

Spectral Synthesis on Direct Products

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Abstract

In our recent work we introduced the concept of localization of ideals in the Fourier algebra of a locally compact abelian group. We showed that localizability of a closed ideal in the Fourier algebra is equivalent to the synthesizability of the variety which corresponds to the respective ideal in the measure algebra. This equivalence provides an effective tool in studying spectral synthesis on locally compact abelian groups. Recently we used localization to prove that, when investigating synthesizability of a variety, compact elements of the underlying group may essentially be disregarded.

Although, in general, direct products do not preserve synthesizability, in the present paper we apply the localization method to show that if spectral synthesis holds on a locally compact abelian group, then it also holds on the direct product of this group with the group of integers. This result, combined with our earlier work, yields a complete characterization of locally compact abelian groups on which spectral synthesis holds, as presented in [9].

1 Introduction

Let G be a locally compact abelian group. Spectral synthesis concerns the structure of uniformly closed, translation-invariant linear subspaces of $\mathcal{C}(G)$, the space of continuous complex-valued functions on G . Such a subspace is called a *variety*. A variety is said to be *synthesizable* if its finite-dimensional subvarieties span a dense subspace in the compact-open topology. If every subvariety of a given variety is synthesizable, we say that *spectral synthesis holds* for that variety. If every variety on G is synthesizable, then we say that *spectral synthesis holds on G* .

On commutative topological groups, finite-dimensional varieties are completely understood: they consist precisely of exponential polynomials. An *exponential polynomial* is a finite linear combination of *exponential monomials* or *m-exponential monomials* of the form

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$$x \mapsto P(a_1(x), \dots, a_n(x)) m(x),$$

where P is a complex polynomial, $a_1, \dots, a_n$ are continuous additive functions, i.e. homomorphisms of G into the additive group of complex numbers, and m is a continuous exponential, i.e. a homomorphism of G into the multiplicative group of nonzero complex numbers. For background on spectral analysis and synthesis on groups, see [5, 6].

The problem of determining which groups admit spectral synthesis has a long history. In [4], the authors characterized the discrete abelian groups on which spectral synthesis holds: this occurs precisely when the group has finite torsion-free rank. In particular, for discrete groups G and H , spectral synthesis holds on $G \times H$ whenever it holds on both G and H .

In the non-discrete setting, however, the situation is more subtle. By the classical theorem of L. Schwartz [1], spectral synthesis holds on $\mathbb{R}$, yet D. I. Gurevich showed in [3] that spectral synthesis fails on $\mathbb{R} \times \mathbb{R}$. The apparent discrepancy between the discrete and continuous cases has remained somewhat mysterious.

The purpose of this paper is to clarify this phenomenon. We prove that if spectral synthesis holds on a locally compact abelian group G , then it also holds on the direct product $G \times \mathbb{Z}$. In contrast, replacing $\mathbb{Z}$ by $\mathbb{R}$ does not preserve spectral synthesis, as shown in [3]. Combining this result with our earlier work [7], where we proved that spectral synthesis is preserved under extension by compact abelian groups, we obtain a structural description of all locally compact abelian groups on which spectral synthesis holds.

Our main tool throughout is the concept of localizability of ideals in the Fourier algebra, introduced in [8]. Localization provides a flexible and powerful method for transferring synthesizability properties between groups and their extensions.

2 Localization

In our earlier paper [8] we introduced the concept of localization of ideals in the Fourier algebra of a locally compact abelian group. For the reader's convenience we briefly recall the relevant definitions.

Let G be a locally compact abelian group, and let $\mathcal{A}(G)$ denote its Fourier algebra, that is, the algebra of all Fourier–Laplace transforms (shortly: Fourier transforms) of compactly supported complex Borel measures on G which are defined as

$$\hat{\mu}(m) = \int \tilde{m} d\mu$$

for each exponential m on G . This algebra is topologically isomorphic to the measure algebra $\mathcal{M}_c(G)$. For simplicity, we say that an ideal $\hat{I}$ in $\mathcal{A}(G)$ is *synthesizable* if the corresponding ideal I in $\mathcal{M}_c(G)$ has synthesizable annihilator $\text{Ann } I$.

Let D be a polynomial derivation on $\mathcal{A}(G)$ (see [8]) of the form

$$D\hat{\mu}(m) = P_D(\partial_{a_1}, \partial_{a_2}, \dots, \partial_{a_n})\hat{\mu}(m) = \int \check{P}_D(a_1(x), a_2(x), \dots, a_n(x))\check{m}(x) d\mu(x),$$

where P_D is a complex polynomial in n variables, the a_j 's are additive functions, i.e. continuous homomorphisms of G into the additive group of the complex numbers, and m is an exponential on G . Following [8], we define $\hat{I}_{D,m}$ to be the set of all $\hat{\mu} \in \mathcal{A}(G)$ such that $\partial^\alpha P_D(\partial_{a_1}, \partial_{a_2}, \dots, \partial_{a_n})\hat{\mu}(m) = 0$ for each multi-index α in $\mathbb{N}^n$. It is shown in [8] that $\hat{I}_{D,m}$ is a closed ideal of $\mathcal{A}(G)$.

For a family $\mathcal{D}$ of derivations we set

$$\hat{I}_{\mathcal{D},m} = \bigcap_{D \in \mathcal{D}} \hat{I}_{D,m},$$

which is again a closed ideal. Thus $\hat{I}_{\mathcal{D},m}$ consists precisely of those functions in $\mathcal{A}(G)$ that are annihilated at m by every derivation in $\mathcal{D}$.

The dual notion is defined as follows. Given an ideal $\hat{I}$ in $\mathcal{A}(G)$ and an exponential m , we denote by $\mathcal{D}_{\hat{I},m}$ the set of all derivations on $\mathcal{A}(G)$ that annihilate $\hat{I}$ at m . The subset of $\mathcal{D}_{\hat{I},m}$ consisting of polynomial derivations is denoted by $\mathcal{P}_{\hat{I},m}$. These notions satisfy the inclusions

$$\hat{I} \subseteq \bigcap_m \hat{I}_{\mathcal{D}_{\hat{I},m},m} \subseteq \bigcap_m \hat{I}_{\mathcal{P}_{\hat{I},m},m}. \quad (1)$$

If m is not a root of $\hat{I}$, then $\mathcal{D}_{\hat{I},m} = \mathcal{P}_{\hat{I},m} = \{0\}$, and consequently

$$\hat{I}_{\mathcal{D}_{\hat{I},m},m} = \hat{I}_{\mathcal{P}_{\hat{I},m},m} = \mathcal{A}(G),$$

so such terms do not affect the intersections in (1).

An ideal $\hat{I}$ is called *localizable* if equality holds throughout (1). The main result of [8] asserts that $\hat{I}$ is synthesizable if and only if it is localizable. This characterization will be used repeatedly in the sequel.

3 The Fourier algebra of $G \times \mathbb{Z}$

It is well known that every exponential $e : \mathbb{Z} \rightarrow \mathbb{C}$ has the form

$$e(k) = \lambda^k, \quad k \in \mathbb{Z},$$

where λ is a nonzero complex number uniquely determined by e . We denote this exponential by e_λ . Consequently, for any commutative topological group G , the exponentials on $G \times \mathbb{Z}$ are precisely the functions of the form

$$m \otimes e_\lambda : (g, k) \mapsto m(g) e_\lambda(k),$$

where m is an exponential on G and $\lambda \in \mathbb{C} \setminus \{0\}$. Thus the Fourier transforms in $\mathcal{A}(G \times \mathbb{Z})$ may be regarded as functions of two variables, defined on pairs (m, λ) with m an exponential on G and $\lambda \neq 0$.

Let G be a locally compact abelian group. For each measure $\mu \in \mathcal{M}_c(G \times \mathbb{Z})$ and each $k \in \mathbb{Z}$ we define

$$S_k(\mu) = \{g \in G : (g, k) \in \text{supp } \mu\}.$$

This is the projection of $\text{supp } \mu$ onto G along the second coordinate. Since μ is compactly supported, only finitely many sets $S_k(\mu)$ are nonempty. We have

$$\text{supp } \mu = \bigcup_{k \in \mathbb{Z}} (S_k(\mu) \times \{k\}),$$

and

$$S_k(\mu) \times \{k\} = (G \times \{k\}) \cap \text{supp } \mu.$$

Thus the sets $S_k(\mu) \times \{k\}$ are pairwise disjoint compact subsets of $G \times \mathbb{Z}$, and only finitely many of them are nonempty. For each $k \in \mathbb{Z}$ we denote by μ_k the measure defined by

$$\langle \mu_k, f \rangle = \int f \cdot \chi_k d\mu,$$

for every $f \in \mathcal{C}(G \times \mathbb{Z})$, where χ_k is the characteristic function of $S_k(\mu) \times \{k\}$. Equivalently,

$$\int f d\mu_k = \int f(g, k) d\mu(g, l),$$

for all $f \in \mathcal{C}(G \times \mathbb{Z})$ and all $k \in \mathbb{Z}$. Clearly,

$$\mu = \sum_{k \in \mathbb{Z}} \mu_k,$$

and this sum is finite.

Let $\mu \in \mathcal{M}_c(G \times \mathbb{Z})$, then for each $k \in \mathbb{Z}$ we have $\mu_k = \mu_0 * \delta_{(0, k)}$, where $\delta_{(0, k)}$ denotes the Dirac measure at $(0, k) \in G \times \mathbb{Z}$. Indeed, for any $f \in \mathcal{C}(G \times \mathbb{Z})$ we compute:

$$\begin{aligned} \langle \mu_0 * \delta_{(0, k)}, f \rangle &= \int \int f(g + h, l + n) d\mu_0(g, l) d\delta_{(0, k)}(h, n) \\ &= \int f(g, l + k) d\mu_0(g, l) = \int f(g, k) d\mu(g, l) = \langle \mu_k, f \rangle. \end{aligned}$$

Given $\mu \in \mathcal{M}_c(G \times \mathbb{Z})$, we define a measure $\mu_G \in \mathcal{M}_c(G)$ by

$$\langle \mu_G, \varphi \rangle = \int \varphi(g) d\mu(g, l),$$

for all $\varphi \in \mathcal{C}(G)$. Since every $\varphi \in \mathcal{C}(G)$ may be regarded as a function in $\mathcal{C}(G \times \mathbb{Z})$, this definition is consistent. Moreover,

$$\langle \mu_G, \varphi \rangle = \int \varphi(g) d\mu_0(g, l).$$

Lemma 1. *If I is a closed ideal in $\mathcal{M}_c(G \times \mathbb{Z})$, then the set*

$$I_G = \{\mu_G : \mu \in I\}$$

is a closed ideal in $\mathcal{M}_c(G)$.

Proof. Linearity is immediate: $\mu_G + \nu_G = (\mu + \nu)_G$ and $\lambda\mu_G = (\lambda\mu)_G$. Let $\mu_G \in I_G$ and $\xi \in \mathcal{M}_c(G)$. For any $\varphi \in \mathcal{C}(G)$,

$$\langle \xi * \mu_G, \varphi \rangle = \int \int \varphi(g + h) d\xi(g) d\mu_G(h) = \int \int \varphi(g + h) d\xi(g) d\mu(h, l).$$

Extend ξ to a measure $\tilde{\xi} \in \mathcal{M}_c(G \times \mathbb{Z})$ by

$$\langle \tilde{\xi}, f \rangle = \int f(g, 0) d\xi(g),$$

for all $f \in \mathcal{C}(G \times \mathbb{Z})$. Then

$$\langle \tilde{\xi}_G, \varphi \rangle = \int \varphi(g) d\tilde{\xi}_0(g, l) = \int \varphi(g) d\xi(g) = \langle \xi, \varphi \rangle,$$

so $\tilde{\xi}_G = \xi$. A direct computation shows that

$$\langle \xi * \mu_G, \varphi \rangle = \langle (\tilde{\xi} * \mu)_G, \varphi \rangle.$$

Since $\tilde{\xi} * \mu \in I$, we conclude that $\xi * \mu_G \in I_G$. To show that I_G is closed, let (μ_α) be a net in I such that $\mu_{\alpha, G} \rightarrow \xi$ in $\mathcal{M}_c(G)$. Then for all $\varphi \in \mathcal{C}(G)$,

$$\lim_\alpha \int \varphi(g) d\mu_{\alpha, G}(g) = \int \varphi(g) d\xi(g).$$

In particular, for every exponential m on G ,

$$\lim_\alpha \int \tilde{m}(g) d\mu_{\alpha, 0}(g, l) = \int \tilde{m}(g) d\xi(g) = \int \tilde{m}(g) d\tilde{\xi}_0(g, l).$$

Thus $\hat{\mu}_{\alpha, 0} \rightarrow \hat{\tilde{\xi}}_0$, which implies $\mu_{\alpha, 0} \rightarrow \tilde{\xi}_0$. Hence

$$\tilde{\xi}_k = \tilde{\xi}_0 * \delta_{(0, k)} = \lim_\alpha \mu_{\alpha, 0} * \delta_{(0, k)} = \lim_\alpha \mu_{\alpha, k}.$$

Therefore,

$$\tilde{\xi} = \sum_k \tilde{\xi}_k = \sum_k \lim_\alpha \mu_{\alpha, k} = \lim_\alpha \sum_k \mu_{\alpha, k} = \lim_\alpha \mu_\alpha,$$

where all sums are finite. Since I is closed, $\tilde{\xi} \in I$, and thus $\tilde{\xi}_G = \xi \in I_G$. This proves that I_G is closed. $\square$

4 Main Theorem

Now we can derive the following theorem.

Theorem 1. *Let G be a locally compact abelian group. If spectral synthesis holds on G , then it also holds on $G \times \mathbb{Z}$.*

Proof. If spectral synthesis holds on G , then by [8], every closed ideal in the Fourier algebra $\mathcal{A}(G)$ is localizable. We must show that the same property holds for every closed ideal of $\mathcal{A}(G \times \mathbb{Z})$.

Let $\hat{I}$ be a closed ideal in $\mathcal{A}(G \times \mathbb{Z})$, and assume that $\hat{I}$ is not localizable. Then there exists a measure $\nu \in \mathcal{M}_c(G \times \mathbb{Z})$ such that $\hat{\nu}$ is annihilated by every polynomial derivation in $\mathcal{P}_{\hat{I}, m, \lambda}$ for all exponentials m on G and all $\lambda \neq 0$, yet $\hat{\nu} \notin \hat{I}$. We show that $\hat{\nu}_G \in \hat{I}_G$, which will imply $\hat{\nu} \in \hat{I}$, a contradiction.

Suppose that a polynomial derivation d annihilates $\hat{I}_G$ at an exponential m on G . Then for every $\hat{\mu} \in \hat{I}_G$,

$$d\hat{\mu}_G(m) = \int p_d(g) \check{m}(g) d\mu_G(g) = \int p_d(g) \check{m}(g) d\mu(g, l) = 0,$$

where $p_d : G \rightarrow \mathbb{C}$ is the generating polynomial of d . Define a polynomial derivation D on $\mathcal{A}(G \times \mathbb{Z})$ by

$$D\hat{\mu}(m, \lambda) = \int p_d(g) \check{m}(g) \lambda^{-l} d\mu(g, l).$$

If $\hat{\mu} \in \hat{I}$, then for each $k \in \mathbb{Z}$,

$$D\hat{\mu}_k(m, \lambda) = \int p_d(g) \check{m}(g) \lambda^{-l} d\mu_k(g, l) = \lambda^{-k} \int p_d(g) \check{m}(g) d\mu(g, l) = 0.$$

Since $\hat{\mu} = \sum_{k \in \mathbb{Z}} \hat{\mu}_k$, it follows that $D\hat{\mu}(m, \lambda) = 0$ for all $\hat{\mu} \in \hat{I}$. Thus $D \in \mathcal{P}_{\hat{I}, m, \lambda}$ for all m and λ . In particular,

$$D\hat{\nu}(m, \lambda) = \int p_d(g) \check{m}(g) \lambda^{-l} d\nu(g, l) = 0.$$

Taking the $k = 0$ component yields

$$d\hat{\nu}_G(m) = D\hat{\nu}_0(m, \lambda) = \int p_d(g) \check{m}(g) d\nu(g, l) = 0.$$

Since d was an arbitrary polynomial derivation annihilating $\hat{I}_G$ at m , we conclude that $\hat{\nu}_G$ is annihilated by $\mathcal{P}_{\hat{I}_G, m}$ for every exponential m on G . As spectral synthesis holds on G , the ideal $\hat{I}_G$ is localizable. Hence $\hat{\nu}_G \in \hat{I}_G$, which implies $\hat{\nu} \in \hat{I}$, contradicting our assumption. This completes the proof. $\square$

Using this theorem and the structure theory of locally compact abelian groups, the following results were established in [9]. Together they give a complete description of all locally compact abelian groups on which spectral synthesis holds.

Corollary 1. *Let G be a compactly generated locally compact abelian group. Then spectral synthesis holds on G if and only if G is topologically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times F$, where $a \leq 1$ and b are nonnegative integers, and F is an arbitrary compact abelian group.*

Corollary 2. *Let G be a locally compact abelian group, and let B denote the closed subgroup of all compact elements of G . Then spectral synthesis holds on G if and only if G/B is topologically isomorphic to $\mathbb{R}^n \times F$, where $n \leq 1$ is a nonnegative integer, and F is a discrete torsion-free abelian group of finite rank.*

5 Statements and Declarations

Data sharing is not applicable to this article, as no datasets were generated or analysed during the current study. The author declares no financial or non-financial interests related to this work.

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