JP Journal of Algebra, Number Theory and Applications

Published Online: March 2015
http://dx.doi.org/10.17654/JPANTAFeb2015_029_042
Volume 36, Number 1, 2015, Pages 29-42 ISSN: 0972-5555

SOLVING BINOMIAL THUE EQUATIONS

István Gaál and László Remete

Mathematical Institute
University of Debrecen
H-4010 Debrecen Pf.12, Hungary
e-mail: igaal@science.unideb.hu
remetel42@gmail.com

Abstract

We consider binomial Thue equations of type $x^n - my^n = \pm 1$ in $x, y \in \mathbb{Z}$. Optimizing the method of Pethő [7] we perform an extensive calculation by a high performance computer to determine all solutions with $\max(|x|, |y|) < 10^{500}$ of binomial Thue equations for $m < 10^7$ for exponents $n = 3, 4, 5, 7, 11, 13, 17, 19, 23, 29$.

1. Introduction

The method of Pethő [7] (see also [4]) gives a fast algorithm to calculate "small" solutions of Thue equations. The method is based on the continued fraction algorithm. By "small" solutions we mean those with absolute values

Received: October 4, 2014; Accepted: November 12, 2014

2010 Mathematics Subject Classification: Primary 11D59; Secondary 11Y50.

Keywords and phrases: binomial Thue equations, supercomputers.

This work was partially supported by the European Union and the European Social Fund through project Supercomputer, the national virtual lab (grant no.: TAMOP-4.2.2.C-11/1/KONV-2012-0010) and also supported in part by K100339 and K75566 from the Hungarian National Foundation for Scientific Research.

Communicated by K. K. Azad

less than, say 10^{500}. Nobody believes that such equations have larger solutions.

All our experiences show that such equations usually only have a few very small solutions.

In this paper, we make this algorithm more efficient in order to calculate "small" solutions of a special type of Thue equations, the binomial Thue equations of type

$$x^n - my^n = \pm 1 \text{ in } x, y \in \mathbb{Z}. \tag{1}$$

Using sharp estimates and utilizing the specialties of these equations we perform an extensive calculation by a high performance computer to determine solutions with $\max(|x|, |y|) < 10^{500}$ of binomial Thue equations for $1 < m < 10^7$ (assuming that the left hand side is irreducible) for the exponents $n = 3, 4, 5, 7, 11, 13, 17, 19, 23, 29$. Our data contains all solutions of these equations with high probability. These results complete several results on the solutions of binomial Thue equations [2], [1], [6].

2. Sharper Estimates

Let n be one of 3, 5, 7, 11, 13, 17, 19, 23, 29, in fact our arguments are valid for any odd primes (the case $n = 4$ we shall deal with later). Assume that for m the left hand side of (1) is irreducible (we skip those m for which the left hand side of (1) is reducible). Our purpose is to determine all solutions $x, y \in \mathbb{Z}$ of (1) with $\max(|x|, |y|) < C$. We shall perform our calculation with $C = 10^{500}$.

Let $\zeta = \exp(2\pi i/n)$. Let x, y be an arbitrary solution of (1). Set

$$\beta_j = x - \zeta^{j-1}\sqrt[n]{m}\, y$$

for $j = 1, \ldots, n$, then equation (1) can be written as

$$\beta_1 \cdots \beta_n = \pm 1. \tag{2}$$

We may assume $y \geq 0$ since on the right side we have ± 1 in our equation. Also, for $y = 0$ we only have the trivial solution $x = \pm 1$. Therefore in the following let $y \geq 1$.

For $n \neq 4$ the $\beta_2, \ldots, \beta_n$ are complex, therefore

$$|\beta_j| = |x - \zeta^{j-1}\sqrt[n]{m}\,y| \geq |\operatorname{Im}(\zeta^{j-1})|\sqrt[n]{m}\,y$$

for $j = 1, \ldots, n$. This yields that

$$|\beta_1| = |x - \sqrt[n]{m}\,y| \leq \frac{1}{c_1 y^{n-1}}$$

that is

$$\left|\sqrt[n]{m} - \frac{x}{y}\right| \leq \frac{1}{c_1 y^n} \tag{3}$$

with

$$c_1 = (\sqrt[n]{m})^{n-1} c_2, \text{ where } c_2 = \prod_{j=2}^{n} |\operatorname{Im}(\zeta^{j-1})|.$$

Thus, we arrived at the crucial point of the method of [7]. Obviously $(x, y) = 1$. If the upper estimate on the right hand side of (3) satisfies

$$\frac{1}{c_1 y^n} < \frac{1}{2y^2}$$

that is

$$y > (2c_1)^{1/(n-2)}, \tag{4}$$

then (3) implies

$$\left|\sqrt[n]{m} - \frac{x}{y}\right| < \frac{1}{2y^2}$$

and appling Legenre's theorem we confer that x/y is a convergent h_j/k_j to $\sqrt[n]{m}$, that is $x = h_j$, $y = k_j$. For the corresponding partial quotients a_j it is well known that

$$\frac{1}{(a_{j+1}+2)k_j^2} < \left| \sqrt[n]{m} - \frac{x}{y} \right|.$$

Let $A = \max_{1 \le i \le s} a_i$, where k_s is the first denominator of a partial quotient exceeding *C*. Combining the above estimate with (3) we get

$$\frac{1}{(A+2)k_j^2} < \left| \sqrt[n]{m} - \frac{x}{y} \right| \le \frac{1}{c_1 k_j^n}$$

whence we obtain

$$y = k_j < c_3 = (c_1(A+2))^{n-2}.$$

We calculate all denominators of partial quotients up to *C*, we take their maximum, calculate the above bound and check all possible $x = h_j$, $y = k_j$ for $k_j < c_3$ running again the continued fraction algorithm.

Now we return to the validity of (4). Simple calculation shows that (4) is satisfied if

$$y > \left(\frac{2}{c_2}\right)^{\frac{1}{n-2}} m^{-\frac{n-1}{n(n-2)}}. \tag{5}$$

If m is large enough, then condition (5) is satisfied for $y \ge 1$. Some small values of *y* must be tested for the following values of *m*:

n	
3	$2 \le m \le 4$
5	$2 \le m \le 10$
7	$2 \le m \le 29$
11	$2 \le m \le 314$
13	$2 \le m \le 1078$
17	$2 \le m \le 13489$
19	$2 \le m \le 48699$
23	$2 \le m \le 652798$
29	$2 \le m \le 7960210$

For all these values of m we have to test all y with

$$y \le \left(\frac{2}{c_2}\right)^{\frac{1}{n-2}} m^{-\frac{n-1}{n(n-2)}}.$$

Easy calculation shows that this merely yields testing $y = 1$ for the values of m contained in the table. This can be done very fast.

Remark. The case $n = 4$ was considered in [5]. Remark that in that case we may assume $x > 0$, $y > 0$. We have

$$|\, x - \sqrt[4]{m}\, y \,| \le 1, \quad |\, x \pm i \sqrt[4]{m}\, y \,| \ge \sqrt[4]{m}\, y,$$

therefore

$$|\, x + \sqrt[4]{m}\, y \,| \ge 2\sqrt[4]{m}\, y - 1 \ge \sqrt[4]{m}\, y$$

whence

$$\left| \sqrt[4]{m} - \frac{x}{y} \right| \le \frac{1}{(\sqrt[4]{m})^3} \frac{1}{y^4} < \frac{1}{2y^2},$$

where the last inequality is valid for all $y \ge 1$.

3. Computational Aspects

We were executing the algorithm of [7] with $C = 10^{500}$ for $n = 3, 4, 5, 7, 11, 13, 17, 19, 23, 29$ (the case $n = 4$ was dealt with in [5]) using the above sharp estimates that made the procedure for binomial Thue equations much more efficient. This efficient algorithm allowed us to perform the calculations for all those $2 \le m \le 10^7$ for which the left side of (1) is irreducible.

The procedure was implemented in Maple [3], we used 1200 digits accuracy. For each exponent this calculation involved almost 10^7 binomial Thue equations. The routines were running on the supercomputer (high performance computer) network situated in Debrecen-Budapest-Pécs-Szeged in Hungary under Linux. For each exponent the total running time was about 120-200 hours calculated for a single node which yields a few hours using parallel computing with a couple of nodes.

4. Solutions

In this chapter, we list the results of our computation. The triples (m, x, y) in our table mean that for the m there is a solution x, y of equation (1). The trivial solution $(m, x, y) = (m, 1, 0)$ is not listed. The solutions are displayed up to sign, that is we include only one of (m, x, y) and $(m, -x, -y)$.

For each exponent m, we list the solutions with

$$\max(|x|, |y|) < 10^{500}$$

of all equations with $2 \le m \le 10^7$ for which the left hand side of equation (1) is irreducible.

4.1. Solutions for $n = 3$

m	x	y	m	x	y	m	x	y	m	x	y	m	x	y
2	1	1	3155	44	3	18745	1036	39	54873	38	1	117650	49	1
7	2	1	3374	15	1	18963	80	3	57811	116	3	118681	344	7
9	2	1	3376	15	1	19441	242	9	59318	39	1	123506	249	5
17	18	7	3605	46	3	19682	27	1	59320	39	1	124999	50	1
19	8	3	3724	31	2	19684	27	1	60853	118	3	125001	50	1
20	19	7	3907	63	4	19927	244	9	61630	79	2	126506	251	5
26	3	1	4095	16	1	20421	82	3	63999	40	1	130067	152	3
28	3	1	4097	16	1	20797	55	2	64001	40	1	132650	51	1
37	10	3	4291	65	4	21951	28	1	66430	81	2	132652	51	1
43	7	2	4492	33	2	21953	28	1	68920	41	1	135269	154	3
63	4	1	4912	17	1	23149	57	2	68922	41	1	136591	103	2
65	4	1	4914	17	1	24388	29	1	72338	125	3	137160	361	7
91	9	2	5080	361	21	24390	29	1	74087	42	1	138303	362	7
124	5	1	5514	53	3	26110	89	3	74089	42	1	140607	52	1
126	5	1	5831	18	1	26999	30	1	75866	127	3	140609	52	1
182	17	3	5833	18	1	27001	30	1	79506	43	1	144703	105	2
215	6	1	6162	55	3	27910	91	3	79508	43	1	148876	53	1
217	6	1	6858	19	1	29790	31	1	82313	87	2	148878	53	1
254	19	3	6860	19	1	29792	31	1	85183	44	1	154566	161	3
342	7	1	7415	39	2	31256	63	2	85185	44	1	156494	485	9
344	7	1	7999	20	1	32006	127	4	87866	9825	221	157463	54	1
422	15	2	8001	20	1	32042	667	21	88121	89	2	157465	54	1
511	8	1	8615	41	2	32767	32	1	89115	134	3	158438	487	9
513	8	1	8827	62	3	32769	32	1	91124	45	1	160398	163	3
614	17	2	9260	21	1	33542	129	4	91126	45	1	166374	55	1
635	361	42	9262	21	1	34328	65	2	93165	136	3	166376	55	1
651	26	3	9709	64	3	34859	98	3	97335	46	1	170954	111	2
728	9	1	10647	22	1	35936	33	1	97337	46	1	175615	56	1
730	9	1	10649	22	1	35938	33	1	99161	324	7	175617	56	1
813	28	3	12166	23	1	37037	100	3	100082	325	7	180362	113	2
999	10	1	12168	23	1	39303	34	1	103822	47	1	181963	170	3
1001	10	1	12978	47	2	39305	34	1	103824	47	1	185192	57	1
1330	11	1	13256	71	3	42874	35	1	107172	95	2	185194	57	1
1332	11	1	13538	143	6	42876	35	1	108304	143	3	188461	172	3
1521	23	2	13823	24	1	44739	71	2	108873	191	4	195111	58	1
1588	35	3	13825	24	1	45372	107	3	109444	287	6	195113	58	1
1657	71	6	14114	145	6	46011	215	6	110017	575	12	205378	59	1
1727	12	1	14408	73	3	46655	36	1	110591	48	1	205380	59	1
1729	12	1	14706	49	2	46657	36	1	110593	48	1	210645	119	2
1801	73	6	15253	124	5	47307	217	6	111169	577	12	212420	179	3
1876	37	3	15624	25	1	47964	109	3	111748	289	6	214205	359	6
1953	25	2	15626	25	1	48627	73	2	112329	193	4	215999	60	1
2196	13	1	16003	126	5	48949	4097	112	112912	145	3	216001	60	1
2198	13	1	17145	361	14	50652	37	1	114084	97	2	217805	361	6
2743	14	1	17575	26	1	50654	37	1	116623	342	7	219620	181	3
2745	14	1	17577	26	1	54871	38	1	117648	49	1	221445	121	2

m	x	y	m	x	y	m	x	y	m	x	y	m	x	y
226980	61	1	389018	73	1	592705	84	1	884737	96	1	1225042	107	1
226982	61	1	391592	4097	56	596239	505	6	887042	1153	12	1225044	107	1
238327	62	1	405223	74	1	599788	253	3	889352	577	6	1242297	215	2
238329	62	1	405225	74	1	603351	169	2	891666	385	4	1248084	323	3
246099	188	3	416275	224	3	614124	85	1	893984	289	3	1253889	647	6
250046	63	1	418509	374	5	614126	85	1	898632	193	2	1255828	971	9
250048	63	1	420751	1124	15	636055	86	1	902629	1353	14	1257769	1943	18
254037	190	3	421874	75	1	636057	86	1	912672	97	1	1259711	108	1
256048	127	2	421876	75	1	650963	260	3	912674	97	1	1259713	108	1
259084	255	4	423001	1126	15	658502	87	1	937082	685	7	1261657	1945	18
260611	511	8	425259	376	5	658504	87	1	941191	98	1	1263604	973	9
262143	64	1	427525	226	3	666101	262	3	941193	98	1	1265553	649	6
262145	64	1	430369	151	2	669922	175	2	945314	687	7	1271412	325	3
263683	513	8	438975	76	1	681471	88	1	960531	296	3	1277289	217	2
265228	257	4	438977	76	1	681473	88	1	970298	99	1	1295028	109	1
268336	129	2	447697	153	2	693154	177	2	970300	99	1	1295030	109	1
274624	65	1	456532	77	1	704968	89	1	980133	298	3	1306421	9948	91
274626	65	1	456534	77	1	704970	89	1	980838	1391	14	1330999	110	1
283162	197	3	468494	233	3	710467	1160	13	985075	199	2	1331001	110	1
287495	66	1	474551	78	1	712306	1161	13	994012	499	5	1355347	332	3
287497	66	1	474553	78	1	720930	269	3	997003	999	10	1367630	111	1
291874	199	3	480662	235	3	728999	90	1	999999	100	1	1367632	111	1
300762	67	1	493038	79	1	729001	90	1	1000001	100	1	1379989	334	3
300764	67	1	493040	79	1	737130	271	3	1003003	1001	10	1386196	223	2
307547	135	2	502460	159	2	753570	91	1	1006012	501	5	1395541	447	4
314431	68	1	506115	1036	13	753572	91	1	1015075	201	2	1404927	112	1
314433	68	1	507215	319	4	766061	183	2	1017241	704	7	1404929	112	1
321419	137	2	507582	1037	13	778687	92	1	1021582	705	7	1414357	449	4
323771	206	3	511999	80	1	778689	92	1	1030300	101	1	1423828	225	2
328508	69	1	512001	80	1	791453	185	2	1030302	101	1	1442896	113	1
328510	69	1	516815	321	4	795739	278	3	1050838	305	3	1442898	113	1
333293	208	3	521660	161	2	804356	93	1	1061207	102	1	1468586	341	3
342999	70	1	524907	242	3	804358	93	1	1061209	102	1	1481543	114	1
343001	70	1	529257	728	9	813037	280	3	1071646	307	3	1481545	114	1
356057	4040	57	531440	81	1	830583	94	1	1092726	103	1	1494578	343	3
357910	71	1	531442	81	1	830585	94	1	1092728	103	1	1520874	115	1
357912	71	1	533631	730	9	857374	95	1	1108718	207	2	1520876	115	1
365526	143	2	538029	244	3	857376	95	1	1124863	104	1	1530341	2420	21
368088	215	3	551367	82	1	865134	667	7	1124865	104	1	1540799	231	2
370662	431	6	551369	82	1	869031	668	7	1141166	209	2	1560895	116	1
373247	72	1	571786	83	1	870984	191	2	1146635	314	3	1560897	116	1
373249	72	1	571788	83	1	875552	287	3	1157624	105	1	1581167	233	2
375846	433	6	582183	167	2	877842	383	4	1157626	105	1	1587963	350	3
378456	217	3	585676	251	3	880136	575	6	1168685	316	3	1601612	117	1
381078	145	2	589183	503	6	882434	1151	12	1191015	106	1	1601614	117	1
389016	73	1	592703	84	1	884735	96	1	1191017	106	1	1615341	352	3

m	x	y	m	x	y	m	x	y	m	x	y	m	x	y
1643031	118	1	2130091	386	3	2823149	424	3	3370502	2249	15	4251527	162	1
1643033	118	1	2146688	129	1	2863287	142	1	3374999	150	1	4251529	162	1
1685158	119	1	2146690	129	1	2863289	142	1	3375001	150	1	4260282	1459	9
1685160	119	1	2163373	388	3	2888177	5554	39	3379502	2251	15	4277826	487	3
1695638	7751	65	2187357	2726	21	2924206	143	1	3388518	751	5	4321265	6841	42
1706490	239	2	2196999	130	1	2924208	143	1	3397550	451	3	4330746	163	1
1713640	359	3	2197001	130	1	2954988	287	2	3442950	151	1	4330748	163	1
1720810	719	6	2248090	131	1	2965296	431	3	3442952	151	1	4370723	327	2
1727999	120	1	2248092	131	1	2970459	575	4	3477266	303	2	4410943	164	1
1728001	120	1	2273931	263	2	2975628	863	6	3511807	152	1	4410945	164	1
1735210	721	6	2282588	395	3	2980803	1727	12	3511809	152	1	4426331	3448	21
1742440	361	3	2291267	791	6	2985983	144	1	3546578	305	2	4451411	329	2
1749690	241	2	2299967	132	1	2985985	144	1	3558219	458	3	4464955	494	3
1767571	1330	11	2299969	132	1	2991171	1729	12	3581576	153	1	4492124	165	1
1771560	121	1	2308691	793	6	2996364	865	6	3581578	153	1	4492126	165	1
1771562	121	1	2317436	397	3	3001563	577	4	3605037	460	3	4519405	496	3
1775557	1332	11	2326203	265	2	3003793	1010	7	3652263	154	1	4574295	166	1
1815847	122	1	2352636	133	1	3006768	433	3	3652265	154	1	4574297	166	1
1815849	122	1	2352638	133	1	3012724	1011	7	3723874	155	1	4657462	167	1
1845779	368	3	2406103	134	1	3017196	289	2	3723876	155	1	4657464	167	1
1860866	123	1	2406105	134	1	3048624	145	1	3760029	311	2	4699422	335	2
1860868	123	1	2442195	404	3	3048626	145	1	3772132	467	3	4713464	503	3
1876037	370	3	2454305	1214	9	3112135	146	1	3784261	935	6	4727534	1007	6
1883653	247	2	2460374	135	1	3112137	146	1	3796415	156	1	4741631	168	1
1906623	124	1	2460376	135	1	3132736	4097	28	3796417	156	1	4741633	168	1
1906625	124	1	2466455	1216	9	3141461	5126	35	3808597	937	6	4755758	1009	6
1922636	3233	26	2478645	406	3	3154963	440	3	3820804	469	3	4769912	505	3
1929781	249	2	2487814	271	2	3167271	1028	7	3833037	313	2	4784094	337	2
1943765	624	5	2515455	136	1	3173437	3086	21	3869892	157	1	4820221	2196	13
1953124	125	1	2515457	136	1	3176522	147	1	3869894	157	1	4826808	169	1
1953126	125	1	2543302	273	2	3176524	147	1	3944311	158	1	4826810	169	1
1962515	626	5	2571352	137	1	3179611	3088	21	3944313	158	1	4833403	2198	13
1984542	377	3	2571354	137	1	3185793	1030	7	3994451	476	3	4912999	170	1
2000375	126	1	2609074	413	3	3198181	442	3	4019678	159	1	4913001	170	1
2000377	126	1	2628071	138	1	3209047	295	2	4019680	159	1	4931602	2383	14
2016294	379	3	2628073	138	1	3241791	148	1	4045013	478	3	4971027	512	3
2048382	127	1	2647162	415	3	3241793	148	1	4057720	319	2	5000210	171	1
2048384	127	1	2685618	139	1	3266062	2819	19	4076830	639	4	5000212	171	1
2072672	255	2	2685620	139	1	3269539	2820	19	4095999	160	1	5029509	514	3
2084888	511	4	2714705	279	2	3274759	297	2	4096001	160	1	5044201	343	2
2091014	1023	8	2743999	140	1	3307948	149	1	4115230	641	4	5088447	172	1
2097151	128	1	2744001	140	1	3307950	149	1	4134520	321	2	5088449	172	1
2097153	128	1	2773505	281	2	3346154	1047	7	4173280	161	1	5132953	345	2
2103302	1025	8	2783387	422	3	3352550	449	3	4173282	161	1	5177716	173	1
2109464	513	4	2803220	141	1	3355751	1048	7	4225338	485	3	5177718	173	1
2121824	257	2	2803222	141	1	3361518	749	5	4242786	1457	9	5237806	521	3

m	x	y	m	x	y	m	x	y
5268023	174	1	6434855	186	1	7513084	1371	7
5268025	174	1	6434857	186	1	7521307	2743	14
5298358	523	3	6469514	559	3	7529535	196	1
5341021	874	5	6539202	187	1	7529537	196	1
5359374	175	1	6539204	187	1	7537771	2745	14
5359376	175	1	6591797	375	2	7546012	1373	7
5377771	876	5	6644671	188	1	7587307	393	2
5405444	351	2	6644673	188	1	7645372	197	1
5428577	703	4	6697829	377	2	7645374	197	1
5451775	176	1	6715611	566	3	7723254	593	3
5451777	176	1	6739369	1700	9	7762391	198	1
5475041	705	4	6751268	189	1	7762393	198	1
5498372	353	2	6751270	189	1	7801662	595	3
5513963	530	3	6763183	1702	9	7829793	1390	7
5545232	177	1	6787053	568	3	7846704	1391	7
5545234	177	1	6858999	190	1	7880598	199	1
5576621	532	3	6859001	190	1	7880600	199	1
5639751	178	1	6967870	191	1	7926126	93311	468
5639753	178	1	6967872	191	1	7940150	399	2
5712483	3754	21	7022736	383	2	7976024	999	5
5735338	179	1	7041088	575	3	7988006	1999	10
5735340	179	1	7050276	767	4	7999999	200	1
5783535	359	2	7059472	1151	6	8000001	200	1
5799660	539	3	7064073	1535	8	8012006	2001	10
5815815	1079	6	7068676	2303	12	8024024	1001	5
5831999	180	1	7073281	4607	24	8060150	401	2
5832001	180	1	7077887	192	1	8080267	602	3
5848215	1081	6	7077889	192	1	8120600	201	1
5864460	541	3	7082497	4609	24	8120602	201	1
5880735	361	2	7087108	2305	12	8161069	604	3
5929740	181	1	7091721	1537	8	8242407	202	1
5929742	181	1	7096336	1153	6	8242409	202	1
6028567	182	1	7105572	769	4	8341522	6287	31
6028569	182	1	7114816	577	3	8345503	6288	31
6095059	548	3	7133328	385	2	8365426	203	1
6128486	183	1	7189056	193	1	8365428	203	1
6128488	183	1	7189058	193	1	8427393	407	2
6162037	550	3	7221032	1353	7	8448116	611	3
6178858	367	2	7237055	1354	7	8468873	1223	6
6229503	184	1	7301383	194	1	8489663	204	1
6229505	184	1	7301385	194	1	8489665	204	1
6280426	369	2	7376915	584	3	8510489	1225	6
6331624	185	1	7414874	195	1	8531348	613	3
6331626	185	1	7414876	195	1	8552241	409	2
6400322	557	3	7452965	586	3	8615124	205	1
6402292	6499	35	7472059	391	2	8615126	205	1

m	x	y
8741815	206	1
8741817	206	1
8826963	620	3
8869742	207	1
8869744	207	1
8912661	622	3
8934172	415	2
8966503	831	4
8998911	208	1
8998913	208	1
9031399	833	4
9063964	417	2
9114217	5431	26
9129328	209	1
9129330	209	1
9216970	629	3
9260999	210	1
9261001	210	1
9305170	631	3
9393930	211	1
9393932	211	1
9460871	423	2
9528127	212	1
9528129	212	1
9595703	425	2
9603769	17855	84
9606402	4039	19
9613539	4040	19
9618299	638	3
9663596	213	1
9663598	213	1
9709037	640	3
9800343	214	1
9800345	214	1
9938374	215	1
9938376	215	1

4.2. Solutions for $n = 4$

m	x	y	m	x	y	m	x	y	m	x	y
2	1	1	65535	16	1	707282	29	1	3580610	87	2
5	3	2	65537	16	1	757335	59	2	3748095	44	1
15	2	1	69729	65	4	809999	30	1	3748097	44	1
17	2	1	74120	33	2	810001	30	1	3851367	443	10
39	5	2	83520	17	1	865365	61	2	3921390	89	2
80	3	1	83522	17	1	923520	31	1	4100624	45	1
82	3	1	93789	35	2	923522	31	1	4100626	45	1
150	7	2	104975	18	1	984560	63	2	4285935	91	2
255	4	1	104977	18	1	1016190	127	4	4477455	46	1
257	4	1	114240	239	13	1048575	32	1	4477457	46	1
410	9	2	117135	37	2	1048577	32	1	4675325	93	2
624	5	1	130320	19	1	1081730	129	4	4879680	47	1
626	5	1	130322	19	1	1115664	65	2	4879682	47	1
915	11	2	144590	39	2	1185920	33	1	5090664	95	2
1295	6	1	159999	20	1	1185922	33	1	5198685	191	4
1297	6	1	160001	20	1	1259445	67	2	5308415	48	1
1785	13	2	176610	41	2	1336335	34	1	5308417	48	1
2400	7	1	194480	21	1	1336337	34	1	5419875	193	4
2402	7	1	194482	21	1	1416695	69	2	5533080	97	2
3164	15	2	213675	43	2	1500624	35	1	5764800	49	1
4095	8	1	234255	22	1	1500626	35	1	5764802	49	1
4097	8	1	234257	22	1	1588230	71	2	6003725	99	2
5220	17	2	256289	45	2	1679615	36	1	6249999	50	1
6560	9	1	279840	23	1	1679617	36	1	6250001	50	1
6562	9	1	279842	23	1	1755519	182	5	6503775	101	2
7140	239	26	304980	47	2	1774890	73	2	6765200	51	1
8145	19	2	331775	24	1	1874160	37	1	6765202	51	1
9999	10	1	331777	24	1	1874162	37	1	7034430	103	2
10001	10	1	360300	49	2	1977539	75	2	7311615	52	1
12155	21	2	390624	25	1	2085135	38	1	7311617	52	1
14640	11	1	390626	25	1	2085137	38	1	7596914	105	2
14642	11	1	422825	51	2	2197065	77	2	7890480	53	1
17490	23	2	456975	26	1	2313440	39	1	7890482	53	1
20735	12	1	456977	26	1	2313442	39	1	8192475	107	2
20737	12	1	493155	53	2	2434380	79	2	8295040	161	3
24414	25	2	505679	80	3	2559999	40	1	8398565	323	6
28560	13	1	518440	161	6	2560001	40	1	8503055	54	1
28562	13	1	531440	27	1	2690420	81	2	8503057	54	1
33215	27	2	531442	27	1	2825760	41	1	8608519	325	6
38415	14	1	544685	163	6	2825762	41	1	8714960	163	3
38417	14	1	558175	82	3	2966145	83	2	8822385	109	2
44205	29	2	571914	55	2	3111695	42	1	9150624	55	1
50624	15	1	614655	28	1	3111697	42	1	9150626	55	1
50626	15	1	614657	28	1	3262539	85	2	9487940	111	2
57720	31	2	659750	57	2	3418800	43	1	9834495	56	1
61535	63	4	707280	29	1	3418802	43	1	9834497	56	1

4.3. Solutions for $n = 5$

m	x	y	m	x	y	m	x	y	m	x	y
2	1	1	32767	8	1	537825	14	1	3199999	20	1
31	2	1	32769	8	1	759374	15	1	3200001	20	1
33	2	1	59048	9	1	759376	15	1	4084100	21	1
242	3	1	59050	9	1	894661	31	2	4084102	21	1
244	3	1	99999	10	1	1048575	16	1	5153631	22	1
1023	4	1	100001	10	1	1048577	16	1	5153633	22	1
1025	4	1	161050	11	1	1222981	33	2	6436342	23	1
3124	5	1	161052	11	1	1419856	17	1	6436344	23	1
3126	5	1	248831	12	1	1419858	17	1	7962623	24	1
7775	6	1	248833	12	1	1889567	18	1	7962625	24	1
7777	6	1	371292	13	1	1889569	18	1	9765624	25	1
16806	7	1	371294	13	1	2476098	19	1	9765626	25	1
16808	7	1	537823	14	1	2476100	19	1			

4.4. Solutions for $n = 7$

m	x	y
2	1	1
127	2	1
129	2	1
2186	3	1
2188	3	1
16383	4	1
16385	4	1
78124	5	1
78126	5	1
279935	6	1
279937	6	1
823542	7	1
823544	7	1
2097151	8	1
2097153	8	1
4782968	9	1
4782970	9	1
9999999	10	1

4.5. Solutions for $n = 11$

m	x	y
2	1	1
2047	2	1
2049	2	1
177146	3	1
177148	3	1
4194303	4	1
4194305	4	1

4.6. Solutions for $n = 13$

m	x	y
2	1	1
8191	2	1
8193	2	1
1594322	3	1
1594324	3	1

4.7. Solutions for $n = 17$

m	x	y
2	1	1
131071	2	1
131073	2	1

4.8. Solutions for $n = 19$

m	x	y
2	1	1
524287	2	1
524289	2	1

4.9. Solutions for $n = 23$

m	x	y
2	1	1
8388607	2	1
8388609	2	1

4.10. Solutions for $n = 29$

m	x	y
2	1	1

References

[1] A. Bazsó, A. Bérczes, K. Győry and Á. Pintér, On the resolution of equations $Ax^n - By^n = C$ in integers x, y and $n \geq 3$, II, Publ. Math. (Debrecen), 76 (2010), 227-250.

[2] M. A. Bennett, Rational approximation to algebraic numbers of small height: The Diophantine equation $|ax^n - by^n| = 1$, J. Reine Angew. Math. 535 (2001), 1-49.

[3] B. W. Char, K. O. Geddes, G. H. Gonnet, M. B. Monagan and S. M. Watt, (eds.) MAPLE, Reference Manual, Watcom Publications, Waterloo, Canada, 1988.

[4] I. Gaál, Diophantine Equations and Power Integral Bases, Boston, Birkhäuser, 2002.

[5] I. Gaál and L. Remete, Binomial Thue equations and power integral bases in pure quartic fields, JP Journal of Algebra, Number Theory and Applications 32(1) (2014), 49-61.

[6] K. Győry and Á. Pintér, Binomial Thue equations, ternary equations, and power values of polynomials, J. Math. Sciences 180 (2012), 569-580.

[7] A. Pethő, On the resolution of Thue inequalities, J. Symbolic Comput. 4 (1987), 103-109.