Alexandria Engineering Journal 82 (2023) 186–194

Contents lists available at ScienceDirect

Alexandria Engineering Journal

journal homepage: www.elsevier.com/locate/aej

Original Article

Exploring the solitary wave solutions of Einstein's vacuum field equation in the context of ambitious experiments and space missions

Mustafa Inc [a,b,c], Muhammad S. Iqbal [d,e], Muhammad Z. Baber [f], Muhammad Qasim [g], Zafar Iqbal [f], Muhammad Akhtar Tarar [h], Ali Hasan Ali [i,j,k,*]

[a] *Department of Computer Engineering, Biruni University, 34010 Istanbul, Turkey*
[b] *Department of Mathematics, Firat University, 23119 Elazig, Turkey*
[c] *Department of Medical Research, China Medical University, 40402 Taichung, Taiwan*
[d] *Department of Humanities & Basic Science, Military College of Signals, NUST, Islamabad, Pakistan*
[e] *Department of Academic Affairs, School of Leadership and Business, Oryx Universal College With Liverpool John Moores University (UK), 12253 Doha, Qatar*
[f] *Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan*
[g] *Royal College of Science, Narowal, Pakistan*
[h] *Civil Engineering Department, The University of Lahore, Pakistan*
[i] *Institute of Mathematics, University of Debrecen, Pf. 400, H-4002 Debrecen, Hungary*
[j] *College of Engineering Technology, National University of Science and Technology, Dhi Qar 64001, Iraq*
[k] *Technical Engineering College, Al-Ayen University, Dhi Qar 64001, Iraq*

ARTICLE INFO

Keywords:
Soliton solutions
Einstein equations
Stability analysis
GERF method
MAE method
Nonlinear phenomena

ABSTRACT

In the present work, Einstein's vacuum field equation is investigated analytically to explore the solitary wave solutions. This equation arises in mathematical physics, having meaningful applications in the general theory of relativity. This concept is crucial for numerous challenging experiments and space missions. The generalized exponential rational function and modified auxiliary equation approaches are used to obtain the exact solitary wave solution. Various types of solutions are extracted, including exponential functions, hyperbolic functions, trigonometric functions, and rational forms. Additionally, a stability analysis for the Einstein vacuum field equation is conducted. Appropriate parameters are chosen to draw 3-D and corresponding contour plots of some solutions, which clearly demonstrate the solitary wave behaviors. The obtained results support the idea that applying these approaches is the most effective strategy for resolving any nonlinear issues that may arise in science and technology.

1. Introduction

Nonlinear partial differential equations arise in various types of physical problems in biological science, physics, mathematics, and engineering [1,2]. In this work, we investigate Einstein's vacuum field equation. General relativity's fundamental equations, known as Einstein's field equations, are essential for several space missions and experiments. In general relativity, this model has a wide range of applications that occur in different types of phenomena. We extract the exact solitary wave solution of this nonlinear model because the advantage of exact solutions is that they do not contain any approximation errors present in numerical solutions. Einstein's field equations are of utmost importance in understanding various behaviors in different phenomena. For empty space-time, Einstein's field equations admit solutions in the linear approximation that can be interpreted as a description of plane gravitational waves. For further information about the exact solutions, the reader may refer to [3]. Jyoti [4] investigated the exact non-static solutions of Einstein's vacuum field equations, obtaining an invariant solution using the classical symmetry method. Kaur [5] discussed the exact solution by means of Lie point symmetries and Painlevé analysis. Quevedo [6] investigated the general static axisymmetric solution of Einstein's vacuum field equations. Moreover, Friedrich [7] discussed the initial boundary value problem for Einstein's Vacuum Field Equation. Rutz [8] described a generalization of the vacuum field equations

---

* Corresponding author.
E-mail addresses: minc@firat.edu.tr (M. Inc), sajid606@gmail.com, m_sajid_iqbal@mcs.edu.pk (M.S. Iqbal), zafarullah8883@gmail.com (M.Z. Baber), qasim.uognwl@gmail.com (M. Qasim), zafar.iqbal@math.uol.edu.pk, zafariqbal0092@yahoo.com (Z. Iqbal), akhtartarar2000@yahoo.com (M.A. Tarar), ali.hasan@science.unideb.hu (A.H. Ali).

https://doi.org/10.1016/j.aej.2023.09.071
Received 8 June 2023; Received in revised form 17 August 2023; Accepted 26 September 2023

in general relativity to Finsler spaces, obtaining a non-Riemannian solution. Heinicke [9] discussed the Schwarzschild and Kerr solutions of Einstein's field equation. Hoffman [10] investigated the stationary non-canonical solutions of the Einstein vacuum field equations. Vishwakarma [11] investigated a new solution of Einstein's vacuum field equations that describe its source of curvature, known as the well-known Ozsváth–Schücking solution. Corda [12] investigated the solution of linearized Einstein field equations in vacuum, computing the linearized Ricci tensor and Riemann tensor. The governing model of general relativity is crucial for many large-scale investigations and space missions. Symmetry considerations [13–16] for Einstein's field equations are of significant importance. In Einstein's theory of gravitation, the Riemann curvature tensor plays a crucial role. For Einstein's vacuum equation, Guilfoyle [17] considers the metric as

$$ds^2 = -(au^2 - 2u_{xx})dt^2 + 2\left(\frac{3}{2t}\right)^{\frac{4}{3}} dx^2 - 2udxdt + dy^2 \\ + 2\left(\frac{3}{2t}\right)^{\frac{2}{3}} dxdy + dtdz, \tag{1}$$

where a is a constant. For the metric Eq.(1), the non exiting components of the Ricci tensor and the curvature tensor are as

$$\begin{aligned} R_{txtx} &= \frac{1}{2}\frac{-2u_{xt}t^{\frac{2}{3}} + 2au_x^2t^{\frac{2}{3}} + 2auu_{xx}t^{\frac{2}{3}} - 2uu_{xxxx}t^{\frac{2}{3}}3^{\frac{1}{3}}2^{\frac{2}{3}}}{t^{\frac{2}{3}}}, \\ R_{tt} &= -\frac{3^{\frac{2}{3}}2^{\frac{4}{3}}(-u_{xt} + au_x^2 + auu_{xx} - u_{xxxx})}{9t^{\frac{4}{3}}}, \\ R_{txty} &= \frac{3^{\frac{2}{3}}2^{\frac{4}{3}}}{9t^{\frac{4}{3}}}, \\ R_{tyty} &= \frac{1}{9t^2}, \end{aligned} \tag{2}$$

where R_{tt} is the only existing element of Ricci tensor for the metric Eq. (1) thus the respective Einstein's vacuum field equation for Eq. (1), where $R_{tt} = 0$ results the following NLPDE as

$$u_{xt} - auu_{xx} - au_x^2 + u_{xxxx} = 0. \tag{3}$$

The physical meaning of the Ricci tensor is that it describes how much the spacetime volume of an object changes due to gravitational tides in general relativity. Geometrically, the Ricci tensor represents volume changes caused by curvature, and spacetime curvature is associated with tidal forces. Mass (which can be considered equivalent to energy) curves spacetime, causing the spacetime around it to become curved. Consequently, the energy required for ambitious experiments and space missions is carried in the form of waves. These solutions are effective for ambitious experiments and space missions to control the speed of waves in the dimension of spacetime.

Now, these days, exact solitary wave solutions are very effective in the field of nonlinear phenomena. In recent times, numerous researchers worked on finding exact solitary wave solutions for NLPDEs using various techniques. Hosseini, K. et al., have generalized a method to solve the nonlinear Konno–Oono model [18]. Kumar, S. et al., explored the abundant different types of exact soliton solution to the (4+1)-dimensional Fokas [19]. Xu, C., et al. explored the bifurcation of the dynamical systems [20–22], and Iqbal, M. S., worked for these types of solutions under the effect of noise [23–26].

In this article, we have explored the exact solitary wave solutions of Einstein's vacuum field equation using the GERF approach and MAE approach. There are various methods for finding exact solutions of NLPDEs, such as the ϕ^6 model expansion approach [27], the auxiliary equation approach [28], the sine-cosine approach [29], the generalized Kudryashov approach [30], the He's semi-inverse approach [31], the exp-function approach [32], and the extended direct algebraic approach [33]. Additionally, we have utilized the amplitude ansatz approach [34]. In this study, the generalized exponential rational functional (GERF) method is utilized. This method is a more general form of the modified exponential rational function method. The GERF method provides us with a wider range of families of solutions compared to the modified form. These families of solutions are presented in the form of exponential, hyperbolic, trigonometric, and rational functions. The novelty of this work lies in obtaining exact solitary wave solutions for Einstein's vacuum field equation, which arises in mathematical physics and holds significant applications in the general theory of relativity. This concept is crucial for numerous challenging experiments and space missions, making the exact solitary wave solutions valuable for researchers in the field of general relativity. To achieve these exact solutions, we employ two techniques, namely the GERF method and the MAE method. These methods provide us with different types of solutions, including exponential functions, hyperbolic functions, trigonometric functions, and rational forms. Furthermore, the stability analysis of these solutions is also presented. Additionally, we illustrate the solutions through graphical representations in the form of 3-Dimensional and 2-Dimensional plots, along with their corresponding contour plots. These visualizations effectively demonstrate the traveling wave structures associated with these solutions. Consequently, GERF method is more suitable than several other methods, and the following section is devoted to present these solutions for the Einstein's vacuum field equation.

2. GERF method

In this section, to find the exact soliton solutions of Eqs. (3), we employ the transformation $\phi(x,t) = u(\rho)$, where $\rho = x + ct$. By using the above transformation, we can express Eq. (3) as follows:

$$c\phi'' - a\phi\phi'' - a(\phi')^2 + \phi'''' = 0. \tag{4}$$

The solution of Eq. (4) can be represented as follows [35],

$$u(\rho) = \eta_0 + \sum_{k=1}^{N} \eta_k \sigma(\rho)^k + \sum_{k=1}^{N} \lambda_k \sigma(\rho)^{-k}. \tag{5}$$

We get $N = 2$ by using homogenous balance principle, then we have the following polynomial for obtaining solutions of Eq. (4) as,

$$u(\rho) = \eta_0 + \eta_1\sigma(\rho) + \eta_2\sigma(\rho)^2 + \lambda_1\sigma(\rho)^{-1} + \lambda_2\sigma(\rho)^{-2}, \tag{6}$$

where

$$\sigma(\rho) = \frac{p_1e^{q_1\rho} + p_2e^{q_2\rho}}{p_3e^{q_3\rho} + p_4e^{q_4\rho}}. \tag{7}$$

Find the derivatives of Eq. (6) using Eq. (7), and substitute them into Eq. (4), resulting in a polynomial in the form of $w^{z(\rho)k}$. Set all the coefficients equal to zero to obtain a system of equations. Solve this system of equations simultaneously using $Mathematica$ to find the values of the unknown constants, such as:

Case 1: When choosing $p_i = [-1,-1,1,-1]$ and $q_i = [1,-1,1,-1]$, where $i = (1,2,3,4)$ then a new form of Eq. (4) is obtained as,

$$\sigma(\rho) = -\frac{\cosh(\rho)}{\sinh(\rho)}. \tag{8}$$

Putting Eq. (6) with Eq. (8) into Eq. (4), we have:

Set 1: By using the corresponding value, we get the solitons solution of Eq. (3) as,

$\eta_0 = \frac{c-8}{a}, \quad \eta_1 = 0, \quad \eta_2 = 0, \quad \lambda_1 = 0, \quad \lambda_2 = \frac{12}{a},$

$$u_1(x,t) = \frac{1}{a}(12\tanh^2(ct+x) + c - 8). \tag{9}$$

Set 2: By using the corresponding value, we get the solitons solution of Eq. (3) as,

$\eta_0 = \frac{c-8}{a}, \quad \eta_1 = 0, \quad \eta_2 = \frac{12}{a}, \quad \lambda_1 = 0, \quad \lambda_2 = \frac{12}{a},$

$$u_2(x,t) = \frac{1}{a}(12\tanh^2(ct+x) + 12\coth^2(ct+x) + c - 8). \tag{10}$$

Case 2: When choosing $p_i = [i, -i, 1, 1]$ and $q_i = [i, -i, i, -i]$, where $i = (1, 2, 3, 4)$ then a new form of Eq. (4) is obtained as,

$$\sigma(\rho) = -\frac{\sin(\rho)}{\cos(\rho)}. \tag{11}$$

Putting Eq. (6) with Eq. (11) into Eq. (4), we have:
Set 1: By using the corresponding value, we get the solitary wave solution of Eq. (3) as,

$$\eta_0 = \frac{c+8}{a}, \quad \eta_1 = 0, \quad \eta_2 = 0, \quad \lambda_1 = 0, \quad \lambda_2 = \frac{12}{a},$$

$$u_3(x,t) = \frac{1}{a}(12\cot^2(ct+x) + c + 8). \tag{12}$$

Set 2: By using the corresponding value, we get the solitary wave solution of Eq. (3) as,

$$\eta_0 = \frac{c+8}{a}, \quad \eta_1 = 0, \quad \eta_2 = \frac{12}{a}, \quad \lambda_1 = 0, \quad \lambda_2 = \frac{12}{a},$$

$$u_4(x,t) = \frac{1}{a}(12\tan^2(ct+x) + 12\cot^2(ct+x) + c + 8). \tag{13}$$

Case 3: When choosing $p_i = [1, 0, 1, 1]$ and $q_i = [1, 0, 1, 0]$, where $i = (1, 2, 3, 4)$ then a new form of Eq. (4) is obtained as,

$$\sigma(\rho) = \frac{e^{\rho}}{e^{\rho} + 1}. \tag{14}$$

Putting Eq. (6) with Eq. (14) into Eq. (4), we have:
Set 1: By using the corresponding value, we get the exponential form solution of Eq. (3) as,

$$\eta_1 = -\frac{12}{a}, \quad \eta_2 = \frac{12}{a}, \quad \lambda_1 = 0, \quad \lambda_2 = 0, \quad c = a\eta_0 - 1,$$

$$u_5(x,t) = \frac{a\eta_0\left(e^{a\eta_0 t + x} + e^{t}\right)^2 - 12e^{a\eta_0 t + t + x}}{a\left(e^{a\eta_0 t + x} + e^{t}\right)^2}. \tag{15}$$

Set 2: By using the corresponding value, we get the exponential form solution of Eq. (3) as,

$$\eta_1 = 0, \quad \eta_2 = 0, \quad \lambda_1 = -2\lambda_2, \quad a = 0, \quad c = -4,$$

$$u_6(x,t) = \eta_0 + \lambda_2 e^{-2(x-4t)}\left(e^{x-4t} + 1\right)^2 - 2\lambda_2 e^{4t-x}\left(e^{x-4t} + 1\right). \tag{16}$$

Case 4: When choosing $p_i = [1 - i, 1 + i, 1, 1]$ and $q_i = [i, -i, i, -i]$, where $i = (1, 2, 3, 4)$ then a new form of Eq. (4) is obtained as,

$$\sigma(\rho) = \frac{\sin(\rho) + \cos(\rho)}{\cos(\rho)}. \tag{17}$$

Putting Eq. (6) with Eq. (17) into Eq. (4), we have:
Set 1: By using the corresponding value, we get the solitary wave solution of Eq. (3) as,

$$\eta_0 = \frac{c+20}{a}, \quad \eta_1 = 0, \quad \eta_2 = 0, \quad \lambda_1 = -\frac{48}{a}, \quad \lambda_2 = \frac{48}{a},$$

$$u_7(x,t) = \frac{48\cos^2(ct+x)}{a(\sin(ct+x) + \cos(ct+x))^2} - \frac{48\cos(ct+x)}{a(\sin(ct+x) + \cos(ct+x))} + \frac{c+20}{a}. \tag{18}$$

Set 2: By using the corresponding value, we get the solitary wave solution of Eq. (3) as,

$$\eta_0 = \frac{c+20}{a}, \quad \eta_1 = -\frac{24}{a}, \quad \eta_2 = \frac{12}{a}, \quad \lambda_1 = 0, \quad \lambda_2 = 0,$$

$$u_8(x,t) = \frac{c+20}{a} + \frac{12\sec^2(ct+x)(\sin(ct+x) + \cos(ct+x))^2}{a} - \frac{24\sec(ct+x)(\sin(ct+x) + \cos(ct+x))}{a}. \tag{19}$$

Case 5: When choosing $p_i = [-3, -1, 1, 1]$ and $q_i = [1, -1, 1, -1]$, where $i = (1, 2, 3, 4)$ then a new form of Eq. (4) is obtained as,

$$\sigma(\rho) = \frac{-\sinh(\rho) - 2\cosh(\rho)}{\cosh(\rho)}. \tag{20}$$

Putting Eq. (6) with Eq. (20) into Eq. (4), we have:
Set 1: By using the corresponding value, we get the solitons solution of Eq. (3) as,

$$\eta_0 = \frac{c+40}{a}, \quad \eta_1 = 0, \quad \eta_2 = 0, \quad \lambda_1 = \frac{144}{a}, \quad \lambda_2 = \frac{108}{a},$$

$$u_9(x,t) = \frac{108\cosh^2(ct+x)}{a(-\sinh(ct+x) - 2\cosh(ct+x))^2} + \frac{144\cosh(ct+x)}{a(-\sinh(ct+x) - 2\cosh(ct+x))} + \frac{c+40}{a}. \tag{21}$$

Set 2: By using the corresponding value, we get the solitons solution of Eq. (3) as,

$$\eta_0 = \frac{c+40}{a}, \quad \eta_1 = \frac{48}{a}, \quad \eta_2 = \frac{12}{a}, \quad \lambda_1 = 0, \quad \lambda_2 = 0,$$

$$u_{10}(x,t) = \frac{c+40}{a} + \frac{12\operatorname{sech}^2(ct+x)(-\sinh(ct+x) - 2\cosh(ct+x))^2}{a} + \frac{48\operatorname{sech}(ct+x)(-\sinh(ct+x) - 2\cosh(ct+x))}{a}. \tag{22}$$

Case 6: When choosing $p_i = [-1, 0, 1, 1]$ and $q_i = [0, 1, 0, 1]$, where $i = (1, 2, 3, 4)$ then a new form of Eq. (4) is obtained as,

$$\sigma(\rho) = -\frac{1}{e^{\rho} + 1}. \tag{23}$$

Putting Eq. (6) with Eq. (23) into Eq. (4), we have:
Set 1: By using the corresponding value, we get the exponential form solution of Eq. (3) as,

$$\eta_1 = 0, \quad \eta_2 = 0, \quad \lambda_1 = 2\lambda_2, \quad a = 0, \quad c = -4,$$

$$u_{11}(x,t) = \eta_0 + \lambda_2\left(e^{x-4t} + 1\right)^2 + 2\lambda_2\left(-e^{x-4t} - 1\right). \tag{24}$$

Set 2: By using the corresponding value, we get the solitons solution of Eq. (3) as,

$$\eta_1 = \frac{12}{a}, \quad \eta_2 = \frac{12}{a}, \quad \lambda_1 = 0, \quad \lambda_2 = 0, \quad c = a\eta_0 - 1,$$

$$u_{12}(x,t) = \eta_0 - \frac{12}{a\left(e^{t(a\eta_0 - 1) + x} + 1\right)} + \frac{12}{a\left(e^{t(a\eta_0 - 1) + x} + 1\right)^2}. \tag{25}$$

Case 7: When choosing $p_i = [-1, 1, 1, 1]$ and $q_i = [1, -1, 1, -1]$, where $i = (1, 2, 3, 4)$ then a new form of Eq. (4) is obtained as,

$$\sigma(\rho) = -\frac{\sinh(\rho)}{\cosh(\rho)}. \tag{26}$$

Putting Eq. (6) with Eq. (26) into Eq. (4), we have:
Set 1: By using the corresponding value, we get the solitons solution of Eq. (3) as,

$$\eta_0 = \frac{c-8}{a}, \quad \eta_1 = 0, \quad \eta_2 = \frac{12}{a}, \quad \lambda_1 = 0, \quad \lambda_2 = 0,$$

$$u_{13}(x,t) = \frac{1}{a}(12\tanh^2(ct+x) + c - 8). \tag{27}$$

Set 2: By using the corresponding value, we get the solitons solution of Eq. (3) as,

$$\eta_0 = \frac{c-8}{a}, \quad \eta_1 = 0, \quad \eta_2 = \frac{12}{a}, \quad \lambda_1 = 0, \quad \lambda_2 = \frac{12}{a},$$

$$u_{14}(x,t) = \frac{1}{a}(12\tanh^2(ct+x) + 12\coth^2(ct+x) + c - 8). \tag{28}$$

Case 8: When choosing $p_i = [2 - i, 2 + i, 1, 1]$ and $q_i = [i, -i, i, -i]$, where $i = (1, 2, 3, 4)$ then a new form of Eq. (4) is obtained as,

$$\sigma(\rho) = \frac{\sin(\rho) + 2\cos(\rho)}{\cos(\rho)}. \tag{29}$$

Putting Eq. (6) with Eq. (29) into Eq. (4), we have:
Set 1: By using the corresponding value, we get the solitary wave solution of Eq. (3) as,

$$\eta_0 = \frac{c+56}{a}, \quad \eta_1 = 0, \quad \eta_2 = 0, \quad \lambda_1 = -\frac{240}{a}, \quad \lambda_2 = \frac{300}{a},$$

$$u_{15}(x,t)=\frac{300\cos^2(ct+x)}{a(\sin(ct+x)+2\cos(ct+x))^2}-\frac{240\cos(ct+x)}{a(\sin(ct+x)+2\cos(ct+x))}+\frac{c+56}{a}. \tag{30}$$

Set 2: By using the corresponding value, we get the solitary wave solution of Eq. (3) as,

$$\eta_0=\frac{c+56}{a},\quad \eta_1=-\frac{48}{a},\quad \eta_2=\frac{12}{a},\quad \lambda_1=0,\quad \lambda_2=0,$$

$$u_{16}(x,t)=\frac{c+56}{a}+\frac{12\sec^2(ct+x)(\sin(ct+x)+2\cos(ct+x))^2}{a}-\frac{48\sec(ct+x)(\sin(ct+x)+2\cos(ct+x))}{a}. \tag{31}$$

Case 9: When choosing $p_i=[1,2,1,1]$ and $q_i=[1,0,1,0]$, where $i=(1,2,3,4)$ then new form of Eq. (4) is obtained as,

$$\sigma(\rho)=\frac{e^\rho+2}{e^\rho+1}. \tag{32}$$

Putting Eq. (6) with Eq. (32) into Eq. (4), we have:

Set 1: By using the corresponding value, we get the exponential form solution of Eq. (3) as,

$$\eta_0=\frac{c+25}{a},\quad \eta_1=0,\quad \eta_2=0,\quad \lambda_1=-\frac{72}{a},\quad \lambda_2=\frac{48}{a},$$

$$u_{17}(x,t)=\frac{48\left(e^{ct+x}+1\right)^2}{a\left(e^{ct+x}+2\right)^2}-\frac{72\left(e^{ct+x}+1\right)}{a\left(e^{ct+x}+2\right)}+\frac{c+25}{a}. \tag{33}$$

Set 2: By using the corresponding value, we get the exponential form solution of Eq. (3) as,

$$\eta_0=\frac{c+25}{a},\quad \eta_1=-\frac{36}{a},\quad \eta_2=\frac{12}{a},\quad \lambda_1=0,\quad \lambda_2=0,$$

$$u_{18}(x,t)=\frac{12\left(e^{ct+x}+2\right)^2}{a\left(e^{ct+x}+1\right)^2}-\frac{36\left(e^{ct+x}+2\right)}{a\left(e^{ct+x}+1\right)}+\frac{c+25}{a}. \tag{34}$$

Case 10: When choosing $p_i=[2,1,1,1]$ and $q_i=[1,0,1,0]$, where $i=(1,2,3,4)$ then a new form of Eq. (4) is obtained as,

$$\sigma(\rho)=\frac{2e^\rho+1}{e^\rho+1}. \tag{35}$$

Putting Eq. (6) with Eq. (35) into Eq. (4), we have:

Set 1: By using the corresponding value, we get the exponential form solution of Eq. (3) as,

$$\eta_0=\frac{c+25}{a},\quad \eta_1=0,\quad \eta_2=0,\quad \lambda_1=-\frac{72}{a},\quad \lambda_2=\frac{48}{a},$$

$$u_{19}(x,t)=\frac{48\left(e^{ct+x}+1\right)^2}{a\left(2e^{ct+x}+1\right)^2}-\frac{72\left(e^{ct+x}+1\right)}{a\left(2e^{ct+x}+1\right)}+\frac{c+25}{a}. \tag{36}$$

Set 2: By using the corresponding value, we get the exponential form solution of Eq. (3) as,

$$\eta_0=\frac{c+25}{a},\quad \eta_1=-\frac{36}{a},\quad \eta_2=\frac{12}{a},\quad \lambda_1=0,\quad \lambda_2=0,$$

$$u_{20}(x,t)=\frac{12\left(2e^{ct+x}+1\right)^2}{a\left(e^{ct+x}+1\right)^2}-\frac{36\left(2e^{ct+x}+1\right)}{a\left(e^{ct+x}+1\right)}+\frac{c+25}{a}. \tag{37}$$

Case 11: When choosing $p_i=[1,1,1,1]$ and $q_i=[0,0,1,-1]$, where $i=(1,2,3,4)$ then a new form of equation (4) is obtained as,

$$\sigma(\rho)=\frac{2}{e^{-\rho}+e^\rho}. \tag{38}$$

Putting Eq. (6) with Eq. (38) into Eq. (4), we have:

Set 1: By using the corresponding value, we get the exponential form solution of Eq. (3) as,

$$\eta_0=\frac{c+4}{a},\quad \eta_1=0,\quad \eta_2=-\frac{12}{a},\quad \lambda_1=0,\quad \lambda_2=0,$$

$$u_{21}(x,t)=\frac{c+4}{a}-\frac{48}{a\left(e^{-ct-x}+e^{ct+x}\right)^2}. \tag{39}$$

Set 2: By using the corresponding value, we get the exponential form solution of Eq. (3) as,

$$\eta_0=\frac{c+196}{a},\quad \eta_1=0,\quad \eta_2=-\frac{24}{a},\quad \lambda_1=0,\quad \lambda_2=-\frac{192}{a},$$

$$u_{22}(x,t)=-\frac{48\left(e^{-ct-x}+e^{ct+x}\right)^2}{a}-\frac{96}{a\left(e^{-ct-x}+e^{ct+x}\right)^2}+\frac{c+196}{a}. \tag{40}$$

3. Modified auxiliary equation method

In this section, the modified auxiliary equation (MAE) method is applied to find the exact solitary wave solutions. The MAE method assumes the general solution of Eq. (4) in the form of a polynomial, such as:

$$\phi(\rho)=b_0+\sum_{k=1}^{N}\left(b_k w^{z(\rho)k}+c_k w^{-z(\rho)k}\right), \tag{41}$$

where b_0, b_k, and c_k are constants, and $z'(\rho)$ satisfies the auxiliary equation such that

$$z'(\rho)=\frac{\alpha_0+\alpha_1\omega^{-z(\zeta)}+\nu\omega^{z(\zeta)}}{\log(\omega)}, \tag{42}$$

where α_0, α_1 and ν are arbitrary constants with $z>0$, $z\neq 1$. Putting the value $N=2$ in Eq. (41) such as,

$$\phi(\rho)=b_0+b_1w^{z(\rho)}+b_2w^{2z(\rho)}+c_1w^{-z(\rho)}+c_2w^{-2z(\rho)}. \tag{43}$$

Finding the derivatives of Eq. (43) by using the Eq. (42) and putting in Eq. (4), resulting in a polynomial in the form of $w^{z(\rho)k}$. Setting all the coefficients equal to zero to obtain a system of equations. Then, solving this system of equations simultaneously using *Mathematica* to find the values of the unknown constants, such as:

$$b_0=\frac{8\alpha_1\nu+\alpha_0^2+c}{a},b_1=\frac{12\alpha_0\nu}{a},b_2=\frac{12\nu^2}{a},c_1=0,c_2=0.$$

Putting these constants in the Eq. (43) and by the help of general solutions of MAE method, we obtain different solitary wave solutions of Eq. (3) such as:

Case 1: When $\alpha_0^2-4\alpha_1\nu<0$ and $\nu\neq 0$, we gain the solitary wave solutions such as,

$$u_{23}(x,t)=\frac{8\alpha_1\nu+\alpha_0^2+c}{a}+\frac{6\alpha_0}{a}\left(\sqrt{4\alpha_1\nu-\alpha_0^2}\tan\left(\frac{1}{2}\sqrt{4\alpha_1\nu-\alpha_0^2}(x-ct)\right)-\alpha_0\right)+\frac{3}{a}\left(\sqrt{4\alpha_1\nu-\alpha_0^2}\tan\left(\frac{1}{2}\sqrt{4\alpha_1\nu-\alpha_0^2}(x-ct)\right)-\alpha_0\right)^2, \tag{44}$$

$$u_{24}(x,t)=\frac{8\alpha_1\nu+\alpha_0^2+c}{a}-\frac{6\alpha_0}{a}\left(\alpha_0+\sqrt{4\alpha_1\nu-\alpha_0^2}\cot\left(\frac{1}{2}\sqrt{4\alpha_1\nu-\alpha_0^2}(x-ct)\right)\right)+\frac{3}{a}\left(\alpha_0+\sqrt{4\alpha_1\nu-\alpha_0^2}\cot\left(\frac{1}{2}\sqrt{4\alpha_1\nu-\alpha_0^2}(x-ct)\right)\right)^2. \tag{45}$$

Case 2: When $\alpha_0^2-4\alpha_1\nu>0$ and $\nu\neq 0$, we gain the shock and singular soliton solutions such as,

$$u_{25}(x,t)=\frac{8\alpha_1\nu+\alpha_0^2+c}{a}-\frac{6\alpha_0}{a}\left(\alpha_0+\sqrt{\alpha_0^2-4\alpha_1\nu}\tanh\left(\frac{1}{2}\sqrt{\alpha_0^2-4\alpha_1\nu}(x-ct)\right)\right)+\frac{3}{a}\left(\alpha_0+\sqrt{\alpha_0^2-4\alpha_1\nu}\tanh\left(\frac{1}{2}\sqrt{\alpha_0^2-4\alpha_1\nu}(x-ct)\right)\right)^2, \tag{46}$$

$$u_{26}(x,t)=\frac{8\alpha_1\nu+\alpha_0^2+c}{a}$$

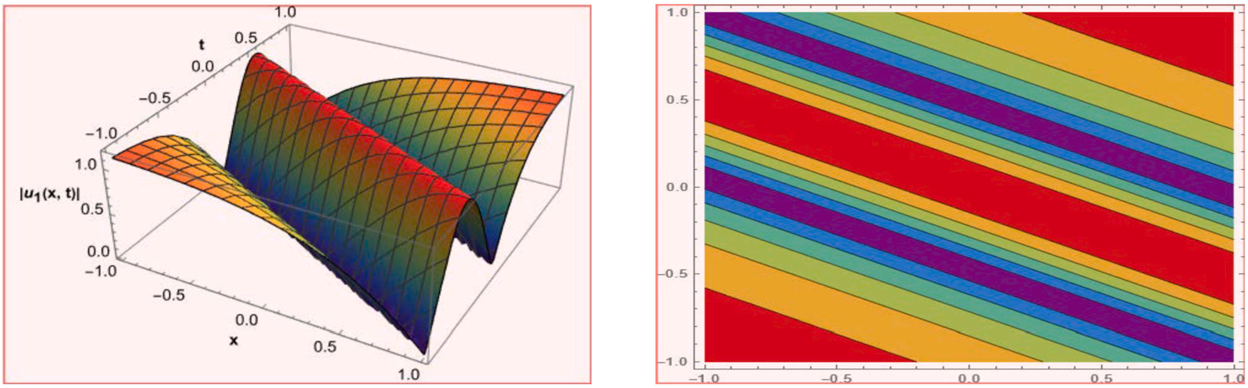

Fig. 1. The physical behavior of $u_1(x,t)$ and its contour is depicted by choosing parameters as $c = 1.9$ and $a = 5.2$.

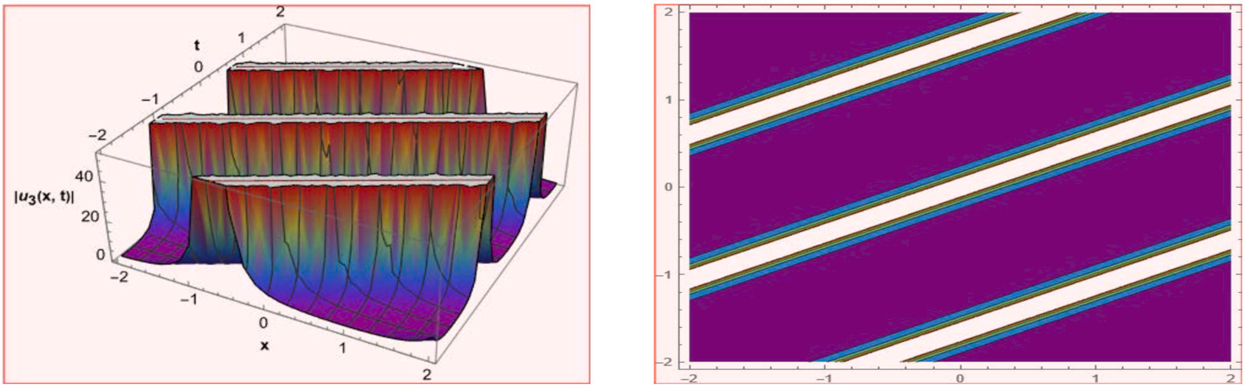

Fig. 2. The physical behavior of $u_3(x,t)$ and its contour is depicted by choosing parameters as $c = -1.9$ and $a = 5.5$.

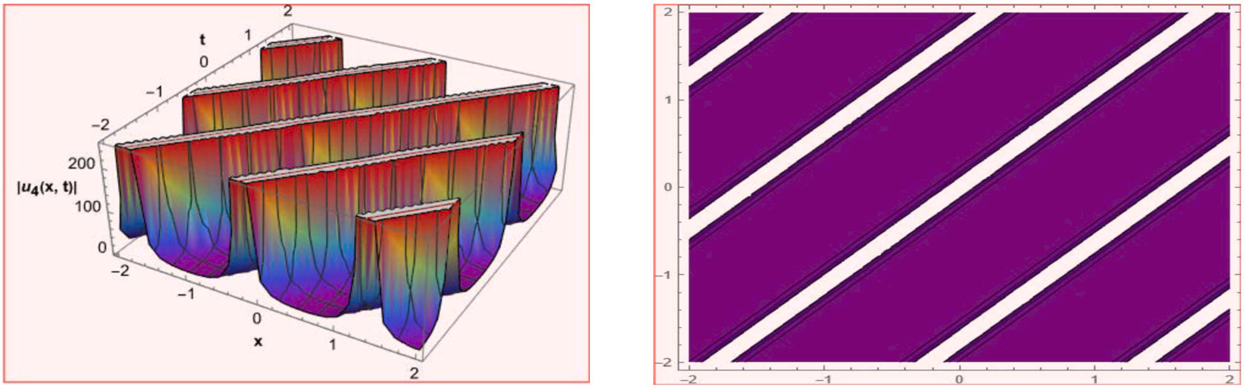

Fig. 3. The physical behavior of $u_4(x,t)$ and its contour is depicted by choosing parameters as $c = -0.9$ and $a = 5.5$.

$$-\frac{6\alpha_0}{a}\left(\alpha_0+\sqrt{\alpha_0^2-4\alpha_1\nu}\coth\left(\frac{1}{2}\sqrt{\alpha_0^2-4\alpha_1\nu}(x-ct)\right)\right)$$
$$+\frac{3}{a}\left(\alpha_0+\sqrt{\alpha_0^2-4\alpha_1\nu}\coth\left(\frac{1}{2}\sqrt{\alpha_0^2-4\alpha_1\nu}(x-ct)\right)\right)^2. \quad (47)$$

Case 3: When $\alpha_0^2-4\alpha_1\nu=0$ and $\nu\neq 0$, we gain the plane wave solution such as,

$$u_{27}(x,t)=\frac{8\alpha_1\nu+\alpha_0^2+c}{a}-\frac{6\alpha_0\left(\alpha_0(x-ct)+2\right)}{a(x-ct)}+\frac{3\left(\alpha_0(x-ct)+2\right)^2}{a(x-ct)^2}. \quad (48)$$

4. Stability analysis

In this section, the stability analysis of Eq. (4) is investigated and explored. For this, suppose that the perturbed solution as follows [35]

$$\psi(x,t)=P+\epsilon w(x,t), \quad (49)$$

where P represent any steady state solution of Eq. (4). Now, by plugging Eq. (49) into Eq. (4), we get

$$-aP\epsilon w_{xx}-a\epsilon w_x^2-aw\epsilon^2 w_{xx}+\epsilon w_{xt}+\epsilon w_{xxxx}=0. \quad (50)$$

Linearizing Eq. (50), we obtain

$$-aP\epsilon w_{xx}+\epsilon w_{xt}+\epsilon w_{xxxx}=0. \quad (51)$$

Suppose that Eq. (51) has solution in the following form

$$w=\gamma e^{i(kx+\sigma t)}. \quad (52)$$

Normalized wave numbers are represented by k and dispersion relations represented by $\sigma=\sigma(k)$. Plugging Eq. (52) into Eq. (51), we get

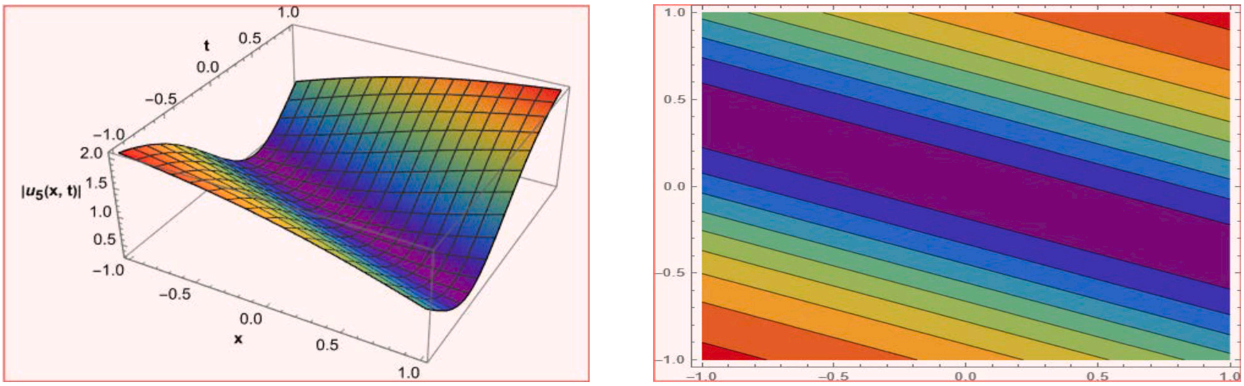

Fig. 4. The physical behavior of $u_5(x,t)$ and its contour is depicted by choosing parameters as $\eta_0 = -2.3$ and $a = -1.5$.

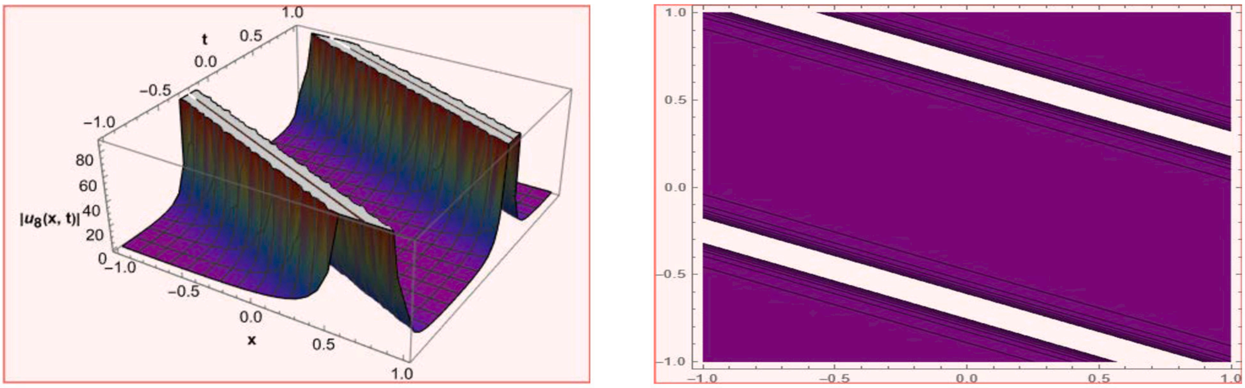

Fig. 5. The physical behavior of $u_8(x,t)$ and its contour is depicted by choosing parameters as $c = 2.3$ and $a = -5.5$.

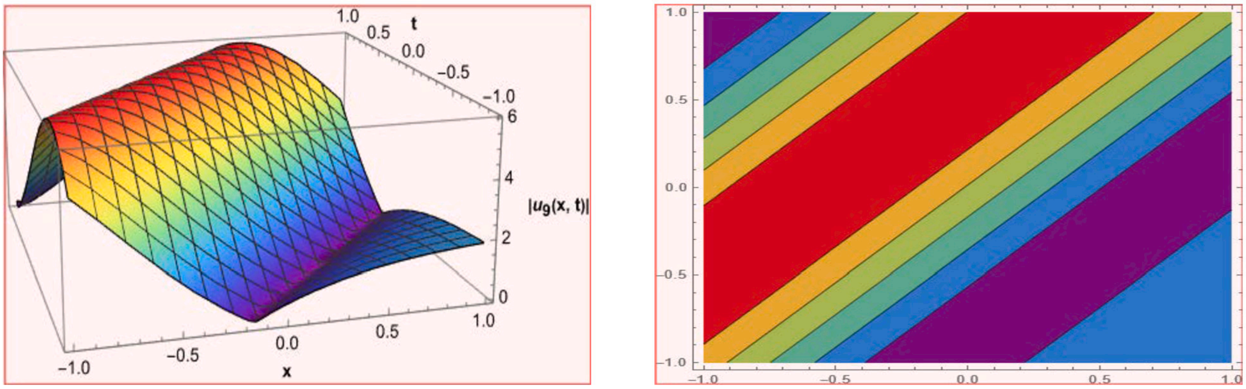

Fig. 6. The physical behavior of $u_9(x,t)$ and its contour is depicted by choosing parameters as $c = -0.9$ and $a = 1.5$.

$$a\gamma k^2 P\epsilon + \gamma k^4\epsilon - \gamma k\sigma\epsilon = 0. \tag{53}$$

Solve the above equation for σ, we get

$$\sigma(k) = akP + k^3. \tag{54}$$

From Eq. (53) any superposition of the solutions will appear as decay if the real portion is found to be negative for all values of k. Because of this, the dispersion is stable.

5. Results and discussions

In this section, we discuss the physical behavior of the results successfully obtained for Einstein's vacuum field equation. These results represent solitary wave solutions achieved using two well-known approaches, namely the GERF and MAE methods. These solutions have physical applicability in the field of general relativity. They are expressed in the form of exponential functions, hyperbolic functions, trigonometric functions, and rational functions. In general relativity (GR), Einstein's field equations (EFEs) play a crucial role, serving as essential equations for space missions and pioneering experiments. As famously quoted by Wheeler, "Space tells matter how to shift. Matter tells space how to curve." This mutual influence is determined by the combination of Einstein's equation and the dynamical equations of the matter present. Physically, these exact solitary wave solutions may enable scientists to control spaceships and conduct pioneering experiments. Fig. 1 illustrates the soliton behavior, while Figs. 2, 3, and 5 display the solitary waves. Fig. 4 represents the dark soliton solution, and Figs. 6 and 7 depict the singular and dark solitons. Additionally, Figs. 8, 9, 10, 11, and 12 provide us with different forms of soliton solutions. These solutions are highly effective and valuable for further

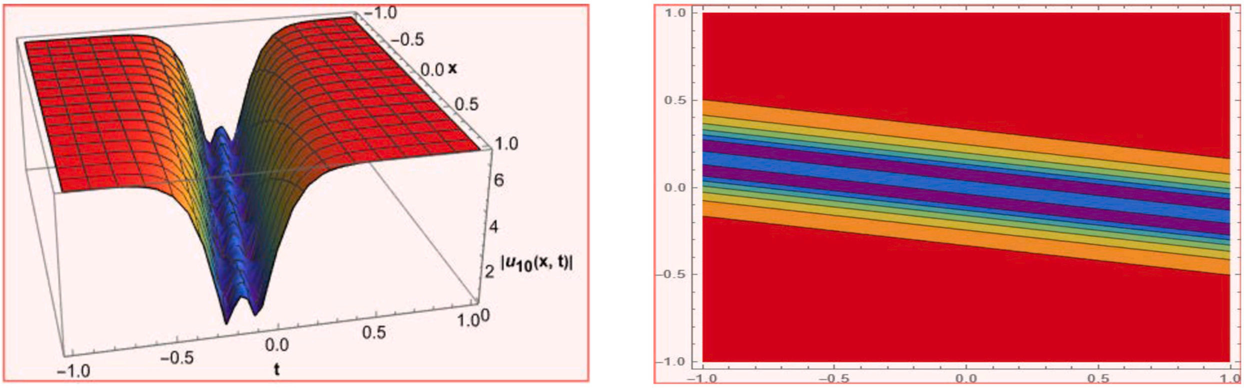

Fig. 7. The physical behavior of $u_{10}(x,t)$ and its contour is depicted by choosing parameters as $c = 5.9$ and $a = 1.5$.

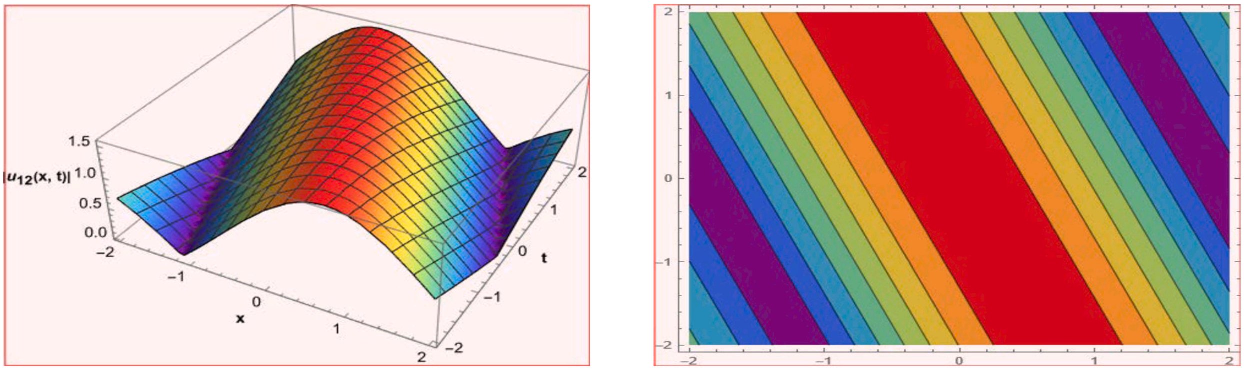

Fig. 8. The physical behavior of $u_{12}(x,t)$ and its contour is depicted by choosing parameters as $\eta_0 = -1.3$ and $a = -1.05$.

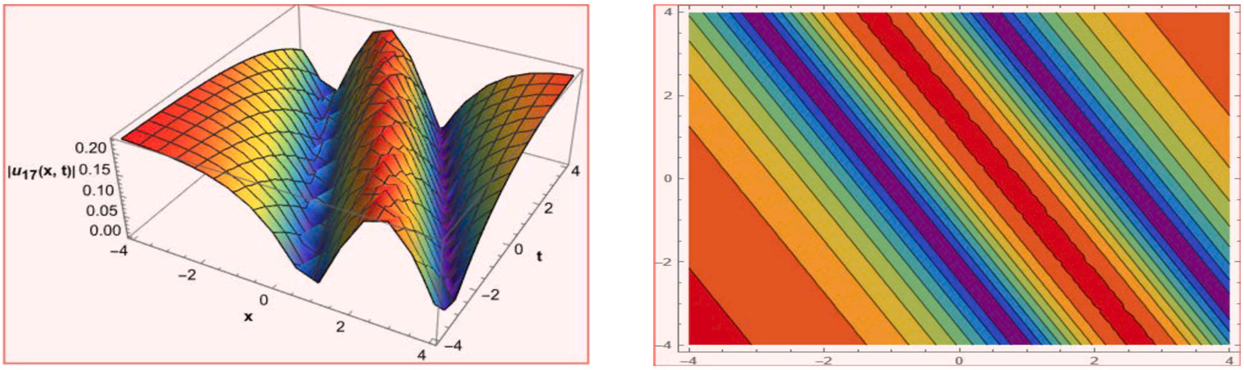

Fig. 9. The physical behavior of $u_{17}(x,t)$ and its contour is depicted by choosing parameters as $c = 0.5$ and $a = -6.5$.

dynamical studies of Einstein's vacuum field equations and for analytical comparisons of the solutions.

6. Conclusion

In this manuscript, we investigated the solitary wave solutions for Einstein's vacuum field equation. This equation arises in mathematical physics, and it has meaningful applications in the general theory of relativity. Different families of solitary wave structures were constructed using the generalized exponential rational function (GERF) method and the modified auxiliary equation (MAE) method. These solutions were obtained in the form of exponential functions, hyperbolic functions, trigonometric functions, and rational functions. Moreover, the stability analysis of Einstein's vacuum field equation was also presented. 3-D and corresponding contour plots of some solutions were constructed, clearly showing the solitary wave behaviors. These plots illustrate the physical behavior of our results, demonstrating dark, singular, solitary wave, and some mixed types of wave solutions. The obtained results confirm that the proposed method is the best tool for solving any nonlinear problems arising in science and technology. These results are very fruitful for further studies of nonlinear dynamics to gain different types of soliton solutions and explore their applications in the general theory of relativity.

Ethical approval

Not applicable.

Funding

Not applicable.

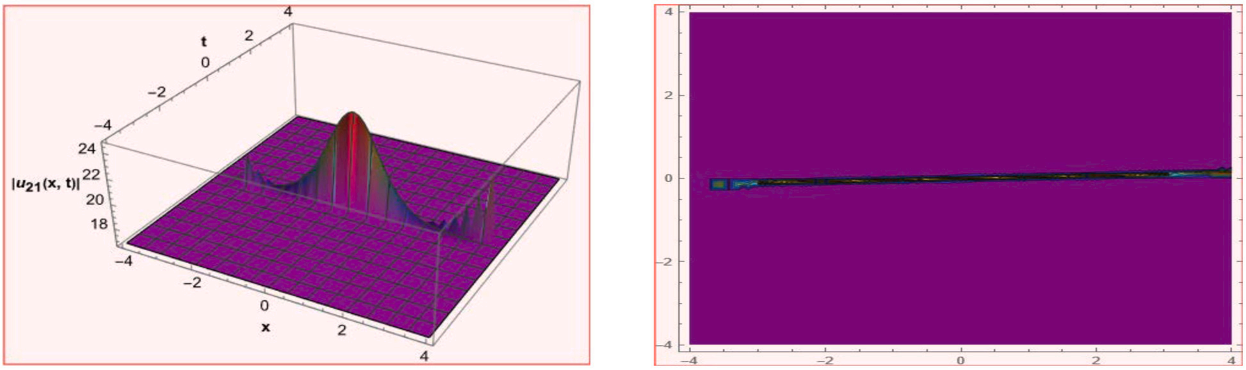

Fig. 10. The physical behavior of $u_{21}(x,t)$ and its contour is depicted by choosing parameters as $c = -29.05$ and $a = 1.5$.

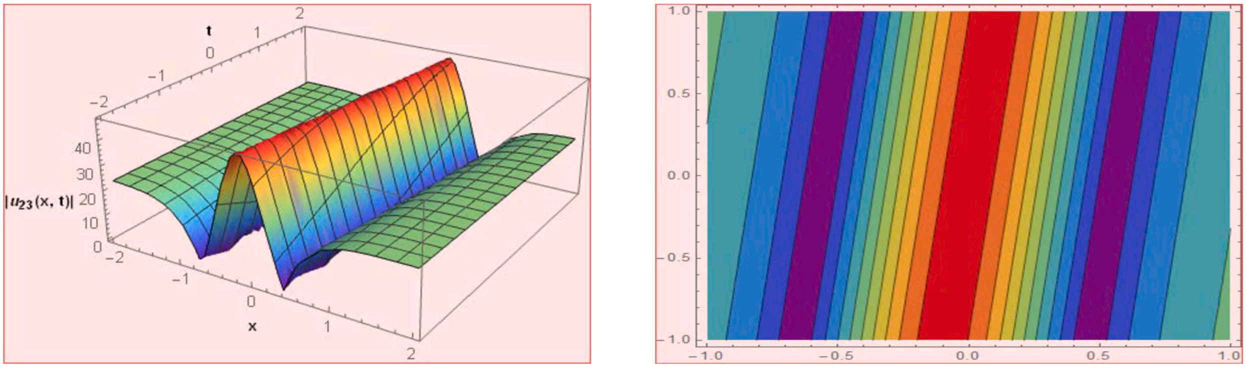

Fig. 11. The physical behavior of $u_{23}(x,t)$ and its contour is depicted by choosing parameters as $\alpha_0 = 4.2, \alpha_1 = 2.2, a = 0.7, c = 0.1$, and $\nu = 0.1$.

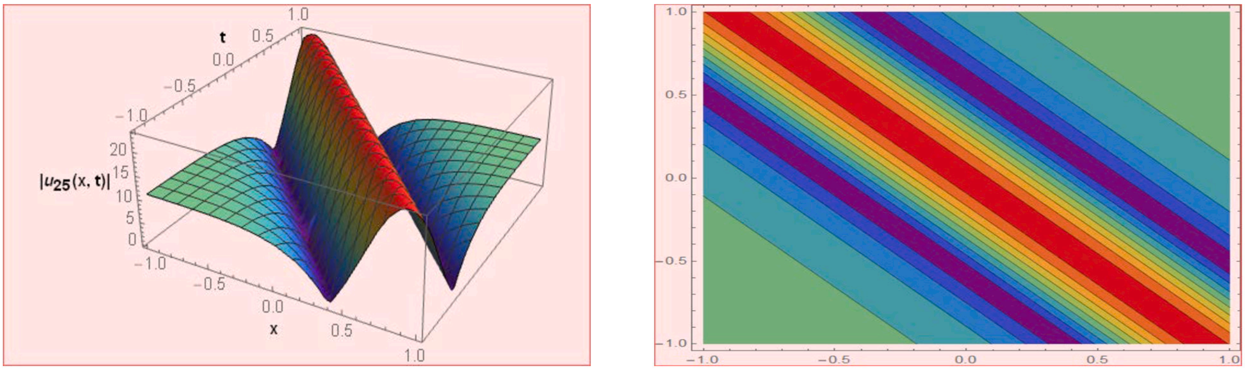

Fig. 12. The physical behavior of $u_{25}(x,t)$ and its contour is depicted by choosing parameters as $\alpha_0 = 3.1, \alpha_1 = -1.2, a = 1.7, c = -0.91$, and $\nu = 2.1$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and materials

Data sharing not applicable to this article as no data-sets were generated or analyzed during this study.

References

[1] F. Mainardi, Fractional calculus: some basic problems in continuumand statisticalmechanics, in: Fractals and Fractional Calculus in ContinuumMechanics, in: CISMCourses and Lectures, vol. 378, Springer, Vienna, Austria, 1997, pp. 291–348.

[2] A. Atangana, E. Alabaraoye, Solving system of fractional partial differential equations arisen in the model of HIV infection of CD4+ cells and attractor one-dimensional Keller-Segel equation, Adv. Differ. Equ. 2013 (2013) 94.

[3] H. Stephani, et al., Exact Solutions of Einstein's Field Equations, 2nd ed., Cambridge University Press, Cambridge, 2003.

[4] D. Jyoti, S. Kumar, Exact non-static solutions of Einstein vacuum field equations, Chin. J. Phys. 68 (2020) 735–744.

[5] L. Kaur, A.M. Wazwaz, Einstein's vacuum field equation: Painlevé analysis and Lie symmetries, Waves Random Complex Media 31 (2) (2021) 199–206.

[6] H. Quevedo, General static axisymmetric solution of Einstein's vacuum field equations in prolate spheroidal coordinates, Phys. Rev. D 39 (10) (1989) 2904.

[7] H. Friedrich, G. Nagy, The initial boundary value problem for Einstein's vacuum field equation, Commun. Math. Phys. 201 (1999) 619–655.

[8] S.F. Rutz, A Finsler generalisation of Einstein's vacuum field equations, Gen. Relativ. Gravit. 25 (1993) 1139–1158.

[9] H. Christian, F.W. Hehl, Schwarzschild and Kerr solutions of Einstein's field equation: an introduction, Int. J. Mod. Phys. D 24 (02) (2015) 1530006.

[10] R.B. Hoffman, Stationary "noncanonical" solutions of the Einstein vacuum field equations, J. Math. Phys. 10 (5) (1969) 953–956.

[11] R.G. Vishwakarma, A new solution of Einstein's vacuum field equations, Pramana 85 (2015) 1101–1110.

[12] C. Corda, A solution of linearized Einstein field equations in vacuum used for the detection of the stochastic background of gravitational waves, Astropart. Phys. 27 (6) (2007) 539–549.

[13] O.P. Bhutani, K. Singh, Generalized similarity solutions for the type D fluid in five dimensional flat space, J. Math. Phys. 39 (1998) 3203–3212.

[14] S.K. Attallaha, M.F. El-Sabbagh, A.T. Ali, Isovector fields and similarity solutions of Einstein vacuum equations for rotating fields, Commun. Nonlinear Sci. Numer. Simul. 12 (2007) 1153–1161.

[15] L. Kaur, R.K. Gupta, On symmetries and exact solutions of the Einstein–Maxwell field equations via the symmetry approach, Phys. Scr. 87 (3) (2013) 035003.

[16] L. Kaur, A.M. Wazwaz, Similarity solutions of field equations with an electromagnetic stress tensor as source, Rom. Rep. Phys. 70 (114) (2018) 1–12.

[17] B.S. Guilfoyle, Interior Weyl-type solutions to the Einstein-Maxwell field equations, Gen. Relativ. Gravit. 31 (11) (1999) 1645–1673.

[18] K. Hosseini, K. Sadri, E. Hincal, A. Abbasi, D. Baleanu, S. Salahshour, Periodic and solitary waves of the nonlinear Konno–Oono model: generalized methods, Opt. Quantum Electron. 55 (6) (2023) 564.

[19] S. Kumar, M. Niwas, M.S. Osman, M.A. Abdou, Abundant different types of exact soliton solution to the (4+1)-dimensional Fokas and (2+1)-dimensional breaking soliton equations, Commun. Theor. Phys. 73 (10) (2021) 105007.

[20] W. Ou, C. Xu, Q. Cui, Z. Liu, Y. Pang, M. Farman, et al., Mathematical study on bifurcation dynamics and control mechanism of tri-neuron bidirectional associative memory neural networks including delay, Math. Methods Appl. Sci. (2023).

[21] Changjin Xu, Qingyi Cui, Zixin Liu, Yuanlu Pan, Xiaohan Cui Wei Ou, Mati ur Rahman, Muhammad Farman, Shabir Ahmad, Zeb Anwar, Extended hybrid controller design of bifurcation in a delayed chemostat model, MATCH Commun. Math. Comput. Chem. 90 (3) (2023) 609–648.

[22] C. Xu, D. Mu, Y. Pan, C. Aouiti, L. Yao, Exploring bifurcation in a fractional-order predator-prey system with mixed delays, J. Appl. Anal. Comput. 13 (2023) 1119–1136.

[23] P. Li, Y. Lu, C. Xu, J. Ren, Insight into Hopf bifurcation and control methods in fractional order BAM neural networks incorporating symmetric structure and delay, Cogn. Comput. (2023) 1–43.

[24] M.S. Iqbal, A.R. Seadawy, M.Z. Baber, M. Qasim, Application of modified exponential rational function method to Jaulent–Miodek system leading to exact classical solutions, Chaos Solitons Fractals 164 (2022) 112600.

[25] T.S. Shaikh, M.Z. Baber, N. Ahmed, N. Shahid, A. Akgül, M. De la Sen, On the soliton solutions for the stochastic Konno–Oono system in magnetic field with the presence of noise, Mathematics 11 (6) (2023) 1472.

[26] M.Z. Baber, N. Ahmed, M.W. Yasin, M.S. Iqbal, A. Akgül, M.B. Riaz, et al., Comparative analysis of numerical with optical soliton solutions of stochastic Gross–Pitaevskii equation in dispersive media, Results Phys. 44 (2023) 106175.

[27] U. Younas, J. Ren, M.Z. Baber, M.W. Yasin, T. Shahzad, Ion-acoustic wave structures in the fluid ions modeled by higher dimensional generalized Korteweg-de Vries–Zakharov–Kuznetsov equation, J. Ocean Eng. Sci. 2022 (2022) 1–12.

[28] C. Liu, X. Liu, A note on the auxiliary equation method for solving nonlinear partial differential equations, Phys. Lett. A 348 (3–6) (2006) 222–227.

[29] A.M. Wazwaz, A sine-cosine method for handlingnonlinear wave equations, Math. Comput. Model. 40 (5–6) (2004) 499–508.

[30] M. Kaplan, A. Akbulut, The analysis of the soliton-type solutions of conformable equations by using generalized Kudryashov method, Opt. Quantum Electron. 53 (2021) 1–21.

[31] M. Najafi, S. Arbabi, M. Najafi, He's semi-inverse method for Camassa–Holm equation and simplified modified Camassa–Holm equation, Int. J. Phys. Res. 1 (2013) 1–6.

[32] J.H. He, X.H. Wu, Exp-function method for nonlinear wave equations, Chaos Solitons Fractals 30 (3) (2006) 700–708.

[33] M.S. Ahmed, A.S. Zaghrout, H.M. Ahmed, A.H. Arnous, Optical soliton perturbation of the Gerdjikov–Ivanov equation with spatio-temporal dispersion using a modified extended direct algebraic method, Optik 259 (2022) 168904.

[34] M. Inc, A.I. Aliyu, A. Yusuf, Optical solitons to the nonlinear Shrödinger's equation with spatio-temporal dispersion using complex amplitude ansatz, J. Mod. Opt. 64 (21) (2017) 2273–2280.

[35] B. Ghanbari, M. Inc, A. Yusuf, D. Baleanu, New solitary wave solutions and stability analysis of the Benney-Luke and the Phi-4 equations in mathematical physics, AIMS Math. 4 (6) (2019) 1523–1539.