1949

EFFECTIVE RESULTS IN THE THEORY OF DIOPHANTINE EQUATIONS

Egyetemi doktori (PhD) értekezés

Rábai Zsolt

Témavezető: Dr. Pink István

DEBRECENI EGYETEM

Természettudományi és Informatikai Doktori Tanács

Matematika- és Számítástudományok Doktori Iskola

Debrecen, 2020.

Ezen értekezést a Debreceni Egyetem Természettudományi és Informatikai Doktori Tanács Matematika- és Számítástudományok Doktori Iskola *Diofantikus és konstruktív számelmélet* programja keretében készítettem a Debreceni Egyetem természettudományi doktori (PhD) fokozatának elnyerése céljából. Nyilatkozom arról, hogy a tézisekben leírt eredmények nem képezik más PhD disszertáció részét.

Debrecen, 2020. január 6.

Rábai Zsolt
jelölt

Tanúsítom, hogy Rábai Zsolt doktorjelölt 2011-2014 között a fent megnevezett Doktori Iskola *Diofantikus és konstruktív számelmélet* programjának keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Nyilatkozom továbbá arról, hogy a tézisekben leírt eredmények nem képezik más PhD disszertáció részét. Az értekezés elfogadását javasolom.

Debrecen, 2020. január 6.

Dr. Pink István
témavezető

EFFECTIVE RESULTS IN THE THEORY OF DIOPHANTINE EQUATIONS

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében a matematika tudományágban

Írta: Rábai Zsolt okleveles matematikus

Készült a Debreceni Egyetem Matematika- és Számítástudományok Doktori Iskolája (Diofantikus és konstruktív számelmélet programja) keretében

Témavezető: Dr. Pink István

A doktori szigorlati bizottság:

elnök:	Dr.	....
tagok:	Dr.	....
	Dr.	....

A doktori szigorlat időpontja: 2019. február

Az értekezés bírálói:

Dr.	....
Dr.	....
Dr.	....

A bírálóbizottság:

elnök:	Dr.	....
tagok:	Dr.	....
	Dr.	....
	Dr.	....
	Dr.	....

Az értekezés védésének időpontja: 2020.

Köszönetnyilvánítás

Ezúton szeretnék köszönetet mondani mindazoknak, akik közvetlen vagy közvetett módon hozzájárultak a disszertációm elkészüléséhez. Mindenekelőtt témavezetőmnek, Dr. Pink Istvánnak, aki támogatásával és tanácsaival hozzájárult szakmai fejlődésemhez és sikeres munkámhoz. Ugyanígy szeretném megköszönni Győry Kálmán professzor úr segítő megjegyzéseit és támogatását, Gaál István és Szalay László professzoroknak pedig a dolgozatom alapos és kritikus átnézését

Szintén szeretnék köszönetet mondani kollégáimnak az Algebra és Számelmélet tanszéken és Számelméleti kutatócsoportban a sok segítséget és támogatást.

Végül szeretném megköszönni szüleimnek, feleségemnek és barátaimnak a rengeteg támogatást, bizalmat és türelmet

Acknowledgements

To start, the author would like to express his deepest gratitude to his supervisor, Dr. István Pink for his constant support and encouragements. Also, he would like to thank for Professor Kálmán Győry and all colleagues in the Department of Algebra and Number theory for their helpful comments and support over the years. He would also like to thank Professors István Gaál and László Szalay for reading this paper carefully, and for their helpful comments and encouragements.

Finally, the author would like to thank his parents, his wife and friends for their constant support.

Contents

Introduction

In 1900, Hilbert published 23 problems in mathematics. These were all unsolved at that time, and many of them were later very influential for 20th-century mathematics. The tenth of these problems was to provide a general algorithm which for any given Diophantine equation, can decide whether the equation has a solution with all unknowns taking integer values. It was later proved by Matiyasevich, that such algorithm does not exist. This posed the need for methods which can be used to solve large families of Diophantine equations. A major breakthrough was the application Baker's method to give effective finiteness results for several types of equations. However, the bounds obtained with Baker-type arguments were often too high for practical applications. For particular equations reduction methods (such as the result of Baker and Davenport [15]) can be used to determine all solutions.

In our PhD dissertation, we will combine the latest effective methods with our own observations to give effective results for families of diophantine equations and inequalities with interesting number theoretic backgrounds. In all our chapters, we will combine several methods to compute the solutions to these equations. In the introduction, we will focus on one particular method, however the details can be found in the appropriate chapter.

In our first chapter, we will show, how elementary considerations and modular arithmetic can be applied to show, that a certain family of polynomial-exponential diophantine equations have only the trivial solution. Suppose that a, b and c are known positive integer numbers, and consider the exponential diophantine equation

$$a^x + b^y = c^z, \tag{1}$$

in positive integer unknowns x, y and z. The application of Baker's theorem on effective lower bounds on linear forms of logarithms led to many exciting results concerning such equations (see for example [103]). The triple of positive integers (a, b, c) is called a Pythagorean triple, if

$$a^2 + b^2 = c^2.$$

Also, (a, b, c) is called a *primitive Pythagorean triple*, if a, b and c are co-prime.

The study of equation (1) with Pythagorean triples as bases has a long history. In 1955, Sierpiński proved that for the smallest and most famous Pythagorean triple $(a, b, c) = (3, 4, 5)$, the corresponding equation (1) has the unique solution $(x, y, z) = (2, 2, 2)$ (see [105]). Similar results were given by Jeśmanowicz in 1956. He showed that if

$$(a, b, c) \in \{(5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)\},$$

then the only solution of (1) is again $(x, y, z) = (2, 2, 2)$. Based on his results he proposed the following conjecture (also known as *Jeśmanowicz's conjecture*).

Conjecture 1. *Let* (a, b, c) *be a primitive Pythagorean triple such that* $a^2 + b^2 = c^2$. *Then the only solution of* (1) *is* $(x, y, z) = (2, 2, 2)$.

Conjecture 1 and its generalizations have received a great deal of attention over the years, however the problem in its general form is still open. It is well known that for any primitive Pythagorean triple (a, b, c), we can write

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2, \tag{2}$$

where m and n are positive co-prime integers of different parities with $m > n$. In 1959, Lu [67], and in 1965 Dem'janenko [40] proved Conjecture 1 for

$$n = 1; \ \ (a, b, c) = (m^2 - 1,\, 2m,\, m^2 + 1)$$

and

$$n = m - 1; \ \ (a, b, c) = (2m - 1,\, 2m(m - 1),\, 2m^2 - 2m + 1),$$

respectively. Since 1990 a lot of progress has been made towards the proof of Conjecture 1. In 1993, Takakuwa and Asaeda, and Takakuwa (See [112], [114], [113],) proved Conjecture 1 for various infinite families of triples (a, b, c). In several papers between 1995 and 2009 Le ([60], [61], [63]) applied the theory of linear forms in logarithms to give quantitative results, and prove Conjecture 1 for many triples. In 1994, Terai [118] introduced a generalization of Conjecture 1 (known as Terai's conjecture). In the following years he proved it for several special cases (see for example [119], [116], [117]). In the last few years, Miyazaki made many important contributions to this field. He proved both Conjecture 1 and Terai's conjecture for various infinite families of triples (see for example [79], [81]). A comprehensive collection of classical and recent results on Jeśmanovicz' conjecture, and its generalizations can be found in [82].
In our work [98], we will extend a result of Miyazaki [80], and prove a modified version of the Jeśmanowicz conjecture for an infinite number of triplets.

In the second chapter, we will show, how recurrent sequences can be applied to give sharp bounds for the size of the solutions of some hyperelliptic diophantine equations of special shape. We will consider the generalized Ramanujan-Nagell equation

$$x^2 + D = y^n, \tag{3}$$

where $D > 0$ is a given integer and x, y, n are positive integer unknowns with $n \geq 3$. Results obtained for general superelliptic equations clearly provide effective finiteness results for this equation, too (see for example [2], [101], [103], and the references given there).

The first result concerning the above equation was due to V. A. Lebesque [64] who proved that there are no solutions for $D = 1$. Ljunggren [65] solved (3) for $D = 2$, and Nagell [89], [91] solved it for $D = 3, 4$ and 5. In his elegant paper [34], Cohn gave a fine summary of the earlier results on equation (3). Further, he developed a method by which he found all solutions of the above equation for 77 positive values of $D \leq 100$. For $D = 74$ and $D = 86$, equation (3) was solved by Mignotte and de Weger [78]. By using the theory of Galois representations and modular forms Bennett and Skinner [25] solved (3) for $D = 55$ and $D = 95$. On combining the theory of linear forms in logarithms with Bennett and Skinner's method and with several additional ideas, Bugeaud, Mignotte and Siksek [126] gave all the solutions of (3) for the remaining 19 values of $D \leq 100$.

Let $S = \{p_1, \ldots, p_s\}$ denote a set of distinct primes and $\mathbf{S}$ the set of non-zero integers composed only of primes from S. Put $P := \max\{p_1, \ldots, p_s\}$ and denote by Q the product of the primes of S. In recent years, equation (3) has been considered also in the more general case when D is no longer fixed but $D \in \mathbf{S}$ with $D > 0$. It follows from Theorem 2 of [111] that in (3) n can be bounded from above by an effectively computable constant depending only on P and s. In [54] an effective upper bound was derived for n which depends only on Q. Cohn [33] showed that if $D = 2^{2k+1}$ then equation (3) has solutions only when $n = 3$ and in this case there are three families of solutions. The case $D = 2^{2k}$ were considered by Arif and Abu Muriefah [4]. They conjectured that the only solutions are given by $(x, y) = (2^k, 2^{2k+1})$ and $(x, y) = (11 \cdot 2^{k-1}, 5 \cdot 2^{2(k-1)/3})$, with the latter solution existing only when $(k, n) = (3M + 1, 3)$ for some integer $M \geq 0$. Partial results towards this conjecture were obtained in [4] and [35] and it was finally proved by Arif and Abu Muriefah [7]. Arif and Abu Muriefah [5] proved that if $D = 3^{2k+1}$

then (3) has exactly one infinite family of solutions. The case $D = 3^{2k}$ has been solved by Luca [68] under the additional hypothesis that x and y are coprime. In fact in [69] Luca solved completely equation (3) if $D = 2^a 3^b$ and $\gcd(x, y) = 1$. Abu Muriefah [86] established that equation (3) with $D = 5^{2k}$ may have a solution only if 5 divides x and p does not divide k for any odd prime p dividing n. The case $D = 2^a 3^b 5^c 7^d$ with $\gcd(x, y) = 1$, where a, b, c, d are non-negative integers was studied by Pink [93]. The cases when $D = 7^{2k}$ and $D = 2^a 5^b$ were also considered by Luca and Togbe [70], [71]. For the case $D = 2^a 5^b 13^c$, see Goins, Luca and Togbe [45], while if $D = 5^a 13^b$, see [48]. The cases $D = 2^a 11^b$ and $D = 5^a 11^b$ have been recently considered in [88] and [52], respectively. Let $p \geq 5$ be an odd prime with $p \not\equiv 7 \pmod 8$. Arif and Abu Muriefah [8] determined all solutions of the equation $x^2 + p^{2k+1} = y^n$, where $\gcd(n, 3h_0) = 1$ and $n \geq 3$. Here h_0 denotes the class number of the field $\mathbb{Q}(\sqrt{-p})$. They also obtained partial results [6] if $D = p^{2k}$, where p is an odd prime. In the particular case when $\gcd(x, y) = 1$, $D = p^2$, p prime with $3 \leq p < 100$, Le [62] gave all the solutions of equation (3). The case $D = p^{2k}$ with $2 \leq p < 100$ prime and $\gcd(x, y) = 1$ was considered by Bérczes and Pink [30]. If in (3) $D = a^2$ with $3 \leq a \leq 501$ and a is odd then Tengely [115] solved completely equation (3) under the assumption $(x, y) \in \mathbb{N}^2, \gcd(x, y) = 1$. The equation $A^4 + B^2 = C^n$ for $AB \neq 0$ and $n \geq 4$ was completely solved by Bennett, Ellenberg and Nathan [73] (see also Ellenberg [46]). For more related results concerning equation (3) see [99], [100] and the references given there. For a survey concerning equation (3) see [125].
In our work [94], we gave all solutions for (3) with $D = 5^k 17^l$ with k and l being non-negative integers.

In the third chapter we will show how to apply the combination of Baker's method with approximation techniques to give bounds for the number of solutions of a family of parametric Thue inequalities, and to completely solve a sub-family of such equations. A classical problem in number theory is the approximation of algebraic numbers by rationals, underlying which one has a theorem of Liouville:

Theorem 1. *(Liouville, 1844) If α is a given algebraic number of degree $n \geq 2$, then there exists a constant $c(\alpha)$ such that, for every $\frac{x}{y} \in \mathbb{Q}$ with $y > 0$, we have*

$$\left|\alpha - \frac{x}{y}\right| > \frac{c(\alpha)}{y^n}.$$

For applications to Diophantine equations, it is of utmost importance to reduce the exponent n here, i.e. to deduce like inequalities with some exponent $\lambda < n$. In full generality, the first such result was due to Thue [120] who proved the following theorem.

Theorem 2. *(Thue, 1909) If α is an algebraic number of degree $n \geq 3$, then, given $\varepsilon > 0$, there exists a constant $c(\alpha, \varepsilon)$ such that for all integers x and $y > 0$ we have*

$$\left|\alpha - \frac{x}{y}\right| > \frac{c(\alpha, \varepsilon)}{y^{\frac{n}{2}+1+\varepsilon}}.$$

From this result, Thue deduced that if $F(x, y) \in \mathbb{Z}[x, y]$ is an irreducible binary form of degree $n \geq 3$, and m is a fixed nonzero integer then the corresponding *Thue equation*

$$F(x, y) = m \tag{4}$$

has at most finitely many solutions in integers x and y. This result is, however, ineffective in the sense that it does not provide any way to actually compute $c(\alpha, \varepsilon)$, and hence cannot be applied to determine the solutions of the corresponding equations.

Whilst there is now a well-developed literature on effective solution of Thue equations, based upon a variety of techniques (including, for instance, lower bounds for linear forms in logarithms of algebraic numbers; see e.g. [13]), in our work, we concentrated on bounding the number of solutions to such equations, rather than their heights. In this regard, it is known that the number of solutions to equation (4) in integers is bounded above in terms of only the degree of F and the number of distinct prime divisors of m (see e.g. Bombieri and Schmidt [27]). We will restrict our attention to what is, in some sense, the simplest possible case, that of binomial Thue equations and inequalities. For these equations, the number of such solutions is bounded in terms of m alone (see Mueller and Schmidt [85]). In particular, we will consider equations of the form

$$|ax^n - by^n| = c, \tag{5}$$

where a, b and c are given positive integers, and x, y and n are unknown integers. Siegel [104], refining earlier work of Thue, showed that if the coefficients a and b are large enough compared to c and n, then (5) has at most one positive solution. Later, Evertse [47] was able to substantially sharpen Siegel's theorem (see our Lemma 12). Both results depend on the so-called hypergeometric method. Related work in this area, including applications and generalizations to cases where a and b are taken to be S-units rather than fixed, may be found in, for example, Mahler [75], [76], Baker [11], [10], [12], Chudnovsky [32] and many, many other papers, including [1]. [16], [17], [18], [19], [20], [21], [74], [28], [29], [56], [55], [49], [50], [51], [77] and [121]. In our work [72], we will extend a result of Bennett and De Weger [23] and Bennett [20], and prove that except for some triples (a, b, n), with $c \leq 3$, (5) has only the trivial solution.

In the final chapter, we will discuss how the theory of elliptic logarithms can be applied to solve certain genus 1 equations. Let m be a fixed integer with $m \geq 3$. Then the number

$$\mathrm{Pyr}_m(x) = \frac{x(x+1)((m-2)x + 5 - m)}{6} \tag{6}$$

is called the pyramidal number with parameters m and x. Interesting aspects of pyramidal numbers are the binomial coefficients $\mathrm{Pyr}_3(x) = \binom{x}{3}$ with integers

$x \geq 3$, and the successive partial sum of the series of triangular numbers. According to Dickson [44], the first mention of pyramidal numbers dates back to the ancient Greece. For detailed historical background, please refer to [44]. Pyramidal numbers and their generalizations, figurate numbers, play an important role in discrete mathematics and number theory. (For a detailed introduction into figurate numbers, consult [43].) The diophantine and arithmetic properties of pyramidal and figurate numbers have been widely investigated over the years. Dickson [44] proved, that every sufficiently large integer is the sum of eight pyramidal numbers. Numerical results due to Richmond [97] and Deng and Yang [41] make it is plausible that the result of Dickson can be improved.

There are also several classical results related to the equal values of pyramidal and other combinatorial numbers. In 1962, Segal [102] proved, that 10 is the only pyramidal number whose double is also a pyramidal number. In 1998, Brindza, Pintér and Turjányi [9] investigated the equal values of pyramidal and polygonal numbers. They considered the equation

$$\mathrm{Poly}_m(x) = \mathrm{Pyr}_n(y),$$

where $\mathrm{Poly}_m(x)$ denotes the sequence of polygonal numbers (for details please refer to [9]) and proved that for all but a finite, computable set of pairs (m, n), $\max(x, y)$ is effectively bounded. In 2012, Dujella, Győry and Pintér [3] studied the power values of pyramidal numbers. Recently, in two papers Pintér and Varga [87] and Hajdu, Tengely, Pintér and Varga [58] used various effective methods to investigate the equal values of general figurate numbers. In our work [57], we consider the equation

$$\mathrm{Pyr}_m(u) = \mathrm{Pyr}_n(v)$$

for given positive integers m and n in positive integer unknowns u and v. We give an effective upper bound for the size of the solutions u and v, and also present a method to solve the equation completely for given m and n.

Our dissertation is based on the results mentioned in articles [98], [94], [72] and [57].

Chapter 1

The shuffle variant of Jeśmanovicz' conjecture

In this section, we will combine elementary methods and modular arithmetic to show that a certain family of polynomial-exponential equations have no solution. Recall that if a, b and c are known positive integer numbers and (a, b, c) is a primitive Pythagorean triple, than it was conjectured by Jeśmanowicz that the only solution of the equation

$$a^x + b^y = c^z, \tag{1.1}$$

is $(x, y, z) = (2, 2, 2)$ in unknown integers x, y and z (See Conjecture 1 in the Introduction). We will state several results of Miyazaki which will play an important role in the section.

For any positive integer N, denote by rad(N) the radical of N (i.e. the product of the distinct prime divisors of N), and ord$_2(N)$ the 2-order of N (i.e. the largest non-negative integer k, such that $2^k|N$). In their recent papers, Miyazaki [83] and Miyazaki, Yuan and Wu [110] proved (among others) the following theorems.

Theorem 3. *If $c \equiv 1 \pmod{b}$, then Conjecture 1 is true.*

Theorem 4. *Let b_0 be a divisor of b, such that b_0 is divisible by $\mathrm{rad}(b)$. Suppose that Conjecture 1 is true for*

$$c \equiv 1 \pmod{b_0}.$$

Then Conjecture 1 is true for all $c \equiv 1 \pmod{b_0/2}$.

Theorem 5. *If $c \equiv 1 \pmod{b/2^{\mathrm{ord}_2(b)}}$, then Conjecture 1 is true.*

Note that here b is always even thus Theorem 5 is an improvement of Theorem 3. It was noted by Miyazaki in [80] that, if (a, b, c) is a primitive Pythagorean triple and $c = b + 1$, then

$$c + b = a^2.$$

From this, he proposed the following problem. Let (a, b, c) be a given primitive Pythagorean triple such that $a^2 + b^2 = c^2$, and consider the equation

$$c^x + b^y = a^z \tag{1.2}$$

in positive integer unknowns x, y and z.

Conjecture 2. *With the above conditions, equation* (1.2) *has the only solution* $(x, y, z) = (1, 1, 2)$ *if $c = b + 1$. If $c > b + 1$ then* (1.2) *has no solutions.*

This is referred to as the *shuffle* variant of Jeśmanovicz' problem. In [80], Miyazaki proved that Conjecture 2 is true if $c \equiv 1 \pmod{b}$. This result is stated as the following lemma.

Lemma 1. *If $c \equiv 1 \pmod{b}$, then Conjecture 2 is true.*

In June 2014, during a visit to Hungary, Miyazaki proposed the following problem. Is it possible to give a generalization of Lemma 1, similar to the way Theorem 5 generalizes Theorem 3? In our current chapter, we give a positive answer to this question.

1.1 Results

Consider the equation

$$c^x + b^y = a^z \tag{1.3}$$

in positive integer unknowns x, y and z. Our main results are the following.

Theorem 6. *Let b_0 be a divisor of b, such that b_0 is divisible by $\mathrm{rad}(b)$. Suppose that Conjecture 2 is true for all Pythagorean triples (a, b, c) with*

$$c \equiv 1 \pmod{b_0}. \tag{1.4}$$

Then Conjecture 2 is true for all Pythagorean triples (a, b, c) with

$$c \equiv 1 \pmod{b_0/2}. \tag{1.5}$$

Theorem 7. *Conjecture 2 is true for all Pythagorean triples (a, b, c) with*

$$c \equiv 1 \pmod{b/2^{\mathrm{ord}_2(b)}}.$$

Combining Lemma 1 and Theorem 6, it is easy to verify Theroem 7. We will give a proof of Theorem 6 in sections 2 and 3. In the last section, we will report about numerical results concerning (1.3) about cases, that are not covered by Lemma 1 and Theorem 7, giving some further evidence for Conjecture 2.

1.2 Preliminaries and auxiliary results

By (2), we can rewrite (1.3) into the form

$$(m^2 + n^2)^x + (2mn)^y = (m^2 - n^2)^z, \tag{1.6}$$

where m and n are given co-prime positive integers of different parities with $m > n$, and x, y and z are unknown positive integers.

Our proof of Theorem 6 will closely follow the work of Miyazaki, Yuan and Wu in [110]. We start with several auxiliary results and general observations. In the proof, the parities of the exponents x, y and z will play a crucial rule. Thus first we give some preliminary remarks about the exponents. The following notation was previously established by Miyazaki in [83]. By Lemma 1, we may suppose that in (1.3) $c \neq b+1$ and $n \neq 1$. Define integers α, β and e with $\alpha \geq 1$, $\beta \geq 2$ and $e = \pm 1$ and odd positive integers i and j as follows:

$$\begin{aligned} &m = 2^{\alpha} i, && n = 2^{\beta} j + e && \text{if } m \text{ is even,} \\ &m = 2^{\beta} j + e, && n = 2^{\alpha} i && \text{if } m \text{ is odd.} \end{aligned} \tag{1.7}$$

Now, assume that Conjecture 2 holds with (1.4), and suppose that it does not hold for (1.5) (or in other words (1.3) has a solution with (1.5)). We will show that this will result in a contradiction. Again it is clear that both b and b_0 are even. By (1.5), we have $c \equiv 1 \pmod{b_0/2}$ that is $c = 1 + t \cdot b_0/2$ for some positive integer t. Since $b_0/2$ is a divisor of $b/2 = mn$, we can write

$$b_0/2 = m_0 n_0,$$

where $\gcd(m_0, n_0) = 1$, $m_0 | m$ and $n_0 | n$. Moreover, m_0 and n_0 are uniquely determined. Since $c = m^2 + n^2$ we have

$$m^2 + n^2 = 1 + m_0 n_0 t. \tag{1.8}$$

If $2 || b_0$, then $b_0/2$ is odd. However, since $c = m^2 + n^2$ is odd, we have that t is even. Thus we have $c = 1 + (t/2) b_0$, which means that $c \equiv 1 \pmod{b_0}$, for which Conjecture 2 is true by assumption. Thus, in what follows, we can assume that $4 | b_0$. We may also assume that t is odd, else we have again $c = 1 + (t/2) b_0$, for which Conjecture 2 is true. Then (1.8) implies that m_0 or n_0 is even, and

$$\text{rad}(m_0) = \text{rad}(m), \qquad \text{rad}(n_0) = \text{rad}(n).$$

From (1.8), we have that

$$m^2 \equiv 1 \pmod{n_0}, \qquad n^2 \equiv 1 \pmod{m_0}. \tag{1.9}$$

Next, we present some lemmas, which will be used in the proof.

Lemma 2. *With the above notation, we have*

$$c - 1 \equiv 0 \pmod{2^{\min(2\alpha,\beta+1)}} \tag{1.10}$$

and

$$\begin{aligned} a - 1 &\equiv 0 \pmod{2^{\min(2\alpha,\beta+1)}}, \quad \textit{if } m \textit{ is odd}, \\ a + 1 &\equiv 0 \pmod{2^{\min(2\alpha,\beta+1)}}, \quad \textit{if } m \textit{ is even}. \end{aligned} \tag{1.11}$$

Proof. This lemma can be proven similarly to Lemma 4 in [110], by simply substituting (1.7) into (1.6).

□

Lemma 3. *With the above notations, we have* $2\alpha \neq \beta + 1$. *Moreover, we have* $\alpha \geq \beta + 1$.

Proof. By Lemma 2, and (1.8), we have

$$\min(2\alpha, \beta + 1) \leq \text{ord}_2(c - 1) = \text{ord}_2(m_0 n_0 t) \leq \text{ord}_2(mn) = \alpha.$$

This implies our lemma.

□

Lemma 4. *Let* $d > 1$ *and let* u, v *be non-zero co-prime integers. Let* p *be a prime factor of* $u - v$. *If* p *is odd, or* $p = 2$ *and* 4 *divides* $u - v$, *then*

$$\text{ord}_p(u^d - v^d) = \text{ord}_p(u - v) + \text{ord}_p(d).$$

Proof. See for example on p. 11 in [95].

□

The next lemma is similar to Lemma 3.1 in [80]. However, we prove it in detail, because we want to emphasize a somewhat different conclusion. We will use this alternate statement to avoid Baker's method during the proof of Theorem 6.

Lemma 5. *Assume that $\alpha > 1$, $\alpha \neq \beta$ and $2\alpha \neq \beta + 1$. Let (x, y, z) be a solution of* (1.6)*. Then both x and z are even.*

Proof. Set

$$M = \begin{cases} 4, & \text{if } m \text{ is even,} \\ m_0, & \text{if } m \text{ is odd.} \end{cases} \tag{1.12}$$

It is clear that $M \geq 3$. Taking (1.6) modulo M and using (1.9), we see that

$$1 \equiv (-1)^z \pmod{M}.$$

Since $M \geq 3$, we conclude that z is even. Now, assume that x is odd and m is even. Then from (1.6) we have

$$(2mn)^y \equiv -m^2(zn^{2z-2} + xn^{2x-2}) + n^{2z} - n^{2x} \pmod{2^{2\alpha+1}}.$$

Write

$$A = -m^2(zn^{2z-2} + xn^{2x-2}), \qquad B = n^{2z} - n^{2x}.$$

Since x is odd, $zn^{2z-2} + xn^{2x-2}$ is odd, thus by Lemma 4

$$\begin{aligned} \operatorname{ord}_2(A) &= \operatorname{ord}_2(m^2) = 2\alpha, \\ \operatorname{ord}_2(B) &= \operatorname{ord}_2(n^{2|x-z|} - 1) = \operatorname{ord}_2(n^2 - 1) = \beta + 1. \end{aligned}$$

Since $\operatorname{ord}_2((2mn)^y) = (\alpha + 1)y$, and $2\alpha \neq \beta + 1$, we have

$$(\alpha + 1)y = \begin{cases} 2\alpha & \text{if } \ 2\alpha < \beta + 1 \\ \beta + 1 & \text{if } \ 2\alpha > \beta + 1 \end{cases}$$

which means that either $\alpha = 1$ and $y = 1$ or $\alpha = \beta$ and $y = 1$ holds. The case, where m is odd can be treated similarly.

$\square$

Lemma 6. *Assume that* $2\alpha \neq \beta + 1$. *Let* (x, y, z) *be a solution of* (1.6). *If* $y > 1$ *and* x *and* z *are even, then* $X \equiv Z \pmod 2$*, where* $x = 2X$ *and* $z = 2Z$ *for some* $X, Z \geq 1$.

Proof. See Lemma 3.1 and Lemma 3.2 in [80].

$\square$

1.3 Proof of Theorem 6

We are now ready to prove Theorem 6. It follows from Lemmas 3 and 5 that both x and z are even. So, we can write $x = 2X$, $z = 2Z$ with integers $X, Z > 1$, and

$$(2mn)^y = D \cdot E$$

with

$$D = (m^2 - n^2)^Z + (m^2 + n^2)^X, \;\; E = (m^2 - n^2)^Z - (m^2 + n^2)^X.$$

Now, if $y = 1$, then

$$(m - n)^2 = m^2 + n^2 - 2mn \leq (m^2 + n^2)^X - 2mn = \frac{D - E}{2} - DE \leq 0,$$

which is a contradiction, since $m \neq n$. Thus, in what follows, we can assume that $y > 1$ holds.

By Lemma 6, we have

$$X \equiv Z \pmod 2.$$

Suppose that X and Z are both even. Then the congruences

$$D \equiv 2 \pmod 4, \;\; D \equiv 2 \pmod{m_0}, \;\; D \equiv 2 \pmod{n_0}$$

are obtained by (1.9). These imply that $D/2$ is odd, and co-prime to m_0n_0, thus to mn. Therefore we get $D = 2$ which is impossible. Hence both X and Z are odd. Then we compute

$$(D, E) \equiv \begin{cases} (0, 2) & \pmod 4 \quad \text{if } m \text{ is even,} \\ (2, 0) & \pmod 4 \quad \text{if } m \text{ is odd,} \end{cases}$$

and

$$D \equiv 2 \pmod{n_0}, \quad E \equiv -2 \pmod{n_0}$$

which yield the equality

$$(D, E) = \begin{cases} (2^{y-1}m^y, 2n^y) & \text{if } m \text{ is even,} \\ (2m^y, 2^{y-1}n^y) & \text{if } m \text{ is odd.} \end{cases}$$

Now, we discuss the two cases separately.

The case that m is even;

If m is even, then we have

$$\frac{D - E}{2} = 2^{y-2}m^y - n^y = (1 + m_0n_0t)^X.$$

Reducing both sides modulo m_0, we get

$$n^y \equiv -1 \pmod{m_0}.$$

If y is even, then

$$-1 \equiv n^y \equiv \left(n^2\right)^{y/2} \equiv 1 \pmod{m_0},$$

which is a contradiction, if $m_0 \geq 3$. Thus, either y is odd, or $m_0 = 2$. In both cases we have $n \equiv -1 \pmod{m_0}$. However, using this we get

$$\begin{aligned} \operatorname{ord}_2(m_0) &\leq \operatorname{ord}_2(n + 1) < \operatorname{ord}_2(n^2 - 1) \\ &= \operatorname{ord}_2(-m^2 + m_0n_0t) = \operatorname{ord}_2(m_0) + \operatorname{ord}_2(-m^2/m_0 + n_0t) = \operatorname{ord}_2(m_0), \end{aligned}$$

which is a contradiction. Thus, neither of the above cases are possible.

The case that m is odd;

Proceeding in a similar way, we get

$$m^y - 2^{y-2}n^y = (1 + m_0 n_0 t)^X,$$

which yields

$$m^y \equiv 1 \pmod{n_0}.$$

Suppose now that y is odd. Then $m \equiv 1 \pmod{n_0}$. This yields a contradiction as in the previous case by estimating $\mathrm{ord}_2(n_0)$. Thus, we now have that m is odd, and $y = 2Y$, with some integer Y. We complete the proof of Theorem 6 by proving the following proposition.

Proposition 1. *Let m and n be co-prime positive integers with n even, m odd and $m > n$. Then the system of equations*

$$\begin{cases} (m^2 - n^2)^Z + (m^2 + n^2)^X = 2m^{2Y}, \\ (m^2 - n^2)^Z - (m^2 + n^2)^X = 2^{2Y-1}n^{2Y} \end{cases} \tag{1.13}$$

has no solution in positive integers X, Y and Z.

Proof. Note that the equations are equivalent to

$$\begin{cases} (m^2 - n^2)^Z = m^{2Y} + 2^{2Y-2}n^{2Y}, \\ (m^2 + n^2)^X = m^{2Y} - 2^{2Y-2}n^{2Y}, \end{cases} \tag{1.14}$$

simultaneously. Assume that there are positive integer solutions X, Y and Z. First we shall show

$$1 < X < Y.$$

Indeed, the inequality $X < Y$ is obtained by

$$m^{2X} < (m^2 + n^2)^X = m^{2Y} - 2^{2Y-2}n^{2Y} < m^{2Y}.$$

Further, if $X = 1$, then $Y \geq 2$ and

$$m^2 + n^2 = m^{2Y} - 2^{2Y-2} n^{2Y} \geq m^Y + 2^{Y-1} n^Y \geq m^2 + 2n^2$$

that is impossible. Next we claim that

$$n \equiv 0 \pmod 4.$$

If not, then we have $\pm n^2 \equiv 4 \pmod 8$ and

$$5^X \equiv 5^Z \equiv 1 + 2^{2Y-2} 4^Y = 1 + 4^{2Y-1} \equiv 1 \pmod 8.$$

Therefore both X and Z are even. Multiplying the left and right hand sides of (1.13) respectively, we get a solution of the equation $S^4 - T^4 = U^2$. But it is well-known that this has no non-trivial solutions, and the congruence $n \equiv 0 \pmod 4$ has been shown. Now, from the second equation of (1.14), we get

$$(m^2 + n^2)^X = m^{2Y} - 2^{2Y-2} n^{2Y} = (m^Y + 2^{Y-1} n^Y)(m^Y - 2^{Y-1} n^Y).$$

Since $\gcd(m^Y + 2^{Y-1} n^Y, m^Y - 2^{Y-1} n^Y) = 1$, there are co-prime positive integers s, t satisfying

$$st = m^2 + n^2, \;\; s^X = m^Y + 2^{Y-1} n^Y, \;\; t^X = m^Y - 2^{Y-1} n^Y.$$

Note that $X > 1$ and $s - t \equiv 0 \pmod 4$. Thus we can apply Lemma 4 so that

$$\text{ord}_2(s-t) + \text{ord}_2(X) = \text{ord}_2((2n)^Y) = (1 + \text{ord}_2(n))Y \geq 3Y,$$

by $n \equiv 0 \pmod 4$, while we can confirm that $\text{ord}_2(X) < Y$, using $X < Y < 2^Y$. Then we get $\text{ord}_2(s-t) > 2Y$, in particluar,

$$2^{2Y} \leq s - t < st = m^2 + n^2.$$

On the other hand, since $n^2 \equiv -m^2 \pmod{m^2 + n^2}$, we have from (1.14) again,

$$0 \equiv m^{2Y} - 2^{2Y-2} n^{2Y} \equiv (1 \pm 2^{2Y-2}) m^{2Y} \pmod{m^2 + n^2}.$$

Then it follows from $\gcd(m, m^2+n^2) = 1$ that $2^{2Y-2} \pm 1$ is divisible by m^2+n^2. Note that $2^{2Y-2} - 1 > 0$, since $Y > X \geq 2$. Hence

$$m^2 + n^2 \leq 2^{2Y-2} \pm 1 < 2^{2Y},$$

which is inconsistent with the inequality shown above. This completes the proof of Proposition 1, and thus the proof of Theorem 6.

□

1.4 Examples

In this section we show how to utilize Lemma 1 and Theorem 7 combined with some elementary calculation to prove Conjecture 2 for a finite set of triples. For this purpose we will consider all primitive Pythagorean triples (a, b, c) for which

$$a^2 + b^2 = c^2 \tag{1.15}$$

and

$$5 \leq c \leq 100, \tag{1.16}$$

and prove the following proposition.

Proposition 2. *If (a, b, c) is a primitive Pythagorean triple with $a^2 + b^2 = c^2$ and $5 \leq c \leq 100$, then Conjecture 2 is true.*

Proof. Altogether there are sixteen triples with (1.15) and (1.16), ten of these are covered by either Lemma 1 or Theorem 7. The remaining six cases are

$(a, b, c) \in \{(21, 20, 29), (45, 28, 53)(33, 56, 65), (39, 80, 89), (77, 36, 85), (65, 72, 97)\}.$

Since the bases are thus fixed in (1.3), it is possible to use the classical theory of S-unit equations. However we will apply here a more recent approach based on a paper of Bertók and Hajdu [26]. In this paper the authors use basic search for small solutions and modular arithmetic to give very good upper bounds for the size of the solutions, and also provide a program code written in SAGE to do the calculations. Consider first the triple $(a, b, c) = (21, 20, 29)$. This gives us the equation

$$29^x + 20^y = 21^z, \tag{1.17}$$

where x, y and z are positive unknown integers. Since $(x, y, z) = (0, 1, 1)$ is a solution of (1.17), it is impossible to find a suitable integer M, such that the congruence

$$29^x + 20^y \equiv 21^z \pmod{M}$$

is not solvable. However using the program of Bertók and Hajdu we get that if we choose

$$M = 3^2 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 97 \cdot 109 \cdot 163 \cdot 193 \cdot 257 \cdot 433 \cdot 487 \cdot 577 \cdot 769,$$

then the congruence

$$29^x + 20^y \equiv 21^2 \cdot 21^{z_0} \pmod{M}$$

is not solvable for any non-negative integers x, y and z_0. Thus in (1.17) we have that $z \leq 1$, that is

$$29^x + 20^y = 21,$$

which has no solutions in positive integers (and the obvious solution $(x, y, z) = (0, 1, 1)$ in non-negative integers). The remaining five cases do not possess trivial solution, and can be dealt with similarly. We omit the details, and only list the results in the following table.

(a, b, c)	Modulus	Result
$(45, 28, 53)$	$13 \cdot 19 \cdot 37 \cdot 73 \cdot 109$	No solutions
$(33, 56, 65)$	$17 \cdot 19 \cdot 37 \cdot 73$	No solutions
$(39, 80, 89)$	$3^2 \cdot 7 \cdot 13^2$	No solutions
$(77, 36, 85)$	$13 \cdot 19 \cdot 37 \cdot 73$	No solutions
$(65, 72, 97)$	$17 \cdot 19 \cdot 37 \cdot 73 \cdot 577$	No solutions

Thus we covered all the six cases, proving Proposition 2.

$\square$

Chapter 2

The generalized Ramanujan-Nagell Equation

In this chapter, consider the equation

$$x^2 + 5^k 17^l = y^n \tag{2.1}$$

in integer unknowns x, y, k, l, n satisfying

$$x \geq 1, y > 1, n \geq 3,\ k \geq 0,\ l \geq 0 \text{ and } \gcd(x, y) = 1. \tag{2.2}$$

We will combine a deep result of Bilu, Hanrot and Voutier [124] with Ljunggren-type and Elliptic equations to compute all solutions of (2.1).

2.1 Results

Our main result is the following.

Theorem 8. *Consider equation* (2.1) *satisfying* (2.2). *Then all solutions of equation (2.1) are:*

$$(x, y, k, l, n) \in \{(94, 21, 2, 1, 3), (2034, 161, 3, 2, 3), (8, 3, 0, 1, 4)\}.$$

Remark 1. We may assume without loss of generality that in (2.1) $n \geq 5$ prime or $n \in \{3, 4\}$. The proof of our Theorem 8 is organized as follows. If $n \geq 5$ prime we combine some results concerning the general properties of Lucas-sequences with a deep , result of Bilu, Hanrot and Voutier [124] concerning the existence of primitive prime divisors in Lucas-sequences to derive a sharp upper bound for n (see also Pink [93], Theorem 2).
If $n \in \{3, 4\}$ there is a general method for giving all solutions of equations of the form $x^2 + p^k q^l = y^n$. Namely the problem is reduced to finding S-integral points on several elliptic curves, where $S = \{p, q\}$. This works well, but in some cases the computation of the rank and the Mordell-Weil group becomes very time consuming so we need another approach. By using the parametrization provided by a theorem of Cohn (see Lemma 7) we get several equations of the form

$$X \pm Y = 3u^2,$$

where X, Y are S-units and $S = \{p, q\}$. These equations are considered locally to get a contradiction or are transformed to Ljunggren-type equations. In fact, we have to give all S-integral points on the resulting Ljunggren-type curves. Then, using the program package MAGMA we solve completely the equations under consideration.

2.2 Auxiliary results

Let $S = \{p_1, \dots, p_s\}$ be a set of distinct primes and denote by **S** the set of non-zero integers composed only of primes from S. Equation (2.1) is a special case of an equation of the type

$$X^2 + D = Y^n, \tag{2.3}$$

where

$$\gcd(X, Y) = 1 \tag{2.4}$$

and

$$D \in \mathbf{S},\ D > 0,\ X \geq 1,\ Y > 1,\ n \geq 3. \tag{2.5}$$

The next lemma provides a parametrization for the solutions of equation (2.3).

Lemma 7. *Suppose that equation* (2.3) *has a solution under the assumptions* (2.4) *and* (2.5) *with* $n \geq 3$ *prime. Denote by* $d > 0$ *the square-free part of* $D = dc^2$ *and let* h *be the class number of the field* $\mathbb{Q}(\sqrt{-d})$*. Then equation* (2.3) *has a solution with* $d \not\equiv 7 \pmod 8$ *in one of the following cases:*

(a) *there exist* $u, v \in \mathbb{Z}$ *such that* $X + c\sqrt{-d} = (u + v\sqrt{-d})^n$ *and* $Y = u^2 + dv^2$.

(b) $d \equiv 3 \pmod 8$ *and there exist* $U, V \in \mathbb{Z}$ *with* $U \equiv V \equiv 1 \pmod 2$ *such that* $X + c\sqrt{-d} = \left(\frac{U + V\sqrt{-d}}{2}\right)^3$ *and* $Y = \frac{U^2 + dV^2}{4}$.

(c) $n = 3$ *if* $D = 3u^2 \pm 8$ *or if* $D = 3u^2 \pm 1$ *for some* $u \in \mathbb{Z}$.

(d) $n = 5$ *if* $D \in \{19, 341\}$.

(e) $p \mid h$.

Proof. This is a theorem of Cohn [36].

□

Recall that a *Lucas-pair* is a pair (α, β) of algebraic integers such that $\alpha + \beta$ and $\alpha\beta$ are non-zero coprime rational integers and α/β is not a root of unity. Given a Lucas-pair (α, β) one defines the corresponding sequence of *Lucas numbers* by

$$L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ (n = 0, 1, 2...).$$

A prime number p is called a *primitive divisor* of L_n if p divides L_n but does not divide $(\alpha - \beta)^2 L_1 \cdots L_{n-1}$.

The next lemma gives a necessary condition for an odd prime p to be a primitive prime divisor of the n-th term of a Lucas-sequence if n is an odd prime. Namely we have the following.

Lemma 8. *Let $L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ be a Lucas-sequence and suppose that n is an odd prime. Further, let $A = (\alpha - \beta)^2$. If p is a primitive prime divisor of L_n then $n \mid p - \left(\frac{A}{p}\right)$, where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre-symbol with respect to the prime p.*

Proof. See Carmichael [31].

□

The next lemma is a deep result of Bilu, Hanrot and Voutier [124] concerning the existence of primitive prime divisors in a Lucas sequence.

Lemma 9. *Let $L_n = L_n(\alpha, \beta)$ be a Lucas sequence. If $n \geq 5$ is a prime then L_n has a primitive prime divisor except for finitely many pairs (α, β) which are explicitly determined in Table 1 of [124].*

Proof. This follows from Theorem 1.4 of [124] and Theorem 1 of [122].

□

The following lemma of Holzer gives a criterium for the existence of solutions of ternary quadratic equations.

Lemma 10. *Let a, b, c be coprime integers, and consider the equation*

$$ax^2 + by^2 + cz^2 = 0 \tag{2.6}$$

where x, y, z are unknown integers. If there is a non-trivial solution for (2.6), then there is one satisfying

$$\mid x \mid \leq \sqrt{\mid bc \mid}, \ \mid y \mid \leq \sqrt{\mid ac \mid}, \ \mid z \mid \leq \sqrt{\mid ab \mid}$$

Proof. See [84].

□

2.3 Proof of Theorem 8

We introduce some notations which will be used in the course of the proof of our Theorem. Consider equation (2.1) satisfying the assumptions (2.2). Denote by $d > 0$ the square-free part of $5^k 17^l$ that is $5^k 17^l = d(5^a 17^b)^2$ where $d \in \{1, 5, 17, 85\}$ and $a, b \in \mathbb{Z}_{\geq 0}$. Further, let $\mathbb{K}$ be the imaginary quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ and denote by h the class number of $\mathbb{K}$. As was mentioned in Remark 1, we have to distinguish essentially three cases without loss of generality. Namely, we may assume that in equation (2.1) $n \geq 5$ prime or $n \in \{3, 4\}$.

Case 1: $n \geq 5$ *prime.*

Suppose first that (2.1) holds satisfying (2.2) with $n \geq 5$ prime. If in (2.1) $y > 1$ is even we obviously have that x is odd. Since for any odd integer t we have $t^2 \equiv 1 \pmod 8$ we get that $1 + d \equiv 0 \pmod 8$ by reducing (2.1) modulo 8. This leads to $d \equiv 7 \pmod 8$ for $d \in \{1, 5, 17, 85\}$ which is clearly a contradiction. Hence in what follows we may assume that in (2.1) $y > 1$ is odd (and hence $x \geq 1$ is even). Since for $d \in \{1, 5, 17, 85\}$ the class number of the field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ is 1 or 2^m, $(m \geq 1)$ we get by Lemma 7 that equation (2.1) can have a solution under assumption (2.2) with $n \geq 5$ prime only in the cases (a) and (d). Since $k \geq 0$ and $l \geq 0$ we see that in (2.1) $D = 19$ cannot occur. Further, if $D = 341 = 11 \cdot 31$ then since $D = 5^k \cdot 17^l$ this choice for D is impossible, too. Hence equation (2.1) can have a solution only in case (a) of Lemma 7. Namely, using the parametrization provided by Lemma 7 and taking complex conjugation, we get

$$(x + 5^a 17^b \sqrt{-d}) = (u + v\sqrt{-d})^n \text{ and } (x - 5^a 17^b \sqrt{-d}) = (u - v\sqrt{-d})^n \quad (2.7)$$

for some $u, v \in \mathbb{Z}$. Further, we also have $y = u^2 + dv^2$. By (2.7) we see that $u \mid x$ and since $y > 1$ is odd and $\gcd(x, y) = 1$ we get that $\gcd(2u, y) = 1$. Let $\alpha = u + v\sqrt{-d}$ and $\beta = u - v\sqrt{-d}$. Then $\gcd(\alpha\beta, \alpha + \beta) = \gcd(y, 2u) = 1$. If α/β is a root of unity then since $n \geq 5$ is prime we have $\alpha/\beta \in \{\pm 1, \pm i\}$ if $d = 1$. This leads to $u = 0$ or $u = \pm v$. Now $u = 0$ yields $x = 0$ which is a contradiction by (2.3). If $u = \pm v$ then $2 \mid y = u^2 + v^2$ which contradicts the fact that y is odd. If $d \in \{5, 17, 85\}$, then α/β is a root of unity if $\alpha/\beta \in \{\pm 1\}$, which leads to either $u = 1, v = 0$ or $u = 0, v = 1$. If $u = 1, v = 0$, then we get a contradiction with $y \geq 3$. If $u = 0, v = 1$, then $y = d$ holds, which leads to a contradiction with $\gcd(x, y) = 1$. Thus

$$L_n = \frac{(u + v\sqrt{-d})^n - (u - v\sqrt{-d})^n}{2v\sqrt{-d}} \tag{2.8}$$

is a Lucas sequence.

Further, by (2.8) we have

$$L_n = \frac{5^a 17^b}{v}$$

for some non-negative integers a, b. By Lemma 9 we get that L_n has a primitive divisor for $n \geq 5$ prime. Also the only prime divisors of L_n can be 5 or 17. By Lemma 8 we get that if p is a primitive divisor of L_n, then $p \equiv \pm 1 \pmod{n}$, so $n \mid p \pm 1$ holds. Since $p \in \{5, 17\}$, we have that one of the following cases holds:

$$n \mid 4 = 2^2,\ n \mid 6 = 2 \cdot 3,\ n \mid 16 = 2^4,\ n \mid 18 = 2 \cdot 3^2$$

Since $n \geq 5$ we get a contradiction for all cases, which implies that (2.1) does not have a solution for $n \geq 5$.

Case 2: $n = 3$.

At first, we point out that the usual method concerning the search for S-integral points on certain elliptic curves proves to be time consuming in this case, so we show a different approach.

By Lemma 7, we see that

$$x + 5^a 17^b \sqrt{-d} = (u + v\sqrt{-d})^3 \tag{2.9}$$

holds, where $d \in \{1, 5, 17, 85\}$ and $u, v \in \mathbb{Z}$. After expanding the right handside of equation (2.9), and comparing the imaginary parts, we get that

$$5^a 17^b = v(3u^2 - dv^2). \tag{2.10}$$

In (2.10) $\gcd(v, 3u^2 - dv^2) = 1$ holds, since otherwise we would get $\gcd(u, v) \neq 1$, which implies $\gcd(x, y) \neq 1$, which is clearly a contradiction. From this, we get the following type of equations:

$$\begin{cases} 3u^2 - dv^2 = f \\ v = g, \end{cases} \tag{2.11}$$

where

$$(f, g) \in \{(\pm 1, \pm 5^a 17^b), (\pm 5^a, \pm 17^b), (\pm 17^b, \pm 5^a), (\pm 5^a 17^b, \pm 1)\}.$$

Since $d \in \{1, 5, 17, 85\}$, we get a total of 16 cases, we have to deal with. We will illustrate the method in one of the more interesting cases, all the others can be done in the same way. Let $d = 5$, $f = \pm 17^b$, $g = \pm 5^a$. From this, we get that

$$3u^2 - 5^{2a+1} = \pm 17^b \tag{2.12}$$

holds. Our main goal is to transform this to Ljunggren-type curves. To reduce the number of curves, and so the time of the computation we write (2.12) to the form of $Ax^2 + By^2 + Cz^2 = 0$. Now using Holzer's theorem (see Lemma 10) we get, that (2.12) has a nontrivial solution if and only if b is odd and $3u^2 - 5^{2a+1} = -17^b$ holds. Now we transform this to the following type.

$$3\left(\frac{u}{17^{2b_1}}\right)^2 = 5^{i+1}\left(\frac{5^{a_1}}{17^{b_1}}\right)^4 - 17^{j+1} \tag{2.13}$$

where $i, j \in \{0, 2\}$, and $a = 4a_1 + i + 1$, $b = 4b_1 + j + 1$. So, the problem is reduced to finding all the $\{17\}$-integral points on quartics of the form of

$$3Y^2 = 5^{i+1}X^4 - 17^{j+1}, \ \ i, j \in \{0, 2\}, \text{ where } X = \frac{5^{a_2}}{17^{b_2}} \text{ and } Y = \frac{u}{17^{2b_2}}.$$

Now, we can use MAGMA to determine all the solutions of the above equations. Repeating this for all the 16 cases we get that all the solutions of (2.1) with $n = 3$ are:

$$(x, y, k, l, n) \in \{(94, 21, 2, 1, 3), (2034, 161, 3, 2, 3)\}.$$

We point out that, in many of the above cases the method used can be combined with local methods to simplify the computations.

Case 3: $n = 4$.

If $n = 4$ holds, then we can write the following:

$$y^4 - x^2 = 5^k 17^l$$

which can be factored as

$$(y^2 - x)(y^2 + x) = 5^k 17^l. \tag{2.14}$$

In (2.14) $\gcd(y^2 - x, y^2 + x) = 1$ holds, else we would get a contradiction with $\gcd(x, y) = 1$. So, we get that

$$\begin{cases} y^2 - x = f \\ y^2 + x = g \end{cases}$$

where $(f, g) \in \{(1, 5^k 17^l), (5^k, 17^l), (17^l, 5^k), (5^k 17^l, 1)\}$. Now, by adding the first equation to the second, we get, that

$$2y^2 = f + g$$

holds. Using a similar method as in the $n = 3$ case we get that with $n = 4$ all the solutions of (2.1) are

$$(x, y, k, l, n) \in \{(8, 3, 0, 1, 4)\}.$$

Chapter 3

Binomial Thue Inequalities

In this chapter we will apply Baker's method combined with hypergeometric approximation techniques to give effective (and computable) upper bounds for the number of solutions of binomial Thue inequalities. Despite the fact that the situation we will consider is a very specialized one, we believe it is instructive to see what can be said explicitly, as a test of the current state of refinement of computational and analytic techniques. As a starting point, we note that, implicit in the techniques of [20] and [23] is the following result.

Theorem 9. *Let c be a positive integer. Then there exists an effectively computable finite set S_c of triples of positive integers a, b and n with the property that if a, b and $n \geq 3$ are any positive integers for which the Diophantine inequality*

$$|ax^n - by^n| \leq c \tag{3.1}$$

has more than a single solution in positive integers x and y, then $(a, b, n) \in S_c$.

The main result of [20] is that the set S_1 is empty.

3.1 Results

Extending the aforementioned theorem, our main result is the following.

Theorem 10. *With S_c defined above, we have $S_3 \subseteq S_3^* \cup T_3$, where*

$$S_3^* = \{(1,2,3),(2,1,3),(1,3,3),(3,1,3),(2,5,3),(5,2,3)\}$$

and

$$T_3 = \{(1,3,n),(3,1,n),(2,5,n),(5,2,n) \ \textit{with} \ 37 \le n \le 347, \ n \textit{ prime} \}.$$

For $(a,b,n) \in S_3^*$, the solutions in positive integers to inequality (3.1) *with $c = 3$ are, in each case, $(x,y) = (1,1)$, and also*

(a,b,n)	$(1,2,3)$	$(2,1,3)$	$(1,3,3)$	$(3,1,3)$	$(2,5,3)$	$(5,2,3)$
(x,y)	$(5,4)$	$(4,5)$	$(3,2)$	$(2,3)$	$(19,14)$	$(14,19)$

In case $n = 3$, this theorem represents a slight sharpening of a classical result of Ljunggren [66], who considered equation (5) with $n = 3$ and $c \in \{1,3\}$. It is very likely that $S_3 = S_3^*$ (which should be provable with a finite but currently infeasible amount of computation). We can, in any case, certainly prove a sharpened version of Theorem 10, with T_3 replaced by a somewhat smaller set, through more careful application of the hypergeometric method; in our opinion the effort involved would somewhat exceed the payoff.

3.2 Some lemmata

In this section, we collect a number of lemmata that we use in the proof of Theorem 10. The first is a state-of-the-art lower bound for linear forms in the logarithms of two algebraic numbers, due to Laurent (Theorem 2 of [59]). For any algebraic number α of degree d over $\mathbb{Q}$, we define as usual the *absolute logarithmic height* of α by the formula

$$h(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\max\left(1, |\alpha^{(i)}|\right)\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$ and the $\alpha^{(i)}$s are the conjugates of α in the field of complex numbers.

Lemma 11. *Let α_1 and α_2 be multiplicatively independent algebraic numbers, h, ρ and μ be real numbers with $\rho > 1$ and $1/3 \leq \mu \leq 1$. Set*

$$\sigma = \frac{1+2\mu-\mu^2}{2}, \quad \lambda = \sigma\log\rho, \quad H = \frac{h}{\lambda} + \frac{1}{\sigma}$$
$$\omega = 2\left(1 + \sqrt{1 + \frac{1}{4H^2}}\right), \quad \theta = \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H}.$$

Consider the linear form $\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1$, where b_1 and b_2 are positive integers. Put

$$D = [\mathbb{Q}(\alpha_1,\alpha_2) : \mathbb{Q}] \,/\, [\mathbb{R}(\alpha_1,\alpha_2) : \mathbb{R}]$$

and assume that

$$h \geq \max\left\{D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.75\right) + 0.06, \lambda, \frac{D\log 2}{2}\right\}, \tag{3.2}$$

$$a_i \geq \max\left\{1, \rho|\log\alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i)\right\} \qquad (i = 1, 2), \tag{3.3}$$

and

$$a_1 a_2 \geq \lambda^2. \tag{3.4}$$

Then

$$\log|\Lambda| \geq -C\left(h + \frac{\lambda}{\sigma}\right)^2 a_1 a_2 - \sqrt{\omega\theta}\left(h + \frac{\lambda}{\sigma}\right) - \log\left(C'\left(h + \frac{\lambda}{\sigma}\right)^2 a_1 a_2\right) \tag{3.5}$$

with

$$C = \frac{\mu}{\lambda^3\sigma}\left(\frac{\omega}{6} + \frac{1}{2}\sqrt{\frac{\omega^2}{9} + \frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1a_2}H^{1/2}} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\lambda\omega}{H}}\right)^2 \tag{3.6}$$

and

$$C' = \sqrt{\frac{C\sigma\omega\theta}{\lambda^3\mu}}. \tag{3.7}$$

The next lemma is a result of Evertse (Theorem 2.1 of [47]) and, as mentioned earlier, represents a refinement of prior work of Siegel on the hypergeometric method.

Lemma 12. *Suppose that a, b, c and n are positive integers with $n \geq 3$. Define*

$$T_n = 3^{-\frac{n-2}{n}} n \prod_{p|n} p^{\frac{1}{p-1}}, \quad \mu_3 = T_3^{11/2}, \quad \mu_n = T_n^{\max\left\{\frac{n+2}{2(n-3)}, \frac{n}{n-2}\right\}} \text{ if } n \geq 4,$$

and

$$\alpha_3 = 9, \quad \alpha_n = \max\left\{\frac{3n-2}{2(n-3)}, \frac{2(n-1)}{n-2}\right\} \text{ if } n \geq 4.$$

Then the inequality (3.1) *has at most one solution in positive coprime integers* x *and* y *satisfying*

$$\max\{ax^n, by^n\} \geq \mu_n c^{\alpha_n}.$$

The final three lemmata we will use are results of the first author [18], [19], [20] and [22]. To be precise, they are a combination of Theorem 5.2 of [20] with Theorem 5.2 of [22], a special case of Theorem 1.1 of [18], and a special case of Theorem 1.1 of [19], respectively. We will use them to treat inequality (3.1) for "small" values of n.

Lemma 13. *Suppose $b > a$ are coprime positive integers and $m = \left[\frac{n+1}{3}\right]$. Let n, $c_1(n)$ and $d(n)$ be as given in the the following table.*

n	$c_1(n)$	$d(n)$	n	$c_1(n)$	$d(n)$	n	$c_1(n)$	$d(n)$
17	8.93	13.06	107	83.55	50.84	227	201.15	116.91
19	9.40	15.46	109	84.18	58.97	229	202.11	100.61
23	13.03	17.66	113	89.22	77.93	233	207.50	102.49
29	17.39	29.95	127	100.47	72.61	239	213.74	105.66
31	17.92	30.55	131	105.34	71.51	241	214.95	95.14
37	21.2	–	137	111.44	79.94	251	226.83	115.64
41	25.83	36.08	139	112.15	77.27	257	233.75	113.23
43	26.62	33.95	149	122.53	85.82	263	240.15	119.49
47	30.46	40.16	151	123.41	89.04	269	246.54	124.75
53	34.78	35.37	157	129.07	81.61	271	247.72	134.21
59	39.18	48.34	163	134.80	93.64	277	254.62	119.17
61	39.96	55.93	167	139.95	82.87	281	260.46	116.79
67	44.76	43.56	173	146.07	87.71	283	261.67	118.21
71	48.36	54.80	179	151.40	83.92	293	274.23	129.73
73	52.83	48.11	181	152.20	91.69	307	289.00	124.89
79	58.27	54.65	191	163.78	84.40	311	294.70	130.14
83	62.70	49.64	193	164.81	91.51	313	296.38	130.18
89	67.56	60.29	197	170.17	104.53	317	302.73	134.63
97	73.71	62.14	199	170.80	110.41	331	317.41	147.69
101	78.29	50.36	211	183.12	124.02	337	324.63	139.95
103	79.16	60.85	223	195.74	112.93	347	338.02	133.98

If

$$(\sqrt[m]{b} - \sqrt[m]{a})^m e^{c_1(n)} < 1, \tag{3.8}$$

then, for all x and $y > 0$ integers, we have

$$\left| \left(\frac{b}{a}\right)^{1/n} - \frac{x}{y} \right| > (C_2(\sqrt[m]{b} + \sqrt[m]{a})^m)^{-1} y^{-\lambda_1},$$

where

$$C_2 = \begin{cases} 3.15 \cdot 10^{24}(m-1)^2 n^{m-1} e^{c_1(n)+d(n)} & \text{if } n \neq 37 \\ 5 \cdot 10^{75} & \text{if } n = 37 \end{cases},$$

and

$$\lambda_1 = (m-1)\left\{1 - \frac{\log((\sqrt[m]{b} + \sqrt[m]{a})^m e^{c_1(n)+1/20})}{\log((\sqrt[m]{b} - \sqrt[m]{a})^m e^{c_1(n)})}\right\}.$$

Lemma 14. *Let $c \in \{1, 2, 3\}$ and a be a positive integer which satisfies*

$$8\left(\sqrt{a} + \sqrt{a+c}\right)^2 > c^4 \cdot (\kappa(c))^3, \tag{3.9}$$

where

$$\kappa(c) = \begin{cases} 3\sqrt{3} & \textit{for } c = 1, 2 \\ \sqrt{3} & \textit{for } c = 3. \end{cases}$$

Then, for all positive integers x and y,

$$\left| \sqrt[3]{1 + \frac{c}{a}} - \frac{x}{y} \right| > (4 \cdot a \cdot \kappa(c))^{-1} \left(10^4 y\right)^{-\lambda_3}, \tag{3.10}$$

where

$$\lambda_3 = 1 + \frac{\log\left(\frac{\kappa(c)}{2}(\sqrt{a} + \sqrt{a+c})^2\right)}{\log\left(\frac{2}{c^2 \cdot \kappa(c)}(\sqrt{a} + \sqrt{a+c})^2\right)}.$$

Lemma 15. *Let a be a positive integer, $c \in \{1, 2, 3\}$ and $n \in \{4, 5, 7, 11, 13\}$. If*

$$\left(\sqrt{a} + \sqrt{a+c}\right)^{2(n-2)} > c^{2(n-1)} \left(\frac{\kappa(c,n)}{c_2(n)}\right)^n, \tag{3.11}$$

then for all positive integers x and y,

$$\left| \sqrt[n]{1 + \frac{c}{a}} - \frac{x}{y} \right| > \frac{1}{a} \cdot (10^{10} y)^{-\lambda_4}, \tag{3.12}$$

where

$$\lambda_4 = 1 + \frac{\log\left(\frac{\kappa(c,n)}{c_2(n)}\left(\sqrt{a} + \sqrt{a+c}\right)^2\right)}{\log\left(\frac{c_2(n)}{c^2 \kappa(c,n)}\left(\sqrt{a} + \sqrt{a+c}\right)^2\right)}, \quad \kappa(c,n) = \prod_{p|n} p^{\max\left\{ord_p\left(\frac{n}{c}\right) + \frac{1}{p-1}, 0\right\}},$$

$c_2(4) = 1.62$, $c_2(5) = 1.84$, $c_2(7) = 1.76$, $c_2(11) = 1.67$ *and* $c_2(13) = 1.65$.

3.3 Proof of Theorem 10

We will consider the inequality

$$|ax^n - by^n| \leq 3 \tag{3.13}$$

in integer unknowns x, y, a, b and n which satisfy, without loss of generality,

$$b > a \geq 1, \; n \geq 3, \; x \geq 1, \; y \geq 1. \tag{3.14}$$

We may further assume, again without loss of generality, that in (3.13) the exponent n is either 4 or an odd prime. By Lemma 12, it follows that if

$$x^n \geq \mu_n \cdot 3^{\alpha_n},$$

then (3.13) has at most one solution in positive integers x and y. This implies that, apart from when $n \in \{3, 4, 5\}$, inequality (3.13) has at most one positive solution with $x \geq 2$. We may thus distinguish two cases.

Case I : The inequality (3.13) has $(x, y) = (1, 1)$ as a solution. We thus have $b = a + c$ for $c \in \{1, 2, 3\}$ and hence are led to consider the inequality

$$|ax^n - (a + c)y^n| \leq 3, \tag{3.15}$$

where $c \in \{1, 2, 3\}$ and a, x, y and n are positive integers with $n \geq 3$.

Case II : We have $n \in \{3, 4, 5\}$, $b - a > 3$ and inequality (3.13) has a solution in positive integers x and y with $x \geq 2$.

We first deal with *Case I*.

3.3.1 Linear forms in two logarithms

The main purpose of this subsection is to prove the following.

Theorem 11. *If there is a solution to inequality* (3.15) *in positive integers* x *and* y *with* $(x, y) \neq (1, 1)$*, then* $n \leq 347$.

To prove this, we will have use of the following technical lemma.

Lemma 16. *If inequality (3.15) has a solution in positive integers* $(x, y) \neq (1, 1)$ *then* $x > \frac{na}{c}$.

Proof of Lemma 16 *:* If $x \leq y$ and $y > 1$, then

$$|ax^n - (a+c)y^n| \geq cy^n > 3,$$

contradicting (3.15). We may thus suppose that $x \geq y+1$, which by (3.15) yields

$$ax^n - (a+c)y^n \geq a(y+1)^n - (a+c)y^n.$$

By the binomial theorem, the right hand side of this is

$$nay^{n-1} + a\left(\binom{n}{2}y^{n-2} + \cdots + \binom{n}{n-1}y + 1\right) - cy^n.$$

Since

$$a\left(\binom{n}{2}y^{n-2} + \cdots + \binom{n}{n-1}y + 1\right) > 3,$$

it follows from (3.15) that

$$nay^{n-1} - cy^n < 0, \tag{3.16}$$

which in turn implies that $x > y > \frac{na}{c}$. $\square$

Proof of Theorem 11 Suppose that inequality (3.15) has a positive solution $(x, y) \neq (1, 1)$ with $n > 347$. By Lemma 16, it follows that $x > na/c$. We consider the linear form

$$|\Lambda| = \left|\log\left(1 + \frac{c}{a}\right) - n\log\left(\frac{x}{y}\right)\right|. \tag{3.17}$$

Since (3.15) is equivalent to the inequality

$$\left|1 - \left(1 + \frac{c}{a}\right)\left(\frac{y}{x}\right)^n\right| \leq \frac{3}{ax^n},$$

and since, for every $z \in \mathbb{C}$ with $|z-1| < 0.795$, we have $|\log(z)| < 2|z-1|$, it follows that

$$|\Lambda| < \frac{6}{x^n}. \tag{3.18}$$

We write

$$\alpha_1 = \frac{x}{y},\ \alpha_2 = 1 + \frac{c}{a},\ b_1 = n,\ b_2 = 1,\ \mu = 0.63,\ \sigma = 0.93155,\ D = 1,$$

$$\rho = 1 + \frac{\log(a+c)}{\log(1+\frac{c}{a})},\quad \text{and choose } a_1 = 2.003\log(x) \ \text{ and } \ a_2 = 3\log(a+c).$$

Applying Lemma 11, one may readily check that (3.4) holds. We distinguish two cases according to whether $a \geq 14$ or $a \leq 13$, respectively.

If $a \geq 14$ then, by calculus, we find that there exist absolute constants c_1, c_2 such that

$$c_1\,\sigma\log(a+c) < \lambda < c_2\,\sigma\log(a+c) \tag{3.19}$$

Here we may choose $c_2 = 1.3646$ if $c = 1$, $c_2 = 1.1835$ if $c = 2$ and $c_2 = 1.1226$ if $c = 3$. The corresponding values of c_1 are $c_1 = 1$ if $c \in \{1, 2\}$, $(c_1, a) = (0.96, 14), (0.98, 16)$, or $(0.99, 17)$, if $c = 3$ and $14 \leq a \leq 17$, and $c_1 = 1$ if $c = 3$ and $a \geq 18$. Since $n > 347$ and $x > \frac{na}{c}$, it follows that $\frac{\log(a+c)}{\log(x)} < 1$ and, via (3.19),

$$\log\left(\frac{n}{3\log(a+c)} + \frac{1}{2.003\log(x)}\right) + \log(\lambda) + 1.81 < \log\left(\frac{c_2\sigma n}{3} + \frac{c_2\sigma}{2.003}\right) + 1.81.$$

Hence, for $a \geq 14$, we may take

$$h = \max\left\{\log\left(\frac{c_2\sigma n}{3} + \frac{c_2\sigma}{2.003}\right) + 1.81, \lambda\right\}.$$

Suppose first that $h = \log\left(\frac{c_2\sigma n}{3} + \frac{c_2\sigma}{2.003}\right) + 1.81$. Then, by (3.19) and the assumption that $a \geq 14$,

$$\frac{h}{\lambda} + \frac{1}{\sigma} \leq A := \frac{\log\left(\frac{c_2\sigma n}{3} + \frac{c_2\sigma}{2.003}\right) + 1.81}{\sigma c_1\log(a+c)} + \frac{1}{\sigma}. \tag{3.20}$$

Lemma 11 and (3.20) together imply that

$$\log|\Lambda| > -C\lambda^2 a_1 a_2 A^2 - \sqrt{\omega\theta}\lambda A - \log(C' a_1 a_2 \lambda^2 A^2) \tag{3.21}$$

and hence, comparing (3.18) and (3.21), we have

$$n < C\lambda^2 A^2 \frac{a_1 a_2}{\log(x)} + \sqrt{\omega\theta}\frac{\lambda}{\log(x)} A + \frac{\log(2cC' a_1 a_2 \lambda^2 A^2)}{\log(x)}. \tag{3.22}$$

Write $C = \frac{\mu}{\lambda^3 \sigma}\tilde{C}$. Then, from the definitions of a_1 and a_2, and from (3.19), necessarily

$$C\lambda^2 \frac{a_1 a_2}{\log(x)} < \frac{\tilde{C}\, 6.009\, \mu}{c_1 \sigma^2}.$$

Since $x > na/c$ and $n > 347$, we have $\frac{\log(a+c)}{\log(x)} < 1$. Combining this with (3.19) we obtain that $\frac{\lambda}{\log(x)} < c_2\sigma$ and, further,

$$\frac{\log(2cC' a_1 a_2 \lambda^2 A^2)}{\log(x)} < 0.421 \log(A) + 1.858.$$

Inequality (3.22) thus implies

$$n < \left(\frac{\mu}{\sigma^2 c_1}\tilde{C} \cdot 6.009\right) A^2 + c_2\sigma\sqrt{\omega\theta} A + 0.421 \log(A) + 1.858. \tag{3.23}$$

Since in Lemma 11 we have $H \geq 1 + \frac{1}{\sigma}$, necessarily $H > 2.0734$, whence $\omega < 4.058$ and $\theta < 1.27$. Further, since $\frac{\lambda}{\sqrt{a_1 a_2}} < \frac{c_2\sigma}{\sqrt{6.009}}$ and $\lambda\left(\frac{1}{a_1} + \frac{1}{a_2}\right) < c_2\sigma\left(\frac{1}{2.003} + \frac{1}{3}\right)$, we have $\tilde{C} < 5.262$ if $c = 1$, $\tilde{C} < 4.853$ if $c = 2$ and $\tilde{C} < 4.735$ if $c = 3$. By combining these estimates with (3.23), we obtain, for $a \geq 14$, that

$$n < \left(6.009\tilde{C} \cdot \frac{\mu}{\sigma^2}\frac{1}{c_1}\right) A^2 + 2.271 c_2\sigma A + 0.421 \log(A) + 1.858. \tag{3.24}$$

To remove the dependence on a in this bound, we appeal to the inequalities $\log(a+c) \geq \log(15)$ for $c = 1$, $\log(a+c) \geq \log(16)$ and $a \geq 14$, $\log(a+c) \geq \log(21)$ for $a \geq 18$ and $c = 3$ and $\log(a+c) = \log(a+3)$ for $c = 3$ and $a \in \{14, 16, 17\}$. Hence we obtain $n \leq 347$ for $c \in \{1, 2, 3\}$ and $a \geq 14$, provided $h = \log\left(\frac{c_2 n\sigma}{3} + \frac{c_2\sigma}{2.003}\right) + 1.81$. If $h = \lambda$, inequality (3.24) actually implies a stronger bound upon n.

For $a \leq 13$ and $c \in \{1, 2, 3\}$, we omit the general estimates and use exact values for a. We will provide details in case $a = 3$ and $c = 2$; the other cases proceed in a similar fashion. We first note that direct calculation of the bounds in Lemma 11 with the same parameters as previously, and with $a = 3$, $c = 2$, $x > 347a/c$, yields an initial upper bound for n of the shape $n < 446$. For each prime n between 347 and 446 we apply an algorithm of Pethő [92] (essentially nothing more than an analysis of convergents in the infinite simple continued fraction expansions to $\sqrt[n]{b/a}$) to search for solutions to our Thue inequality with $x \leq 10^{500}$. After a short computation, we find that the only such solution is $(x, y) = (1, 1)$. We may thus assume that $x > 10^{500}$. Using this, (3.22) now yields $n \leq 326$, as desired. □

3.3.2 The hypergeometric method

Theorem 11 leaves us with only finitely many fixed exponents to treat in (3.15). In this subsection, we will assume that n is either 4 or an odd prime between 3 and 347. We first apply Lemma 12 to (3.15). Observe, that

$$\max\{ax^n, (a+c)y^n\} \geq a,$$

so if

$$a \geq \mu_n c^{\alpha_n},$$

then (3.15) has at most one solution. Put $a_0(n) = \mu_n 3^{\alpha_n}$. We remark here, that $a_0(3) = 22678753$, $a_0(4) = 23943$ and $a_0(n) \leq 1103$ for all other values of n. We thus need to consider (3.15) only with $a \leq a_0(n)$. Note that (3.15) implies the inequality

$$\left| \sqrt[n]{1 + \frac{c}{a}} - \frac{x}{y} \right| \leq \frac{3}{any^n}. \tag{3.25}$$

To deduce an upper bound for y in (3.15) we combine (3.25) with Lemmata 13, 14 and 15. We thus have

- for $n = 3$:

$$y < \left(\frac{12 \cdot \kappa(c) \cdot 10^{4\lambda_3}}{n} \right)^{\frac{1}{n-\lambda_3}},$$

- for $n \in \{4, 5, 7, 11, 13\}$:

$$y < \left(\frac{3 \cdot 10^{10\lambda_4}}{n} \right)^{\frac{1}{n - \lambda_4}},$$

- for $17 \leq n \leq 347$:

$$y < \left(\frac{3C_2 \left(\sqrt[m]{a+c} + \sqrt[m]{a} \right)^m}{an} \right)^{\frac{1}{n - \lambda_1}}.$$

If we assume that

$$(a, c) \notin \{(1, 1), (1, 2), (1, 3), (2, 3)\},$$

routine computations in MAPLE show that these bounds are less then 10^{1000}, except for some “small”' values of a and n, where we can appeal to PARI/GP to solve the corresponding Thue equations directly. By a well known theorem of Legendre, we have that in (3.15) the ratio x/y is a convergent in the continued fraction expansion of $\sqrt[n]{1 + \frac{c}{a}}$. We can thus apply the aforementioned algorithm of Pethõ [92] to compute all solutions of the occurring inequalities. The exceptional cases here which do not satisfy the requirements of Lemmata 13, 14 and 15 (again, all with “small” values of a and n) may also be treated via PARI/GP. It remains to deal with the pairs

$$(a, c) \in \{(1, 1), (1, 2), (1, 3), (2, 3)\},$$

for $n = 4$ or prime n, $3 \leq n \leq 347$. In case $(a, c) = (1, 1)$, the desired result is an immediate consequence of Proposition 5.1 of [22]; we find an additional solution with $n = 3$ and $(x, y) = (5, 4)$. Suppose next that $(a, c) = (1, 3)$. The Diophantine equations

$$x^n - 4y^n = \pm 1, \pm 2$$

can be shown to have no solutions in positive integers for $n \geq 3$ by combining work of Ribet [96] with elementary arguments, while

$$x^n - 4y^n = \pm 3$$

has no solutions in integers x and y with $|xy| > 1$, provided n has a prime divisor $p \geq 7$ (see Theorem 1.2 of [24]). It remains, therefore, to treat inequality (3.15) with $(a, c) = (1, 2)$ or $(2, 3)$ and $n \in \{3, 4, 5, 7, 11, 13, 17\}$, and $(a, c) = (1, 3)$, $n \in \{3, 4, 5\}$. We appeal to PARI/GP and find no further nontrivial solutions to (3.15), unless $(a, c, n) = (1, 2, 3)$ (where there is the additional solution $(x, y) = (3, 2)$) or $(a, c, n) = (2, 3, 3)$ (where we have $(x, y) = (19, 14)$). This completes the proof of *Case I*.

Case II can be handled similarly. We can assume, for the remainder of the proof, that for any positive solution (x, y) of (3.13), we have $x \geq 2$. Denote by (x_0, y_0) a known solution of (3.13). As previously, we may conclude from Lemma 12 that if $\max(x_0, y_0)$ is larger than a computable constant X_n, then the only positive solution of (3.13) is (x_0, y_0). Hence, we have only to consider (3.13) with $n \in \{3, 4, 5\}$ and with a given finite set $\mathcal{X}$ of the pairs (x_0, y_0). By way of example, if $a = 1$ and $n = 3$, we have $2 \leq x_0 \leq 283$, and determine by_0^3 by factoring $ax_0^3 + t$ for $t \in \{\pm 1, \pm 2, \pm 3\}$. In general, applying Lemma 12 to our set of pairs $\mathcal{X}$, we arrive at a finite set of possible pairs (a, b), with corresponding finite set of Thue inequalities (really, in this case, equations) to solve. In most cases, we can carry this out easily via the hypergeometric method. Assume that (x_0, y_0) is given and that $ax_0^n - by_0^n = -t$, with $t \in \{\pm 1, \pm 2, \pm 3\}$. Then b can be written as $\frac{ax_0^n + t}{y_0^n}$ and, after substituting this into (3.13), we find that

$$\left| ax^n - \frac{ax_0^n + t}{y_0^n} y^n \right| \leq 3.$$

Applying Lemmata 14 and 15, we are led to inequalities of the shape

$$\frac{c_1}{(x_0 y)^\lambda} < \left| \frac{x y_0}{x_0 y} - \sqrt[n]{1 + \frac{t}{a x_0^n}} \right| \leq \frac{3 \cdot y_0^n}{a (x_0 y)^n},$$

where the constant c_1 can be deduced from the statements of Lemmata 14 or 15. This yields, in a similar fashion to *Case I*, that y is bounded by some absolute constant (usually around 10^{500}). From (3.13),

$$\left| \frac{x}{y} - \sqrt[n]{\frac{b}{a}} \right| < \frac{3}{a n y^n}$$

and hence, via Legendre's theorem, we have that x/y is a convergent in the simple continued fraction expansion of $\sqrt[n]{b/a}$. Thus, we may again apply Pethõ's algorithm [92] to compute all solutions of the corresponding inequalities. Repeating this procedure for all $(x_0, y_0) \in \mathcal{X}$, and using PARI/GP for some exceptional equations with small coefficients which we are unable to handle via the hypergeometric method, we conclude that (3.13) has at most one solution for each triple (a, b, n) in *Case II*. This completes the proof of Theorem 10. Full details of these computations are available from the authors upon request.

Chapter 4

Equations concerning pyramidal numbers

In the last chapter we will apply linear forms in elliptic logarithms to solve a family of genus 1 equations. Set $\mathrm{Pyr}_m(x) = \dfrac{x(x+1)((m-2)x+5-m)}{6}$ and consider the equation

$$\mathrm{Pyr}_m(u) = \mathrm{Pyr}_n(v), \tag{4.1}$$

in positive integers u and v for given m and n. In what follows, we give effective upper bounds for the size of the solutions of (4.1). We apply the so-called Elliptic Logarithm method, which was developed by Stroeker and Tzanakis [108], and independently by Gebel, Pethő and Zimmer [53] and later improved by Stroeker and Tzanakis [109]. Two interesting special cases are studied by computational number-theoretic tools.

Before stating the main results, we would like to introduce another form of the problem. It is easy to see that (4.1) is equivalent to the equation

$$(m-2)u^3 + 3u^2 + (5-m)u = (n-2)v^3 + 3v^2 + (5-n)v \tag{4.2}$$

in positive integer unknowns u and v.

4.1 Results

With this latter form, the main results are the following.

Theorem 12. *Let m and n be given positive integers with $3 \leq \min(m, n)$ and $m \neq n$. Then the equation* (4.2) *has at most finitely many solutions in integer unknowns u and v. In fact* $\max(u, v) < C_1$*, where C_1 is an effectively computable positive constant depending only on m and n.*

Remark We would like to mention here, that Theorem 12 is also a direct consequence of the celebrated result of Baker and Coates (see [14]). However, the currently discussed Elliptic Logarithm method gives more practical bounds. Sadly, due to the nature of the method, it is currently not possible to make C_1 explicit in terms of m and n.

Using the techniques mentioned above and the program packages MAGMA [123], SAGE [42] and MAPLE, we prove

Theorem 13. *For given m and n with $3 \leq n < m \leq 10$, all solutions of* (4.2) *in (u, v) integers with $(u, v) \notin \{(0, 0)(-1, -1), (-1, 0), (0, -1), (1, 1)\}$ are given in the following table.*

(m,n)	(u,v)
$(4,3)$	$(0,-2),(-1,-2)$
$(5,3)$	$(0,-2),(-1,-2),(-35,-51)$
$(6,3)$	$(0,-2),(-1,-2),(-16,-26)$
$(7,3)$	$(0,-2),(-1,-2),(-2,-4)$
$(7,5)$	$(-5,-6),(6,7)$
$(8,3)$	$(0,-2),(-1,-2),(7,12)$
$(8,4)$	$(-2,-3),(3,4)$
$(8,6)$	$(-276,-316)$
$(9,3)$	$(0,-2),(-1,-2),(-8,-16),(2,3)$
$(9,4)$	$(-13,-20)$
$(9,7)$	$(152,170)$
$(10,3)$	$(0,-2),(-1,-2)$
$(10,4)$	$(55,87)$
$(10,6)$	$(35,44)$

(4.3)

As a direct corollary to Theorem 13, we can state the following.

Corollary 1. *For given m and n with $3 \leq n < m \leq 10$, all solutions of (4.1) in positive integers (u,v) with $(u,v) \neq (1,1)$ are given by*

$$(m,n,u,v) \in \left\{ \begin{matrix} (8,3,7,12),(9,3,2,3),(8,4,3,4),(10,4,55,87), \\ (7,5,6,7),(10,6,35,44),(9,7,152,170). \end{matrix} \right\} \tag{4.4}$$

Before proceeding with the proofs, we would like to make some preliminary remarks.

Remarks. Easy substitution shows that the elements of the excluded set

$$\{(0,0),(-1,-1),(-1,0),(0,-1),(1,1)\}$$

are solutions of (4.2) for all m and n.

As the computational data shows, giving all the solutions with unknown m, n, u and v is hopeless, as there does not seem to be any pattern in the solutions. If we consider (4.2) for given u and v in integer unknowns m and n, we get a linear equation. In this case (4.2) has either no solutions or infinitely many. The latter is the case if and only if

$$\gcd\left(u^3-u, v^3-v\right) \bigm| \left(2u^3-3u^2-5u-2v^3+3v^2+5v\right)$$

We would also like to mention here, that in the case where both $m-2$ and $n-2$ are perfect cubes, we can apply elementary calculations to deduce an upper bound for $\max\{u,v\}$. Indeed, suppose that in (4.2), $m-2=k^3$ and $n-2=l^3$. Set

$$U=k^3l^2u+l^2, \;\; V=k^2l^3v+k^2.$$

Then we have

$$U^3-V^3=l^4((k^3-3)k^3+3)U-k^4((l^3-3)l^3+3)V+l^6(3k^3-2)-k^6(3l^3-2).$$

Here $U=V$ cannot occur. Using the triangle inequality, we get

$$(\max\{U,V\})^2 \le U^2+UV+V^2 \le C_0 \max\{U,V\},$$

with

$$C_0=k^4((l^3-3)l^3+3)+l^4((k^3-3)k^3+3)+k^6(3l^3-2)+l^6(3k^3-2),$$

and so

$$\max\{U,V\} \le C_0.$$

4.2 Auxiliary Results

Lemma 17. *Let m and n be given positive integers with $m \neq n$. Then equation* (4.2) *is birationally equivalent to the Weierstrass curve*

$$y^2=x^3+c(m,n)x+d(m,n), \tag{4.5}$$

where

$$\begin{aligned} c(m,n) = \quad & -48n^2m^2 + 336nm^2 + \\ & +4368m - 624m^2 - 624n^2 + \\ & +4368n - 8112 - 2352nm + 336mn^2, \end{aligned}$$

and

$$\begin{aligned} d(m,n) = \quad & 281216 - 227136m - 227136n - 4352n^3 - 4352m^3 + 256m^4 - \\ & -52544nm^2 + +57424n^2 + 57424m^2 + 194656nm + 64n^4m^2 - \\ & -52544mn^2 + 4352mn^3 + 4352nm^3 - 256n^4m - 256nm^4 - \\ & -1088n^2m^3 + 64n^2m^4 + 256n^4 + 14592n^2m^2 - 1088n^3m^2. \end{aligned}$$

Moreover, there exist mutually invertible birational transformations $\Phi(x,y)$ *and* $\Psi(x,y)$*, under which*

$$u = \Phi(x,y), \ \ \textit{and} \ \ v = \Psi(x,y). \tag{4.6}$$

Proof. We prove Lemma 17 using an algorithm due to Nagell [90]. We will closely follow the method described by Connell in [37]. Let us start with (4.1), which has the rational point $(u,v) = (0,0)$.

Step 1. Substitute $u = U/W$ and $v = V/W$ in (4.2), and clear the denominators to get the homogenous form

$$F = F_3 + F_2W + F_1W^2 = 0, \tag{4.7}$$

where

$$\begin{aligned} F_3 &= (m-2)U^3 - (n-2)V^3, \\ F_2 &= 3U^2 - 3V^2, \\ F_1 &= (5-m)U - (5-n)V. \end{aligned}$$

The rational point P with $(u,v) = (0,0)$ has the projective coordinates $[U : V : W] = [0 : 0 : 1]$. The tangent line to (4.7) in P, given by $F_1 = 0$ meets (4.7) in

$Q = [e_2(5-n), e_2(5-m), e_3]$, where

$$e_2 = F_2(-(5-n), -(5-m)) = 3(5-n)^2 - 3(5-m)^2,$$
$$e_3 = F_3(-(5-n), -(5-m)) = -(m-2)(5-n)^3 + (n-2)(5-m)^3.$$

The aim of this step is to bring Q into the origin with a suitable change of coordinates. Before we can do this, we have to examine e_2 and e_3 a little further. It is easy to see, that e_2 can only be 0 if and only if $|5-m| = |5-n|$, which means that (m, n) comes from the set

$$S = \{(1,9), (2,8), (3,7), (4,6), (6,4), (7,3), (8,2), (9,1)\}.$$

On the other hand, e_3 cannot be 0, since that would mean, that

$$\frac{(5-m)^3}{m-2} = \frac{(5-n)^3}{n-2} \tag{4.8}$$

holds. But the derivative of $\frac{(5-x)^3}{x-2}$ is negative for every positive integer x other than 5. Thus, for $x \neq 5$, the function

$$\frac{(5-x)^3}{x-2}$$

is monotonous which means, that for $m \neq n \neq 5$, (4.8) cannot occur. By choosing $m = 5$ in (4.8), we get, that $n = 5$, which contradicts $m \neq n$.

Thus we can distinguish two cases: either we have $e_2 \neq 0$ and $e_3 \neq 0$ with $(m, n) \notin S$, or we have $e_2 = 0$, $e_3 \neq 0$ with $(m, n) \in S$. If (m, n) is in S, then Q is the origin, thus Q is a flex. If this is the case, one can jump directly to *Step* 2 .

We make the coordinate transformation

$$U = U' + \frac{(5-n)e_2}{e_3} W',$$
$$V = V' + \frac{(5-m)e_2}{e_3} W',$$
$$W = W'$$

to send Q to the origin. We can now return to affine coordinates $u' = U'/W'$, $v' = V'/W'$.

Step 2. Now our equation is of the form $f' = f_1' + f_2' + f_3' = 0$, where f_i' denotes the homogenous part of $f'(u', v')$ of degree i, $i \in \{1, 2, 3\}$. Introduce $t = \frac{v'}{u'}$, and denote $f_i'(1, t)$ by ϕ_i to get the quadratic equation

$$u'^2\phi_3 + u'\phi_2 + \phi_1 = 0. \tag{4.9}$$

Let $\delta = \phi_2^2 - 4\phi_3\phi_1$. Then

$$u' = \frac{-\phi_2 \pm \sqrt{\delta}}{2\phi_3}, \text{ and } v' = tu' \tag{4.10}$$

The zeros of δ are the slopes of the tangents to the curve in the (u', v')-plane that pass through Q. One such value is

$$t_0 = \frac{5 - m}{5 - n}.$$

Thus $(t - t_0)$ is a linear factor of δ. Write $\tau = (t - t_0)^{-1}$, and let $\rho = \tau^4\delta$. Clearly, ρ is a cubic polynomial in τ.

Step 3. Finally, if

$$\rho = c_1\tau^3 + c_2\tau^2 + c_3\tau + c_4,$$

then substitute $\tau = \frac{x'}{c_1}, \rho = \frac{y^2}{c_1^2}$ to get the elliptic equation

$$y^2 = x'^3 + c_2x'^2 + c_1c_3x' + c_1^2c_4.$$

Substituting $x' = x - \frac{c_2}{3}$ yields (4.5). The transformations Φ and Ψ can be traced back starting with (4.10).

$\square$

4.3 Outline of the proof of Theorem 12

By Lemma 17, our initial equation is equivalent to the Weierstrass curve (4.5). Our goal is to apply the Elliptic Logarithm method to deduce the upper bound c_1. To do this, we have to make sure first, that (4.5) is non-singular. The discriminant of (4.5) is

$$D(m,n) = -6912 \cdot D_1(m,n) \cdot (m-n)^2 \tag{4.11}$$

where

$$D_1(m,n) =$$

$$\begin{aligned}
&-12303200 - 4987111n^2 - 4987111m^2 - 14047282nm + 4409272nm^2 + \\
&+4409272mn^2 - 90912mn^3 - 90912nm^3 + 13540280m + 13540280n + 546080n^3 + \\
&+546080m^3 + 48224m^4 - 8704m^5 + 256m^6 + 48224n^4 - 8704n^5 + 256n^6 - \\
&-637744n^2m^2 - 436552n^3m^2 - 158176n^4m + 154792n^4m^2 - 158176nm^4 - \\
&-436552n^2m^3 + 154792n^2m^4 + 17920n^5m - 14080n^5m^2 + 287712n^3m^3 \\
&-65376n^3m^4 - 512n^6m + 384n^6m^2 - 65376n^4m^3 + 12176n^4m^4 + 17920m^5n \\
&-14080m^5n^2 - 512m^6n + 384m^6n^2 + 5120n^5m^3 - 800n^5m^4 + 5120n^3m^5 - \\
&-128n^3m^6 - 128n^6m^3 - 800n^4m^5 + 32n^5m^5 + 16m^4n^6 + 16m^6n^4.
\end{aligned}$$

Clearly, $D(m,n) = 0$ if and only if $D_1(m,n) = 0$. Write

$$
\begin{aligned}
Pol_0 = \;& 256n^6 - 8704n^5 + 48224n^4 + 546080n^3 - 4987111n^2 + 13540280n- \\
& -12303200, \\
Pol_1 = \;& -512n^6 + 17920n^5 - 158176n^4 - 90912n^3 + 4409272n^2 - 14047282n+ \\
& +13540280 + m(384n^6 - 14080n^5 + 154792n^4 - 436552n^3 - 637744n^2+ \\
& +4409272n - 4987111), \\
Pol_2 = \;& -128n^6 + 5120n^5 - 65376n^4 + 287712n^3 - 436552n^2 - 90912n + 546080+ \\
& +m\left(16n^6 - 800n^5 + 12176n^4 - 65376n^3 + 154792n^2 - 158176n + 48224\right), \\
Pol_3 = \;& 32n^5 - 800n^4 + 5120n^3 - 14080n^2 + 17920n - 8704, \\
Pol_4 = \;& 16n^4 - 128n^3 + 384n^2 - 512n + 256.
\end{aligned}
$$

Thus, $D_1 = Pol_4 m^6 + Pol_3 m^5 + Pol_2 m^3 + Pol_1 m + Pol_0$. It is obvious, that for suitably large m all the polynomials Pol_i are monotonously increasing in n, thus always positive for large enough n. Easy calculation shows that $D_1 > 0$ for $\min\{m, n\} > 30$. Now fix m for $m = m_0 \leq 30$. Then $D_1(m_0, n)$ is a polynomial in n. Searching for the integer roots of $D_1(m_0, n)$ for all $1 \leq m_0 < 30$ we find, that $D_1(m, n) = 0$ can occur only with $m = n = 5$. Repeating this last step with fixing $n = n_0 \leq 30$ we get the same result thus proving that (4.5) is non-singular for all (m, n), where $m \neq n$.

Now, we turn to the Elliptic Logarithms. Here we follow the approach described by Stroeker and Tzanakis in [108] and Stroeker de Weger in [107]. Let r be the rank of the curve (4.5), $P_1, \dots P_r$ a basis of the Mordell-Weil group, and P_{r+1} a torsion pont on E. Then a rational point P on the curve is of the shape $P = m_1 P_1 + \cdots + m_r P_r + P_{r+1}$ with $m_i \in \mathbb{Z}$. Write $M = \max\limits_{1 \leq i \leq r} |m_i|$. Then according to [107] the linear form in elliptic logarithms has the form

$$
L(P) = m_0 \omega + m_1 u_1 + \cdots + m_r u_r + u_{r+1} - u_0,
$$

where u_i are the elliptic logarithms of the points P_i, u_0 is the elliptic logarithm of a well-chosen Q_0 point on E, and m_0 is a scaling factor. Using this notation, we have $\max\{M, |m_0|\} \leq rM + 1$.
On one hand, we have an upper bound for this linear form:

$$|L(P)| < \exp(c_1 - c_2 M^2), \tag{4.12}$$

where the constants c_1 and c_2 are effectively determinable. On the other hand a result by David [38] provides a lower bound for the linear form $L(P)$. Combining this with (4.12) provides an upper bound for M. Thus x and y are bounded in terms of m and n, which combined with (4.6) yields an upper bound for $\max\{u, v\}$ completing the proof of Theorem 12.

4.4 Examples

In what follows, we will illustrate the method described in Section 4.3 in two interesting special cases. Iterating the steps described here for $3 \leq n \leq m \leq 10$, we get the set of solutions (4.3), thus proving Theorem 13. First, we consider (4.2) with $(m, n) = (9, 7)$. More precisely, we prove the following theorem.

Theorem 14. *The only positive integer, which is both* 9*-pyramidal and* 7*-pyramidal, is* 4108560.

Theorem 14 is the direct consequence of the following lemma.

Lemma 18. *All solutions of the equation*

$$7u^3 + 3u^2 - 4u = 5v^3 + 3v^2 - 2v \tag{4.13}$$

in integer unknowns u *and* v *are*

$$(u, v) \in \{(-1, -1), (-1, 0), (0, -1), (1, 1), (152, 170)\}. \tag{4.14}$$

Proof. Equation (4.13) is birationally equivalent to the minimal Weierstrass curve

$$Y^2 = X^3 - 1209X + 19361, \tag{4.15}$$

under the transformation

$$(X,Y) = \left(\frac{20u - 18 - 43v}{2u - v}, \frac{3(231u^3 + 129u^2 - 26u - 165v^3 - 156v^2 - 14v)}{(2u - v)^2} \right).$$

Using the program package MAGMA, we get that the rank of (4.15) is 4, and the torsion subgroup of (4.15) is $\mathcal{O}$. The generators of the Mordell-Weil group are:

$$P_1 = (19, 57), P_2 = (25, -69), P_3 = (-5, -159), P_4 = (-41, 3).$$

Let $P = m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4$ be a rational point on (4.15) which is the image of an integer point on (4.13). Then the linear form $L(P)$ is of the shape

$$L(P) = m_0\omega + m_1u_1 + m_2u_2 + m_3u_3 + m_4u_4 - u_0,$$

where ω is the fundamental real period of (4.15), and u_i $(i = 0, \ldots, 4)$ are the elliptic logarithms of Q_0 and P_i, $(i = 1, \ldots, 4)$. After some calculation we have

$$Q_0 = (-31.8884\ldots, 159.6485\ldots),$$

$$\omega = 2.1510\ldots,\ u_0 = 0.9728\ldots,\ u_1 = 0.5717\ldots,$$

$$u_2 = 1.6797\ldots,\ u_3 = 1.3289\ldots,\ u_4 = 1.0739\ldots.$$

Also, in this particular case, (4.12) reads as

$$|L(P)| < \exp(10.168 - 1.23M^2).$$

Combining this with the aforementioned result of David, we get $M < 0.384{\cdot}10^{116}$. To reduce the upper bound, we apply de Weger's [39] method based on the LLL-algorithm. After a few iterations we get $M < 13$. Searching for all points on (4.15) with $M < 13$, and applying the inverse of the birational transformations mentioned above, we are able to calculate the set of solutions (4.14).

$\square$

As a second application for the method, we consider the following problem. For given integers $x \geq 1$ and $y \geq 1$, what are the solutions of the diophantine equation

$$\binom{x+2}{3} = 1^2 + 2^2 + \cdots + y^2. \tag{4.16}$$

Using the definition of the binomial coefficients, and some well-known properties of sums of squares, we get, that (4.16) is equivalent to (4.2) with $(m, n) = (4, 3)$. We have the following theorem.

Theorem 15. *The only solution of* (4.16) *in integers* $x \geq 1$ *and* $y \geq 1$ *is the trivial solution* $(x, y) = (1, 1)$.

This is the direct consequence of the following lemma.

Lemma 19. *The diophantine equation*

$$2u^3 + 3u^2 + u = v^3 + 3v^2 + 2v \tag{4.17}$$

has no solutions in integers (u, v) *other than the trivial solutions* $(u, v) \in \{(-1, -1), (-1, 0), (0, -1)(1, 1)\}$.

Proof. We proceed as in the previous case. Equation (4.17) is birationally equivalent to the minimal Weierstrass curve

$$Y^2 = X^3 - 48X + 272, \tag{4.18}$$

under the transformation

$$(X, Y) = \left(\frac{-4(5u + 9 + 5v)}{u - 2v}, \frac{12(30u^3 + 66u^2 + 41u - 15v^3 - 39v^2 - 28v)}{(u - 2v)^2} \right).$$

The rank of (4.18) is 1, and the torsion subgroup of (4.18) is trivial. The generator of the Mordell-Weil group is

$$P_1 = (16, 60).$$

Denote by $P = m_1 P_1$ a rational point on (4.18), which is the image of an integer solution of (4.17). Then we have the following linear form

$$L(P) = m_0 \omega + m_1 u_1 - u_0,$$

where ω is the fundamental real period of (4.17), and u_0, u_1 are the elliptic logarithms of the points P_1 and $Q_0 = (29.7388\ldots, 158.5738\ldots)$. After some calculation, we have

$$\omega = 3.7814\ldots,\ u_0 = 0.3685\ldots,\ u_1 = 0.5074\ldots.$$

In this case, (4.12) reads as

$$|L(P)| < \exp(8.02852 - 0.05909M^2).$$

Combining this with David's result, we get $M < 0.2919 \cdot 10^{52}$. After a few iterations of de Weger's algorithm, we arrive at $M < 18$. Checking for solutions with this condition, we find that the only solution of (4.17) is the trivial one.

$\square$

4.5 Computational Remarks

We would like to make some remarks concerning the practical details of the computation. Due to MAPLE's powerful symbolic computational capabilities, the birational transformation between (4.2) and the Weierstrass model was computed using a simple implementation of Nagell's algorithm in MAPLE 18. The calculation of the elliptic logarithms, the upper bound and the reduction was also done in MAPLE 18, using MAPLE's built-in LLL routine. The computation of the group of rational points on the curves and the search using the bounds from the reduction were carried out in both MAGMA and SAGE. The runtimes on a personal computer equipped with an AMD A10-7800 CPU ranged from being only a few seconds to few minutes depending on the curves. The most time consuming steps were the computation of the Mordell-Weil generators, the reduction process and the exhaustive search below the reduced bound. Compared to the run times presented for example in [106], no big difference is seen. This may be caused by the small absolute value of the parameters. The goal of the present paper is to investigate an interesting family of diophantine equations, as the run times show however one can extend these results to higher values of m and n if desired. Also, we think that an algorithm could be written in MAGMA (or any other of the three), that gives a list of all solutions for given m and n, similarly to some other special cases of genus 1 equation solvers already present in MAGMA.

Chapter 5

Summary

In our dissertation, we combine the latest effective methods with our own observations to give effective results for infinite families of diophantine equations and inequalities with interesting number theoretic backgrounds. The dissertation consists of four chapters. In the first, we use elementary methods with modular arguments to give all solutions to an infinite family of equations. Let (a, b, c) be a given primitive Pythagorean triple such that $a^2 + b^2 = c^2$, and consider the equation

$$c^x + b^y = a^z \tag{5.1}$$

in positive integer unknowns x, y and z.

Conjecture 3. *With the above conditions, equation* (5.1) *has the only solution* $(x, y, z) = (1, 1, 2)$ *if* $c = b + 1$*. If* $c > b + 1$ *then* (5.1) *has no solutions.*

This is referred to as the *shuffle* variant of Jeśmanovicz' problem. In [80], Miyazaki proved that Conjecture 3 is true if $c \equiv 1 \pmod{b}$. In our work, we extend his work with the following results.

Theorem 16. *Let* b_0 *be a divisor of* b*, such that* b_0 *is divisible by* $\mathrm{rad}(b)$*. Suppose* *that Conjecture 3 is true for all Pythagorean triples* (a,b,c) *with*

$$c \equiv 1 \pmod{b_0}. \tag{5.2}$$

Then Conjecture 3 is true for all Pythagorean triples (a,b,c) *with*

$$c \equiv 1 \pmod{b_0/2}. \tag{5.3}$$

Theorem 17. *Conjecture 3 is true for all Pythagorean triples* (a,b,c) *with*

$$c \equiv 1 \pmod{b/2^{\mathrm{ord}_2(b)}}.$$

In the second chapter, we combine a deep result of Bilu, Hanrot and Voutier [124] with results concerning Ljunggren-type and elliptic curves to give all solutions to the equation

$$x^2 + 5^k 17^l = y^n \tag{5.4}$$

in integer unknowns x, y, k, l, n satisfying

$$x \geq 1, y > 1, n \geq 3,\ k \geq 0,\ l \geq 0 \text{ and } \gcd(x,y) = 1. \tag{5.5}$$

The latter equation is often called the generalized Ramanujan-Nagell equation. The first results concerning equations similar to (5.4) were given by Lebesque [64], Ljunggren [65] and Nagell [89], [91]. In our work, we prove an analogues results such as by Luca and Togbe [70], [71]. Our main result is the following.

Theorem 18. *Consider equation* (5.4) *satisfying* (5.5)*. Then all solutions of equation (5.4) are:*

$$(x,y,k,l,n) \in \{(94,21,2,1,3),(2034,161,3,2,3),(8,3,0,1,4)\}.$$

In the third chapter, we combine a refined version of Baker's method with hypergeometric approximation methods to effectively bound the number of solutions of a family of binomial Thue inequalities. Here we extend a former result of Bennett [20] and prove the following.

Theorem 19. *Let c be a positive integer. Then there exists an effectively computable finite set S_c of triples of positive integers a, b and n with the property that if a, b and $n \geq 3$ are any positive integers for which the Diophantine inequality*

$$|ax^n - by^n| \leq c \tag{5.6}$$

has more than a single solution in positive integers x and y, then $(a, b, n) \in S_c$.

Theorem 20. *With S_c defined above, we have $S_3 \subseteq S_3^* \cup T_3$, where*

$$S_3^* = \{(1,2,3), (2,1,3), (1,3,3), (3,1,3), (2,5,3), (5,2,3)\}$$

and

$$T_3 = \{(1,3,n), (3,1,n), (2,5,n), (5,2,n) \ \textit{with} \ 37 \leq n \leq 347, \ n \textit{ prime} \}.$$

For $(a, b, n) \in S_3^*$, the solutions in positive integers to inequality (5.6) with $c = 3$ are, in each case, $(x, y) = (1, 1)$, and also

(a,b,n)	$(1,2,3)$	$(2,1,3)$	$(1,3,3)$	$(3,1,3)$	$(2,5,3)$	$(5,2,3)$
(x,y)	$(5,4)$	$(4,5)$	$(3,2)$	$(2,3)$	$(19,14)$	$(14,19)$

In the final chapter, we apply the so-called Ellog method which was developed by Stroeker and Tzanakis [108], and independently by Gebel, Pethő and Zimmer [53] and later improved by Stroeker and Tzanakis [109] to solve a problem concerning pyramidal numbers. Set $\mathrm{Pyr}_m(x) = \dfrac{x(x+1)((m-2)x+5-m)}{6}$ and consider the equation

$$\mathrm{Pyr}_m(u) = \mathrm{Pyr}_n(v), \tag{5.7}$$

in positive integers u and v for given m and n. We prove the following results.

Theorem 21. *Let m and n be given positive integers with $3 \leq \min(m, n)$ and $m \neq n$. Then the equation (5.7) has at most finitely many solutions in integer unknowns u and v. In fact $\max(u, v) < C_1$, where C_1 is an effectively computable positive constant depending only on m and n.*

Theorem 22. *For given m and n with $3 \leq n < m \leq 10$, all solutions of (5.7) in positive integers (u, v) with $(u, v) \neq (1, 1)$ are given by*

$$(m, n, u, v) \in \left\{ \begin{array}{c} (8, 3, 7, 12), (9, 3, 2, 3), (8, 4, 3, 4), (10, 4, 55, 87), \\ (7, 5, 6, 7), (10, 6, 35, 44), (9, 7, 152, 170). \end{array} \right\} \tag{5.8}$$

Chapter 6

Összefoglaló

Jelen disszertációban effektív végességi tételeket kombináltunk saját észrevételeinkkel, melyekkel végességi eredményeket tudtunk bizonyítani érdekes számelméleti háttérrel rendelkező diofantikus egyenletek végtelen családjaira. A disszertáció négy fejezetből áll. Az elsőben elemi módszerek és a lokális módszer kombinálásával megadjuk egy végtelen egyenletcsalád összes megoldását. Legyen (a, b, c) egy primitív pitagorarszi számhármas, melyre $a^2 + b^2 = c^2$, és tekintsük a

$$c^x + b^y = a^z \tag{6.1}$$

egyenletet pozitív egész x, y és z ismeretlenekben.

1. Sejtés A fenti feltételekkel a (6.1) egyenlet egyetlen megoldása $(x, y, z) = (1, 1, 2)$, ha $c = b + 1$. Ha $c > b + 1$ akkor (6.1)-nak nincs megoldása.

Ez utóbbi eredményt szokás a kevert Jeśmanovicz problémának hívni. Miyazaki [80] bizonyította, hogy az 1. Sejtés igaz, ha $c \equiv 1 \pmod b$. Disszertációnkban ez utóbbi eredményét általánosítjuk. A fő eredményeink a következők.

1. Tétel Legyen b_0 egy osztója b-nek, melyre b_0 osztható b radikáljával. Tegyük fel hogy az 1. Sejtés igaz minden olyan (a, b, c) pitagoraszi hármasra, melyre

$$c \equiv 1 \pmod{b_0}. \tag{6.2}$$

Ekkor az 1. Sejtés igaz minden olyan (a, b, c) hármas esetén, melyre

$$c \equiv 1 \pmod{b_0/2}. \tag{6.3}$$

2. Tétel Az 1. Sejtés igaz minden olyan (a, b, c) hármas esetén, melyre

$$c \equiv 1 \pmod{b/2^{\mathrm{ord}_2(b)}}.$$

A második fejezetben Bilu, Hanrot és Voutier [124] egy Lucas-sorozatokra vonatkozó mély eredményét kombináljuk Ljunggren típusú és elliptikus görbékre vonatkozó eredményekkel, hogy meghatározzuk az

$$x^2 + 5^k 17^l = y^n \tag{6.4}$$

egyenlet összes megoldását x, y, k, l, n egész ismeretlenekben, melyekre

$$x \geq 1, y > 1, n \geq 3,\ k \geq 0,\ l \geq 0 \text{ and } \gcd(x, y) = 1. \tag{6.5}$$

Az utóbbi egyenletet gyakran hívják általánosított Ramanujan-Nagell egyenletnek. Az első, (6.4)-hez hasonló egyenletekre vonatkozó eredmények Lebesquehez [64], Ljunggrenhez [65] és Nagellhez [89], [91] köthetők. Munkánkban néhány szerző (például Luca és Togbe [70], [71]) friss eredményeivel analóg eredményeket bizonyítunk. Fő eredményünk a következő.

3. Tétel Az (6.4) egyenlet (6.5) feltételnek eleget tevő összes megoldása

$$(x, y, k, l, n) \in \{(94, 21, 2, 1, 3), (2034, 161, 3, 2, 3), (8, 3, 0, 1, 4)\}.$$

A harmadik fejezetben a Baker módszer egy Laurent [59] által kidolgozott változatát kombináljuk hipergeometrikus approximációs technikákkal, hogy effektív korlátot adjunk binom Thue egyenlőtlenésgek egy végtelen családjának megoldásszámára. Ezzel Bennett egy korábbi eredményét [20] általánosítjuk. Fő eredményünk a következő.

4. Tétel Legyen c egy pozitív egész. Ekkor létezik a, b és n egészekből álló hármasok egy effektíven meghatározható S_c halmaza azzal a tulajdonsággal, hogy ha a, b és $n \geq 3$ olyan pozitív egészek, melyekre az

$$|ax^n - by^n| \leq c \tag{6.6}$$

egyenlőtlenségnek egynél több megoldása van x és y pozitív egészekben, akkor $(a, b, n) \in S_c$.

5. Tétel Ha S_c a fenti módon adott, akkor $S_3 \subseteq S_3^* \cup T_3$, ahol

$$S_3^* = \{(1,2,3), (2,1,3), (1,3,3), (3,1,3), (2,5,3), (5,2,3)\}$$

and

$$T_3 = \{(1,3,n), (3,1,n), (2,5,n), (5,2,n) \text{ ahol } 37 \leq n \leq 347, \ n \text{ prím} \}.$$

Ha $(a, b, n) \in S_3^*$, akkor a (6.6) egyenlőtlenség összes megoldása $c = 3$ esetén minden esetben $(x, y) = (1, 1)$, valamint

(a,b,n)	$(1,2,3)$	$(2,1,3)$	$(1,3,3)$	$(3,1,3)$	$(2,5,3)$	$(5,2,3)$
(x,y)	$(5,4)$	$(4,5)$	$(3,2)$	$(2,3)$	$(19,14)$	$(14,19)$

Az utolsó fejezetben az úgynevezett Elliptikus logaritmusok módszerét alkalmazzuk egy piramidális számok egyenlő értékeire vonatkozó probléma megoldására. Ezt a módszert Stroeker és Tzanakis [108] fejlesztette ki, illetve tőlük függetlenül Gebel, Pethő és Zimmer [53]. Legyenek m és n adott pozitív egész számok, valamint $\mathrm{Pyr}_m(x) = \dfrac{x(x+1)((m-2)x+5-m)}{6}$. Tekintsük a

$$\mathrm{Pyr}_m(u) = \mathrm{Pyr}_n(v), \tag{6.7}$$

egyenletet u és v pozitív egész ismeretlenekben. Fő eredményeink a következők.

6. Tétel Legyenek m és n adott pozitív egészek, melyekre $3 \leq \min(m, n)$ és $m \neq n$. Ekkor a (6.7) egyenletnek csak véges sok megoldása van u és v pozitív egészekben. Továbbá $\max(u, v) < C_1$, ahol C_1 egy effektíven meghatározható, csak m-től és n-től függő konstans.

7. Tétel Adott m és n egészekre, melyekre $3 \leq n < m \leq 10$, a (6.7) egyenlet összes megoldása (u, v) pozitív egészekben, melyre $(u, v) \neq (1, 1)$,

$$(m, n, u, v) \in \left\{ \begin{array}{c} (8,3,7,12), (9,3,2,3), (8,4,3,4), (10,4,55,87), \\ (7,5,6,7), (10,6,35,44), (9,7,152,170). \end{array} \right\} \tag{6.8}$$

Bibliography

[1] K. Győry A. Bazsó, A. Bérczes and Á. Pintér. On the resolution of equations $ax^n - by^n = c$ in integers x, y and $n \geq 3$, ii. *Publ. Math. Debrecen*, 76:227–25, 2010.

[2] B. Brindza A. Bérczes and L. Hajdu. On power values of polynomials. *Publ. Math. Debrecen,*, 53:375–381, 1998.

[3] K. Győry A. Dujella and Á. Pintér. On power values of pyramidal numbers, I. *Acta Arith.*, 155(2):217–226, 2012.

[4] S. A. Arif and F .S. Muriefah. On the diophantine equation $x^2 + 2^k = y^n$. *Internat. J. Math. Math. Sci.*, 20:299–304, 1997.

[5] S. A. Arif and F. S. Muriefah. The diophantine equation $x^2 + 3^m = y^n$. *Internat. J. Math. Math. Sci.*, 21:619–620, 1998.

[6] S. A. Arif and F. S. Muriefah. The diophantine equation $x^2 + q^{2k} = y^n$. *Arab. J. Sci. Sect. A Sci.*, 26:53–62, 2001.

[7] S. A. Arif and F. S. Muriefah. On the diophantine equation $x^2 + 2^k = y^n$ ii. *Arab J. Math. Sci.*, 7:67–71, 2001.

[8] S. A. Arif and F. S. Muriefah. On the diophantine equation $x^2 + q^{2k+1} = y^n$. *J. Number Theory*, 95:95–100, 2002.

[9] Á. Pintér B. Brindza and S. Turjányi. On equal values of pyramidal and polygonal numbers. *Indag. Math. (N.S.)*, 9(2):183–185, 1998.

[10] A. Baker. Rational approximations to $\sqrt[3]{2}$ and other algebraic numbers. *Quart. J. Math. Oxford Ser. (2)*, 15:375–383, 1964.

[11] A. Baker. Rational approximations to certain algebraic numbers. *Proc. London Math. Soc. (3)*, 14:385–398, 1964.

[12] A. Baker. Simultaneous rational approximations to certain algebraic numbers. *Proc. Cambridge Phil. Soc*, 63:693–702, 1967.

[13] A. Baker. Contributions to the theory of diophantine equations. *Phil. Trans. Roy. Soc. London,*, 263:173–208, 1968.

[14] A. Baker and J. Coates. Integer points on curves of genus 1. *Mathematical Proceedings of the Cambridge Philosophical Society*, 67(3):595–602, 1970.

[15] A. Baker and H. Davenport. The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$. *Quart. J. Math. Oxford Ser. (2)*, 20:129–137, 1969.

[16] A. Bazsó. On binomial thue equations and ternary equations with s-unit coefficients. *Publ. Math. Debrecen*, 77:499–516, 2010.

[17] M. A. Bennett. Simultaneous rational approximation to binomial functions. *J. Austral. Math. Soc.*, 348:1717–1738, 1996.

[18] M. A. Bennett. Effective measures of irrationality for certain algebraic numbers. *J. Austral. Math. Soc.*, 62:329–344, 1997.

[19] M. A. Bennett. Explicit lower bounds for rational approximation to algebraic numbers. *Proc. London Math. Soc.*, 75:63–78, 1997.

[20] M. A. Bennett. Rational approximation to algebraic numbers of small height: the diophantine equation $|ax^n - by^n| = 1$. *J. Reine Angew. Math.*, 535:1–49, 2001.

[21] M. A. Bennett. Products of consecutive integers. *Bull. London Math. Soc.*, 36:683–694, 2004.

[22] M. A. Bennett. The diophantine equation $(x^k - 1)(y^k - 1) = (z^k - 1)^t$. *Indag. Math. (N.S.)*, 18(4):507–525, 2007.

[23] M. A. Bennett and B. M. M. de Weger. On the diophantine equation $|ax^n - by^n| = 1$. *Math. Comp.*, 67:413–438, 1998.

[24] M. A. Bennett and C. Skinner. Ternary diophantine equations via galois representations and modular forms. *Canad. J. Math.*, 56:23–54, 2004.

[25] M. A. Bennett and C. M. Skinner. Ternary diophantine equations via galois representations and modular forms. *Canad. J. Math.*, 56(1):23–54, 2004.

[26] Cs. Bertók and L. Hajdu. A Hasse-type principle for exponential diophantine equations and its applications. *Mathematics of Computation*, 85(298):849–860, 2016.

[27] E. Bombieri and W. Schmidt. On thue's equation. *Invent. Math.*, 88:69–81, 1987.

[28] A. Bérczes and A. Pethő. On norm form equations with solutions forming arithmetic progressions. *Publ. Math. Debrecen*, 65(3–4):281–290, 2004.

[29] A. Bérczes and A. Pethő. Computational experiences on norm form equations with solutions from an arithmetic progression. *Glas. Mat. Ser. III*, 41(61):1–8, 2006.

[30] A. Bérczes and I. Pink. On the diophantine equation $x^2 + p^{2k} = y^n$. *Archiv der Mathematik,*, 91:505–517, 2008.

[31] R. D. Carmichael. On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$. *Ann. Math. (2),*, 15:30–70, 1913.

[32] G. V. Chudnovsky. On the method of thue-siegel. *Ann. of Math. (2)*, 117:325–382, 1983.

[33] J. H. E. Cohn. The diophantine equation $x^2 + 2^k = y^n$. *Arch. Math (Basel)*, 59:341–344, 1992.

[34] J. H. E. Cohn. The diophantine equation $x^2 + c = y^n$. *Acta Arith.*, 65:367–381, 1993.

[35] J. H. E. Cohn. The diophantine equation $x^2 + 2^k = y^n.ii.$. *Int. J. Math. Math. Sci.,*, 22:459–462, 1999.

[36] J. H. E. Cohn. The diophantine equation $x^2 + c = y^n$ ii. *Acta Arith.*, 109:205–206, 2003.

[37] I. Connell. Elliptic curve handbook. *Preprint*, 1996.

[38] S. David. Minorations de formes linéaires de logarithmes elliptiques. *Mémoires de la Societé Mathématique de France*, 62:1–143, 1995.

[39] B. M. M. de Weger. *Algorithms for Diophantine equations*. CWI-Tract No. 65, 1989.

[40] V. A. Dem'janenko. On Jeśmanowicz' problem for pythagorean numbers. *Izv. Vysš. Učebn. Zaved. Mat.*, 48:52–56, 1965.

[41] Y. F. Deng and C. H. Yang. Waring's problem for pyramidal numbers. *Sci. China Ser. A*, 37(3):277–283, 1994.

[42] The Sage Developers. *Sage Mathematics Software (Version 6.4.1)*, 2015. http://www.sagemath.org.

[43] E. Deza and M. Deza. *Figurate numbers*. World Scientific, 2012.

[44] L. E. Dickson. *History of the Theory of Numbers, Volume II: Diophantine Analysis*. Courier Dover Publications, 2012.

[45] F. Luca E. Goins and A. Togbé. On the diophantine equation x2+ $2\alpha 5\beta 13\gamma$= yn, algorithmic number theory. *Lecture Notes in Comput. Sci*, 5011:430–442, 2008.

[46] J. S. Ellenberg. Galois representations attached to q-curves and the generalized fermat equation a^4+ b^2 =c^P. *American journal of mathematics*, pages 763–787, 2004.

[47] J. H. Evertse. *Upper Bounds for the numbers of solutions of diophantine equations*. PhD thesis, Leiden, 1983.

[48] F. Luca F. S. Muriefah and A. Togbe. On the diophantine equation $x^2 + 5^a 13^b = y^n$. *Glasgow Math. J*, 50:175–181, 2008.

[49] K. Győry and Á. Pintér. Almost perfect powers in products of consecutive integers. *Monatsh. Math.*, 145:19–33, 2005.

[50] K. Győry and Á. Pintér. On the resolution of equations $ax^n - by^n = c$ in integers x, y and $n \geq 3$, i. *Publ. Math. Debrecen*, 70:483–501, 2007.

[51] K. Győry and Á. Pintér. Binomial thue equations, ternary equations and power values of polynomials (in russian). *Fundam. Prikl. Mat.*, 16(5):61–77, 2010.

[52] G. Soydan N. Tzanakis I.N. Cangül, M. Demirci. On the diophantine equation $x^2 + 5^a 11^b = y^n$. *Funct. Approx. Comment. Math.*, 43:209–225, 2010.

[53] A. Pethő J. Gebel and H. G. Zimmer. Computing integral points on elliptic curves. *Acta Arith.*, 68(2):171–192, 1994.

[54] I. Pink K. Győry and Á. Pintér. Power values of polynomials and binomial thue-mahler equations. *Publ. Math. Debrecen*, 65:341–362, 2004.

[55] I. Pink K. Győry and Á. Pintér. Power values of polynomials and binomial thue-mahler equations. *Publ. Math. Debrecen*, 65:341–362, 2004.

[56] L. Hajdu K. Győry and N. Saradha. On the diophantine equation $n(n + d) \cdots (n + (k-1)d = by^l)$. *Canad. Math. Bull.*, 47:373–388, 2004.

[57] T. Kovács and Zs. Rábai. Equal values of pyramidal numbers. *Indag. Math.*, 29(5):1157–1166, 2018.

[58] Sz. Tengely L. Hajdu, Á Pintér and N. Varga. Equal values of figurate numbers. *J. Number Th.*, 137:130–141, 2014.

[59] M. Laurent. Linear forms in two logarithms and interpolation determinants ii. *Acta Arith.*, 133(4):325–348, 2008.

[60] M. Le. A note on Jeśmanowicz' conjecture. *Colloq. Math.*, 69(1):47–51, 1995.

[61] M. Le. On Jeśmanowicz' conjecture concerning Pythagorean numbers. *Proc. Japan Acad. Ser. A Math. Sci.*, 72(5):97–98, 1996.

[62] M. Le. On the diophantine equation $x^2 + p^2 = y^n$. *Publ. Math. Debrecen*, 63:27–78, 2003.

[63] M. Le. A note on Jeśmanowicz' conjecture concerning primitive Pythagorean triplets. *Acta Arith.*, 138(2):137–144, 2009.

[64] V. A. Lebesgue. Sur l'impossibilité en nombres entiers de l'équation xm= y2+ 1, nouv. *Ann. Math*, 9(9):178–181, 1850.

[65] W. Ljunggren. *Über einige Arcustangensgleichungen, die auf interessante unbestimmte Gleichungen führen*. Almqvist & Wiksell, 1943.

[66] W. Ljunggren. On an improvement of a theorem of t. nagell concerning the diophantine equation $ax^3 + by^3 = c$. *Math. Scand.*, 1:297–309, 1953.

[67] W. Lu. On the Pythagorean numbers $4n^2 - 1$, $4n$, $4n^2 + 1$. *Journal of Sichuan University (Natural Science Edition)*, 2:39–42, 1959.

[68] F. Luca. On a diophantine equation. *Bull. Austral. Math. Soc.*, 61:241–246, 2000.

[69] F. Luca. On the equation $x^2 + 2^a 3^b = y^n$. *Int. J. Math. Sci.*, 29:239–244, 2002.

[70] F. Luca and A. Togbe. On the diophantine equation $x^2 + 7^{2k} = y^n$. *Fibonacci Quarterly*, 54:322–326, 2007.

[71] F. Luca and A. Togbe. On the diophantine equation $x^2 + 2^a 5^b = y^n$. *Int. J. Number Theory 4,*, 6:973–979, 2008.

[72] I. Pink M. A. Bennett and Zs. Rábai. On the number of solutions of binomial thue inequalities. *Publ. Math. Debrecen*, 83(1-2):241–256, 2013.

[73] J. S. Ellenberg M. A. Bennett and N. C. Ng. The diophantine equation a4+ 2δb2= cn. *International Journal of Number Theory*, 6(02):311–338, 2010.

[74] M. Mignotte M. A. Bennett, K. Győry and Á. Pintér. Binomial thue equations and polynomial powers. *Compos. Math.*, 142:1103–1121, 2006.

[75] K. Mahler. Ein beweis des thue-siegelschen satzes über die approximation algebraischer zahlen für binomische gleichungen. *Math. Ann.*, 105:267–276, 1931.

[76] K. Mahler. Zur approximation algebraischer zahlen, i: Über den grössten primteiler binärer formen. *Math. Ann.*, 107:691–730, 1933.

[77] M. Mignotte. A note on the equation $ax^n - by^n = c$. *Acta Arith.*, 75:287–295, 1996.

[78] M. Mignotte and B. M. M. de Weger. On the equations $x^2 + 74 = y^5$ and $x^2 + 86 = y^5$. *Glasgow Math. J.*, 8(1):77–85, 1996.

[79] T. Miyazaki. On the conjecture of Jeśmanowicz concerning Pythagorean triples. *Bull. Aust. Math. Soc.*, 80(3):413–422, 2009.

[80] T. Miyazaki. The shuffle variant of Jeśmanowicz' conjecture concerning Pyzhagorean triples. *Journal of the Australian Mathematical Society*, 90(03):355–370, 2011.

[81] T. Miyazaki. Terai's conjecture on exponential Diophantine equations. *Int. J. Number Theory*, 7(4):981–999, 2011.

[82] T. Miyazaki. *On the exponential Diophantine equation $a^x + b^y = c^z$*. PhD thesis, Tokyo Metropolitan University, 2012.

[83] T. Miyazaki. Generalizations of classical results on Jeśmanowicz' conjecture concerning Pythagorean triples. *Journal of Number Theory*, 133(2):583–595, 2013.

[84] L. J. Mordell. *Diophantine Equations*. Academic Press, 1969.

[85] J. Mueller and W. Schmidt. Thue's equations and a conjecture of siegel. *Acta Math.*, 160:207–247, 1988.

[86] F. S. Muriefah. On the diophantine equation $x^2 + 5^{2k} = y^n$. *Demonstratio Mathematica*, 319(2):285–289, 2006.

[87] Á. Pintér N. and Varga. Resolution of a nontrivial diophantine equation without reduction methods. *Publ. Math. Debrecen*, 79(3-4):605–610, 2011.

[88] F. Luca Á. Pintér G. Soydan N. Cangül, M. Demirci. On the diophantine equation $x^2 + 2^a 11^b = y^n$. *Fibonacci Quart.*, 48:39–46, 2010.

[89] T. Nagell. Sur l'impossibilit$\acute{e}$ de quelques $\acute{e}$quations a deux indetermin$\acute{e}$es. *Norsk. Mat. Forensings Skifter*, 13:65–82, 1923.

[90] T. Nagell. Sur les propriétés arithmétiques des cubiques planes du premier genre. *Acta Math.*, 52(1):93–126, 1929.

[91] T. Nagell. *Contributions to the theory of a category of Diophantine equations of the second degree with two unknowns*. Almqvist & Wiksells boktr., 1955.

[92] A. Pethő. On the resolution of thue inequalities. *J. Symbolic Comput.*, 4:103–109, 1987.

[93] I. Pink. On the diophantine equation $x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n$. *Publ. Math. Debrecen*, 70(1-2):149–166, 2007.

[94] I. Pink and Zs. Rábai. On the diophantine equation $x^2 + 5^k 17^l = y^n$. *Commun. Math.*, 19(1):1–9, 2011.

[95] P. Ribenboim. *Catalan's conjecture: are 8 and 9 the only consecutive powers?* Academic Press Boston, 1994.

[96] K. A. Ribet. On the equation $a^p + 2^\alpha b^p + c^p = 0$. *Acta Arith.*, 79:7–16, 1997.

[97] H. W. Richmond. Notes on a problem of the "Waring" type. *J. London Math. Soc.*, 19:38–41, 1944.

[98] Zs. Rábai. A note on the shuffle variant of jeśmanowicz' conjecture. *Tokyo J. Math.*, 40(1):153–163, 2017.

[99] N. Saradha and A. Srinivasan. Solutions of some generalized ramanujan-nagell equations. *Indag. Math. (N.S.)*, 17(1):103–114, 2006.

[100] N. Saradha and A. Srinivasan. Solutions of some generalized ramanujan-nagell equations via binary quadratic forms. *Publ. Math. Debrecen*, 71(3-4):349–374, 2007.

[101] A. Schinzel and R. Tijdeman. On the equation $y^m = p(x)$. *Acta Arith.*, 31:199–204, 1976.

[102] S. L. Segal. Mathematical Notes: A Note on Pyramidal Numbers. *Amer. Math. Monthly*, 69(7):637–638, 1962.

[103] T. N. Shorey and R. Tijdeman. *Exponential diophantine equations*. Cambridge–New York, 1986.

[104] C. L. Siegel. Die gleichung $ax^n - by^n = c$. *Math. Ann.*, 114:57–68, 1937.

[105] W. Sierpiński. On the equation $3^x + 4^y = 5^z$. *Wiadom. Mat.*, 1:194–195, 1955/56.

[106] R. Stroeker and B. M. M. de Weger. Elliptic binomial diophantine equations. *Math. Comput. Amer. Math. Soc.*, 68(227):1257–1281, 1999.

[107] R. J. Stroeker and B. M. M. de Weger. Solving elliptic diophantine equations: the general cubic case. *Acta Arith.*, 87:339–365, 1998.

[108] R. J. Stroeker and N. Tzanakis. Solving elliptic diophantine equations by estimating linear forms in elliptic logarithms. *Acta Arith.*, 67(2):177–196, 1994.

[109] R. J. Stroeker and N. Tzanakis. Computing all integer solutions of a genus 1 equation. *Math. Comput. Amer. Math. Soc.*, 72(244):1917–1933, 2003.

[110] P. Yuan T. Miyazaki and D. Wu. Generalizations of classical results on Jeśmanowicz' conjecture concerning Pythagorean triples, II. *Journal of Number Theory*, 141(1):184–201, 2014.

[111] R. Tijdeman T. N. Shorey, A. J. Van der Poorten and A. Schinzel. Applications of the gel'fond-baker method to diophantine equations. *Transcendence theory: advances and applications*, pages 59–77, 1977.

[112] K. Takakuwa and Y. Asaeda. On a conjecture on Pythagorean numbers. *Proc. Japan Acad. Ser. A Math. Sci.*, 69(7):252–255, 1993.

[113] Kei Takakuwa. On a conjecture on Pythagorean numbers. III. *Proc. Japan Acad. Ser. A Math. Sci.*, 69(9):345–349, 1993.

[114] Kei Takakuwa and You Asaeda. On a conjecture on Pythagorean numbers. II. *Proc. Japan Acad. Ser. A Math. Sci.*, 69(8):287–290, 1993.

[115] Sz. Tengely. On the diophantine equation $x^2 + a^2 = 2y^p$. *Indag. Math. (N.S.)*, 15:291–304, 2004.

[116] N. Terai. The Diophantine equation $a^x + b^y = c^z$. III. *Proc. Japan Acad. Ser. A Math. Sci.*, 72(1):20–22, 1996.

[117] N. Terai and K. Takakuwa. A note on the Diophantine equation $a^x + b^y = c^z$. *Proc. Japan Acad. Ser. A Math. Sci.*, 73(9):161–164, 1997.

[118] Nobuhiro Terai. The Diophantine equation $a^x + b^y = c^z$. *Proc. Japan Acad. Ser. A Math. Sci.*, 70(1):22–26, 1994.

[119] Nobuhiro Terai. The Diophantine equation $a^x + b^v = c^z$. II. *Proc. Japan Acad. Ser. A Math. Sci.*, 71(6):109–110, 1995.

[120] A. Thue. Über annäherungswerte algebraischer zahlen. *J. Reine Angew. Math.*, 135:284–305, 1909.

[121] R. Tijdeman. Some applications of baker's sharpened bounds to diophantine equations. *Séminaire Delange-Pisot-Poitou. Théorie des nombres*, 16(2):1–7, 1974.

[122] P. M. Voutier. Primitive divisors of lucas and lehmer sequences. *mathematics of computation*, 64(210):869–888, 1995.

[123] J. Cannon W. Bosma and C. Playoust. The magma algebra system I: The user language. *J. Symbolic Comput.*, 24(3):235–265, 1997.

[124] G. Hanrot Y. Bilu and P. M. Voutier. Existence of primitive divisors of lucas and lehmer numbers. with an appendix by m. mignotte. *J. Reine Angew. Math.*, 539:75–122, 2001.

[125] F.S. Abu Muriefah Y. Bugeaud. The diophantine equation $x^2 + c = y^n$: a brief overview. *Revista Colombiana de Matematicas,*, 40:31–37, 2006.

[126] M. Mignotte Y. Bugeaud and S. Siksek. Classical and modular approaches to exponential and diophantine equations ii. the lebesque-nagell equation. *Compos. Math.*, 142(1):31–62, 2006.