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Teaching
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On a special class of generalized conics with infinitely many focal points

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Abstract. Let a continuous, piecewise smooth curve in the Euclidean space be given. We are going to investigate the surfaces formed by the vertices of generalized cones with such a curve as the common directrix and the same area. The basic geometric idea in the background is when the curve runs through the sides of a non-void triangle ABC . Then the sum of the areas of some triangles is constant for any point of such a surface. By the help of a growth condition we prove that these are convex compact surfaces in the space provided that the points A , B and C are not collinear. The next step is to introduce the general concept of *awnings* spanned by a curve. As an important example awnings spanned by a circle will be considered. Estimations for the volume of the convex hull will be also given.

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1. Introduction

The concept of generalized conics as the set of points $\vec{x} \in \mathbb{R}^n$ such that

$$\int_G \alpha(\vec{g}) d(\vec{g}, \vec{x}) dg \leq \rho$$

can be found in [2], p. 742. Here $G \subset \mathbb{R}^n$ is the set of foci, $\alpha: G \rightarrow \mathbb{R}$ is a weight function and ρ is a real constant. As an open question the authors posed the

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problem whether *which results* (of the classical case) *can be extended to the cases of infinitely many focal points or to continuous set of foci*. We are going to deal with a similar problem based on special choices of G and a more general weight function admitting the points of the space in its argument. Our geometric idea can be formulated as follows: let three (or more, but finite many) points A , B and C be given in the three-dimensional euclidean space and consider the set of the points P such that

$$\mathcal{A}(ABP) + \mathcal{A}(APC) + \mathcal{A}(PBC) = \text{const.} \quad (1)$$

where $\mathcal{A}(ABP)$, $\mathcal{A}(APC)$ and $\mathcal{A}(PBC)$ denote the area of the triangles ABP , APC and PBC , respectively. We will show that (1) defines convex compact surfaces in the space provided that the points A , B and C are not collinear. In case of collinear points we have unbounded cylindrical surfaces. The global minimum of the left hand side as the function of P is attained at any point contained by the convex hull of the triangle ABC . The next step is the substitution of the sides of the triangle ABC with a continuous, piecewise smooth curve. We are going to investigate the niveau surfaces of the function $P \rightarrow \mathcal{A}(P)$ measuring the area of generalized cones with such a curve as the common directrix and the vertex P . They are called *awnings* spanned by the curve. As an important special case awnings spanned by a circle will be considered. We give estimations for the volume of the convex hull.

2. The basic geometric idea

Let the points A , B and C be given in the three-dimensional Euclidean space and consider the function

$$\begin{aligned} \Sigma(P) := & \frac{1}{2}|(P-A) \times (P-B)| + \frac{1}{2}|(P-C) \times (P-A)| \\ & + \frac{1}{2}|(P-B) \times (P-C)| \end{aligned}$$

measuring the sum of the areas of the triangles ABP , APC and PBC .

LEMMA 1. Σ is a convex function.

PROOF. It is enough to prove the convexity of the expression

$$|(P-A) \times (P-B)| = |P \times (A-B) + A \times B|$$

as the function of P . Here

$$f(P) := P \times (A - B) + A \times B$$

is an affine mapping preserving the affine, especially the convex combination of the points, i.e.

$$f(\lambda P + (1 - \lambda)Q) = \lambda f(P) + (1 - \lambda)f(Q).$$

The triangle inequality implies that Σ is convex. $\square$

LEMMA 2. *If A , B and C are not collinear then the global minimum is attained at the points of the convex hull of the triangle ABC .*

PROOF. According to the triangle inequality it can be easily seen that Σ is great than or equal to the norm of the sum

$$\frac{1}{2}(P - A) \times (P - B) + \frac{1}{2}(P - C) \times (P - A) + \frac{1}{2}(P - B) \times (P - C)$$

which is just the area of the triangle ABC . If P is not in the plane of ABC then the orthogonal projection decreases the sum of the areas. If P is not in the convex hull of the triangle ABC then P and, for example, the point C are separated by the line AB . According to the *Plane separation postulate* the segment PC meets the line AB which implies that ABC is covered by the union of the triangles ABP , APC and PBC . $\square$

LEMMA 3. *If a function satisfies the growth condition*

$$\lim_{|P| \rightarrow \infty} \frac{\Sigma(P)}{|P|} := \lim_{r \rightarrow \infty} \inf \left\{ \frac{\Sigma(P)}{|P|} \mid |P| > r \right\} > 0$$

then it has bounded level sets.

PROOF. Suppose, in contrary, that the level set

$$M := \{P \mid \Sigma(P) \leq c\}$$

is unbounded. Then we can choose a sequence P_n of the elements of M such that the norm $|P_n|$ tends to the infinity. Therefore

$$\lim_{|P| \rightarrow \infty} \frac{\Sigma(P)}{|P|} \leq \lim_{n \rightarrow \infty} \frac{\Sigma(P_n)}{|P_n|} \leq \lim_{n \rightarrow \infty} \frac{c}{|P_n|} = 0,$$

which is a contradiction. $\square$

REMARK 1. Requiring a function to have bounded level sets is also a kind of growth conditions. For convex functions these conditions are actually equivalent to each other as it was stated in [1] but the proof is left as an exercise.

THEOREM 1. *If A , B and C are not collinear then the levels of the function Σ are convex and compact.*

PROOF. Since Σ is continuous the levels are closed subset of the space. In order to see the compactness we prove that it satisfies the growth condition

$$\lim_{|P| \rightarrow \infty} \frac{\Sigma(P)}{|P|} := \lim_{r \rightarrow \infty} \inf \left\{ \frac{\Sigma(P)}{|P|} \mid |P| > r \right\} > 0$$

which implies that the levels are bounded. First of all note that

$$\begin{aligned} & \frac{1}{4} |(P - A) \times (P - B)|^2 + \frac{1}{4} |(P - C) \times (P - A)|^2 \\ & + \frac{1}{4} |(P - B) \times (P - C)|^2 \leq \Sigma^2(P). \end{aligned}$$

The left hand side can be written into the form

$$L_2(P) + L_1(P) + L_0$$

with a quadratic form, a linear functional and a real constant denoted by L_2 , L_1 and L_0 , respectively. The steps of the proof can be summarized as follows:

- We prove that L_2 is positive definite.
- Taking $L_1(P) = \langle L, P \rangle$ we have that

$$-|L| \leq \frac{L_1(P)}{|P|} \leq |L|.$$

- If

$$k := \min \left\{ \frac{L_2(P)}{|P|^2} \mid |P| \neq 0 \right\} > 0$$

then

$$\frac{\Sigma^2(P)}{r^2} \geq k - \frac{|L|}{r} + \frac{L_0}{r^2}, \quad \text{where } r = |P|.$$

Taking the limit $r \rightarrow \infty$ it follows that

$$\lim_{|P| \rightarrow \infty} \frac{\Sigma(P)}{|P|} \geq \sqrt{k} > 0$$

as was to be stated.

Since

$$(P - A) \times (P - B) = P \times (A - B) + A \times B = P \times A - P \times B + A \times B$$

we have that

$$\begin{aligned} |(P - A) \times (P - B)|^2 &= \langle P \times A, P \times A \rangle + \langle P \times B, P \times B \rangle + \langle A \times B, A \times B \rangle \\ &\quad - 2\langle P \times A, P \times B \rangle + 2\langle P \times A, A \times B \rangle - 2\langle P \times B, A \times B \rangle. \end{aligned}$$

Using the identity

$$\langle X \times Y, Z \times W \rangle = \langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle$$

to separate the quadratic term of the sum, it follows that

$$\begin{aligned} |(P - A) \times (P - B)|^2 &= |A|^2 |P|^2 - \langle A, P \rangle^2 + 2\langle A, P \rangle \langle B, P \rangle \\ &\quad - 2|P|^2 \langle A, B \rangle + |P|^2 |B|^2 - \langle P, B \rangle^2 + \text{linear} + \text{const.} \end{aligned}$$

and, consequently,

$$\begin{aligned} |(P - A) \times (P - B)|^2 &= |P|^2 (|A|^2 - 2\langle A, B \rangle + |B|^2) \\ &\quad + (\langle A, P \rangle - \langle B, P \rangle)^2 + \text{linear} + \text{const.} = \end{aligned}$$

Therefore

$$|(P - A) \times (P - B)|^2 = |P|^2 |A - B|^2 - \langle P, A - B \rangle^2 + \text{linear} + \text{const.}$$

and the quadratic form L_2 can be expressed as

$$\begin{aligned} L_2(P) &= \frac{1}{4} (|P|^2 |A - B|^2 - \langle P, A - B \rangle^2) + \frac{1}{4} (|P|^2 |C - A|^2 - \langle P, C - A \rangle^2) \\ &\quad + \frac{1}{4} (|P|^2 |B - C|^2 - \langle P, B - C \rangle^2). \end{aligned}$$

The Cauchy-Schwarz inequality shows that $L_2(P)$ vanishes if and only if P is linear dependent with each of the vectors $A - B$, $C - A$ and $B - C$. Therefore P is equal to zero or the points A , B and C are collinear. $\square$

REMARK 2. The linear term can be expressed as

$$\begin{aligned} L_1(P) &= \frac{1}{2} \langle P, A \rangle (\langle A, B + C \rangle - |B|^2 - |C|^2) + \frac{1}{2} \langle P, B \rangle (\langle B, A + C \rangle - |A|^2 - |C|^2) \\ &\quad + \frac{1}{2} \langle P, C \rangle (\langle C, A + B \rangle - |A|^2 - |B|^2) \end{aligned}$$

and we have that

$$L_0 = \frac{1}{4} |A \times B|^2 + \frac{1}{4} |B \times C|^2 + \frac{1}{4} |C \times A|^2.$$

COROLLARY 1. *If A , B and C are not collinear then the niveau surfaces of the function Σ are convex and compact.*

3. A special class of generalized conics with infinitely many focal points

Let $\gamma: [a, b] \rightarrow \mathbb{R}^3$ be a continuous, piecewise smooth curve under the partition $a = t_0 < t_1 < \dots < t_{n-1} = t_n = b$ and consider the function

$$P \rightarrow \mathcal{A}(P)$$

measuring the area of generalized cones with such a curve as the common directrix and the vertex P .

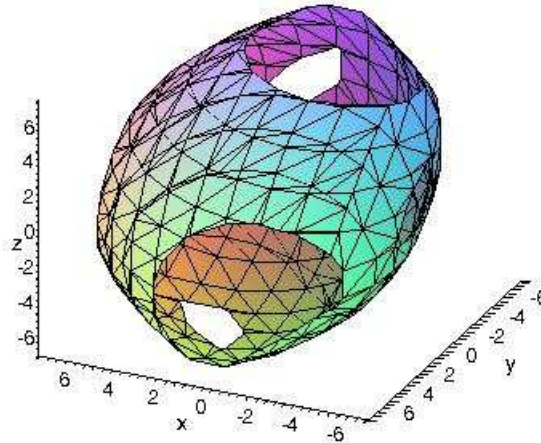


Figure 1. A niveau surface with $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$.

DEFINITION 1. Niveau surfaces of the form $\mathcal{A}(P) = \text{const.}$ are called *awnings* spanned by γ .

Using the parameterization

$$\vec{r}: [0, 1] \times [a, b] \rightarrow \mathbb{R}^3, \quad \vec{r}(u, v) := uP + (1 - u)\gamma(v)$$

an easy calculation shows that

$$D_1 \vec{r}(u, v) = P - \gamma(u), \quad D_2 \vec{r}(u, v) = (1 - u) \gamma'(v)$$

and, consequently,

$$D_1 \vec{r}(u, v) \times D_2 \vec{r}(u, v) = (1 - u)(P - \gamma(v)) \times \gamma'(v)$$

provided that v is not one of the parameters t'_i s. Then the area function can be calculated by the formula

$$\mathcal{A}(P) = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_0^1 |D_1 \vec{r} \times D_2 \vec{r}| du dv.$$

Particularly,

$$\mathcal{A}(P) = \frac{1}{2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |(P - \gamma) \times \gamma'| dv.$$

THEOREM 2. *Any awning is the boundary of a generalized conic with the set of foci $G := \{\vec{g} \mid \vec{g} = \gamma(v), \text{ where } v \in [a, b]\}$ and the weight function*

$$\alpha(\vec{g}, P) := \frac{1}{2} \sin(\text{the angle of } P - \vec{g} \text{ and the tangent line of } G \text{ at } \vec{g}).$$

PROOF. Let $\vec{g} = \gamma(v)$ be a point in G . Then

$$\frac{1}{2} |(P - \gamma(v)) \times \gamma'(v)| = |P - \vec{g}| |\gamma'(v)| \alpha(\vec{g}, P) = d(\vec{g}, P) \alpha(\vec{g}, P) |\gamma'(v)|.$$

Integrating by v we have that

$$\mathcal{A}(P) = \int_G d(\vec{g}, P) \alpha(\vec{g}, P) dg$$

and the niveau surfaces of $\mathcal{A}$ are the boundary of generalized conics of the form

$$\int_G d(\vec{g}, P) \alpha(\vec{g}, P) dg \leq \text{const.}$$

as was to be stated. $\square$

LEMMA 4. *The function $P \rightarrow \mathcal{A}(P)$ is convex.*

PROOF. The proof is similar to that of Lemma 1 taking the affine term

$$f(P) = P \times \gamma'(v) - \gamma(v) \times \gamma'(v)$$

for some fixed parameter v . $\square$

REMARK 3. If the curve γ runs through the sides of the triangle ABC then we get back to the basic geometric idea.

REMARK 4. Using the parameterization

$$\vec{r}(u, v_1, \dots, v_{n-2}) = uP + (1 - u)\gamma(v_1, \dots, v_{n-2}),$$

where $\gamma: H \rightarrow \mathbb{R}^n$ is a parameterized surface of dimension $n - 2$, the awnings can be given by

$$\frac{1}{n-1} \int_H |(P - \gamma) \times D_1\gamma \times \dots \times D_{n-2}\gamma| dv_1 \dots dv_{n-2} = \text{const.}$$

in case of higher dimensional spaces.

4. Awnings spanned by a circle

Let γ be a parameterized circle, i.e.

$$\gamma(v) := (r \cos v, r \sin v, 0),$$

where $0 \leq v \leq 2\pi$. Awnings spanned by γ can be given by the integral equation

$$\frac{r}{2} \int_0^{2\pi} \sqrt{z^2 + (x \cos v + y \sin v - r)^2} dv = \text{const.}$$

They are obviously invariant under the rotation around the z -axis. In what follows we investigate them in details.

LEMMA 5. *Awnings spanned by a circle are convex compact surfaces in the space. The global minimum of the function*

$$\mathcal{A}(P) := \frac{r}{2} \int_0^{2\pi} \sqrt{z^2 + (x \cos v + y \sin v - r)^2} dv, \quad P = P(x, y, z)$$

is attained at any point contained by the convex hull of the circle.

PROOF. Let $n \geq 3$ and consider the vertices $A_0, \dots, A_n = A_0$ of the inscribed regular n -gon with a fixed initial point A_0 as the set of foci. It can be easily seen that

$$\lim_{n \rightarrow \infty} \Sigma_n(P) = \mathcal{A}(P)$$

and the set of the extremal points is just the limit

$$\lim_{n \rightarrow \infty} \text{conv}\{A_0, \dots, A_n\}$$

of the convex hulls. $\square$

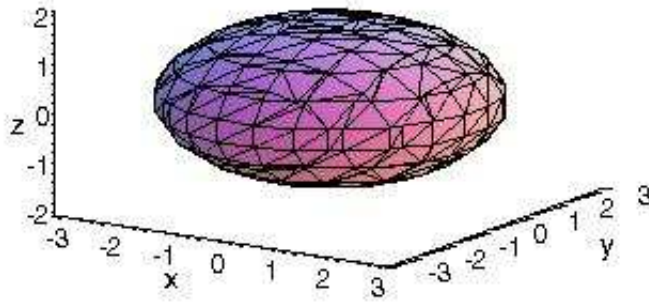


Figure 2. An awning spanned by the unit circle.

5. Approximation of awnings by quadrics

THEOREM 3. *The volume enclosed by the awning*

$$\frac{r}{2} \int_0^{2\pi} \sqrt{z^2 + (x \cos v + y \sin v - r)^2} dv = c$$

is greater than or equal to

$$V_1 := \frac{8}{3}\pi \left(\frac{c^2}{r^2\pi^2} - r^2 \right)^{\frac{3}{2}}.$$

PROOF. Since

$$\left(\int_0^{2\pi} f dv \right)^2 \leq 2\pi \int_0^{2\pi} f^2 dv$$

we have that

$$c^2 \leq \frac{r^2}{2} \pi \int_0^{2\pi} z^2 + (x \cos v + y \sin v - r)^2 dv.$$

Therefore the surface defined by the equation

$$c^2 = \frac{r^2}{2} \pi \int_0^{2\pi} z^2 + (x \cos v + y \sin v - r)^2 dv$$

is contained by the convex hull of the awning. Here

$$c^2 = \frac{r^2}{2} \pi^2 (x^2 + y^2 + 2r^2 + 2z^2),$$

which gives an ellipsoid

$$1 = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2}$$

with

$$\alpha^2 = \beta^2 = 2 \left(\frac{c^2}{r^2 \pi^2} - r^2 \right) \quad \text{and} \quad \gamma^2 = \frac{c^2}{r^2 \pi^2} - r^2.$$

Its volume is stated as a lower bound. □

REMARK 5. The surfaces has a contact point of first order with

$$x_0 = y_0 = 0 \quad \text{and} \quad z_0 = \pm \left(\frac{c^2}{r^2 \pi^2} - r^2 \right)^{\frac{1}{2}}.$$

To find an upper bound for the volume enclosed by the awning we can use the vertices $A_0, \dots, A_n = A_0$ of the inscribed regular n -gon with a fixed initial point A_0 . According to the convexity, we have that

$$\Sigma_n(P) \leq \mathcal{A}(P)$$

and, consequently,

$$\frac{1}{4} \sum_i |(P - A_i) \times (P - A_{i+1})|^2 \leq (\Sigma_n(P))^2 \leq c^2.$$

Therefore the surface defined by the equation

$$\frac{1}{4} \sum_i |(P - A_i) \times (P - A_{i+1})|^2 = c^2$$

contains the awning in its convex hull. To simplify the calculation as far as possible consider the case $n = 3$ with $r = \sqrt{\frac{2}{3}}$. Rotating in the euclidean sense if necessary we can suppose that

$$A_0 = (1, 0, 0), \quad A_1 = (0, 1, 0) \quad \text{and} \quad A_2 = (0, 0, 1);$$

see Figure 1. Using the notations as above

$$\frac{1}{4} \sum_i |(P - A_i) \times (P - A_{i+1})|^2 = L_2(x, y, z) + L_1(x, y, z) + L_0.$$

By the help of a straightforward calculation

$$L_2(x, y, z) = x^2 + y^2 + z^2 + \frac{1}{2}(xy + yz + zx)$$

$$L_1(x, y, z) = -(x + y + z) \quad \text{and} \quad L_0 = \frac{3}{4}.$$

Taking the matrix

$$\begin{pmatrix} 1 & 1/4 & 1/4 \\ 1/4 & 1 & 1/4 \\ 1/4 & 1/4 & 1 \end{pmatrix}$$

of the quadratic form, the eigenvalues are $3/2$ and $3/4$ with a system of eigenvectors

$$b_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad b_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad b_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Using the coordinate transformation

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

it follows that

$$L_2(\tilde{x}, \tilde{y}, \tilde{z}) + L_1(\tilde{x}, \tilde{y}, \tilde{z}) + L_0 = \frac{3}{2}\tilde{x}^2 + \frac{3}{4}\tilde{y}^2 + \frac{3}{4}\tilde{z}^2 - \sqrt{3}\tilde{x} + \frac{3}{4}$$

and the left hand side of the equation

$$L_2(x, y, z) + L_1(x, y, z) + L_0 = c^2$$

can be written into the form

$$\frac{(\tilde{x} - \tilde{x}_0)^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} + \frac{\tilde{z}^2}{\gamma^2} = 1$$

with

$$\alpha^2 = \frac{2c^2}{3} - \frac{1}{6} \quad \text{and} \quad \beta^2 = \gamma^2 = 2\left(\frac{2c^2}{3} - \frac{1}{6}\right).$$

THEOREM 4. *The volume enclosed by the awning*

$$\frac{r}{2} \int_0^{2\pi} \sqrt{z^2 + (x \cos v + y \sin v - r)^2} dv = c, \quad r = \sqrt{\frac{2}{3}}$$

is less than or equal to

$$V_2 := \frac{8}{3}\pi \left(\frac{2c^2}{3} - \frac{1}{6}\right)^{\frac{3}{2}}.$$

THEOREM 5. *The volume enclosed by the awning*

$$\Sigma_3(P) = c, \quad A_0 = (1, 0, 0), \quad A_1 = (0, 1, 0) \quad \text{and} \quad A_2 = (0, 0, 1)$$

is between

$$V_1 := \frac{8}{3}\pi \left(\frac{2c^2}{9} - \frac{1}{6}\right)^{\frac{3}{2}} \quad \text{and} \quad V_2 := \frac{8}{3}\pi \left(\frac{2c^2}{3} - \frac{1}{6}\right)^{\frac{3}{2}}.$$

PROOF. According to Theorem 4 it is enough to investigate the lower bound. The Cauchy–Schwarz inequality shows that

$$c^2 = (\Sigma_3(P))^2 \leq 3(L_2(x, y, z) + L_1(x, y, z) + L_0).$$

Therefore the surface defined by the equation

$$c^2 = 3(L_2(x, y, z) + L_1(x, y, z) + L_0)$$

is contained by the convex hull of the awning. Similar calculations as above give rise to the ellipsoid

$$\frac{(\tilde{x} - \tilde{x}_0)^2}{\alpha^2} + \frac{\tilde{y}^2}{\beta^2} + \frac{\tilde{z}^2}{\gamma^2} = 1$$

with

$$\alpha^2 = \frac{2c^2}{9} - \frac{1}{6} \quad \text{and} \quad \beta^2 = \gamma^2 = 2\left(\frac{2c^2}{9} - \frac{1}{6}\right).$$

Its volume is stated as a lower bound. $\square$

References

- [1] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization*, Springer-Verlag, New York, Inc, 2000.
- [2] C. Groß and T.-K. Stempel, On Generalization of Conics and on a Generalization of the Fermat–Torricelli problem, *Amer. Math. Monthly* **105**, no. 8 (1998), 732–743.

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