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On the optimal designs for the prediction of complex Ornstein-Uhlenbeck processes

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ABSTRACT

Physics, chemistry, biology or finance are just some examples out of the many fields where complex Ornstein-Uhlenbeck (OU) processes have various applications in statistical modeling. They play role e.g. in the description of the motion of a charged test particle in a constant magnetic field or in the study of rotating waves in time-dependent reaction diffusion systems, whereas Kolmogorov used such a process to model the so-called Chandler wobble, the small deviation in the Earth's axis of rotation. A common problem in these applications is deciding how to choose a set of a sample locations in order to predict a random process in an optimal way. We study the optimal design problem for the prediction of a complex OU process on a compact interval with respect to integrated mean square prediction error (IMSPE) and entropy criteria. We derive the exact forms of both criteria, moreover, we show that optimal designs based on entropy criterion are equidistant, whereas the IMSPE based ones may differ from it. Finally, we present some numerical experiments to illustrate selected cases of optimal designs for small number of sampling locations.

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1. Introduction

Physics, chemistry, biology or finance are just some examples out of the many fields where random processes have various applications in statistical modeling. In the current article we study the problem of optimal design for the prediction of a complex Ornstein-Uhlenbeck (OU) process on a compact interval with respect to the Integrated Mean Square Prediction Error (IMSPE) and entropy criteria. A complex OU process (see e.g. Arató 1982), defined in detail in Section 2, is used in the study of rotating waves in time-dependent reaction diffusion systems (Bayn and Lorenz 2008; Otten 2015), it can also describe the motion of a charged test particle in a constant magnetic field (Balescu 1997), and has several applications in financial mathematics as well (see e.g. Barndorff-Nielsen and Shephard 2001). An important application of the complex OU is the so-called Chandler wobble (CW), the small deviation in the Earth's axis of rotation relative to the solid earth (Lambeck 1980). It is a famous example of motion performed by any spinning sphere which is not entirely spherical, and the investigation

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of the properties of the CW helps in the understanding of the physical processes in the Earth. The uniqueness of this motion is the change of about 9 meters in the point where the axis intersects the Earth's surface and it has a period of 435 days. Since 1899 the International Latitude Service (ILS) has been measuring the wobble. Recently monitoring of the polar motion is done by the International Earth Rotation Service (IERS). Since its discovery in 1891, a huge number of articles deal with the analysis of the CW and among them Jeffreys (1942) used first the stochastic difference equation method to estimate the period and damping parameters of the motion. Later Kolmogorov introduced more realistic approaches and described the CW using the model

$$Z(t) = Z_1(t) + iZ_2(t) = me^{i2\pi t} + Y(t), \quad t > 0 \quad (1.1)$$

where $Z_1(t)$ and $Z_2(t)$ are the coordinates of the deviation of the instantaneous pole from the North Pole (Arató, Kolmogorov, and Sinay 1962). In this model the first term is a periodical component, whereas the second one $Y(t)$ is a complex OU process.

For the complex OU process Baran, Szák-Kocsis, and Stehlík (2018) derived the exact form of the Fisher information matrix (FIM) and investigated the properties of D-optimal designs for estimation of model parameters. Here we derive exact form of the IMSPE already studied e.g. in Baran, Sikolya, and Stehlík (2013), Baran, Sikolya, and Stehlík (2015) or Cray (2002), Cray (2016) and show that in contrast to the case of classical real OU process on a compact interval investigated by Baldi Antognini and Zagoraiou (2010), the equidistant design is usually not optimal. We also investigate the properties of the optimal design with respect to the entropy criterion (Shewry and Wynn 1987; Baldi Antognini and Zagoraiou 2010). Note that in some situations the latter design is equivalent to a D-optimal one, see e.g. Sebastiani and Wynn (2000) or Wynn (2004). The article is organized as follows. Section 2 gives the necessary definitions connected to the complex OU process, Sections 3 and 4 contain the results about IMSPE and entropy criteria, respectively, whereas in Section 5 some simulation results are presented. The article ends with the concluding Section 6 and to maintain the continuity of the explanation, proofs are provided in the Appendix.

2. Complex Ornstein-Uhlenbeck process with a trend

Let $Y(t) = Y_1(t) + iY_2(t)$, $t \geq 0$, be a complex valued stationary OU process defined by the stochastic differential equation (SDE)

$$dY(t) = -\gamma Y(t)dt + \sigma d\mathcal{W}(t), \quad Y(0) = \xi \quad (2.1)$$

where $\gamma = \lambda - i\omega$ with $\lambda > 0$, $\omega \in \mathbb{R}$, $\sigma > 0$, $\mathcal{W}(t) = \mathcal{W}_1(t) + i\mathcal{W}_2(t)$, $t \geq 0$, is a standard complex Brownian motion, with $\mathcal{W}_1(t)$ and $\mathcal{W}_2(t)$ being independent standard Brownian motions, and $\xi = \xi_1 + i\xi_2$, where ξ_1 and ξ_2 are centered normal random variables that are chosen according to stationarity (Arató 1982). In the current article we consider the shifted complex stochastic process $Z(t) = Z_1(t) + iZ_2(t)$, defined as

$$Z(t) = m + Y(t), \quad t \geq 0 \quad (2.2)$$

where $m = m_1 + im_2$, $m_1, m_2 \in \mathbb{R}$, and $Y(t)$, $t \geq 0$, is a complex valued stationary OU process, observed in design points taken from the non-negative half-line \mathbb{R}_+ .

In order to simplify calculations, instead of the complex process $Y(t)$ one can use the two-dimensional real valued stationary OU process $(Y_1(t), Y_2(t))^T$ defined by the SDE

$$\begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = A \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} dt + \sigma \begin{bmatrix} d\mathcal{W}_1(t) \\ d\mathcal{W}_2(t) \end{bmatrix}, \quad \text{where } A := \begin{bmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{bmatrix} \quad (2.3)$$

Note that in physics (2.3) is a Langevin equation, see e.g. Balescu (1997). The real and imaginary parts of a complex OU process form a two-dimensional real OU process satisfying (2.3), and conversely, if $(Y_1(t), Y_2(t))^T$ satisfies (2.3) then $Y_1(t) + iY_2(t)$ will be a complex OU process which solves (2.1). Naturally, $EY_1(t) = EY_2(t) = 0$, and the covariance matrix function of the process $(Y_1(t), Y_2(t))^T$ is given by

$$\mathcal{R}(\tau) := E \begin{bmatrix} Y_1(t+\tau) \\ Y_2(t+\tau) \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}^T = \frac{\sigma^2}{2\lambda} e^{A\tau} = \frac{\sigma^2}{2\lambda} e^{-\lambda\tau} \begin{bmatrix} \cos(\omega\tau) & \sin(\omega\tau) \\ -\sin(\omega\tau) & \cos(\omega\tau) \end{bmatrix}, \quad \tau \geq 0 \quad (2.4)$$

This results in a complex covariance function of the complex OU process $Y(t)$ defined by (2.1) of the form

$$C(\tau) := EY(t+\tau)\overline{Y(t)} = \frac{\sigma^2}{\lambda} e^{-\lambda\tau} (\cos(\omega\tau) - i\sin(\omega\tau)), \quad \tau \geq 0$$

behaving like a damped oscillation with frequency ω .

In the next steps we will consider the damping parameter λ , frequency ω and standard deviation σ as known. Nevertheless a promising direction moving forward could be the examination of models where the above mentioned parameters should also be estimated. Note that the estimation of σ can be done on the basis of a single realization of the complex process, see e.g. Arató (1982, Chapter 4). Now, without loss of generality, one can set the variances of $Y_1(t)$ and $Y_2(t)$ to be equal to one, that is $\sigma^2/(2\lambda) = 1$, which reduces $\mathcal{R}(\tau)$ to a correlation matrix function. In Arató, Baran, and Ispány (1999) we can find more results on the maximum-likelihood estimation of the covariance parameters.

3. Optimal design with respect to the IMSPE criterion

Assume we observe our complex process $Z(t)$ at design points $0 \leq t_1 < t_2 < \dots < t_n$ resulting in the $2n$ -dimensional real vector $Z = (Z_1(t_1), Z_2(t_1), Z_1(t_2), Z_2(t_2), \dots, Z_1(t_n), Z_2(t_n))^T$, where

$$Z_1(t) = m_1 + Y_1(t), \quad Z_2(t) = m_2 + Y_2(t)$$

The main aim of the kriging technique consists of predicting the output of the investigated process or field on the experimental region, and for any untried location $x \in \mathcal{X}$, which in our case lies in a closed interval $\mathcal{X} \subset \mathbb{R}$, the estimation procedure is focused on the best linear unbiased estimator (BLUE) $\hat{Z}(x)$ of $Z(x)$. Considering again the two-dimensional vector form of the complex process, the real and imaginary parts $\hat{Z}_1(x)$ and $\hat{Z}_2(x)$, respectively, of the BLUE are given as

$$\begin{bmatrix} \hat{Z}_1(x) \\ \hat{Z}_2(x) \end{bmatrix} = \begin{bmatrix} \hat{m}_1 \\ \hat{m}_2 \end{bmatrix} + Q(x)C^{-1}(n) \left(\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} - H(n)^T \begin{bmatrix} \hat{m}_1 \\ \hat{m}_2 \end{bmatrix} \right)$$

where $(Z_1 \ Z_2)^\top := (Z_1(t_1), Z_2(t_1), Z_1(t_2), Z_2(t_2), \dots, Z_1(t_n), Z_2(t_n))$ is the real observation vector, $H(n)$ is the $2 \times 2n$ matrix

$$H(n) := \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix}$$

$C(n)$ is the $2n \times 2n$ covariance matrix of the observations, $\mathcal{Q}(x)$ is the $2 \times 2n$ matrix of correlations between $Z(x) = (Z_1(x), Z_2(x))$ and $\{(Z_1(t_j), Z_2(t_j)), j = 1, 2, \dots, n\}$ defined by $\mathcal{Q}(x) = (Q(x, t_1), \dots, Q(x, t_n))$ with

$$Q(x, t_i) := \frac{\sigma^2}{2\lambda} e^{-\lambda|x-t_i|} \begin{bmatrix} \cos(\omega(x-t_i)) & \sin(\omega(x-t_i)) \\ -\sin(\omega(x-t_i)) & \cos(\omega(x-t_i)) \end{bmatrix}$$

and $(\hat{m}_1, \hat{m}_2)^\top$ is the generalized least squares estimator of $(m_1, m_2)^\top$, that is

$$\begin{bmatrix} \hat{m}_1 \\ \hat{m}_2 \end{bmatrix} = \left(H(n)C(n)^{-1}H(n)^\top \right)^{-1} H(n)C(n)^{-1} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

Thus, a natural criterion is to minimize suitable functionals of the mean squared prediction error (MSPE) given by

$$\text{MSPE}(\hat{Z}(x)) := \frac{\sigma^2}{2\lambda} \text{tr} \left[\mathbb{I}_2 - (\mathbb{I}_2, \mathcal{Q}(x)) \begin{bmatrix} \mathbb{O}_2 & \vdots & H_n \\ \vdots & \ddots & \vdots \\ H_n^\top & \vdots & C(n) \end{bmatrix}^{-1} (\mathbb{I}_2, \mathcal{Q}(x))^\top \right] \quad (3.1)$$

where \mathbb{I}_k and $\mathbb{O}_k, k \in \mathbb{N}$, denote the k -dimensional unit and null matrices, respectively. Since the prediction accuracy is often related to the entire prediction region \mathcal{X} , the design criterion IMSPE is given by

$$\text{IMSPE}(\hat{Z}) := \frac{2\lambda}{\sigma^2} \int_{\mathcal{X}} \text{MSPE}(\hat{Z}(x)) dx$$

Instead of an arbitrary interval \mathcal{X} , without loss of generality, we may consider $\mathcal{X} = [0, 1]$. Further, as extrapolative prediction should be treated with caution in kriging, we can set $t_1 = 0$ and $t_n = 1$, which results in a reduction of free parameters and simplifies our formula.

Theorem 3.1. *In our setup,*

$$\text{IMSPE}(\hat{Z}) = 2(1 - A_n + G(n)^{-1}B_n) \quad (3.2)$$

where

$$\begin{aligned} G(n) &= 1 + \sum_{\ell=1}^{n-1} g(d_\ell), \quad \text{with} \quad g(x) := \frac{1 - 2e^{-\lambda x} \cos(\omega x) + e^{-2\lambda x}}{1 - e^{-2\lambda x}}, \quad x > 0, \quad \text{and} \quad g(0) := 0 \\ A_n &= \varrho_{n,n} + \sum_{i=1}^{n-1} \frac{\varrho_{i,i} - 2\pi_i \varrho_{i+1,i} + \pi_i^2 \varrho_{i+1,i+1}}{1 - \pi_i^2} \end{aligned} \quad (3.3)$$

$$B_n = 1 - 2v_{n,n} + \varrho_{n,n} - 2 \sum_{i=1}^{n-1} \frac{(v_{i,i} - \pi_i v_{i+1,i}) - \pi_i (v_{i+1,i} - \pi_i v_{i+1,i+1})}{1 - \pi_i^2} \quad (3.4)$$

$$\begin{aligned}
& + 2 \sum_{i=1}^{n-1} \frac{(\varrho_{n,i} - \pi_i \varrho_{n,i+1}) (\cos(\omega(d_i + \dots + d_{n-1})) - \pi_i \cos(\omega(d_{i+1} + \dots + d_{n-1})))}{1 - \pi_i^2} \\
& + \sum_{i=1}^{n-1} \frac{(\varrho_{i,i} - 2\pi_i \varrho_{i+1,i} + \pi_i^2 \varrho_{i+1,i+1}) (1 - 2\pi_i \cos(\omega d_i) + \pi_i^2)}{(1 - \pi_i^2)^2} \\
& + 2 \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} (\varrho_{i,j} - \pi_i \varrho_{i+1,j} - \pi_j \varrho_{i,j+1} + \pi_i \pi_j \varrho_{i+1,j+1}) \\
& \times \left(\frac{\cos(\omega(d_j + \dots + d_{i-1})) - \pi_i \cos(\omega(d_j + \dots + d_i)) - \pi_j \cos(\omega(d_{j+1} + \dots + d_{i-1}))}{(1 - \pi_i^2)(1 - \pi_j^2)} \right. \\
& \left. + \frac{\pi_i \pi_j \cos(\omega(d_{j+1} + \dots + d_i))}{(1 - \pi_i^2)(1 - \pi_j^2)} \right)
\end{aligned}$$

where for $i \wedge j := \min\{i, j\}$, $i \vee j := \max\{i, j\}$, $i, j \in \mathbb{N}$,

$$\begin{aligned}
\varrho_{i,j} &:= \frac{1}{2\lambda} (2e^{-\lambda(d_j + \dots + d_{i-1})} - e^{-\lambda(2d_1 + \dots + 2d_{j-1} + d_j + \dots + d_{i-1})} - e^{-\lambda(d_j + \dots + d_{i-1} + 2d_i + \dots + 2d_{n-1})}) \\
&\quad + (d_j + \dots + d_{i-1}) e^{-\lambda(d_j + \dots + d_{i-1})}, \quad 1 \leq j \leq i \leq n \\
v_{i,j} &:= \frac{2\lambda}{\lambda^2 + \omega^2} \cos(\omega(d_{i \wedge j} + \dots + d_{i-1 \vee j})) \\
&\quad + \frac{e^{-\lambda(d_1 + \dots + d_{i-1})}}{\lambda^2 + \omega^2} (\omega \sin(\omega(d_1 + \dots + d_{j-1})) - \lambda \cos(\omega(d_1 + \dots + d_{j-1}))) \\
&\quad + \frac{e^{-\lambda(d_i + \dots + d_{n-1})}}{\lambda^2 + \omega^2} (\omega \sin(\omega(d_j + \dots + d_{n-1})) - \lambda \cos(\omega(d_j + \dots + d_{n-1})))
\end{aligned}$$

with the empty sum to be defined as zero, and $\pi_i := \exp(-\lambda d_i)$ with $d_i := t_{i+1} - t_i$, $i = 1, 2, \dots, n-1$.

Example 3.2. As an illustration consider a three-point design, that is $n=3$, $t_1 = 0, t_2 := d, t_3 = 1$. Figure 1a shows the mean squared prediction error (MSPE) function for $\lambda = 1, \omega = 4$ together with the corresponding contour plot (Figure 1b). In Figures 1c and 1d the IMSPE corresponding to the equidistant three-point design for the prediction as function of λ and ω and its contour plot, respectively, are given. Straightforward calculation shows that in this case $d = \frac{1}{2}$ is a minimizer of $\text{IMSPE}(\hat{Z})$ for all possible values of λ and ω , that is the optimal design is equidistant.

Example 3.3. Consider now the four-point design $\{0, d_1, d_1 + d_2, 1\}$. In this case the partial derivatives of $\text{IMSPE}(\hat{Z})$ with respect to d_1 and d_2 at $d_1 = d_2 = 1/3$ are not necessarily zero, that is in general, one cannot state that the equidistant design is optimal.

4. Optimal information gain for complex OU process

Another approach to optimal design is to find locations which maximize the amount of obtained information. Following the ideas of Shewry and Wynn (1987) one has to maximize the entropy $\text{Ent}(Z)$ of the observations corresponding to the chosen design which

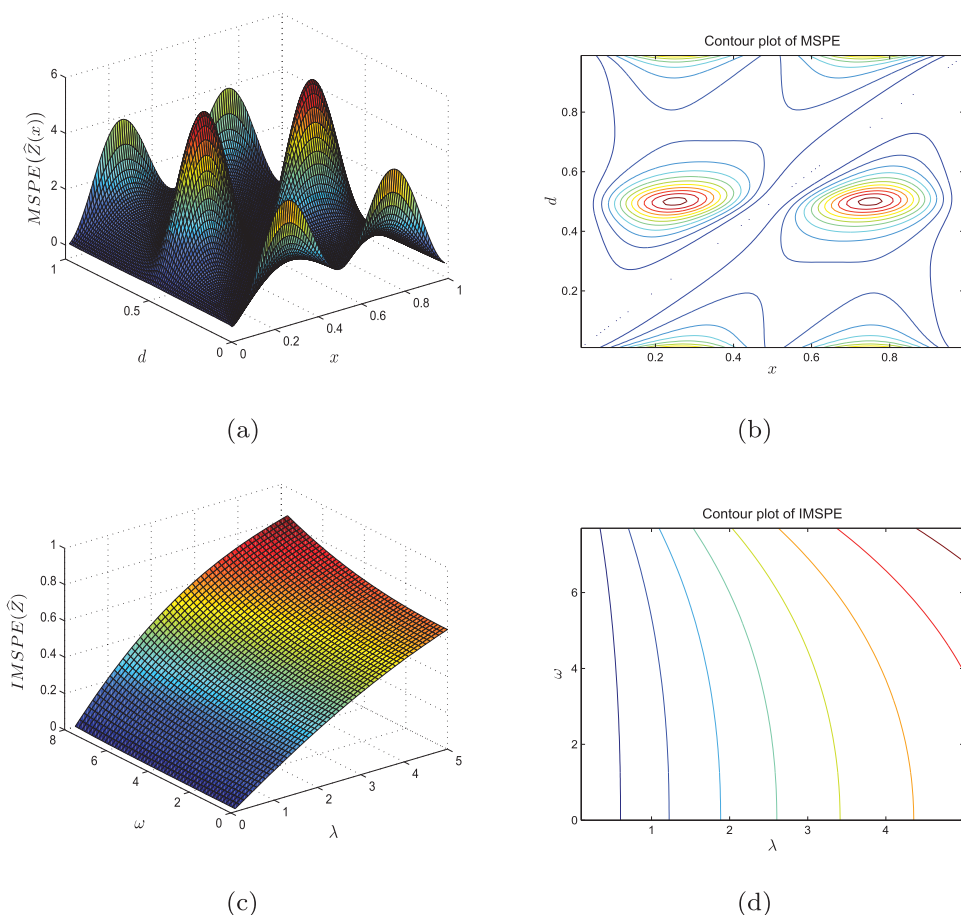


Figure 1. Mean squared prediction error (MSPE) for the three-point design $\{0, d, 1\}$ for $\lambda = 1$, $\omega = 4$ (a) and IMSPE corresponding to the equidistant three-point design for the prediction as function of λ and ω (c), together with the corresponding contour plots (b) and (d), respectively.

in our Gaussian case forms a $2n$ -dimensional normal vector with covariance matrix $\frac{\sigma^2}{2\lambda} C(n)$, that is

$$\text{Ent}(Z) = n \left(1 + \ln \left(\frac{\pi \sigma^2}{\lambda} \right) \right) + \frac{1}{2} \ln \det C(n)$$

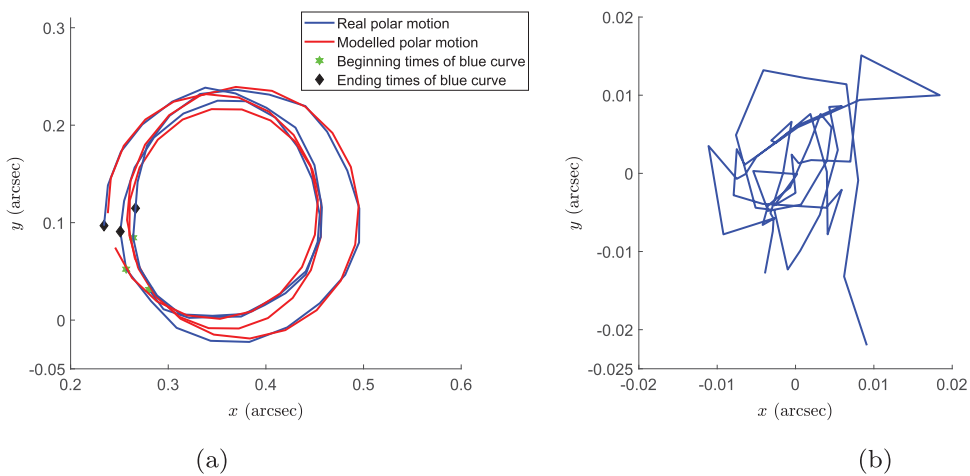
Theorem 4.1. Under conditions of Theorem 3.1 entropy $\text{Ent}(Z)$ has the form

$$\text{Ent}(Z) = n \left(1 + \ln \left(\frac{\pi \sigma^2}{\lambda} \right) \right) + \frac{1}{2} \sum_{i=1}^{n-1} \ln (1 - 2\pi_i^2) \quad (4.1)$$

For any sample size the equidistant design $d_1 = d_2 = \dots = d_{n-1}$ is optimal with respect to the entropy criterion.

Table 1. IMSPE values (in arcsec²) corresponding to the optimal and to the equispaced design and relative efficiency of the equispaced design.

		$\lambda = 2.452, \omega = -4.127$	$\lambda = 4.997, \omega = -0.356$	$\lambda = 4.937, \omega = -5.777$
		(estimates from Y2017)	(estimates from Y2016)	(estimates from Y2015)
$n = 3$	optimal	0.8327	1.3179	1.5010
	equispaced	0.8327	1.3179	1.5010
	rel. eff. (%)	100	100	100
$n = 4$	optimal	1.0740	0.9981	1.3844
	optimal design	(0, 0.344, 0.75, 1)	(0, 0.334, 0.671, 1)	(0, 0.394, 0.794, 1)
	equispaced	1.1284	0.9982	1.4675
$n = 5$	rel. eff. (%)	95.18	99.99	94.34
	optimal	1.0323	0.6389	0.9621
	optimal design	(0, 0.019, 0.196, 0.661, 1)	(0, 0.245, 0.495, 0.739, 1)	(0, 0.206, 0.422, 0.791, 1)
	equispaced	1.4753	0.6395	1.0792
	rel. eff. (%)	69.97	99.91	89.15

**Figure 2.** Real and modeled yearly polar motion (a) for the three-year period 2015–2017 based on data from IERS EOP C01 IAU2000, together with the corresponding plot of deviation of the model from the observation (b).

5. Numerical experiments

In order to compare the performances of the two examined criteria we compare the optimal values of $\text{IMSPE}(\hat{Z})$ calculated using `fmincon` function of Matlab to its values corresponding to the equispaced design which is optimal for the entropy criterion. In Table 1 the values of IMSPE are given for both designs together with the relative efficiency of the equispaced monotonic design with respect to the optimal one for various sample sizes and combinations of damping parameter λ and frequency ω . Observing that for the three-point design ($n = 3$), the efficiency of the equispaced design is 100%, so in this case the equidistant design is optimal. In the other cases it is notable, that the optimal values of $\text{IMSPE}(\hat{Z})$ are better than the optimal values of entropy criterion. The damping parameter λ and frequency ω used for the simulations are estimated based on public pole coordinates from the IERS EOP C01 IAU2000. This database contains a series of the Earth Orientation Parameters given at a 0.05 year time interval (since 1890), and we consider the x and y

coordinates of the deviation of the North Pole (in arcsec). We make yearly estimates of λ and ω based on data from the previous three years. As a first step we estimate the regression model that fits best the real polar motion in the least squares sense, thus we obtain the coordinates of modeled polar motion (\hat{x} and \hat{y} coordinates). Then, according to Kolmogorov's model (1.1) of the CW, deviation of the real and fitted yearly polar motion results in the required complex OU process ($\tilde{x} = x - \hat{x}$ and $\tilde{y} = y - \hat{y}$ coordinates). The maximum likelihood estimators for the parameters λ and ω of the complex OU process are given in Arató (1968) or in Arató (1982, Chapter 4, pp. 221–223). The required integrals of the realizations of the OU process are calculated numerically with a time step of 0.05 year corresponding to the data frequency. As an example, Figure 2 shows the real and modeled yearly polar motion based on data from the three-year period 2015–2017, together with the corresponding deviation modeled as a complex OU process.

6. Conclusions

We derive the exact form of the IMSPE for a shifted complex OU process on a compact interval and show that optimal design for prediction based on IMSPE may well differ from the equidistant one. This is in contrast both to the D-optimal design for estimation (Baran, Szák-Kocsis, and Stehlík 2018) and to the case of the classical real OU process (Baldi Antognini and Zagoraiou 2010), but similar to the optimal design for the prediction of OU sheets on a monotonic set (Baran, Sikolya, and Stehlík 2013). We also investigate the properties of the optimal design with respect to entropy criterion and we show that these optimal designs are equidistant. Simulations illustrate selected cases of optimal designs for small number of sampling locations. The damping parameter λ and frequency parameter ω used for the simulations are estimated based on real data (pole coordinates from the IERS EOP C01 IAU2000), which is a well known application of the complex OU process, namely Kolmogorov's model (1.1) of the Chandler wobble. Since the above discussed designs depend on values of damping and frequency parameters, obtained optimal designs are only locally optimal. Such knowledge may be crucial for experiments to increase efficiency of design in practical setups.

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A. Appendix

A.1 proof of Theorem 3.1

To shorten our formulae, in what follows instead of $Q(x, t_i)$ we are using simply $Q_i, i = 1, 2, \dots, n$. According to the results of Baran, Szák-Kocsis, and Stehlík (2018)

$$C(n) = \begin{bmatrix} \mathbb{I}_2 & \mathbf{e}^{A^\top d_1} & \mathbf{e}^{A^\top(d_1+d_2)} & \mathbf{e}^{A^\top(d_1+d_2+d_3)} & \dots & \dots & \mathbf{e}^{A^\top\left(\sum_{j=1}^{n-1} d_j\right)} \\ \mathbf{e}^{Ad_1} & \mathbb{I}_2 & \mathbf{e}^{A^\top d_2} & \mathbf{e}^{A^\top(d_2+d_3)} & \dots & \dots & \mathbf{e}^{A^\top\left(\sum_{j=2}^{n-1} d_j\right)} \\ \mathbf{e}^{A(d_1+d_2)} & \mathbf{e}^{Ad_2} & \mathbb{I}_2 & \mathbf{e}^{A^\top d_3} & \dots & \dots & \mathbf{e}^{A^\top\left(\sum_{j=3}^{n-1} d_j\right)} \\ \mathbf{e}^{A(d_1+d_2+d_3)} & \mathbf{e}^{A(d_2+d_3)} & \mathbf{e}^{Ad_3} & \mathbb{I}_2 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \mathbf{e}^{A^\top d_{n-1}} \\ \mathbf{e}^{A\left(\sum_{j=1}^{n-1} d_j\right)} & \mathbf{e}^{A\left(\sum_{j=2}^{n-1} d_j\right)} & \mathbf{e}^{A\left(\sum_{j=3}^{n-1} d_j\right)} & \dots & \dots & \mathbf{e}^{Ad_{n-1}} & \mathbb{I}_2 \end{bmatrix}$$

and the inverse of $C(n)$ is given by

$$C^{-1}(n) = \begin{bmatrix} U_1 & -\mathbf{e}^{A^\top d_1} U_1 & 0 & 0 & \dots & \dots & 0 \\ -\mathbf{e}^{Ad_1} U_1 & V_2 & -\mathbf{e}^{A^\top d_2} U_2 & 0 & \dots & \dots & 0 \\ 0 & -\mathbf{e}^{Ad_2} U_2 & V_3 & -\mathbf{e}^{A^\top d_3} U_3 & \dots & \dots & 0 \\ 0 & 0 & -\mathbf{e}^{Ad_3} U_3 & V_4 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & V_{n-1} & -\mathbf{e}^{A^\top d_{n-1}} U_{n-1} \\ 0 & 0 & 0 & \dots & \dots & -\mathbf{e}^{Ad_{n-1}} U_{n-1} & U_{n-1} \end{bmatrix}$$

where $U_k := [\mathbb{I}_2 - \mathbf{e}^{(A+A^\top)d_k}]^{-1}$, $k = 1, 2, \dots, n-1$, and $V_k := U_k + \mathbf{e}^{(A+A^\top)d_{k-1}} U_{k-1}$, $k = 2, 3, \dots, n-1$.

Consider first the $\text{MSPE}(\hat{Z}(x))$ given by (3.1). Short matrix algebraic calculations show

$$\begin{aligned} \begin{bmatrix} \mathbb{O}_2 & \vdots & H_n \\ \vdots & \ddots & \vdots \\ H_n^\top & \vdots & C(n) \end{bmatrix}^{-1} &= \begin{bmatrix} \mathbb{O}_2 & \vdots & \mathbb{O}_{2n \times 2}^\top \\ \vdots & \ddots & \vdots \\ \mathbb{O}_{2n \times 2} & \vdots & C^{-1}(n) \end{bmatrix} - \left(H(n) C(n)^{-1} H(n)^\top \right)^{-1} \\ &\times \begin{bmatrix} \mathbb{I}_2 & \vdots & -H(n) C^{-1}(n) \\ \vdots & \ddots & \vdots \\ -C^{-1}(n) H(n)^\top & \vdots & C^{-1}(n) H(n)^\top H(n) C^{-1}(n) \end{bmatrix} \end{aligned}$$

and according to the results of Baran, Szák-Kocsis, and Stehlík (2018) we have

$$H(n) C(n)^{-1} H(n)^\top = G(n) \mathbb{I}_2 = \left(1 + \sum_{\ell=1}^{n-1} g(d_\ell) \right) \mathbb{I}_2, \quad \text{where} \quad g(x) := \frac{1 - 2e^{-\lambda x} \cos(\omega x) + e^{-2\lambda x}}{1 - e^{-2\lambda x}}$$

$x > 0$, and $g(0) := 0$. We remark that $H(n) C(n)^{-1} H(n)^\top$ is the FIM on parameter vector $(m_1, m_2)^\top$ based on observations $\{(Z_1(t_j), Z_2(t_j)), j = 1, 2, \dots, n\}$. In this way, we obtain

$$\begin{aligned} \text{MSPE}(\hat{Z}(x)) &= \frac{\sigma^2}{2\lambda} \text{tr} [\mathbb{I}_2 - Q(x) C^{-1}(n) Q^\top(x) + G(n)^{-1} (\mathbb{I}_2 - H(n) C^{-1}(n) Q^\top(x) \\ &\quad - Q(x) C^{-1}(n) H(n)^\top + Q(x) C^{-1}(n) H(n)^\top H(n) C^{-1}(n) Q^\top(x))] \\ &= \frac{\sigma^2}{2\lambda} \text{tr} \left[\mathbb{I}_2 - Q(x) C^{-1}(n) Q^\top(x) + G(n)^{-1} (\mathbb{I}_2 - Q(x) C^{-1}(n) H(n)^\top) \right. \\ &\quad \left. \times (\mathbb{I}_2 - Q(x) C^{-1}(n) H(n)^\top)^\top \right] \end{aligned}$$

Examining member by member the above expression

$$\begin{aligned}
 & \text{tr}(\mathcal{Q}(x)C^{-1}(n)\mathcal{Q}^\top(x)) = \\
 &= \text{tr}\left(Q_n Q_n^\top + \sum_{i=1}^{n-1} U_i \left(Q_i Q_i^\top - Q_i e^{A^\top d_i} Q_{i+1}^\top - Q_{i+1} e^{A d_i} Q_i^\top + Q_{i+1} e^{(A+A^\top)d_i} Q_{i+1}^\top\right)\right) \\
 &= 2p_{n,n} + 2 \sum_{i=1}^{n-1} \frac{p_{i,i} - 2\pi_i p_{i+1,i} + \pi_i^2 p_{i+1,i+1}}{1 - \pi_i^2} \\
 & \text{tr}\left(H(n)C^{-1}(n)\mathcal{Q}^\top(x) + \mathcal{Q}(x)C^{-1}(n)H(n)^\top\right) = \\
 &= \text{tr}\left(Q_n + Q_n^\top + \sum_{i=1}^{n-1} (Q_i - Q_{i+1} e^{A d_i}) U_i (\mathbb{I}_2 - e^{A^\top d_i}) + \sum_{i=1}^{n-1} U_i (\mathbb{I}_2 - e^{A d_i}) (Q_i^\top - e^{A^\top d_i} Q_{i+1}^\top)\right) \\
 &= 4q_{n,n} + 4 \sum_{i=1}^{n-1} \frac{(q_{i,i} - \pi_i q_{i+1,i}) - \pi_i (q_{i+1,i} - \pi_i q_{i+1,i+1})}{1 - \pi_i^2} \\
 & \text{tr}(\mathcal{Q}(x)C^{-1}(n)H(n)^\top H(n)C^{-1}(n)\mathcal{Q}^\top(x)) = \\
 &= \text{tr}\left(\left(\sum_{i=1}^{n-1} (Q_i - Q_{i+1} e^{A d_i}) U_i (\mathbb{I}_2 - e^{A^\top d_i}) + Q_n\right) \left(Q_n^\top + \sum_{i=1}^{n-1} U_i (\mathbb{I}_2 - e^{A d_i}) (Q_i^\top - e^{A^\top d_i} Q_{i+1}^\top)\right)\right) \\
 &= 2p_{n,n} + 4 \sum_{i=1}^{n-1} \frac{(p_{n,i} - \pi_i p_{n,i+1}) (\cos(\omega(t_n - t_i)) - \pi_i \cos(\omega(t_n - t_{i+1})))}{1 - \pi_i^2} \\
 &\quad + 2 \sum_{i=1}^{n-1} \frac{(p_{i,i} - 2\pi_i p_{i+1,i} + \pi_i^2 p_{i+1,i+1}) (1 - 2\pi_i \cos(\omega d_i) + \pi_i^2)}{(1 - \pi_i^2)^2} \\
 &\quad + 4 \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} (p_{i,j} - \pi_i p_{i+1,j} - \pi_j p_{i,j+1} + \pi_i \pi_j p_{i+1,j+1}) \\
 &\quad \times \left(\frac{\cos(\omega(t_i - t_j)) - \pi_i \cos(\omega(t_{i+1} - t_j)) - \pi_j \cos(\omega(t_i - t_{j+1})) + \pi_i \pi_j \cos(\omega(t_{i+1} - t_{j+1}))}{(1 - \pi_i^2)(1 - \pi_j^2)} \right)
 \end{aligned}$$

where $\pi_i := \exp(-\lambda d_i)$, with $d_i := t_{i+1} - t_i$, $p_{ij} := \exp(-\lambda(|x - t_i| + |x - t_j|))$, and $q_{i,j} := \exp(-\lambda|x - t_i|) \cos(\omega(x - t_j))$, $i, j = 1, 2, \dots, n-1$. Further, according to the definition of the IMSPE criterion, we can write

$$\text{IMSPE}(\hat{Z}) = 2(1 - A_n + G(n)^{-1} B_n)$$

where

$$\begin{aligned}
 A_n &:= \frac{1}{2} \int_{\mathcal{X}} \text{tr}(\mathcal{Q}^\top(x)C^{-1}(n)\mathcal{Q}(x)) dx \\
 B_n &:= \frac{1}{2} \int_{\mathcal{X}} \text{tr}\left(\mathbb{I}_2 - H(n)C^{-1}(n)\mathcal{Q}(x) - \mathcal{Q}^\top(x)C^{-1}(n)H(n)^\top\right. \\
 &\quad \left.+ \mathcal{Q}^\top(x)C^{-1}(n)H(n)^\top H(n)C^{-1}(n)\mathcal{Q}(x)\right) dx
 \end{aligned}$$

In this way, using that for $i \wedge j := \min\{i, j\}$, $i \vee j := \max\{i, j\}$, $i, j \in \mathbb{N}$, we have

$$\begin{aligned}
 \int_{\mathcal{X}} p_{i,j} dx &= \int_0^1 e^{-\lambda(|x - t_i| + |x - t_j|)} dx \\
 &= \frac{1}{2\lambda} (2e^{-\lambda(t_i - t_j)} - e^{-\lambda(t_i + t_j)} - e^{-\lambda(1 - t_i + 1 - t_j)}) + (t_i - t_j)e^{-\lambda(t_i - t_j)}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\lambda} \left(2e^{-\lambda(d_j + \dots + d_{i-1})} - e^{-\lambda(2d_1 + \dots + 2d_{j-1} + d_j + \dots + d_{i-1})} - e^{-\lambda(d_j + \dots + d_{i-1} + 2d_i + \dots + 2d_{n-1})} \right) \\
&\quad + (d_j + \dots + d_{i-1}) e^{-\lambda(d_j + \dots + d_{i-1})} := q_{i,j}, \quad 1 \leq j \leq i \leq n, \\
\int_{\mathcal{X}} q_{i,j} \, dx &= \int_0^1 e^{-\lambda|x-t_i|} \cos(\omega(x-t_j)) dx = \frac{2\lambda}{\lambda^2 + \omega^2} \cos(\omega(t_i - t_j)) \\
&\quad + \frac{e^{-\lambda t_i}}{\lambda^2 + \omega^2} (\omega \sin(\omega t_j) - \lambda \cos(\omega t_j)) + \frac{e^{-\lambda(1-t_i)}}{\lambda^2 + \omega^2} (\omega \sin(\omega(1-t_j)) - \lambda \cos(\omega(1-t_j))) \\
&= \frac{2\lambda}{\lambda^2 + \omega^2} \cos(\omega(d_{i \wedge j} + \dots + d_{i \vee j} - 1)) \\
&\quad + \frac{e^{-\lambda(d_1 + \dots + d_{i-1})}}{\lambda^2 + \omega^2} (\omega \sin(\omega(d_1 + \dots + d_{j-1})) - \lambda \cos(\omega(d_1 + \dots + d_{j-1}))) \\
&\quad + \frac{e^{-\lambda(d_i + \dots + d_{n-1})}}{\lambda^2 + \omega^2} (\omega \sin(\omega(d_j + \dots + d_{n-1})) - \lambda \cos(\omega(d_j + \dots + d_{n-1}))) := v_{i,j}
\end{aligned}$$

with the empty sum to be defined as zero, short calculation leads to (3.3) and (3.4). \square

A.2. Proof of Theorem 4.1

Following the idea of Baldi Antognini and Zagoraiou (2010, Lemma 3.1) the covariance matrix $C(n)$ can be written as

$$C(n) = LDU$$

where L is a lower, U is an upper block triangular matrix with \mathbb{I}_2 as blocks in the main diagonal and D is block diagonal matrix with

$$\left(\mathbb{I}_2, \mathbb{I}_2 - e^{(A+A^\top)d_1}, \mathbb{I}_2 - e^{(A+A^\top)d_2}, \dots, \mathbb{I}_2 - e^{(A+A^\top)d_{n-1}} \right)$$

as main block matrices. In this way

$$\det C(n) = \det \mathbb{I}_2 \times \det(\mathbb{I}_2 - e^{(A+A^\top)d_1}) \times \dots \times \det(\mathbb{I}_2 - e^{(A+A^\top)d_{n-1}}) = \prod_{i=1}^{n-1} (1 - 2\pi_i^2)$$

which proves (4.1). Now, from (4.1) we have

$$\ln \det C(n) = \sum_{i=1}^{n-1} f(d_i), \quad \text{where } f(x) := \ln(1 - 2e^{-2\lambda x})$$

Since

$$\frac{\partial^2 f(d)}{\partial d^2} = -\frac{8\lambda^2 e^{2\lambda d}}{(e^{2\lambda d} - 2)^2} < 0 \quad \text{for any } d \in (0, 1)$$

$f(x)$ is a concave function of x , and the result follows from Schur-concavity of the entropy criterion. \square