

Multiplicative loops of 2-dimensional topological quasifields

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Abstract

We determine the algebraic structure of the multiplicative loops for locally compact 2-dimensional topological connected quasifields. In particular, our attention turns to multiplicative loops which have either a normal subloop of positive dimension or which contain a 1-dimensional compact subgroup. In the last section we determine explicitly the quasifields which coordinatize locally compact translation planes of dimension 4 admitting an at least 7-dimensional Lie group as collineation group.

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1. Introduction

Locally compact connected topological non-desarguesian translation planes have been a popular subject of geometrical research since the seventies of the last century ([18], [2]-[9], [13], [15]). These planes are coordinatized by locally compact quasifields Q such that the kernel of Q is either the field \mathbb{R} of real numbers or the field \mathbb{C} of complex numbers (cf. [11], IX.5.5 Theorem, p. 323). If the quasifield Q is 2-dimensional, then its kernel is \mathbb{R} .

The classification of topological translation planes \mathcal{A} was accomplished by reconstructing the spreads corresponding to \mathcal{A} from the translation complement which is the stabilizer of a point in the collineation group of \mathcal{A} . In this way all planes \mathcal{A} having an at least 7-dimensional collineation group have been determined ([3]-[8], [15]).

Although any spread gives the lines through the origin and hence the multiplication in a 2-dimensional quasifield Q coordinatizing the plane \mathcal{A} , to the algebraic structure of the multiplicative loop Q^* of a proper quasifield Q

is not given special attention apart from the facts that the group topologically generated by the left translations of Q^* is the connected component of $GL_2(\mathbb{R})$, the group topologically generated by the right translations of Q^* is an infinite-dimensional Lie group (cf. [14], Section 29, p. 345) and any locally compact 2-dimensional semifield is the field of complex numbers ([17]).

Since in the meantime some progress in the classification of compact differentiable loops on the 1-sphere has been achieved (cf. [10]), we believe that loops could have more space in the research concerning 4-dimensional translation planes. Using the images of differentiable sections $\sigma : G/H \rightarrow G$, where $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$, we classify the C^1 -differentiable multiplicative loops Q^* of 2-dimensional locally compact quasifields Q by functions, the Fourier series of which are described in [10].

The multiplicative loops Q^* of 2-dimensional locally compact left quasifields Q for which the set of the left translations of Q^* is the product \mathcal{TK} with $|\mathcal{T} \cap \mathcal{K}| \leq 2$, where \mathcal{T} is the set of the left translations of a 1-dimensional compact loop and \mathcal{K} is the set of the left translations of Q^* corresponding to the kernel K_r of Q , form an important subclass of loops, that we call decomposable loops. Namely, if Q^* has a normal subloop of positive dimension or if it contains the group $SO_2(\mathbb{R})$, then Q^* is decomposable. Moreover, we show that any 1-dimensional C^1 -differentiable compact loop is a factor of a decomposable multiplicative loop of a locally compact connected quasifield coordinatizing a 4-dimensional translation plane. A 2-dimensional locally compact quasifield Q is the field of complex numbers if and only if the multiplicative loop Q^* contains a 1-dimensional normal compact subloop.

Till now mainly those simple loops have been studied for which the group generated by their left translations is a simple group. If the group generated by the left translations of a loop L is simple, then L is also simple (cf. Lemma 1.7 in [14]). The multiplicative loops Q^* of 2-dimensional locally compact quasifields show that there are many interesting 2-dimensional locally compact quasi-simple loops for which the group generated by their left translations has a one-dimensional centre.

In the last section we use Betten's classification to determine in our framework the multiplicative loops Q^* of the quasifields which coordinatize the 4-dimensional non-desarguesian translation planes \mathcal{A} admitting an at least seven-dimensional collineation group and to study their properties. The results obtained there yield the following

Theorem *Let \mathcal{A} be a 4-dimensional locally compact non-desarguesian translation plane which admits an at least 7-dimensional collineation group Γ . If the quasifield Q coordinatizing \mathcal{A} is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the*

1-dimensional orbit of Γ or contain the set of the fixed points of Γ , then the multiplicative loop Q^* of Q is decomposable if and only if one of the following cases occurs:

- (a) Γ is 8-dimensional, the translation complement C is the group $\text{GL}_2(\mathbb{R})$ and acts reducibly on the translation group \mathbb{R}^4 ;
- (b) Γ is 7-dimensional, the translation complement C fixes two distinct lines of \mathcal{A} and leaves on one of them, one or two 1-dimensional subspaces invariant;
- (c) Γ is 7-dimensional, the translation complement C fixes two distinct lines $\{S, W\}$ through the origin and acts transitively on the spaces P_S and P_W but does not act transitively on the product space $P_S \times P_W$, where P_S and P_W are the sets of all 1-dimensional subspaces of S , respectively of W .

2. Preliminaries

A binary system (L, \cdot) is called a quasigroup if for any given $a, b \in L$ the equations $a \cdot y = b$ and $x \cdot a = b$ have unique solutions which we denote by $y = a \backslash b$ and $x = b / a$. If a quasigroup L has an element 1 such that $x = 1 \cdot x = x \cdot 1$ holds for all $x \in L$, then it is called a loop and 1 is the identity element of L . The left translations $\lambda_a : L \rightarrow L, x \mapsto a \cdot x$ and the right translations $\rho_a : L \rightarrow L, x \mapsto x \cdot a$, $a \in L$, are bijections of L . Two loops (L_1, \circ) and $(L_2, *)$ are called isotopic if there exist three bijections $\alpha, \beta, \gamma : L_1 \rightarrow L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \circ y)$ holds for all $x, y \in L_1$. A binary system (K, \cdot) is called a subloop of (L, \cdot) if $K \subset L$, for any given $a, b \in K$ the equations $a \cdot y = b$ and $x \cdot a = b$ have unique solutions in K and $1 \in K$. The kernel of a homomorphism $\alpha : (L, \cdot) \rightarrow (L', *)$ of a loop L into a loop L' is a normal subloop N of L , i.e. a subloop of L such that

$$x \cdot N = N \cdot x, (x \cdot N) \cdot y = x \cdot (N \cdot y), (N \cdot x) \cdot y = N \cdot (x \cdot y) \quad (1)$$

hold for all $x, y \in L$. A loop L is called simple if $\{1\}$ and L are its only normal subloops.

A loop L is called topological, if it is a topological space and the binary operations $(a, b) \mapsto a \cdot b$, $(a, b) \mapsto a \backslash b$, $(a, b) \mapsto b / a : L \times L \rightarrow L$ are continuous. Then the left and right translations of L are homeomorphisms of L . If L is a connected differentiable manifold such that the loop multiplication and the left division are continuously differentiable mappings, then we call L an almost \mathcal{C}^1 -differentiable loop. If also the right division of L is continuously differentiable, then L is a \mathcal{C}^1 -differentiable loop. A connected topological loop is quasi-simple if it contains no normal subloop of positive dimension. Every topological, respectively almost \mathcal{C}^1 -differentiable, connected loop L having a Lie group G as the group topologically generated by the left trans-

lations of L corresponds to a sharply transitive continuous, respectively \mathcal{C}^1 -differentiable section $\sigma : G/H \rightarrow G$, where $G/H = \{xH | x \in G\}$ consists of the left cosets of the stabilizer H of $1 \in L$ such that $\sigma(H) = 1_G$ and $\sigma(G/H)$ generates G . The section σ is sharply transitive if the image $\sigma(G/H)$ acts sharply transitively on the factor space G/H , i.e. for given left cosets xH, yH there exists precisely one $z \in \sigma(G/H)$ which satisfies the equation $zxH = yH$.

A (left) quasifield is an algebraic structure $(Q, +, \cdot)$ such that $(Q, +)$ is an abelian group with neutral element 0, $(Q \setminus \{0\}, \cdot)$ is a loop with identity element 1 and between these operations the (left) distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$ holds. A locally compact connected topological quasifield is a locally compact connected topological space Q such that $(Q, +)$ is a topological group, $(Q \setminus \{0\}, \cdot)$ is a topological loop, the multiplication $\cdot : Q \times Q \rightarrow Q$ is continuous and the mappings $\lambda_a : x \mapsto a \cdot x$ and $\rho_a : x \mapsto x \cdot a$ with $0 \neq a \in Q$ are homeomorphisms of Q . If for any given $a, b, c \in Q$ the equation $x \cdot a + x \cdot b = c$ with $a + b \neq 0$ has precisely one solution, then Q is called planar. A translation plane is an affine plane with transitive group of translations; this is coordinatized by a planar quasifield (cf. [16], Kap. 8).

The kernel K_r of a (left) quasifield Q is a skewfield defined by

$$K_r = \{k \in Q; (x+y) \cdot k = x \cdot k + y \cdot k \text{ and } (x \cdot y) \cdot k = x \cdot (y \cdot k) \text{ for all } x, y \in Q\}.$$

In this paper we consider left quasifields Q . Then Q is a right vector space over K_r . Moreover, for all $a \in Q$ the map $\lambda_a : Q \rightarrow Q, x \mapsto a \cdot x$ is K_r -linear. According to [12], Theorem 7.3, p. 160, every quasifield that has finite dimension over its kernel is planar.

Let F be a skewfield and let V be a vector space over F . A collection \mathcal{B} of subspaces of V with $|\mathcal{B}| \geq 3$ is called a spread of V if for any two different elements $U_1, U_2 \in \mathcal{B}$ we have $V = U_1 \oplus U_2$ and every vector of V is contained in an element of \mathcal{B} .

If S and W are different subspaces of the spread \mathcal{B} , then V can be coordinatized in such a way that $S = \{0\} \times X$ and $W = X \times \{0\}$. Any spread of $V = X \times X$ can be described by a collection \mathcal{M} of linear mappings $X \rightarrow X$ satisfying the following conditions:

- (M_1) For any $\omega_1 \neq \omega_2 \in \mathcal{M}$ the mapping $\omega_1 - \omega_2$ is bijective.
 - (M_2) For all $x \in X \setminus \{0\}$ the mapping $\phi_x : \mathcal{M} \rightarrow X : \omega \mapsto \omega(x)$ is surjective.
- Namely, if \mathcal{M} is a collection of linear mappings satisfying (M_1) and (M_2), then the sets $U_\omega = \{(x, \omega(x)), x \in X\}$ and $\{0\} \times X$ yield a spread of $V = X \times X$. Conversely, every component $U \in \mathcal{B} \setminus \{S\}$ of V is the graph of a linear mapping $\omega_U : W \rightarrow S$ and the set of ω_U gives a collection \mathcal{M} of linear mappings of X satisfying (M_1) and (M_2) (cf. [13], Proposition 1.11.).

The mapping ω_W is the zero mapping. For this reason any collection \mathcal{M} of linear mappings of X which satisfy (M_1) and M_2 is called a spread set of X . Every translation plane can be obtained from a spread set of a suitable vector space $V = X \times X$ (cf. [13], Theorem 1.5, p. 7, and [1]). As every translation plane can be coordinatized by a quasifield and a quasifield contains 0 and 1, the associated spread set contains the zero endomorphism and the identity map. This is not true for arbitrary spread sets \mathcal{M} , but if $\omega_0, \omega_1 \in \mathcal{M}$ are distinct, then $\mathcal{M}' = \{(\omega - \omega_0)(\omega_1 - \omega_0)^{-1}, \omega \in \mathcal{M}\}$ is a normalized spread set of X , i.e. a spread set which contains the zero and the identity map. The translation planes obtained from \mathcal{M} and \mathcal{M}' are isomorphic (cf. [13], Lemma 1.15, p. 13). Let \mathcal{M} be a normalized spread set of X , $e \in X \setminus \{0\}$ and let $\phi_e : \mathcal{M} \rightarrow X$ be defined by $\phi_e(\omega) = \omega(e)$. Then the multiplication $\circ : X \times X \rightarrow X$ defined by $m \circ x = (\phi_e^{-1}(m))(x)$ yields a multiplicative loop of a left quasifield Q coordinatizing the translation plane \mathcal{A} belonging to the spread \mathcal{M} of X .

If we fix a basis of Q over its kernel K_r and identify X with the vector space of pairs $\{(x, y)^t, x, y \in K_r\}$, then the set \mathcal{M} consists of matrices $C(\alpha, \beta, \gamma, \delta) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \alpha, \beta, \gamma, \delta \in K_r$. If $e = (1, 0)^t$, then we get $\phi_e(C(\alpha, \beta, \gamma, \delta)) = C(\alpha, \beta, \gamma, \delta)(e) = (\alpha, \gamma)^t$. Since \mathcal{M} is a spread of X the set of vectors $(\alpha, \gamma)^t$ consists of all vectors of X . Hence if $(\alpha, \gamma)^t$ is an element of X , then there exists a unique matrix of \mathcal{M} having $(\alpha, \gamma)^t$ as the first column.

We consider multiplicative loops of locally compact connected topological quasifields Q of dimension 2 coordinatizing 4-dimensional non-desarguesian topological translation planes. Then the kernel K_r of Q is isomorphic to the field of the real numbers, $(Q, +)$ is the vector group \mathbb{R}^2 and the multiplicative loop $(Q \setminus \{0\}, \cdot)$ is homeomorphic to $\mathbb{R} \times S^1$, where S^1 is the circle.

4. Multiplicative loops of 2-dimensional quasifields

Let $(Q, +, *)$ be a real topological (left) quasifield of dimension 2. Let e_1 be the identity element of the multiplicative loop $Q^* = (Q \setminus \{0\}, *)$ of Q , which generates the kernel $K_r = \mathbb{R}$ of Q as a vector space and let $B = \{e_1, e_2\}$ be a basis of the right vector space Q over K_r . Once we fix B , we identify Q with the vector space of pairs $(x, y)^t \in \mathbb{R}^2$ and K_r with the subspace of pairs $(x, 0)^t$. The element $(1, 0)^t$ is the identity element of Q^* . According to [14], Theorem 29.1, p. 345, the group G topologically generated by the left translations of Q^* is the connected component of the group $\text{GL}_2(\mathbb{R})$. As $\dim Q^* = 2$ and the stabilizer of the identity element of Q^* in G does not contain any non-trivial normal subgroup of G we may replace the stabilizer of the identity by the subgroup $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$. The elements g of G have a

unique decomposition as the product

$$g = \begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}$$

with suitable elements $u > 0$, $k > 0$, $l \in \mathbb{R}$, $t \in [0, 2\pi)$. Hence the loop Q^* corresponds to a continuous section $\sigma : G/H \rightarrow G$;

$$\begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} H \mapsto \begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix} \quad (2)$$

where the pair of continuous functions $a(u, t), b(u, t) : \mathbb{R}_{>0} \times [0, 2\pi) \rightarrow \mathbb{R}$, where $\mathbb{R}_{>0}$ is the set of positive numbers, satisfies the following conditions:

$$a(u, t) > 0, \quad a(1, 0) = 1, \quad b(1, 0) = 0.$$

As Q is a left quasifield, any $(x, y)^t \in Q^*$ induces a linear transformation $M_{(x, y)} \in \sigma(G/H)$. More precisely one has

$$\begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} = M_{(x, y)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r \cos \varphi & r \sin \varphi \\ -r \sin \varphi & r \cos \varphi \end{pmatrix} \begin{pmatrix} a(r, \varphi) & b(r, \varphi) \\ 0 & a^{-1}(r, \varphi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (3)$$

where $x = r \cos(\varphi)a(r, \varphi)$, $y = -r \sin(\varphi)a(r, \varphi)$. The kernel K_r of Q consists of $(0, 0)^t$ and $(r \cos(k\pi)a(r, k\pi), 0)^t$, $r > 0$, $k \in \{0, 1\}$, such that the matrices corresponding to the elements $(r \cos(k\pi)a(r, k\pi), 0)^t$ have the form

$$M(r \cos(k\pi)a(r, k\pi), 0) = \begin{pmatrix} r \cos(k\pi)a(r, k\pi) & r \cos(k\pi)b(r, k\pi) \\ 0 & r \cos(k\pi)a^{-1}(r, k\pi) \end{pmatrix}.$$

The identity matrix I corresponds to the identity $(1, 0)^t$ of Q^* . As to each real number $r \cos(k\pi)a(r, k\pi)$ belongs precisely one matrix

$$M(r \cos(k\pi)a(r, k\pi), 0)$$

the functions $f_1(r) = ra(r, 0)$, $r > 0$ and $f_2(r) = -ra(r, \pi)$, $r > 0$ are strictly monotone. If the functions $a(r, 0)$, $a(r, \pi)$ are differentiable, then for every $r > 0$ the derivatives $a(r, 0) + ra'(r, 0)$ and $-a(r, \pi) - ra'(r, \pi)$ are either always positive or negative. This is equivalent to the fact that the derivatives $[\ln(a(r, 0))]', [\ln(a(r, \pi))]'$ are always greater or smaller than $-r^{-1}$. Since the matrix $M(r \cos(k\pi)a(r, k\pi), 0)$ is isomorphic to the group $(\mathbb{R}, +)$ the function $a(r, 0)$ is a homomorphism and for the function $b(r, 0)$ the identity $b(r_1 r_2, 0) = a(r_1, 0)b(r_2, 0) + b(r_1, 0)a^{-1}(r_2, 0)$ is satisfied for all $r_1, r_2 > 0$.

Remark 1. The set $\mathcal{K} = \{M(r \cos(k\pi)a(r, k\pi), 0); r > 0, k \in \{0, 1\}\}$ of the left translations of Q^* corresponding to the kernel K_r of Q is

$$\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, r \in \mathbb{R} \setminus \{0\} \right\}$$

if and only if one has $a(r, k\pi) = 1$, $b(r, k\pi) = 0$ for all $r > 0$, $k \in \{0, 1\}$.

The section σ given by (2) is sharply transitive precisely if for all pairs $(u_1, t_1), (u_2, t_2)$ in $\mathbb{R}_{>0} \times [0, 2\pi)$ there exists precisely one $(u, t) \in \mathbb{R}_{>0} \times [0, 2\pi)$ and $k > 0, l \in \mathbb{R}$ such that

$$\begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix} \begin{pmatrix} u_1 \cos t_1 & u_1 \sin t_1 \\ -u_1 \sin t_1 & u_1 \cos t_1 \end{pmatrix} = \\ \begin{pmatrix} u_2 \cos t_2 & u_2 \sin t_2 \\ -u_2 \sin t_2 & u_2 \cos t_2 \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}. \quad (4)$$

As the determinant of the matrices on both sides of (4) are equal we get that $u = u_1^{-1}u_2$. Therefore the system (4) of equations is uniquely solvable if and only if for any fixed $u > 0$ the mapping

$$\sigma_u : \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} H \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix}$$

determines a quasigroup F_u homeomorphic to S^1 . One may take as the points of F_u the vectors $(ua(u, t)a^{-1}(u, 0) \cos t, -ua(u, t)a^{-1}(u, 0) \sin t)^t$ and as the section the mapping

$$\sigma_u : \begin{pmatrix} ua(u, t)a^{-1}(u, 0) \cos t \\ -ua(u, t)a^{-1}(u, 0) \sin t \end{pmatrix} \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u, t)a^{-1}(u, 0) & b(u, t) \\ 0 & a^{-1}(u, t)a(u, 0) \end{pmatrix} = \\ \begin{pmatrix} a(u, t)a^{-1}(u, 0) \cos t & b(u, t) \cos t + a^{-1}(u, t)a(u, 0) \sin t \\ -a(u, t)a^{-1}(u, 0) \sin t & -b(u, t) \sin t + a^{-1}(u, t)a(u, 0) \cos t \end{pmatrix}. \quad (5)$$

In this way we see that the quasigroup F_u has the right identity $(u, 0)^t$ since

$$\sigma_u \begin{pmatrix} ua(u, t)a^{-1}(u, 0) \cos t \\ -ua(u, t)a^{-1}(u, 0) \sin t \end{pmatrix} \cdot \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} ua(u, t)a^{-1}(u, 0) \cos t \\ -ua(u, t)a^{-1}(u, 0) \sin t \end{pmatrix}.$$

The quasigroup F_u is a loop, i.e. $(u, 0)^t$ is the left identity of F_u , if and only if

$$\sigma_u \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} a(u, 0)a^{-1}(u, 0) \cos 0 & b(u, 0) \cos 0 \\ 0 & a^{-1}(u, 0)a(u, 0) \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & b(u, 0) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which means $b(u, 0) = 0$ for all $u > 0$. The almost \mathcal{C}^1 -differentiable loop Q^* belonging to the sharply transitive \mathcal{C}^1 -differentiable section σ given by (2) is \mathcal{C}^1 -differentiable precisely if the mapping $(xH, yH) \mapsto z : G/H \times G/H \rightarrow \sigma(G/H)$ determined by $zxH = yH$ is \mathcal{C}^1 -differentiable (cf. [14], p. 32), i.e. the solutions $u > 0, t \in [0, 2\pi)$ of the matrix equation (4) are continuously differentiable functions of $u_1, u_2 \in \mathbb{R}_{>0}, t_1, t_2 \in [0, 2\pi)$. The function $u = u_1^{-1}u_2$ is continuously differentiable. If for each fixed $u > 0$ the section σ_u given by (5) yields a 1-dimensional \mathcal{C}^1 -differentiable compact loop, then the function $t(u_1, u_2, t_1, t_2) = t_{(u_1, u_2)}(t_1, t_2)$ is continuously differentiable (cf. [14], Examples 20.3, p. 258). Indeed, the function $t_{(u_1, u_2)}(t_1, t_2)$ is determined implicitly by equations which depend continuously differentiably also on the parameters u_1 and u_2 . Applying the above discussion we can prove the following:

Theorem 2. *Let Q^* be the \mathcal{C}^1 -differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield Q . Then Q^* is diffeomorphic to $S^1 \times \mathbb{R}$ and belongs to a \mathcal{C}^1 -differentiable sharply transitive section σ of the form*

$$\begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} H \mapsto \begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} \cdot \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix},$$

with $b(u, 0) = 0$ for all $u > 0$ if and only if for each fixed $u > 0$ the function $a_u^{-1}(t) := a(u, 0)a^{-1}(u, t)$ has the shape

$$a_u^{-1}(t) = e^t \left(1 - \int_0^t R(s) e^{-s} ds \right)$$

where $R(s)$ is a continuous function, the Fourier series of which is contained in the set \mathcal{F} of Definition 1 in [10] and converges uniformly to R . Moreover, $b_u(t) := b(u, t)$ is a periodic \mathcal{C}^1 -differentiable function with $b_u(0) = b_u(2\pi) = 0$ such that

$$b_u(t) > -a_u(t) \int_0^t \frac{(a_u^2(s) - a_u'^2(s))}{a_u^4(s)} ds \quad \text{for all } t \in (0, 2\pi).$$

Proof. The section σ_u given by (5) yields a 1-dimensional \mathcal{C}^1 -differentiable compact loop having the group $\text{SL}_2(\mathbb{R})$ as the group topologically generated by its left translations if and only if for each fixed $u > 0$ the continuously differentiable functions $a(u, 0)a^{-1}(u, t) := \bar{a}_u(t)$, $-b(u, t) := \bar{b}_u(t)$ satisfy the conditions

$$\bar{a}_u'^2(t) + \bar{b}_u(t)\bar{a}_u'(t) + \bar{b}_u'(t)\bar{a}_u(t) - \bar{a}_u^2(t) < 0, \quad \bar{b}_u'(0) < 1 - \bar{a}_u'^2(0) \quad (6)$$

(cf. [14], Section 18, (C), p. 238, [10], pp. 132-139). The solution of the differential inequalities (6) is given by Theorem 6 in [10], pp. 138-139. This proves the assertion. \square

Proposition 3. *Let Q^* be the \mathcal{C}^1 -differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield Q . Assume that for each fixed $u > 0$ the function $a_u(t) := a^{-1}(u, 0)a(u, t)$ is the constant function 1 and that $b(u, 0) = 0$ is satisfied for all $u > 0$. Then Q^* belongs to a \mathcal{C}^1 -differentiable sharply transitive section σ of the form (2) if and only if for each fixed $u > 0$ one has $b_u(t) := b(u, t) > -t$ for all $0 < t < 2\pi$.*

Proof. If for each fixed $u > 0$ the function $a(u, 0)a^{-1}(u, t) = a_u^{-1}(t) = \bar{a}_u(t)$ is constant with value 1, then the section σ_u given by (5) yields a \mathcal{C}^1 -differentiable compact loop L if and only if for each fixed $u > 0$ the continuously differentiable function $\bar{b}_u(t) := -b_u(t)$ satisfies the differential inequality $\bar{b}'_u(t) < 1$ with the initial condition $\bar{b}'_u(0) < 1$ (cf. (6)). This is the case precisely if one has $b_u(t) > -t$ for all $0 < t < 2\pi$. \square

Proposition 4. *Let Q^* be the \mathcal{C}^1 -differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield Q . Assume that for each fixed $u > 0$ the function $b(u, t)$ is the constant function 0. Then Q^* belongs to a \mathcal{C}^1 -differentiable sharply transitive section σ of the form (2) precisely if for each fixed $u > 0$ one has $e^{-t} < a(u, t)a^{-1}(u, 0) < e^t$ for all $0 < t < 2\pi$.*

Proof. If for each fixed $u > 0$ the function $b(u, t) = -\bar{b}_u(t)$ is constant with value 0, then the section σ_u given by (5) determines a \mathcal{C}^1 -differentiable compact loop L if and only if for each fixed $u > 0$ the following inequalities are satisfied:

$$(\bar{a}'_u(t) - \bar{a}_u(t))(\bar{a}'_u(t) + \bar{a}_u(t)) < 0, \quad 0 < 1 - \bar{a}_u^2(0),$$

where $\bar{a}_u(t) = a(u, 0)a^{-1}(u, t)$. This is the case precisely if either one has $\bar{a}'_u(t) - \bar{a}_u(t) < 0$ and $\bar{a}'_u(t) + \bar{a}_u(t) > 0$ or one has $\bar{a}'_u(t) - \bar{a}_u(t) > 0$ and $\bar{a}'_u(t) + \bar{a}_u(t) < 0$. Now we consider the first case. Then the function $\bar{a}_u(t)$ determines a loop if and only if for each fixed $u > 0$ it is a subfunction of a differentiable function $h_u(t) := h(u, t)$ with $h_u(0) = 1$, $h_u^2(0) = 1$, $h'_u(t) = h_u(t)$ and an upper function of a differentiable function $l_u(t) := l(u, t)$ with $l_u(0) = 1$, $l_u^2(0) = 1$, $l'_u(t) = -l_u(t)$ (cf. [19], p. 66). Hence for each fixed $u > 0$ the function $\bar{a}_u(t)$ is a subfunction of the function e^t and an upper function of the function e^{-t} for all $t \in (0, 2\pi)$. Therefore, any continuously differentiable function $\bar{a}_u(t)$ such that for each fixed $u > 0$ and for all $t \in (0, 2\pi)$ one has $e^{-t} < \bar{a}_u(t)^{-1} < e^t$ determines a \mathcal{C}^1 -differentiable compact loop L .

In the second case an analogous consideration as in the first case gives that for all fixed $u > 0$ the function $a(u, t)a^{-1}(u, 0)$ must be a subfunction of the function e^{-t} and an upper function of the function e^t for all $t \in (0, 2\pi)$. Hence in this case the function $a(u, t)a^{-1}(u, 0)$ does not exist. \square

Proposition 5. *Let*

$$\begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} \xrightarrow{H} \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1, t) & b(u, t) \\ 0 & a^{-1}(1, t) \end{pmatrix}, \quad u > 0, t \in [0, 2\pi) \quad (7)$$

with $b(u, 0) = 0$ for all $u > 0$ be a section belonging to a multiplicative loop Q^ of a locally compact 2-dimensional connected topological quasifield Q . Then Q^* contains for any $u > 0$ a 1-dimensional compact subloop.*

Proof. The image of the section (7) acts sharply transitively on the point set $\mathbb{R}^2 \setminus \{(0,0)^t\}$. Since the subgroup $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, u > 0 \right\}$ leaves any line through $(0,0)^t$ fixed, the subset

$$\mathcal{T} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix}, t \in [0, 2\pi) \right\} \quad (8)$$

acts sharply transitively on the oriented lines through $(0,0)^t$ for any $u > 0$. Therefore \mathcal{T} corresponds to a 1-dimensional compact loop since $b(u,0) = 0$ for all $u > 0$. \square

As \mathcal{T} given by (8) is the image of a section corresponding to a 1-dimensional compact subloop of Q^* , every element of \mathcal{T} is elliptic.

Proposition 6. *Every element of the set \mathcal{T} given by (8) is elliptic if and only if the following holds: $a(1, k\pi) = 1$ for $k \in \{0, 1\}$:*

1) *if for all $t \in [0, 2\pi)$ and $u > 0$ one has $b(u, t) = 0$, then the function $a(1, t)$ satisfies the inequalities:*

$$\frac{1 - |\sin(t)|}{|\cos(t)|} \leq a(1, t) \leq \frac{1 + |\sin(t)|}{|\cos(t)|}, \quad (9)$$

2) *if the function $b(u, t)$ is different from the constant function 0, then for $\sin(t) > 0$ one has*

$$\frac{(a(1, t) + a(1, t)^{-1}) \cos(t) - 2}{\sin(t)} < b(u, t) < \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) + 2}{\sin(t)}, \quad (10)$$

for $\sin(t) < 0$ we have

$$\frac{(a(1, t) + a(1, t)^{-1}) \cos(t) + 2}{\sin(t)} < b(u, t) < \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) - 2}{\sin(t)}. \quad (11)$$

Proof. Any element of (8) is elliptic if and only if the inequality

$$|\cos(t)(a(1, t) + a(1, t)^{-1}) - \sin(t)b(u, t)| \leq 2 \quad (12)$$

holds, where the equality sign occurs only for $t = k\pi$, $k \in \{0, 1\}$. Hence $a(1, k\pi) = 1$. If $b(u, t) = 0$, then inequality (12) reduces to $a^2(1, t)|\cos(t)| - 2a(1, t) + |\cos(t)| \leq 0$ which is equivalent to inequalities (9). If $b(u, t) \neq 0$, then inequality (12) is equivalent for all $t \neq k\pi$, $k \in \{0, 1\}$, to

$$(a(1, t) + a(1, t)^{-1})^2 \cos^2(t) - 2(a(1, t) + a(1, t)^{-1}) \sin(t) \cos(t) b(u, t) + \sin^2(t) b^2(u, t) < 4. \quad (13)$$

Solving the quadratic equation

$$(a(1, t) + a(1, t)^{-1})^2 \cos^2(t) - 2(a(1, t) + a(1, t)^{-1}) \sin(t) \cos(t)x + \sin^2(t)x^2 = 4 \quad (14)$$

we get

$$x = \frac{2(a(1, t) + a(1, t)^{-1}) \cos(t) \sin(t) \pm 4 \sin(t)}{2 \sin^2(t)} = \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) \pm 2}{\sin(t)}.$$

Comparing (13) and (14) one obtains

$$\left(b(u, t) - \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) - 2}{\sin(t)} \right) \left(b(u, t) - \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) + 2}{\sin(t)} \right) < 0$$

which yields inequalities (10) and (11). \square

Proposition 7. *Let Q^* be the multiplicative loop of a locally compact quasifield Q of dimension 2 containing a 1-dimensional compact normal subloop. The quasifield Q is the field \mathbb{C} of complex numbers if and only if \mathcal{T} is a normal subset in the set of all left translations of Q^* .*

Proof. If Q is the field of complex numbers, then Q^* is the group $\text{SO}_2(\mathbb{R}) \times \mathbb{R}$ and the assertion is true. If the set \mathcal{T} is a normal subset in the set of the left translations of a proper loop Q^* , then it is normal in the connected component $\text{GL}_2^+(\mathbb{R})$ of the group $\text{GL}_2(\mathbb{R})$ because $\text{GL}_2^+(\mathbb{R})$ is the group topologically generated by the left translations of Q^* (cf. [14], Section 29, p. 345). If \mathcal{T} is normal in $\text{GL}_2^+(\mathbb{R})$, then for $D = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$ one has that

$$D^{-1} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} D D^{-1} \begin{pmatrix} a(1, t) & b(1, t) \\ 0 & a^{-1}(1, t) \end{pmatrix} D =$$

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} D^{-1} \begin{pmatrix} a(1, t) & b(1, t) \\ 0 & a^{-1}(1, t) \end{pmatrix} D = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1, t) & b(1, t) \\ 0 & a^{-1}(1, t) \end{pmatrix}$$

is satisfied for all $\varphi \in [0, 2\pi)$ if and only if $a(1, t) = 1$ and $b(1, t) = 0$ or equivalently $\mathcal{T} = \text{SO}_2(\mathbb{R})$. But the compact group $\text{SO}_2(\mathbb{R})$ is not normal in $\text{GL}_2^+(\mathbb{R})$. Hence Q^* is not proper and the assertion follows. \square

Lemma 8. *If the multiplicative loop Q^* of a 2-dimensional locally compact connected topological quasifield Q is not quasi-simple, then the set $\mathcal{K} = \{M(r \cos(k\pi)a(r, k\pi), 0); r > 0, k \in \{0, 1\}\}$ of the left translations of Q^* belonging to the kernel K_r of Q has the form $\mathcal{K} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$, which is a normal subgroup of the set Λ_{Q^*} of all left translations of Q^* .*

Proof. If Q is the field of complex numbers, then the assertion is true. If the loop Q^* is proper and not quasi-simple, then by Lemma 1.7, p. 19, in [14], the left translations of a normal subloop of Q^* generate a normal subgroup N of $\text{GL}_2^+(\mathbb{R})$ which is the group topologically generated by all left translations

of Q^* . Hence the set Λ_{Q^*} of the left translations of Q^* must contain the group $C = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, 0 < u \in \mathbb{R} \right\}$ as a normal subgroup. It follows that the set \mathcal{K} of the left translations corresponding to the elements of the kernel K_r of Q contained in Λ_{Q^*} has the form given in the assertion and $\mathcal{K} \cap \Lambda_{Q^*}$ is a normal subgroup of Λ_{Q^*} . \square

Assume that the set \mathcal{K} of the left translations of the loop Q^* having $(1, 0)^t$ as identity corresponding to the elements of the kernel K_r of Q has the form given in Lemma 8. According to (3) the element

$$\begin{pmatrix} ra(r, \varphi) \cos \varphi & rb(r, \varphi) \cos \varphi + ra^{-1}(r, \varphi) \sin \varphi \\ -ra(r, \varphi) \sin \varphi & -rb(r, \varphi) \sin \varphi + ra^{-1}(r, \varphi) \cos \varphi \end{pmatrix}$$

corresponds to the left translation of $(ra(r, \varphi) \cos \varphi, -ra(r, \varphi) \sin \varphi)^t$, $r > 0$, $\varphi \in [0, 2\pi)$. Let N^* be the subgroup of Q^* corresponding to the normal subgroup \mathcal{K} of Λ_{Q^*} . We show that $N^* := \{(\hat{s}, 0)^t, \hat{s} \in \mathbb{R} \setminus \{0\}\}$ is normal in Q^* . For all elements $x := (\cos \varphi, -\sin \varphi)^t$, $y := (u, v)^t$ of Q^* the condition $(N^* * x) * y = N^* * (x * y)$ of (1) is satisfied if and only if we have

$$\left[\begin{pmatrix} \hat{s} \\ 0 \end{pmatrix} * \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} \right] * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \hat{s}' \\ 0 \end{pmatrix} * \left[\begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} \right]$$

for all $\varphi \in [0, 2\pi)$, $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ with suitable $\hat{s}, \hat{s}' \in \mathbb{R} \setminus \{0\}$ such that $\hat{s} = s \cos(k\pi)$, $\hat{s}' = s' \cos(k\pi)$, where $s, s' > 0$, $k \in \{0, 1\}$. This is the case precisely if one has

$$\begin{pmatrix} usa(s, \varphi + k\pi) \cos(\varphi + k\pi) + vsb(s, \varphi + k\pi) \cos(\varphi + k\pi) + vsa^{-1}(s, \varphi + k\pi) \sin(\varphi + k\pi) \\ -usa(s, \varphi + k\pi) \sin(\varphi + k\pi) - vsb(s, \varphi + k\pi) \sin(\varphi + k\pi) + vsa^{-1}(s, \varphi + k\pi) \cos(\varphi + k\pi) \end{pmatrix} =$$

$$\begin{pmatrix} s' \cos(k\pi)(ua(1, \varphi) \cos \varphi + vb(1, \varphi) \cos \varphi + va^{-1}(1, \varphi) \sin \varphi) \\ s' \cos(k\pi)(-ua(1, \varphi) \sin \varphi - vb(1, \varphi) \sin \varphi + va^{-1}(1, \varphi) \cos \varphi) \end{pmatrix}$$

or equivalently for all $\varphi \in [0, 2\pi)$, $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ we have

$$[(ua(s, \varphi + k\pi) + vb(s, \varphi + k\pi)) \cos(\varphi + k\pi) + va^{-1}(s, \varphi + k\pi) \sin(\varphi + k\pi)] \cdot [(-ua(1, \varphi) - vb(1, \varphi)) \sin \varphi + va^{-1}(1, \varphi) \cos \varphi] =$$

$$[(-ua(s, \varphi + k\pi) - vb(s, \varphi + k\pi)) \sin(\varphi + k\pi) + va^{-1}(s, \varphi + k\pi) \cos(\varphi + k\pi)] \cdot [(ua(1, \varphi) + vb(1, \varphi)) \cos \varphi + va^{-1}(1, \varphi) \sin \varphi].$$

The last equation holds if and only if one has

$$(a(s, \varphi + k\pi)a^{-1}(1, \varphi) - a^{-1}(s, \varphi + k\pi)a(1, \varphi))uv + (b(s, \varphi + k\pi)a^{-1}(1, \varphi) - a^{-1}(s, \varphi + k\pi)b(1, \varphi))v^2 = 0.$$

Therefore we obtain $a(s, \varphi + k\pi)a^{-1}(1, \varphi) - a^{-1}(s, \varphi + k\pi)a(1, \varphi) = 0$ and $b(s, \varphi + k\pi)a^{-1}(1, \varphi) - a^{-1}(s, \varphi + k\pi)b(1, \varphi) = 0$. As $a(s, \varphi)$ is positive we have $a(s, \varphi + k\pi) = a(1, \varphi)$, $b(s, \varphi + k\pi) = b(1, \varphi)$ for all $s > 0$, $\varphi \in [0, 2\pi)$, $k \in \{0, 1\}$. By (1) the group N^* is a normal subgroup of Q^* if and only if for all $\varphi \in [0, 2\pi)$, $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ one has

$$\left[\begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} * \begin{pmatrix} \hat{s} \\ 0 \end{pmatrix} \right] * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} * \left[\begin{pmatrix} \hat{s}' \\ 0 \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} \right] \quad \text{or}$$

$$\begin{pmatrix} sa(1, \varphi)a(1, \varphi)\cos(\varphi + k\pi) & sa(1, \varphi)b(1, \varphi)\cos(\varphi + k\pi) + ssin(\varphi + k\pi) \\ -sa(1, \varphi)a(1, \varphi)\sin(\varphi + k\pi) & -sa(1, \varphi)b(1, \varphi)\sin(\varphi + k\pi) + scos(\varphi + k\pi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} =$$

$$\begin{pmatrix} a(1, \varphi)\cos\varphi & b(1, \varphi)\cos\varphi + a^{-1}(1, \varphi)\sin\varphi \\ -a(1, \varphi)\sin\varphi & -b(1, \varphi)\sin\varphi + a^{-1}(1, \varphi)\cos\varphi \end{pmatrix} \begin{pmatrix} s'\cos(k\pi)u \\ s'\cos(k\pi)v \end{pmatrix}$$

for suitable $\hat{s}, \hat{s}' \in \mathbb{R} \setminus \{0\}$ such that $\hat{s} = s\cos(k\pi)$, $\hat{s}' = s'\cos(k\pi)$, where $s, s' > 0$, $k \in \{0, 1\}$. This is equivalent to

$$\begin{pmatrix} sua(1, \varphi)^2\cos(\varphi + k\pi) + sv[a(1, \varphi)b(1, \varphi)\cos(\varphi + k\pi) + \sin(\varphi + k\pi)] \\ -sua(1, \varphi)^2\sin(\varphi + k\pi) + sv[-a(1, \varphi)b(1, \varphi)\sin(\varphi + k\pi) + \cos(\varphi + k\pi)] \end{pmatrix} =$$

$$\begin{pmatrix} us'a(1, \varphi)\cos(\varphi + k\pi) + s'v[b(1, \varphi)\cos(\varphi + k\pi) + a^{-1}(1, \varphi)\sin(\varphi + k\pi)] \\ -us'a(1, \varphi)\sin(\varphi + k\pi) + s'v[-b(1, \varphi)\sin(\varphi + k\pi) + a^{-1}(1, \varphi)\cos(\varphi + k\pi)] \end{pmatrix}.$$

A direct computation yields that the above equality is true for all $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $\varphi \in [0, 2\pi)$, $k \in \{0, 1\}$.

Using Proposition 7, Lemma 8 and the discussion above we have the following

Theorem 9. *The multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q with $(1, 0)^t$ as identity of Q^* is not quasi-simple if and only if for all $r > 0$, $\varphi \in [0, 2\pi)$, $k \in \{0, 1\}$ one has $a(r, k\pi) = 1$, $b(r, k\pi) = 0$, $a(r, \varphi + k\pi) = a(1, \varphi)$ and $b(r, \varphi + k\pi) = b(1, \varphi)$. Then Q^* is a split extension of a 1-dimensional normal subgroup N^* by a subloop homeomorphic to the 1-sphere. Moreover, one has*

- a) N^* is isomorphic to $\mathbb{R} \times Z_2$, where Z_2 is the group of order 2.
- b) This extension is the direct product precisely if Q is the field \mathbb{C} .

Proof. We have only to prove a) and b). According to Lemma 8 and the above discussion the only possibility for a normal subloop of positive dimension is the group N^* . The intersection of a compact subloop of Q^* with N^* has cardinality 2 (cf. Proposition 5 and Lemma 8). Hence the claim a) is proved. The claim of b) follows from Proposition 7. \square

From Proposition 7 and Theorem 9 it follows:

Corollary 10. *The multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q with $(1, 0)^t$ as identity of Q^* is the direct product of the group \mathbb{R} and a subloop homeomorphic to the 1-sphere if and only if Q is the field of complex numbers.*

The set Λ_{Q^*} of the left translations of Q^* with a normal subloop of positive dimension and with $(1, 0)^t$ as identity can be written into the form

$$\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(1, t) & ub(1, t) \\ 0 & ua^{-1}(1, t) \end{pmatrix}, u > 0, t \in [0, 2\pi) \right\} \quad (15)$$

with $a(1, k\pi) = 1$, $b(1, k\pi) = 0$, $k \in \{0, 1\}$.

5. Decomposable multiplicative loops of 2-dimensional quasifields

Definition 1. We call the multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q decomposable, if the set of all left translations of Q^* is a product \mathcal{TK} with $|\mathcal{T} \cap \mathcal{K}| \leq 2$, where \mathcal{T} is the set of all left translations of a 1-dimensional compact loop of the form (8) and \mathcal{K} is the set of all left translations of Q^* belonging to the kernel K_r of Q .

If the loop Q^* is decomposable, then it contains compact subloops for any $u > 0$ corresponding to the section (7). From now on we choose $u = 1$. Then the compact loop F_u of Q^* has identity $(1, 0)^t$ and one has

$$\begin{pmatrix} \cos ta(1, t) & \cos tb(1, t) + \sin ta^{-1}(1, t) \\ -\sin ta(1, t) & -\sin tb(1, t) + \cos ta^{-1}(1, t) \end{pmatrix} \left[\begin{pmatrix} r \cos(k\pi)a(r, k\pi) & r \cos(k\pi)b(r, k\pi) \\ 0 & r \cos(k\pi)a^{-1}(r, k\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \\ = \begin{pmatrix} r \cos(t + k\pi)a(r, t + k\pi) & r \cos(t + k\pi)b(r, t + k\pi) + r \sin(t + k\pi)a^{-1}(r, t + k\pi) \\ -r \sin(t + k\pi)a(r, t + k\pi) & -r \sin(t + k\pi)b(r, t + k\pi) + r \cos(t + k\pi)a^{-1}(r, t + k\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (16)$$

Equation (16) yields that $a(r, t + k\pi) = a(1, t)a(r, k\pi)$.

Now we give sufficient and necessary conditions for the loop Q^* to be decomposable.

Proposition 11. The multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1, 0)^t$ as identity of Q^* is decomposable if and only if for all $r > 0$, $t \in [0, 2\pi)$, $k \in \{0, 1\}$ one has

$$a(r, t + k\pi) = a(1, t)a(r, k\pi), \quad b(r, t + k\pi) = a(1, t)b(r, k\pi) + a^{-1}(r, k\pi)b(1, t).$$

Proof. The point $(x, y)^t$ is the image of the point $(1, 0)^t$ under the linear mapping $M_{(x, y)}$ and the set $\{M_{(x, y)}; (x, y)^t \in Q^*\}$ acts sharply transitively on Q^* . The matrix equation

$$\begin{pmatrix} \cos ta(1, t) & \cos tb(1, t) + \sin ta^{-1}(1, t) \\ -\sin ta(1, t) & -\sin tb(1, t) + \cos ta^{-1}(1, t) \end{pmatrix} \left[\begin{pmatrix} r \cos(k\pi)a(r, k\pi) & r \cos(k\pi)b(r, k\pi) \\ 0 & r \cos(k\pi)a^{-1}(r, k\pi) \end{pmatrix} \begin{pmatrix} u \cos \varphi a(u, \varphi) \\ -u \sin \varphi a(u, \varphi) \end{pmatrix} \right] = \\ = \begin{pmatrix} r \cos(t + k\pi)a(r, t + k\pi) & r \cos(t + k\pi)b(r, t + k\pi) + r \sin(t + k\pi)a^{-1}(r, t + k\pi) \\ -r \sin(t + k\pi)a(r, t + k\pi) & -r \sin(t + k\pi)b(r, t + k\pi) + r \cos(t + k\pi)a^{-1}(r, t + k\pi) \end{pmatrix} \begin{pmatrix} u \cos \varphi a(u, \varphi) \\ -u \sin \varphi a(u, \varphi) \end{pmatrix} \quad (17)$$

holds precisely if the identities of the assertion are satisfied. \square

If Q^* is decomposable such that its compact subloop has identity $(1, 0)^t$, then $|\mathcal{T} \cap \mathcal{K}| = 2$ because one has $\mathcal{T} \cap \mathcal{K} = \left\{ I, \begin{pmatrix} -1 & -b(1, \pi) \\ 0 & -1 \end{pmatrix} \right\}$. In this case the set of all left translations of Q^* is a product \mathcal{TW} with $\mathcal{T} \cap \mathcal{W} = I$, where \mathcal{W} is the set of all left translations corresponding to the connected component of the kernel K_r of Q .

Theorem 12. If the multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1, 0)^t$ as identity of Q^* is not quasi-simple, then Q^* is decomposable.

Proof. By Theorem 9 the loop Q^* is not quasi-simple if and only if for all $r > 0$, $t \in [0, 2\pi)$, $k \in \{0, 1\}$ one has $a(r, k\pi) = 1$, $b(r, k\pi) = 0$, $a(r, t + k\pi) = a(1, t)$ and $b(r, t + k\pi) = b(1, t)$. Therefore the identities given in the assertion of Proposition 11 are satisfied. \square

Proposition 13. *The \mathcal{C}^1 -differentiable multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1, 0)^t$ as identity of Q^* is decomposable precisely if for the inverse function $\bar{a}(1, t) = a^{-1}(1, t)$ and for $\bar{b}(1, t) = -b(1, t)$ the differential inequalities*

$$\bar{a}'^2(1, t) + \bar{b}(1, t)\bar{a}'(1, t) + \bar{b}'(1, t)\bar{a}(1, t) - \bar{a}^2(1, t) < 0, \quad \bar{b}'(1, 0) < 1 - \bar{a}'^2(1, 0) \quad (18)$$

are satisfied.

Proof. If Q^* is a \mathcal{C}^1 -differentiable multiplicative loop of a quasifield Q with $(1, 0)^t$ as identity, then for $u = 1$ the continuously differentiable functions $\bar{a}_u(t) := a(u, 0)a^{-1}(u, t)$, $\bar{b}_u(t) := -b(u, t)$ satisfy the conditions in (6). The set of all left translations of Q^* is a product \mathcal{TK} if and only if one has $a(u, t + k\pi) = a(u, k\pi)a(1, t)$ and $b(u, t + k\pi) = a(1, t)b(u, k\pi) + a^{-1}(u, k\pi)b(1, t)$ for all $u > 0$, $t \in [0, 2\pi)$, $k \in \{0, 1\}$ (cf. Proposition 11). Putting this into (6) we get for $u = 1$

$$a'^2(1, t) + b(1, t)a'(1, t)a^2(1, t) - b'(1, t)a^3(1, t) - a^2(1, t) < 0 \text{ and} \\ b'(1, 0) > a'^2(1, 0) - 1 \quad (19)$$

since $a(1, k\pi) = 1$, $k \in \{0, 1\}$. Inequalities (19) are equivalent to the inequalities (18) with $\bar{a}(1, t) = a^{-1}(1, t)$ and $\bar{b}(1, t) = -b(1, t)$. \square

Corollary 14. *Let T be any 1-dimensional \mathcal{C}^1 -differentiable connected compact loop such that the set \mathcal{T} of its left translations has the form (8) and let \mathcal{K} be any set of matrices of the form*

$$\mathcal{K} = \left\{ \begin{pmatrix} u \cos(k\pi)a(u, k\pi) & u \cos(k\pi)b(u, k\pi) \\ 0 & u \cos(k\pi)a^{-1}(u, k\pi) \end{pmatrix}, u > 0, k \in \{0, 1\} \right\},$$

where $a(u, k\pi) > 0$ and $b(u, k\pi)$ are continuously differentiable functions such that $ua(u, 0)$, $-ua(u, \pi)$ are strictly monotone. Then the product \mathcal{TK} is the set of all left translations of a \mathcal{C}^1 -differentiable decomposable multiplicative loop Q^ of a 2-dimensional locally compact connected quasifield Q with $(1, 0)^t$ as identity of Q^* .*

Proof. Since one has

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1, t) & b(1, t) \\ 0 & a(1, t)^{-1} \end{pmatrix} \begin{pmatrix} u \cos(k\pi)a(u, k\pi) & u \cos(k\pi)b(u, k\pi) \\ 0 & u \cos(k\pi)a^{-1}(u, k\pi) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(t+k\pi) & \sin(t+k\pi) \\ -\sin(t+k\pi) & \cos(t+k\pi) \end{pmatrix} \begin{pmatrix} ua(u, k\pi)a(1, t) & ub(u, k\pi)a(1, t) + ub(1, t)a^{-1}(u, k\pi) \\ 0 & ua^{-1}(u, k\pi)a(1, t)^{-1} \end{pmatrix} = \\ \begin{pmatrix} \cos(t+k\pi) & \sin(t+k\pi) \\ -\sin(t+k\pi) & \cos(t+k\pi) \end{pmatrix} \begin{pmatrix} ua(u, t+k\pi) & ub(u, t+k\pi) \\ 0 & ua^{-1}(u, t+k\pi) \end{pmatrix},$$

for the continuously differentiable functions $a(1, t)$, $b(1, t)$ the inequalities (19) hold and $a(1, k\pi) = 1$, $k \in \{0, 1\}$, for $u = 1$ the continuously differentiable functions $\bar{a}_u(t+k\pi) = a(u, 0)a^{-1}(u, t+k\pi) = a(u, 0)a^{-1}(u, k\pi)a^{-1}(1, t)$, $\bar{b}_u(t+k\pi) = -b(u, t+k\pi) = -b(u, k\pi)a(1, t) - b(1, t)a^{-1}(u, k\pi)$ satisfy inequalities (6). Hence the product \mathcal{TK} given in the assertion is the image of a \mathcal{C}^1 -differentiable section of a multiplicative loop Q^* of a quasifield Q . \square

Proposition 15. *The set Λ_{Q^*} of all left translations of the multiplicative loop Q^* for a locally compact connected topological 2-dimensional quasifield Q with $(1, 0)^t$ as identity of Q^* contains the group $\text{SO}_2(\mathbb{R})$ if and only if Λ_{Q^*} has the form*

$$\Lambda_{Q^*} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(u, 0) & ub(u, 0) \\ 0 & ua^{-1}(u, 0) \end{pmatrix}, u > 0, t \in [0, 2\pi) \right\} \quad (20)$$

where $a(u, 0)$, $b(u, 0)$ are arbitrary continuous functions with $a(u, 0) > 0$ such that $ua(u, 0)$ is strictly monotone. In this case Q^* is decomposable.

Proof. If the set Λ_{Q^*} contains the group $\text{SO}_2(\mathbb{R})$, then for each fixed $u > 0$ the function $a(u, t)$ is constant with value 1 and the function $b(u, t)$ is constant with value 0. So the functions $a(u, t) = a(u, 0)$, $b(u, t) = b(u, 0)$ do not depend on the variable t . Hence the identities in Proposition 11 are satisfied and the set Λ_{Q^*} has the form \mathcal{TW} as in the assertion.

Conversely, if $ua(u, 0)$ is a strictly monotone continuous function, then for arbitrary continuous functions $a(u, 0)$, $b(u, 0)$ with $a(u, 0) > 0$ the set given by (20) is the set Λ_{Q^*} of all left translations of the multiplicative loop Q^* of a locally compact quasifield such that Λ_{Q^*} contains the group $\text{SO}_2(\mathbb{R})$ and Q^* has identity $(1, 0)^t$. Hence Q^* is decomposable and the assertion is proved. \square

6. Betten's classification of 4-dimensional translation planes

Using 2-dimensional spreads, Betten in [3], [4], [5], [6], [7], [8], see also [13] and [15], has classified all locally compact 4-dimensional translation planes which admit an at least 7-dimensional collineation group. His normalized 2-dimensional spreads are images of sharply transitive sections $\sigma' : G/H' \rightarrow G$, where G is the connected component of the group $\text{GL}_2(\mathbb{R})$, H' is the subgroup $\left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}, d > 0, c \in \mathbb{R} \right\}$ (cf. [2], [3]) and $\sigma'(G/H')$ consists of the elements

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ra(r, t) & 0 \\ 0 & r^{-1}a^{-1}(r, t) \end{pmatrix} \begin{pmatrix} 1 & b(r, t)a^{-1}(r, t) \\ 0 & r^2 \end{pmatrix}.$$

With respect to the stabilizer $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$ the sharply transitive section σ' transforms into a sharply transitive section $\sigma : G/H \rightarrow G$ given by (2), because the elements of $\sigma'(G/H')$ coincide with

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a(r, t) & b(r, t) \\ 0 & a^{-1}(r, t) \end{pmatrix}.$$

Proposition 16. *Let \mathcal{A} be a 4-dimensional non-desarguesian translation plane admitting an 8-dimensional collineation group such that \mathcal{A} is coordinatized by the locally compact topological quasifield Q . Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^* :*

a) Λ_{Q^*} has the form (15) with $a(1, t) = 1$ and $b(1, t) = 0$ for $0 \leq t \leq \pi$, $a(1, t) = 1/\sqrt{\cos^2 t + \frac{\sin^2 t}{w}}$ and $b(1, t) = a(1, t) \frac{1-w}{w} \sin t \cos t$ for $\pi < t < 2\pi$. The quasifields Q_w , $w > 1$, correspond to a one-parameter family of planes \mathcal{A}_w .

b) Λ_{Q^*} is the range of the section given by (2) such that for $\alpha \geq \frac{-3\beta^2}{4}$ one has $a(r, t) = \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha + \beta^2}$ and $b(r, t) = \frac{\beta(-\alpha+1)}{\sqrt{\alpha^2 + \beta^2}}$ with $r \cos(t) = \alpha \frac{\alpha + \beta^2}{\sqrt{\alpha^2 + \beta^2}}$, $r \sin(t) = -\beta \frac{\alpha + \beta^2}{\sqrt{\alpha^2 + \beta^2}}$.

For $\alpha < \frac{-3\beta^2}{4}$ we have $a(r, t) = 3\sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2}}$ and $b(r, t) = \beta\sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2}} + \frac{\beta\alpha}{3\sqrt{\alpha^2(\alpha^2 + \beta^2)}}$ with $r \cos(t) = \frac{\alpha}{3}\sqrt{\frac{\alpha^2}{\alpha^2 + \beta^2}}$, $r \sin(t) = -\frac{\beta}{3}\sqrt{\frac{\alpha^2}{\alpha^2 + \beta^2}}$. The quasifield Q coordinatizes a single plane.

c) Λ_{Q^*} is the range of the section given by (2) such that $a(r, t) = \sqrt{\frac{v^2 + s^2}{\frac{s^4}{3} + s^2 v + v^2}}$,

$$b(r, t) = \frac{-\frac{s^3}{3}v + s^3 + sv}{\sqrt{(\frac{s^4}{3} + s^2 v + v^2)(s^2 + v^2)}} \text{ with}$$

$$r \cos(t) = v\sqrt{\frac{\frac{s^4}{3} + s^2 v + v^2}{s^2 + v^2}}, \quad r \sin(t) = -s\sqrt{\frac{\frac{s^4}{3} + s^2 v + v^2}{s^2 + v^2}}.$$

The quasifield Q coordinatizes a single plane.

In case a) the multiplicative loop Q_w^* is decomposable and a split extension of the normal subgroup $\widetilde{N}^* \cong \mathbb{R}$ corresponding to the connected component of $\tilde{\mathcal{K}} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$ with a subloop homeomorphic to the 1-sphere.

In case c) the set \mathcal{K} of the left translations of Q^* corresponding to the kernel K_r of the quasifield Q has the form $\tilde{\mathcal{K}}$. In case b) the set \mathcal{K} has the form

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \alpha > 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \frac{\alpha}{9} \end{pmatrix}, \alpha < 0 \right\}.$$

In cases b) and c) the multiplicative loops Q^* are not decomposable and quasi-simple.

Proof. If the translation complement of \mathcal{A} is the group $\mathrm{GL}_2(\mathbb{R})$ and acts reducibly on \mathbb{R}^4 , then one obtains the one-parameter family \mathcal{A}_w , $w > 1$, of the non-desarguesian translation planes corresponding to the following spreads:

$$\{S\} \cup \left\{ \begin{pmatrix} s & -v \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v \geq 0 \right\} \cup \left\{ \begin{pmatrix} s & \frac{-v}{w} \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v < 0 \right\},$$

$w > 1$ (cf. [3], Satz 5, p. 144). Any such spread coincides with the set Λ_{Q^*} in (15) with $a(1, t)$ and $b(1, t)$ as in assertion a). By Theorem 9 the multiplicative loop Q_w^* is a split extension of a normal subgroup N^* with a 1-dimensional compact loop. By Theorem 12 the loop Q_w^* is decomposable.

Hence N^* has the form as in the assertion.

If the translation complement $\mathrm{GL}_2(\mathbb{R})$ acts irreducibly on \mathbb{R}^4 , then one obtains a single plane \mathcal{A} generated by the spread

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & -\alpha\beta - \beta^3 \\ \beta & \alpha + \beta^2 \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha \geq \frac{-3\beta^2}{4} \right\} \cup \left\{ \begin{pmatrix} \alpha & \frac{1}{3}\alpha\beta \\ \beta & \frac{\alpha}{9} + \frac{\beta^2}{3} \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha < \frac{-3\beta^2}{4} \right\}, \quad (21)$$

(cf. [5], Satz, p. 553).

If the translation complement is solvable, then one gets a single plane \mathcal{A} generated by the spread

$$\{S\} \cup \left\{ \begin{pmatrix} v & -\frac{s^3}{3} \\ s & s^2 + v \end{pmatrix}, s, v \in \mathbb{R} \right\}, \quad (22)$$

(cf. [4], Satz 2 (b), p. 331).

The spread (21), respectively (22) coincides with the image of the section σ in (2) with the well defined functions $a(r, t)$ and $b(r, t)$ given in assertion b), respectively c). In case c) one has $a(r, k\pi) = 1$, $b(r, k\pi) = 0$, for all $r > 0$, $k \in \{0, 1\}$ hence Remark 1 gives the form \mathcal{K} of the assertion. In case b) we have $a(r, 0) = 1$, $a(r, \pi) = 3$, $b(r, l\pi) = 0$, for all $r > 0$, $l \in \{0, 1\}$. These give the form \mathcal{K} of the assertion.

For decomposable Q^* , the identity $a(r, t + k\pi) = a(1, t)a(r, k\pi)$ holds for all $r > 0$, $t \in [0, 2\pi)$, $k \in \{0, 1\}$ (cf. Proposition 11). In case b) for $-3 \leq \alpha \leq 1$ one has $a(1, t) = \sqrt{\alpha^2 - \alpha + 1}$ which yields a contradiction. In case c) we have $a(r, \frac{\pi}{4} + k\pi) = \sqrt{\frac{2}{1-s+\frac{s^2}{3}}}$, $s \in \mathbb{R} \setminus \{0\}$ and the condition $a(r, \frac{\pi}{4} + k\pi) = a(1, \frac{\pi}{4})$ gives a contradiction. Hence in both cases Q^* is not decomposable and therefore quasi-simple (cf. Theorem 12). \square

If the translation complement of a 4-dimensional topological plane \mathcal{A} has dimension 3, then the point ∞ of the line $S = \{(0, 0, u, v), u, v \in \mathbb{R}\}$ is fixed under the seven-dimensional collineation group Γ of \mathcal{A} .

Proposition 17. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane \mathcal{A} such that the 7-dimensional collineation group Γ of \mathcal{A} acts transitively on the points of $W \setminus \{\infty\}$, where W is the translation axis of \mathcal{A} and the kernel of the action of the translation complement on the line S has dimension 1. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^* :*

a) Λ_{Q^*} is the range of the section (2) such that

$$a(r, t) = \sqrt{\frac{s^2 + v^2}{s^2v + v^2 + \frac{s^4}{3} + s^2}} \text{ and } b(r, t) = \frac{s^3 - \frac{s^3v}{3}}{\sqrt{(s^2v + v^2 + \frac{s^4}{3} + s^2)(s^2 + v^2)}}$$

with $r \cos(t) = v \sqrt{\frac{s^2v + v^2 + \frac{s^4}{3} + s^2}{s^2 + v^2}}$, $r \sin(t) = -s \sqrt{\frac{s^2v + v^2 + \frac{s^4}{3} + s^2}{s^2 + v^2}}$. The quasifield Q corresponds to a single plane.

b) Λ_{Q^*} is the range of the section given by (2) such that

$$a(r, t) = \sqrt{\frac{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2v\gamma \sin(u) - 2\gamma u \cos(u) + 2\gamma u}{v^2 + u^2 - 2\gamma^2 + 2\gamma^2 \cos(u)}} \text{ and}$$

$$b(r, t) = \frac{-2u\gamma \sin u + 2v\gamma \cos u - 2v\gamma}{\sqrt{v^2 + u^2 + 2\gamma^2(1 - \cos u) - 2v\gamma \sin u - 2\gamma u \cos u + 2\gamma u \sqrt{v^2 + u^2 - 2\gamma^2(1 - \cos u)}}}$$

with

$$r \cos(t) = (v - \gamma \sin(u)) \sqrt{\frac{v^2 + u^2 - 2\gamma^2 + 2\gamma^2 \cos u}{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2v\gamma \sin(u) - 2\gamma u \cos(u) + 2\gamma u}},$$

$$r \sin(t) = (u - \gamma(\cos(u) - 1)) \sqrt{\frac{v^2 + u^2 - 2\gamma^2 + 2\gamma^2 \cos u}{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2v\gamma \sin(u) - 2\gamma u \cos(u) + 2\gamma u}}.$$

The quasifields Q_γ coordinatize a one-parameter family of planes $\mathcal{A}_\gamma, 0 < |\gamma| \leq 1$.

In all cases the multiplicative loop Q^* is not decomposable and quasi-simple. The set \mathcal{K} of the left translations of Q^* corresponding to the kernel of the quasifield Q has the form $\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$.

Proof. If the translation complement C leaves a 1-dimensional subspace of S invariant, then one obtains a single plane \mathcal{A} which corresponds to the following spread:

$$\{S\} \cup \left\{ \begin{pmatrix} v & -\frac{s^3}{3} - s \\ s & s^2 + v \end{pmatrix}, s, v \in \mathbb{R} \right\} \quad (23)$$

(cf. [18], 73.10., [4], pp. 330-331).

If the translation complement acts transitively on the 1-dimensional subspaces of S , then one gets a one-parameter family $E_\gamma, 0 < |\gamma| \leq 1$, of planes which are generated by the normalized spread

$$\{S\} \cup \left\{ \begin{pmatrix} v - \gamma \sin u & u + \gamma(\cos u - 1) \\ \gamma(\cos u - 1) - u & v + \gamma \sin u \end{pmatrix}, u, v \in \mathbb{R} \right\}, \quad (24)$$

([8], Satz, p. 128, [13], Proposition 5.8). The spread (23), respectively (24) coincides with the image of the section σ in (2) such that the well defined functions $a(r, t)$ and $b(r, t)$ are given in assertion a), respectively b). Since in both cases one has $a(r, k\pi) = 1, b(r, k\pi) = 0$, for all $r > 0, k \in \{0, 1\}$, Remark 1 gives the form of \mathcal{K} . Moreover, in case a) one has $a(r, \frac{\pi}{4} + k\pi) = \sqrt{\frac{2}{2+v+\frac{v^2}{3}}}, v \in \mathbb{R} \setminus \{0\}$. In case b) for $v = 1$ we get

$$a(r_j, t_j) = \sqrt{\frac{1+u^2+2\gamma^2(1-\cos u)-2\gamma\sin u-2\gamma u\cos u+2\gamma u}{1+u^2-2\gamma^2+2\gamma^2\cos u}}, \quad a(1, t_j) = 1.$$

For decomposable Q^* one has $a(r, t+k\pi) = a(1, t)a(r, k\pi)$ for all $r \in \mathbb{R} \setminus \{0\}, t \in [0, 2\pi), k \in \{0, 1\}$ (cf. Proposition 11) which yields a contradiction. Thus in both cases Q^* is not decomposable and hence quasi-simple (cf. Theorem 12). \square

Proposition 18. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane \mathcal{A} such that the translation complement C of the 7-dimensional collineation group Γ of \mathcal{A} has an orbit of dimension 1 on $W \setminus \{0\}$, C leaves in the set of lines through the origin only S fixed and the kernel of its action on S has positive dimension. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^* :*

a) Λ_{Q^*} is the range of the section (2) such that for $\beta \geq 0$ one has

$$a(r, t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + z\alpha\beta\frac{1}{1+s} - w\beta\frac{2}{1+s}}} \text{ and } b(r, t) = \frac{w\alpha\beta\frac{1-s}{1+s} + \alpha\beta + z\beta\frac{2+s}{1+s}}{\sqrt{\alpha^2 + \beta^2}\sqrt{\alpha^2 + z\alpha\beta\frac{1}{1+s} - w\beta\frac{2}{1+s}}}$$

$$\text{with } r \cos(t) = \alpha \sqrt{\frac{\alpha^2 + z\alpha\beta\frac{1}{1+s} - w\beta\frac{2}{1+s}}{\alpha^2 + \beta^2}}, \quad r \sin(t) = -\beta \sqrt{\frac{\alpha^2 + z\alpha\beta\frac{1}{1+s} - w\beta\frac{2}{1+s}}{\alpha^2 + \beta^2}}.$$

For $\beta < 0$ one gets

$$a(r', t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + q\alpha(-\beta)\frac{1}{1+s} + p(-\beta)\frac{2}{1+s}}} \text{ and } b(r', t) = \frac{p\alpha(-\beta)\frac{1-s}{1+s} + \alpha\beta - q(-\beta)\frac{2+s}{1+s}}{\sqrt{\alpha^2 + \beta^2}\sqrt{\alpha^2 + q\alpha(-\beta)\frac{1}{1+s} + p(-\beta)\frac{2}{1+s}}}$$

with

$$r' \cos(t) = \alpha \sqrt{\frac{\alpha^2 + q\alpha(-\beta)\frac{1}{1+s} + p(-\beta)\frac{2}{1+s}}{\alpha^2 + \beta^2}} \text{ and } r' \sin(t) = -\beta \sqrt{\frac{\alpha^2 + q\alpha(-\beta)\frac{1}{1+s} + p(-\beta)\frac{2}{1+s}}{\alpha^2 + \beta^2}}.$$

The quasifields $Q_{s,w,z,p,q}$ coordinatize a family of planes $\mathcal{A}_{s,w,z,p,q}$ such that the parameters s, w, z, p, q satisfy the conditions $0 < s < 1$, $z^2 + 4w(1 - s^2) \leq 0$, $q^2 - 4p(1 - s^2) \leq 0$.

b) Λ_{Q^*} is the range of the section (2) such that for $\beta \geq 0$ we have

$$a(r, t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + z\alpha\beta - w\beta^2 + 2\alpha\beta \ln \beta + z\beta^2 \ln \beta + \beta^2 (\ln \beta)^2}} \text{ and}$$

$$b(r, t) = \frac{(w+1)\alpha\beta + z\beta^2 - z\alpha\beta \ln \beta - \alpha\beta (\ln \beta)^2 + 2\beta^2 \ln \beta}{\sqrt{\alpha^2 + \beta^2} \sqrt{\alpha^2 + z\alpha\beta + 2\alpha\beta \ln \beta - w\beta^2 + z\beta^2 \ln \beta + \beta^2 (\ln \beta)^2}}$$

with

$$r \cos(t) = \alpha \sqrt{\frac{\alpha^2 + z\alpha\beta - w\beta^2 + 2\alpha\beta \ln \beta + z\beta^2 \ln \beta + \beta^2 (\ln \beta)^2}{\alpha^2 + \beta^2}},$$

$$r \sin(t) = -\beta \sqrt{\frac{\alpha^2 + z\alpha\beta - w\beta^2 + 2\alpha\beta \ln \beta + z\beta^2 \ln \beta + \beta^2 (\ln \beta)^2}{\alpha^2 + \beta^2}}.$$

For $\beta < 0$ we obtain

$$a(r', t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2) \ln(-\beta) + \beta^2 (\ln(-\beta))^2}} \text{ and}$$

$$b(r', t) = \frac{(1-p)\alpha\beta - q\beta^2 + (2\beta^2 + q\alpha\beta) \ln(-\beta) - \alpha\beta (\ln(-\beta))^2}{\sqrt{\alpha^2 + \beta^2} \sqrt{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2) \ln(-\beta) + \beta^2 (\ln(-\beta))^2}}$$

with

$$r' \cos(t) = \alpha \sqrt{\frac{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2) \ln(-\beta) + \beta^2 (\ln(-\beta))^2}{\alpha^2 + \beta^2}},$$

$$r' \sin(t) = -\beta \sqrt{\frac{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2) \ln(-\beta) + \beta^2 (\ln(-\beta))^2}{\alpha^2 + \beta^2}}.$$

The quasifields $Q_{w,z,p,q}$ coordinatize a family of planes $\mathcal{A}_{w,z,p,q}$ such that for the parameters w, z, p, q the relations $\left(\frac{z}{2}\right)^2 \leq -w - 1$, $\left(\frac{q}{2}\right)^2 \leq p - 1$ hold.

c) Λ_{Q^*} is the range of the section given by (2) such that $a(r, k\pi) = 1$ and $b(r, k\pi) = 0$, $k \in \{0, 1\}$ with $r = |\beta|$.

For $u \in \mathbb{R}$, $\beta > 0$, we get

$$a(r, t) = \sqrt{\frac{u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z) \sin(l) \cos(l)}{u^2 + uz - w}},$$

$$b(r, t) = \frac{\cos^2(l)(2uw + 2u + 2z) + \sin(l) \cos(l)(1 - w^2 - z^2 - 2uz) - (u + z + uw)}{\sqrt{(u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z) \sin(l) \cos(l))(u^2 + uz - w)}}$$

with

$$r \cos(t) = \beta \left(u - (w+1) \sin(l) \cos(l) + z \sin^2(l) \right) \sqrt{\frac{u^2 + uz - w}{u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z) \sin(l) \cos(l)}},$$

$$r \sin(t) = \beta \left(w \sin^2(l) + z \sin(l) \cos(l) - \cos^2(l) \right) \sqrt{\frac{u^2 + uz - w}{u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z) \sin(l) \cos(l)}},$$

where $l = \frac{1}{k} \ln \beta$. For $u \in \mathbb{R}$, $\beta < 0$ one obtains

$$a(r', t') = \sqrt{\frac{u^2 + \sin^2(l_1)(q^2 + 2qu + p^2) + \cos^2(l_1) + (2u + 2q - 2up) \sin(l_1) \cos(l_1)}{u^2 + uq + p}}$$

$$b(r', t') = \frac{\sin(l_1) \cos(l_1)(1 - 2uq - p^2 - q^2) + \sin^2(l_1)(2q + 2u - 2up) + (up - q - u)}{\sqrt{(u^2 + \sin^2(l_1)(q^2 + 2qu + p^2) + \cos^2(l_1) + (2q + 2u - 2up) \sin(l_1) \cos(l_1))(u^2 + uq + p)}}$$

with

$$\begin{aligned} r' \cos(t') &= \beta \left((p-1) \sin(l_1) \cos(l_1) - q \sin^2(l_1) - u \right) \cdot \\ &\quad \sqrt{\frac{u^2 + uq + p}{u^2 + \sin^2(l_1)(q^2 + 2qu + p^2) + \cos^2(l_1) + (2u + 2q - 2up) \sin(l_1) \cos(l_1)}}, \\ r' \sin(t') &= -\beta \left(\cos^2(l_1) + q \sin(l_1) \cos(l_1) + p \sin^2(l_1) \right) \cdot \\ &\quad \sqrt{\frac{u^2 + uq + p}{u^2 + \sin^2(l_1)(q^2 + 2qu + p^2) + \cos^2(l_1) + (2u + 2q - 2up) \sin(l_1) \cos(l_1)}}, \end{aligned}$$

where $l_1 = \frac{1}{k} \ln(-\beta)$.

The quasifields $Q_{k,w,z,p,q}$ coordinatize a family of planes $\mathcal{A}_{k,w,z,p,q}$ such that for the parameters k, w, z, p, q one has $k \neq 0$, $(4+k^2)(z^2 + (w+1)^2) \leq k^2(1-w)^2$, $(4+k^2)(q^2 + (p-1)^2) \leq k^2(p+1)^2$, $(w, z, p, q) \neq (-1, 0, 1, 0)$.

In all cases Q^* is not decomposable and quasi-simple. The set of the left translations of Q^* belonging to the kernel of Q is $\mathcal{K} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$.

Proof. If the translation complement C fixes two 1-dimensional subspaces of S , then we have a family of translation planes $\mathcal{A}_{s,w,z,p,q}$ such that the normalized spreads belonging to these planes are given as follows:

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & w\beta \frac{1-s}{1+s} \\ \beta & z\beta \frac{1}{1+s} + \alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta \geq 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & p(-\beta) \frac{1-s}{1+s} \\ \beta & q(-\beta) \frac{1}{1+s} + \alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta < 0 \right\}, \quad (25)$$

(cf. [6], Satz 1, pp. 411-412).

If the translation complement C fixes only one 1-dimensional subspace of S , then there is a family of translation planes $\mathcal{A}_{w,z,p,q}$ such that the corresponding normalized spreads have the form:

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & w\beta - z\beta \ln \beta - \beta(\ln \beta)^2 \\ \beta & \alpha + z\beta + 2\beta \ln \beta \end{pmatrix}, \alpha \in \mathbb{R}, \beta \geq 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & -p\beta - \beta(\ln(-\beta))^2 + q\beta \ln(-\beta) \\ \beta & q(-\beta) + \alpha + 2\beta \ln(-\beta) \end{pmatrix}, \alpha \in \mathbb{R}, \beta < 0 \right\} \quad (26)$$

(cf. Satz 2, [6], pp. 418-419).

If the translation complement C acts transitively on the 1-dimensional subspaces of S , then we have a family of translation planes $\mathcal{A}_{k,w,z,p,q}$ such that the normalized spreads belonging to these planes have the form

$$\begin{aligned} &\{S\} \cup \left\{ \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta \in \mathbb{R} \right\} \cup \\ &\left\{ \beta \begin{pmatrix} u - (w+1) \sin(l) \cos(l) + z \sin^2(l) & w \cos^2(l) - z \sin(l) \cos(l) - \sin^2(l) \\ \cos^2(l) - z \sin(l) \cos(l) - w \sin^2(l) & z \cos^2(l) + (w+1) \sin(l) \cos(l) + u \end{pmatrix}, u \in \mathbb{R}, \beta > 0 \right\} \cup \\ &\left\{ \beta \begin{pmatrix} (p-1) \sin(l_1) \cos(l_1) - q \sin^2(l_1) - u & q \sin(l_1) \cos(l_1) - p \cos^2(l_1) - \sin^2(l_1) \\ \cos^2(l_1) + q \sin(l_1) \cos(l_1) + p \sin^2(l_1) & (1-p) \sin(l_1) \cos(l_1) - q \cos^2(l_1) - u \end{pmatrix}, u \in \mathbb{R}, \beta < 0 \right\}, \quad (27) \end{aligned}$$

where $l = \frac{1}{k} \ln \beta$, $l_1 = \frac{1}{k} \ln(-\beta)$ (cf. [15], Proposition 4.1, p. 6, and [6], Satz 3, pp. 422-423). The spreads (25), respectively (26), respectively (27) coincide with the image of the section σ in (2) such that the well defined functions $a(r, t)$ and $b(r, t)$ are given in assertion a), respectively b), respectively c).

Since in all three cases we have $a(r, k\pi) = 1$, $b(r, k\pi) = 0$, $r > 0$, $k \in \{0, 1\}$, Remark 1 shows that \mathcal{K} has the form as in the assertion. In case a), respectively b) for $\beta > 0$ one gets $a(r, \frac{\pi}{4} + \pi) = \frac{\sqrt{2}\beta}{\sqrt{\beta^2 - z\beta\frac{2+s}{1+s} - w\beta\frac{2}{1+s}}}$, respectively

$a(r, \frac{\pi}{4} + \pi) = \sqrt{\frac{2}{1-z-w-2\ln\beta+z\ln\beta+(\ln\beta)^2}}$. In case c) for $u = 0$, $\beta > 0$ we get that $a(1, t_j)$ is constant. These relations give a contradiction to the condition $a(r, t + k\pi) = a(1, t)$, $r > 0$, $t \in [0, 2\pi)$, $k \in \{0, 1\}$ of Proposition 11. Hence in all cases Q^* is not decomposable and quasi-simple (cf. Theorem 12). \square

Proposition 19. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane \mathcal{A} such that the translation complement C of the 7-dimensional collineation group Γ of \mathcal{A} has an orbit of dimension 1 on $W \setminus \{0\}$, C leaves only S in the set of lines through the origin fixed and the kernel of its action on S is zero-dimensional. Then the set Λ_{Q^*} of all left translations of the multiplicative loop Q^* is given by the range of the section (2) defined as follows: For $\alpha \geq -\frac{\beta^2}{2}$ one has*

$$a(r, t) = \sqrt{\frac{\alpha^2 + \beta^2}{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right) \left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q} \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

$$b(r, t) = \frac{\frac{p}{q}\alpha \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}} - \frac{p}{q}(\alpha^2 + \beta^2) + \frac{1-q}{q}\beta\alpha^2 + \frac{\alpha\beta^3}{6q} - \frac{\beta^3\alpha}{2} + \frac{\beta^3}{2q} + \frac{\beta^3}{2} + \alpha\beta}{\sqrt{\alpha^2 + \beta^2} \sqrt{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right) \left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q} \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

with

$$r \cos(t) = \alpha \sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right) \left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q} \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}},$$

$$r \sin(t) = -\beta \sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right) \left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q} \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}}.$$

For $\alpha < -\frac{\beta^2}{2}$ we get

$$a(r, t) = \sqrt{\frac{\alpha^2 + \beta^2}{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right) \left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q} \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

$$b(r, t) = \frac{\frac{w}{q}\alpha \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{p}{q}(-\alpha^2 - \beta^2) + \left(\frac{z+1}{q}\alpha\beta - \frac{z\beta}{q}\right) \left(\alpha + \frac{\beta^2}{2}\right) - \frac{\alpha\beta^3}{3q} + \frac{\beta^3}{2q}}{\sqrt{\alpha^2 + \beta^2} \sqrt{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right) \left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q} \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

with

$$r \cos(t) = \alpha \sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right) \left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q} \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}},$$

$$r \sin(t) = -\beta \sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right) \left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q} \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}}.$$

The quasifields $Q_{w,z,p,q}$ coordinatize a family of planes $\mathcal{A}_{w,z,p,q}$ such that the parameters w, z, p, q satisfy $(3w)^2 \leq -16z(z+1)$, $(3p)^2 \leq 16q(q-1)$, $q > 0$, $z < 0$ and $(w, z, p, q) \neq (0, -\frac{1}{3}, 0, 3)$.

The multiplicative loops $Q_{w,z,p,q}^*$ of the quasifields $Q_{w,z,p,q}$ are not decomposable and quasi-simple. The left translations of $Q_{w,z,p,q}^*$ corresponding to the kernel of $Q_{w,z,p,q}$ have the form $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$, $0 \neq r \in \mathbb{R}$, if and only if $w = p = 0$, $q = -z = 1$.

Proof. By Satz 5 in [6], the planes $\mathcal{A}_{w,z,p,q}$ are determined by the normalized spreads which have the form

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & -\frac{p}{q}\alpha + \frac{p}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{(1-q)}{q}\beta\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\beta^3}{3q} \\ \beta & -\frac{p}{q}\beta + \frac{\beta^2}{2q} + \left(\alpha + \frac{\beta^2}{2}\right) \end{pmatrix}, \beta \in \mathbb{R}, \alpha \geq -\frac{\beta^2}{2} \right\} \cup$$

$$\left\{ \begin{pmatrix} \alpha & -\frac{p}{q}\alpha + \frac{w}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{(z+1)}{q}\beta\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\beta^3}{3q} \\ \beta & -\frac{p}{q}\beta + \frac{\beta^2}{2q} - \frac{z}{q}\left(\alpha + \frac{\beta^2}{2}\right) \end{pmatrix}, \beta \in \mathbb{R}, \alpha < -\frac{\beta^2}{2} \right\}.$$

These spreads coincide with the image of the section σ in (2) such that the well defined functions $a(r, t)$ and $b(r, t)$ are given in the assertion. One gets that $a(r, 0) = 1$ and $a(r, \pi) = \sqrt{\frac{-q}{z}}$ for all $r > 0$.

For $\beta > 2$ we obtain

$$a\left(r, \frac{\pi}{4} + \pi\right) = \frac{\sqrt{2}\beta}{\sqrt{\frac{\beta^4}{3q} - \frac{\beta^3}{2q} - \frac{w}{q}\beta\left(\beta - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \left(\beta - \frac{\beta^2}{2}\right)\left(\frac{z+1}{q}\beta^2 - \frac{\beta z}{q}\right)}}.$$

The loop $Q_{w,z,p,q}^*$ is not decomposable since we have a contradiction to the condition $a(r, \frac{\pi}{4} + k\pi) = a(1, \frac{\pi}{4})a(r, k\pi)$, $r > 0$, $k \in \{0, 1\}$ (cf. Proposition 11). Hence $Q_{w,z,p,q}^*$ is quasi-simple (cf. Theorem 12). As $a(r, k\pi) = 1$ and $b(r, k\pi) = 0$, $r > 0$, $k \in \{0, 1\}$ holds precisely if $w = p = 0$, $q = -z = 1$ the last assertion follows. \square

Proposition 20. Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane \mathcal{A} such that the translation complement C of the 7-dimensional collineation group \mathcal{A} fixes two distinct lines $\{S, W\}$ through the origin and leaves on S one or two 1-dimensional subspaces invariant. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^* having the form (20):

a)

$$a(r, 0) = r^{\frac{1-w}{1+w}}, \quad b(r, 0) = c \left(r^{\frac{w-1}{w+1}} - r^{\frac{1-w}{1+w}} \right),$$

with $r = s^{\frac{w+1}{2}}$, $s > 0$, $t = -\varphi$, where s and φ are variables of the spreads (28). The quasifields $Q_{w,c}$ coordinatize a family of planes $\mathcal{A}_{w,c}$ such that for the parameters $w \neq 1, c$ one has $0 < w$ and $(w-1)^2 c^2 \leq 4w$.

b)

$$a(r, 0) = 1, \quad b(r, 0) = \frac{\ln r}{d},$$

with $r = e^s$, $t = -\varphi$, where s and φ are variables of the spreads (29). The quasifields Q_d coordinatize a one-parameter family of planes \mathcal{A}_d such that $4d^2 \geq 1$.

In both cases Q^* is decomposable and contains the group $\text{SO}_2(\mathbb{R})$.

Proof. If the group C fixes two 1-dimensional subspaces of S , respectively only one 1-dimensional subspace of S , then one obtains a family of translation planes corresponding to the normalized spreads

$$\{S, W\} \cup \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} s & c(s^w - s) \\ 0 & s^w \end{pmatrix}, s, \varphi \in \mathbb{R}, s > 0 \right\} \quad (28)$$

(cf. [7], Satz 1 and [9], p. 15), respectively

$$\{S, W\} \cup \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} e^s & e^{s \frac{s}{d}} \\ 0 & e^s \end{pmatrix}, s, \varphi \in \mathbb{R} \right\}, \quad (29)$$

(cf. [7], Satz 2 and [9], p. 15). In both cases these spreads coincide with the set $\Lambda = \text{SO}_2(\mathbb{R})\mathcal{K}$ given in (20) such that the set \mathcal{K} corresponding to the kernel K_r of Q is determined by the functions $a(r, 0)$, $b(r, 0)$ as in assertion a), respectively b). \square

Remark 21. In [2] D. Betten constructed 4-dimensional locally compact non-desarguesian planes \mathcal{A}_f corresponding to continuous, non-linear, strictly monotone functions f defined for $0 \leq u \in \mathbb{R}$ with $f(0) = 0$ and $\lim_{u \rightarrow \infty} f(u) = \infty$. The planes \mathcal{A}_f are determined by the normalized spreads

$$\left\{ \begin{pmatrix} u \cos \varphi & -\frac{f(u) \sin \varphi}{f(1)} \\ u \sin \varphi & \frac{f(u) \cos \varphi}{f(1)} \end{pmatrix}, u > 0, \varphi \in [0, 2\pi) \right\}.$$

These spreads coincide with the set $\Lambda = \text{SO}_2(\mathbb{R})\mathcal{K}$ given in (20) such that the set \mathcal{K} corresponding to the kernel K_r of the quasifield Q_f coordinatizing \mathcal{A}_f is determined by the functions $a(r, 0) = \sqrt{\frac{uf(1)}{f(u)}}$, $b(r, 0) = 0$ with $r = \sqrt{\frac{uf(u)}{f(1)}}$, $t = -\varphi$, $u \neq 0$. For $f(u) = f(1)u^w$ these planes are planes in Proposition 20 a) with $c = 0$ and $a(r, 0) = r^{\frac{1-w}{1+w}}$. Otherwise the full collineation group of the planes \mathcal{A}_f has dimension 6.

Proposition 22. Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane \mathcal{A} such that the translation

complement C of the 7-dimensional collineation group of \mathcal{A} fixes two distinct lines $\{S, W\}$ through the origin and acts transitively on the spaces P_S and P_W of all 1-dimensional subspaces of S , respectively W . Then the multiplicative loop Q^* of Q can be given by one of the following sets Λ_{Q^*} of the left translations of Q^* :

a) Λ_{Q^*} is the range of the section (2) with

$$\begin{aligned} a(r, u) &= \sqrt{\frac{dD}{de^{2(qt-ps)} + de^{2q\pi} + e^{qt-ps+q\pi} (2d \cos s \cos t + (c^2 + 1 + d^2) \sin s \sin t)}}, \\ b(r, u) &= \frac{e^{2(qt-ps)} [(-c^2 - 1 + d^2) \cos t \sin t - c(c^2 + 1 + d^2) \sin^2 t]}{\sqrt{dD [d(e^{2(qt-ps)} + e^{2q\pi}) + e^{qt-ps+q\pi} (2d \cos s \cos t + (d^2 + c^2 + 1) \sin s \sin t)]}} + \\ &+ \frac{e^{qt-ps+q\pi} (\cos s \cos t + d \sin s \sin t + c \cos s \sin t)}{\sqrt{dD [d(e^{2(qt-ps)} + e^{2q\pi}) + e^{qt-ps+q\pi} (2d \cos s \cos t + (d^2 + c^2 + 1) \sin s \sin t)]}}, \end{aligned}$$

such that

$$\begin{aligned} r \cos u &= \frac{e^{qt-ps} (\cos s \cos t + c \sin t \cos s + d \sin t \sin s) + e^{q\pi}}{1 + e^{q\pi}} a^{-1}(r, u), \\ r \sin u &= -\frac{e^{qt-ps} (d \cos s \sin t - \sin s \cos t - c \sin s \sin t)}{1 + e^{q\pi}} a^{-1}(r, u), \\ D &= e^{2(qt-ps)} ((\cos t + c \sin t)^2 + d^2 \sin^2 t) + e^{2q\pi} + 2e^{qt-ps+q\pi} (\cos s \cos t + c \cos s \sin t + d \sin s \sin t). \end{aligned}$$

The quasifields $Q_{p,q,c,d}$ coordinatize a family of planes $\mathcal{A}_{p,q,c,d}$ such that the parameters p, q, c, d satisfy the conditions

$$\begin{aligned} p &= q > 0 & \text{and} & \quad -1 \leq d < 0, \\ q > 0, p &= \frac{k-1}{k+1}q, k = 1, 2, 3, \dots & \text{and} & \quad d > 0, \end{aligned}$$

$$-(q+p)^2 A + (q-p)^2 B - 4AB \geq 0, \text{ where } A = \frac{(d-1)^2 + c^2}{4d} \text{ and } B = \frac{(d+1)^2 + c^2}{4d}.$$

The multiplicative loops Q^* of the quasifields $Q_{p,q,c,d}$ are not decomposable and quasi-simple.

b) Λ_{Q^*} has the form (15) with

$$a(1, u) = \sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}, \quad b(1, u) = \frac{\sin nt \cos nt (d^2 - 1 - c^2) - c \sin^2 nt (d^2 + 1 + c^2)}{d \sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}$$

such that

$$r \cos u = \frac{s(\cos nt \cos mt + c \sin nt \cos mt + d \sin nt \sin mt)}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}, \quad r \sin u = \frac{s(d \sin nt \cos mt - \cos nt \sin mt - c \sin nt \sin mt)}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}$$

and $s \geq 0$.

The quasifields $Q_{m,n,c,d}$ coordinatize a family of planes $\mathcal{A}_{m,n,c,d}$ such that the parameters $m, n \in \mathbb{Z}$, $(m, n) = 1$, $c, d \in \mathbb{R}$ satisfy the conditions

$$\begin{aligned} m &= n = 1 & \text{and} & \quad -1 \leq d < 0 \\ m &= 1, 2, 3, \dots & n &= m + 1 & \text{and} & \quad d > 0 \\ m &= 1, 3, 5, \dots & n &= m + 2 & \text{and} & \quad d > 0 \end{aligned}$$

$$(n-m)^2 B \geq (n+m)^2 A, \text{ where } A = \frac{(d-1)^2 + c^2}{4d} \text{ and } B = \frac{(d+1)^2 + c^2}{4d}.$$

The loops $Q_{m,n,c,d}^*$ are split extensions of the normal subgroup $\widetilde{N}^* \cong \mathbb{R}$ corresponding to the connected component of $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, 0 \neq u \in \mathbb{R} \right\}$ with a subloop homeomorphic to the 1-sphere.

Proof. If the translation complement C acts transitively on the product space $P_S \times P_w$, then there is a family of translation planes corresponding to the normalized spreads

$$\{S, W\} \cup \left\{ \left(\begin{array}{cc} \frac{\alpha(s,t)+e^{q\pi}}{1+e^{q\pi}} & \frac{\gamma(s,t)-c\alpha(s,t)}{d(1+e^{q\pi})} \\ \frac{\beta(s,t)}{1+e^{q\pi}} & \frac{\delta(s,t)-c\beta(s,t)+de^{q\pi}}{d(1+e^{q\pi})} \end{array} \right), s, t \in \mathbb{R} \right\}$$

such that $\alpha(s, t) = e^{qt-ps}(\cos s \cos t + c \sin t \cos s + d \sin t \sin s)$,
 $\beta(s, t) = e^{qt-ps}(d \cos s \sin t - \sin s \cos t - c \sin s \sin t)$,
 $\gamma(s, t) = e^{qt-ps}(d \cos t \sin s - \sin t \cos s + c \cos t \cos s)$,
 $\delta(s, t) = e^{qt-ps}(d \cos t \cos s + \sin t \sin s - c \cos t \sin s)$ (cf. [7], Satz 3, pp. 135-136). These spreads coincide with the image of the section σ in (2) with the well defined functions $a(r, u)$ and $b(r, u)$ as in assertion a). For $s = 0$ we get a contradiction to the condition $a(r_j, u_j) = a(r_j, 0)a(1, u_j)$ which must hold for decomposable Q^* . It follows that Q^* is not decomposable and hence quasi-simple (cf. Theorem 12).

If the translation complement C does not act transitively on the product space $P_S \times P_W$, then there is a family of translation planes which correspond to the normalized spreads

$$\{S, W\} \cup \left\{ \left(\begin{array}{cc} s & 0 \\ 0 & s \end{array} \right) \left(\begin{array}{cc} a_{11}(t) & -\frac{c}{d}a_{11}(t) + \frac{1}{d}a_{21}(t) \\ a_{12}(t) & -\frac{c}{d}a_{12}(t) + \frac{1}{d}a_{22}(t) \end{array} \right), s \geq 0, t \in \mathbb{R} \right\}$$

with $a_{11}(t) = \cos nt \cos mt + c \sin nt \cos mt + d \sin nt \sin mt$,
 $a_{12}(t) = d \sin nt \cos mt - \cos nt \sin mt - c \sin nt \sin mt$,
 $a_{21}(t) = d \cos nt \sin mt - \sin nt \cos mt + c \cos nt \sin mt$,
 $a_{22}(t) = d \cos nt \cos mt + \sin nt \sin mt - c \cos nt \sin mt$ (cf. [7], Satz 4, pp. 142-144). These spreads coincide with the set Λ_{Q^*} in (15) such that the periodic functions $a(1, t)$ and $b(1, t)$ are given in assertion b). As in the proof of Proposition 16 a) it follows that the loop $Q_{m,n,c,d}^*$ is a split extension as in the assertion. \square

Corollary 23. *Let \mathcal{A} be a 4-dimensional locally compact non-desarguesian topological plane which admits an at least 7-dimensional collineation group Γ . If the quasifield Q coordinatizing \mathcal{A} is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the 1-dimensional orbit of Γ or contain the set of the fixed points of Γ , then for the multiplicative loop Q^* of Q one of the following holds:*

- a) Q^* is quasi-simple and not decomposable. Such quasifields Q are described by Propositions 16 b), 16 c), 17), 18), 19) and in Proposition 22 a).
- b) Q^* is quasi-simple but decomposable and it is a product $SO_2(\mathbb{R})B$, where B is a 1-dimensional loop homeomorphic to \mathbb{R} . The quasifields Q of this type

are described in Proposition 20.

c) Q^* is a split extension of the group $\widetilde{N}^* \cong \mathbb{R}$ with a loop homeomorphic to the 1-sphere. The quasifields of this type are described in Propositions 16 a) and 22 b).

Proof. A locally compact topological quasifield coordinatizing the translation plane \mathcal{A} and constructed with respect to two lines satisfying the assumptions is isotopic to a quasifield given in Betten's classification (cf. [11], p. 321, [3] Satz 5). For isotopic loops Q_1^* and Q_2^* the following holds: The group generated by their left translations, every subgroup and all nuclei of them are isomorphic (cf. [14], Lemmata 1.9, 1.10, p. 20). From these facts we get: If Q_1 is quasisimple and not decomposable, then also Q_2 is quasisimple and not decomposable. If Q_1 contains the subgroup $SO_2(\mathbb{R})$, then also Q_2 contains the group $SO_2(\mathbb{R})$. If Q_1 is a split extension of \widetilde{N}^* with a 1-dimensional compact loop, then the same holds for Q_2 . \square

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References

- [1] André, J. Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe. *Math. Z.* 60:156-186.
- [2] Betten, D. Nicht-desarguessche 4-dimensionale Ebenen. *Arch. Math.* 21:100-102.
- [3] Betten, D. 4-dimensionale Translationsebenen. *Math. Z.* 128:129-151.
- [4] Betten, D. 4-dimensionale Translationsebenen mit 8-dimensionaler Kollineationsgruppe. *Geom. Ded.* 2:327-339.
- [5] Betten, D. 4-dimensionale Translationsebenen mit irreduzibler Kollineationsgruppe. *Arch. Math.* 24:552-560. *Math. Z.* 132:249-259.
- [6] Betten, D. 4-dimensionale Translationsebenen mit genau einer Fixrichtung. *Geom. Ded.* 3:405-440.
- [7] Betten, D. 4-dimensionale Translationsebenen mit 7-dimensionaler Kollineationsgruppe. *J. Reine Angew. Math.* 285:126-148.

- [8] Betten, D. 4-dimensionale Translationsebenen mit kommutativer Standgruppe. *Math. Z.* 154:125-141.
- [9] Betten, D. (1997) On the classification of 4-Dimensional Flexible Projective Planes. In: Johnson, N. L., ed. *Mostly finite geometries*. Decker, pp. 9-33.
- [10] Figula, Á., Strambach, K. Loops on spheres having a compact-free inner mapping group. *Monatsh. Math.* 156:123-140.
- [11] Grundhöfer, T., Salzmann, H. H. (1990). Locally compact double loops and ternary fields. In: Chein, O., et al., ed. *Quasigroups and Loops: Theory and Applications*. Heldermann, pp. 313-355.
- [12] Hughes, D. R., Piper, F. C. (1973). *Projective Planes*. Springer.
- [13] Knarr, N. (1995). *Translation Planes*. Springer.
- [14] Nagy, P. T., Strambach, K. (2002). *Loops in group theory and Lie theory*. Walter de Gruyter.
- [15] Ortleb, S. A new family of locally compact 4-dimensional translation planes admitting a 7-dimensional collineation group. *Adv. Geom.* 9:1-12.
- [16] Pickert, G. (1955) *Projektive Ebenen*. Springer.
- [17] Plaumann, P., Strambach, K. Zweidimensionale Quasialgebren mit Nullteilern. *Aequationes Math.* 15:249-264.
- [18] Salzmann, H., Betten, D., Grundhöfer, T., Hähl, H., Löwen, R., Stropfel, M. (1995). *Compact projective planes*. Walter de Gruyter.
- [19] Walter, W. (1970). *Differential and Integral Inequalities*. Springer.