Short thesis for the degree of
doctor of philosophy (PhD)

On convexity with respect to Chebyshev systems and Cauchy-Schwarz type inequalities for solutions of Levi–Civita-type functional equations

by

Mahmood Kamil Shihab

Supervisor: Prof. Dr. Zsolt Páles

UNIVERSITY OF DEBRECEN
Doctoral School of Mathematical and Computational Sciences
Debrecen, 2024

Introduction

In this short thesis we summarize the most important our results which achieved during PhD study. We present the interesting lemmas, propositions, theorems, and corollaries based on our research which can be found with full details in the papers [**18**], [**19**] and [**17**].

This dissertation is divided into three chapters. In what follows, we give a brief description for each of them.

Chapter one is devoted to give an elementary proof for the decomposition theorem of Maksa and Páles [**10**] which is an extension of the Ng theorem [**12**], (Ng characterized a Wright convex function as a function which is the sum of a convex and an additive function). Maksa and Páles [**10**] proved that a real function is Wright convex of order n if and only if it is a sum of two functions: a convex function of order n and a generalized polynomial of degree at most n. In our proof we adopt the method of Páles [**15**].

The main purpose of Chapter two is to introduce various convexity notions with respect to a given positive Chebyshev system ω and give relations among them. In one of our main result, we generalize the celebrated theorem of Bernstein–Doetsch [**1**] to the setting of ω-Jensen convexity. From this , we derive that a locally bounded function is ω-Jensen affine if and only if it is the linear combination of the members of the Chebyshev system. In section 3 of this Chapter we extend the notion of Wright convexity to the setting of Chebyshev systems and point out that it is an intermediate convexity property between ω-convexity and ω-Jensen convexity. We also generalize the decomposition theorem of higher-order Wright convex function (obtained by Maksa and Páles [**10**] in 2009) to certain Chebyshev systems.

The main goal of Chapter three is to show that if a real valued function defined on a groupoid satisfies a certain Levi–Civita-type functional equation, then it also fulfills a Cauchy–Schwarz-type functional inequality. In particular, if the groupoid is the multiplicative structure of commutative ring, then we can establish the existence of nontrivial additive functions satisfying inequalities connected to the multiplicative structure.

CHAPTER 1

Decomposition of higher-order Wright convex functions revisited

A celebrated result of C. T. Ng [**12**] established a deeper connection between convexity and Wright convexity. It characterizes Wright convex functions as those functions that are of the form $f = g + a$, where g is convex and a is additive. The original proof of the paper [**12**] applied de Bruijn's theorem [**3**] which is related to functions which have continuous differences. Several subsequent proofs of the result of Ng (c.f., Nikodem [**13**] and Kominek [**8**]) used another approach, which was based on Rode's theorem [**22**]. Basically, all the previously known proofs used transfinite induction for the construction of the additive part a of the decomposition. In a recent paper [**15**], Páles obtained a new proof in which the convex summand g was first constructed in an elementary way. Therefore, there was no transfinite induction involved. In the paper [**10**], Maksa and Páles extended the decomposition theorem of Ng to the context of higher-order convexity notions. The main purpose of this Chapter is to adopt the methods of the paper [**15**] and establish a new and elementary proof for the theorem of Maksa and Páles.

1. Higher-order convex and Wright convex functions

In what follows, we are going to define several higher-order convexity concepts in terms of *difference operators* and *divided differences*.

We recall that, for a fixed real number h, the operator Δ_h, acting on a real function $f : I \to \mathbb{R}$, is defined by

$$\Delta_h f(x) := f(x+h) - f(x) \qquad (x \in I \cap (I - h)).$$

Given a fixed $n \in \mathbb{N}$, a map $f : I \to \mathbb{R}$ is said to be *Jensen convex of order* n (briefly n-Jensen convex) if

$$\Delta_h^{n+1} f(x) \geq 0 \qquad \big(h > 0,\, x \in I \cap (I - (n+1)h)\big). \tag{1.1}$$

A map $f : I \to \mathbb{R}$ is said to be *Wright convex of order* n (briefly n-Wright convex) if it satisfies the functional inequality

$$\begin{aligned} &\Delta_{h_1} \cdots \Delta_{h_{n+1}} f(x) \geq 0 \\ &\qquad \big(h_1, \dots, h_{n+1} > 0,\ x \in I \cap (I - (h_1 + \cdots + h_{n+1}))\big). \end{aligned} \tag{1.2}$$

In the investigation of functional inequalities (1.1) and (1.2), those maps that fulfill these inequalities with equality play a fundamental role in the theory of linear functional equations. Therefore, for $n \in \mathbb{N}$, we consider the equation

$$\Delta_h^{n+1} f(x) = 0 \qquad (h > 0,\ x \in I \cap (I - (n+1)h)),$$

which is termed the *Fréchet functional equation* in this theory. It is well-known (see [**9**], [**24**]) that $f : I \to \mathbb{R}$ satisfies this equation if and only if it is a *polynomial function of degree at most* n, i.e., it has the representation

$$f(x) = a_0 + a_1(x) + \cdots + a_n(x) \qquad (x \in I),$$

where $a_0 \in \mathbb{R}$ and a_k is the *diagonalization* of some k-additive and symmetric function $A_k : \mathbb{R}^k \to \mathbb{R}$, that is, $a_k(x) = A_k(x, \dots, x)$, $(x \in \mathbb{R},\ k = 1, \dots, n)$. Standard polynomials are exactly the continuous polynomial functions. On the other hand, using Hamel bases, it is not difficult to construct non-continuous polynomial functions (see [**9**]).

The *divided difference* of the function $f : I \to \mathbb{R}$ with respect to the pairwise distinct points $x_0, \dots, x_n \in I$ is defined by

$$[x_0, \dots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}.$$

Obviously, divided differences are symmetric functions of their variables, furthermore, it is easy to show that they enjoy the following recursive property

$$[x_0, \dots, x_n; f] = \frac{[x_1, \dots, x_n; f] - [x_0, \dots, x_{n-1}; f]}{x_n - x_0}$$

for all $n \in \mathbb{N}$ and pairwise distinct elements $x_0, \dots, x_n \in I$.

Based on the works of T. Popoviciu [**20**,**21**], given $n \in \mathbb{N}$, a map $f : I \to \mathbb{R}$ is said to be *convex of order* n *on* I (shortly n-convex on I) if the inequality

$$[x_0, x_1, \dots, x_n, x_{n+1}; f] \geq 0$$

holds for all pairwise distinct elements $x_0, x_1, \dots, x_n, x_{n+1} \in I$. Due to the symmetry of divided differences, without loss of generality, we may assume $x_0 < x_1 < \cdots < x_n < x_{n+1}$ here.

2. New proof of decomposition theorem of higher-order

In the paper [**10**], Maksa and Páles extended the decomposition theorem of Ng to the context of higher-order convexity notions. They proved that a real function is Wright convex of order n if and only if it can be decomposed as the sum of a convex function of order n and a polynomial function of order at most n. Their proof was again using transfinite tools in the background. We adopt the methods of the paper [**15**] and establish a new and elementary proof for the theorem of Maksa and Páles.

The following result was obtained in the book [**9**] and in a more general form in the paper [**4**].

LEMMA. *Let $n \in \mathbb{N}$. Then every n-convex function is n-Wright convex, and every n-Wright convex function is n-Jensen convex.*

One of the main results of the paper [**10**] established the following generalization of Ng's decomposition theorem [**12**].

THEOREM. *Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be an n-Wright convex function. Then, there exist an n-convex function $g : I \to \mathbb{R}$ and a polynomial function $P : \mathbb{R} \to \mathbb{R}$ of degree at most n such that*

$$f(x) = g(x) + P(x) \qquad (x \in I).$$

Our aim was to obtain a new and transfinite induction-free proof for this result. To accomplish this goal, we needed the following lemma, which was our most important tool.

LEMMA. *Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be an n-Jensen convex function. Then there exists a continuous n-convex function $g : I \to \mathbb{R}$ such that $g|_{I\cap\mathbb{Q}} = f|_{I\cap\mathbb{Q}}$.*

CHAPTER 2

On convexity properties with respect to a Chebyshev system

The main role of Chapter two is to define the Wright convexity with respect to Chebyshev system and generalize the decomposition theorem of Ng [**12**] to a certain Chebyshev system.

The simplex of strictly ordered n-tuples of a set $H \subset \mathbb{R}$, denoted by $\sigma_n(H)$, is defined by

$$\sigma_n(H) := \{(x_1, \ldots, x_n) \in H^n \quad | \quad x_1 < \cdots < x_n\}.$$

We assume that $|H| \geq n$. Let $\omega = (\omega_1, \ldots, \omega_n) : H \to \mathbb{R}^n$ be a vector-valued function, and define the functional operator $\Phi_\omega := \Phi_{(\omega_1,\ldots,\omega_n)} : \sigma_n(H) \to \mathbb{R}$ by

$$\Phi_\omega(x_1, \ldots, x_n) := \begin{vmatrix} \omega_1(x_1) \ldots \omega_1(x_n) \\ \vdots \quad \ddots \quad \vdots \\ \omega_n(x_1) \ldots \omega_n(x_n) \end{vmatrix} \qquad ((x_1, \ldots, x_n) \in \sigma_n(H)).$$

A continuous function ω is said to be an *n-dimensional positive (respectively negative) Chebyshev system over H* if Φ_ω is strictly positive (respectively, strictly negative) over $\sigma_n(H)$. The system ω is called an *n-dimensional Chebyshev system over H* if it is either a positive or a negative Chebyshev system over H. If $\omega : \mathbb{R} \to \mathbb{R}^n$ equals the *n-dimensional standard or polynomial system $\pi_n : I \to \mathbb{R}^n$*, which is defined by

$$\pi_n(t) := (1, t, \ldots, t^{n-1}) \qquad (t \in \mathbb{R}),$$

and turns out to be a positive Chebyshev system. More generally, if $p_1 < \cdots < p_n$ are given exponents, then the system given by

$$\mathbb{R}_+ \ni t \mapsto (t^{p_1}, \ldots, t^{p_n})$$

is also a positive Chebyshev system on $\mathbb{R}_+$. Important Chebyshev system arise also related to hyperbolic and trigonometric functions. For instance, for all $n \in \mathbb{N}$, the systems given by

$$I \ni t \mapsto (\cos(t), \sin(t), \ldots, \cos(nt), \sin(nt)),$$
$$I \ni t \mapsto (1, \cos(t), \sin(t), \ldots, \cos(nt), \sin(nt))$$

are positive $2n$- and $(2n+1)$-dimensional Chebyshev systems over any nonempty open interval I with length less than or equal to π and 2π, respectively. (For the proof of this statements, see the introduction of the paper [**16**].) There are analogous Chebyshev systems in terms of hyperbolic functions as well. We give the standard examples of convex functions with respect of Chebyshev system.

(i) Polynomial system: $\omega(x) := (1, x, \dots, x^n)$;
(ii) exponential system: $\omega(x) := (1, \exp(x), \dots, \exp(nx))$;
(iii) hyperbolic system: $\omega(x) := (1, \cosh(x), \sinh(x), \dots, \cosh(nx), \sinh(nx))$ and
(iv) trigonometric system: $\omega(x) := (1, \cos(x), \sin(x), \dots, \cos(nx), \sin(nx))$,

where $\omega : I \to \mathbb{R}^n$ for all items (i), (ii) and (iii), and $\omega :]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}$ for (iv). For further standard applications of Chebyshev systems, we refer to the monographs [**2**], [**6**] and [**7**].

In what follows, we recall some definitions from the paper [**16**] (see also the paper [**4**] for these definitions in the polynomial setting). Let $I \subset \mathbb{R}$ be a nonvoid interval, $n \in \mathbb{N}$ and let $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an n-dimensional positive Chebyshev system over I. For a function $f : I \to \mathbb{R}$, the functional operator $\Phi_{(\omega,f)} : \sigma_{n+1}(I) \to \mathbb{R}$ is defined by $\Phi_{(\omega,f)} := \Phi_{(\omega_1,\dots,\omega_n,f)}$.

For a given vector $t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, a function $f : I \to \mathbb{R}$ is said to be (t,ω)-*convex* if

$$\Phi_{(\omega,f)}(x, x + t_1 h, \dots, x + (t_1 + \dots + t_n)h) \geq 0 \tag{2.1}$$

holds for all $h > 0$, $x \in I$ with $x + (t_1 + \dots + t_n)h \in I$. If $T \subseteq \mathbb{R}_+$ and f is (t,ω)-convex for every $t \in T^n$, then f is called (T,ω)-*convex*.

If $t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$ and (2.1) is satisfied with equality, then f is called a (t,ω)-*affine function*. If $T \subseteq \mathbb{R}_+$ and f is (t,ω)-affine for every $t \in T^n$, then f is called (T,ω)-*affine*. In particular, we say that f is ω-*Jensen convex* if it is $(\{1\},\omega)$-convex, i.e., if

$$\Phi_{(\omega,f)}(x, x + h, \dots, x + nh) \geq 0 \tag{2.2}$$

holds for all $h > 0$, $x \in I$ with $x + nh \in I$. If (2.2) is valid with equality instead of inequality, then f is said to be ω-*Jensen affine*.

A function f is termed ω-*convex* if it is $(\mathbb{R}_+,\omega)$-convex. It is easy to see that f is ω-convex on I if and only if

$$\Phi_{(\omega,f)}(x_0, x_1, \dots, x_n) \geq 0 \qquad ((x_0, x_1, \dots, x_n) \in \sigma_{n+1}(I)). \tag{2.3}$$

A function f is called ω-*affine* if (2.3) is satisfied with equality.

We have to mention that in the case when $\omega = \pi_n$, then the concepts of ω-convexity and ω-Jensen convexity, was introduced by Hopf [**5**] and Popoviciu [**21**] (see also the book [**9**] by Kuczma) and these properties were called convexity and Jensen convexity of order $(n-1)$, respectively. If a function $f : I \to \mathbb{R}$ is n times differentiable, then it is convex of order $(n-1)$, i.e., convex with respect to the polynomial system π_n if and only if the nth derivative of f is nonnegative over I. In the particular case when $n = 2$, this is the standard characterization of convexity of twice differentiable functions.

It is a nontrivial question whether or not ω-convex functions form a proper subclass of ω-Jensen convex functions. Depending on the Chebyshev system, the answer could be positive and negative as well. On the other hand, it is well-known that, for all $n \geq 2$, π_n-convex functions form a proper subset of π_n-Jensen convex functions. By [**14**, Theorem 2], it follows that, for any additive function $A : \mathbb{R} \to \mathbb{R}$ and $n \in \mathbb{N}$, the function $f := A^{n-1}$ is Jensen affine (and hence it is Jensen convex) of order $n-1$. On the other hand, f is convex of order $n-1$ if and only if A is continuous. Therefore, if A is discontinuous, then f cannot be convex of order $n-1$. For a construction of a Jensen convex function of order $n-1$ which is not Wright convex of order $n-1$, we refer to the paper [**11**].

In what follows we give a short description of this Chapter. The following result shows that, for any nonempty set $T \subseteq \mathbb{R}_+$, (T,ω)-convexity implies ω-Jensen convexity.

THEOREM. *Let $T \subseteq \mathbb{R}_+$ be a nonempty set. If a function $f : I \to \mathbb{R}$ is (T,ω)-convex (resp. (T,ω)-affine), then it is $(\mathbb{Q}_+,\omega)$-convex (resp. $(\mathbb{Q}_+,\omega)$-affine), in particular, it is ω-Jensen convex (resp. ω-Jensen affine).*

1. Results on ω-Jensen functions

If D is a subset of I and $f : D \to \mathbb{R}$, then f is said to be *compactly uniformly continuous* if, for all compact subintervals $[a,b]$ of I and for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in [a,b] \cap D$ with $|x-y| < \delta$, we have that $|f(x) - f(y)| < \varepsilon$.

LEMMA. *Let D be a subset of I and let $f : D \to \mathbb{R}$ be a compactly uniformly continuous function. Then f admits a continuous extension to I. Provided that D is dense, the extension is unique.*

THEOREM. *Let $n \geq 2$ and let $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an n-dimensional positive Chebyshev system and let $\mathbb{K}$ be a subfield of $\mathbb{R}$. If a function $f : I \to \mathbb{R}$ is $(\mathbb{K}_+,\omega)$-convex (resp. $(\mathbb{K}_+,\omega)$-affine), then there exists a continuous ω-convex (resp. ω-affine) function $g : I \to \mathbb{R}$ such that $g|_{I\cap\mathbb{K}} = f|_{I\cap\mathbb{K}}$.*

(Here, and in the sequel, $\mathbb{K}_+$ denotes the intersection $\mathbb{K} \cap \mathbb{R}_+$.)

COROLLARY. *Let $n \geq 2$ and $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an n-dimensional positive Chebyshev system. If $f : I \to \mathbb{R}$ is ω-convex (resp. ω-affine), then it is continuous on I.*

COROLLARY. *Let $n \geq 2$ and $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be an n-dimensional positive Chebyshev system. If $f : I \to \mathbb{R}$ is ω-Jensen convex (resp. ω-Jensen affine), then there exists a continuous ω-convex (resp. ω-affine) function $g : I \to \mathbb{R}$ such that $g|_{I\cap\mathbb{Q}} = f|_{I\cap\mathbb{Q}}$.*

The following statement is the extension of the celebrated Bernstein–Doetsch theorem [**1**] to the setting of ω-Jensen convexity.

THEOREM. *If $f : I \to \mathbb{R}$ is ω-Jensen convex and bounded on a nonempty open subset of I, then it is continuous on I.*

THEOREM. *Let $f : I \to \mathbb{R}$ be a function which is bounded on a nonempty open subset of I. Then it is ω-Jensen affine if and only if $f = \alpha_1\omega_1 + \cdots + \alpha_n\omega_n$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.*

LEMMA. *Let $\omega : I \to \mathbb{R}^n$ be a positive Chebyshev system. Assume that there exist a positive continuous function $\omega_0 : I \to \mathbb{R}_+$ and a matrix $M = (a_{i,j})_{1\leq i\leq n,\, 0\leq j\leq n-1} \in \mathbb{R}^{n\times n}$ such that*

$$\omega_i(x) = (a_{i,n-1}x^{n-1} + \cdots + a_{i,1}x + a_{i,0}) \cdot \omega_0(x) \qquad (x \in I,\ i \in \{1, \dots, n\}). \tag{2.4}$$

Then $\det(M) > 0$ and, for all $(x_1, \dots, x_n) \in \sigma_n(I)$,

$$\Phi_\omega(x_1, \dots, x_n) = \omega_0(x_1) \cdots \omega_0(x_n) \cdot \det(M) \cdot \Phi_{\pi_n}(x_1, \dots, x_n).$$

Additionally, let $f : I \to \mathbb{R}$. Then, for all $(x_0, \dots, x_n) \in \sigma_{n+1}(I)$,

$$\Phi_{\omega,f}(x_0, \dots, x_n) = \omega_0(x_0) \cdots \omega_0(x_n) \cdot \det(M) \cdot \Phi_{\pi_n,f/\omega_0}(x_0, \dots, x_n).$$

THEOREM. *Assume that there exist a positive continuous function $\omega_0 : I \to \mathbb{R}_+$ and a matrix $M = (a_{i,j})_{1\leq i\leq n,\, 0\leq j\leq n-1} \in \mathbb{R}^{n\times n}$ such that* (2.4) *holds. Then $f : I \to \mathbb{R}$ is an ω-Jensen convex (resp. ω-Jensen affine) function if and only if $\frac{f}{\omega_0}$ is a π_n-Jensen convex (resp. π_n-Jensen affine) function.*

THEOREM. *A function $f : I \to \mathbb{R}$ is π_n-Jensen affine if and only if there exist a constant $A_0 \in \mathbb{R}$, an additive function $A_1 : \mathbb{R} \to \mathbb{R}$, a symmetric biadditive function $A_2 : \mathbb{R}^2 \to \mathbb{R}$, …, and a symmetric $(n-1)$-additive function $A_{n-1} : \mathbb{R}^{n-1} \to \mathbb{R}$ such that*

$$f(x) = A_{n-1}(x, \dots, x) + \cdots + A_2(x, x) + A_1(x) + A_0 \quad (x \in I).$$

COROLLARY. *Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $(a_{i,j})_{1\le i\le n,\, 0\le j\le n-1} \in \mathbb{R}^{n\times n}$ *such that* (2.4) *holds. Then* $f : I \to \mathbb{R}$ *is an* ω*-Jensen affine function if and only if there exist a constant* $A_0 \in \mathbb{R}$*, an additive function* $A_1 : \mathbb{R} \to \mathbb{R}$*, a symmetric biadditive function* $A_2 : \mathbb{R}^2 \to \mathbb{R}$*, ..., and a symmetric* $(n-1)$*-additive function* $A_{n-1} : \mathbb{R}^{n-1} \to \mathbb{R}$ *such that*

$$f(x) = \big(A_{n-1}(x,\dots,x) + \dots + A_2(x,x) + A_1(x) + A_0\big)\omega_0(x) \qquad (x \in I).$$

2. Wright convexity with respect to Chebyshev systems

In 1954, Wright [**25**] introduced a concept of convexity which is stronger than Jensen convexity and weaker than convexity. A function $f : I \to \mathbb{R}$ is called *Wright convex* if

$$f(tx + (1-t)y) + f((1-t)x + ty) \le f(x) + f(y) \quad (x, y \in I,\ t \in [0,1]).$$

One can easily see that convexity implies Wright convexity, and, by putting $t = \frac{1}{2}$ into the above inequality, we can see that Jensen convexity is a consequence of Wright convexity.

A characterization and the ultimate understanding of Wright convexity was established by Ng [**12**], who proved that $f : I \to \mathbb{R}$ is Wright convex if and only if it is of the form $f = g + A|_I$, where $g : I \to \mathbb{R}$ is convex and $A : \mathbb{R} \to \mathbb{R}$ is additive. If A is discontinuous, then f will be discontinuous and hence cannot be convex. On the other hand, if A is a discontinuous additive function, then $|A|$ is Jensen convex but not Wright convex.

The concept of Wright convexity is closely related to Schur convexity, sometimes it is termed ultramodularity and has applications, for instance, in the theory of copulas and t-norms (see [**23**] and the references there in).

A higher-order generalization of Wright convexity was introduced by Gilányi and Páles [**4**] as follows. In this paper, a function $f : I \to \mathbb{R}$ was called *Wright convex of order* $(n-1)$, if

$$\Delta_{h_1} \cdots \Delta_{h_n} f(x) \ge 0$$

holds for all $h_1, \dots, h_n > 0$ and $x \in I \cap (I - (h_1 + \dots + h_n))$. One can easily see that Wright convexity of order 1 is equivalent to Wright convexity in the standard sense.

In what follows, we extend the notion of higher-order Wright convexity to the setting of positive Chebyshev systems. Let $\omega = (\omega_1, \dots, \omega_n) : I \to \mathbb{R}^n$ be a positive n-dimensional Chebyshev system. We say that $\overline{\omega} : I \to \mathbb{R}^{n+1}$ *is an extension of* ω if there exists a continuous function $\omega_{n+1} : I \to \mathbb{R}$ such that

$\overline{\omega} := (\omega_1, \ldots, \omega_n, \omega_{n+1})$ and $\overline{\omega}$ is a positive $(n+1)$-dimensional Chebyshev system.

Let ω be a positive n-dimensional Chebyshev system and $\overline{\omega}$ be an arbitrarily fixed extension of ω. We say that a function $f : I \to \mathbb{R}$ *is $\overline{\omega}$-Wright convex* if, for all $h_1, \ldots, h_n > 0$ and $x \in I \cap (I - (h_1 + \cdots + h_n))$, the inequality

$$\sum_{(i_1,\ldots,i_n)} \frac{\Phi_{(\omega,f)}(x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_n})}{\Phi_{\overline{\omega}}(x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_n})} \geq 0$$

holds, where the summation is taken over all permutation $(i_1, \ldots, i_n)$ of the elements $\{1, \ldots, n\}$.

We establishes the connections between ω-convexity, $\overline{\omega}$-Wright convexity and ω-Jensen convexity.

THEOREM. *Let ω be a positive n-dimensional Chebyshev system and $\overline{\omega}$ be an extension of ω. Then every ω-convex function is $\overline{\omega}$-Wright convex and every $\overline{\omega}$-Wright convex function is ω-Jensen convex.*

The next theorem describes the connection between $\overline{\omega}$-Wright convexity and Wright convexity of order $(n-1)$. In what follows, the symbol $[x_0, x_1, \ldots, x_n, g]$ denotes the standard nth-order divided difference of a function $g : I \to \mathbb{R}$ at the pairwise distinct nodes $x_1, x_1, \ldots, x_n \in I$.

THEOREM. *Assume that there exist a positive continuous function $\omega_0 : I \to \mathbb{R}_+$ and a matrix $M := (a_{i,j})_{1\leq i\leq n,\, 0\leq j\leq n-1} \in \mathbb{R}^{n\times n}$ such that* (2.4) *holds. Define $\omega_{n+1} : I \to \mathbb{R}$ by $\omega_{n+1}(t) := t^n\omega_0(t)$. Then $\overline{\omega} := (\omega, \omega_{n+1})$ is an extension of the Chebyshev system ω. In addition, we have the following assertions:*

(i) For all $(x_0, x_1, \ldots, x_n) \in \sigma_{n+1}(I)$, the equality

$$\frac{\Phi_{(\omega,f)}(x_0, x_1, \ldots, x_n)}{\Phi_{\overline{\omega}}(x_0, x_1, \ldots, x_n)} = [x_0, x_1, \ldots, x_n; f/\omega_0]$$

holds. Furthermore, a function $f : I \to \mathbb{R}$ is ω-convex if and only if f/ω_0 is convex of order $(n-1)$.

(ii) For all $h_1, \ldots, h_n > 0$ and $x \in I \cap (I - (h_1 + \cdots + h_n))$, the equality

$$\sum_{(i_1,\ldots,i_n)} \frac{\Phi_{(\omega,f)}(x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_n})}{\Phi_{\overline{\omega}}(x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_n})} = \frac{\Delta_{h_1} \cdots \Delta_{h_n}(f/\omega_0)(x)}{h_1 \cdots h_n}$$

holds. Furthermore, a function $f : I \to \mathbb{R}$ is $\overline{\omega}$-Wright convex if and only if f/ω_0 is Wright convex of order $(n-1)$.

In the following result, we establish a characterization theorem for $\overline{\omega}$-Wright convexity provided that the underlying Chebyshev system is strongly related to the polynomial one. This result generalizes the decomposition theorem of Maksa and Páles [**10**] which is related to the polynomial system. An alternative and more elementary proof of that theorem has been recently given by the authors in [**18**].

THEOREM. *Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $M := (a_{i,j})_{1\le i\le n,\, 0\le j\le n-1} \in \mathbb{R}^{n\times n}$ *such that* (2.4) *holds. Define* $\omega_{n+1} : I \to \mathbb{R}$ *by* $\omega_{n+1}(t) := t^n\omega_0(t)$ *and set* $\overline{\omega} := (\omega, \omega_{n+1})$. *Then a function* $f : I \to \mathbb{R}$ *is* $\overline{\omega}$*-Wright convex if and only if there exist an* ω*-convex function* $F : I \to \mathbb{R}$ *and, for each* $k \in \{1, \dots, n-1\}$*, a symmetric* k*-additive mapping* $A_k : \mathbb{R}^k \to \mathbb{R}$ *and a real constant* A_0 *such that, for all* $x \in I$*,*

$$f(x) = F(x) + (A_0 + A_1(x) + \cdots + A_{n-1}(x, \dots, x))\omega_0(x).$$

In our subsequent result we will prove that if the extension of the two dimensional polynomial system is not a polynomial of at most second degree, then the convexity with respect to the two dimensional polynomial system (i.e., standard convexity) is equivalent to Wright convexity with respect to this extension. For the proof of this result, we will need the following characterization of a polynomial of at most second degree.

LEMMA. *Let* $\rho : I \to \mathbb{R}$ *be a continuous function which satisfies the functional equation*

$$\frac{\rho(z) - \rho(y)}{z - y} = \frac{\rho(z+u) - \rho(y-u)}{(z+u) - (y-u)},$$
$$u \ge 0,\ y, z \in (I+u) \cap (I-u),\ z + u \ge y.$$

Then ρ *is a polynomial of at most second degree over* I.

THEOREM. *Assume that there exist a positive continuous function* $\omega_0 : I \to \mathbb{R}_+$ *and a matrix* $M := (a_{i,j})_{1\le i\le 2,\, 0\le j\le 1} \in \mathbb{R}^{2\times 2}$ *such that* (2.4) *holds for* $n = 2$. *Assume that* $\omega_3 : I \to \mathbb{R}$ *is a continuous function such that* $\overline{\omega} = (\omega_1, \omega_2, \omega_3)$ *is an extension of* $\omega = (\omega_1, \omega_2)$ *and* ω_3/ω_0 *is not a polynomial of at most second degree. Then every* $\overline{\omega}$*-Wright convex function is* ω*-convex, i.e.,* $\overline{\omega}$*-Wright convexity is equivalent to* ω*-convexity.*

CHAPTER 3

Cauchy–Schwarz-type inequalities for solutions of Levi–Civita-type functional equations

Let $(G, *)$ be a groupoid. (Recall that a pair $(G, *)$ is said to be a groupoid if $\cdot$ is a binary operation on G, i.e., $\cdot : G \times G \to G$.) Let $A : G \to \mathbb{R}$ be a function such that there exist functions $f_1, \dots, f_n, g_1, \dots, g_n : G \to \mathbb{R}$ such that the functional equation

$$A(x * y) = f_1(x)g_1(y) + \cdots + f_n(x)g_n(y) \qquad (x, y \in G)$$

is fulfilled. Under certain assumptions on n and on the functions $f_1, \dots, f_n, g_1, \dots, g_n$, we are going to prove that for all $x, y \in G$ the function A will satisfy either the inequality $A(x * y)^2 \leq A(x * x)A(y * y)$ or the reversed one $A(x * x)A(y * y) \leq A(x * y)^2$. In the important particular case when the groupoid is the multiplicative structure of a commutative ring and A is additive, we will establish the existence of nontrivial additive functions which satisfy one of the above mentioned inequalities.

1. The inequality $A(x * y)^2 \leq A(x * x)A(y * y)$

In our first result we assume that the function A satisfies a Levi–Civita-type functional equation over a groupoid.

PROPOSITION. *Let $(G, *)$ be a groupoid and let $A : G \to \mathbb{R}$ be a function. Assume that there exist $n \in \mathbb{N}$ and functions $f_1, \dots, f_n : G \to \mathbb{R}$ such that A satisfies the Levi–Civita-type functional equation*

$$A(x * y) = f_1(x)f_1(y) + \cdots + f_n(x)f_n(y)$$

for all $x, y \in G$. Then, A fulfills the functional inequality

$$A(x * y)^2 \leq A(x * x)A(y * y)$$

for all $x, y \in G$.

If the groupoid is the multiplicative semigroup of a commutative ring $(R, +, \cdot)$ and A is additive, then we can establish a characterization of the corresponding inequality. Recall that in a ring, the product $x \cdot y$ of the elements $x, y \in R$ is simply denoted by xy, and x^2 is defined to be the product $x \cdot x$.

THEOREM. *Let $(R, +, \cdot)$ be a commutative ring and let $A : R \to \mathbb{R}$ be an additive function. Then A satisfies the inequality*

$$A(xy)^2 \leq A(x^2)A(y^2)$$

for all $x, y \in R$ if and only if at least one of the following conditions hold

(i) $A(x^2) \geq 0$ *for all $x \in R$,*
(ii) $A(x^2) \leq 0$ *for all $x \in R$.*

2. The inequality $A(x*x)A(y*y) \leq A(x*y)^2$

In the subsequent two propositions, we present two Levi–Civita-type functional equations which imply the reversed inequality for A.

PROPOSITION. *Let $(G, *)$ be a groupoid and $A : G \to \mathbb{R}$ be a function. Assume that there exist two functions $f, g : G \to \mathbb{R}$ such that the Levi–Civita-type functional equation*

$$A(x*y) = f(x)f(y) - g(x)g(y)$$

holds for all $x, y \in G$. Then A satisfies the functional inequality

$$A(x*x)A(y*y) \leq A(x*y)^2 \tag{3.1}$$

for all $x, y \in G$.

PROPOSITION. *Let $(G, *)$ be a groupoid. Let $A : R \to \mathbb{R}$ be a function. Assume that there exist $f, g : R \to \mathbb{R}$ such that the Levi–Civita-type functional equation*

$$A(x*y) = f(x)g(y) + g(x)f(y)$$

holds for all $x, y \in G$. Then, for all $x, y \in G$, A satisfies the functional inequality (3.1).

COROLLARY. *Assume that $A : \mathbb{R} \to \mathbb{R}$ satisfies the Leibniz Rule with respect to multiplication, i.e.,*

$$A(xy) = xA(y) + A(x)y \qquad (x, y \in \mathbb{R}).$$

Then, for all $x, y \in \mathbb{R}$, the inequality

$$A(x^2)A(y^2) \leq A(xy)^2 \tag{3.2}$$

holds.

In particular, if $A:\mathbb{R}\to\mathbb{R}$ is a derivation (i.e., A is additive and satisfies the Leibniz Rule with respect to multiplication), then the above corollary implies that it fulfills the inequality (3.2).

If the groupoid is the multiplicative semigroup of a commutative ring $(R,+,\cdot)$ and A is additive, then we can establish a characterization of the inequality (3.1) over a particular subset of the ring.

THEOREM. *Let $(R,+,\cdot)$ be a commutative ring with a multiplicative unit element e and $A:R\to\mathbb{R}$ be an additive function with $A(e)\neq 0$. Let the subset $R_A\subseteq R$ be defined by*

$$R_A := \{x\in R \mid 0\le A(x^2)A(e)\}.$$

Then $e\in R_A$ and A satisfies functional inequality

$$A(x^2)A(y^2)\le A(xy)^2 \tag{3.3}$$

for all $x,y\in R_A$ if and only if

$$A(x^2)A(e)\le A(x)^2 \tag{3.4}$$

for all $x\in R_A$.

In the following example we show that the additivity of the function A in this theorem is necessary.

EXAMPLE. Let $q\in(0,1)$ and let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by

$$f(x)=\begin{cases} x & x\neq 1,\\ q & x=1.\end{cases}$$

Clearly, f is not additive. Therefore, for $x\notin\{1,-1\}$, we have that

$$f(x^2)f(1)=qx^2\le x^2=f(x)^2$$

for $x=\pm 1$,

$$f(1^2)f(1)=q^2=f(1)^2,\qquad f((-1)^2)f(1)=q^2<1=f(-1)^2,$$

which shows that (3.4) is satisfied for all $x\in\mathbb{R}$. On the other hand, for $x,y\in\mathbb{R}\setminus\{1,-1\}$ with $xy=1$, we can conclude that

$$f(x^2)f(y^2)=x^2y^2=1>q^2=f(xy)^2,$$

which shows that the above theorem is not satisfied.

The next example shows that if the function A in the above theorem is non-additive, continuous and satisfies $A(e)=0$, then the conclusion of this theorem may not be valid.

EXAMPLE. Let $A : \mathbb{R} \to \mathbb{R}$ be defined as $A(x) = |x-1|$. Note that A is continuous and not additive. Since $A(1) = 0$ this implies that

$$A(x^2)A(1) = 0 \leq (x-1)^2 = A(x)^2.$$

Thus the inequality (3.4) holds for all $x \in \mathbb{R}$. On the other hand we have that

$$A(x^2)A(y^2) = |x^2-1||y^2-1| \qquad \text{and} \qquad A(xy)^2 = (xy-1)^2.$$

Hence for $x = 2$ and $y = \frac{1}{2}$ we have that $A(x^2)A(y^2) = \frac{9}{4}$ but $A(xy)^2 = 0$, therefore the inequality (3.3) does not hold.

3. Consequences of systems of Levi–Civita-type functional equations

THEOREM. *Let $(G, *)$ be a groupoid and $A, B : G \to \mathbb{R}$ be functions. Assume that there exist $f, g : G \to \mathbb{R}$ such that A and B satisfy the following system of Levi–Civita-type functional equations*

$$\begin{aligned} A(x*y) &= f(x)f(y) - g(x)g(y) \qquad \textit{and} \\ B(x*y) &= f(x)g(y) + g(x)f(y) \end{aligned} \tag{3.5}$$

for all $x, y \in G$. Then the inequalities

$$-B(x*y)^2 \leq A(x*x)A(y*y) \leq A(x*y)^2 \tag{3.6}$$

and

$$-A(x*y)^2 \leq B(x*x)B(y*y) \leq B(x*y)^2 \tag{3.7}$$

hold for all $x, y \in G$.

An interesting consequence of the functional equations in (3.5) is that A and B satisfy the following identity:

$$B(x*y)^2 + A(x*x)A(y*y) = A(x*y)^2 + B(x*x)B(y*y), \ (x, y \in G).$$

Therefore, the inequalities (3.6) and (3.7) can be expressed as the following chain of inequalities

$$\begin{aligned} 0 &\leq A(x*x)A(y*y) + (B(x*y))^2 \\ &= B(x*x)B(y*y) + A(x*y)^2 \leq A(x*y)^2 + B(x*y)^2 \qquad (x, y \in G). \end{aligned}$$

The following result is probably well-known, but we could not find an exact reference for it.

COROLLARY. *For all $x, y \in \mathbb{R}$, we have*

$$\begin{aligned} -\sin(x+y)^2 &\leq \cos(2x)\cos(2y) \leq \cos(x+y)^2 \qquad \textit{and} \\ -\cos(x+y)^2 &\leq \sin(2x)\sin(2y) \leq \sin(x+y)^2. \end{aligned}$$

COROLLARY. *Let $(G, *)$ be a groupoid and let $\varphi : G \to \mathbb{C}$ be a homomorphism into the multiplicative semigroup of complex numbers. Define $A := \Re\varphi$ and $B := \Im\varphi$. Then, for all $x, y \in G$, the inequalities* (3.6) *and* (3.7) *hold.*

COROLLARY. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an automorphism of the field $\mathbb{C}$. Define $A := \Re\varphi$ and $B := \Im\varphi$. Then $A : \mathbb{C} \to \mathbb{R}$ and $B : \mathbb{C} \to \mathbb{R}$ are additive mappings, furthermore, for all $x, y \in \mathbb{C}$,*

$$
\begin{aligned}
&- B(xy)^2 \le A(x^2)A(y^2) \le A(xy)^2 \qquad \textit{and}\\
&- A(xy)^2 \le B(x^2)B(y^2) \le B(xy)^2.
\end{aligned}
$$

THEOREM. *Let $(G, *)$ be a groupoid and $A, B : G \to \mathbb{R}$ be functions. Assume that there exist $f, g : G \to \mathbb{R}$ such that A and B satisfy the following Levi–Civita-type functional equations*

$$
\begin{aligned}
A(x * y) &= f(x)f(y) + g(x)g(y) \qquad \textit{and}\\
B(x * y) &= f(x)g(y) + g(x)f(y)
\end{aligned}
$$

for all $x, y \in G$. Then the inequalities

$$B(x * x)B(y * y) \le A(x * y)^2 \le A(x * x)A(y * y)$$

and

$$B(x * x)B(y * y) \le B(x * y)^2 \le A(x * x)A(y * y).$$

hold for all $x, y \in G$.

An interesting consequence of the functional equations in (3.5) is that A and B satisfy the following identity:

$$B(x * x)B(y * y) + A(x * x)A(y * y) = A(x * y)^2 + B(x * y)^2 \quad (x, y \in G).$$

The following result is probably also well-known, but we could not find a reference for it.

COROLLARY. *For all $x, y \in \mathbb{R}$, we have*

$$\sinh(2x)\sinh(2y) \le \sinh(x + y)^2 < \cosh(x + y)^2 \le \cosh(2x)\cosh(2y)$$

To formulate the next result, let p be a square free positive integer and let $\mathbb{Q}(\sqrt{p})$ denote the subfield of $\mathbb{R}$ generated by $\sqrt{p}$. Then, one can see that $\mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} : a, b \in \mathbb{Q}\}$. In what follows, we equip $\mathbb{Q}(\sqrt{p})$ with the topology inherited from $\mathbb{R}$.

THEOREM. *Let p be a square free positive integer. Then there exist two discontinuous additive functions $A : \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ and $B : \mathbb{Q}(\sqrt{p}) \to \mathbb{R}$ such* *that the functions A and B fulfill the inequalities*

$$\begin{aligned} B(x^2)B(y^2) &\le A(xy)^2 \le A(x^2)A(y^2), \\ B(x^2)B(y^2) &\le B(xy)^2 \le A(x^2)A(y^2) \end{aligned} \tag{3.8}$$

for all $x, y \in \mathbb{Q}(\sqrt{p})$.

We note that there does not exist a discontinuous additive function $A : \mathbb{R} \to \mathbb{R}$ such that the inequality $A(xy)^2 \le A(x^2)A(y^2)$ be valid for all real numbers x, y. Indeed, if A is a discontinuous additive function satisfying this inequality, then $A(u)$ is not zero for some $u > 0$. With the substitution $y := \sqrt{u}$, the inequality shows that A is either nonnegative (if $A(u) > 0$) or nonpositive (if $A(u) < 0$) on the set of positive numbers. This, by classical results on additive functions (see [**9**]), implies that $A(x) = ax$ for some $a \in \mathbb{R}$, and hence A is continuous.

Motivated by the above remark, we could formulate the following open problem: Find a description or characterization of those maximal subrings (or subfields) of $\mathbb{R}$ such that system of inequalities in (3.8) holds for a discontinuous pair (A, B) of additive functions which are defined on this subring (or subfield).

Bibliography

[1] F. Bernstein and G. Doetsch. Zur theorie der konvexen funktionen. *Mathematische Annalen*, 76(4):514–526, 1915.

[2] M. Bessenyei. Hermite–Hadamard-type inequalities for generalized convex functions. *J. Inequal. Pure Appl. Math*, 9(3), 2008.

[3] N. G. de Bruijn. Functions whose differences belong to a given class. *Nieuw Arch. Wiskunde (2)*, 23:194–218, 1951.

[4] A. Gilanyi and Zs. Pales. On convex functions of higher order. *Math. Inequal. Appl*, 11(2):271–282, 2008.

[5] E. Hopf. *Über die zusammenhänge zwischen gewissen höheren differenzen-quotienten reeller funktionen einer reellen variablen und deren differenzierbarkeitseigenschaften*. Norddeutsche Buchdruckerei, 1926.

[6] S. Karlin. *Total positivity. Vol. I*. Stanford University Press, Stanford, CA, 1968.

[7] S. Karlin and J. Studden. *Tchebycheff systems: with applications in analysis and statistics*, volume Vol. XV of *Pure and Applied Mathematics*. Interscience Publishers John Wiley & Sons, New York-London-Sydney, 1966.

[8] Z. Kominek. A continuity result on t-Wright-convex functions. *Publications Mathematicae-Debrecen*, 63(1-2):213–219, 2003.

[9] M. Kuczma. *An introduction to the theory of functional equations and inequalities*. Birkhäuser Verlag, Basel, second edition, 2009. Cauchy's equation and Jensen's inequality, Edited and with a preface by Attila Gilányi.

[10] Gy. Maksa and Zs. Páles. Decomposition of higher-order Wright-convex functions. *J. Math. Anal. Appl.*, 359(2):439–443, 2009.

[11] J. Mrowiec, T. Rajba, and Sz. Wasowicz. On the classes of higher-order Jensen-convex functions and Wright-convex functions, II. *J. Math. Anal. Appl.*, 450(2):1144–1147, 2017.

[12] C. T. Ng. Functions generating Schur-convex sums. In *General inequalities, 5 (Oberwolfach, 1986)*, volume 80 of *Internat. Schriftenreihe Numer. Math.*, pages 433–438. Birkhäuser, Basel, 1987.

[13] K. Nikodem. On some class of midconvex functions. *Ann. Polon. Math.*, 50(2):145–151, 1989.

[14] Zs. Páles. On Wright-but not Jensen-convex functions of higher order. *Ann. Univ. Sci. Budapest. Sect. Comput.*, 41:227–234, 2013.

[15] Zs. Páles. An elementary proof for the decomposition theorem of Wright convex functions. *Ann. Math. Sil.*, 34(1):142–150, 2020.

[16] Zs. Páles and É. Radácsi. Characterizations of higher-order convexity properties with respect to Chebyshev systems. *Aequationes mathematicae*, 90(1):193–210, 2016.

[17] Zs. Páles and M. K. Shihab. Cauchy–Schwarz-type inequalities for additive functions. *Period. Math. Hungar.*, to appear.

[18] Zs. Páles and M. K. Shihab. Decomposition of higher-order Wright convex functions revisited. *Results Math.*, 77(2):Paper No. 73, 11, 2022.

[19] Zs. Páles and M. K. Shihab. On convexity properties with respect to a Chebyshev system. *J. Math. Anal. Appl.*, 530(2):Paper No. 127728, 20, 2024.

[20] T. Popoviciu. *Sur quelques propriétés des fonctions d'une ou de deux variables réelles*. PhD thesis, Institutul de Arte Grafice" Ardealul, 1934.

[21] T. Popoviciu. Les fonctions convexes. Hermann et cie. *Editeurs, Paris*, 1944.

[22] G. Rodé. Eine abstrakte Version des Satzes von Hahn-Banach. *Arch. Math. (Basel)*, 31(5):474–481, 1978/79.

[23] S. Saminger-Platz, A. Kolesárová, R. Mesiar, and E. Klement. The key role of convexity in some copula constructions. *Eur. J. Math.*, 6(2):533–560, 2020.

[24] L. Székelyhidi. *Convolution type functional equations on topological abelian groups*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

[25] E. M. Wright. An inequality for convex functions. *The American Mathematical Monthly*, 61(9):620–622, 1954.

UNIVERSITY of DEBRECEN

UNIVERSITY AND NATIONAL LIBRARY
UNIVERSITY OF DEBRECEN
H-4002 Egyetem tér 1, Debrecen
Phone: +3652/410-443, email: publikaciok@lib.unideb.hu

Registry number: DEENK/64/2024.PL
Subject: PhD Publication List

Candidate: Mahmood Kamil Shihab
Doctoral School: Doctoral School of Mathematical and Computational Sciences
MTMT ID: 10090426

List of publications related to the dissertation

Foreign language scientific articles in international journals (2)

1. Páles, Z., **Shihab, M. K.**: On convexity properties with respect to a Chebyshev system.
   J. Math. Anal. Appl. 530 (2), 1-20, 2024. ISSN: 0022-247X.
   DOI: http://dx.doi.org/10.1016/j.jmaa.2023.127728
   IF: 1.3 (2022)

2. Páles, Z., **Shihab, M. K.**: Decomposition of higher-order Wright convex functions revisited.
   Results Math. 77 (2), 1-11, 2022. ISSN: 1422-6383.
   DOI: http://dx.doi.org/10.1007/s00025-022-01606-7
   IF: 2.2

Total IF of journals (all publications): 3,5
Total IF of journals (publications related to the dissertation): 3,5

The Candidate's publication data submitted to the iDEa Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

27 February, 2024