Diophantine Equations Concerning Linear Recurrences

PhD thesis

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Debrecen, 1999.

Contents

1 Introduction

At the beginning of the XIIIth century, the Italian mathematician FIBONACCI (1170-1250?) asked the following question. Suppose we have a pair of early born rabbits, and after maturing they beget every month a new pair of rabbits that becomes productive at the age of two months. Assuming that the rabbits never die, how many pairs of rabbits are there in the n^{th} months?

FIBONACCI, who was also known as LEONARDO DA PISA, travelled widely with his father in North Africa and Arabia, and in his book *Liber abbaci* [82] he summarized the arithmetical and algebraic knowledge of that era. *Liber abbaci* played a basic role in the spread of the Hindu-Arabic place-valued decimal system in Europe, and among others, dealt with the rule of divisibility by 3 and the Chinese Remainder Theorem. FIBONACCI's other books of major importance, *Practica geometriae* in 1220, contain a large collection of geometry and trigonometry. Also in *Liber quadratorum* in 1225 he approximates a root of a cubic obtaining an answer which in decimal notation is correct to 9 places. LEONARDO drew up the first table of prime numbers, and proved that the zeros of the equation $x^3 + 2x^2 + 10x = 20$ cannot be given in the form $\sqrt{a + \sqrt{b}}$, where a and b are rational numbers.

FIBONACCI applied the sequence $1, 1, 2, 3, 5, 8, \ldots$ to solve the above problem of rabbits. Since FIBONACCI till our days many mathematicians have investigated this so-called FIBONACCI sequence. For example, GIRARD [34] showed that the terms F_n of the FIBONACCI sequence satisfy the equality

$$F_n = F_{n-1} + F_{n-2} \quad , \tag{1. 1}$$

where $F_0 = 0$ and $F_1 = 1$. BINET [12] gave the explicit formula

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \quad , \quad n \geq 0 \tag{1. 2}$$

for the FIBONACCI numbers. (Reputedly, this formula was discovered by MOIVRE in 1718 and proved ten years later by NICOLAS BERNOULLI.)

The FIBONACCI sequence and the number $\frac{1+\sqrt{5}}{2}$ (the golden ratio) appear in nature, in many different areas of life or sciences. Connections exist between the FIBONACCI sequence and music, architecture, painting; see e.g. LOWMAN [59], PREZIOSI [83], HEDIAN [44], respectively. The sequence F occurs, for example, in chemistry (WLODARSKI [118]), in astronomy (READ [84]), in medical sciences (HUNG-SHANNON-THORNTON [46]), in geography (Sharp [100]), in information theory (Magazin [61]), in electro-industry (Arce [3]), in environment science (Deininger [28]). A huge number of papers are devoted to convey that the Fibonacci numbers and (or) the golden ratio can be found in biology (e.g. Davies-Altevogt [26]), and in physics (e.g. Davis [27]).

Although much knowledge has been gathered concerning the Fibonacci sequence, several open problems have remained to be solved. For example, it is not known whether the Fibonacci sequence contains infinitely many primes. However, Graham has shown that the generalized Fibonacci sequence with

$$\begin{cases} G_0 = 1786\ 772701\ 928802\ 632268\ 715130\ 455793 \quad , \\ G_1 = 1059\ 683225\ 053915\ 111058\ 165141\ 686995 \quad , \\ G_n = G_{n-1} + G_{n-2} \quad (n \geq 2) \quad \text{and} \quad \gcd(G_0, G_1) = 1 \end{cases} \tag{1. 3}$$

contains no primes at all! Another unsolved problem is to establish all pure powers in the Fibonacci numbers, but all squares, cubes and fifth powers have already been determinated.

By starting with initial conditions different from $F_0 = 0$, $F_1 = 1$, but keeping the recursion (1. 1), one can obtain a generalization of the Fibonacci sequence. Another type of generalization of the Fibonacci sequence is the second order linear recurrences. The sequence G is called a second order linear recurrence if

$$G_n = AG_{n-1} + BG_{n-2} \quad (n \geq 2) \quad , \tag{1. 4}$$

where $A, B \neq 0$, G_0 and G_1 are fixed rational integers (or real, or complex numbers) with $|G_0|+|G_1| > 0$. Yet another generalization of the Fibonacci numbers allows more than two terms in the recursion (1. 4). In this manner the terms of a k^{th} order linear recurrence are defined as the non-trivial linear combination of the k preceding numbers with certain initial values.

In the following we write about the contents and the structure of this dissertation. In Chapter 2 we introduce the notation which will be used throughout the dissertation, and we recite some important definitions and theorems of the subjects of linear recurrences and estimates of linear forms in logarithms of algebraic numbers. In Chapters 3 and 4 nine theorems of ours are proved; these are new results. Chapter 3 deals with the occurrence of pure powers in the products of the terms of linear recurrences and the occurrence of binomial coefficients in binary recurrences. Chapter 4 contains an effective estimation concerning common terms of products of recursive sequences.

After a short summary of the results on figurate numbers occuring in linear recurrences, in Section 3.2 we consider the mixed exponential-polynomial diophantine equations

$$(2^n-1)(3^n-1)=x^2 \quad , \tag{1. 5}$$

$$(2^n-1)(5^n-1)=x^2 \quad , \tag{1. 6}$$

$$(2^n-1)(6^n-1)=x^2 \tag{1. 7}$$

and

$$(a^n-1)((a^k)^n-1)=x^2 \tag{1. 8}$$

in positive integers n, x as well as $n, x, a > 1, k > 1$. The products $(2^n-1)(3^n-1)$, $(2^n-1)(5^n-1)$, $(2^n-1)(6^n-1)$ and $(a^n-1)((a^k)^n-1)$ satisfy a fourth order linear recursive relation. They can also be interpreted as products of two binary recurrences. Section 3.2 contains Theorems 1-4, and the aim of this section is to give all squares in the four above-mentioned sequences. In [107] and with Hajdu in [43] we solve completely the diophantine equations (1. 5), (1. 6), (1. 7) and (1. 8). By a result of Shorey and Stewart [101], there exists no perfect powers in such sequences, provided that their exponents are large enough. This general result gives no information about the low exponent powers, for example squares, belonging to linear recursive sequences of order four. Only for some classes of Lehmer sequences of first and second kind are there known to be similar results. They were proved by McDaniel [63]. The Lehmer and "associated" Lehmer sequences are examined for the existence of perfect square terms and terms which are twice a square. The author determines all solutions of the corresponding diophantine equations. Thus the results of Theorems 1-4 differ from McDaniel's in the view of the examined fourth order recurrences.

Section 3.3 deals with the occurrence of binomial coefficients of the form $\binom{x}{3}$ in certain binary recurrences. The problem arise from the results of Ming [69, 70] and McDaniel [64], who determined all the triangular numbers (i.e. positive integers of the form $T_x = \frac{x(x+1)}{2}$) in the Fibonacci, the Lucas and the Pell sequences. Since $T_x = \binom{x+1}{2}$, it is natural to ask which terms of a binary recurrence are binomial coefficient of the form $\binom{x}{3}$, $\binom{x}{4}, \ldots$, etc. In [110] we proved that certain type of binary recurrences contains finite number of binomial coefficients of the form $\binom{x}{3}$. This problem leads to the solution of some elliptic equations. Using the program package

SIMATH we could find all integer points of the corresponding curves and have all required binomial coefficients in the FIBONACCI, the LUCAS and the PELL sequences, for demonstrating our method. SIMATH is a computer algebra system, especially for number theoretic purposes. These results are presented in Theorem 5 and 6.

After this the character of the dissertation changes because the proofs of Theorems 7-9 are based on the GEL'FOND-BAKER method concerning linear forms in logarithms of algebraic numbers. These theorems are contained in Section 3.4 and Chapter 4.

Section 3.4, among others, investigates the equation

$$G_x H_y = w^q \tag{1. 9}$$

in x, y, $w > 1$ and q, where G and H are linear recurrences having dominating roots. Earlier RIBENBOIM [88] and RIBENBOIM and MCDANIEL [89] dealt with the diophantine equation $G_x G_y = w^2$ in case of second order recurrences. For general recursive sequences of order larger than two there are fewer results. In [51] KISS showed that, under some conditions, the equation

$$G_n G_x = w^q \tag{1. 10}$$

in integers x, w, q has no solution with $x > n$ and $q > q_0$, where q_0 is an effective computable constant depending only on n and G. In [15] with BRINDZA and LIPTAI we investigate the equation (1. 10) if the suffix n is not fixed and the sequences are distinct. Assuming that the suffixes n and x are not too far from each other we have an effective upper bound for q if some other conditions are also fulfilled (Theorem 7). In our Theorem 8 ([108]) we investigate the equation

$$dG_{x_1}^{(1)} \cdot \ldots \cdot G_{x_\nu}^{(\nu)} = sw^q \tag{1. 11}$$

and give a further extension of Theorem 7.

Chapter 4 concerns the common terms of linear recurrences. After a historical survey we study the equation

$$s_1 G_{x_1}^{(1)} \cdot \ldots \cdot G_{x_\nu}^{(\nu)} = s_2 H_{y_1}^{(1)} \cdot \ldots \cdot H_{y_\mu}^{(\mu)} \tag{1. 12}$$

in x_i $(i = 1, \ldots, \nu)$, y_j $(j = 1, \ldots, \mu)$ and $s_1, s_2 \in S$, the set of all rational integers composed of the fixed primes $p_1, \ldots, p_t$. First MIGNOTTE [67, 68] studied the common terms of two sequences. He proved that the equation $G_x = H_y$ has only finitely many solutions in x and y if the dominating zeros of the characteristic polynomials of the recurrences are multiplicatively indipendent. Later KISS [49] examined the equation

$$s_1 G_x = s_2 H_y \quad , \tag{1. 13}$$

where s_1 and s_2 are in a set S, and gave effective upper bound for $\max\{x, y\}$. Our Theorem 9 gives a certain generalization of the equation (1. 13) and, under some conditions, provides that the suffixes are bounded ([109]).

The results of our papers [15], [108] and [109] have already been appeared and [107] has been accepted for publication. Our articles [43] and [110] are submitted for publication.

Now I would like to express my thanks to Professor Péter Kiss, my thesis supervisor and teacher, who made many valuable suggestions which have helped to improve this volume. He encouraged me, surveyed the manuscript with great care, and I am very grateful to him.

I also thank Professors Kálmán Győry, Attila Pethő and Béla Brindza for the support provided when it was needed. They maintaned the research climate at the Institute of Mathematics of Kossuth Lajos University of Debrecen, when I spent time there, and inspired me to work. I am very indebted for their help.

Thanks also due to PhD. Lajos Hajdu, Tamás Herendi and Attila Bérczes, who gave me good ideas and practical advice.

I am also very grateful to Miss Hazel Perfect and Mrs. Zsuzsanna Németh for carefully review the manuscript.

Finally I thank my parents and my wife for their tolerance and encouragement.

László Szalay

2 Notation and basic results

Now we are turning to draft some basic facts on linear recurrence sequences and linear forms in logarithms.

2.1 Recurrence sequences

A *homogeneous linear recurrence* $G = \{G\}_{n=0}^{\infty}$ *of order* k $(k > 1, k \in \boldsymbol{Z})$ is defined by the recursion

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \ldots + A_k G_{n-k} \qquad (n \geq k) \quad , \tag{2. 1}$$

where the initial values $G_0, \ldots, G_{k-1}$ and the coefficients $A_1, \ldots, A_k$ are complex numbers, $A_k \neq 0$ and $|G_0| + \ldots + |G_{k-1}| > 0$. The formula $G = G(A_1, \ldots, A_k, G_0, \ldots, G_{k-1})$ is often used for denoting the sequence G in (2. 1). A recurrence sequence may satisfy different relations of the form (2. 1), but every recurrence sequence has a unique recurrence relation of minimal order. In the sequel, the order, the coefficients, etc. are meant with respect to this unique recurrence of minimal order. The *companion polynomial* (or *characteristic polynomial*) of the sequence G is the polynomial

$$g(x) = x^k - A_1 x^{k-1} - \ldots - A_k \quad . \tag{2. 2}$$

Denote by $\alpha_1, \ldots, \alpha_t$ the distinct zeros of the companion polynomial $g(x)$, which can there be written in the form

$$g(x) = (x - \alpha_1)^{e_1} \cdots (x - \alpha_t)^{e_t} \quad . \tag{2. 3}$$

The following result plays a basic role in the theory of recurrence sequences (see e.g. [103]).

Theorem A. *Let G be a sequence satisfying the relation (2. 1) with $A_k \neq 0$, and $g(x)$ its companion polynomial with distinct roots $\alpha_1, \ldots, \alpha_t$. Denote by K the extension $\boldsymbol{Q}(\alpha_1, \ldots, \alpha_t, A_1, \ldots, A_k, G_0, \ldots, G_{k-1})$ of the field of rational numbers and let $g(x)$ be given in the form (2. 3). Then* *there exist uniquely determined polynomials $g_i(x) \in K[x]$ of degree less than e_i $(i = 1, \ldots, t)$ such that*

$$G_n = g_1(n)\alpha_1^n + \ldots + g_t(n)\alpha_t^n \qquad (n \geq 0) \quad . \tag{2. 4}$$

If some zero α of the companion polynomial $g(x)$, for instance $\alpha = \alpha_1$, has multiplicity 1 and $|\alpha| > |\alpha_i|$ for $i = 2, \ldots, t$ then we say that α is a dominating zero. If a recurrence G has a dominating zero α then

$$|\alpha| > 1 \tag{2. 5}$$

and the terms of the sequence can be expressed in the form

$$G_n = a\alpha^n + g_2(n)\alpha_2^n \ldots + g_t(n)\alpha_t^n \qquad (n \geq 0) \quad , \tag{2. 6}$$

where $a \in K$.

A recurrence sequence is said to be *degenerate* if its companion polynomial has two distinct zeros whose ratio is a root of unity, and *non-degenerate* otherwise.

This PhD thesis examines *integral recurrences*, i.e. all initial terms and coefficients are rational integers. If the order k is equal to 2 the recurrence is called *binary*. If a binary recurrence $G_n = AG_{n-1} + BG_{n-2}$, $(n \geq 2)$ has initial values G_0 and G_1 then it is denoted by the symbol $G = G(A, B, G_0, G_1)$. Special binary recurrences are the Fibonacci, Lucas, and Pell sequences: $F = F(1, 1, 0, 1)$, $L = L(1, 1, 2, 1)$ and $P = P(2, 1, 0, 1)$, respectively.

For other important concepts and results see e.g. Shorey and Tijdeman [103], Jarden [47], Lucas [60], Brousseau[16], LeVeque [55], Pethő[81], Török [114], Gerőcs [33].

2.2 Linear forms in logarithms

In the next chapters the Baker-Gel'fond method is sometimes used for the proof of certain theorems. This method is based on the effective estimates for linear forms in the logarithms of algebraic numbers. The purpose of this subsection is to summarize all the required preliminaries from the topic of linear forms in logarithms. Moreover we mention some important results from the history of the BAKER-GEL'FOND method to represent its power.

Let $\pi_1, \pi_2, \ldots, \pi_n$ and $\eta_0, \eta_1, \ldots, \eta_n$ $(n \geq 2)$ be non-zero algebraic numbers. We examine the linear form

$$\eta_0 + \eta_1 \log\pi_1 + \ldots + \eta_n \log\pi_n \tag{2. 7}$$

in logarithms. (Here and everywhere in this thesis the logarithms of the complex numbers have their principal values.) In 1934 GEL'FOND [32] and SCHNEIDER [99] showed independently that, if $n = 2$, $\eta_0 = 0$ and η_1/η_2 is irrational, then the sum in (2. 7) never can be zero. BAKER [4, II] generalized the GEL'FOND-SCHNEIDER theorem to arbitrarily many logarithms. Later BAKER [4, III] gave the inhomogeneous version of the above-mentioned result, and in [4, IV] he established a more practicable theorem for the rational case.

These results turned out successful in providing effective upper bounds for the size of all solutions of certain diophantine equations, and getting finiteness theorems. For example in [5] BAKER showed that the elliptic equation

$$y^2 = ax^3 + bx^2 + cx + d \tag{2. 8}$$

has only a finite number of solutions by giving an effective upper bound for the absolute value of the solutions. The finiteness of the number of solutions had been known before (MORDELL [72, 73], SIEGEL [115]), but not in effective form.

BAKER himself, and with other authors made several versions of his earlier works. BAKER and STARK [10] dealt with the special case, when the height of one of the algebraic numbers is much larger (or smaller) than the others. (Here and in the sequel the height $h(\pi)$ of an algebraic number π means its ordinary height, i.e. $h(\pi)$ is equal to the maximum of absolute values of the coefficients of the minimal defining polynomial of π.) This assumption can also be found in [6, I,II,III], where BAKER sharpened the upper bounds for linear forms in logarithms. The following theorem is a special case of his result in [6, II], and we will use it later.

Theorem B. *Let $\pi_1, \ldots, \pi_n$ be non-zero algebraic numbers of heights not exceeding $M_1, \ldots, M_n$, $(M_n \geq 4)$, respectively. Further let $b_1, \ldots, b_{n-1}$*

be rational integers with absolute values at most B and let b_n be a non-zero rational integer with absolute value at most B' $(B' \geq 3)$. Suppose, that $\sum_{i=1}^{n} b_i \log \pi_i \neq 0$. Then there exists an effectively computable constant $C = C(n, M_1, \ldots, M_{n-1}, \pi_1, \ldots, \pi_n)$ such that

$$|\sum_{i=1}^{n} b_i \log \pi_i| > e^{-C(\log M_n \log B' + \frac{B}{B'})} \quad , \tag{2. 9}$$

where the logarithms have their principal values.

In [121], Wüstholz gave a new approach to the subject of linear forms in logarithms. In [11], Baker and Wüstholz proved the following quantitative version of a theorem of Baker [113, Chapter 1], which is fully explicit with respect to all parameters.

Theorem C. *Let $\pi_1, \ldots, \pi_n$ be algebraic numbers not 0 or 1, and let $\Lambda = \eta_1 \log \pi_1 + \ldots + \eta_n \log \pi_n$, where $\eta_1, \ldots, \eta_n$ are rational integers not all 0. We suppose that $B \geq \max_j\{|\eta_j|\}$ and that $A_j \geq h(\pi_j)$ for any π_j $(j = 1, \ldots, n)$. Assume that the field K generated by $\pi_1, \ldots, \pi_n$ over the rationals has degree at most d. If $A_j \geq e$, $B \geq e$ and $\Lambda \neq 0$ then*

$$|\Lambda| > e^{-C \log A_1 \ldots \log A_n \log B} \quad , \tag{2. 10}$$

where $C = (16nd)^{2(n+2)}$.

The application of the Gel'fond-Baker method to diophantine equations has become very useful in providing upper bounds for the solutions. Here we refer, for example, to Bugeaud and Győry [19], [20] (S-unit equations and Thue equations), Győry [37] (discriminant form equations and the index form equations), Győry and Papp [42] (norm form equations), Győry [39] (decomposable form equations), Tijdeman [112] (Catalan equation) Brindza [13], [14] (hyperelliptic equations). For further applications of linear forms in logarithms to diophantine equations we refer to the books of Baker [7] and [8], Baker and Masser [9], Győry [38] and Shorey-Tijdeman [103], as well as to the references given in these works.

Finally we introduce some other notation. Throughout this dissertation we denote by $\nu_p(k)$ the p-adic value of the integer k, where p is a fixed rational prime number. As usual, $\phi(k)$ denotes the Euler function, $d(k)$ denotes the number of divisors, and $\sigma(k)$ the sum of divisors; $\gcd(a, b)$ means the greatest common divisor of the integers a and b.

3 Pure powers in linear recurrences

3.1 Historical survey

In 1963, both Moser and Carlitz [74], and Rollett [96] posed the following problem. In the Fibonacci series the first, second and twelfth terms are squares. Are there any others?

The first result was provided, in the same year, by Wunderlicht [120]. He showed, by an ingenious computational method, that for $3 \leq n \leq 1\,000\,008$, the only square $F_n > 1$ is F_{12}. The problem became very popular, and in a short time several papers dealt with it. Cohn[22, 23], and Wyler[122], applying elementary methods, proved independently that the only square Fibonacci numbers are $F_0 = 0$, $F_1 = F_2 = 1$ and $F_{12} = 144$. In 1964 Buchanan[17] investigated the pure powers in the Fibonacci numbers and he stated that $F_{12} = 144$ and $F_6 = 8$ and the trivial $F_1 = F_2 = 1$ are the only terms which are powers of integers other than first degree. In the same year he retracted his article [18] because it was not complete. Cohn[24] even showed that if $F_n = 2x^2$ then $n = 0, 3$ or 6 ($x = 0, 1$ or 2). Similar results for the Lucas numbers were obtained by Alfred[1] and Cohn[24]: if $L_n = x^2$ then $n = 1$ or 3 ($x = 1$ or 2), and if $L_n = 2x^2$ then $n = 0$ or 6 ($x = 1$ or 3).

The next step concerning the investigation of powers in the Fibonacci and Lucas sequences was made by London and Finkelstein [58]. They established all full cube Fibonacci and Lucas numbers. The problem of determining all the p^{th} powers in the sequences F_n (and L_n) was reduced to solving two superelliptic equations. For $p = 3$ they solved those equations and found that the only cubes in the Fibonacci sequence F_n are $F_0 = 0$, $F_1 = F_2 = 1$ and $F_6 = 8$, and in the Lucas sequence L_n is $L_1 = 1$. Thus they have completed yet another proof of Gauss' conjecture on complex quadratic fields of class number one. Diophantine equations whose solutions must be Fibonacci and Lucas cubes occurred in a proof of Siegel, when he showed that there are exactly nine complex quadratic number fields of class number one. Later Steiner[105] (earlier under the name Finkelstein) showed in a more elementary proof that the equations $F_n = x^3$ and $L_n = x^3$ have only the solutions which had already been found. He also investigated higher powers in both sequences. Lagarias and Weisser[53] specially also gave all the Fibonacci and Lucas cubes by determining the set of the Fibonacci and Lucas numbers of the form $2^a 3^b x^3$ and $2^a x^3$, respectively. Pethő[76] gave a new proof of the theorem of London and Finkelstein, applying Baker's method and computer investigations. His method is applicable to determine all the p^{th} powers in the Fibonacci sequence for small primes p. He gave also the solutions of $F_n = qx^3$ and $F_n = q^2x^3$ for infinitely many primes q. Later all the fifth power were found by Pethő [77] in the Fibonacci sequence.

Many authors have dealt with several types and forms of the Fibonacci, Lucas and Pell numbers. For instance Gryte, Kingsley and Williams [36] used computers to find the Fibonacci numbers F_n of the form $k^2 + 1$, where $5 < n < 10^6$. Finkelstein[29, 30] proved that the numbers of the form $k^2 + 1$ are $F_1 = F_2 = 1$, $F_3 = 2$ and $F_5 = 5$ in the Fibonacci sequence, and $L_0 = 2$ and $L_1 = 1$ in the Lucas sequence. Robbins [92] solved the diophantine equations $F_n = w^2 - 1$, $F_n = w^3 \pm 1$, $L_n = w^2 - 1$ and $F_n = w^3 \pm 1$. He also dealt with the equation $F_n = cx^2$ [93] with several coefficients c, with $P_n = px^2$ [94], $F_n = px^2 \pm 1$ and $F_n = px^3 \pm 1$ [95], where p is a prime number. The equation $L_n = px^2$ was also considered by Goldman [35] with $p = 3, 7, 47$ and 2207. Antoniadis [2] proved that $F_n = 3z^2 + 1$ iff $n = \pm 1, 2, \pm 7$; $F_n = 3z^2 - 1$ iff $n = -2, \pm 3$; $L_n = 3z^2 + 1$ iff $n = 1, 3, 9$; and $L_n = 3z^2 - 1$ iff $n = -1, 0, 5, \pm 8$. Ribenboim [86] has considered the Pell sequence P_n and the associated Pell sequence R_n, and determined the finitely many indices n such that $P_{2n+1} = x^3 \pm 1$, $P_{2n} = x^3 \pm 2$, $R_{2n+1} = x^3 \pm 2$ and $R_{2n} = x^3 \pm 6$. He also showed that for odd n $P_n \neq x^2 \pm 1$ (except for $n = 3$), $P_n \neq x^2 \pm 5$, $R_n \neq x^2 \pm 2$ (except for $n = 3$), $R_n \neq x^2 \pm 14$. For even n he showed that $P_n \neq x^2 \pm 2$, $R_n \neq x^2 \pm 6$. Earlier Pethő [78] found all the powers in the Pell sequence, proving that the equation $P_n = x^q$ has for $q = 2$ only a non-trivial solution, namely $(n, x) = (7, 13)$. After what had gone before, one can think that similar problems have always a finite number of solutions, but it is not true. Nemes and Pethő [75] noted, that the equations $L_n = x^2 + 2$ and $L_n = x^2 - 2$ have infinitely many integer solutions n, x. In the second part of the article they gave necessary conditions for the diophantine equation $G_n = P(x)$ to have infinitely many integer solutions x, n, where G_n is a second order linear recurrence sequence and $P(x) \in \boldsymbol{Z}[x]$.

The k^{th} triangular number is defined by the formula $T_k = \frac{k(k+1)}{2}$. Hogatt conjectured that $F_0 = 0$, $F_1 = F_2 = 1$, $F_4 = 3$, $F_8 = 21$ and $F_{10} = 55$ are the only triangular numbers in the Fibonacci sequence.

This problem was originally posed by TALLMAN [111] in the Fibonacci Quarterly. The first step to this direction is due to WILLIAMS [117] who made a computer search among the first 10^6 terms of the sequence and confirmed HOGATT's conjecture. WALL [116] proved that there are no more triangular numbers among the first billion FIBONACCI numbers. In the end, in 1989, MING [69] solved HOGATT's problem completely by showing that the conjecture is true. The only triangular LUCAS numbers are 1, 3 and 5778 (see [70]), and the only triangular PELL number is 1 (see [64]). MING [71] and MCDANIEL [65, 66] established independently all pronic FIBONACCI and LUCAS numbers. (Pronic is an old-fashioned term meaning the product of two consecutive integers.)

A more general result was obtained by PETHŐ [77, 79], who proved that if $G_n = AG_{n-1} + BG_{n-2}$ is a non-degenerate second order linear recurrence with $B \neq 0$ and $\gcd(A, B) = 1$, then there exists an effectively computable constant $c_1 = c_1(A, B, G_0, G_1, S)$, such that any integer solution n, $|x| > 1$, $q > 1$ and $w \in S$ of

$$G_n = wx^q \tag{3. 1}$$

satisfies $\max\{|x|, q, |w|, n\} < c_1$, where S is a set of integers composed solely of a finite number of primes. SHOREY and STEWART [101] have got a similar result. They showed that any non-degenerate binary recurrence sequence contains only finitely many terms which are pure powers. In the case of linear recurrences of order larger than two, and when its characteristics polynomial has just one root of largest absolute value, SHOREY and STEWART also proved, subject to some hypotheses to avoid degeneracy, that a term of the sequence is not a q^{th} power for q larger than c, where c is an effectively computable positive number. Unfortunately, this general result gives no information about low exponent powers, for example squares, belonging to linear recursive sequences.

In [50] KISS gave two generalizations of the previous result. Polynomial values in linear recurrences were investigated by NEMES and PETHŐ [75]. PETHŐ [80] also worked out an algorithm which can compute all but possibly one integer solutions n, z of the equation $G_n = p^z$, where G_n is a second order linear recursive sequence, which satisfies certain conditions, and p is a prime.

Another type of problem was studied by RIBENBOIM [88]. Two FIBONACCI numbers are said to be in the same square class if there exist nonzero integers x, y such that $F_m x^2 = F_n y^2$, or equivalently $F_m F_n$ is a square. RIBENBOIM called a square class trivial if it consists of only one number. RIBENBOIM showed that the square class of F_n is trivial if $n \neq 1, 2, 3, 6, 12$;

and analogously the square class of L_n is trivial if $n \neq 0, 1, 3, 6$. In their paper RIBENBOIM and MCDANIEL [89] obtained that each square class of a non-degenerate binary recurrence sequence $G = G(A, B, 0, 1)$ with $\gcd(A, B) = 1$ and $A^2 - 4B \neq 0$ is finite, and its elements are effectively computable. KISS [51] gained similar results on the sequences of order k, generally. Under some conditions it was shown that the equation $G_n G_x = w^q$ in positive integers x, w, q has no solutions with $x > n$ and $q > q_0$. The proof is a nice application of the GEL'FOND-BAKER method.

Denote by S the set of non-zero integers which have only a finite number of fixed primes as prime factors. In [41] GYŐRY, KISS and SCHINZEL showed that

$$G_x \in S \tag{3. 2}$$

holds for only finitely many LUCAS or LEHMER sequences G, and for finitely many integers x. GYŐRY [40] improved this result giving an explicit upper bound for x and the constants of the sequences which satisfy (3. 2).

For other details and related papers see, for example, COHN [25], SHOREY and TIJDEMAN [102], ROBBINS [91], STEWART [106], WOLFSKILL [119], RIBENBOIM and MCDANIEL [90], RIBENBOIM [87].

In Section 3.2 and 3.3 our new results are presented.

3.2 Resolution of some diophantine equations

There are several results which investigate the pure powers in the product of linear recurrences, but with few exceptions they deal with effective estimates. In this chapter we provide explicit resolutions of the diophantine equations

$$(2^n - 1)(3^n - 1) = x^2 \quad , \tag{3. 3}$$

$$(2^n - 1)(5^n - 1) = x^2 \tag{3. 4}$$

and

$$(2^n - 1)(6^n - 1) = x^2 \tag{3. 5}$$

in positive integers n and x. In [107] we proved that the first equation has no solution, and the second one has only one solution: $n = 1, x = 2$. Moreover with HAJDU we showed in [43] that equation (3. 5) has no solution.

The main idea of our proofs is to calculate the LEGENDRE symbols of both sides of the equation. The inconsistent results of these evaluations prove the insolubility of the given equation.

We also consider the equation

$$(a^n - 1)\left(\left(a^k\right)^n - 1\right) = x^2 \quad , \tag{3. 6}$$

with integers $a > 1,\ n > 0,\ k > 1,\ x > 0$, and prove that it has only three solutions. This last result is based on the following two well-known theorems.

Theorem D. (LJUNGGREN, [57]) *The diophantine equation*

$$\frac{x^n - 1}{x - 1} = y^2 \quad , \quad (n > 2) \tag{3. 7}$$

is impossible in integers x, y ($|x| > 1$), except when $n = 4$, $x = 7$ and $n = 5$, $x = 3$.

Theorem E. (CHAO KO, [21]) *The equation*

$$x^p + 1 = y^2 \quad , \tag{3. 8}$$

where p is a prime greater than 3, has no solution in integers $x \neq 0$ and y.

The terms $2^n - 1$, $3^n - 1$, $5^n - 1$, $6^n - 1$ and $(a^k)^n - 1$ satisfy the binary recurrence relations $R^{(2)}(3, -2, 0, 1)$, $R^{(3)}(4, -3, 0, 2)$, $R^{(5)}(6, -5, 0, 4)$, $R^{(6)}(7, -6, 0, 5)$ and $R^{(a^k)}(a^k + 1, -a^k, 0, a^k - 1)$, respectively. Also the products $(2^n - 1)(3^n - 1)$, $(2^n - 1)(5^n - 1)$ and $(2^n - 1)(6^n - 1)$ satisfy the fourth order linear recursive relations

$$G^{(3)}\,(12, -47, 72, -36, 0, 2, 24, 182) \quad , \tag{3. 9}$$

$$G^{(5)}\,(18, -97, 180, -100, 0, 4, 72, 868) \tag{3. 10}$$

and

$$G^{(6)}\,(21, -128, 252, -144, 0, 5, 105, 1505) \quad , \tag{3. 11}$$

respectively. Similarly, the sequence $(a^n - 1)\left(a^{kn} - 1\right)$ may have analogous interpretation. Thus to solve the mixed exponential-polynomial diophantine equation (3. 3) (or (3. 4), (3. 5), (3. 6)) is equivalent to the determination of all perfect squares in a fourth order recurrence or in the products of the terms of two binary sequences. This provides the equations

$$G_n^{(3)} = x^2 \qquad \text{or} \qquad R_n^{(2)} \cdot R_n^{(3)} = x^2 \quad , \tag{3. 12}$$

$$G_n^{(5)} = x^2 \qquad \text{or} \qquad R_n^{(2)} \cdot R_n^{(5)} = x^2 \quad , \tag{3. 13}$$

$$G_n^{(6)} = x^2 \qquad \text{or} \qquad R_n^{(2)} \cdot R_n^{(6)} = x^2 \quad , \tag{3. 14}$$

and with $a > 1$, $k > 1$

$$G_n^{(a^k)} = x^2 \qquad \text{or} \qquad R_n^{(a)} \cdot R_n^{(a^k)} = x^2 \quad . \tag{3. 15}$$

In case of the fourth order recurrences only for some classes of Lehmer sequences of first and second kind are there known to be similar results. In [63] McDaniel examined the existence of perfect square terms of Lehmer sequences and gained interesting theorems.

Our precise results are the following ones.

Theorem 1. *The equation*

$$(2^n - 1)(3^n - 1) = x^2 \quad , \tag{3. 16}$$

has no solution in positive integers n and x.

Theorem 2. *The equation*

$$(2^n - 1)(5^n - 1) = x^2 \tag{3. 17}$$

has the only solution $n = 1$, $x = 2$ in positive integers n and x.

Theorem 3. *The equation*

$$(2^n - 1)(6^n - 1) = x^2 \tag{3. 18}$$

has no solution in positive integers n and x.

Theorem 4. *The equation*

$$(a^n - 1)\left(\left(a^k\right)^n - 1\right) = x^2 \quad , \tag{3. 19}$$

has the only solutions $(a,n,k,x)=(2,3,2,21)$, $(3,1,5,22)$ *and* $(7,1,4,120)$ *in positive integers* $a>1$, n, $k>1$ *and* x.

We have the following immediate consequences of Theorems 1 and 2.

Corollary 1. *The equation* $2\cdot\sigma\left(6^n\right)=x^2$ *has no solution, the equation* $\sigma\left(10^n\right)=x^2$ *has only the solution* $n=0$, $x=1$.

To prove Corollary 1 we need to use the well-known result of the summatory function: $\sigma(k)=\prod_{p_i|k}\frac{p_i^{e_i+1}-1}{p_i-1}$, where $\nu_{p_i}(k)=e_i>0$.

Corollary 2. *The equation* $\sum_{i,j=1}^{n}\phi\left(2^i\cdot 3^j\right)=x^2$ *has no solution, the equation* $\sum_{i,j=1}^{n}\phi\left(2^i\cdot 5^j\right)=x^2$ *has only the solution* $n=1$, $x=2$.

These results follow from the multiplicitivity of the Euler ϕ function and from the equality $p^n-1=\phi(p^n)+\phi(p^{n-1})+\ldots+\phi(p)$, where p is a prime number.

It is interesting to observe, that if one replaces the Euler ϕ function by the number of divisors function then for all primes p and q the sum

$$\sum_{i,j=1}^{n}d\left(p^i\cdot q^j\right)=\sum_{i,j=1}^{n}(i+1)(j+1)=\left(\sum_{k=2}^{n+1}k\right)^2=\left(\frac{n(n+3)}{2}\right)^2 \quad (3.\ 20)$$

is always a perfect square number.

The proofs of the theorems even require four lemmas. Lemmas 3 and 4 are very similar, so here we prove only Lemma 3. For the proofs of Lemmas 1 and 2 see e.g. [52].

Let $t>1$ be an arbitrary integer and denote by $(\boldsymbol{Z}/t\boldsymbol{Z})^\star$ the multiplicative group of the reduced residue classes modulo t.

Lemma 1. *Let* $\alpha>1$ *be a rational integer and* p *be an odd prime number. If* g *is a primitive root of* $(\boldsymbol{Z}/p\boldsymbol{Z})^\star$ *then*
a) g *is a primitive root of* $(\boldsymbol{Z}/p^\alpha\boldsymbol{Z})^\star$ *if* $g^{p-1}\not\equiv 1 \pmod{p^2}$, *and*
b) $g(p+1)$ *is a primitive root of* $(\boldsymbol{Z}/p^\alpha\boldsymbol{Z})^\star$ *if* $g^{p-1}\equiv 1 \pmod{p^2}$.

Lemma 1 immediately implies the following results by the choice of

a) $p = 3$, $g = 2$ and $g = 5$;
b) $p = 5$, $g = 2$ and $g = 3$.

Corollary of Lemma 1. *If $\alpha > 1$ is a rational integer then*
a) the numbers 2 and 5 are primitive roots of $(\boldsymbol{Z}/3^\alpha\boldsymbol{Z})^\star$, and
b) the numbers 2 and 3 are primitive roots of $(\boldsymbol{Z}/5^\alpha\boldsymbol{Z})^\star$.

Lemma 2. *Let $\alpha > 2$ be an integer. Then the multiplicative group $G = (\boldsymbol{Z}/2^\alpha\boldsymbol{Z})^\star$ is not cyclic, but*
a) the number 5 generates the subgroup G_1 of G, where
$$G_1 = \{x \in G | x \equiv 1 \pmod 4\}, \text{ and}$$
b) the number 3 generates the subgroup G_2 of G, where
$$G_2 = \{x \in G | x \equiv 1 \text{ or } x \equiv 3 \pmod 8\}.$$

Lemma 3. *Let α and k be positive integers with $k \not\equiv 0 \pmod 5$. If $n = k \cdot 4 \cdot 5^{\alpha-1}$ then*

$$\nu_5\left((2^n - 1)(3^n - 1)\right) = 2\alpha. \tag{3. 21}$$

Lemma 4. *Let α and k be positive integers with $k \not\equiv 0 \pmod 3$. If $n = k \cdot 2 \cdot 3^{\alpha-1}$ then*

$$\nu_3\left((2^n - 1)(5^n - 1)\right) = 2\alpha. \tag{3. 22}$$

Proof of Lemma 3

Consider the congruences

$$2^n \equiv 1 \pmod{5^\alpha} \qquad \text{and} \qquad 3^n \equiv 1 \pmod{5^\alpha} \quad , \tag{3. 23}$$

where α is a fixed positive integer, and n is unknown. According to Corollary b) of Lemma 1 and $\phi(5^\alpha) = 4 \cdot 5^{\alpha-1}$ we obtain the solutions $n = k \cdot 4 \cdot 5^{\alpha-1}$ $(k = 1, 2, \ldots)$ for both congruences. Even if $k \not\equiv 0 \pmod 5$ and (3. 23) is fulfilled then

$$2^n \not\equiv 1 \pmod{5^{\alpha+1}} \qquad \text{and} \qquad 3^n \not\equiv 1 \pmod{5^{\alpha+1}}. \tag{3. 24}$$

Thus $\nu_5(2^n-1)=\alpha=\nu_5(3^n-1)$, which proves Lemma 3. ■

Proof of Theorem 1

Suppose that (n,x) is a solution of the equation (3. 3). Since $2|(3^n-1)$ but $2\not|(2^n-1)$ for every positive integer n, it follows that $2|x$, $4|x^2$ and $4|(3^n-1)$. Consequently n is an even number, but in this case $8|(3^n-1)$ so $4|x$, $16|x^2$ and $16|(3^n-1)$. By Lemma 2b) we know that the number 3 generates the subgroup $G_2\subset\left(\boldsymbol{Z}/2^4\boldsymbol{Z}\right)^\star$, so $n=s\frac{\phi(2^4)}{2}=4s$. Hence n must be divisible by 4, and n can be uniquely written in the form $n=k\cdot 4\cdot 5^{\alpha-1}$, where $1\le\alpha\in\boldsymbol{Z}$ and $k\in\boldsymbol{Z}$, $k\not\equiv 0\pmod 5$. By applying Lemma 3, together with (3. 3) we have

$$\frac{2^n-1}{5^\alpha}\,\frac{3^n-1}{5^\alpha}=x_1^2\quad, \tag{3. 25}$$

where $x_1=\frac{x}{5^\alpha}$ and the prime 5 divides neither the left nor the right hand side of (3. 25). The LEGENDRE symbol $\left(\frac{x_1^2}{5}\right)=1$ because $\gcd(x_1,5)=1$. On the other hand

$$\left(\frac{\frac{2^n-1}{5^\alpha}\,\frac{3^n-1}{5^\alpha}}{5}\right)=A\cdot B\quad, \tag{3. 26}$$

by introducing the notation A and B for the LEGENDRE symbols $\left(\frac{(2^n-1)/5^\alpha}{5}\right)$ and $\left(\frac{(3^n-1)/5^\alpha}{5}\right)$, respectively. We shall show that the calculation of A and B leads to a contradiction because the left side of (3. 25) is not a quadratic residue modulo 5. More exactly, we shall prove that $A=\left(\frac{3k}{5}\right)$, $B=\left(\frac{k}{5}\right)$, so $AB=\left(\frac{3}{5}\right)=-1$. It means that the equation $(2^n-1)(3^n-1)=x^2$ has no solution in positive integers n and x. Now we turn to the calculation of A and B.

Let $R=\alpha-1$ and first let $k=1$ (i.e. $n=4\cdot 5^R$). We are going to compute the residues of the expressions $\frac{2^{4\cdot 5^R}-1}{5^{R+1}}$ and $\frac{3^{4\cdot 5^R}-1}{5^{R+1}}$ after dividing them by 5.

a) If $R=0$ then $\frac{2^4-1}{5}=3\equiv 3\pmod 5$, and $\frac{3^4-1}{5}=16\equiv 1\pmod 5$.

b) If $R = 1$ then

$$\frac{2^{4\cdot 5}-1}{5^2} = \frac{(2^4-1)}{5}\frac{\left(1+2^4+\ldots+\left(2^4\right)^4\right)}{5} = \frac{(2^4-1)}{5}\frac{Q_1}{5} \tag{3. 27}$$

and

$$\frac{3^{4\cdot 5}-1}{5^2} = \frac{(3^4-1)}{5}\frac{\left(1+3^4+\ldots+\left(3^4\right)^4\right)}{5} = \frac{(3^4-1)}{5}\frac{Q_2}{5} \quad . \tag{3. 28}$$

Since $Q_1 \equiv Q_2 \equiv 5 \pmod{5^2}$ therefore $\frac{Q_1}{5} \equiv \frac{Q_2}{5} \equiv 1 \pmod 5$ and $\frac{2^{4\cdot 5}-1}{5^2} \equiv 3\cdot 1 = 3 \pmod 5$, $\frac{3^{4\cdot 5}-1}{5^2} \equiv 1\cdot 1 = 1 \pmod 5$.

c) If $R > 1$ then we replace 2^4 by y in the first case and replace 3^4 by y in the second case. Thus for both cases the expression $\frac{y^{5^R}-1}{5^{R+1}}$ is equal to

$$\frac{(y-1)(1+y+\ldots+y^4)(1+y^5+\ldots+y^{4\cdot 5})\cdots(1+y^{5^{R-1}}+\ldots+y^{4\cdot 5^{R-1}})}{5^{R+1}}. \tag{3. 29}$$

Observe that $y^5 \equiv 1 \pmod{5^2}$, so each factor of the numerator is divisible by 5, but none of them is divisible by 5^2, consequently $\frac{y^{5^R}-1}{5^{R+1}} \equiv m\cdot 1\cdots 1 \pmod 5$, where $m = 3$ if $y = 2^4$ and $m = 1$ if $y = 3^4$.

These results make it possible to calculate the general case, when k is an arbitrary positive integer. Since $\frac{y^{5^R}-1}{5^{R+1}} \equiv m \pmod 5$, therefore

$$y^{5^R} \equiv 1 + m\cdot 5^{R+1} \pmod{5^{R+2}} \quad , \tag{3. 30}$$

so

$$\left(y^{5^R}\right)^k \equiv \left(1+m\cdot 5^{R+1}\right)^k \equiv 1 + k\cdot m\cdot 5^{R+1} \pmod{5^{R+2}} \quad , \tag{3. 31}$$

and

$$\frac{y^{k\cdot 5^R}-1}{5^{R+1}} \equiv k\cdot m \pmod 5 \quad . \tag{3. 32}$$

The truth of the Theorem 1 follows from the congruence (3. 32). ■

Proof of Theorem 2

Suppose that (n, x) is a solution of the equation (3. 4), and distinguish two cases.

a) First we assume that n is even. Then n can uniquely be written in the form $n = k \cdot 2 \cdot 3^{\alpha-1}$, where $1 \leq \alpha \in \boldsymbol{Z}$ and $k \in \boldsymbol{Z}$, $k \not\equiv 0 \pmod 3$. According to Lemma 4 we may transform (3. 4) into the form

$$\frac{2^n - 1}{3^\alpha} \frac{5^n - 1}{3^\alpha} = x_1^2 \quad , \tag{3. 33}$$

where $x_1 = \frac{x}{3^\alpha}$, $\gcd(x_1, 3) = 1$, $\gcd(\frac{2^n-1}{3^\alpha}, 3) = 1$ and $\gcd(\frac{5^n-1}{3^\alpha}, 3) = 1$. In this case to finish the proof we have to use the same method step by step as we did above, during the proof of Theorem 1. We can show the insolubility of the equation (3. 4) for even n by evaluating the Legendre symbols of both sides of (3. 33).

b) Now we suppose that n is odd.
If $n \equiv 3 \pmod 4$ then we may write

$$\left(2^{4k+3} - 1\right)\left(5^{4k+3} - 1\right) = x^2 \quad , \quad (k \geq 0) \quad , \tag{3. 34}$$

and it is easy to see that $2^{4k+3} - 1 \equiv 7 \pmod{10}$ and $5^{4k+3} - 1 \equiv 4 \pmod{10}$, from which it follows that $\left(2^{4k+3} - 1\right)\left(5^{4k+3} - 1\right) \equiv 3 \pmod 5$. But this contradicts to (3. 34) since $\left(\frac{3}{5}\right) = -1$.

Only the case $n \equiv 1 \pmod 4$ remains. If $2 \leq n$ then equation (3. 4) is equivalent to the equation

$$(2^n - 1)(5^{n-1} + \ldots + 5 + 1) = x_1^2 \quad , \tag{3. 35}$$

where $x_1 = \frac{x}{2}$. The corresponding congruence modulo 4 is

$$x_1^2 \equiv 3(1 + \ldots + 1) = 3n \equiv 3 \pmod 4 \quad , \tag{3. 36}$$

which is impossible. It is easy to check that the remainding case $n = 1$ gives the solution $x = 2$ of the equation (3. 4), and this proves Theorem 2. ■

Proof of Theorem 3

Suppose that (n, x) is a solution of equation (3. 5). If n is odd then $(2^n - 1)(6^n - 1) \equiv -1 \pmod 3$ which cannot be a square. Now we can assume that n is even and distinguish two cases.

a) First put $n = 4t$ with some positive integer t. The proof of this case depends on the method of quadratic residues which we have already used at the proof of Theorem 1 and Theorem 2, so here we do not go into details.

b) Now let $n = 4t + 2 = 2(2t + 1)$ where t is a natural number. In this case we must investigate the equation $(4^u - 1)(36^u - 1) = x^2$ for odd $u = 2t + 1$. This last equation is also satisfied $\pmod{18}$, hence it is easy to verify that 3 must divide u. Then we have the equation

$$(64^w - 1)\,(46656^w - 1) = x^2 \tag{3. 37}$$

to solve for odd positive integers $w = \frac{u}{3}$. To see the insolvability of this equation we give two positive integers such that no term of the sequence $(64^w - 1)(46656^w - 1)$ is a quadratic residue for both the given two numbers as moduli. For example, 17 and 97 are such numbers.

To show this, let $I_w = (64^w - 1)(46656^w - 1)$. Then $I_w \equiv ((-4)^w - 1)(8^w - 1) \pmod{17}$. Since $(-4)^4 \equiv 1 \pmod{17}$ and $8^8 \equiv 1 \pmod{17}$ it is sufficient to examine the cases $w = 1, 3, 5, 7$. $I_1 \equiv 16 \pmod{17}$ and $I_7 \equiv 8 \pmod{17}$ which are quadratic residues $\pmod{17}$. $I_3 \equiv 3 \pmod{17}$ and $I_5 \equiv 11 \pmod{17}$ which are not quadratic residues $\pmod{17}$.

On the other hand $I_w \equiv (64^w - 1)((-1)^w - 1) \equiv (64^w - 1)(-2) \pmod{97}$. Since $64^8 \equiv 1 \pmod{97}$ we must investigate the cases $w = 1, 3, 5, 7$. $I_1 \equiv 68 \pmod{97}$ and $I_7 \equiv 5 \pmod{97}$ are not quadratic residues $\pmod{97}$, but $I_3 \equiv 96 \pmod{97}$ and $I_5 \equiv 33 \pmod{97}$ are quadratic residues $\pmod{97}$. This completes the proof of the theorem. ■

Proof of Theorem 4

Suppose that the four-tuple (a, n, k, x) $(a > 1, k > 1)$ is a solution of equation (3. 6). Let $y = a^n$. Now we have the equality

$$x^2 = (y - 1)^2(y^{k-1} + \ldots + y + 1) = (y - 1)^2 \left(\frac{y^k - 1}{y - 1}\right) \quad . \tag{3. 38}$$

Thus $\frac{y^k-1}{y-1}$ must be a square. By Theorem D, if $k>2$ then $k=4$ or $k=5$. Consequently from $y=a^n=7$ it follows that $a=7, n=1, x=120$ and $y=a^n=3$ gives $a=3, n=1, x=22$. These two cases provide the solutions $(a,k,n,x)=(7,4,1,120)$ and $(3,5,1,22)$ of (3. 6).

Now suppose that $k=2$. Then $(y-1)^2(y+1)=x^2$ and

$$y+1=a^n+1 \tag{3. 39}$$

must be a square. Without loss of generality we may assume that n is a prime. If $n=2$ then (3. 39) cannot be a square, and it is well known that if $n=3$ then a for positive integer a (3. 39) is a square only in case of $a=2$. Thus equation (3. 6) has one more solution: $(a,k,n,x)=(2,2,3,21)$. Finally, by Theorem E (3. 39) cannot be a square if $n>3$. This completes the proof of Theorem 4. ■

3.3 Occurrence of the binomial coefficients of the form $\binom{x}{3}$ in binary recurrences

The purpose of this section is to prove that there are finite number of binomial coefficients of the form $\binom{x}{3}$ in certain binary recurrences, and give a simple method for the determination of these coefficients. We illustate our method by the FIBONACCI, the LUCAS and the PELL sequences. First we transform the problem into two elliptic equations and apply a theorem of MORDELL to them. He proved in [72, 73] that the equation $Ey^2 = Ax^3+Bx^2+Cx+D$ $(A,B,C,D,E \in \boldsymbol{Z}\ ,\ E \neq 0)$ has only finitely many solutions if the polynomial Ax^3+Bx^2+Cx+D has three distinct roots. This theorem later was generalized by SIEGEL [115]. All these results are ineffective, that is, their proofs do not provide an algorithm for determining the solutions. The first general effective result on the elliptic equations is due to BAKER [5].

After showing the finiteness we use the program package SIMATH [104] to find all the integer points on the corresponding elliptic curves. SIMATH is a computer algebra system, especially for number theoretic purpose, and its algorithms are based on some deep results of GEBEL, PETHŐ and ZIMMER [31].

Now we introduce some notations we need. Let the sequence $\{U_n\}_{n=0}^{\infty}$ be defined by the initial terms U_0, U_1 and by the recursion

$$U_n = AU_{n-1} + BU_{n-2} \qquad (n \geq 2) \quad , \tag{3. 40}$$

where $U_0, U_1, A, B \in \boldsymbol{Z}$ with the conditions $|U_0| + |U_1| > 0$ and $AB \neq 0$. Moreover, let α and β be the roots of the polynomial

$$p(x) = x^2 - Ax - B \quad , \tag{3. 41}$$

and we denote the discriminant $A^2 + 4B$ of $p(x)$ by D. Suppose that $D \neq 0$ (i.e. $\alpha \neq \beta$), and throughout this paper we assume that $U_0 = 0$ and $U_1 = 1$.

The sequence

$$V_n = AV_{n-1} + BV_{n-2} \qquad (n \geq 2) \quad , \tag{3. 42}$$

with the initial values $V_0 = 2$ and $V_1 = A$ is the associate sequence of U. The recurrences U and V satisfy the relation $V_n^2 - DU_n^2 = 4(-B)^n$. Finally, it is even assumed that $|B| = 1$. Then

$$V_n^2 - DU_n^2 = 4(\pm 1)^n = \pm 4 \quad . \tag{3. 43}$$

The following theorems formulate precisely our results.

Theorem 5. *Both the equations $U_n = \binom{x}{3}$ and $V_n = \binom{x}{3}$ have only a finite number of solutions (n, x) in integers $n \geq 0$ and $x \geq 3$.*

Theorem 6. *All the integer solutions of the equation*
i) $F_n = \binom{x}{3}$ are $(n, x) = (1, 3)$ and $(2, 3)$,
ii) $L_n = \binom{x}{3}$ are $(n, x) = (1, 3)$ and $(3, 4)$,
iii) $P_n = \binom{x}{3}$ is $(n, x) = (1, 3)$.

Proof of Theorem 5

Let U and V be binary recurrences specified above. We distinguish two cases.

a) First we deal with the equation

$$U_n = \binom{x}{3} \tag{3. 44}$$

in integers n and x. Applying (3. 43) together with $y = V_n$ and $x_1 = x - 1$, we have $U_n = \binom{x_1+1}{3}$ and

$$y^2 - D\left(\frac{x_1^3 - x_1}{6}\right)^2 = \pm 4 \quad . \tag{3. 45}$$

Take the 36 times of the equation (3. 45). Let $x_2 = x_1^2$ and $y_1 = 6y$, and using these new variables, from (3. 45) we get

$$y_1^2 = Dx_2^3 - 2Dx_2^2 + Dx_2 \pm 144 \quad . \tag{3. 46}$$

Multiplying by $3^6 D^2$ the equation (3. 46), together with $k = 3^3 D y_1$ and $l = 3D(3x_2 - 2)$ it follows that

$$k^2 = l^3 - 27D^2 l + (54D^3 \pm 104976D^2) \quad . \tag{3. 47}$$

By the above-mentioned theorem of Mordell it is sufficient to show that the polynomial $u(l) = l^3 - 27D^2 l + (54D^3 \pm 104976D^2)$ has three distinct roots. Suppose that the polynomial $u(l)$ has a multiple root $\tilde{l}$. Then $\tilde{l}$ satisfies the equation $u'(l) = 3l^2 - 27D^2 = 0$, i.e. $\tilde{l} = \pm 3D$. Since $u(3D) = \pm 104976D^2$ it follows that $D = 0$ which is impossible. Moreover, $u(-3D) = 108D^3 \pm 104976D^2$ implies that $D = 0$ or $D = \pm 972$. But $D \neq 0$ and by $|B| = 1$ there are no integer A for which $D = A^2 + 4B = \pm 972$. Consequently, $u(l)$ has three distinct zeros.

b) The second case consists of the examination of the diophantine equation

$$V_n = \binom{x}{3} \tag{3. 48}$$

in the integers n and x. Let $y = U_n$ and $x_1 = x - 1$. Applying the method step by step as above in part a), it leads to the elliptic curve

$$k^2 = l^3 - 27D^2 l + cD^3 \quad , \tag{3. 49}$$

where $c = -104922$ if n is even and $c = 105030$ otherwise. The polynomial $v(l) = l^3 - 27D^2 l + cD^3$ has also three distinct roots because $v'(l) = 3l^2 - 27D^2$, $\tilde{l} = \pm 3D$ and $v(\pm 3D) = 0$ implies $D = 0$.

Thus the proof of Theorem 5 is complete. ■

Proof of Theorem 6

All integer points on the corresponding elliptic curves of the equations (3. 47) and (3. 49) (which are in short WEIERSTRASS normal form) can be found by SIMATH.

By (3. 47) and (3. 49) one can compute the coefficients of the elliptic curves in case of the FIBONACCI, the LUCAS and the PELL sequences. The calculations are summarized in `Table 1`, as well as all the integer points belonging to them. Every binary recurrence leads to two elliptic equations because of the even and odd suffixes. For the FIBONACCI and LUCAS sequences $D = 5$, and for the PELL sequence $D = 8$.

Equation	*Transformed equations*	*All the integer solutions* (l, k)
$F_n = \binom{x}{3}$	$k^2 = l^3 - 675l + 2631150$	$(15, 1620)$, $(-30, 1620)$, $(5199, 374868)$, $(735, 19980)$, $(150, 2430)$, $(-129, 756)$
$F_n = \binom{x}{3}$	$k^2 = l^3 - 675l - 2617650$	$(150, 810)$, $(555, 12960)$, $(1014, 32238)$, $(195, 2160)$, $(451, 9424)$, $(4011, 254016)$
$L_n = \binom{x}{3}$	$k^2 = l^3 - 675l - 13115250$	no solution
$L_n = \binom{x}{3}$	$k^2 = l^3 - 675l + 13128750$	$(375, 8100)$, $(-74, 3574)$, $(150, 4050)$, $(-201, 2268)$, $(2391, 116964)$
$P_n = \binom{x}{3}$	$k^2 = l^3 - 1728l + 6746112$	$(-192, 0)$, $(24, 2592)$, $(-48, 2592)$, $(97, 2737)$ $(312, 6048)$, $(564, 13608)$, $(5208, 375840)$
$P_n = \binom{x}{3}$	$k^2 = l^3 - 1728l - 6690816$	$(240, 2592)$, $(609, 14769)$

`Table 1`

The last step is to calculate x and y from the solutions (l, k). By the proof of Theorem 5 it follows that $x = 1 + \sqrt{\frac{l+6D}{9D}}$, $y = \frac{k}{162D}$ in case of the equation (3. 44) and $y = \frac{k}{162D^2}$ in case of the associate sequences. Except some values x and y they are not integer. The exceptions (where $x \geq 3$) provide all the solutions of the equations (3. 47) and (3. 49). Then the proof of Theorem 6 is ended. ■

3.4 Effective estimates concerning recurrences

3.4.1 Lemmas

As in section 2.1, let G be a k^{th} order linear, recursive sequence of rational integers. Suppose that the dominating zero of the companion polynomial $g(x)$ is α, then as in (2. 6), we can write

$$G_n = a\alpha^n + g_2(n)\alpha_2^n \ldots + g_s(n)\alpha_s^n \qquad (n \geq 0) \quad . \tag{3. 50}$$

In the sequel we will use the following lemmas.

Lemma 5. *Let G and H be linear recurrences with dominating zeros α and β, respectively, with*

$$G_x = a\alpha^x + g_2(x)\alpha_2^x \ldots + g_s(x)\alpha_s^x = a\alpha^x\left(1 + \varepsilon_g\right) \qquad (x \geq 0) \quad , \tag{3. 51}$$

$$H_y = b\beta^y + h_2(y)\beta_2^y \ldots + h_t(y)\beta_t^y = b\beta^y\left(1 + \varepsilon_h\right) \qquad (y \geq 0) \quad . \tag{3. 52}$$

Suppose that $ab \neq 0$.

a) If x is sufficient large then there exist effectively computable positive real numbers c_1 and c_2 depending only on the recurrence G for which

$$e^{c_1 x} < |G_x| < e^{c_2 x} \quad . \tag{3. 53}$$

b) There exists an effectively computable positive real number c_3 depending only on the recurrence G for which

$$\left|\log\left(1 + \varepsilon_g\right)\right| < e^{-c_3 x} \quad , \tag{3. 54}$$

if x is sufficient large.

c)

$$\left|\log\left(\frac{1+\varepsilon_g}{1+\varepsilon_h}\right)\right| \le 2\left(|\varepsilon_g| + |\varepsilon_h|\right) \quad , \tag{3. 55}$$

$$|\log\left(1+\varepsilon_g\right)\left(1+\varepsilon_h\right)| \le 2\left(|\varepsilon_g| + |\varepsilon_h|\right) \quad , \tag{3. 56}$$

if x and y are sufficient large.

The proof of Lemma 5a) mainly depends on the fact, that ε_g exponentially tends to 0 as x tends to ∞. Moreover, to prove the parts b) and c) use the inequality $|\log(1+z)| \le 2|z|$ if $|z| \le \frac{1}{3}$, and the properties of the absolute value function.

Lemma 6. *Let $c_1 > 0$, $c_2 > 0$ and $0 < \delta < 1$ be real numbers, further let x and y be positive rational integers. If*

a) $x > x_0$, $x \ge y$ and $y > \delta x$

or

b) $y > y_0$, $y \ge x$ and $x > \delta y$

then there exists a real number $c > 0$ which depends on δ, c_1, c_2, x_0 or y_0 such that

$$e^{-c_1 x} + e^{-c_2 y} < e^{-c(x+y)} \quad . \tag{3. 57}$$

Proof of Lemma 6

We deal with case a) first. It is easy to see that there are positive constants c_3 , c_4 such that

$$e^{-c_1 x} + e^{-c_2 y} < e^{-c_1 x} + e^{-c_2 \delta x} = e^{-c_1 x} + e^{-c_3 x} \le 2e^{-c_4 x} = e^{\log 2 - c_4 x} \quad . \tag{3. 58}$$

Since $x > x_0$, it follows that

$$e^{\log 2 - c_4 x} < e^{-c_5 x} = e^{-c_6 (2x)} \le e^{-c(x+y)} \tag{3. 59}$$

with suitable constants c_5, c_6, c. The proof of case b) is very similar. ■

3.4.2 On products of the terms of linear recurrences

Let G and H be k^{th} and l^{th} order linear recursive sequences of rational integers, respectively, (see (2. 1)):

$$G_n = A_1 G_{n-1} + \ldots + A_k G_{n-k} \quad (n \geq k), \tag{3. 60}$$

$$H_n = B_1 H_{n-1} + \ldots + B_l H_{n-l} \quad (n \geq l). \tag{3. 61}$$

Suppose that the companion polynomials $g(x)$ and $h(x)$ have dominating zeros, say α and β, respectively, then as in (2. 6), we can write that

$$G_n = a\alpha^n + g_2(n)\alpha_2^n \ldots + g_s(n)\alpha_s^n \qquad (n \geq 0) \quad . \tag{3. 62}$$

$$H_n = b\beta^n + h_2(n)\beta_2^n \ldots + h_t(n)\alpha_t^n \qquad (n \geq 0) \quad . \tag{3. 63}$$

Moreover assume that $ab \neq 0$. With Brindza and Liptai, we showed in [15] that if the product of G_x and H_y with x, y, which are not too far from each other, is a q-th power then q is less than a bound which is effectively computable. More precisely, using Theorem B and Baker's method we proved

Theorem 7. *Let G and H be linear recursive sequences with dominating zeros, which satisfy the above conditions, and let δ be a real number with $0 < \delta < 1$. Assume that $\alpha \notin \boldsymbol{Z}$ or $\beta \notin \boldsymbol{Z}$. Then the equation*

$$G_x H_y = w^q \tag{3. 64}$$

in positive integers $w > 1$, x , y , q for which $\delta x < y < \frac{1}{\delta}x$, implies that $q < q_0$, where q_0 is an effectively computable number depending only on G , H and δ.

The idea of examination of the above problem was derived from a paper of Kiss [51], where the author showed that the equation $G_n G_x = w^q$ in positive integers x, w, q has no solution with $x > n$ and $q > q_0$. Although in the equation (3. 64) both indices are unknown and the recurrences may be distinct, Theorem 7 does not generalize the result of Kiss because there are differences between the assumptions in the two theorems.

Proof of Theorem 7

Denote by $c_1, c_2, \ldots$ positive real numbers which are effectively computable, and which depend on the number δ and the sequences G and H. We may assume without loss of generality that $|\alpha| \geq |\beta|$, and that the terms of the sequences G and H are positive.

In [101] Shorey and Stewart proved that for any d the equation $dH_y = w^q$ (resp. $dG_x = w^q$) implies that $q < q_0$. Hence, in what follows, we may assume that $x > n_0$ and $y > n_0$ for some n_0.

Let x, y, w and q (with the given conditions) be integers satisfying (3. 64). Then by (3. 62) and (3. 63) with x and y from (3. 64) we have

$$w^q = a\alpha^x\,(1+\varepsilon_x)\,b\beta^y\,(1+\varepsilon_y) \quad , \tag{3. 65}$$

where

$$\varepsilon_x = \left(\frac{g_2(x)}{a}\left(\frac{\alpha_2}{\alpha}\right)^x + \ldots + \frac{g_s(x)}{a}\left(\frac{\alpha_s}{\alpha}\right)^x\right) \tag{3. 66}$$

and

$$\varepsilon_y = \left(\frac{h_2(y)}{b}\left(\frac{\beta_2}{\beta}\right)^y + \ldots + \frac{h_t(y)}{b}\left(\frac{\beta_t}{\beta}\right)^y\right) \quad . \tag{3. 67}$$

Since ε_x (and also ε_y) tends to zero as x (and y) tends to infinity, then there exist real numbers $0 < \tau_1, \tau_2 < 1$ such that

$$|a||\alpha|^x|b||\beta|^y(1-\tau_1)(1-\tau_2) < w^q < |a||\alpha|^x|b||\beta|^y(1+\tau_1)(1+\tau_2) \quad , \tag{3. 68}$$

from which

$$c_1|\alpha|^x|\beta|^y < w^q < c_2|\alpha|^x|\beta|^y \tag{3. 69}$$

follows with some $c_1, c_2 > 0$. Using the condition $|\alpha| \geq |\beta|$ we get

$$\log c_1 + (x+y)\log|\beta| < q\log w < \log c_2 + (x+y)\log|\alpha| \quad , \tag{3. 70}$$

from which

$$c_3\frac{x+y}{q} < \log w < c_4\frac{x+y}{q} \tag{3. 71}$$

with some $c_3, c_4 > 0$.

We distinguish two cases. First we suppose that $w^q = ab\alpha^x\beta^y$, moreover, without loss of generality we may assume that, for example, $\alpha \notin \boldsymbol{Z}$. Let $\alpha' \neq \alpha$ be any conjugate of α and let φ be an automorphism of $\overline{\boldsymbol{Q}}$ with

$\varphi(\alpha) = \alpha'$. Then $\varphi(\beta) = \beta'$ is a conjugate of β and $|\beta'| \le |\beta|$, $|\alpha'| < |\alpha|$. Moreover,

$$ab\alpha^x\beta^y = \varphi(ab)(\varphi(\alpha))^x(\varphi(\beta))^y \quad . \tag{3. 72}$$

Thus

$$\left|\frac{\alpha}{\varphi(\alpha)}\right|^x = \left|\frac{\varphi(ab)}{(ab)}\left(\frac{\varphi(\beta)}{\beta}\right)^y\right| \le \left|\frac{\varphi(ab)}{(ab)}\right| \quad , \tag{3. 73}$$

whence x is bounded. Using again the above-mentioned theorem of Shorey and Stewart in [101] we obtain the statement of the theorem.

Now we can suppose that $w^q \ne ab\alpha^x\beta^y$, i.e. $Q = \left|\log\frac{w^q}{a\alpha^x b\beta^y}\right| \ne 0$. Applying Theorem B with $n = 4$, $\pi_1 = w$, $\pi_2 = ab$, $\pi_3 = \alpha$, $\pi_4 = \beta$, $B = x + y$, $B' = q$ and $M_4 = w$, it follows that

$$Q = |q\log w - \log ab - x\log\alpha - y\log\beta| > e^{-C(\log w\log q + \frac{x+y}{q})} \quad . \tag{3. 74}$$

On the other hand, (3. 65) and Lemma 5c), and afterwards Lemma 6, provides that

$$Q = \left|\log\frac{w^q}{a\alpha^x b\beta^y}\right| = |\log(1+\varepsilon_x)(1+\varepsilon_y)| < e^{-c_5 x} + e^{-c_6 y} < e^{-c_7(x+y)} \quad . \tag{3. 75}$$

Combining (3. 74) and (3. 75) we obtain the inequality

$$c_7(x+y) < C(\log w\log q + \frac{x+y}{q}) \quad . \tag{3. 76}$$

By (3. 71) it follows that $\frac{x+y}{q} < \frac{1}{c_3}\log w < \frac{1}{c_3}\log w\log q$, consequently

$$c_7(x+y) < c_8\log w\log q \quad . \tag{3. 77}$$

We now apply (3. 71) again for the left side of (3. 77), and we conclude that

$$\frac{c_7}{c_4}q\log w < c_8\log w\log q \quad , \tag{3. 78}$$

and

$$q < c_9\log q \quad . \tag{3. 79}$$

Hence $q < q_0$ and Theorem 7 is proved. ■

In the following we shall investigate an extended version of Theorem 7, where the number of recurrences is at least 2. In [108] we studied the problem of the product of a finite number of linear recurrences, and we obtained Theorem 8.

We shall use the same notation as before, but we need to update it for ν ($\nu \in \boldsymbol{N}, \nu > 1$) variables.

Let $G^{(i)} = \{G_n^{(i)}\}_{n=0}^{\infty}$ ($i = 1, 2, \ldots, \nu$) be linear recurrences of order k_i (≥ 2) defined by

$$G_n^{(i)} = A_1^{(i)} G_{n-1}^{(i)} + \ldots + A_{k_i}^{(i)} G_{n-k_i}^{(i)} \quad (n \geq k_i) \quad , \tag{3. 80}$$

where the initial values $G_j^{(i)}$ ($j = 0, 1, \ldots, k_i - 1$) and the coefficients $A_l^{(i)}$ ($l = 1, 2, \ldots, k_i$) of the sequences are rational integers. We suppose, as in Section 2.1, that $A_{k_i}^{(i)} \neq 0$ and that there is at least one non-zero initial value for each recurrence.

By $\alpha_1^{(i)} = \gamma_i, \alpha_2^{(i)}, \ldots, \alpha_{t_i}^{(i)}$ we denote the distinct roots of the characteristic polynomial

$$p_i(x) = x^{k_i} - A_1^{(i)} x^{k_i - 1} - \ldots - A_{k_i}^{(i)} \tag{3. 81}$$

of the sequence $G^{(i)}$, and we assume that $t_i > 1$ and $|\gamma_i| > |\alpha_j^{(i)}|$ for $j > 1$. Consequently $|\gamma_i| > 1$. Further suppose that the multiplicity of each roots γ_i is 1. Then the terms of the sequences $G^{(i)}$ ($i = 1, 2, \ldots, \nu$) can be written in the form

$$G_n^{(i)} = a_i \gamma_i^n + p_2^{(i)}(n) \left(\alpha_2^{(i)}\right)^n + \ldots + p_{t_i}^{(i)}(n) \left(\alpha_{t_i}^{(i)}\right)^n \quad (n \geq 0) \quad , \tag{3. 82}$$

where $p_j^{(i)}(x)$ ($j = 1, \ldots, t_i$) are polynomials from the ring

$$\boldsymbol{Q}(\gamma_i, \alpha_2^{(i)}, \ldots, \alpha_{t_i}^{(i)})[x] \quad . \tag{3. 83}$$

Assume that $\prod_{i=1}^{\nu} a_i \neq 0$.

Let $d \in \boldsymbol{Z}$ be a fixed non-zero rational integer, and let $p_1, \ldots, p_t$ be given rational primes. Denote by S the set of all rational integers composed solely of $p_1, \ldots, p_t$:

$$S = \{s \in \boldsymbol{Z} | s = \pm p_1^{e_1} \ldots p_t^{e_t} \ , \ e_i \in \boldsymbol{N}\} \quad . \tag{3. 84}$$

In particular $1 \in S$. Let

$$\mathcal{G}(x_1, \ldots, x_\nu) = G_{x_1}^{(1)} \ldots G_{x_\nu}^{(\nu)} \tag{3. 85}$$

be a function defined on the set $\boldsymbol{N}^\nu$. By the definitions of the sequences $G^{(i)}$ $(i = 1, \ldots, \nu)$, $\mathcal{G}$ takes integer values. With the given integer d let us consider the equation

$$d\mathcal{G}(x_1, \ldots, x_\nu) = sw^q \tag{3. 86}$$

in positive integers $w > 1$, q , x_i $(i = 1, 2, \ldots, \nu)$ and $s \in S$. We will show that under some conditions for $\mathcal{G}$, $q < q_0$ is also fulfilled if q satisfies the equation above. More precisely, using the Gel'fond-Baker method, we will prove

Theorem 8. *Let $\mathcal{G}(x_1, \ldots, x_\nu)$ be the function defined in (3. 85). Further let $d \in \boldsymbol{Z}$ be a fixed non-zero integer, and let $\delta < 1$ be a positive real number. Suppose that $\mathcal{G}(x_1, \ldots, x_\nu) \neq \prod_{i=1}^{\nu} a_i \gamma_i^{x_i}$ if $x_i > n_0$ $(i = 1, 2, \ldots, \nu)$ for some positive n_0. Then every positive integer solution w, q, s, $x_1, \ldots, x_\nu$ of the equation*

$$d\mathcal{G}(x_1, \ldots, x_\nu) = sw^q \tag{3. 87}$$

with $w > 1$, $s \in S$, $x_j > \delta \max_i\{x_i\}$ $(j = 1, 2, \ldots, \nu)$ and $x_i > n_0$ $(i = 1, 2, \ldots, \nu)$, satisfies

$$q < q_0 \quad , \tag{3. 88}$$

where q_0 is a computable number depending on $n_0, \delta, d, S, G^{(1)}, \ldots, G^{(\nu)}$.

For the proof of Theorem 8 we need the following lemma, which is a generalization of Lemma 6.

Lemma 7. *Let $\delta < 1, c_1, \ldots, c_k$ and x_0 be positive real numbers. Further let $x_1, \ldots, x_k$ be natural numbers with maximum value $x_m = \max_i\{x_i\}$ $(i \in \{1, \ldots, k\})$. If $x_j > \delta x_m$ is satisfied for every x_j and $x_m > x_0$ for some suitable $x_0 \in \boldsymbol{R}$ then there exists a real number $c > 0$, which depends on $k, \delta, \max_i\{c_i\}$ and x_0, for which*

$$\sum_{i=1}^{k} e^{-c_i x_i} < e^{-c(x_1 + \ldots + x_k)} = e^{-cx} \quad , \tag{3. 89}$$

where $x = x_1 + \ldots + x_k$.

Proof of Lemma 7

Using the conditions of Lemma 7 we have

$$\sum_{i=1}^{k} e^{-c_i x_i} < \sum_{i=1}^{k} e^{-c_i \delta x_m} = \sum_{i=1}^{k} e^{-d_i x_m} \quad , \tag{3. 90}$$

where $d_i = \delta c_i$. If $d_m = \min_i\{d_i\}$ then

$$\sum_{i=1}^{k} e^{-d_i x_m} \leq k e^{-d_m x_m} = e^{\log k - d_m x_m} \quad . \tag{3. 91}$$

Since $x_m \geq x_0$, it follows that

$$e^{\log k - d_m x_m} \leq e^{-d_m^\star x_m} = e^{-ckx_m} \leq e^{-cx} \tag{3. 92}$$

with a suitable constant $d_m^\star$ and $c = \frac{d_m^\star}{k}$. ■

Proof of Theorem 8

By $c_0, c_1, \ldots$ we denote positive real numbers which are effectively computable, depending on $n_0, \delta, d, S, G^{(1)}, \ldots, G^{(\nu)}$. We may assert, without loss of generality, that the terms of the recurrence $G^{(i)}$ $(i = 1, \ldots, \nu)$ are positive, $d > 0$ and that the inequalities

$$|\gamma_1| \geq |\gamma_2| \geq \ldots \geq |\gamma_\nu| \tag{3. 93}$$

also hold.

Let $x_1, \ldots, x_\nu, w, q$ and $s \in S$ be integers satisfying (3. 87). Further let $x = x_1 + \ldots + x_\nu$. We may assume that $s > 0$ and if

$$s = p_1^{e_1} \cdots p_t^{e_t} \tag{3. 94}$$

then $e_j < q$, otherwise a part of s can be merged into w^q. With some positive real numbers $u_1, \ldots, u_\nu$ and by Lemma 5a) we have

$$p_i^{e_i} \leq d\mathcal{G}(x_1, \ldots, x_\nu) < de^{u_1 x_1 + \ldots + u_\nu x_\nu} \leq e^{ux + \log d} \quad , \tag{3. 95}$$

where $u = \max_i\{u_i\} \in \boldsymbol{R}$. Hence

$$e_i < c_0 x \tag{3. 96}$$

with some positive real number c_0.

Observe that it is sufficient to consider the case $x_i > n_0$ $(i = 1, 2, \ldots, \nu)$. Otherwise, if we suppose that some $x_j \le n_0$ $(j \in \{1, 2, \ldots, \nu\})$ then $x_m = \max_i\{x_i\}$ cannot be arbitrarily large because of the assertion $x_j > \delta x_m$. Whence we have finitely many possibilities to choose ν-tuples $(x_1, \ldots, x_\nu)$, and the range of $\mathcal{G}(x_1, \ldots, x_\nu)$ is finite. Thus if the equality (3. 87) is satisfied with a fixed d then q must be bounded. In the sequel we suppose that $x_i > n_0$ $(i = 1, 2, \ldots, \nu)$, (i.e. $x > \nu n_0$).

Using (3. 82) with $n = x_1, \ldots, x_\nu$, we can re-write (3. 87) in the form

$$sw^q = d \prod_{i=1}^{\nu} a_i \gamma_i^{x_i} \left(1 + \varepsilon_i\right) \quad , \tag{3. 97}$$

where

$$\varepsilon_i = \frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i}\right)^{x_i} + \ldots + \frac{p_{t_i}^{(i)}(x_i)}{a_i} \left(\frac{\alpha_{t_i}^{(i)}}{\gamma_i}\right)^{x_i} \quad . \tag{3. 98}$$

Since $\lim\limits_{x_i \to \infty} \varepsilon_i = 0$ then there exist real constants $0 < \tau_1, \ldots, \tau_\nu < 1$ such that

$$d \prod_{i=1}^{\nu} |a_i||\gamma_i|^{x_i} |1 - \tau_i| < sw^q < d \prod_{i=1}^{\nu} |a_i||\gamma_i|^{x_i} |1 + \tau_i| \quad , \tag{3. 99}$$

and by collecting the constants in (3. 99)

$$c_1 \prod_{i=1}^{\nu} |\gamma_i|^{x_i} < sw^q < c_2 \prod_{i=1}^{\nu} |\gamma_i|^{x_i} \tag{3. 100}$$

is established. Let $x = x_1 + \ldots + x_\nu$ and apply (3. 93) for both sides of (3. 100) to conclude that $c_1|\gamma_\nu|^x < sw^q < c_2|\gamma_1|^x$. Taking logarithms, we obtain

$$\log c_1 + x\log|\gamma_\nu| < \log s + q\log w < \log c_2 + x\log|\gamma_1| \quad . \tag{3. 101}$$

Since $e_j < q$, we have

$$\log s + q\log w < c_3 q + q\log w \le q\log w \left(1 + \frac{c_3}{\log 2}\right) \quad , \tag{3. 102}$$

and so by (3. 101) we can find positive constants c_4 and c_5 such that

$$c_4 x < q\log w < c_5 x \tag{3. 103}$$

if x is large enough. But it may be assumed otherwise there are only finitely many possibilities for $x_1, \ldots, x_\nu$ and q is bounded.

By (3. 103), it follows that

$$c_4 \frac{x}{q} < \log w < c_5 \frac{x}{q} \quad . \tag{3. 104}$$

By (3. 97),

$$\frac{sw^q}{d\prod\limits_{i=1}^{\nu} a_i\gamma_i^{x_i}} = \prod_{i=1}^{\nu}(1+\varepsilon_i) \quad , \tag{3. 105}$$

and taking the absolute value of the logarithms of both sides of (3. 105) we have

$$Q := \left|\log\frac{sw^q}{d\prod_{i=1}^{\nu} a_i\gamma_i^{x_i}}\right| = \left|\log\prod_{i=1}^{\nu}(1+\varepsilon_i)\right| \quad . \tag{3. 106}$$

By combining Lemma 5b) and the right side of (3. 106), we establish

$$Q < \sum_{i=1}^{\nu} |\log(1+\varepsilon_i)| < \sum_{i=1}^{\nu} e^{-c_i^\star x_i} \quad , \tag{3. 107}$$

where $c_i^\star$ is a suitable positive constant ($i = 1, 2, \ldots, \nu$). Applying Lemma 7 and using the notation $x = x_1 + \ldots x_\nu$, we conclude that

$$Q < e^{-c_6(x_1+\ldots+x_\nu)} = e^{-c_6 x} \quad . \tag{3. 108}$$

On the other hand

$$Q = \left|\log s + q\log w - \log d - \log\prod_{i=1}^{\nu}|a_i| - x_1\log|\gamma_1| - \ldots - x_\nu\log|\gamma_\nu|\right|, \tag{3. 109}$$

where $\log s = e_1\log p_1 + \ldots + e_t\log p_t$ (see (3. 94)).

Since $sw^q = \mathcal{G}(x_1, \ldots, x_\nu) \neq \prod_{i=1}^{\nu} a_i\gamma_i^{x_i}$ if $x_i > n_0$ ($i = 1, 2, \ldots, \nu$), it follows that $\prod_{i=1}^{\nu}(1+\varepsilon_i) \neq 1$ and consequently $Q \neq 0$. Accordingly, Theorem B can be applied with the condition $n = \nu + t + 3$. The ordinary heights of p_j ($j = 1, 2, \ldots, t$), d, $\prod_{i=1}^{\nu} a_i$ and γ_i ($i = 1, 2, \ldots, \nu$) are constants. So $\pi_n = w = M_n$ and $B' = q$. Recalling that $e_i < c_0 x$ and examining the absolute values of the integer coefficients of the logarithms in (3. 109), $B = cx := \max\{x, c_0 x\}$ is a good choice. By (3. 109) and Theorem B, it follows that

$$Q > e^{-c_7(\log w \log q + \frac{cx}{q})} \quad . \tag{3. 110}$$

Now combine (3. 108) and (3. 110) to obtain the inequality

$$c_6 x < c_7(\log w \log q + \frac{cx}{q}) \quad , \tag{3. 111}$$

which together with (3. 104) implies that

$$c_6 x < c_7(\log w \log q + \frac{c}{c_4}\log w) < c_8 \log w \log q \tag{3. 112}$$

with some $c_8 > 0$. Applying (3. 104) again, we conclude that $\frac{1}{c_5} q \log w < x$. Futher, by (3. 112),

$$c_9 q < \log q \quad . \tag{3. 113}$$

Finally (3. 113) implies that $q < q_0$, which proves the theorem. ■

4 Common terms of linear recurrences

4.1 Historical survey

In this section we recite some interesting and important results on common terms of linear recurrences. At the start our main purpose is to investigate the diophantine equation

$$G_x = H_y \tag{4. 1}$$

in non-negative integers x and y, where G and H are linear recurrences.

First, binary recurrences are treated. Let A and B be arbitrarily fixed non-zero integers, $G = G(A, B, G_0, G_1)$ and $H = H(A, B, H_0, H_1)$ be second order linear recursive sequences for which both $|G_0|+|G_1|$ and $|H_0|+|H_1|$ are positive. The sequences G and H are called *equivalent* if there exist integers k and l such that $G_{n+k} = H_{n+l}$ or $G_{n+k} = -H_{n+l}$ for every integer $n \geq 0$.

It follows from a theorem of Revuz [85] that the equation (4. 1) has at most only finite number of solutions if G and H are non-equivalents. Obviously, equivalent recurrences G and H have infinitely many common terms. In the case of $A = 1$ and $B = 1$ (generalized Fibonacci sequence) Hirsch [45] proved the following. If G and H are not equivalent sequences then all common terms of them are less than $(\sqrt{5} - 1) \cdot \max\{|G_1G_3 - G_2^2|, |H_1H_3 - H_2^2|\}$.

This effective result was generalized by Kiss [48]. Denote the roots of polynomial $x^2 - Ax - B$ by α and β. Suppose that $\frac{\alpha}{\beta}$ is not a root of unity, $|\alpha| > |\beta|$ and the discriminant $D = A^2 + 4B > 0$. Then $G_n = \frac{a\alpha^n - b\beta^n}{\alpha - \beta}$ and $H_n = \frac{c\alpha^n - d\beta^n}{\alpha - \beta}$, where $a = G_1 - G_0\beta$, $b = G_1 - G_0\alpha$, $c = H_1 - H_0\beta$ and $d = H_1 - H_0\alpha$. Kiss showed for non-equivalent sequences that if $|\beta| < 1$ then (4. 1) has no solutions with the condition

$$x, y > 1 + n_0 \quad , \quad n_0 = \frac{1}{\log|\beta|} \log \frac{|\alpha|}{2 \cdot \max\{|b|, |d|\}} \quad . \tag{4. 2}$$

He also proved that the equation $G_x = H_y$ has no solutions with

$$|G_x| \geq 6 \cdot \max\{|G_0|, |G_1|, |H_0|, |H_1|\} \cdot \max\{|b|, |d|\} \cdot |B|^{n_0} \tag{4. 3}$$

if $B < 0$ and $|\beta| < 1$. Furthermore if $ac \neq 0$ then the equation (4. 1) has no solutions for which

$$x, y > \max\left\{\frac{\log\varepsilon - \log|\frac{b}{a}|}{\log|\frac{b}{a}|}, \frac{\log\varepsilon - \log|\frac{d}{c}|}{\log|\frac{d}{c}|}\right\} \tag{4. 4}$$

with some ε. A counterexample shows that the last statement is not true if the assumption $ac \neq 0$ is given up.

Explicit upper bounds for the solutions of the equation (4. 1) were provided by Mátyás [62] in the case of various coefficients A and B. Liptai [56] gained an ineffective finiteness result for the sequences $G(2u_0, 1, 0, 1)$ and $H(2v_0, 1, 0, 1)$, where $1 \neq u_0 < v_0$ are integers.

Several results are known concerning linear recurrences of higher order. Mignotte [67, 68] studied the equation (4. 1) if the companion polynomials of G and H have multiplicatively independent dominating zeros, and proved that the above equation has only finitely many integer solutions x, y. He also showed that if (4. 1) has infinitely many solutions then the two dominating zeros are multiplicatively dependent, and the set of the solutions is the union of a finite set and a finite number of arithmetical progressions.

Let S be the set of integers which can be written in the form $\pm p_1^{e_1} \dots p_s^{e_s}$, where p_i are fixed primes and $e_i \geq 0$, $(i = 1, \dots, s)$. The following theorem is due to Kiss [49]. Under certain conditions, if

$$s_1 G_x = s_2 H_y \tag{4. 5}$$

with $s_1, s_2 \in S$ then $\max\{x, y\}$ is less than an explicitely computable constant. In [109] we investigated a generalization of this statement. In [50] Kiss gave a lower bound for the values of $||s_1 G_x| - |s_2 H_y||$ too.

In [54] Laurent obtained conditions for two linear recurrences to have a finite number of common terms. In [97, 98] Schlickewei and Schmidt reformulated Laurent's Theorem 3 of [54], and they gave new conditions for the finiteness and infiniteness of the number of the common terms. In addition, in certain cases their theorems provide the indices of the common terms.

4.2 Theorem on the products of recurrences

Let $G^{(i)} = \{G_n^{(i)}\}_{n=0}^{\infty}$ $(i = 1, 2, \ldots, \nu)$ be linear recurrences of orders k_i (≥ 2) defined by the recursions

$$G_n^{(i)} = A_1^{(i)} G_{n-1}^{(i)} + \ldots + A_{k_i}^{(i)} G_{n-k_i}^{(i)} \quad (n \geq k_i), \tag{4. 6}$$

where the initial values $G_j^{(i)}$ $(j = 0, 1, \ldots, k_i - 1)$ and the coefficients $A_l^{(i)}$ $(l = 1, 2, \ldots, k_i)$ of the sequences are rational integers. We suppose, that $A_{k_i}^{(i)} \neq 0$ and there is at least one non-zero initial value for any of the recurrences.

By $\alpha_1^{(i)} = \gamma_i, \alpha_2^{(i)}, \ldots, \alpha_{t_i}^{(i)}$ we denote the distinct roots of the characteristic polynomial

$$p_i(x) = x^{k_i} - A_1^{(i)} x^{k_i - 1} - \ldots - A_{k_i}^{(i)} \tag{4. 7}$$

of the sequence $G^{(i)}$, and we assume that $t_i > 1$ and $|\gamma_i| > |\alpha_j^{(i)}|$ for $j > 1$. Consequently $|\gamma_i| > 1$. Suppose that the multiplicities of the roots γ_i are 1. It is known that the terms of the sequences $G^{(i)}$ $(i = 1, 2, \ldots, \nu)$ can be written in the form

$$G_n^{(i)} = a_i \gamma_i^n + p_2^{(i)}(n) \left(\alpha_2^{(i)}\right)^n + \ldots + p_{t_i}^{(i)}(n) \left(\alpha_{t_i}^{(i)}\right)^n \quad (n \geq 0), \tag{4. 8}$$

where $a_i \neq 0$ are fixed numbers and $p_j^{(i)}(x)$ $(j = 1, 2 \ldots, t_i)$ are polynomials in

$$\mathbb{Q}(\gamma_i, \alpha_2^{(i)}, \ldots, \alpha_{t_i}^{(i)})[x] \quad . \tag{4. 9}$$

Similarly, we define the linear recursive sequences $H^{(j)} = \{H_n^{(j)}\}_{n=0}^{\infty}$ $(j = 1, 2, \ldots, \mu)$ of orders l_j (≥ 2) given by

$$H_n^{(j)} = B_1^{(j)} H_{n-1}^{(j)} + \ldots + B_{l_j}^{(j)} H_{n-l_j}^{(j)} \quad (n \geq l_j), \tag{4. 10}$$

where the conditions for the initial values and coefficients of $H^{(j)}$ are the same as in the case of $G^{(i)}$. Let $\beta_1^{(j)}, \beta_2^{(j)}, \ldots, \beta_{u_j}^{(j)}$ be the zeros of the characteristic polynomials

$$q_j(x) = x^{l_j} - B_1^{(j)} x^{l_j - 1} - \ldots - B_{l_j}^{(j)}, \quad (j = 1, \ldots \mu). \tag{4. 11}$$

We assume, that there is a dominating zero, say the first, among them and for simplicity it is denoted by δ_j, that is $|\delta_j| > |\beta_i^{(j)}|$ for $2 \leq i \leq u_j$.

Suppose also that the multiplicities of the roots δ_j $(j = 1, 2, \ldots, \mu)$ are 1. Consequently

$$H_n^{(j)} = b_j \delta_j^n + q_2^{(j)}(n) \left(\beta_2^{(j)}\right)^n + \ldots + q_{u_j}^{(j)}(n) \left(\beta_{u_j}^{(j)}\right)^n \quad (n \geq 0), \quad (4.\ 12)$$

where $|\delta_i| > 1$ and $q_i^{(j)}(x)$ $(i = 2, \ldots, u_j)$ are polynomials. Further, assume that $b_j \neq 0$ $(j = 1, \ldots, \mu)$.

In the sequel we give an extension of the equation (4. 5). As in (3. 84), denote by S the set of all rational integers composed solely of certain primes. Let $\mathcal{G}(x_1, \ldots, x_\nu) = G_{x_1}^{(1)} \cdots G_{x_\nu}^{(\nu)}$ be a function defined on the set $\boldsymbol{N}^\nu$, as in (3. 85). Let the function $\mathcal{H}$ be defined on the set $\boldsymbol{N}^\mu$ by

$$\mathcal{H}(y_1, \ldots, y_\mu) = H_{y_1}^{(1)} \cdots H_{y_\mu}^{(\mu)} \quad . \quad (4.\ 13)$$

Then we examine the equation

$$s_1 \mathcal{G}(x_1, \ldots, x_\nu) = s_2 \mathcal{H}(y_1, \ldots, y_\mu) \quad (4.\ 14)$$

in $s_1, s_2 \in S$ and $x_i, y_j \in \boldsymbol{N}$ $(i = 1, 2, \ldots, \nu; j = 1, 2, \ldots, \mu)$. It will be proved that if the sequences satisfy certain conditions then the solutions x_i and y_j of (4. 14) are bounded. More precisely, we shall prove

Theorem 9. *Let $\mathcal{G}(x_1, \ldots, x_\nu)$ and $\mathcal{H}(y_1, \ldots, y_\mu)$ be the functions defined in (3. 85) and (4. 13). Futher let $\delta < 1$ and n_1 be positive real numbers. Suppose that $G_{l_i}^{(i)} \neq a_i \gamma_i^{l_i}$, $H_{k_j}^{(j)} \neq b_j \delta_j^{k_j}$ and*

$$s_1 \prod_{i=1}^{\nu} a_i \gamma_i^{l_i} \neq s_2 \prod_{j=1}^{\mu} b_j \delta_j^{k_j} \quad (4.\ 15)$$

for $l_i, k_j > n_1$ $(i = 1, 2, \ldots, \nu; j = 1, 2, \ldots, \mu)$ and for any integer $s_1, s_2 \in S$. If

$$s_1 \mathcal{G}(x_1, \ldots, x_\nu) = s_2 \mathcal{H}(y_1, \ldots, y_\mu) \quad (4.\ 16)$$

is satisfied with $s_1, s_2 \in S$ and positive integers $x_1, \ldots, x_\nu, y_1, \ldots, y_\mu$ for which $min_{i,j}\{x_i, y_j\} > \delta \cdot max_{i,j}\{x_i, y_j\}$ then $max_{i,j}\{x_i, y_j\} < n_2$, where n_2 is an effectively computable constant depending only on δ , n_1 and the functions $\mathcal{G}$ and $\mathcal{H}$.

Proof of Theorem 9

The proof depends on Theorem C and Lemma 7. Suppose that the equation (4. 16) holds for some $s_1 = p_1^{e_1} \dots p_t^{e_t}$, $s_2 = p_1^{f_1} \dots p_t^{f_t}$ $(s_1, s_2 \in S)$ and $x_i, y_j \in \boldsymbol{N}$. We may assume that $\gcd(s_1, s_2) = 1$. Without loss of generality, we may suppose that the terms of the recurrences are positive and $s_1, s_2 > 0$.

Since the sequences $G^{(i)}$ and $H^{(j)}$ have dominating zeros, by Lemma 5a), we find that

$$G_{x_i}^{(i)} < e^{c_i^\star x_i} \quad , \quad H_{y_j}^{(j)} < e^{d_j^\star y_j} \tag{4. 17}$$

with suitable positive constants $c_i^\star$ and $d_j^\star$.

By the estimations (4. 17) it follows that there exist positive constants c_1 and d_1 such that

$$\mathcal{G}(x_1, \dots, x_\nu) = G_{x_1}^{(1)} \cdots G_{x_\nu}^{(\nu)} < e^{c_1 x} \tag{4. 18}$$

and

$$\mathcal{H}(y_1, \dots, y_\mu) = H_{y_1}^{(1)} \cdots H_{y_\mu}^{(\mu)} < e^{d_1 y} \quad , \tag{4. 19}$$

where $x = x_1 + \dots + x_\nu$ and $y = y_1 + \dots + y_\mu$.

By (4. 16), $p_i^{e_i}$ divides $\mathcal{H}(y_1, \dots, y_\mu)$ and $p_j^{f_j}$ divides $\mathcal{G}(x_1, \dots, x_\nu)$, and together with (4. 18) and (4. 19) this implies that

$$p_i^{e_i} < e^{d_1 y} \quad \text{and} \quad p_j^{f_j} < e^{c_1 x} \quad (i, j = 1, 2, \dots, t) \quad . \tag{4. 20}$$

Consequently

$$e_i < d_2 y \quad \text{and} \quad f_j < c_2 x \tag{4. 21}$$

with some positive real numbers c_2 and d_2.

Transform the equality (4. 16) into the form

$$\frac{s_1 \prod_{i=1}^{\nu} a_i \left(\gamma_i\right)^{x_i}}{s_2 \prod_{j=1}^{\mu} b_j \left(\delta_j\right)^{y_j}} = \frac{(1+\xi_1) \cdots (1+\xi_\mu)}{(1+\varepsilon_1) \cdots (1+\varepsilon_\nu)} \tag{4. 22}$$

where

$$\varepsilon_i = \left(\frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i} \right)^{x_i} + \dots + \frac{p_{t_i}^{(i)}(x_i)}{a_i} \left(\frac{\alpha_{t_i}^{(i)}}{\gamma_i} \right)^{x_i} \right) \tag{4. 23}$$

and

$$\xi_j = \left(\frac{q_2^{(j)}(y_j)}{b_j} \left(\frac{\beta_2^{(j)}}{\delta_j} \right)^{y_j} + \dots \frac{q_{u_j}^{(j)}(y_j)}{b_j} \left(\frac{\beta_{u_j}^{(j)}}{\delta_j} \right)^{y_j} \right) \quad , \tag{4. 24}$$

$(i = 1, \ldots, \nu\ ,\ j = 1, \ldots, \mu)$. It was earlier mentioned that

$$|\varepsilon_i| < e^{-c'_i x_i} \quad \text{and} \quad |\xi_j| < e^{-d'_j y_j} \tag{4. 25}$$

for $x_i, y_j > n_3$, with some $c'_i, d'_j > 0$.

Take the logarithm of the right side of (4. 22), and consider the following inequalities:

$$Q = \left| \sum_{j=1}^{\mu} \log(1+\xi_j) - \sum_{i=1}^{\nu} \log(1+\varepsilon_i) \right| \quad \leq \tag{4. 26}$$

$$\leq \sum_{j=1}^{\mu} |\log(1+\xi_j)| + \sum_{i=1}^{\nu} |\log(1+\varepsilon_i)| \quad < \quad 2\left(\sum_{j=1}^{\mu} |\xi_j| + \sum_{i=1}^{\nu} |\varepsilon_i| \right) <$$

$$< 2\left(\sum_{j=1}^{\mu} e^{-d'_j y_j} + \sum_{i=1}^{\nu} e^{-c'_i x_i} \right) \quad < \quad 2e^{-c_3(x+y)} < e^{-c_4(x+y)} \quad ,$$

where $x+y = \sum\limits_{i=1}^{\nu} x_i + \sum\limits_{j=1}^{\mu} y_j$. Here the first inequality is a consequence of the properties of the absolute value function, the second one follows from the well-known inequality $|\log(z+1)| < 2|z|$, holding for any complex number z satisfying $|z| \leq \frac{1}{3}$, the third one comes from (4. 25). The next inequality is a consequence of Lemma 7, and the fifth one is easy to see.

Taking the logarithm of the left side of (4. 22), we have

$$Q = \left| \log s_1 - \log s_2 + \log\left(\frac{\prod_{i=1}^{\nu} a_i}{\prod_{j=1}^{\mu} b_j} \right) + \sum_{i=1}^{\nu} x_i \log\gamma_i - \sum_{j=1}^{\mu} y_j \log\delta_j \right| \quad . \tag{4. 27}$$

By the condition (4. 15) in Theorem 9 it follows that the left side of (4. 22) is not equal to 1, i.e. $Q \neq 0$. Hence we can apply Theorem C to (4. 27). The decomposition $\log s_1 - \log s_2 = (e_1 - f_1)\log p_1 + \ldots + (e_t - f_t)\log p_t$ shows that the ordinary heights of all arguments of the logarithms in (4. 27) are constant numbers, therefore $\prod_j \log A_j = c_5$. The coefficients of the logarithms, together with (4. 21), gives $B = c_6(x+y)$. Now we apply Theorem C to obtain the estimation

$$Q > e^{-Cc_5 \log(c_6(x+y))} > e^{-c_7 \log(x+y)} \quad . \tag{4. 28}$$

To derive the inequality

$$c_4(x+y) < c_7 \log(x+y) \quad , \tag{4. 29}$$

(4. 27) and (4. 28) have to be compared. The proof of the theorem is complete, because (4. 29) is valid only if $x+y<n_2$. ■

Summary

The dissertation deals with some types of diophantine equations concerning linear recurrences.

After the preface (Chapter 1), in Chapter 2 we introduce some notation which will be used throughout the dissertation, and we recite some important definitions and theorems of the subjects of linear recurrences and the estimates of linear forms in logarithms of algebraic numbers.

Section 3.1 is a short account of the results on the figurate numbers in linear recurrences. The new results begin in Section 3.2, where we consider the mixed exponential-polynomial diophantine equations

$$(2^n-1)(3^n-1)=x^2 \quad , \tag{1}$$

$$(2^n-1)(5^n-1)=x^2 \quad , \tag{2}$$

$$(2^n-1)(6^n-1)=x^2 \tag{3}$$

and

$$(a^n-1)((a^k)^n-1)=x^2 \tag{4}$$

in positive integers n,x as well as $n,x,a>1,k>1$. The products $(2^n-1)(3^n-1)$, $(2^n-1)(5^n-1)$, $(2^n-1)(6^n-1)$ and $(a^n-1)((a^k)^n-1)$ satisfy a fourth order linear recursive relation. They can also be interpreted as products of two binary recurrences. In [107] and with Hajdu in [43] we solve completely the diophantine equations (1), (2), (3) and (4). By a result of Shorey and Stewart [101], there exists no perfect powers in such sequences, provided that their exponents are large enough. This general result gives no information about the low exponent powers, for example squares, belonging to linear recursive sequences of order four. Only for some classes of Lehmer sequences of first and second kind are there known to be similar results, which were proved by McDaniel [63].

Section 3.3 deals with the occurrence of binomial coefficients of the form $\binom{x}{3}$ in certain binary recurrences. The problem arise from the results of Ming [69, 70] and McDaniel [64], who determined all the triangular numbers (i.e. positive integers of the form $T_x=\frac{x(x+1)}{2}$) in the Fibonacci, the Lucas and the Pell sequences. Since $T_x=\binom{x+1}{2}$, it is natural to ask which terms of a binary recurrence are binomial coefficient of the form $\binom{x}{3}$, $\binom{x}{4},\ldots$, etc. In [110] we proved that a certain type of the binary recurrences contains finite number of binomial coefficients of the form $\binom{x}{3}$. This problem leads to the solution of some elliptic equations. Using the program

package SIMATH we could find all integer points of the corresponding curves and have all required binomial coefficients, for example, in the FIBONACCI, the LUCAS and the PELL sequences.

Section 3.4, among others, investigates the equation

$$G_x H_y = w^q \tag{5}$$

in x, y, $w > 1$ and q, where G and H are linear recurrences having dominating roots. Earlier RIBENBOIM [88] and RIBENBOIM and MCDANIEL [89] dealt with the diophantine equation $G_x G_y = w^2$ in case of second order recurrences. For general recursive sequences of order larger than two there are fewer results. In [51] KISS showed that, under some conditions, the equation

$$G_n G_x = w^q \tag{6}$$

in integers x, w, q has no solution with $x > n$ and $q > q_0$, where q_0 is an effective computable constant depending only on n and G. In [15] with BRINDZA and LIPTAI we examine the equation (6) if the suffix n is not fixed and the sequences are distinct. Assuming that the suffixes n and x are not too far from each other we have an effective upper bound for q if some other conditions are also fulfilled. Later in [108] $G^{(i)}$'s $(i = 1, \dots, \nu)$ are given recurrences with dominating zeros, d is a fixed integer, and we investigate the equation

$$dG^{(1)}_{x_1} \cdot \ldots \cdot G^{(\nu)}_{x_\nu} = sw^q \tag{7}$$

in integers w, $q > 1$, x_i $(i = 1, \dots, \nu)$ and $s \in S$, the set of all rational integers composed of the fixed primes $p_1, \dots, p_t$, and give a further extension of the previous theorem.

Chapter 4 concerns the common terms of linear recurrences. After a historical survey we study the equation

$$s_1 G^{(1)}_{x_1} \cdot \ldots \cdot G^{(\nu)}_{x_\nu} = s_2 H^{(1)}_{y_1} \cdot \ldots \cdot H^{(\mu)}_{y_\mu} \tag{8}$$

in x_i $(i = 1, \dots, \nu)$, y_j $(j = 1, \dots, \mu)$ and $s_1, s_2 \in S$. First MIGNOTTE [67, 68] studied the common terms of two sequences. Later KISS [49] examined the equation

$$s_1 G_x = s_2 H_y \quad , \tag{9}$$

where x and y are integers, s_1 and s_2 are in a set S, and gave effective upper bound for $\max\{x, y\}$. Our theorem gives a certain generalization of the equation (9) and, under some conditions, provides that the suffixes are bounded ([109]).

Összefoglaló

Előzmények, vizsgálati módszerek és új eredmények áttekintése

Az értekezés négy egymáshoz kapcsolódó fejezetből áll. Az új eredményeket a 3. és a 4. fejezet tartalmazza. Az előkészítő részek után következő 3. fejezet a lineáris rekurzív sorozatokban előforduló teljes hatványokkal ill. bizonyos bináris rekurzív sorozatokban előforduló $\binom{x}{3}$ alakú binomiális együtthatókkal foglalkozik. A rövidebb lélegzetvételű 4. fejezet egy a lineáris rekurzív sorozatok közös elemeivel kapcsolatos eredményt, valamint annak előzményeit mutatja be.

A 60-as években – főleg a *Fibonacci Quarterly* folyóirat megjelenése után – kezdett előtérbe kerülni a különböző alakú számok keresése rekurzív sorozatokban. 1964-ben Cohn[22, 23] and Wyler[122] egymástól függetlenül bizonyították, hogy a Fibonacci sorozatban csak az $F_0 = 0$, $F_1 = F_2 = 1$ és $F_{12} = 144$ tagok négyzetszámok. Sok matematikus tevékenysége nyomán váltak ismertté a Fibonacci- ill. Lucas-sorozatban pl. az összes $2x^2$, $x^2 \pm 1$, $3x^2 \pm 1$, $x(x+1)$, $\frac{x(x+1)}{2}$, x^3, $x^3 \pm 1$, ... stb. alakú számok. Más sorozatokat is vizsgáltak, pl. Pethő[78] megadta az összes teljes hatványt a Pell-sorozatban.

A megoldási módszerek eleinte jobbára elemiek voltak, de egyre gyakrabban alkalmazták az ún. Baker-módszert, amely algebrai számok logaritmusainak lineáris formáira nyert becsléseken alapszik.

Shorey és Stewart[101], valamint Pethő[79, 77] munkái – szintén a Baker-módszert alkalmazva – általánosabb eredményeket hoztak. Belátták, hogy egy nem-degenerált rekurzív sorozatban nem fordulhat elő tetszőlegesen nagy kitevőjű teljes hatvány, ha a sorozat karakterisztikus polinomjának van egy egyszeres multiplicitású legnagyobb abszolút értékű ún. domináns gyöke. Ezen eredmények effektívek, azaz meg lehet adni egy, a kitevőre vonatkozó, a sorozat paramétereitől függő felső korlátot. Sajnos az előző tételek nem szolgáltatnak információt a kis kitevőjű hatványokról, így pl. a négyzetszámokról sem. A négyzetszámokkal kapcsolatban másodrendű rekurzív sorozatok esetében több fontos eredmény született, de negyedrendű sorozatoknál csak a Lehmer-sorozatokra van néhány érdekes tétel (McDaniel [63]).

A 3.2. fejezetben az 1-4. Tételben megadjuk bizonyos (nem Lehmer típusú) negyedrendű lineáris rekurzív sorozatokban az előforduló teljes négyzeteket. Az 1-4. Tételben szereplő diofantikus egyenletek értelmezhetők úgy is, hogy két másodrendű rekurzív sorozat szorzatában keressük a négyzetszámokat.

A 3.3 fejezet bizonyos típusú bináris rekurziókban előforduló $\binom{x}{3}$ alakú binomiális együtthatókkal foglakozik. MORDELL egy tételét felhasználva bizonyítjuk, hogy a

$$G_n = AG_{n-1} \pm G_{n-2} \quad , \quad G_0 = 0 \quad , \quad G_1 = 1 \tag{1}$$

rekurzióval megadott másodrendű sorozatok, valamint ezek asszociáltjai véges sok $\binom{x}{3}$ alakú binomiális együtthatót tartalmaznak (5. tétel). A bizonyítás során elliptikus egyenletekre vezetjük vissza a problémát, így lehetőség nyílik arra, hogy a SIMATH számítógépes programcsomagot felhasználva – rögzített sorozat esetén – megkeressük a megoldásokat. Mindezek demonstrálására a Fibonacci, a Lucas ill. a Pell sorozatokban megadjuk a fenti alakú binomiális együtthatókat. (6. tétel) Ezen eredményeket [110] tartalmazza.

[15]-ben BRINDZÁval és LIPTAIval közösen beláttuk, hogy bizonyos feltételek mellett két sorozat tagjainak szorzata sem lehet tetszőlegesen nagy kitevőjű teljes hatvány (7. Tétel). Korábban KISS [51] bizonyított hasonlókat, de ő egy adott sorozat két tagjával foglalkozik. Az 7. Tételnél szigorúbb feltételrendszer mellett [108]-ben megmutattuk, hogy ha véges sok lineáris rekurzió tagjai szorzatának konstansszorosa sw^q alakú, akkor q kisebb egy felső korlátnál, ahol s egy olyan egész számot jelöl, amelynek a prímfaktorizációjában csak rögzített prímek lehetnek (8. Tétel).

MIGNOTTE [67, 68] vetette fel, és bizonyította, hogy ha két sorozat karakterisztikus polinomjainak domináns gyökei multiplikatív értelemben függetlenek, akkor a szóban forgó sorozatoknak csak véges sok közös eleme lehet. KISS [49] vizsgálta az

$$s_1 G_x = s_2 H_y \tag{2}$$

egyenletet, ahol G és H rekurzív sorozatok és s_2, s_2 olyan egész számok, amelyek prímfaktorizációjában rögzített prímek vannak. A szerző megmutatta, hogy bizonyos feltételek teljesülése esetén az (2) egyenlet x, y megoldásaira $\max\{x, y\}$ kisebb egy explicite meghatározható konstansnál. A 9. Tétel az (2) egyenlethez hasonló problémával foglalkozik azzal a különbséggel, hogy az egyenlet bal- és jobb oldalán is tetszőleges sok , de véges számú rekurzív sorozat tagjainak szorzata áll. A kapott eredmény [109]-ben jelent meg, és analóg KISS előbb említett tételével.

A 7-9. Tételek bizonyításaiban a BAKER-módszert alkalmaztuk.

A disszertáció kilenc tételéből hat társszerző nélkül elért eredmény. A kilenc tétel hat cikkben jelent ill. jelenik meg, és a doktori iskola alatt, ill. az abszolutórium megszerzése után készültek.

A következő fejezetben bevezetünk néhány szükséges jelölést, majd azután részletesen ismertetjük a fent vázolt eredményeket.

Jelölések, elnevezések

Legyen $G = \{G\}_{n=0}^{\infty}$ egy k-adrendű ($k > 1, k \in \boldsymbol{Z}$) homogén lineáris rekurzív sorozat, amelyet a

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \ldots + A_k G_{n-k} \qquad (n \geq k) \tag{3}$$

összefüggés definiál, ahol az $A_1, A_2, \ldots, A_k$ paraméterek ($A_k \neq 0$) és a nem mind nulla $G_0, G_1, \ldots, G_{k-1}$ kezdőelemek rögzített egészek. A G sorozat jelölésére gyakran használjuk a $G = G(A_1, \ldots, A_k, G_0, \ldots, G_{k-1})$ alakot, továbbá ν számú lineáris rekurzív sorozat tagjainak $G_{x_1}^{(1)} \ldots G_{x_\nu}^{(\nu)}$ szorzatát pedig jelölje a

$$\mathcal{G}(x_1, \cdots, x_\nu) \tag{4}$$

szimbólum.

A G sorozat karakterisztikus polinomjának nevezzük a

$$g(x) = x^k - A_1 x^{k-1} - \ldots - A_k \tag{5}$$

polinomot, amelynek különböző gyökei legyenek $\alpha_1, \ldots, \alpha_t$, multiplicitásuk legyen rendre $e_1, \ldots, e_t$. A rekurzív sorozatok alaptétele szerint a G sorozat tagjai explicit módon előállíthatók

$$G_n = g_1(n)\alpha_1^n + \ldots + g_t(n)\alpha_t^n \qquad (n \geq 0) \tag{6}$$

formában, ahol $g_i(x)$ egy legfeljebb $(e_i - 1)$-edfokú polinom, melynek együtthatói a $\boldsymbol{Q}(\alpha_1, \ldots, \alpha_t)$ számtest elemei.

Ha a G sorozat $g(x)$ karakterisztikus polinomjának van olyan egyszeres multiplicitású $\alpha = \alpha_1$ gyöke amelyre $|\alpha| > |\alpha_i|$ ($i = 2, \ldots, t$) akkor az α gyököt domináns gyöknek mondjuk. Ebben az esetben az (6) előállítás

$$G_n = a\alpha^n + g_2(n)\alpha_2^n \ldots + g_t(n)\alpha_t^n \qquad (n \geq 0) \tag{7}$$

alakra módosul. A továbbiakban feltesszük, hogy dominán gyök esetén az a szám (vagy az a-t helyettesítő, más betűvel jelölt érték) nem zérus, valamint hogy G nem degenerált, azaz a karakterisztikus polinomja bármely két gyökének hányadosa nem egységgyök.

A későbbiekben szükségünk lesz még az alábbi halmaz bevezetésére. Rögzített $p_1, \ldots, p_m$ prímszámok esetén legyen

$$S = \{s \in \boldsymbol{Z} \mid s = \pm p_1^{u_1} \ldots p_m^{u_m}, u_i \in \boldsymbol{Z}, u_i \geq 0\} \quad . \tag{8}$$

Végül, a $\nu_p(n)$ szimbólum egy n egész szám p-adikus értékelését jelöli valamely p prímszám esetén.

Az új eredmények ismertetése

A disszertáció 3.2 fejezetében a

$$(2^n - 1)(3^n - 1) = x^2 \quad , \tag{9}$$

$$(2^n - 1)(5^n - 1) = x^2 \quad , \tag{10}$$

$$(2^n - 1)(6^n - 1) = x^2 \tag{11}$$

és az

$$(a^n - 1)\left(a^{kn} - 1\right) = x^2 \tag{12}$$

egyenletek egész megoldásait keressük az n és x, valamint (12) esetén még az a és k ismeretlenekben. Legyenek

$$G^{(3)}(12, -47, 72, -36, 0, 2, 24, 182) \quad , \tag{13}$$

$$G^{(5)}(18, -97, 180, -100, 0, 4, 72, 868) \tag{14}$$

és

$$G^{(6)}(21, -128, 252, -144, 0, 5, 105, 1505) \tag{15}$$

negyedrendű lineáris rekurziók. Hasonló módon értelmezhető a $G^{(a^k)}$ sorozat is. Legyenek továbbá $R^{(2)}(3, -2, 0, 1)$, $R^{(3)}(4, -3, 0, 2)$,
$R^{(5)}(6, -5, 0, 4)$, $R^{(6)}(7, -6, 0, 5)$ valamint $R^{(a^k)}(a^k+1, -a^k, 0, a^k-1)$ lineáris másodrendű rekurzív sorozatok. Könnyen látható, hogy

$$G_n^{(3)} = R_n^{(2)} \cdot R_n^{(3)} = (2^n - 1)(3^n - 1) \quad , \tag{16}$$

$$G_n^{(5)} = R_n^{(2)} \cdot R_n^{(5)} = (2^n - 1)(5^n - 1) \quad , \tag{17}$$

$$G_n^{(6)} = R_n^{(2)} \cdot R_n^{(6)} = (2^n - 1)(6^n - 1) \quad , \tag{18}$$

$$G_n^{(a^k)} = R_n^{(a)} \cdot R_n^{(a^k)} = (a^n - 1)\big((a^{kn} - 1\big) \quad . \tag{19}$$

Tehát a $(2^n - 1)(3^n - 1) = x^2$, a $(2^n - 1)(5^n - 1) = x^2$, a $(2^n - 1)(6^n - 1) = x^2$, valamint az $(a^n - 1)\big((a^k)^n - 1\big) = x^2$ polinomiális-exponenciális egyenleteket megoldani ugyanazt jelenti mint rendre megkeresni a $G^{(3)}$, $G^{(5)}$, $G^{(6)}$ és $G^{(a^k)}$ negyedrendű sorozatokban a négyzetszámokat, vagy másképp fogalmazva megkeresni az $R_n^{(2)} \cdot R_n^{(3)}$, $R_n^{(2)} \cdot R_n^{(5)}$, $R_n^{(2)} \cdot R_n^{(6)}$ és $R_n^{(a)} \cdot R_n^{(a^k)}$ szorzatokban a teljes négyzeteket. A következő tételek az előbbiekben felvetett problémák megoldását adják.

1. Tétel.(SZALAY[107]) *A*

$$(2^n - 1)(3^n - 1) = x^2 \tag{20}$$

diofantikus egyenlet nem oldható meg pozitív egészekben.

2. Tétel.(SZALAY[107]) *A*

$$(2^n - 1)(5^n - 1) = x^2 \tag{21}$$

diofantikus egyenletnek az $n = 1, x = 2$ az egyetlen pozitív egész megoldása.

3. Tétel.(HAJDU-SZALAY[43]) *A*

$$(2^n - 1)(6^n - 1) = x^2 \tag{22}$$

diofantikus egyenletnek nincs pozitív egész megoldása.

4. Tétel.(HAJDU-SZALAY[43]) *Az*

$$(a^n - 1)\big((a^k)^n - 1\big) = x^2 \tag{23}$$

diofantikus egyenletnek $k > 1$, $a > 1$ esetén csak az alábbi a pozitív egész megoldásai vannak: $(a, n, k, x) = (2, 3, 2, 21)$, $(3, 1, 5, 22)$ és $(7, 1, 4, 120)$.

Az 1-2. Tételek bizonyításához – többek között – szükség van az alábbi lemmákra.

3. Lemma(SZALAY[107]) *Ha α és k pozitív egész számok, $k \not\equiv 0 \pmod 5$, és $n = k \cdot 4 \cdot 5^{\alpha-1}$ akkor*

$$\nu_5\left((2^n-1)(3^n-1)\right) = 2\alpha. \tag{24}$$

4. Lemma(SZALAY[107]) *Ha α és k pozitív egész számok, $k \not\equiv 0 \pmod 3$, és $n = k \cdot 2 \cdot 3^{\alpha-1}$ akkor*

$$\nu_3\left((2^n-1)(5^n-1)\right) = 2\alpha. \tag{25}$$

Az 1-3. Tételek bizonyítása során lényegében azt látjuk be, hogy $(2^n-1)(3^n-1)$, $(2^n-1)(6^n-1)$ és egy kivételtől eltekintve $(2^n-1)(5^n-1)$ nem lehet kvadratikus maradék alkalmasan megválasztott modulusokra.

A 4. Tétel bizonyítása LJUNGGREN [57] és CHAO KO [21] egy-egy tételén alapszik.

A 3.3 fejezetben belátjuk, hogy bizonyos típusú bináris rekurzív sorozatokban az $\binom{x}{3}$ alakú binomiális együtthatók csak véges sokszor fordulhatnak elő (5. Tétel). A bizonyítás során használt átalakítások lehetővé tették, hogy egy konkrét rekurzív sorozat esetén a SIMATH számítógépes algebrai rendszert felhasználva megkeressük a fenti típusú binomiális együtthatókat bizonyos sorozatokban. Az eredeti probléma elliptikus egyenletek megoldására vezet, amely sikeresen kezelhető a SIMATH-tal. A módszer demonstrálására a Fibonacci- (F_n), a Lucas- (L_n) ill. a Pell-sorozat (P_n) esetén adjuk meg az említett binomiális együtthatókat (6. Tétel). A pontos eredmények a következőek. Legyen

$$U_n = AU_{n-1} + BU_{n-2} \quad (n \geq 2) \tag{26}$$

egy másodrendű rekurzív sorozat, ahol $U_0 = 0$ és $U_1 = 1$, $A \neq 0$ egész együttható, és $|B| = 1$. Legyen továbbá a V sorozat az U asszociált sorozata. Ekkor igaz az

5. Tétel.(SZALAY[110]) *Az $U_n = \binom{x}{3}$ és $V_n = \binom{x}{3}$ diofantikus egyenleteknek csak véges sok $n \geq 0$, $x \geq 3$ egész megoldása van.*

A nevezetes sorozatokra az alábbi tételt nyertük.

6. Tétel.(SZALAY[110]) *Az*
i) $F_n = \binom{x}{3}$ *egyenlet megoldásai:* $(n, x) = (1, 3)$, $(2, 3)$;
ii) $L_n = \binom{x}{3}$ *egyenlet megoldásai:* $(n, x) = (1, 3)$, $(3, 4)$;
iii) $P_n = \binom{x}{3}$ *egyenlet egyetlen megoldása:* $(n, x) = (1, 3)$.

A 3.4 fejezettől fogva az eredmények jellege megváltozik, a tételek effektív becslésekkel foglalkoznak és a BAKER-módszerrel igazoltuk őket. Legyenek

$$G = G(A_1, \dots, A_k, G_0, \dots, G_{k-1}) \tag{27}$$

valamint

$$H = H(B_1, \dots, B_l, H_0, \dots, H_{l-1}) \tag{28}$$

rendre k-adrendű ill. l-edrendű lineáris rekurziók, amelyek karakterisztikus polinomjai rendre $g(x)$ ill. $h(x)$, rendre α ill. β domináns gyökökkel. Tegyük fel, hogy (7)-ben G esetén $a \neq 0$ és H esetén $b \neq 0$. Az alábbi tétel szerint ha nem túl távol eső x és y indexekre a $G_x H_y$ szorzat egy q-adik hatvány, akkor a q kitevő nem lehet tetszőleges nagy.

7. Tétel.(BRINDZA, LIPTAI, SZALAY[15]) *Legyenek G és H az előző bekezdésben meghatározott lineáris rekurzív sorozatok, valamint legyen δ olyan valós szám, amelyre $0 < \delta < 1$. Tegyük fel, hogy az α vagy β domináns gyökök valamelyike nem egész. Ekkor a*

$$G_x H_y = w^q \tag{29}$$

azon pozitív egész $w > 1$, x , y , q megoldásaira, amelyekre $\delta x < y < \frac{1}{\delta} x$, következik, hogy $q < q_0$, ahol $q_0 = q_0(G, H, \delta)$ egy effektíve kiszámítható konstans.

A következő tétel kapcsolódik a (29) egyenlethez, de a bal oldalon szereplő két sorozat helyett tetszőleges számú sorozat tagjainak szorzatát vizsgálja az előzőtől különböző feltétellel. Jelölje γ_i a $G^{(i)}$ sorozat karakterisztikus polinomjának domináns gyökét, a_i pedig γ_i nem-nulla együtthatóját $G^{(i)}_{x_i}$ explicit előállításában. A pontos állítás a következő.

8. Tétel.(SZALAY[108]) *Legyen $\mathcal{G}(x_1, \dots, x_\nu)$ a (3. 85)-ben definiált ν változós függvény, d egy nem nulla egész szám. Legyen továbbá δ pozitív valós szám a $\delta < 1$ feltétellel. Tegyük fel, hogy $\mathcal{G}(x_1, \dots, x_\nu) \neq \prod_{i=1}^{\nu} a_i \gamma_i^{x_i}$ ha $x_i > n_0$ $(i = 1, 2, \dots, \nu)$ valamely n_0 esetén. Ha az $x_1, \dots, x_\nu, s, w, q$ egész számok megoldásai a*

$$d\mathcal{G}(x_1, \dots, x_\nu) = sw^q \tag{30}$$

egyenletnek, úgy hogy $w > 1$, $s \in S$ és $x_j > \delta \max_i\{x_i\}$ $(j = 1, 2, \dots, \nu)$, akkor

$$q < q_0 \quad , \tag{31}$$

ahol q_0 egy effektíve kiszámítható n_0-tól, δ-tól, d-től, S-től és $\mathcal{G}$-től függő konstans.

A 4. fejezetben lineáris rekurzív sorozatok közös tagjaival foglalkozunk. Legyenek $\mathcal{G}(x_1, \dots, x_\nu)$ és $\mathcal{H}(y_1, \dots, y_\mu)$ (4) alapján definiált függvények. Az

$$s_1\mathcal{G}(x_1, \dots, x_\nu) = s_2\mathcal{H}(y_1, \dots, y_\mu) \tag{32}$$

diofantikus egyenletet vizsgáljuk, ahol $s_1, s_2 \in S$. Jelölje a vizsgált $G^{(i)}$ ill. $H^{(j)}$ rekurzív sorozatok karakterisztikus polinomjainak domináns gyökeit rendre γ_i ill. δ_j, azok együtthatóit a sorozatok explicit előállításában rendre a_i ill. b_i. A következő effektív tétel a vizsgált egyenlettel kapcsolatos.

9. Tétel.(SZALAY[109]) *Legyenek $\mathcal{G}(x_1, \dots, x_\nu)$ és $\mathcal{H}(y_1, \dots, y_\mu)$ az előbb értelmezett függvények. Legyen továbbá $\delta < 1$ pozitív valós szám, valamint n_1 olyan pozitív valós szám, amelyre $G_{l_i}^{(i)} \neq a_i\gamma_i^{l_i}$, $H_{k_j}^{(j)} \neq b_j\delta_j^{k_j}$ és bármely $s_1, s_2 \in S$ esetén*

$$s_1 \prod_{i=1}^{\nu} a_i \gamma_i^{l_i} \neq s_2 \prod_{j=1}^{\mu} b_j \delta_j^{k_j} \tag{33}$$

ha $l_i, k_j > n_1$ $(i = 1, 2, \dots, \nu; j = 1, 2, \dots, \mu)$. Ha

$$s_1\mathcal{G}(x_1, \dots, x_\nu) = s_2\mathcal{H}(y_1, \dots, y_\mu) \tag{34}$$

teljesül olyan $s_1, s_2 \in S$, és $x_1, \dots, x_\nu, y_1, \dots, y_\mu$ pozitív egészekre, amelyekre $min_{i,j}\{x_i, y_j\} > \delta \cdot max_{i,j}\{x_i, y_j\}$, akkor $max_{i,j}\{x_i, y_j\} < n_2$, ahol n_2 egy effektív konstans.

A 4-7. Tételek az algebrai számok logaritmusainak lineáris kombinációira adott alsó becslések segítségével igazolhatók.

Az értekezés eredményei elméleti jellegűek, ezért a felhasználásuk az alapkutatásban, a matematikán belül lehetséges.

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