac{k-3}{2}} \left(\left(\frac{1}{u_{2j}} + u_{2j+1}\right)^r + \left(\frac{1}{u_{2j+1}} + u_{2j+2}\right)^r\right) + u_k^r + \frac{1}{u_k^r} \\
&\geq u_1^r + \frac{1}{u_1^r} + \sum_{j=1}^{\frac{k-3}{2}} \left(u_{2j+1}^r + \frac{1}{u_{2j+1}^r}\right) + u_k^r + \frac{1}{u_k^r} \\
&= \sum_{j=0}^{\frac{k-1}{2}} \left(u_{2j+1}^r + \frac{1}{u_{2j+1}^r}\right) \geq \sum_{j=0}^{\frac{k-1}{2}} 2 = k + 1.
\end{aligned}$$

If r tends to zero in (2.23), then the limit of the lower estimate is $2\sqrt{2} + k - 2$, which is smaller than $k + 1$, showing that (2.25) provides a better lower estimate than (2.23) for small positive values of r. □

Proposition 2.4.3. [MP22] *Let* $r > 0$. *Then*

$$C_{H_{r,m-n-1}} \geq \begin{cases} \frac{1}{2} & \textit{if } m = n + 3, \\ \left(\frac{2^{\frac{1-r}{2}} + (m-n-4) + 2^{\frac{1-r}{2}}}{m-n-1}\right)^{\frac{1}{r}} & \textit{if } m \geq n + 4,\ r \leq 1, \\ \left(\frac{m-n-2}{m-n-1}\right)^{\frac{1}{r}} \cdot \frac{2^{\frac{1-r}{2r}} + (m-n-4) + 2^{\frac{1-r}{2r}}}{m-n-2} & \textit{if } m \geq n + 4,\ 1 \leq r. \end{cases} \tag{2.26}$$

and the constant on the left hand side is the best possible if either $m \in \{n+3, n+4\}$ *or* $r = 1$*. In addition, if* $m - n$ *is odd, then*

$$C_{H_{r,m-n-1}} \geq \frac{1}{2}. \tag{2.27}$$

Proof. If $m - n = 2$, that is, $m = n + 2$, then the left hand side of (2.22) equals zero, thus, the inequality is trivial. On the other hand, for $(p_n, p_{n+1}, p_{n+2}) = (0, 1, 0)$ equality holds in (2.22). Thus, in the rest of the proof, we may assume that $m - n > 2$.

To prove (2.22), let $p \in \mathcal{S}(n|m)$ with $p_n, p_m \geq 0$ and $p_{n+1}, \dots, p_{m-1} > 0$. Then

$$\begin{aligned}
&2^r \sum_{i=n+1}^{m-1} \left(\frac{p_{i-1} + p_{i+1}}{2p_i}\right)^r \\
&\qquad = \left(\frac{p_n + p_{n+2}}{p_{n+1}}\right)^r + \sum_{i=n+2}^{m-2} \left(\frac{p_{i-1} + p_{i+1}}{p_i}\right)^r + \left(\frac{p_{m-2} + p_m}{p_{m-1}}\right)^r \\
&\qquad \geq \left(\frac{p_{n+2}}{p_{n+1}}\right)^r + \sum_{i=n+2}^{m-2} \left(\frac{p_{i-1}}{p_i} + \frac{p_{i+1}}{p_i}\right)^r + \left(\frac{p_{m-2}}{p_{m-1}}\right)^r \\
&\qquad = F_{r,m-n-2}\left(\frac{p_{n+2}}{p_{n+1}}, \dots, \frac{p_{m-3}}{p_{m-2}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\frac{1}{m-n-1} \sum_{i=n+1}^{m-1} \left(\frac{p_{i-1} + p_{i+1}}{2p_i}\right)^r\right)^{\frac{1}{r}} \\
&\qquad\qquad \geq \left(\frac{1}{2^r(m-n-1)} F_{r,m-n-2}\left(\frac{p_{n+2}}{p_{n+1}}, \dots, \frac{p_{m-3}}{p_{m-2}}\right)\right)^{\frac{1}{r}}.
\end{aligned}$$

If $m = n + 3$, then, by the $k = 1$ case of Lemma 2.4.2, we get

$$\left(\frac{1}{2} \sum_{i=n+1}^{n+2} \left(\frac{p_{i-1} + p_{i+1}}{2p_i}\right)^r\right)^{\frac{1}{r}} \geq \frac{1}{2}.$$

Applying Lemma 2.4.2, for $k := m-n-2 \geq 2$ and $0 < r \leq 1$, we get

$$\left(\frac{1}{m-n-1}\sum_{i=n+1}^{m-1}\left(\frac{p_{i-1}+p_{i+1}}{2p_i}\right)^r\right)^{\frac{1}{r}} \geq \left(\frac{2^{\frac{1-r}{2}}+(m-n-4)+2^{\frac{1-r}{2}}}{m-n-1}\right)^{\frac{1}{r}}.$$

Similarly, for $k := m-n-2 \geq 2$ and $r \geq 1$, it follows that

$$\left(\frac{1}{m-n-1}\sum_{i=n+1}^{m-1}\left(\frac{p_{i-1}+p_{i+1}}{2p_i}\right)^r\right)^{\frac{1}{r}} \geq \left(\frac{m-n-2}{m-n-1}\right)^{\frac{1}{r}} \cdot \frac{2^{\frac{1-r}{2r}}+(m-n-4)+2^{\frac{1-r}{2r}}}{m-n-2}.$$

To prove (2.27), assume that $m-n$ is odd. Then, applying the inequality (2.25) for $k = m-n-2$, we get

$$\left(\frac{1}{m-n-1}\sum_{i=n+1}^{m-1}\left(\frac{p_{i-1}+p_{i+1}}{2p_i}\right)^r\right)^{\frac{1}{r}} \geq \left(\frac{(m-n-2)+1}{2^r(m-n-1)}\right)^{\frac{1}{r}} = \frac{1}{2},$$

which was to be shown. □

In the case when M is the $(m-n-1)$-variable geometric mean G, we can establish the following result in which we will get an exact formula for the constant $C_{G_{m-n-1}}$.

Proposition 2.4.4. [MP22] $C_{G_{m-n-1}} = \frac{1+(-1)^{m-n-1}}{4}$, *that is, for all sequences* $p \in \mathcal{S}(n|m)$ *with* $p_n, p_m \geq 0$ *and* $p_{n+1}, \ldots, p_{m-1} > 0$,

$$\frac{1+(-1)^{m-n-1}}{4} \leq \sqrt[m-n-1]{\prod_{i=n+1}^{m-1}\frac{p_{i-1}+p_{i+1}}{2p_i}}. \tag{2.28}$$

and the constant on the left hand side is the best possible.

Proof. Assume first that $m-n$ is even. Then the left hand side of (2.28) equals zero, thus, the inequality is trivial. To show that the left hand side is optimal, define the sequence $p \in \mathcal{S}(n|m)$ by

$$p_{n+2i} := \varepsilon \quad (i \in \{0, \dots, \tfrac{m-n}{2}\}) \quad \text{and} \quad p_{n+2i+1} := 1,$$

where $i \in \{0, \dots, \frac{m-n-2}{2}\}$ and $\varepsilon > 0$ is an arbitrary positive number. Then, using that $m-n$ is even, we can obtain that

$$\prod_{i=n+1}^{m-1} \frac{p_{i-1} + p_{i+1}}{2p_i} = \prod_{i=n+1}^{m-1} \varepsilon^{(-1)^{i-(n+1)}} = \varepsilon.$$

Therefore, the rights hand side of (2.28) equals $\sqrt[m-n-1]{\varepsilon}$, which can be arbitrarily small. Hence, in this case, we obtain that $C_G = 0$.

Consider now the case when $m-n$ is odd and $m-n \geq 3$. Using that the product has an even number of factors, we get

$$\begin{aligned}\prod_{i=n+1}^{m-1} \frac{p_{i-1} + p_{i+1}}{2p_i} &= \prod_{j=0}^{\frac{m-n-3}{2}} \frac{p_{n+2j} + p_{n+2+2j}}{2p_{n+1+2j}} \cdot \frac{p_{n+1+2j} + p_{n+3+2j}}{2p_{n+2+2j}} \\ &\geq \prod_{j=0}^{\frac{m-n-3}{2}} \frac{p_{n+2+2j}}{2p_{n+1+2j}} \cdot \frac{p_{n+1+2j}}{2p_{n+2+2j}} = \frac{1}{2^{m-n-1}}.\end{aligned}$$

Taking the $(m-n-1)$th root of this inequality side by side, we obtain that (2.28) is also true in the case when $m-n$ is odd and $m-n \geq 3$.

To verify the sharpness of the left hand side of (2.28), let $\varepsilon > 0$ be arbitrary and, for $i \in \{n, \dots, m\}$, define

$$p_i := \begin{cases} \varepsilon^{\frac{m-i-1}{2}} & \text{if } i-n \text{ is even,} \\ \varepsilon^{\frac{i-n-1}{2}} & \text{if } i-n \text{ is odd.} \end{cases}$$

Then

$$\begin{aligned}\prod_{i=n+1}^{m-1}\frac{p_{i-1}+p_{i+1}}{2p_i} &= \prod_{j=0}^{\frac{m-n-3}{2}}\frac{p_{n+2j}+p_{n+2+2j}}{2p_{n+1+2j}}\cdot\frac{p_{n+1+2j}+p_{n+3+2j}}{2p_{n+2+2j}}\\ &= \prod_{j=0}^{\frac{m-n-3}{2}}\frac{\varepsilon^{\frac{m-n-2j-1}{2}}+\varepsilon^{\frac{m-n-2j-3}{2}}}{2\varepsilon^{j}}\cdot\frac{\varepsilon^{j}+\varepsilon^{j+1}}{2\varepsilon^{\frac{m-n-2j-3}{2}}}\\ &= \prod_{j=0}^{\frac{m-n-3}{2}}\left(\frac{\varepsilon+1}{2}\cdot\frac{1+\varepsilon}{2}\right)=\left(\frac{1+\varepsilon}{2}\right)^{m-n-1}.\end{aligned}$$

By taking ε arbitrarily small, we can see that the right hand side of the above equality can be arbitrarily close to $\frac{1}{2^{m-n-1}}$, which shows that the left hand side of (2.28) is a sharp lower bound for the right hand side. □

Proposition 2.4.5. [MP22] *$C_{H_{\infty,m-n-1}}=\cos\left(\frac{\pi}{m-n}\right)$, that is, for all sequences $p\in\mathcal{S}(n|m)$ with $p_n,p_m\geq 0$ and $p_{n+1},\dots,p_{m-1}>0$,*

$$\cos\left(\frac{\pi}{m-n}\right)\leq\max_{n+1\leq i\leq m-1}\frac{p_{i-1}+p_{i+1}}{2p_i}. \tag{2.29}$$

Moreover, with $p_i:=\sin\left(\frac{i-n}{m-n}\pi\right)$, the inequality (2.29) holds with equality.

Proof. Let

$$q:=\max_{n+1\leq i\leq m-1}\frac{p_{i-1}+p_{i+1}}{2p_i}.$$

Then, using the positivity of $p_1,\dots,p_n$, it follows that the sequence p is q-concave.

In the first part of the proof, we show that, for $k\in\{n,\dots,m-1\}$,

$$0\leq U_{k-n}(q)\qquad\text{and}\qquad U_{k-n-1}(q)p_{k+1}\leq U_{k-n}(q)p_k. \tag{2.30}$$

These inequalities are obvious for $k=n$ because $U_0(q)=1$ and $U_{-1}(q)=0\leq p_n$. Assume that we have proved (2.30) for some $k\in\{n,\dots,m-2\}$. Then, by the q-concavity of p, we have that

$$p_k+p_{k+2}\leq 2qp_{k+1}$$

Multiplying this inequality by $U_{k-n}(q) \geq 0$ and adding it to the second inequality in (2.30) side by side, we get

$$U_{k-n-1}(q)p_{k+1} + U_{k-n}(q)p_{k+2} \leq 2qU_{k-n}(q)p_{k+1},$$

which, by applying (2.3), implies

$$U_{k-n}(q)p_{k+2} \leq (2qU_{k-n}(q) - U_{k-n-1}(q))p_{k+1} = U_{k-n+1}(q)p_{k+1}.$$

This inequality shows that $U_{k-n+1}(q)$ is nonnegative and the second inequality in (2.30) is valid for $k+1$ (instead of k).

Based on the first inequality in (2.30), for $k \in \{n, \ldots, m-1\}$, we now show that

$$\cos\left(\frac{\pi}{k+1-n}\right) \leq q. \tag{2.31}$$

This is obvious if $k = n$, since q is nonnegative. If $k = n+1$, then (2.30) gives that $0 \leq U_1(q) = 2q$ and hence $q \geq 0 = \cos\left(\frac{\pi}{2}\right)$, which proves (2.31) in this case.

Now assume that (2.31) holds for some $k \in \{n+1, \ldots, m-2\}$. The two largest zeroes of U_{k+1-n} are $\cos\left(\frac{2\pi}{k+2-n}\right)$ and $\cos\left(\frac{\pi}{k+2-n}\right)$, furthermore $U_{k+1-n}(t) < 0$ if $\cos\left(\frac{2\pi}{k+2-n}\right) < t < \cos\left(\frac{\pi}{k+2-n}\right)$ and $U_{k+1-n}(t) \geq 0$ if $t \geq \cos\left(\frac{\pi}{k+2-n}\right)$. Observe that $\frac{\pi}{k+2-n} < \frac{\pi}{k+1-n} < \frac{2\pi}{k+2-n}$. Therefore, $\cos\left(\frac{2\pi}{k+2-n}\right) < \cos\left(\frac{\pi}{k+1-n}\right) < \cos\left(\frac{\pi}{k+2-n}\right)$. If q were smaller than $\cos\left(\frac{\pi}{k+2-n}\right)$, then, by the inductive assumption, $\cos\left(\frac{\pi}{k+1-n}\right) \leq q < \cos\left(\frac{\pi}{k+2-n}\right)$ and hence $U_{k+1-n}(q) < 0$, which contradicts (2.30) (if it is applied for $k+1$ instead of k. Thus must be nonsmaller than $\cos\left(\frac{\pi}{k+2-n}\right)$, which shows that (2.31) is valid for $k+1$.

Finally, applying (2.31) for $k = m-1$, we can see that $\cos\left(\frac{\pi}{m-n}\right) \leq q$, which proves that (2.29) holds.

To verify that (2.29) is sharp, let $p_i := \sin\left(\frac{i-n}{m-n}\pi\right)$ for $i \in \{n, \ldots, m\}$. Then, for $i \in \{n+1, \ldots, m-1\}$,

$$\begin{aligned}
\frac{p_{i-1} + p_{i+1}}{2p_i} &= \frac{\sin\left(\frac{i-1-n}{m-n}\pi\right) + \sin\left(\frac{i+1-n}{m-n}\pi\right)}{2\sin\left(\frac{i-n}{m-n}\pi\right)} \\
&= \frac{2\sin\left(\frac{i-n}{m-n}\pi\right)\cos\left(\frac{\pi}{m-n}\right)}{2\sin\left(\frac{i-n}{m-n}\pi\right)} = \cos\left(\frac{\pi}{m-n}\right),
\end{aligned}$$

which shows that (2.29) holds with equality for this particular sequence p. □

As a curiosity, we can obtain the following inequality for the cosine function.

Corollary 2.4.6. [MP22] *For* $m \geq 3$*,*

$$\frac{m-4+\sqrt{2}}{m-2} \leq \cos\left(\frac{\pi}{m}\right), \tag{2.32}$$

and equality holds if $m = 4$*.*

Proof. If $m = 3$, then the inequality is equivalent to $\sqrt{2} - 1 \leq \frac{1}{2}$, which is obviously true.

If $m \geq 4$ and $r \geq 1$, then, in view of Proposition 2.4.3, for all sequences $p \in \mathcal{S}(0|m)$ with $p_0, p_m \geq 0$ and $p_1, \ldots, p_{m-1} > 0$, we have that

$$\begin{aligned}
&\left(\frac{m-2}{m-1}\right)^{\frac{1}{r}} \cdot \frac{2^{\frac{1-r}{2r}} + (m-4) + 2^{\frac{1-r}{2r}}}{m-2} \\
&\quad \leq C_{H_{r,m-1}}\left(\frac{p_0+p_2}{2p_1}, \ldots, \frac{p_{i-1}+p_{i+1}}{2p_i}, \ldots, \frac{p_{m-2}+p_m}{2p_{m-1}}\right) \\
&\quad \leq C_{H_{\infty,m-1}}\left(\frac{p_0+p_2}{2p_1}, \ldots, \frac{p_{i-1}+p_{i+1}}{2p_i}, \ldots, \frac{p_{m-2}+p_m}{2p_{m-1}}\right) \\
&\quad = \max_{1 \leq i \leq m-1} \frac{p_{i-1}+p_{i+1}}{2p_i}.
\end{aligned}$$

By taking the limit $r \to \infty$, it follows that

$$\frac{m-4+\sqrt{2}}{m-2} \leq \max_{1 \leq i \leq m-1} \frac{p_{i-1}+p_{i+1}}{2p_i}.$$

In particular, with $p_i := \sin\left(\frac{i}{m}\pi\right)$, we get that

$$\frac{m-4+\sqrt{2}}{m-2} \leq \cos\left(\frac{\pi}{m}\right),$$

which was to be shown.

For $m = 4$, both sides of the inequality are equal to $\frac{\sqrt{2}}{2}$ and hence equality holds in (2.32). □

2.5 An application of q-concave sequences

In this section, we consider a selfmap of the space $\mathbb{R}^n$ which originates from the investigation of approximately convex real functions. Our main aim here is to prove that it has a unique fixed point.

In what follows, we will adopt the following convention: For an arbitrary sequence $a \in \mathcal{S}(1|n)$, let a be extended to be in $\mathcal{S}(0|n+1)$ by setting $a_0 := 0$ and $a_{n+1} := 0$. For $n \in \mathbb{N}$ and for a vector $\gamma = \big(\gamma_1, \dots, \gamma_{\lfloor \frac{n+1}{2} \rfloor}\big) \in \mathbb{R}^{\lfloor \frac{n+1}{2} \rfloor}$, we define the map $\mathcal{T}_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\big(\mathcal{T}_\gamma(a)\big)_i := \min_{1 \le j \le \min(i, n+1-i)} \Big(\frac{a_{i-j} + a_{i+j}}{2} + \gamma_j\Big),$$

where $a \in \mathbb{R}^n$ and $i \in \{1, \dots, n\}$).

In order to make the map $\mathcal{T}_\gamma$ a contraction with respect to a suitable norm on $\mathbb{R}^n$, we construct new norms in terms of positive sequences. Let $|\cdot|_\infty$ denote the maximum norm on $\mathbb{R}^n$, which is defined as $|a|_\infty := \max_{1 \le i \le n} |a_i|$. If $p \in \mathcal{S}(1|n)$ is a sequence with positive members, then we define $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}$ by

$$\|a\|_p := \max_{1 \le i \le n} p_i^{-1} |a_i| = |p^{-1} a|_\infty \qquad (a \in \mathbb{R}^n).$$

It is easy to check that $\|\cdot\|_p$ is a norm, and hence $\mathbb{R}^n$ is a Banach space with respect to $\|\cdot\|_p$.

Theorem 2.5.1. [MP22] *Let $p \in \mathcal{S}(1|n)$ be a sequence with positive members and define*

$$q := \max_{1 \le i \le n} \frac{p_{i-1} + p_{i+1}}{2p_i} \qquad \text{and} \qquad q^* := \begin{cases} q & \text{if } q \le 1, \\ T_{\lfloor \frac{n+1}{2} \rfloor}(q) & \text{if } q > 1. \end{cases} \tag{2.33}$$

Then, for all $\gamma \in \mathbb{R}^{\lfloor \frac{n+1}{2} \rfloor}$, the mapping $\mathcal{T}_\gamma$ is q^*-Lipschitzian on the normed space $(\mathbb{R}^n, \|\cdot\|_p)$. In particular, if p is strictly concave, then $\mathcal{T}_\gamma$ is a contraction on the normed space $(\mathbb{R}^n, \|\cdot\|_p)$.

Proof. First of all, for all $k \in \mathbb{N}$, we prove that the function $\min\colon \mathbb{R}^k \to \mathbb{R}$ is Lipschitzian with respect to the maximum norm $|\cdot|_\infty$ with Lipschitz modulus $L = 1$. Indeed, if $x, y \in \mathbb{R}^k$, then

$$\min(x) = \min_{1\le i\le k} x_i \le \min_{1\le i\le k} \big(y_i + |x_i - y_i|\big) \le \min_{1\le i\le k} \big(y_i + |x-y|_\infty\big) \le \min_{1\le i\le k} y_i + |x-y|_\infty = \min(y) + |x-y|_\infty.$$

Interchanging the roles of x and y in the above argument and then combining the two inequalities so obtained, we get that

$$\big|\min(x) - \min(y)\big| \le |x-y|_\infty,$$

which proves our statement.

The definition of the number q in (2.33) implies that $p \in \mathcal{S}(0|n+1)$ is a q-concave sequence, and according to Proposition 2.4.5, $q \ge \cos\left(\frac{\pi}{n+1}\right)$. Then, $q > \cos\left(\frac{\pi}{j}\right)$ for all $j \in \{1,\dots,n\}$. Therefore, applying the last inequality of Proposition 2.3.3, we obtain that, for all $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,\min(i, n+1-i)\}$, (2.18) holds. Hence, on the same domain,

$$\frac{p_{i-j} + p_{i+j}}{2p_i} \le T_j(q). \tag{2.34}$$

If $i \in \{1,\dots,n\}$, then $\min(i, n+1-i) \le \frac{i+(n+1-i)}{2} = \frac{n+1}{2}$, which shows that the maximal value of j is $\left\lfloor\frac{n+1}{2}\right\rfloor$. Therefore, (2.34) implies that, for all $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,\min(i, n+1-i)\}$,

$$\frac{p_{i-j} + p_{i+j}}{2p_i} \le \max\left\{T_1(q),\dots,T_{\lfloor\frac{n+1}{2}\rfloor}(q)\right\}. \tag{2.35}$$

In what follows, we show that the right hand side of this inequality equals q^*.

If $\cos\left(\frac{\pi}{n+1}\right) \le q < 1$, then $0 < \arccos(q) \le \frac{\pi}{n+1}$. Therefore, according to the first part of Lemma 2.2.1, the sequence $T_j(q)$ is decreasing for $j \in \{0,\dots,n+1\}$ and hence $T_j(q) \le T_1(q) = q = q^*$ for all $j \in \{1,\dots,\left\lfloor\frac{n+1}{2}\right\rfloor\}$. If $q = 1$, then $T_j(q) = 1 = q^*$ for all $j \in \mathbb{N}$. On the other hand, $1 < q$, then according to the second part of Lemma 2.2.1, the sequence $(T_i(q))_{i=1}^\infty$ is increasing and hence $T_j(q) \le T_{\lfloor \frac{n+1}{2} \rfloor}(q) = q^*$ holds for all $j \in \{1, \dots, \lfloor \frac{n+1}{2} \rfloor\}$.

Observe that, by the definition of the norm $\|\cdot\|_p$, for every $a \in \mathbb{R}^n$, we have that $|a_i| \le p_i \|a\|_p$ is valid for $i \in \{0, 1, \dots, n, n+1\}$. Now let $i \in \{1, \dots, n\}$ be fixed. Using the Lipschitz property of the minimum function with $k := \min(i, n+1-i)$ and the inequality (2.35), for all $a, b \in \mathbb{R}^n$, we get

$$\begin{aligned}
&p_i^{-1} \big| \big(\mathcal{T}_\gamma(a)\big)_i - \big(\mathcal{T}_\gamma(b)\big)_i \big| \\
&\quad = p_i^{-1} \bigg| \min_{1 \le j \le \min(i,n+1-i)} \Big(\frac{a_{i-j} + a_{i+j}}{2} + \gamma_j \Big) \\
&\qquad\quad - \min_{1 \le j \le \min(i,n+1-i)} \Big(\frac{b_{i-j} + b_{i+j}}{2} + \gamma_j \Big) \bigg| \\
&\quad \le p_i^{-1} \max_{1 \le j \le \min(i,n+1-i)} \bigg| \Big(\frac{a_{i-j} + a_{i+j}}{2} + \gamma_j \Big) - \Big(\frac{b_{i-j} + b_{i+j}}{2} + \gamma_j \Big) \bigg| \\
&\quad \le \max_{1 \le j \le \min(i,n+1-i)} \frac{|a_{i-j} - b_{i-j}| + |a_{i+j} - b_{i+j}|}{2p_i} \\
&\quad \le \max_{1 \le j \le \min(i,n+1-i)} \frac{p_{i-j} + p_{i+j}}{2p_i} \|a - b\|_p \\
&\quad \le \max_{1 \le j \le \min(i,n+1-i)} T_j(q) \|a - b\|_p \le q^* \|a - b\|_p.
\end{aligned}$$

Now, upon taking the maximum with respect to $i \in \{1, \dots, n\}$, we arrive at

$$\big\| \mathcal{T}_\gamma(a) - \mathcal{T}_\gamma(b) \big\|_p \le q^* \|a - b\|_p,$$

which completes the proof of the q^*-Lipschitz property of $\mathcal{T}_\gamma$ on the space $(\mathbb{R}^n, \|\cdot\|_p)$.

If the sequence p is strictly concave, then it is q-concave with some $q < 1$. Therefore, the q-Lipschitz property of $\mathcal{T}_\gamma$ shows that the map $\mathcal{T}_\gamma$ is a q-contraction on $(\mathbb{R}^n, \|\cdot\|_p)$. □

Corollary 2.5.2. [MP22] *For all $\gamma \in \mathbb{R}^{\lfloor \frac{n+1}{2} \rfloor}$, the mapping $\mathcal{T}_\gamma \colon \mathbb{R}^n \to \mathbb{R}^n$ has a unique fixed point in $\mathbb{R}^n$.*

Proof. Let $p_i := i(n+1-i)$ for $i \in \{0, \ldots, n+1\}$. Then, by the geometric mean-arithmetic mean inequality, we have that $p_i \leq \left(\frac{n+1}{2}\right)^2$. Thus, for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, \min(i, n+1-i)\}$, we have

$$\begin{aligned}
\frac{p_{i-j} + p_{i+j}}{2p_i} &= \frac{(i-j)(n+1-i+j) + (i+j)(n+1-i-j)}{2i(n+1-i)} \\
&= \frac{2i(n+1) - 2i^2 - 2j^2}{2i(n+1-i)} = \frac{i(n+1) - i^2 - j^2}{i(n+1-i)} \\
&\leq \frac{i(n+1-i) - 1}{i(n+1-i)} = \frac{p_i - 1}{p_i} \\
&\leq \frac{\left(\frac{n+1}{2}\right)^2 - 1}{\left(\frac{n+1}{2}\right)^2} = \frac{n^2 + 2n - 3}{n^2 + 2n + 1} = \frac{(n-1)(n+3)}{(n+1)^2}.
\end{aligned}$$

Therefore, the sequence $p \in \mathcal{S}(0|n+1)$ is q-concave with $q = \frac{(n-1)(n+3)}{(n+1)^2} < 1$. According to Theorem 2.5.1, the mapping $\mathcal{T}_\gamma$ is a q-contraction on $(\mathbb{R}^n, \|\cdot\|_p)$. Therefore, by the Banach Fixed Point theorem, it possesses a unique fixed point. □

Summary

Summary of Chapter 1

An extension of the Rådström cancellation theorem to cornets

Introduction

In the theory of convex sets, a basic Cancellation Principle was discovered by Rådström [R52b] in 1952. The Lemma 2 of his paper states that the inclusion

$$A + B \subseteq C + B$$

implies $A \subseteq C$ provided that A, B, C are nonempty subsets of a normed space X, C is closed and convex and B is bounded.

This lemma turned out to be a basic tool in various fields and hundreds of papers have used it by now. For instance, in nonsmooth analysis [CGT10,GGM16,GM18,HN17,D15], optimization theory [CKR14], theory of convex sets and functions [DMM11,GKKU14,GPPU12,GPU10a, GP15,GU14,VN12,VN15,K14], set-valued analysis [CT13,KPR15,K19], [LMNS14,P09,O17,P13], set-valued differential equations [BG09,PS14, M15], set-valued functional equations [BMP18,M12,S13,S19], iteration theory [AN16,AGMM10,SS12,XNZ11,G12], etc.

Cornets and convexity properties in cornets

Definition 1.1. [MP21]. An ordered triplet $(X, +, \preceq)$ is called an *ordered commutative semigroup* if

(i) $(X, +)$ is a commutative unital semigroup with a unit element 0;
(ii) $(X, \preceq)$ is a partially ordered set, that is, $\preceq$ is a reflexive, antisymmetric and transitive binary relation on X;
(iii) For all $x, y, z \in X$ with $x \preceq y$, the inequality $x + z \preceq y + z$ holds.

If the partially ordered set $(X, \preceq)$ is complete, i.e., every nonempty lower bounded subset of X has a greatest lower bound, then $(X, +, \preceq)$ is called a *complete ordered commutative semigroup*.

In a semigroup $(X, +)$, we naturally have the multiplication by natural numbers which is defined recursively by

$$1 \cdot x := x, \qquad (n+1) \cdot x := n \cdot x + x \qquad (n \in \mathbb{N}).$$

If the semigroup is unital, then we also define $0 \cdot x := 0$.

In the next definition we present the central concept of our paper.

Definition 1.2. [MP21]. An ordered quadruple $(X, +, *, \preceq)$ is called a *cornet* if $(X, +, \preceq)$ is an ordered commutative semigroup and "$*$" is a multiplication of the elements of X by positive integers such that the following conditions hold:

(i) For all $n, m \in \mathbb{N}$ and $x \in X$, $(nm) * x = n * (m * x)$;
(ii) For all $n \in \mathbb{N}$ and $x, y \in X$, $n * (x + y) = n * x + n * y$;
(iii) For all $n, m \in \mathbb{N}$ and $x \in X$, $(n + m) * x \preceq n * x + m * x$;
(iv) For all $n \in \mathbb{N}$ and $x, y \in X$, the inequality $x \preceq y$ holds if and only if $n * x \preceq n * y$;
(v) $1 * x = x$;
(vi) $n * 0 = 0$.

If the partially ordered set $(X, \preceq)$ is complete, then $(X, +, *, \preceq)$ is called a *complete cornet*. A unital subsemigroup $(S, +)$ of a cornet $(X, +, *, \preceq)$ which is also closed with respect to the multiplication $*$ is called a *subcornet* of $(X, +, *, \preceq)$ with the ordering restricted to S.

It is obvious that if $(X, +)$ is a commutative unital semigroup such that, for all $n \in \mathbb{N}$, the mapping $n \mapsto n \cdot x$ is injective, then $(X, +, \cdot, =)$ is a cornet.

Definition 1.3. [MP21]. Let $(X, +, *, \preceq)$ be a cornet and $n \in \mathbb{N}$. An element $x \in X$ will be called *n-convex* if it fulfills the equality $n * x = n \cdot x$. For fixed elements $x \in X$ and $n \in \mathbb{N}$, we introduce the notations

$$C_x := \{n \in \mathbb{N} \mid x \text{ is } n\text{-convex}\} \quad \text{and} \quad C^n := \{x \in X \mid x \text{ is } n\text{-convex}\},$$

respectively. If $C_x = \mathbb{N}$, i.e., if x is n-convex for all $n \in \mathbb{N}$, then we say that x is *convex*. The *n-convex hull* of an element $x \in X$, denoted as $\mathrm{conv}_n(x)$, is an element $y \in C^n$ such that $x \preceq y$ and, whenever $x \preceq z \in C^n$, then $y \preceq z$.

In general, the n-convex hull of an element may not exist, but if it exists, then it is unique. In order to formulate conditions which are sufficient for the existence, we say that the $*$-multiplication in a complete cornet $(X, +, *, \preceq)$ is *n-continuous* (with respect to the ordering "$\preceq$") if, for all nonempty lower bounded subsets $H \subseteq X$, we have

$$\inf(n * H) = n * \inf(H).$$

Proposition 1.4. [MP21]. *Let $n \in \mathbb{N}$ and let $(X, +, *, \preceq)$ be a complete cornet in which the $*$-multiplication is n-continuous. Then $(C^n, +, *, \preceq)$ is a complete subcornet of the cornet $(X, +, *, \preceq)$. Furthermore, for every element $x \in X$, x admits an n-convex hull if and only if it has an n-convex majorant.*

In a cornet $(X, +, *, \preceq)$, let K^n denote the collection of those elements which have an n-convex hull and let M^n denote the set of those elements that have an n-convex majorant. Obviously, we have $C^n \subseteq$

$K^n \subseteq M^n$. Using this terminology, the previous proposition asserts that if $(X, +, *, \preceq)$ is a complete cornet in which the $*$-multiplication is n-continuous, then $K^n = M^n$.

To illustrate the rich applicability of the above concepts, we provide the most basic examples for cornets in the subsequent three propositions. For these definitions, we introduce the notion of wedge in abelian group setting.

Definition 1.5. [MP21]. If $(G, +)$ is an abelian semigroup and $n \in \mathbb{N}$, then for a subset $S \subseteq G$, define

$$n^{-1}(S) := \{x \in G \mid n \cdot x \in S\}.$$

A subsemigroup S of the group $(G, +)$ is said to be *n-divisible* if, for all $x \in S$, the set $n^{-1}(\{x\}) \cap S$ is nonempty. If this set is a singleton, then S is called *uniquely n-divisible* and its unique element will be denoted by x/n.

In a unital abelian semigroup G, a subset $W \subseteq G$ is called a *wedge* if the following properties are satisfied:

(i) W is a unital subsemigroup of G.
(ii) If $u, v \in W$ such that $u + v = 0$, then $u = v = 0$.
(iii) For all $n \in \mathbb{N}$, the inverse image $n^{-1}(W)$ is contained in W.

In terms of a wedge $W \subseteq G$, we can define a partial order $\preceq_W$ in the following way: For $x, y \in G$, we say that $x \preceq_W y$ if $y \in x + W$. It immediately follows that $\preceq_W$ is a reflexive, and transitive relation on G. If, in addition, G is cancellative (which is always the case if G is group), then $\preceq_W$ is antisymmetric and hence it is a partial order on G.

Proposition 1.6. [MP21]. *Let $(G, +)$ be an abelian group and let $W \subseteq G$ be a wedge. Then, for a subsemigroup S of G containing W, the quadruple $(S, +, \cdot, \preceq_W)$ is a cornet in which every element is n-convex for all $n \in \mathbb{N}$. In particular, by taking $W := \{0\}$, it follows that $(G, +, \cdot, =)$ is a cornet.*

Proposition 1.7. [MP21]. *Let* $(G,+)$ *be an abelian group,* W *be a wedge and let* S *be a subsemigroup of* G *containing* W*. Let* $P_W(S)$ *denote the collection of all nonempty* W*-invariant subsets* A *of* S*, which means that* $A+W\subseteq A$ *holds. Define the operations* $+$ *and* $*$ *by:*

$$\begin{aligned} A+B &:= \{a+b \mid a\in A,\, b\in B\} && (A,B\in P_W(S)),\\ n*A &:= \{n\cdot a+w \mid a\in A,\, w\in W\} && (A\in P_W(S),\, n\in\mathbb{N}). \end{aligned} \tag{1.1}$$

Then $(P_W(S),+,*,\subseteq)$ *is a complete cornet with the unit element* W*. Furthermore, the mapping*

$$\varphi(x) := x+W \qquad (x\in S)$$

is an injective order reversing homomorphic mapping of $(S,+,\cdot,\preceq_W)$ *into the cornet* $(P_W(S),+,*,\subseteq)$*. In addition, if* $n\in\mathbb{N}$ *and* W *is* n*-divisible, then* $A\in P_W(S)$ *is* n*-convex if and only if, for all* $x_1,\dots,x_n\in A$*, we have*

$$n^{-1}(\{x_1+\cdots+x_n\})\cap A\neq\emptyset.$$

Proposition 1.8. [MP21]. *Let* $(G,+)$ *be an abelian group,* W *be a wedge and let* S *be a uniquely divisible subsemigroup of* G *which contains* W*. Let, for* $p\in\,]0,1]$*,*

$$F_W^p(S) := \{f: S\to[0,1] \mid \sup f\geq p \text{ and } f \text{ is } W\text{-nondecreasing}\} \tag{1.2}$$

and define the addition and the scalar multiplication in $F_W^p(S)$ *by*

$$\begin{aligned} (f\oplus g)(x) &:= \sup_{\substack{u,v\in S\\ u+v=x}} \min\big(f(u),g(v)\big),\\ (n\odot f)(x) &:= f\left(\frac{x}{n}\right) && (f,g\in F_W^p(S),\, x\in S,\, n\in\mathbb{N}). \end{aligned} \tag{1.3}$$

Finally, let $\leq$ *denote the pointwise ordering in* $F_W^p(S)$*. Then the quadruple* $(F_W^p(S),\oplus,\odot,\leq)$ *is a complete cornet whose unit element is the characteristic function of the wedge* W*. Furthermore, the mapping*

$$\Phi(A) := \chi_A \qquad (A\in P_W(S))$$

is an injective cornet-preserving mapping of $(P_W(S), +, *, \subseteq)$ *into the quadruple* $(F_W^1(S), \oplus, \odot, \leq)$. *In addition, a function* $f \in F_W^p(S)$ *is* n*-convex if and only if it is* n*-quasiconcave, i.e., for all* $x_1, \dots, x_n \in S$,

$$\min(f(x_1), \dots, f(x_n)) \leq f\Big(\frac{x_1 + \cdots + x_n}{n}\Big).$$

Topological notions and boundedness in cornets

In a natural way, we can introduce the notions of nonnegative and Archimedean elements in a cornet with the following definition.

Definition 1.9. [MP21]. In a cornet $(X, +, *, \preceq)$ an element $x \in X$ is said to be *nonnegative* if $0 \preceq x$ holds. The element x is called *Archimedean*, denoted by $0 \prec x$, if, for all $u \in X$, there exists $n_0 \in \mathbb{N}$ such that $0 \preceq u + n * x$ for all $n_0 \leq n$. The set of all nonnegative and Archimedean elements in X will be denoted by $X_\preceq$ and $X_\prec$, respectively.

The properties of nonnegative and Archimedean elements are established in the following assertion.

Proposition 1.10. [MP21]. *Let* $(X, +, *, \preceq)$ *be a cornet. Then* $X_\prec$ *is contained in* $X_\preceq$ *and*

$$X_\prec + X_\preceq \subseteq X_\prec.$$

In addition, $X_\prec$ *and* $X_\preceq$ *are subcornets of* $(X, +, *, \preceq)$.

In what follows, we introduce the notions of continuity of the addition, boundedness and closedness with respect to a subsemigroup of Archimedean elements. For comparison, we recall first the standard topological concepts for abelian groups.

Definition 1.11. [MP21]. If $(G, +)$ is an abelian group and $\mathcal{T}$ is a Hausdorff topology on G, then we say that $(G, \mathcal{T}, +)$ is a *topological group* if the $(x, y) \mapsto x - y$ is a continuous map of $G \times G$ into G. A subset $U \subseteq G$ is said to be *convex* if, for all $n \in \mathbb{N}$,

$$\{u_1 + \cdots + u_n \mid u_1, \dots, u_n \in U\} = \{n \cdot u \mid u \in U\}.$$

We say that G is *locally convex* if every neighborhood of 0 contains a convex neighborhood of 0. A subset $H \subseteq G$ is said to be *topologically bounded* if, for all neighborhood U of 0, there exists $n \in \mathbb{N}$ such that $H \subseteq \{u_1 + \cdots + u_n \mid u_1, \ldots, u_n \in U\}$.

Definition 1.12. [MP21]. Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$. We say that the *addition is $\mathcal{A}$-continuous* if, for all $a \in \mathcal{A}$, there exists $b \in \mathcal{A}$ such that $b + b \preceq a$ holds. We say that an element $x \in X$ is *$\mathcal{A}$-bounded* if, for all $a \in \mathcal{A}$, there exists $n_0 \in \mathbb{N}$ such that $x \preceq n * a$ for all $n \geq n_0$.

Proposition 1.13. [MP21]. *Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$ such that the addition is $\mathcal{A}$-continuous. Then the $\mathcal{A}$-bounded elements form a subcornet of $(X, +, *, \preceq)$.*

Definition 1.14. [MP21]. Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$. Given an element $x \in X$, we say that $y \in X$ is the *$\mathcal{A}$-closure* of x if $y \preceq x + a$ holds for all $a \in A$ and, y is the largest element of X with is property, i.e., if $z \preceq x + a$ holds for all $a \in A$, then $z \preceq y$. It is clear that the $\mathcal{A}$-closure of an element, if exists, is unique and is denoted by $\mathrm{cl}_{\mathcal{A}}(x)$. An element x is called *$\mathcal{A}$-closed* if $x = \mathrm{cl}_{\mathcal{A}}(x)$. The set of all elements of X which has an $\mathcal{A}$-closure will be denoted by $\mathrm{Cl}_{\mathcal{A}}$.

In the subsequent propositions, we consider the cornets that were introduced in Propositions 1.6, 1.7 and 1.8, and we determine all bounded, closed and Archimedean elements in these structures.

Proposition 1.15. [MP21]. *Let $(G, +)$ be a topological abelian group such that there is no proper open subgroup of G. Let $W \subseteq G$ be a wedge with $W^\circ \neq \emptyset$ and let S be a subsemigroup of G containing W. Then we have the following claims:*

(i) In the cornet $(S, +, \cdot, \preceq_W)$ the set of nonnegative elements is W.
(ii) The set W° is a subsemigroup of the Archimedean elements.
(iii) Every element of S is W°-bounded.
(iv) If, in addition, G is locally convex and W is topologically closed and $W^\circ = W^\circ + W^\circ$, then every element of S is also W°-closed.

Proposition 1.16. [MP21]. *Let $(G,+)$ be a topological abelian group such that there is no proper open subgroup of G. Let W be a wedge and let S be a subsemigroup of G containing W. Let $P_W(S)$ denote the collection of all nonempty W-invariant subsets of S. Define the operations $+$ and $*$ by* (1.1)*. Then the following statements hold:*

(i) The set of nonnegative elements of the cornet $(P_W(S),+,*,\subseteq)$ consists of those W-invariant subsets of S that contain 0 (which denotes the neutral element of G).
(ii) The collection $\mathcal{A}$ of those W-invariant subsets which contain an open convex neighborhood $C \in P_W(S)$ of 0 is a subsemigroup of the Archimedean elements.
(iii) An element of $P_W(S)$ is $\mathcal{A}$-bounded if it is the sum of a topologically bounded subset of S and W.
(iv) If, in addition, G is locally convex, then any topologically closed element of $P_W(S)$ is also $\mathcal{A}$-closed and the addition is $\mathcal{A}$-continuous.

Proposition 1.17. [MP21]. *Let $(G,+)$ be a topological abelian group such that there is no proper open subgroup of G. Let W be a wedge and let S be a uniquely divisible subsemigroup of G containing W. Let, for $p \in]0,1]$, the set $F_W^p(S)$ be defined by* (1.2) *and define the operations $\oplus$ and $\odot$ by* (1.3)*. Then, the following statements hold.*

(i) The set of nonnegative elements of the cornets $(F_W^p(S),\oplus,\odot,\leq)$ consists of those W-invariant functions f such that $f(0)=1$ (here 0 denotes the neutral element of G).
(ii) The cornet $(F_W^p(S),\oplus,\odot,\leq)$ has no Archimedean elements for $p \in]0,1[$. On the other hand, the set $\mathcal{A}$ of those $a \in F_W^1(S)$ or which there exists an open convex neighborhood C of 0 such that $a|_C = 1$ is a subsemigroup of the Archimedean elements of $(F_W^1(S),\oplus,\odot,\leq)$.
(iii) Any $f \in F_W^1(S)$ is $\mathcal{A}$-bounded if $\operatorname{supp}(f) := \{u \in S \mid f(u) > 0\}$ is covered by the sum of a topologically bounded subset of S and W.
(iv) If, in addition, G is locally convex, then any upper semicontinuous element of $F_W^1(S)$ is also $\mathcal{A}$-closed and the addition $\oplus$ is $\mathcal{A}$-continuous.

Main results

We now present the extension of the Rådström Cancellation Theorem.

Theorem 1.18. [MP21]. *Let $(X, +, *, \preceq)$ be a cornet and let $\mathcal{A}$ be a subsemigroup of $X_{\prec}$ such that the addition is $\mathcal{A}$-continuous. Let $x, y, z \in X$ such that z is $\mathcal{A}$-bounded and y is $\mathcal{A}$-closed and m-convex for some $m \geq 2$. If*

$$x + z \preceq y + z$$

holds, then we have $x \preceq y$.

Corollary 1.19. [MP21]. *Let $(G, +)$ be a locally convex topological abelian group such that there is no proper open subgroup of G. Let W be a wedge and let S be a subsemigroup of G containing W. Let $P_W(S)$ denote the collection of W-invariant subsets of S and define the operations $+$ and $*$ by (1.1). Let $A, B, C \in P_W(S)$ such that B is covered by the sum of a topologically bounded subset of S and W and C is a topologically closed m-convex subset of S for some $m \geq 2$. If*

$$A + B \subseteq C + B$$

holds, then $A \subseteq C$.

Corollary 1.20. [MP21]. *Let $(G, +)$ be a locally convex topological abelian group such that there is no proper open subgroup of G. Let W be a wedge and let S be a uniquely divisible subsemigroup of G containing W. Let the set $F^1_W(S)$ be defined by (1.2) and define the operations $\oplus$ and $\odot$ by (1.3). Let $f, g, h \in F^1_W(S)$ such that $\operatorname{supp}(h)$ is covered by the sum of a topologically bounded subset of S and W and g is upper semicontinuous and m-quasiconcave for some $m \geq 2$. If*

$$f \oplus h \leq g \oplus h$$

holds, then $f \leq g$.

Summary of Chapter 2
On convex and concave sequences and their applications

Introduction

In the theory of convexity, the investigation of convex functions play a fundamental role. We refer to the following monographs for the details: Hardy–Littlewood–Pólya [HLP34], Kuczma [K85], Mitrinović [M70], Mitrinović–Pečarić–Fink [MPF91, MPF93], Niculescu–Persson [NP06], Popoviciu [P44], and Roberts–Varberg [RV73]. The investigation of convex sequences probably started in the book Mitrinović [M70]. This subfield is still very active, some recent results and applications have been obtained by Krasniqi [K16], Niezgoda [N11,N17a,N17b], Sofonoea–Ţincu–Acu [STA18], Tabor–Tabor–Żoldak [TTZ12], Wu–Debnath [WD07], Yıldız [Y18]. In this chapter we introduce the notions of q-convex, q-affine and q-concave sequences and some basic results on them are also presented. Then we establish their surprising connection to Chebyshev polynomials of the first and of the second kind. Finally, an application of them to fixed point theory is presented.

Given $n, m \in \mathbb{Z}$ with $2 \leq m - n$, let $\mathcal{S}(n|m)$ denote the linear space $\mathbb{R}^{\{n,\dots,m\}}$ of all real sequences, i.e., the collection of all functions $p : \{n, \dots, m\} \to \mathbb{R}$. It is natural to define the notions of concavity, convexity and affinity for the elements of $\mathcal{S}(n|m)$. A sequence $p = (p_n, \dots, p_m) \in \mathcal{S}(n|m)$ is called *convex* if, for all $i \in \{n+1, \dots, m-1\}$,

$$2p_i \leq p_{i-1} + p_{i+1}. \tag{2.1}$$

If, for all $i \in \{n+1, \dots, m-1\}$, the reversed inequality holds in (2.1), then the sequence is termed *concave*. Finally, if a sequence is simultaneously convex and concave, then it is said to be *affine*. If the inequality (2.1) holds with strict inequality sign, then we speak about strict convexity and concavity, respectively.

In what follows, a sequence $p = (p_n, \dots, p_m) \in \mathcal{S}(n|m)$ is called *q-convex* if, for all $i \in \{n+1, \dots, m-1\}$,

$$2qp_i \leq p_{i-1} + p_{i+1}. \tag{2.2}$$

If, for all $i \in \{n+1, \dots, m-1\}$, the reversed inequality holds in (2.2), then the sequence is termed *q-concave*. If a sequence is simultaneously q-convex and q-concave, then it is said to be *q-affine*.

We can easily see that the strict convexity of a positive (or negative) sequence implies its q-convexity for some q. Analogously, $p \in \mathcal{S}(n|m)$ is a negative strictly convex sequence, then it is q-convex with $0 < q < 1$.

The subclasses of q-convex and q-concave sequences in $\mathcal{S}(n|m)$ will be denoted $\mathcal{C}_q^{\cup}(n|m)$ and $\mathcal{C}_q^{\cap}(n|m)$, respectively. Finally, $\mathcal{A}_q(n|m)$ will stand for the subclass of q-affine sequences, that is,

$$\mathcal{A}_q(n|m) := \mathcal{C}_q^{\cup}(n|m) \cap \mathcal{C}_q^{\cap}(n|m).$$

It is easy to see that $\mathcal{A}_q(n|m)$ is a linear subspace of $\mathcal{S}(n|m)$, and both $\mathcal{C}_q^{\cup}(n|m)$ and $\mathcal{C}_q^{\cap}(n|m)$ are convex cones in $\mathcal{S}(n|m)$, i.e., they are closed with respect to linear combinations with nonnegative coefficients.

Auxiliary results for Chebyshev polynomials

For $k \in \mathbb{Z}$, let $T_k \colon \mathbb{R} \to \mathbb{R}$ and $U_k \colon \mathbb{R} \to \mathbb{R}$ denote the Chebyshev polynomials of the first and of the second kind of order k, which are defined by the system of equations for $k \in \mathbb{Z}$

$$\begin{aligned} &T_0(x) := 1, \quad T_1(x) := x, \quad\ \ T_{k-1}(x) + T_{k+1}(x) = 2xT_k(x), \\ &U_0(x) := 1, \quad U_1(x) := 2x, \quad U_{k-1}(x) + U_{k+1}(x) = 2xU_k(x), \end{aligned} \tag{2.3}$$

respectively. The last equalities in (2.3) rewritten as

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x),$$

can be used to compute T_k and U_k for $k \geq 2$ recursively. If we rewrite them as

$$T_{k-1}(x) = 2xT_k(x) - T_{k+1}(x), \quad U_{k-1}(x) = 2xU_k(x) - U_{k+1}(x),$$

then T_k and U_k can be determined for $k \leq -1$. One can easily prove that, for $k \in \mathbb{Z}$,

$$T_{-k} = T_k \quad \text{and} \quad U_{-k} = -U_{k-2}.$$

In particular, $U_{-1} = 0$. It is clear that, for $k \geq 0$, the degree of T_k and U_k equals k. It is well-known that these polynomials satisfy, for all $u \in \mathbb{R}$ and $k \in \mathbb{Z}$, the equalities

$$T_k(\cos(u)) = \cos(ku), \qquad T_k(\cosh(u)) = \cosh(ku)$$

and

$$U_k(\cos(u)) = \frac{\sin((k+1)u)}{\sin(u)}, \quad U_k(\cosh(u)) = \frac{\sinh((k+1)u)}{\sinh(u)}.$$

From these representations it easily follows that the roots of T_k (for $k \neq 0$) and U_{k-1} (for $k \notin \{-1, 0, 1\}$) are given by

$$\left\{ \cos\left(\frac{2i-1}{2k}\pi\right) \mid i \in \{1, \ldots, |k|\} \right\} \quad \text{and}$$
$$\left\{ \cos\left(\frac{i}{k}\pi\right) \mid i \in \{1, \ldots, |k|-1\} \right\},$$

respectively. Therefore, the largest root of T_k (for $k \neq 0$) and U_{k-1} (for $k \notin \{-1, 0, 1\}$) are given by $\cos\left(\frac{\pi}{2k}\right)$ and $\cos\left(\frac{\pi}{k}\right)$ respectively.

q-concave, q-convex and q-affine sequences

The next proposition shows that $\mathcal{A}_q(n|m)$ is a two dimensional subspace of $\mathcal{S}(n|m)$.

Proposition 2.1. [MP22]. *A sequence $p \in \mathcal{S}(n|m)$ is q-affine if and only if there exist $a, b \in \mathbb{R}$ such that*

$$p_i := aU_{i-n}(q) + bT_{i-n}(q) \qquad (i \in \{n, \dots, m\}).$$

In addition, if $p \in \mathcal{A}_q(n|m)$, then, for all $i, j, k \in \{n, \dots, m\}$,

$$U_{k-j-1}(q)p_i + U_{j-i-1}(q)p_k = U_{k-i-1}(q)p_j.$$

In particular, for $i \in \{n, \dots, m\}$ and $j \in \{1, \dots, \min(i-n, m-i)\}$,

$$p_{i-j} + p_{i+j} = 2T_j(q)p_i.$$

In the following statement, we establish some properties of the class of q-concave (and hence of q-convex) sequences.

Proposition 2.2. [MP22]. *The cone $\mathcal{C}_q^{\cap}(n|m)$ is closed with respect to the pointwise minimum and the cone $\mathcal{C}_q^{\cup}(n|m)$ is closed with respect to the pointwise maximum.*

As q-affine sequences are q-concave and also q-convex, we obtain that the pointwise minimum and maximum of a finite family of q-affine sequences are q-concave and also q-convex, respectively.

Proposition 2.3. [MP22]. *Let $i, j, k \in \{n, \dots, m\}$ with $i < j < k$. Assume that*

$$q \geq \cos\left(\frac{\pi}{\max(j-i, k-j)}\right).$$

Then, for all $p \in \mathcal{C}_q^{\cap}(n|m)$,

$$U_{k-j-1}(q)p_i + U_{j-i-1}(q)p_k \leq U_{k-i-1}(q)p_j.$$

In particular, if $i \in \{n+1, \dots, m-1\}$ and $j \in \{1, \dots, \min(i-n, m-i)\}$ and

$$q > \cos\left(\frac{\pi}{j}\right),$$

then

$$p_{i-j} + p_{i+j} \leq 2T_j(q)p_i.$$

Proposition 2.4. [MP22]. *Let $j, k \in \{n, \dots, m\}$ with $j < k$. In addition, assume that*

$$q > \cos\left(\frac{\pi}{k-j}\right).$$

Let $p \in \mathcal{C}^\cap(n|m)$ and define

$$r_i := p_k \frac{U_{i-j-1}(q)}{U_{k-j-1}(q)} + p_j \frac{U_{k-i-1}(q)}{U_{k-j-1}(q)} \qquad (i \in \{n, \dots, m\}).$$

Then, $r = (r_n, \dots, r_m)$ is a q-affine sequence and, for $i \in \{n, \dots, m\}$,

$$r_i \begin{cases} \geq p_i & \text{if } i < j \text{ or } k < i. \\ = p_i & \text{if } i \in \{j, k\}. \\ \leq p_i & \text{if } j < i < k. \end{cases}$$

In the following proposition, we establish a characterization of q-concave sequences.

Proposition 2.5. [MP22]. *Let $p \in \mathcal{S}(n|m)$. Then p is q-concave if and only if, for all $j \in \{n, \dots, m-1\}$, there exists $r \in \mathcal{A}_q(n|m)$ such that*

$$p_j = r_j, \quad p_{j+1} = r_{j+1}, \quad \text{and} \quad p_i \leq r_i \quad \text{for} \quad i \in \{n, \dots, m\}.$$

An application of q-concave sequences

In what follows, we will adopt the following convention: For an arbitrary sequence $a \in \mathcal{S}(1|n)$, let a be extended to be in $\mathcal{S}(0|n+1)$ by setting $a_0 := 0$ and $a_{n+1} := 0$. For $n \in \mathbb{N}$ and for a vector $\gamma = \left(\gamma_1, \dots, \gamma_{\lfloor \frac{n+1}{2} \rfloor}\right) \in \mathbb{R}^{\lfloor \frac{n+1}{2} \rfloor}$, we define the map $\mathcal{T}_\gamma \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\left(\mathcal{T}_\gamma(a)\right)_i := \min_{1 \leq j \leq \min(i, n+1-i)} \left(\frac{a_{i-j} + a_{i+j}}{2} + \gamma_j\right),$$

where $a \in \mathbb{R}^n$ and $i \in \{1, \dots, n\}$).

In order to make the map $\mathcal{T}_\gamma$ a contraction with respect to a suitable norm on $\mathbb{R}^n$, we construct new norms in terms of positive sequences. Let

$|\cdot|_\infty$ denote the maximum norm on $\mathbb{R}^n$. If $p \in \mathcal{S}(1|n)$ is a sequence with positive members, then we define $\|\cdot\|_p \colon \mathbb{R}^n \to \mathbb{R}$ by

$$\|a\|_p := \max_{1\le i\le n} p_i^{-1}|a_i| = |p^{-1}a|_\infty \qquad (a \in \mathbb{R}^n).$$

Then $\|\cdot\|_p$ is a norm and $\mathbb{R}^n$ is a Banach space with respect to $\|\cdot\|_p$.

Theorem 2.6. [MP22]. *Let $p \in \mathcal{S}(1|n)$ be a sequence with positive members and define*

$$q := \max_{1\le i\le n} \frac{p_{i-1}+p_{i+1}}{2p_i} \qquad \textit{and} \qquad q^* := \begin{cases} q & \textit{if } q \le 1, \\ T_{\lfloor \frac{n+1}{2} \rfloor}(q) & \textit{if } q > 1. \end{cases}$$

Then, for all $\gamma \in \mathbb{R}^{\lfloor \frac{n+1}{2} \rfloor}$, the mapping $\mathcal{T}_\gamma$ is q^*-Lipschitzian on the normed space $(\mathbb{R}^n, \|\cdot\|_p)$. In particular, if p is strictly concave, then $\mathcal{T}_\gamma$ is a contraction on the normed space $(\mathbb{R}^n, \|\cdot\|_p)$.

Corollary 2.7. [MP22]. *For all $\gamma \in \mathbb{R}^{\lfloor \frac{n+1}{2} \rfloor}$, the mapping $\mathcal{T}_\gamma \colon \mathbb{R}^n \to \mathbb{R}^n$ has a unique fixed point in $\mathbb{R}^n$.*

Összefoglaló

Az 1. Fejezet összefoglalója

A Rådström-féle egyszerűsítési szabály kiterjesztése tölcsérekre

Bevezetés

A konvex halmazok elméletéhez tartozó Egyszerűsítési Szabályt Rådström [R52b] fedezte fel 1952-ben. Az eredeti cikkében a második Lemma szerint az

$$A + B \subseteq C + B$$

tartalmazásból következik, hogy $A \subseteq C$, feltéve, hogy az A, B, C nemüres halmazok részhalmazai egy X normált térnek, C zárt, konvex és B pedig korlátos.

Később ez a lemma alapvető eszköz lett a matematika különféle területein, és mára cikkek százai felhasználták ezt az eredményt. Például nemsima analízis [CGT10, GGM16, GM18, HN17, D15], optimalizálás [CKR14], konvex halmazok és függvények elmélete [DMM11, GKKU14, GPPU12, GPU10a, GP15, GU14, VN12, VN15, K14], halmazértékű analízis [CT13, KPR15, LMNS14, P09, K19, O17, P13], halmazértékű differenciálegyenletek [BG09,PS14,M15], halmazértékű függvényegyenletek [BMP18, M12, S13, S19], iterációelmélet [AN16, AGMM10, SS12, XNZ11, G12], stb.

Tölcsérek és konvexitási tulajdonságok tölcsérekben

1.1. Definíció. [MP21]. Az $(X, +, \preceq)$ rendezett hármast *rendezett kommutatív félcsoport*nak nevezzük, ha

(i) $(X, +)$ egy kommutatív egységelemes félcsoport, aminek az egységeleme a 0;
(ii) $(X, \preceq)$ egy részbenrendezett halmaz, azaz $\preceq$ egy reflexív, antiszimmetrikus és tranzitív reláció X-en;
(iii) Bármely $x, y, z \in X$ és $x \preceq y$ esetén az $x+z \preceq y+z$ egyenlőtlenség fennáll.

Ha az $(X, \preceq)$ részbenrendezett halmaz teljes, azaz X bármely nemüres alulról korlátos részhalmazának létezik pontos felső korlátja, akkor az $(X, +, \preceq)$ hármast *teljes rendezett kommutatív félcsoportnak* hívjuk.

Egy $(X, +)$ félcsoportban természetes úton definiálható a természetes számokkal való szorzás az alábbi rekurzív módon

$$1 \cdot x := x, \qquad (n+1) \cdot x := n \cdot x + x \qquad (n \in \mathbb{N}).$$

Ha a félcsoport egységelemes, akkor definiáljuk továbbá az egységgel való szorzást is: $0 \cdot x := 0$.

A következő definíció jelen fejezetnek a központi fogalma.

1.2. Definíció. [MP21]. Egy rendezett $(X, +, *, \preceq)$ négyest *tölcsér*nek nevezünk, ha $(X, +, \preceq)$ egy rendezett kommutatív félcsoport és "$*$" az X halmaz elemeinek pozítív egész számokkal való szorzását jelenti, amely teljesíti az alábbi feltételeket:

(i) Bármely $n, m \in \mathbb{N}$ és $x \in X$ esetén $(nm) * x = n * (m * x)$;
(ii) Bármely $n \in \mathbb{N}$ és $x, y \in X$ esetén $n * (x + y) = n * x + n * y$;
(iii) Bármely $n, m \in \mathbb{N}$ és $x \in X$ esetén $(n + m) * x \preceq n * x + m * x$;
(iv) Bármely $n \in \mathbb{N}$ és $x, y \in X$ esetén az $x \preceq y$ egyenlőtlenség pontosan akkor teljesül, ha $n * x \preceq n * y$;
(v) $1 * x = x$;

(vi) $n * 0 = 0$.

Ha az $(X, \preceq)$ részbenrendezett halmaz teljes, akkor az $(X, +, *, \preceq)$ négyest *teljes tölcsér*nek nevezzük. Egy $(S, +)$ egységelemes részfélcsoportját az $(X, +, *, \preceq)$ tölcsérnek, amely zárt a "$*$" műveletre nézve az $(X, +, *, \preceq)$ tölcsér *résztölcsér*jének hívjuk, ahol a rendezés alatt a rendezés S halmazra való leszűkítését értjük.

Világos, hogy ha $(X, +)$ egy olyan kommutatív egységelemes félcsoport, hogy bármely $n \in \mathbb{N}$ esetén az $n \mapsto n \cdot x$ leképezés injektív, akkor az $(X, +, \cdot, =)$ négyes tölcsér.

1.3. Definíció. [MP21]. Legyen $(X, +, *, \preceq)$ egy tölcsér és $n \in \mathbb{N}$. Az $x \in X$ elemet n-*konvex*nek nevezzük, ha $n * x = n \cdot x$. Rögzített $x \in X$ és $n \in \mathbb{N}$ esetén, bevezetjük az alábbi jelöléseket:

$$C_x := \{n \in \mathbb{N} \mid x \ n\text{-konvex}\} \quad \text{és} \quad C^n := \{x \in X \mid x \ n\text{-konvex}\}.$$

Ha $C_x = \mathbb{N}$, azaz ha x n-konvex bármely $n \in \mathbb{N}$ esetén, akkor azt mondjuk, hogy x *konvex*. Az $x \in X$ elem n*-convex burka* egy olyan $y \in C^n$ elem, hogy $x \preceq y$, továbbá ha $x \preceq z \in C^n$, akkor $y \preceq z$. Az x elem konvex burkát $\operatorname{conv}_n(x)$-szel jelöljük

Általában egy elem n-konvex burka nem feltétlenül létezik. Ha viszont létezik az n-convex burok, akkor az egyértelmű. A létezés biztosításához további feltételekre van szükségünk. Azt mondjuk, hogy a $*$-szorzás művelet egy $(X, +, *, \preceq)$ teljes tölcséreben n*-folytonos* (a "$\preceq$" rendezésre nézve), ha bármely nemüres alulról korlátos $H \subseteq X$ részhalmaz esetén fennáll, hogy

$$\inf(n * H) = n * \inf(H).$$

1.4. Állítás. [MP21]. *Legyen $n \in \mathbb{N}$ és $(X, +, *, \preceq)$ egy teljes tölcsér, amiben a $*$ szorzás n-folytonos. Ekkor $(C^n, +, *, \preceq)$ egy teljes résztölcsére az $(X, +, *, \preceq)$ tölcsérnek. Továbbá bármely $x \in X$ esetén egy x elemnek pontosan akkor létezik az n-konvex burka, ha van n-konvex majoránsa.*

Egy $(X, +, *, \preceq)$ tölcsérben jelölje K^n azon elemek halmazát, amelyeknek létezik n-konvex burka és jelölje M^n azon elemek halmazát, amelyeknek létezik n-konvex majoránsa. Ekkor világos, hogy $C^n \subseteq K^n \subseteq M^n$. Ezen jelölésekkel az előző állítás szerint, ha $(X, +, *, \preceq)$ egy teljes tölcsér, amiben a $*$-szorzás n-folytonos, akkor $K^n = M^n$.

A fenti fogalmak alkalmazhatóságának illusztrálására a következő három állításban alapvető példákat mutatunk tölcsérekre. Ehhez előbb bevezetjük az ék fogalmát Abel-csoportokban.

1.5. Definíció. [MP21]. Legyen $(G, +)$ egy Abel-félcsoport, $S \subseteq G$ és $n \in \mathbb{N}$. Definiáljuk

$$n^{-1}(S) := \{x \in G \mid n \cdot x \in S\}.$$

Egy S részfélcsoportját $(G, +)$-nak *n-oszthatónak* nevezzük, ha bármely $x \in S$ esetén az $n^{-1}(\{x\}) \cap S$ nemüres. Ha ez a halmaz egyelemű, akkor S-t *egyértelműen n-oszthatónak* nevezzük és ezt az egyértelmű elemet x/n-nel jelöljük. Egy G egységelemes Abel-félcsoportban egy $W \subseteq G$ részhalmazt *éknek* hívjuk, ha az alábbi tulajdonságok teljesülnek:

(i) W egy egységelemes részfélcsoportja G-nek.
(ii) Ha $u, v \in W$ olyan, hogy $u + v = 0$, akkor $u = v = 0$.
(iii) Bármely $n \in \mathbb{N}$ esetén, az $n^{-1}(W)$ ősképet tartalmazza W.

Egy $W \subseteq G$ ék esetén definiálhatunk egy részbenrendezést, $\preceq_W$-t, az alábbi módon: Az $x, y \in G$ elemek esetén azt mondjuk, hogy $x \preceq_W y$, ha $y \in x + W$. Azonnal látható, hogy $\preceq_W$ egy reflexív és tranzitív reláció G-n. Ha még G ráadásul kancellatív is (ami mindig teljesül, ha G csoport), akkor $\preceq_W$ antiszimmetrikus is, tehát részbenrendezés G-n.

1.6. Állítás. [MP21]. *Legyen $(G, +)$ egy Abel-csoport és $W \subseteq G$ egy ék. Ekkor, ha S olyan részfélcsoportja G-nek, amely tartalmazza W-t, akkor az $(S, +, \cdot, \preceq_W)$ négyes tölcsér, amelyben minden elem n-konvex bármely $n \in \mathbb{N}$ esetén. Speciálisan az $W := \{0\}$ választással $(G, +, \cdot, =)$ tölcsér.*

1.7. Állítás. [MP21]. *Legyen $(G,+)$ egy Abel-félcsoport, W egy ék, és legyen S részfélcsoportja G-nek. Jelölje $P_W(S)$ az S halmaz összes W-invariáns A részhalmazát, azaz az olyan halmazokat, amelyekre $A+W\subseteq A$ fennáll. Definiáljuk a $+$ és $*$ műveleteket az alábbi módon:*

$$\begin{aligned} A+B &:= \{a+b \mid a\in A,\, b\in B\} && (A,B\in P_W(S)),\\ n*A &:= \{n\cdot a+w \mid a\in A,\, w\in W\} && (A\in P_W(S),\, n\in\mathbb{N}). \end{aligned} \tag{1.1}$$

Ekkor $(P_W(S),+,*,\subseteq)$ egy teljes tölcsér, amelynek egységeleme W. Továbbá az

$$\varphi(x) := x+W \qquad (x\in S)$$

leképezés egy injektív rendezésváltó homomorf leképezése $(S,+,\cdot,\preceq_W)$-nek a $(P_W(S),+,*,\subseteq)$ tölcsérbe. Ha még ráadásul W n-osztható, ha $n\in\mathbb{N}$, akkor $A\in P_W(S)$ pontosan akkor n-konvex, ha bármely $x_1,\dots,x_n\in A$ esetén

$$n^{-1}(\{x_1+\dots+x_n\})\cap A\neq\emptyset.$$

1.8. Állítás. [MP21]. *Legyen $(G,+)$ egy Abel-csoport, W egy ék, és S egy W-t tartalmzaó, egyértelműen osztható részfélcsoportja G-nek. Legyen, $p\in\,]0,1]$ esetén,*

$$F_W^p(S) := \{f\colon S\to[0,1] \mid \sup f\geq p \text{ és } f\ W\text{-nemcsökkenő}\} \tag{1.2}$$

és definiáljuk az összeadást és a számmal való szorzást $F_W^p(S)$-ben a következőképp:

$$\begin{aligned} (f\oplus g)(x) &:= \sup_{\substack{u,v\in S\\ u+v=x}} \min\big(f(u),g(v)\big),\\ (n\odot f)(x) &:= f\left(\frac{x}{n}\right) && (f,g\in F_W^p(S),\, x\in S,\, n\in\mathbb{N}). \end{aligned} \tag{1.3}$$

Jelölje $\leq$ a pontonkénti rendezést $F_W^p(S)$-ben. Ekkor $(F_W^p(S),\oplus,\odot,\leq)$ egy teljes tölcsér, aminek az egységeleme a W ék karakterisztikus függvénye. Továbbá a

$$\Phi(A) := \chi_A \qquad (A\in P_W(S))$$

módon adott leképezés injektív tölcsértartó leképezése $(P_W(S), +, *, \subseteq)$*-nek* $(F_W^1(S), \oplus, \odot, \leq)$*-be. Ráadásul az* $f \in F_W^p(S)$ *függvény pontosan akkor* n*-konvex, ha* n*-kvázikonkáv, azaz bármely* $x_1, \dots, x_n \in S$ *esetén*

$$\min(f(x_1), \dots, f(x_n)) \leq f\Big(\frac{x_1 + \cdots + x_n}{n}\Big).$$

Topologikus fogalmak és korlátosság tölcsérekben

A nemnegatív és Archimedesi elemek természetes módon vezetkezők be tölcsérekben a következő definícióval.

1.9. Definíció. [MP21]. Legyen $(X, +, *, \preceq)$ egy tölcsér. Azt mondjuk, hogy az $x \in X$ elem *nemnegatív*, ha $0 \preceq x$ fennáll. Az x elemet *Archimedesi*nek mondjuk és $0 \prec x$-szel jelöljük, ha bármely $u \in X$ esetén van olyan $n_0 \in \mathbb{N}$, hogy ha $n_0 \leq n$, akkor $0 \preceq u + n * x$ teljesül. Az X halmaz nemnegatív és Archimedesi elemeinek halmazát rendre $X_{\preceq}$ és $X_{\prec}$-szel jelöljük.

A következő állítás a nemnegatív és Archimedesi elemek tulajdonságait mutatja meg.

1.10. Állítás. [MP21]. *Legyen* $(X, +, *, \preceq)$ *egy tölcsér. Ekkor* $X_{\preceq}$ *tartalmazza* $X_{\prec}$*-et és*

$$X_{\prec} + X_{\preceq} \subseteq X_{\prec}.$$

Ráadásul, $X_{\prec}$ *és* $X_{\preceq}$ *résztölcsérei* $(X, +, *, \preceq)$*-nek.*

A következőkben bevezetjük az összeadás folytonosságának fogalmát, valamint Archimedesi elemeket tartalmazó részfélcsoport korlátosságát és zártságát. Összehasonlításul először idézzük fel a klasszikus topologikus fogalmakat Abel-csoportokra.

1.11. Definíció. [MP21]. Ha $(G, +)$ egy Abel-csoport és $\mathcal{T}$ Hausdorff-topológia G-n, akkor azt mondjuk, hogy $(G, \mathcal{T}, +)$ egy *topologikus csoport*, ha $(x, y) \mapsto x - y$ egy folytonos leképezése $G \times G$-nek G-be. Egy $U \subseteq G$ halmazt *konvex*nek nevezünk, ha bármely $n \in \mathbb{N}$ esetén

$$\{u_1 + \cdots + u_n \mid u_1, \dots, u_n \in U\} = \{n \cdot u \mid u \in U\}.$$

Azt mondjuk, hogy G *lokálisan konvex*, ha a 0 bármely környezete tartalmaz konvex környezetét 0-nak. A $H \subseteq G$ részhalmazt *topologikusan korlátos*nak nevezzük, ha 0 bármely U környezete esetén van olyan $n \in \mathbb{N}$, hogy $H \subseteq \{u_1 + \cdots + u_n \mid u_1, \ldots, u_n \in U\}$.

1.12. Definíció. [MP21]. Legyen $(X, +, *, \preceq)$ egy tölcsér és $\mathcal{A}$ részfélcsoportja $X_\prec$-nek. Azt mondjuk, hogy az összeadás $\mathcal{A}$*-folytonos*, ha bármely $a \in \mathcal{A}$ esetén van olyan $b \in \mathcal{A}$, hogy $b + b \preceq a$. Az $x \in X$ elemet $\mathcal{A}$*-zárt*nak nevezzük, ha bármely $a \in \mathcal{A}$ esetén van olyan $n_0 \in \mathbb{N}$, hogy ha $n \geq n_0$, akkor $x \preceq n * a$ fennáll.

1.13. Állítás. [MP21]. *Legyen* $(X, +, *, \preceq)$ *egy tölcsér és legyen* $\mathcal{A}$ *olyan részfélcsoportja* $X_\prec$*-nek, amelyben az összeadás* $\mathcal{A}$*-folytonos. Ekkor az* $\mathcal{A}$*-korlátos elemek résztölcsért alkotnak* $(X, +, *, \preceq)$*-ben.*

1.14. Definíció. [MP21]. Legyen $(X, +, *, \preceq)$ egy tölcsér és $\mathcal{A}$ egy részfélcsoportja $X_\prec$-nek. Adott $x \in X$ esetén azt mondjuk, hogy $y \in X$ az x elem $\mathcal{A}$*-lezárt*ja, ha $y \preceq x + a$ teljesül minden $a \in \mathcal{A}$ esetén, és y a legnagyobb ilyen tulajdonságú elem X-ben, azaz ha $z \preceq x + a$ bármely $a \in \mathcal{A}$ esetén, akkor $z \preceq y$. Világos, hogy ha egy elemnek létezik az $\mathcal{A}$-lezártja, akkor az egyértelmű. Ezt az elemet $\mathrm{cl}_{\mathcal{A}}(x)$-val jelöljük. Egy x elemet $\mathcal{A}$*-zárt*nak nevezünk, ha $x = \mathrm{cl}_{\mathcal{A}}(x)$. Az összes $\mathcal{A}$-lezárható elemek halmazát $\mathrm{Cl}_{\mathcal{A}}$-val jelöljük.

A soron következő állításokban megvizsgáljuk az 1.6, 1.7 és 1.8 Állításokban bemutatott tölcséreket és megkeressük az összes korlátos, zárt és Archimedesi elemeket ezekben a struktúrákban.

1.15. Állítás. [MP21]. *Legyen* $(G, +)$ *egy olyan topologikus Abel-csoport, amelynek nincs valódi nyílt részcsoportja. Legyen* $W \subseteq G$ *egy olyan ék, amelyre* $W^\circ \neq \emptyset$*, és legyen* S *egy* W*-t tartalmazó részfélcsoportja* G*-nek. Ekkor az alábbiak teljesülnek:*

(i) *Az* $(S, +, \cdot, \preceq_W)$ *tölcsér nemnegatív elemeinek halmaza* W*.*

(ii) *Az* W° *halmaz részfélcsoportja az Archimedesi elemek halmazának.*

(iii) S *minden eleme* W°*-korlátos.*

(iv) Ha még ráadásul G lokálisan konvex, W topologikusan zárt és $W^\circ = W^\circ + W^\circ$, akkor S minden eleme egyidejűleg W°-zárt.

1.16. Állítás. [MP21]. *Legyen $(G, +)$ egy olyan topologikus Abel-csoport, amelynek nincs valódi nyílt részcsoportja. Legyen W egy ék és S egy W-t tartalmazó részfélcsoportja G-nek. Jelölje $P_W(S)$ az S halmaz W-invariáns részhalmazait és definiáljuk a $+$ és $*$ műveleteket* (1.1) *szerint. Ekkor az alábbi állítások teljesülnek.*

(i) A $(P_W(S), +, *, \subseteq)$ tölcsér nemnegatív elemei S azon W-invariáns részhalmazaiból áll, amelyek tartalmazzák a 0 elemet (ami G neutrális eleme).
(ii) Azon W-invariáns részhalmazok $\mathcal{A}$ családja, amelyek tartalmaznak nyílt konvex környezetét 0-nak, részfélcsoportját alkotják az Archimedesi elemeknek.
(iii) A $P_W(S)$ halmaz egy eleme pontosan akkor $\mathcal{A}$-korlátos, ha előáll S és W topologikusan korlátos részhalmazai összegeként.
(iv) Ha még ráadásul G lokálisan konvex, akkor $P_W(S)$ bármely topologikusan zárt eleme $\mathcal{A}$-zárt is, valamint az összeadás $\mathcal{A}$-folytonos.

1.17. Állítás. [MP21]. *Legyen $(G, +)$ egy olyan topologikus Abel-csoport, amelynek nincs valódi nyílt részcsoportja. Legyen W egy ék és S egy W-t tartalmazó egyértelműen osztható részfélcsoportja G-nek. Definiáljuk $p \in]0, 1]$ esetén az $F_W^p(S)$ halmazt* (1.2) *alapján, a műveleteket pedig* (1.3) *szerint. Ekkor teljesülnek az alábbi állítások.*

(i) Az $(F_W^p(S), \oplus, \odot, \leq)$ tölcsér nemnegatív elemei azok a W-invariáns f függvények, amelyekre $f(0) = 1$ (ahol 0 jelöli G neutrális elemét).
(ii) Az $(F_W^p(S), \oplus, \odot, \leq)$ tölcsérnek nincs Archimedesi eleme $p \in]0, 1[$ esetén. Másrészt az olyan $a \in F_W^1(S)$ elemek $\mathcal{A}$ halmaza, amelyekhez létezik a 0-nak, C-vel jelölt, nyílt konvex környezete, hogy $a|_C = 1$, részfélcsoportját alkotja az $(F_W^1(S), \oplus, \odot, \leq)$ tölcsér Archimedesi elemeinek.
(iii) Egy $f \in F_W^1(S)$ elem $\mathcal{A}$-zárt, ha $\operatorname{supp}(f) := \{u \in S \mid f(u) > 0\}$ lefedhető S és W topologikusan zárt részhalmazának összegével.

(iv) Ha még ráadásul G lokálisan konvex, akkor bármely felülről félig folytonos eleme $F_W^1(S)$-nek $\mathcal{A}$-zárt is, az $\oplus$ összeadás művelet pedig $\mathcal{A}$-folytonos.

Fő eredmények

Ebben a részben kimondjuk a Rådström-féle Egyszerűsítési Szabály általánosítását tölcsérekre.

1.18. Tétel. [MP21]. *Legyen $(X, +, *, \preceq)$ egy tölcsér és $\mathcal{A}$ olyan részfélcsoportja $X_{\prec}$-nek, hogy benne az összeadás $\mathcal{A}$-folytonos. Legyenek továbbá $x, y, z \in X$ olyanok, hogy z $\mathcal{A}$-korlátos, y $\mathcal{A}$-zárt és m-konvex, valamely $m \geq 2$-re. Ekkor, ha*

$$x + z \preceq y + z$$

teljesül, akkor $x \preceq y$.

1.19. Következmény. [MP21]. *Legyen $(G, +)$ egy olyan lokálisan konvex Abel-csoport, amelynek nincs valódi nyílt részcsoportja. Legyen W egy ék és S egy W-t tartalmazó részfélcsoportja G-nek. Jelölje $P_W(S)$ az S halmaz W-invariáns részhalmazait és definiáljuk a $+$ és $*$ műveleteket* (1.1) *szerint. Legyen $A, B, C \in P_W(S)$ olyan, hogy B lefedhető S és W topologikusan korlátos részhalmazainak összegével, C pedig topologikusan zárt m-konvex részhalmaza S-nek, valamely $m \geq 2$ esetén. Ekkor, ha*

$$A + B \subseteq C + B$$

teljesül, akkor $A \subseteq C$.

1.20. Következmény. [MP21]. *Legyen $(G, +)$ egy olyan topologikus Abel-csoport, amelynek nincs valódi nyílt részcsoportja. Legyen W egy ék és S egy W-t tartalmazó egyértelműen osztható részfélcsoportja G-nek. Definiáljuk $p \in]0, 1]$ esetén az $F_W^1(S)$ halmazt* (1.2) *alapján, a műveleteket pedig* (1.3) *szerint. Legyenek $f, g, h \in F_W^1(S)$ olyanok, hogy* $\operatorname{supp}(h)$

lefedhető S és W topologikusan korlátos részhalmazainak összegével, valamint g pedig felülről félig folytonos és m-kvázikonkáv valamely $m \geq 2$ esetén. Ekkor, ha

$$f \oplus h \leq g \oplus h$$

teljesül, akkor $f \leq g$.

A 2. Fejezet összefoglalója

Konvex és konkáv sorozatokról és alkalmazásaikról

Bevezetés

A konvexitás elméletében a konvex függvények alapvető szerepet játszanak. A téma további tanulmányozásához az alábbi monográfokat javasoljuk: Hardy–Littlewood–Pólya [HLP34], Kuczma [K85], Mitrinović [M70], Mitrinović–Pečarić–Fink [MPF91, MPF93], Niculescu–Persson [NP06], Popoviciu [P44], és Roberts–Varberg [RV73]. A konvex sorozatok vizsgálata valószínűleg Mitrinović [M70] könyvével kezdődött. A matematika ezen ága még most is nagyon aktív. Néhány friss eredmény található a következőkben: Krasniqi [K16], Niezgoda [N11, N17a, N17b], Sofonoea–Ţincu–Acu [STA18], Tabor–Tabor–Żoldak [TTZ12], Wu–Debnath [WD07], Yıldız [Y18]. Ebben a fejezetben bevezetjük a q-konvex, q-affin és q-konkáv konkáv sorozatok fogalmát és néhány alapvető eredményt mutatunk be róluk. Továbbá megmutatjuk a meglepő kapcsolatukat az első- és másodfajú Csebisev-polinomokkal. Végül egy alkalmazást mutatunk a fixponttételek elméletében.

Legyenek $n, m \in \mathbb{Z}$ olyanok, hogy $2 \leq m - n$ és jelölje $\mathcal{S}(n|m)$ az n-edik indextől az m-edik indexig tartó valós sorozatok lineáris terét, azaz az összes $p : \{n, \dots, m\} \to \mathbb{R}$ függvényt. A konkávítás, konvexitás és affinitás fogalmak természetes módon definiálhatók az $\mathcal{S}(n|m)$ halmazon. Azt mondjuk, hogy egy $p = (p_n, \dots, p_m) \in \mathcal{S}(n|m)$ *konvex*, ha bármely

$i \in \{n+1, \dots, m-1\}$ esetén

$$2p_i \le p_{i-1} + p_{i+1}. \tag{2.1}$$

Ha minden $i \in \{n+1, \dots, m-1\}$ esetén a fordított irányú egyenlőtlenség teljesül (2.1)-ben, akkor a sorozatot *konkáv*nak mondjuk. Végül, ha egy sorozat egyidejűleg konvex és konkáv, akkor *affin*nak nevezzük. Ha az (2.1) egyenlőtlenség szigorú, akkor rendre szigorúan konvex és szigorúan konkáv sorozatról beszélünk.

Ebben a részben egy $p = (p_n, \dots, p_m) \in \mathcal{S}(n|m)$ sorozatra a q-*konvex* elnevezést fogjuk használni, ha minden $i \in \{n+1, \dots, m-1\}$ esetén

$$2qp_i \le p_{i-1} + p_{i+1}. \tag{2.2}$$

Ha bármely $i \in \{n+1, \dots, m-1\}$ esetén (2.2)-ben fordított irányú egyenlőtlenség teljesül, akkor azt mondjuk, hogy a sorozat q-*konkáv*. Ha egy sorozat egyidejűleg q-konvex és q-konkáv, akkor a sorozatot q-*affin*nak nevezzük.

Könnyen belátható, hogy egy pozitív (vagy negatív) tagú sorozat szigorú konvexitásából következik a q-konvexitása, valamilyen q-ra. Hasonlóan, ha $p \in \mathcal{S}(n|m)$ egy negatív tagú, szigorúan konvex sorozat, akkor p q-konvex valamely $0 < q < 1$ esetén.

A q-konvex, illetve q-konkáv sorozatok halmazát $\mathcal{S}(n|m)$-ben rendre $\mathcal{C}_q^{\cup}(n|m)$ és $\mathcal{C}_q^{\cap}(n|m)$ jelöli. Végül a q-affin sorozatok halmazát $\mathcal{A}_q(n|m)$-mel jelöljük, azaz

$$\mathcal{A}_q(n|m) := \mathcal{C}_q^{\cup}(n|m) \cap \mathcal{C}_q^{\cap}(n|m).$$

Könnyen látható, hogy $\mathcal{A}_q(n|m)$ lineáris altere $\mathcal{S}(n|m)$-nek, továbbá hogy $\mathcal{C}_q^{\cup}(n|m)$ és $\mathcal{C}_q^{\cap}(n|m)$ konvex kúpok $\mathcal{S}(n|m)$-ben, azaz zártak a lineáris kombináció képzésre nemnegatív együtthatókkal.

Segéderedemények Csebisev-polinomokról

Jelölje $k \in \mathbb{Z}$ esetén $T_k : \mathbb{R} \to \mathbb{R}$ az elsőfajú-, $U_k : \mathbb{R} \to \mathbb{R}$ pedig a másodfajú k-adrendű Csebisev-polinomok halmazát, amelyek rendre a következő egyenletrendszerekkel definiálhatók:

$$\begin{aligned} T_0(x) &:= 1, \quad T_1(x) := x, \quad\; T_{k-1}(x) + T_{k+1}(x) = 2xT_k(x), \\ U_0(x) &:= 1, \quad U_1(x) := 2x, \quad U_{k-1}(x) + U_{k+1}(x) = 2xU_k(x). \end{aligned} \tag{2.3}$$

Az utolsó egyenlőségeket (2.3)-ban átírva, felhasználhatók T_k és U_k ($k \geq 2$) kiszámítására:

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x).$$

Átírva a fenti egyenleteket

$$T_{k-1}(x) = 2xT_k(x) - T_{k+1}(x), \quad U_{k-1}(x) = 2xU_k(x) - U_{k+1}(x),$$

alakba T_k és U_k kiszámítható $k \leq -1$ esetén. Könnyen igazolható, hogy ha $k \in \mathbb{Z}$, akkor

$$T_{-k} = T_k \quad \text{és} \quad U_{-k} = -U_{k-2}.$$

Speciálisan $U_{-1} = 0$. Világos, hogy ha $k \geq 0$, akkor T_k és U_k is k-adfokú. Jól ismert, hogy ezek a polinomok eleget tesznek bármely $u \in \mathbb{R}$ és $k \in \mathbb{Z}$ esetén az alábbi egyenleteknek:

$$T_k(\cos(u)) = \cos(ku), \qquad T_k(\cosh(u)) = \cosh(ku)$$

és

$$U_k(\cos(u)) = \frac{\sin((k+1)u)}{\sin(u)}, \quad U_k(\cosh(u)) = \frac{\sinh((k+1)u)}{\sinh(u)}.$$

Ezekből az alakokból jól látható, hogy T_k (ha $k \neq 0$) és U_{k-1} (ha $k \notin \{-1, 0, 1\}$) gyökei rendre megadhatók az alábbiak szerint:

$$\left\{ \cos\left(\frac{2i-1}{2k}\pi\right) \mid i \in \{1, \ldots, |k|\} \right\} \quad \text{és}$$
$$\left\{ \cos\left(\frac{i}{k}\pi\right) \mid i \in \{1, \ldots, |k|-1\} \right\}.$$

Tehát T_k (ha $k \neq 0$) és U_{k-1} (ha $k \notin \{-1, 0, 1\}$) legnagyobb gyöke rendre $\cos\left(\frac{\pi}{2k}\right)$ és $\cos\left(\frac{\pi}{k}\right)$ alakban adható meg.

q-konkáv, q-konvex és q-affin sorozatok

A következő állítás szerint az affin sorozatok $\mathcal{A}_q(n|m)$ halmaza kétdimenziós alterét alkotja $\mathcal{S}(n|m)$-nek.

2.1. Állítás. [MP22]. *Egy $p \in \mathcal{S}(n|m)$ sorozat pontosan akkor q-affin, ha léteznek olyan $a, b \in \mathbb{R}$ valós számok, hogy*

$$p_i := aU_{i-n}(q) + bT_{i-n}(q) \qquad (i \in \{n, \dots, m\}).$$

Továbbá, ha $p \in \mathcal{A}_q(n|m)$, akkor bármely $i, j, k \in \{n, \dots, m\}$ esetén

$$U_{k-j-1}(q)p_i + U_{j-i-1}(q)p_k = U_{k-i-1}(q)p_j.$$

Speciálisan, ha $i \in \{n, \dots, m\}$ és $j \in \{1, \dots, \min(i-n, m-i)\}$, akkor

$$p_{i-j} + p_{i+j} = 2T_j(q)p_i.$$

A következő állításban meghatározzuk a q-konkáv (és ezzel együtt a q-konvex) sorozatok néhány tulajdonságát.

2.2. Állítás. [MP22]. *A $\mathcal{C}_q^{\cap}(n|m)$ kúp zárt a pontonkénti minimumképzésre, a $\mathcal{C}_q^{\cup}(n|m)$ kúp pedig zárt a pontonkénti maximumképzésre.*

Mivel a q-affin sorozatok egyidejűleg q-konkávak és q-konvexek, ezért azt is kapjuk, hogy q-affin sorozatok véges családjának pontonkénti minimuma q-konkáv, a pontonkénti maximuma pedig q-konvex.

2.3. Állítás. [MP22]. *Legyenek $i, j, k \in \{n, \dots, m\}$ olyanok, hogy $i < j < k$ és tegyük fel, hogy*

$$q \geq \cos\left(\frac{\pi}{\max(j-i, k-j)}\right).$$

Ekkor bármely $p \in \mathcal{C}_q^{\cap}(n|m)$ esetén

$$U_{k-j-1}(q)p_i + U_{j-i-1}(q)p_k \leq U_{k-i-1}(q)p_j.$$

Speciálisan, ha $i \in \{n+1,\dots,m-1\}$ *és* $j \in \{1,\dots,\min(i-n,m-i)\}$, *valamint*

$$q > \cos\left(\frac{\pi}{j}\right),$$

akkor

$$p_{i-j} + p_{i+j} \le 2T_j(q)p_i.$$

2.4. Állítás. [MP22]. *Legyenek* $j, k \in \{n,\dots,m\}$ *olyanok, hogy* $j < k$. *Továbbá tegyük fel, hogy*

$$q > \cos\left(\frac{\pi}{k-j}\right).$$

Legyen $p \in \mathcal{C}^\cap(n|m)$ *és tekintsük a következő kifejezést*

$$r_i := p_k \frac{U_{i-j-1}(q)}{U_{k-j-1}(q)} + p_j \frac{U_{k-i-1}(q)}{U_{k-j-1}(q)} \qquad (i \in \{n,\dots,m\}).$$

Ekkor $r = (r_n,\dots,r_m)$ *egy* q*-affin sorozat és* $i \in \{n,\dots,m\}$ *esetén*

$$r_i \begin{cases} \ge p_i & \text{ha } i < j \text{ vagy } k < i. \\ = p_i & \text{ha } i \in \{j,k\}. \\ \le p_i & \text{ha } j < i < k. \end{cases}$$

A soron következő állításban megfogalmazzuk a q-konkáv sorozatok egy karakterizációját.

2.5. Állítás. [MP22]. *A* $p \in \mathcal{S}(n|m)$ *sorozat pontosan akkor* q*-konkáv, ha bármely* $j \in \{n,\dots,m-1\}$ *esetén van olyan* $r \in \mathcal{A}_q(n|m)$, *hogy*

$$p_j = r_j, \quad p_{j+1} = r_{j+1}, \quad \text{és} \quad p_i \le r_i \qquad (i \in \{n,\dots,m\}).$$

A q-konkáv sorozatok egy alkalmazása

A következőkben egy tetszőleges $a \in \mathcal{S}(1|n)$ sorozatot kiterjeszthetünk úgy, hogy $\mathcal{S}(0|n+1)$-beli legyen, ha $a_0 := 0$ és $a_{n+1} := 0$. Tetszőleges

$n \in \mathbb{N}$ és $\gamma = \big(\gamma_1, \dots, \gamma_{\lfloor \frac{n+1}{2} \rfloor}\big) \in \mathbb{R}^{\lfloor \frac{n+1}{2} \rfloor}$ vektor esetén definiáljuk a $\mathcal{T}_\gamma \colon \mathbb{R}^n \to \mathbb{R}^n$ leképezést az alábbi módon

$$\big(\mathcal{T}_\gamma(a)\big)_i := \min_{1 \le j \le \min(i, n+1-i)} \Big(\frac{a_{i-j} + a_{i+j}}{2} + \gamma_j\Big),$$

ahol $a \in \mathbb{R}^n$ és $i \in \{1, \dots, n\}$).

Azt szeretnénk megmutatni, hogy fenti $\mathcal{T}_\gamma$ leképezés kontrakció egy megfelelő $\mathbb{R}^n$-beli normában. Ehhez jelölje $|\cdot|_\infty$ a maximumnormát $\mathbb{R}^n$-ben. Legyen, egy $p \in \mathcal{S}(1|n)$ pozitív tagú sorozat esetén, $\|\cdot\|_p \colon \mathbb{R}^n \to \mathbb{R}$ a következőképpen definiálva

$$\|a\|_p := \max_{1 \le i \le n} p_i^{-1} |a_i| = |p^{-1} a|_\infty \qquad (a \in \mathbb{R}^n).$$

Ekkor $\|\cdot\|_p$ norma $\mathbb{R}^n$-ben, sőt ezzel a normával $\mathbb{R}^n$ Banach-tér.

2.6. Tétel. [MP22]. *Legyen $p \in \mathcal{S}(1|n)$ egy pozitív tagú sorozat, valamint*

$$q := \max_{1 \le i \le n} \frac{p_{i-1} + p_{i+1}}{2p_i} \qquad \text{és} \qquad q^* := \begin{cases} q & \text{ha } q \le 1, \\ T_{\lfloor \frac{n+1}{2} \rfloor}(q) & \text{ha } q > 1. \end{cases}$$

Ekkor a $\mathcal{T}_\gamma$ leképezés q^*-Lipschitz az $(\mathbb{R}^n, \|\cdot\|_p)$ normált téren ($\gamma \in \mathbb{R}^{\lfloor \frac{n+1}{2} \rfloor}$). Speciálisan, ha p szigorúan konkáv, akkor $\mathcal{T}_\gamma$ kontrakció.

2.7. Következmény. [MP22]. *Bármely $\gamma \in \mathbb{R}^{\lfloor \frac{n+1}{2} \rfloor}$ esetén a $\mathcal{T}_\gamma \colon \mathbb{R}^n \to \mathbb{R}^n$ leképezésnek egyértelműen létezik fixpontja $\mathbb{R}^n$-ben.*

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