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Some New Findings on Mixed Bertrand-Edgeworth Duopolies

PhD Thesis

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A doktori értekezés betétlapja

SOME NEW FINDINGS ON MIXED BERTRAND-EDGEWORTH DUOPOLIES

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Nyilatkozat

Alulírott Balogh Tamás László nyilatkozom, hogy

a) értekezésemet korábban más intézményben nem nyújtottam be és azt nem utasították el;
b) nem állok doktori fokozat visszavonására irányuló eljárás alatt, illetve 5 éven belül nem vontak vissza tőlem korábban odaítélt doktori fokozatot;
c) a disszertáció önálló munkám, az irodalmi hivatkozások egyértelműek és teljesek.

Kelt: Debrecen, 2015. február 9.

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Balogh Tamás László

A doktori értekezés összefoglalója

SOME NEW FINDINGS ON MIXED BERTRAND-EDGEWORTH DUOPOLIES (VEGYES BERTRAND-EDGEWORTH DUOPÓLIUMOK JÁTÉKELMÉLETI MODELLEZÉSE)

A mindennapi életben gyakran találkozunk duopol piaci szituációkkal. Ezért fontosnak tartjuk, hogy egzakt és alkalmazható modelleket építsünk a vállalatok döntéseinek megértése és előrejelzése érdekében. Minden modell, melynek a gyakorlatra alkalmazható kiindulási feltevései vannak, hozzájárul ahhoz, hogy jobban megértsük egy termék piacának kínálati oldalát.

A disszertáció fő célja, hogy bemutassa, hogyan döntenek a vállalatok egyensúlyban a vegyes Bertrand-Edgeworth duopólium-modellekben. Feltételezve állami tulajdon jelenlétét a piacon, az értekezés részletesen elemzi a vállalatok ár- és mennyiségi döntéseit, az időzítési játékot, valamint az állami tulajdon társadalmi jólétre gyakorolt hatásait. A disszertáció újdonságértéke a Bertrand-Edgeworth duopóliumokkal kapcsolatos ismeretek kibővítésében rejlik.

A disszertáció fókuszában a Bertrand-Edgeworth duopóliumok azon típusa áll, amelyben az állam egy versengő vállalat részbeni vagy kizárólagos tulajdonosként megjelenik a piacon. Az értekezés célja az ilyen vegyes duopol piacok játékelméleti elemzése.

A disszertáció hozzáadott értéke a következőképp foglalható össze:

1. Választ adunk arra a kérdésre, hogy meglehetősen gyenge feltevések mellett a vegyes Bertrand-Edgeworth duopol modelleknek milyen feltételek mellett létezik tiszta Nash-egyensúlyi pontja. Ezen kívül meg is adjuk a tiszta Nash-egyensúlyi pontokat.
2. Feltéve, hogy a vállalatok döntési sorrendje endogén (azaz a vállalatok időzítési játékot is játszanak), megadjuk az egyensúlyi döntési sorrendet az összes vizsgált vegyes- és félig-vegyes Bertrand-Edgeworth duopóliumra.
3. Vizsgáljuk a vállalaton belüli állami tulajdon össztársadalmi jólétre gyakorolt hatását is a tisztán magánvállalatos esettel történő összehasonlításban.

Summary of the doctoral thesis

SOME NEW FINDINGS ON MIXED BERTRAND-EDGEWORTH DUOPOLIES

Duopolies are present in our everyday life. Therefore, it is important to build sophisticated models that explain and predict oligopolist behavior. Every model that has valid assumptions in describing or helping in the description of a real life market structure contributes to knowledge on producer behavior.

The dissertation aims at presenting equilibrial firm behavior in a mixed Bertrand-Edgeworth duopoly environment. Assuming partial public ownership on the market, we provide a thorough analysis concerning equilibrium prices and quantities, endogenous timing and social welfare effects. The contribution of the thesis to knowledge is broadening the theory of Bertrand-Edgeworth models for a better understanding of duopolistic markets with public ownership.

The main contribution of the thesis is summarized as follows:

1. The existence of a pure strategy Nash equilibrium in a mixed Bertrand-Edgeworth duopoly is analyzed. Furthermore, the equilibrium profiles are characterized for all the given subcases.
2. Provided that the ordering of firms' decisions is endogenous (i.e. we consider the timing game), the equilibrial ordering of moves is given for mixed and semi-mixed Bertrand-Edgeworth duopolies.
3. A social welfare analysis is carried out for the mixed Bertrand-Edgeworth duopoly, i.e. we characterize the difference the presence of a public firm makes in social welfare compared to the standard case with only private firms competing on the market.

Maminak

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My special thanks go to Attila Tasnádi, professor at Corvinus University of Budapest. I really appreciate his enormous efforts he exerts in our joint projects. Being a member of his MTA "Lendület" research group means I have the opportunity to share ideas in a challenging atmosphere.

I wish to thank Prof. Christian Ewerhart for having me at the University of Zurich during the 2012/13 academic year. Under his leadership I had the exceptional opportunity to experience the scientific excellence as well as the everyday life of a leading European university, and, of course, the Swiss way of life.

I am also very grateful to the official reviewers of the first version of my dissertation. The comments of Ferenc Forgó and József Gáll have helped me a lot to improve the dissertation. I also wish to thank my mom for the grammatical correction. I take the responsibility for any remaining mistakes.

I would like to thank all my fellow PhD students and colleagues from the University of Debrecen, Corvinus University of Budapest and the University of Zurich for a friendly and productive atmosphere. I feel very lucky to have such great colleagues around me.

Last, but not least, I wish to thank my family and friends for their constant support and trust in me.

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1 Introduction

Why are there fewer available units if only two or three firms control the market of a given product? Why do monopolies set higher prices than sellers on a perfect competition market? Questions that might arise in people's everyday life. Whether they are natural persons or legal entities, as consumers, everyone is affected by the different market structures that emerge on the supplier side of a desired product or service. As far as prices and available quantities are concerned, it is by far not the same, whether the buyer faces a perfect competition, an oligopoly or a monopoly on the market.

Since the birth of microeconomic theory, the discipline has been trying to explain consumer and supplier behavior by building mathematical models. Of course, one does not have to assume that the participants of a given market are robots who execute commands based on exact models. However, models have proved to be useful in describing and understanding the reasons why certain decisions are preferred to others in most of the real-life market situations.

Modelling consumer and producer behavior is separated already in most of the bachelor microeconomics textbooks. The field of Industrial Organization (IO) focuses on problems on the supplier side. Needless to say, IO literature has become widespread in the last decades. It gave birth to lots of models discussing different market structures. Within IO, oligopoly theory discusses markets with at least two firms on the producer side. The number of firms can grow to any positive integer. Oligopolies form a link between monopolies and perfect competition markets. They are present in our everyday life and have several specific features. If there are only two firms that control the market of a homogenous good, we arrive at a duopoly. The theory of duopolies has been investigated since the 19th century. Although there are many problems that have been solved since then, there have remained some open questions. A survey on duopoly theory is presented later on in the dissertation. The main focus of the thesis lies in some specific open questions of duopoly theory.

The most frequently used modelling tool in duopoly (and oligopoly) literature is game theory. If there are several producers of a given homogenous good, then all the firms have to keep an eye on all competitors and build the others' behavior in their own decisions. In other words, they are in a strategic interaction with each other. Game theory has useful tools to deal with all kinds of strategic interaction situations.[1] The main task is to find an equilibrium of a situation: according to the most frequently used approach - the Nash equilibrium concept - an equilibrium stands for a strategy profile that none of the interacting participants have an incentive to alter. If such a profile is found in the model, one may predict that this profile also emerges in real life situations. The methodology of the dissertation builds strongly on the above mentioned Nash equilibrium concept.

The oldest examples of duopoly models date back to the 19th century. Named after Antoine Augustin Cournot and Joseph Louis Francois Bertrand, Cournot-duopoly and Bertrand-duopoly are the classical duopoly models. In the Cournot framework, the decision variable is the quantity. The market price is obtained by substituting the quantities into an exogenously given market demand function. On the contrary, the Bertrand model assumes that the firms' decision variable is the price.

These two simple models provide evidence why duopoly prices and quantities lie between monopoly and perfect competition prices and quantities. However, simple models often ignore the important assumption of capacity constraints, i.e. the assumption that firms may be capable of (or interested in) producing only a limited amount of the given good.

Edgeworth criticised the lack of the previous assumption and built capacity constraints in the Bertrand duopoly model, giving birth to Bertrand-Edgeworth duopolies (Edgeworth [1925]). Capacity constraints have been employed also for the Cournot-model, for an early contribution in this direction we refer the reader to Burger [1966].

Nowadays, Bertrand-Edgeworth duopoly models, that lie in our focus, have sev-

[1] Among many others, we refer the reader to an interesting application by Balogh and Kormos [2014].

eral versions. As far as the decision variables are concerned, the Bertrand-Edgeworth framework allows that both price and quantity can become decision variables. This leads to a more realistic modelling tool than Cournot and Bertrand duopolies. In other versions, price is the primary decison variable, but taking into account the capacity constraints, production limitation is a built-in feature of a Bertrand-Edgeworth duopoly.

There are numerous situations where the state is the exclusive owner or has an interest in one of the competitors of an oligopoly market. Oligopolies with a state-owned firm are called mixed oligopolies, while standard oligopolies refer to a market with only private firms. The same terminology is valid for duopolies.

Game theory assumes that all participants (players) aim at maximizing their own payoff. Similarly, if a firm has full private ownership, then it is assumed that its main goal is to maximize its profit. However, if the state is present on the market as the owner of a firm, then the objective of the state-owned firm is not to maximize the firm's profit any more, but to set the total social welfare as high as possible. Thus, the payoff-maximizing behavior of a public firm is equivalent to maximizing welfare.

1.1 Contribution of the thesis and applied methodology

In general, modeling duopol markets by means of game theory has the following steps.

1. Setting the framework and assumptions.
2. Setting the participants (players), their decision sets (strategy sets) and their objective functions (that they want to maximize).
3. Solving the model (finding the equilibrial behavior of the players) using an equilibrium concept (most frequently the Nash equilibrium).
4. Discussion of the results.

We will follow these steps in our analysis. Because of its frequent use in the present dissertation, we give the formal definition of the pure-strategy Nash equilibrium as it can be found for example in Forgó, Pintér, Simonovits, and Solymosi [2005].

Definition 1.1. Let us denote by $G = \{1, ..., n; S_1, ..., S_n; f_1, ..., f_n\}$ a game with n players, where S_i denotes the strategy set of player i, while f_i stands for the payoff function of player i. A strategy profile s^* is a pure-strategy Nash equilibrium, if $f_i(s_i^*, s_{-i}^*) \geq f_i(s_i, s_{-i}^*)$ holds true for every $s_i \in S_i$ and every $i = 1, ..., n$.

In other words, a strategy profile is a Nash equilibrium if and only if none of the players can increase her payoff by a unilateral strategy modification. The definition can be trivially restricted to the two-player case.

Clearly, the solution of a game depends strongly on the objective functions of the participants. Provided that there is a difference in the objective functions between the standard and the mixed models, the equilibria of the two model versions will not remain the same.

The dissertation focuses on Bertrand-Edgeworth duopolies where the state is present on the market as the exclusive or partial owner of a firm. The overall aim of the thesis is to provide a thorough game theoretic analysis of such markets.

The main contribution of the thesis is summarized as follows:

1. The existence of a pure strategy Nash equilibrium in a mixed Bertrand-Edgeworth duopoly is analyzed. Furthermore, the equilibrium profiles are characterized for all the given subcases.

2. Provided that the ordering of firms' decisions is endogenous (i.e. we consider the timing game), the equilibrial ordering of moves is given for mixed and semi-mixed Bertrand-Edgeworth duopolies.

3. A social welfare analysis is carried out for the mixed Bertrand-Edgeworth duopoly, i.e. we characterize the difference the presence of a public firm makes

in social welfare compared to the standard case with only private firms competing on the market.

1.2 Motivation

Oligopolies are present in our everyday life. Therefore, it is important to build sophisticated models that explain and predict oligopolist behavior. For a thorough analysis of earlier models in oligopoly theory we refer the reader to the seminal work of Friedman [1983].

Every model that has valid assumptions in describing or helping in the description of a real life market structure contributes to knowledge on producer behavior.

To illustrate the importance of the topic, we provide evidence from practical life. We present the following incomplete list of famous duopolies and oligopolies on the global and on the Hungarian national markets:

- credit card market: competition between VISA and MasterCard (and American Express in the U.S.)
- airplane industry: competition between Airbus and Boeing
- market of computers' graphics processing units: competition between AMD and Nvidia
- sports equipment industry: competition between the leading Nike and Adidas
- oil industry: the competition between the Norwegian Statoil and the Russian Gazprom in certain countries
- commercial banks in New Zealand and Hungary: one of the competitors (Kivibank) is state-owned in New Zealand, while Budapest Bank has recently been purchased by the Hungarian state
- Hungarian mobile phone network supplier market: competition between Telekom, Telenor and Vodafone

- Hungarian petrol station suppliers: the partially state-owned MOL competes with OMV, Shell, Agip and a few smaller suppliers

- Hungarian hypermarket suppliers: Tesco, Auchan and InterSpar

A market can be considered a duopoly (or oligopoly) if two (or three, four, etc...) firms have large control over the whole market, even if there exist other smaller suppliers. Still, a game theoretic framework needs strict assumptions about the participants, strategies and objective functions.

As listed above, there exist oligopolies where the state is also present on the market as an owner. These oligopolies provide motivation for the analysis of mixed oligopoly models. An initial building block of oligopoly theory is duopoly theory. There are many different ways of modelling a mixed duopoly. As we will present later on, the Bertrand-Edgeworth framework has proved to grasp some important features of this market structure that appear in real life. Thus, adding new results to the theory of Bertrand-Edgeworth duopolies is a step towards a better understanding of duopolist behavior.

1.3 Key research questions

When doing research by means of game theory in general, one of the most important issues to be discussed is the existence of a pure-strategy equilibrium in the given game. If exactly one pure strategy equilibrium exists, then the game is theoretically easy to handle, rational players are expected to choose their one and only strategy that leads to equilibrium.[2] If there are multiple pure strategy equilibria, then several strategy profiles can emerge in equilibrium. If the number of pure strategy equilibria equals zero, then the concept of mixed equilibrium is needed to give a solution to the game. There are several ways to prove that a mixed Nash equilibrium profile exists for any game with some specific features. One can use for example a fixed point theorem, such as Kakutani's or Brouwer's (see e.g. Fudenberg and Tirole [1991]).

[2]We have to note that it cannot be guaranteed in practice that people really choose their equlibrial strategy.

Shubik [1955] was the first to state that in the case of Bertrand-Edgeworth duopolies, the existence of a pure-strategy Nash equilibrium cannot be guaranteed. Therefore, when observing this type of duopolies, the first research question refers to the existence of pure-strategy Nash equilibrium.

Research question 1. *Under what conditions does a pure-strategy Nash equilibrium in a Bertrand-Edgeworth duopoly with public ownership exist?*

When solving the existence problem, the characterization of the pure strategy Nash equilibria also has to be dealt with. The goal of this analysis is stated in the next research question.

Research question 2. *Given entire or partial public ownership in one of the competitors in a Bertrand-Edgeworth duopoly and provided that a pure Nash equilibrium exists, what are the equilibrium prices and quantities of both firms for the simultaneous and the sequential versions of the game?*

As far as the timing of decisions is concerned, there exist simultaneous and sequential games. In the simultaneous case, all the players make their strategic decisions without knowing any other player's choice, while in the sequential case, players choose strategies in sequence, therefore, the latter players can observe the others' choice before making the decision.

The timing of decisions is often endogenized in the oligopoly literature. For seminal contributions in this field, we refer the reader to Hamilton and Slutsky [1990], Deneckere and Kovenock [1992] and Matsumura [1995]. Endogenous timing means that the ordering of firm decisions is not exogenously given. In this framework the two firms' equilibrium payoffs for different orderings will lead to an equilibrial timing of decisions. Of course, a lack of pure strategy equilibrium can also occur in the timing game. The third research question involves the problem of endogenous timing.

Research question 3. *Which ordering of decisions emerges if a private and a purely or partially public firm compete on the market in a Bertrand-Edgeworth duopoly provided that timing is endogenous?*

The main objective of the state is assumed to maximize social welfare. In mixed oligopoly frameworks, the state is not assumed to be a regulator, but it is a market participant that tries to maximize welfare by its price and production decisions, see e.g. Merrill and Schneider [1966], Harris and Wiens [1980], or Brandao and Castro [2007]. When the state enters a duopoly market by acquiring some or all shares of one of the competing firms, the level of social welfare generated on the market may not remain the same, as there is a modification in one firm's objective function. Therefore, it is necessary to state a question concerning the change in social welfare the presence of a public firm may cause.

Research question 4. *What is the direction and magnitude of social welfare change the appearance of a purely or partially public firm generates in a Bertrand-Edgeworth duopoly framework?*

The four main research questions are linked to each other. The links are as follows. Firstly, we will give constructive proofs to the existence of pure strategy Nash equilibria, wherever it is possible. This means that when proving the existence, we present the equilibrium profile. Secondly, when analyzing the timing game, we strongly rely on the results of the exogenous orderings of moves. Thirdly, the social welfare effect of public ownership can be examined by comparing the public firm's equilibrium payoff (objective function value) in the mixed duopoly case to the calculated social welfare in the standard case.

1.4 Structure of the thesis

The dissertation consists of seven sections. We aim at providing a clear structure where background, related work, detailed analysis of the topic and own work is presented. The remainder of the thesis is organized as follows.

Section 2 recalls relevant results from the field of duopoly theory. We present classical models from both price-setting and quantity-setting types. Furthermore, Bertrand-Edgeworth duopolies are introduced. We present some different approaches of the model. The key assumptions are introduced, the existence of a game theoretic equilibrium is investigated, and some interesting contributions are recalled.

Section 3 offers a survey on mixed oligopolies, i.e. models with a state-owned competitor. The difference in the assumptions of the standard and mixed versions of the model is highlighted. We also specify the models used in the main analysis. The survey follows a unified way of presenting the results: the main assumptions from some important contributions are collected, the outcomes of the models are presented and the differences from other models are highlighted.

Section 4 provides a detailed analysis of the mixed Bertrand-Edgeworth duopoly in the so-called production-to-order (PTO) framework. In this section the public firm is assumed to be fully owned by the state. The PTO setting assumes that production takes place after the amount of sales are fixed in a contract. This setting models among others the airplane industry. We give the pure-strategy Nash equilibrium for all the three possible orderings of moves, solve the timing game and discuss the public firm's social welfare effect.

Section 5 deals with the mixed Bertrand-Edgeworth duopoly in the production-in-advance (PIA) case. In the PIA case we assume that production takes place already before sales are realized. Markets of perishable goods can be modelled with this setting. Just like the previous section, this section also contains the results on the existence and characterization of pure-strategy Nash equilibria for all possible orderings of moves. Furthermore, the timing game and the social welfare issue is also discussed.

In Section 6, the assumption of pure public ownership is relaxed: we will assume that one of the firms is only partially owned by the state. The objective function of the firm with both public and private ownership changes compared to the case of pure public ownership. Its objective function becomes a weighted sum of total social welfare and its own profit, where the weights are the proportions of public and private ownership. This modification leads to new model outcomes. Within Section 6, we refer to the results on the production-to-order case and present the analysis of the production-in-advance case.

Finally, Section 7 concludes and restates the contribution of the dissertation. The main results are also collected in Section 7.

2 Standard duopolies

Throughout the dissertation - in accordance with the literature -, a standard duopoly or oligopoly will stand for a model with only pure profit maximizer firms. on the contrary, mixed duopolies and oligopolies involve at least one firm that is fully or partially owned by the state, and therefore its objective differs from pure profit maximization.

This section offers a collection of relevant results from the literature on duopolies and oligopolies. Our aim is to present how oligopoly theory had developed until the most important results of Bertrand-Edgeworth models were published.

Thus, first, we present the classical Cournot and Bertrand duopoly models. We also offer an introduction to mixed duopolies and a collection of the relevant contributions from this field. Mixed duopolies are introduced later on in Section 3.

From the game theoretic perspective, each duopoly model can be considered as a game. A game in strategic (normal) form is well-defined if the following ingredients are given (see any game theory textbooks, e.g. Fudenberg and Tirole [1991]):

- the set of players,
- the strategy sets of all players,
- the payoff functions of all players.

There exist several different approaches to determine the outcome of a game. In most of the cases the Nash equilibrium concept is used due to its simplicity and wide applicability. The dissertation follows this practice. Other concepts include the dynamic approach (see e.g. Okuguchi and Szidarovszky [1990]) and the successive elimination of strictly dominated strategies (see e.g. Forgó [2013]).

Finding the Nash equilibria of a game has the following steps.

1. Determining the best response correspondance for all the players.
2. Finding all the strategy profiles that contain strategies that are the best responses to each other.

3. Proving that no other Nash equilibrium profile exists.

For most of the referred contributions, we will omit the detailed solution concept and will focus only on the results.

2.1 Cournot duopoly

The birth of Cournot duopoly (sometimes - when more than two firms are involved - referred to as Cournot competition, see e.g. Anderson and Neven [1991]) dates back to the 19th century (Sandmo [2011]). This model is a classical framework predicting two firms' behavior in equilibrium, if there are no more firms on the market of a given, homogenous good. According to Varian [2006], the assumptions of the basic model are as follows.

Assumption 2.1. The market demand function of the homogenous good is linear, i.e. $D(p) = a - bQ$, where p refers to the price, $Q = q_1 + q_2$, the total production of the two firms, while a and b are parameters. It is also assumed that $a > c$.

We note that the strict assumption of linear demand will be relaxed later on.

Assumption 2.2. Both firms have identical unit costs denoted by c.

Assumption 2.3. The payoff functions of the firms are: $\pi_i = (a - bQ)q_i - cq_i$ for $i = \{1, 2\}$.

Assumption 2.4. Both firms simultaneously decide on their quantities. The market price is then determined by substituting the total production into the market demand function.

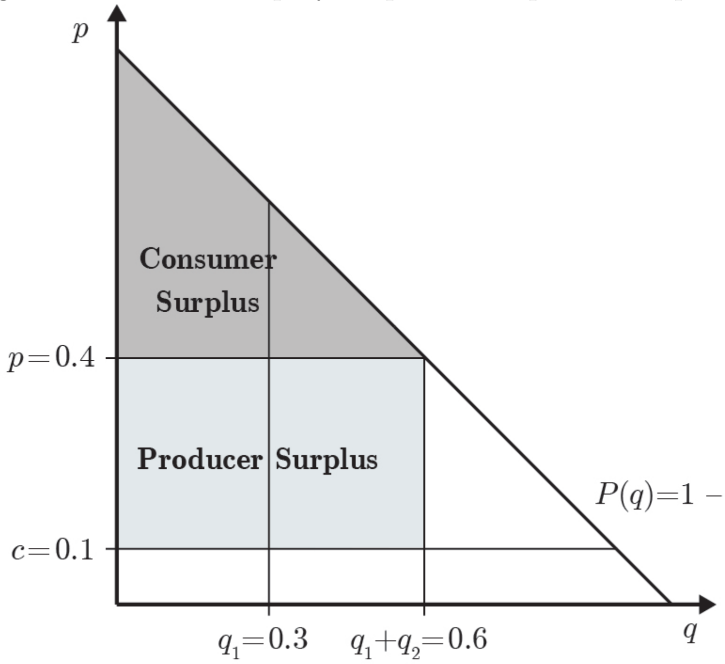
In the Cournot game the players are the two firms, their strategy sets are the quantities (any non-negative number), and their payoff functions are their profit functions. Thus, the game is well defined.

The only Nash equilibrium price and quantities of the Cournot competition are the following.

$$q_1^* = q_2^* = \frac{a-c}{3b}; \quad p_1^* = p_2^* = \frac{a+2c}{3} \tag{1}$$

The equilibrium for a specific demand function and cost level is illustrated below in Figure 1.

Figure 1: Cournot duopoly - equilibrium price and quantities



Comparing this result to a quantity-setting monopoly and a perfect competition on a market with the same parameters, we obtain that both Counot production and Cournot price lies between monopolist and perfect competition production and price. Hence, from the point of view of social welfare, Cournot duopoly is less effective than perfect competition, but more effective than monopoly. This simple model is the first example that predicts duopolist behavior and places a duopoly in a social welfare ranking of different market structures on a homogenous good's market.

An important finding has confirmed the content of the last paragraph. Namely, if the number of competitors in a Cournot competition (oligopoly) converges to infinity, then the equilibrium price converges to $p^* = MC = c$, the marginal cost. This is called the "Folk theorem", which is precisely stated and proved in Novshek [1980]. This finding links Cournot competition even more to monopoly and perfect competition.

We emphasize that only the simultaneous moves case of this simple model is named after Cournot. The sequential game with the same assumptions is another classical model in the field and is called Stackelberg duopoly. According to Stackelberg [1934], the model's outcome is that in equilibrium the first mover (the Stackelberg leader) sets its output equal to what a monopolist would set on the same market, while the second mover (the Stackelberg follower) produces half of the monopolist amount:

$$q_1^* = \frac{a-c}{2b}; \quad q_2^* = \frac{a-c}{4b}; \quad p_1^* = p_2^* = \frac{a+3c}{4} \tag{2}$$

Thus, Stackelberg competition results in higher social welfare than monopoly, but lower than perfect competition. Comparing Stackelberg to Cournot equilibrium, one can conclude that the social welfare generated on the Stackelberg market is somewhat higher (due to lower market price and higher production in equilibrium).

2.2 Bertrand duopoly

Named after the French mathematician Joseph Louis Francois Bertrand, the other classical two-firm model is called Bertrand duopoly (or Bertrand competition). To state the assumptions of the basic homogenous good Bertrand model, we refer the reader to Assumptions 2.1-2.3 of Cournot competition. The main difference is that in the Bertrand model the firms' decision variable is their price level. A main finding of the general Bertrand oligopoly model is that the firm with the lowest price offered will serve the entire market. When turning to the Nash equilibrium analysis of the game with two firms, we arrive at the following unique Nash equilibrium[3]:

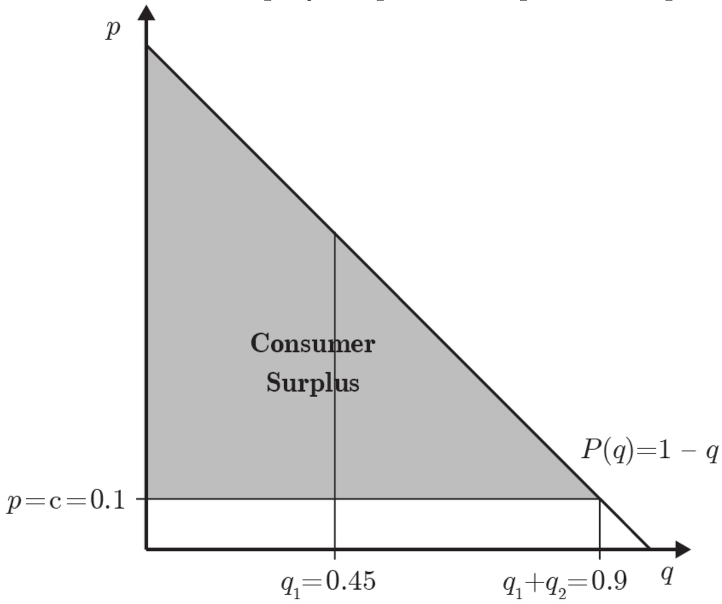
$$q_1^* = q_2^* = \frac{a-c}{2b}; \quad p^* = MC = c \tag{3}$$

[3]For both Bertrand and Cournot models it is important to note that the birth of the Nash equilibrium concept Nash [1950] dates to much later than that of the classical duopoly concepts. Consequently, Cournot and Bertrand could not use the definition of Nash in their models. Still, nowadays Nash equilibrium is the most frequently used equilibrium concept, and the results of Bertrand and Cournot are referred to in this framework.

The argument that supports this equilibrium is that neither of the firms can set its price above c, because then the competitor has an incentive to undercut the price and serve the entire demand at a lower price. On the contrary, setting the price below c would mean negative profits.

The Bertrand outcome for a specific cost level is illustrated in Figure 2 below.

Figure 2: Bertrand duopoly - equilibrium price and quantities



The outcome is somewhat unrealistic as the market price level remains the same as that of the perfect competition market. Thus, the Bertrand model predicts that there is no decrease in social welfare compared to perfect competition. This is frequently referred to as the Bertrand paradox, see e.g. Tirole [1988] or Hehenkamp [2002].

As far as the ordering of decisions is concerned, the outcome of the Bertrand competition remains the same for the simultaneous moves case and for the sequential moves cases.

After going through the Cournot and Bertrand competition, we can conclude that there are several features of real-life markets that these models do not grasp. The following list contains a selection of shortcomings:

- The firms' capacities are unlimited, they can always satisfy the demand they face. On the contrary, firms have limited capacities in practice.
- In the original framework the market demand function is linear, which might not be the case on several markets in practice.[4]
- The decision variable of the firms is either price or quantity, the other value is obtained by a substitution into the market demand function. However, on certain markets, there might be left-over supplies that cannot be sold.

The shortcomings of the models have led the way to the birth of more realistic frameworks. As we will present in the next subsection, the Bertrand-Edgeworth model tackles some problematic issues of Cournot and Bertrand duopolies. On the other hand, some major problems arise, including the existence of equilibrium.

2.3 Bertrand-Edgeworth competition

Francis Ysidro Edgeworth [1925] was the first to criticize the outcome of the Bertrand competition. In his paper originally dating from 1897[5], the assumption that the firm with the lowest price can satisfy the entire demand on that specific price level is considered to be unrealistic.

Bertrand-Edgeworth competition is a price- (and quantity-) setting oligopoly that is specified by the fact that production of the firms is restricted by their respective capacity constraints. (Osborne and Pitchik [1986]) We will give the formal definition of capacity constraints later on.

When introducing capacity constraints to the model where the demand side is represented by an aggregated demand function[6], we have to face two major problems that have not yet been present in the Cournot and Bertrand models:

[4] There are contributions in the field of Cournot oligopolies that relax strict assumptions towards market demand and cost functions, e.g. Forgó [1995].

[5] The original English version of the paper was lost, the Italian paper was translated back to English in 1925. For further details visit the following website:
http://cruel.org/econthought/texts/edgeworth/edgepapers.html

[6] i.e. the way the basic Bertrand and Cournot games are specified

1. The model is not sufficiently specified, unless the firm offering the lowest price is capable of satisfying the entire demand.

2. The existence of a pure-strategy Nash equilibrium cannot be guaranteed.

We discuss these two problems in the following subsections.

2.3.1 Rationing rules

If we assume that consumers choose the firm with the lowest price and we also assume that the firm with the lowest price cannot satisfy all the consumers, then the simple question arises: which consumers should be served? This question needs to be answered by an extra assumption that is called *rationing rule*. A rationing rule aims at determining the ordering according to which consumers are served if demand exceeds supply at a certain price level.

There are several types of rationing rules. The most frequently used version is called efficient rationing. Efficient rationing in the duopoly case intuitively means the following: if the firm with lower price cannot satisfy the entire demand, then it serves consumers in a decreasing order of their reservation prices. Thus, the first served consumer is the one with the highest reservation price for the good. The firm with the lower price serves residual demand. The efficient rationing rule is given formally in the following definition.

Definition 2.1. Function Δ denotes an efficient rationing rule if the demand of firm $i \in \{1, 2\}$ is the following:

$$\Delta_i (D, p_1, q_1, p_2, q_2) = \begin{cases} D(p_i) & \text{if} \quad p_i < p_j \\ T_i(p, q_1, q_2) & \text{if} \quad p = p_i = p_j \\ (D(p_i) - q_j)^+ & \text{if} \quad p_i > p_j \end{cases}$$

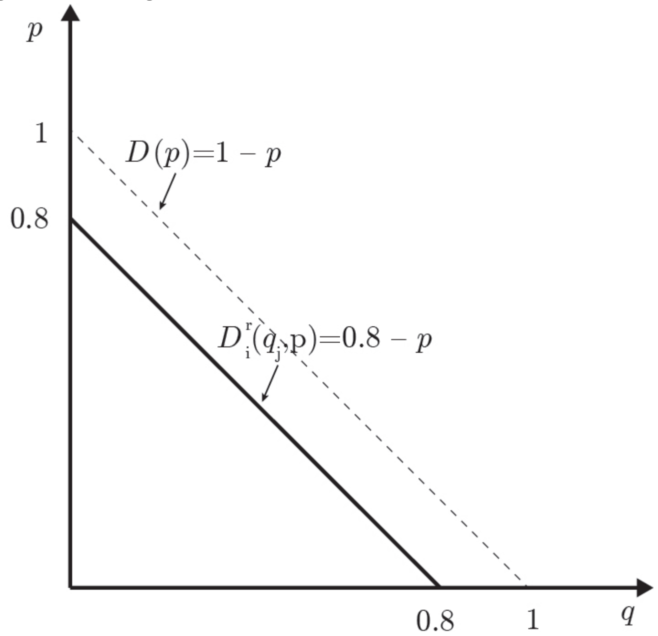
for all $i \in \{1, 2\}$, where T_i stands for a tie-breaking rule that dictates the shares of firms from the demand given equal prices.[7]

[7] We note that $(D(p_i) - q_j)^+ = \max\{D(p_i) - q_j; 0\}$.

The efficient rationing rule means that the demand function of the higher-price firm (i.e. the residual demand function, D^r) is generated by a parallel leftward shift of function D by q_j units where q_j stands for the quantity sold by the lower-price firm.

An example for the original and the residual demand curves are depicted in Figure 3 below.

Figure 3: Original and residual demand curves



For contributions that make use of efficient rationing, we refer the reader to Kreps and Scheinkman [1983], Dixon [1990], Tasnádi [1999b], Boccard and Waughty [2000], and Lepore [2008].

Another well-known rationing rule is called proportional rationing. The idea behind proportional rationing is that the firm with the lower price can only satisfy the proportion of its capacity and the demand it faces. For further discussion of proportional rationing and for other rationing rules we refer the reader to Tasnádi [2001].

Adding a rationing rule to the set of players (firms), the payoff functions (profits) and the market demand function, the Bertrand-Edgeworth game becomes well-defined.

2.3.2 Existence of pure-strategy Nash equilibria

The existence of pure-strategy Nash equilibria is not guaranteed for Bertrand-Edgeworth oligopolies (see e.g. Shubik [1955]). We summarize the related results from Tirole [1988] and Levitan and Shubik [1972]. For equal unit costs and exogenously given capacity constraints the following theorem holds true.

Theorem 2.1. *In an n-firm Bertrand-Edgeworth oligopoly with capacity constraints, depending on the shape of D and the capacity constraints of the competing firms (k_i for $\forall i \in \{1...n\}$) one of the following cases holds true:*

1. *Firms' prices equal their unit costs.*
2. *There is no pure-strategy Nash equilibrium.*
3. *Firms set the Cournot competition equilibrium price.*

The first case of the previous theorem emerges if the sum of firms' capacities is large enough for the Bertrand equilibrium (i.e. perfect competition). That is, $D^{-1}(\sum_{i=1}^{n} k_i) = 0$.[8] The third case occurs if $D^{-1}(\sum_{i=1}^{n} k_i) > 0$ and firms maximize profits by setting $p_i^* = D^{-1}(\sum_{i=1}^{n} k_i)$ price level.

The second, intermediate case is the most interesting and the most frequently observed one. That is, $D^{-1}(\sum_{i=1}^{n} k_i) > 0$, but $p_i^* > D^{-1}(\sum_{i=1}^{n} k_i)$, i.e. the profit-maximizing price levels of the firms exceed the inverse demand function value.

In their seminal paper, Levitan and Shubik [1972] established that for a Bertrand-Edgeworth model with efficient rationing and linear demand a Nash equilibrium in

[8] We denote by D^{-1} the inverse demand function whenever it exists. We will restrict our analysis to demand functions that have an inverse. The exact specification of the demand function is given in Section 4.

mixed strategies exists for the intermediate case. Furthermore, the mixed equilibrium was characterized.

To establish more general results on the existence of Nash equilibria (taking into account the possible non-linear shape of the demand function), an applicable existence theorem was needed. One main shortage of the Bertrand-Edgeworth competition is that the competitors' profit functions are not continuous and not quasiconcave (Tasnádi [2001]). The existence theorem that tackled these shortcomings was published by Dasgupta and Maskin [1986a]. In their paper, the authors proved that a Nash equilibrium in mixed strategies exists for games with a special family of discontinuous payoff functions. Some applications of the theorem is presented in Dasgupta and Maskin [1986b]. Later on, there appeared more general versions of the existence theorem (see e.g. Simon [1987] or Reny [1999]).

As far as the characterization of the mixed Nash equilibrium is concerned, the result of Levitan and Shubik [1972] was developed further for a more general setting - the case of strictly decreasing demand functions - by Davidson and Deneckere [1986], although this paper contained only an implicit formula of the mixed equilibrium. Later on, the explicit formula was presented by Allen and Hellwig [1993].

Thus, from a game theoretic perspective, a general standard Bertrand-Edgeworth model was solved. However, provided that for certain games there is no Nash equilibrium in pure strategies, practical relevance of the results has become questionable.

In the related literature we find examples of two directions concerning the way of handling the lack of pure-strategy Nash equilibria. The first approach avoids the concept of Nash equilibrium and uses other concepts, such as ϵ-equilibrium or iterated elimination of strictly dominated strategies. For contributions we refer the reader to Dixon [1987] and Börgers [1992]. The second approach is the characterization of pure and/or mixed Nash equilibria, including the reconstruction of models so that the new model has a pure-strategy Nash equilibrium. Examples of this approach are, among others, Deneckere and Kovenock [1992], Deneckere and Kovenock [1996], Reynolds and Wilson [2000], Chowdhury [2005], and Lepore [2008].

In our main analysis in Sections 4, 5 and 6 we will present models that always have pure-strategy Nash equilibria.[9] Before that, it is necessary to present results on oligopolies with a state-owned competitor. Section 3 offers an introduction to the theory of mixed oligopolies and collects the most relevant contributions.

[9] We note that for models in Section 6 the existence of pure Nash equilibria will depend on the parameters.

3 Mixed duopolies

This section considers mixed duopolies and oligopolies. The mixed structure refers to the fact that at least one of the competitors is owned by the state (partially or exclusively). From the game theoretic point of view, this means that one of the players has a different payoff function. When dealing with state-owned firms, the state's social welfare maximizing behavior is captured through their payoff function. It is important to emphasize that in such models the state does not play the role of an external regulator, but as a competitor, it tries to control the market towards a socially optimal equilibrium.

In what follows, we offer a survey on mixed oligopolies and characterize the model of our main analysis.

3.1 Contributions on mixed oligopolies and endogenous timing

The very first contribution on mixed oligopolies was published in 1966. Merrill and Schneider [1966] mention three ways of ownership and control in an industry: complete private ownership, complete government ownership, and private ownership restricted by close government control. They introduce and build a simple model for a fourth way of ownership: public control of some, and private control of the other competing firms in the same industry. This is in fact the structure that nowadays is called mixed oligopolies.

Thereafter, until the 1980s, mixed oligopolies have not been investigated in detail. The early literature considered games with a predetermined order of moves and investigated market regulation possibilities through a public firm, especially, in order to increase social welfare. The first contributions considered quantity-setting models. For instance, Beato and Mas-Colell [1984] investigate possible ways to regulate a market: they analyze the marginal cost pricing rule in detail. Cremer, Marchand, and Thisse [1989] analyze the regulation of a simultaneous-move quantity-setting oligopoly market. Their main research question is whether the government should nationalize some of the competing firms - departing from a Cournot-Nash equilibrium - on an oligopolistic market in order to maximize social welfare. They report that the market outcome is equivalent to the solution, in which the central authority maximizes total surplus subject to the constraints that the public firms must break even. Fraja and Delbono [1989] show for a mixed quantity-setting homogeneous good oligopoly that public leadership leads to higher social welfare than the simultaneous-move mixed or standard oligopolies. However, comparing the simultaneous-move case with or without a public firm leads to an ambivalent result: It can happen that if there are sufficiently many identical firms on the market, then the public firm is better advised to maximize its profits in order to increase social welfare. Remaining in the quantity-setting framework, Corneo and Jeanne [1994] derive conditions under which the presence of a public firm can be beneficial. Anderson, de Palma, and Thisse [1997] consider an oligopoly with one public and several private firms. They investigate whether the privatization of the public firm is harmful. In their model, the market demand is given by a representative consumer's utility function. They report that even if the public firm makes a loss, total social welfare might be higher if there is public ownership in the long run. Therefore, according to their framework, a single public firm on an oligopolistic market should not be privatized. A more recent study by Fujiwara [2007] deals with partial privatization in a differentiated mixed oligopoly.

The analysis of endogenous timing of decisions in mixed oligopolies began later. Endogenous timing means that firstly firms choose the time to announce their price or production[10] (relatively to each other), then, secondly, prices or productions are announced. Thus, timing of decisions becomes an endogenized problem of the model, and therefore, an equilibrium analysis of the timing game can be carried out. The Nash equilibrium of the timing game lies at an ordering of moves where it is of none of the players' interest to deviate from[11].

The timing question in purely private oligopolies has been investigated since the

[10]Depending on the decision variable of the given model.

[11]Based on their payoff function values for the different orderings.

mid 1980s. We will not go into a detailed survey, but list some contributions. Earlier research was carried out among others by Gal-Or [1985], Dowrick [1986],Boyer and Moreaux [1987], Hamilton and Slutsky [1990], Robson [1990], Deneckere and Kovenock [1992], Anderson and Engers [1992], Anderson and Engers [1994], Matsumura [1995] and Pal [1996]. More recent contributions on the purely private case include Huck, Müller, and Norman [2002], Matsumura [2002], Tasnádi [2003], Fonseca, Muller, and Normann [2006], Etro [2008], and Matsumura and Ogawa [2009].

Timing issues have also been addressed in the mixed oligopoly framework. In a seminal paper Pal [1998] investigates the endogenous emergence of certain orders of moves for mixed oligopolies. Assuming linear demand and constant marginal costs, he shows for a quantity-setting oligopoly with one public firm that the simultaneous-move case does not emerge, the public firm as a first mover emerges just in the two-period duopolistic case, while the private firms moving simultaneously in the first period followed by the public firm in the second period always constitutes a subgame-perfect Nash equilibrium of the timing game. His main observations are that incorporating a public firm substantially changes the outcome of the timing game and that the presence of a public firm increases social welfare.

Referring to Pal [1998], Jacques [2004] noted that in the duopolistic case with more than two periods the public firm producing in the first period and the private firm producing in the last period is also a subgame-perfect Nash equilibrium. In addition, Lu [2007] shows for the duopolistic case there are even more subgame-perfect Nash equilibria in which the private firm can produce in any period with the exception of the last one and the public firm has to produce in a subsequent period. Matsumura [2003] relaxes the assumptions of linear demand and identical marginal costs employed by Pal [1998] and considers a two-period mixed model. The paper states that there are several pure-strategy Nash equilibria of the model, including simultaneous Cournot-type and sequential Stackelberg-type equilibria with the public firm being the follower. By introducing small inventory costs, the only remaining equilibrium is the Stackelberg solution with private leadership. The case of increasing marginal costs in Pal's [1998] framework has recently been investigated by

Tomaru and Kiyono [2010]. Their main achievement is that unlike in earlier results, both private and public leadership can emerge in equilibrium. Another direction to extend the results in Pal [1998] can be found in Anam, Basher, and Chiang [2007] that assumes uncertain demand. They find that under demand uncertainty both private and public leadership can emerge in equilibrium of the timing game. Whether the public or the private firm becomes the leader, depends on the degree of privatization and market volatility. Finally, Lu and Poddar [2009] consider a two-stage private and mixed duopoly. In the first stage, firms reveal their capacities, then they decide upon production. Assuming linear demand, they obtain that there are far more Nash equilibria in the mixed version than in the standard version of the game. Furthermore, equilibria are characterized for both settings.

There is less literature on price-setting mixed oligopolies. Ogawa and Kato [2006] consider mixed duopolies in the framework of a homogeneous good price-setting game. They investigate a symmetric Bertrand duopolistic setting in which the firms have to serve the entire demands they face. Assuming linear demand and identical quadratic cost functions, they find for the two sequential-move games that the presence of a public firm may be either harmful or beneficial, while for the simultaneous-move game the outcome remains the same as shown by Dastidar [1995] for the purely private case. Dastidar [2011] extends these results to strictly convex cost functions and decreasing demand functions. In another recent work Chowdhury [2009] considers a price-setting mixed Bertrand duopoly (that is, firms have to serve the demands they face) and a mixed semi-Bertrand duopoly (in which only the public firm has to meet all demand). However, he focuses on the simultaneous-move case, and thus, he does not solve the timing problem. For a heterogeneous goods price-setting mixed duopoly timing game Bárcena-Ruiz [2007] obtained the endogenous emergence of simultaneous moves.

Other questions that have been addressed in the literature of capacity-constrained Bertrand-Edgeworth framework are partial privatization (as initiated by Matsumura [1998]), free entry (as investigated for the quantity-setting case, for instance, by Ino and Matsumura [2010]), the presence of foreign private firms (see for example, Fjell and Pal [1996]) and the endogenous emergence of capacity differentials between private and public firms (like by Corneo and Rob [2003], in which the efficiency gap between private and public firms is derived from workers' incentives).

Our main analysis in Sections 4, 5 and 6 will consider a both price- and quantity-setting framework. However, in the production-to-order framework, the interaction situation will reduce to a price-setting game. The difference between production-to-order and production-in-advance frameworks needs to be highlighted. The next subsection deals with this distinction.

3.2 Production-to-order and production-in-advance frameworks

We can distinguish between production-in-advance (PIA) and production-to-order (PTO) concerning how the firms organize their production in order to satisfy the consumers' demand.[12] In the former case production takes place before sales are realized, while in the latter one sales are determined before production takes place. Markets of perishable goods are usually mentioned as examples of advance production on a market. On the other hand, if a consumer (be it the state, a company or even a natural person) shows demand for a certain product that is only available upon order - as producers are not willing to keep stocks -, the market can be modelled by PTO. Phillips, Menkhaus, and Krogmeier [2001] emphasized that there are also goods which can be traded both in a PIA and in a PTO environment since PIA markets can be regarded as a kind of spot market, whereas PTO markets as a kind of forward market. For example, coal and electricity are sold in both types of environments.

The comparison of PIA and PTO environments have been made in experimental and theoretical frameworks for standard oligopolies. For instance, assuming strictly increasing marginal cost functions Mestelman, Welland, and Welland [1987] found

[12] The PIA game is also frequently called the price-quantity game (briefly PQ-game) or even production-to-stock (PTS game). Throughout the thesis we use the PIA abbreviation.

that in an experimental posted offer market the firms' profits are lower in case of PIA. A more recent experimental analysis of the PIA environment can be found in Cracau and Franz [2012]. The authors point out that the mixed strategy equilibrium of the linear-demand oligopoly game predicts producer behavior better than Bertrand and Cournot models do. It is also demonstrated that prices and profits in the experiment exceed the theoretically predicted values, but the level of collusion remains lower than that in the theory. In another recent experimental analysis Davis [2013] observes similar equilibrium prices given PIA and PTO settings and lower profits in PIA.

Much earlier, in a theoretical paper, Shubik [1955] [13] investigated the pure-strategy equilibrium of the PIA game and conjectured that profits will be lower in the case of PIA than in the case of PTO. Davis [2013] can be considered as an experimental evidence of Shubik [1955]. Levitan and Shubik [1978] and Gertner [1986] (Essay one) focus on determining the mixed-strategy equilibrium for the constant unit cost case. Gertner [1986] also derived some important properties of the mixed-strategy equilibrium of the PIA game for strictly convex cost functions. In a later contribution van den Berg and Bos [2011] deal with collusions in oligopolies in both PIA and PTO frameworks.

Assuming constant unit costs and identical capacity constraints, Tasnádi [2004] found that profits are identical in the two environments and that prices are higher under PIA than under PTO. In a recent paper Zhu, Wu, and Sun [2014] showed for the case of strictly convex cost functions that PIA equilibrium profits are higher than PTO equilibrium profits. In addition, considering different orders of moves and asymmetric cost functions Zhu, Wu, and Sun [2014] demonstrated that the leader-follower PIA game leads to higher profits than the simultaneous-move PIA game. From the mentioned papers on the constant unit cost and identical capacity constraint-case only Zhu, Wu, and Sun [2014] considered sequential orders of moves. More leader-follower games are analyzed in Boyer and Moreaux [1987], Deneckere and Kovenock [1992] and Tasnádi [2003] in the Bertrand-Edgeworth framework.

[13] We referred to the same paper in Section 2 where we deal with the existence of pure-strategy Nash equilibria in Bertrand-Edgeworth oligopolies.

3.3 Setting the framework of the analysis

After an introduction to both Bertrand-Edgeworth competition and mixed oligopolies, we set the assumptions of the model that lies in the focus of the dissertation.

In the following sections, we will carry out thorough game theoretic analyses of four mixed Bertrand-Edgeworth duopoly models with capacity constraints.

Due to the fact that models might differ slightly from each other, the exact model specifications are given at the beginning of each chapter. However, for a clear view of the thesis as a whole, we collected the common features of the models in the following list:

1. There are two firms competing on the market of a homogenous good.
2. The decision variables of the firms are both price and quantity.
3. The consumer side is given by a market demand function.
4. The market demand function is monotone, strictly decreasing and twice continuously differentiable.
5. The two firms cannot produce a higher amount than their respective capacity constraints.
6. Both firms have constant and identical unit costs.
7. One of the competing firms has pure private ownership, while there is a certain share of the state in the other one.
8. There is no information asymmetry, the demand function, the unit cost, the capacities and the payoff functions are common knowledge.

The exact scope of our models can be divided according to two axes. The first axis is the degree of state ownership in one of the firms. We discuss separately the case where there is pure state ownership and the case where the share of the state is limited.[14]. The second axis is the timing of production. We will distinguish between production-to-order and production-in-advance frameworks and carry out analyses for both types. Thus, we have four models all together.

The following Table 1 illustrates these two ways of model differentiation and indicates the sections which the separate models are discussed in.

Table 1: Discussed models

Models	**Pure public ownership**	**Limited public ownership**
PTO	Section 4	Section 6.2
PIA	Section 5	Section 6.3

We note that results for models with limited public ownership ratio are derived from the results of the models with pure public ownership.

A close paper to our theoretical setting is Tasnádi [2004], since we will investigate the constant unit cost case with capacity constraints. The main difference is that we will replace one profit-maximizing firm with a purely or partially social welfare maximizing firm.

The PTO mixed duopoly case to be discussed in Section 4 is published in Balogh and Tasnádi [2012]. Our main findings for this case are the following:

1. There is payoff equivalence between the games with exogenously given order of moves.
2. The social welfare increases in equilibrium compared to the standard version.
3. An equilibrium in pure strategies exists for all possible orderings of moves.

[14]We note that the other firm is always considered purely private.

However, only one of these strong results remains valid for the production-in-advance case, namely, that a pure-strategy equilibrium exists for all possible orderings of moves. The analysis of the PIA case will be presented in Section 5 and can also be found in Balogh and Tasnádi [2014]. We will demonstrate for the PIA mixed duopoly the existence of an equilibrium in pure strategies, (weakly) lower social welfare than in the case of the PTO mixed duopoly and the emergence of simultaneous moves as a solution of a timing game.

As far as the models with limited public ownership are concerned, even the existence of a pure strategy Nash equilibrium will depend on the parameters. Therefore, as demonstrated in Section 6, not as many positive results can be stated as in Sections 4 and 5. Some results presented in Section 6 are published in Balogh [2014].

4 Mixed Bertrand-Edgeworth duopolies - Production-to-order framework

In this section we determine the pure-strategy Nash equilibria and the endogenous order of moves for the production-to-order (PTO) version of the mixed Bertrand-Edgeworth duopoly. We establish the payoff equivalence of the games with an exogenously given order of moves if the most plausible equilibrium is realized on the market. Hence, in this case it does not matter whether one becomes a leader or a follower. We also show that replacing a private firm by a public firm in the standard Bertrand-Edgeworth game with capacity constraints increases social welfare and that a pure-strategy equilibrium always exists.

As introduced in Section 3, we will investigate a homogeneous good mixed Bertrand-Edgeworth duopoly with capacity constraints. In contrast with the literature on endogenous timing in purely private firm oligopolies as well as mixed oligopolies we find that the order of moves does not matter (Corollary 4.1). In addition, social welfare is higher in the mixed version of the Bertrand-Edgeworth game than in its standard version with only private firms (Corollary 4.2). We also obtain that the mixed version of the Bertrand-Edgeworth duopoly game has an equilibrium in pure strategies for any capacity levels (Corollary 4.3).

The remainder of this section is organized as follows. In Section 4.1 we present our exact framework of the PTO case and introduce the necessary notations. In Section 4.2 we recall the results of Deneckere and Kovenock [1992] on the capacity-constrained Bertrand-Edgeworth duopoly game with purely private firms, which we will need in comparing the results of the mixed version of the Bertrand-Edgeworth game to its standard version. Section 4.3 determines the equilibria of three games with exogenously given orderings of moves. Section 4.4 gathers the main consequences of our analysis carried out in Section 4.3. Section 4.5 presents an illustrative example of our results in the case of linear demand. Finally, we draw the conclusions of Section 4.

4.1 The framework

The demand is given by function D on which we impose the following restrictions:

Assumption 4.1. The demand function D intersects the horizontal axis at quantity a (where $a > 0$) and the vertical axis at price b. D is strictly decreasing, concave and twice continuously differentiable on $(0, a)$; moreover, D is right-continuous at 0, left-continuous at b and $D(p) = 0$ for all $p \geq b$.

Clearly, any price-setting firm will not set its price above b. Let us denote by P the inverse demand function. Thus, $P(q) = D^{-1}(q)$ for $0 < q \leq a$, $P(0) = b$, and $P(q) = 0$ for $q > a$.

On the producers' side we have a public firm and a private firm, that is, we consider a so-called mixed duopoly. We label the public firm with 1 and the private firm with 2. Our assumptions imposed on the firms' cost functions are as follows:

Assumption 4.2. The two firms have zero unit costs up to their respective positive capacity constraints k_1 and k_2.[15]

We shall denote by p^c the market clearing price and by p^M the price set by a monopolist without capacity constraints, i.e. $p^c = P(k_1 + k_2)$ and $p^M = \arg\max_{p \in [0,b]} pD(p)$. In what follows $p_1, p_2 \in [0, b]$ stand for the prices set by the firms.

For all $i \in \{1, 2\}$ we shall denote by p_i^m the unique revenue maximizing price on the firms' residual demand curves with respect to the capacity constraint of firm i (i.e. the high-price firm produces the quantity given by its residual demand curve). $D_i^r(p) = (D(p) - k_j)^+$, where $j \in \{1, 2\}$ and $j \neq i$, i.e. $p_i^m = \arg\max_{p \in [0,b]} pD_i^r(p)$. The inverse residual demand curves will be denoted by R_1 and R_2. Clearly, p^c and p_i^m are well defined whenever Assumptions 6.1 and 6.2 are satisfied.

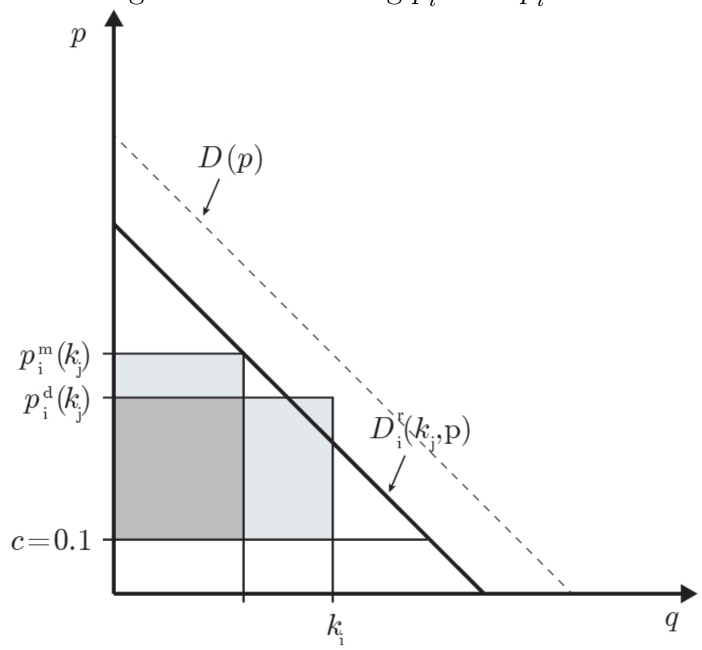
Let us denote by p_i^d the price for which $p_i^d \min\{k_i, D(p_i^d)\} = p_i^m D_i^r(p_i^m)$, when-

[15] The real assumption here is that firms have identical unit costs since in case of production-to-order, as will be assumed later, this is just a matter of normalization.

ever this equation has a solution.[16] Provided that the private firm has 'sufficient' capacity (i.e. $p^c < p_2^m$), then if it is a profit-maximizer, it is indifferent whether it is serving residual demand at price level p_i^m or selling $\min\{k_i, D\left(p_i^d\right)\}$ at the lower price level p_i^d.

In the following Figure 4 we illustrate the way p_i^m and p_i^d are determined.

Figure 4: Determining p_i^m and p_i^d



Concerning the employed rationing rule, we impose the following assumption.

Assumption 4.3. We assume efficient rationing on the market.

Thus, the firms' demands equal

$$\Delta_i\left(D, p_1, q_1, p_2, q_2\right) = \begin{cases} D\left(p_i\right) & \text{if} \quad p_i < p_j \\ T_i(p, q_1, q_2) & \text{if} \quad p = p_i = p_j \\ \left(D\left(p_i\right) - q_j\right)^+ & \text{if} \quad p_i > p_j \end{cases}$$

[16]The equation defining p_i^d has a solution if, for instance, $p_i^m \geq p^c$, which will be the case in our analysis when we will refer to p_i^d.

for all $i \in \{1,2\}$, where T_i stands for a tie-breaking rule.[17] We will consider two sequential-move games (one with the public firm as the first mover and one with the private firm as the first-mover) and a simultaneous-move game. An appropriate tie-breaking rule will be specified for each of the three cases.

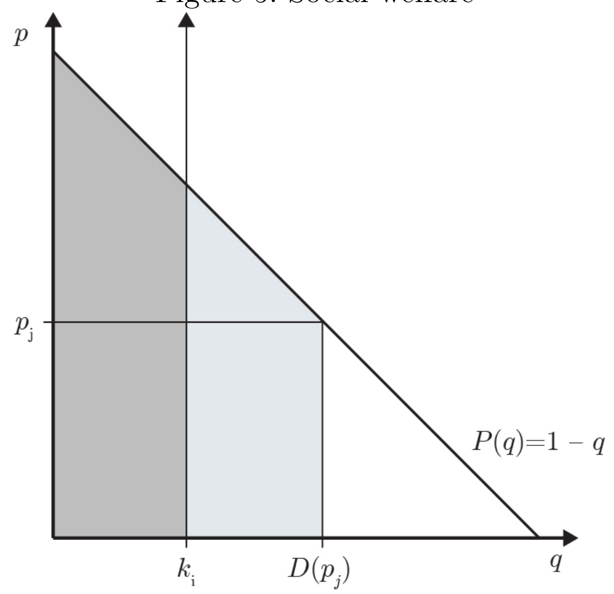
We assume that the firms play the production-to-order type Bertrand-Edgeworth game, and therefore, the game reduces to a price-setting game since the firms can adjust their productions to the demands they face. Now we are ready to specify the firms objective functions. The public firm aims to maximize total surplus, that is,

$$\pi_1(p_1,p_2) = \int_0^{\min\left\{(D(p_j)-k_i)^+,k_j\right\}} R_j(q)dq + \int_0^{\min\{a,k_i\}} P(q)dq, \tag{4}$$

where $0 \le p_i \le p_j \le b$.

Figure 5 illustrates the social welfare, where the area indicated by dark gray corresponds to the second integral, while the area indicated by light gray corresponds to the first integral in expression (4).

Figure 5: Social welfare



[17] The selection of the appropriate tie-breaking rule will ensure the existence of a Nash equilibrium or subgame-perfect Nash equilibrium in order to avoid the consideration of ε-equilibria implying a more difficult analysis without substantial gain.

It should be mentioned that the first integral is not directly taken based on R_j, but by a rightward shift of the vertical axis by k_i units, which enables us to illustrate the social welfare in one figure. It can easily be seen that in determining social welfare the low price does not play a role unless both prices are too high.

The private firm is a profit-maximizer, and therefore,

$$\pi_2(p_1, p_2) = p_2 \min \{k_2, \Delta_2 (D, p_1, k_1, p_2, k_2)\} . \tag{5}$$

4.2 The benchmark

We will compare our price-setting mixed-oligopoly games with the price-setting games with purely private firms, that is, both firms' profit functions take the form of the expression given by (40). The purely private case was investigated by Deneckere and Kovenock [1992] from which we recall the interesting case in which the simultaneous-move game does not have an equilibrium in pure strategies, i.e., $p_2^m > p^c$ in case of $k_2 \geq k_1$.[18] We shall emphasize that Deneckere and Kovenock [1992] assume for the sequential-move games that the demand is allocated first to the second mover[19] and for the simultaneous-move case that the demand is allocated in proportion of the firms' capacities. [20]

Proposition 4.1 (Deneckere and Kovenock,1992). *Under $k_2 \geq k_1$, $p_2^m > p^c$ and Assumptions 4.1-4.3, the results below are valid about the equilibrium of the three games.*

1. *The simultaneous-move game only has an equilibrium in mixed strategies with common support $[p_2^d, p_2^m]$ and equilibrium profits are equal to $\pi_1^S = p_2^d k_1$ and $\pi_2^S = p_2^m D_2^r(p_2^m) = p_2^d \min\{k_2, D\left(p_2^d\right)\}$.*[21]

[18] For other cases the game reduces to the standard Cournot and Bertrand games.

[19] This distinction ensures that the second mover does not need to slightly undercut the first mover.

[20] It should be emphasized that the obtained results remain valid for a large class of other tie-breaking rules employed in the simultaneous-move game.

[21] The results gathered in case 1 summarize the results obtained by Levitan and Shubik [1972], Kreps and Scheinkman [1983] and Osborne and Pitchik [1986].

2. *If firm 2 moves first and firm 1 second, then in a subgame-perfect equilibrium the equilibrium prices are given by* $p_2 = p_1 = p_2^m$ *and the respective equilibrium profits equal* $\pi_2^L = p_2^m D_2^r(p_2^m)$ *and* $\pi_1^F = p_2^m k_1$. *In addition, if* $k_1 = k_2$, *then we also have a second subgame-perfect equilibrium with equilibrium prices given by* $p_2 = p_2^d$ *and* $p_1 = p_2^m$.

3. *If firm 1 moves first and firm 2 second, then in a subgame-perfect equilibrium the equilibrium prices are given by* $p_1 = p_2^d$ *and* $p_2 = p_2^m$ *and the respective equilibrium profits equal* $\pi_1^L = p_2^d k_1$ *and* $\pi_2^F = p_2^m D_2^r(p_2^m)$. *In addition, if* $k_1 = k_2$, *then we also have a second subgame-perfect equilibrium with equilibrium prices given by* $p_1 = p_2^m$ *and* $p_2 = p_2^m$.

Deneckere and Kovenock [1992] find that firm 2 moving first and firm 1 moving second constitutes an equilibrium of the timing game, which can be verified by looking at the payoff table of a two-period timing game shown in Table 2, where in the case of $k_2 > k_1$ we have $\pi_1^F > \pi_1^S = \pi_1^L$ and $\pi_2^L = \pi_2^S = \pi_2^F$.

Table 2: Payoffs for the two period timing game.

	First period	Second period
First period	$(\pi_1^S, \quad \pi_2^S)$	$(\pi_1^L, \quad \pi_2^F)$
Second period	$(\pi_1^F, \quad \pi_2^L)$	$(\pi_1^S, \quad \pi_2^S)$

In addition, by introducing more time periods and discounting, Deneckere and Kovenock [1992] show that this ordering of moves is the one and only emerging endogenously.

4.3 Exogenously given order of moves

In our analysis we will discuss the above mentioned three games with exogenously given ordering of moves for the mixed duopoly version of the Bertrand-Edgeworth game. We are restricting ourselves again to the interesting case in which the simultaneous-move purely private version of the Bertrand-Edgeworth game does not have an equilibrium in pure strategies, i.e. $\max\{p_1^m, p_2^m\} > p^c$.[22] Note that as there are two different types of firms (as far as their payoffs are concerned), it is not the same whether the private or the public firm has the higher capacity. In particular, we have to distinguish between the following two cases: (i) $p_2^m > p^c$ and (ii) $p_1^m > p^c \geq p_2^m$. We will also refer to the first case as the *strong private firm* case and to the second case as the *weak private firm* case. Therefore, we would have to analyze both cases separately for all the three orderings of moves.

4.3.1 The strong private firm case

Firstly, let us make it clear that the price p_2^d having been defined earlier always exists since $p_2^m > p^c$. Therefore, we make no mistake if we use this price in our results in this section.

Now we collect some basic results in the following lemmas.

Lemma 4.1. *Under Assumptions 4.1-4.3 and $p_2^m > p^c$, it is in none of the timing games with an exogenously given order of moves optimal for the private firm to declare $p_2 < p_2^d$.*

Proof. We get the result directly from the definition of p_2^d. By setting $p_2 = p_2^m$, even if the private firm serves only residual demand, its profit will be higher than at a price p_2 less than p_2^d. □

Lemma 4.2. *Under Assumptions 4.1-4.3 and $p_2^m > p^c$, the social welfare in an equilibrium cannot be larger than the social welfare associated with the case in which both firms set price p_2^d.*

Proof. The definition of the public firm's profit implies that if a firm serves residual demand, then the lower price it sets the higher the social welfare. On the other

[22] Otherwise, the ordinary price-setting game results in a market-clearing or a competitive outcome, which also remains the outcome of the mixed duopoly game. For more details we refer to the concluding remarks in Section 4.5.

hand, if a company is the low-price firm, and it produces at its capacity limit, then changing its price, as long as it remains a capacity constrained low-price firm, does not alter social welfare. The private firm will not set its price below p_2^d by Lemma 4.1, while the public firm cannot increase social welfare by setting a price below p_2^d. Hence, comparing all cases satisfying $\min\{p_1, p_2\} \geq p_2^d$, social welfare is maximized when both firms set price p_2^d. □

Lemmas 4.1 and 4.2 show that deleting strictly dominated strategies from the private firms strategy set excludes prices below p_2^d and after that p_2^d turns out to be a weakly dominant strategy of the public firm.

Now that we are aware of these results, we begin to discuss the three games with an exogenously given ordering of moves. We will start with specifying the tie-breaking rules for these games. The most common assumption, if two firms set the same price, is that they share the consumers in proportion to their capacities (i.e. firms i's sales equal $\min\{k_i, D(p)k_i/(k_1 + k_2)\}$. Clearly, any of the two firms has the right to let its competitor serve a certain portion of its consumers. Such an act would seem to be irrational at first sight, but we will see below that if it is carried out by the public firm, it can drive the market to a socially better equilibrium. A more complete specification of the game would allow the public firm to select the consumers freely, strategically leaving them to the private firm. However, we fix a tie-breaking rule, which turns out to be compatible with the public firms incentives, and after determining the equilibria of the three games it can be easily verified that the public firm could not have selected a better tie-breaking rule.

Assumption 4.4. We specify the tie-breaking rules for the three games in the following way:

- In the *simultaneous-move* game we assume that if $p_1 = p_2 \leq p_2^d$, then the demand is allocated first to the private firm (in other words: the public firm allows the private to serve the entire demand up to its previously given capacity k_2). Otherwise the two firms share the consumers proportionally, i.e. firm i's sales equal $\min\{k_i, D(p)k_i/(k_1 + k_2)\}$.

- When *the public firm is the first mover*, we assume that the entire demand is allocated first to the private firm (i.e. to the second mover) at any price level in the case of price ties.

- Provided that *the private firm moves first*, if $p_1 = p_2 \leq p_2^d$, then the demand is allocated first to the private firm; otherwise the entire demand is allocated first to the public firm (i.e. to the second mover).

One can observe that for prices higher than p_2^d we employ the same tie-breaking rule as Deneckere and Kovenock [1992] in establishing Proposition **??**. For the simultaneous-move game we could have selected many other tie-breaking rules for prices not equal to p_2^d. The main requirement for prices above p_2^d is that none of the firms has a priority to serve consumers up to its capacity constraint in case of price ties. Turning to the game with a public leader, it is easy to see that in case of other tie-breaking rules the subgames do not have a solution since the private firm just wants to undercut the public firm's price for prices above p_2^d. Hence, allocating demand first to the second mover restores solvability without changing the nature of the game.[23] Finally, for the game with a private leader we should remark that the public firm has many optimal replies since matching or undercutting the private firm's price does not change social welfare. However, if the public firm wants to punish the private firm for setting high prices, then it should commit itself to undercutting the private firm's price. This commitment is credible since for prices higher than p_2^d the public firm has no incentive to deviate from undercutting the private firm's price in the second period, which explains why our tie-breaking rule gives priority to the public firm.

Now we will give all the equilibria of the different cases in separate propositions. We start with the simultaneous-move case.

[23] Another argument for our tie-breaking rule is as follows: one of the main corollaries of the section is that the appearance of the public firm is beneficial from the point of view of social welfare. Applying a tie-breaking rule that favors the private firm as a second mover makes this result even stronger.

Proposition 4.2 (Simultaneous moves). *Assume that $p_2^m > p^c$ and Assumptions 4.1-4.4 hold. Then the simultaneous-move game has the following two types of Nash-equilibria in pure strategies:*

$$p_1^* = p_2^* = p_2^d \ (NE_1) \quad \textit{and} \quad p_1^* \leq p_2^d, p_2^* = p_2^m \ (NE_2),$$

where the continuum of NE_2 equilibria are payoff equivalent. Moreover, if $k_1 \leq k_2$ and $k_1 \leq D(p^M)$, then the simultaneous-move game has in addition the following set of Nash-equilibria

$$p_1^* > \max\left\{p^M, P(k_2)\right\}, \ p_2^* = \max\left\{p^M, P(k_2)\right\} \quad (NE_3).$$

Finally, no other equilibrium in pure strategies exists.

Proof. In obtaining a better understanding of the simultaneous-move game, the best response correspondences B_1 and B_2 of the two firms will be helpful. In deriving B_1, note that the tie-breaking rule can be neglected since in the case of equal prices social welfare does not depend on the allocation of consumers to the firms. We will consider three cases. First, if the private firm sets price p_2 such that none of the two firms can solely serve the entire demand, then the public firm maximizes social welfare by not setting a higher price than the private firm. We have multiple best responses since decreasing the public firm's price within $[0, p_2)$ results in converting its income to consumer surplus. However, the sum (i.e. the payoff of the public firm) remains the same. Raising its price above p_2 would mean that the public firm faces residual demand and achieves a lower level of social welfare than when it sets $p_1 = p_2$. Second, if the private firm sets price p_2 such that it can serve at least the entire demand, while the public firm can serve at most the entire demand, then the public firm loses its influence on social welfare and can set its price arbitrarily. Third, if the private firm sets a sufficiently high price such that the public firm assures the best social outcome by offering its whole capacity at price $P(k_1)$, then any price

$p_1 \leq P(k_1)$ maximizes social welfare. Summarizing our findings,

$$B_1(p_2) = \begin{cases} [0, p_2] & \text{if } p_2 < \min\{P(k_1), P(k_2)\}, \\ [0, b] & \text{if } P(k_2) \leq p_2 \leq P(k_1), \\ [0, P(k_1)] & \text{if } P(k_1) < p_2 \text{ or } p_2 = P(k_1) < P(k_2). \end{cases} \tag{6}$$

Turning to B_2, if the public firm sets a price below p_2^d, then the private firm reacts with p_2^m. If the public firm sets price p_2^d, then the private firm has two optimal replies: p_2^d and p_2^m because of Assumption 4.4. If $p_1 > p_2^d$, then the private firm will undercut the public firm's price and an optimal reply does not exist as long as it has to share the demand with the public firm at price p_1. Finally, the public firm's price can be large enough not to be followed by the private firm. In particular, if p_1 is larger than $P(k_2)$ and larger than p^M, then the private firm will either set a price to sell its entire capacity or the monopolist's price. Thus, we have obtained

$$B_2(p_1) = \begin{cases} \{p_2^m\} & \text{if } p_1 < p_2^d, \\ \{p_2^d, p_2^m\} & \text{if } p_1 = p_2^d, \\ \emptyset & \text{if } p_2^d < p_1 \leq \max\{P(k_2), p^M\}, \\ \{\max\{P(k_2), p^M\}\} & \text{if } \max\{P(k_2), p^M\} < p_1. \end{cases} \tag{7}$$

Now one can verify directly or by relying on the best response correspondences that NE_1, NE_2 and NE_3 are equilibrium profiles.[24]

It remains to be shown that no other equilibrium exists. Take an equilibrium profile (p_1^*, p_2^*). By Lemma 5.1 $p_2^* \geq p_2^d$. Let $\overline{p} = P(k_1)$. Since the public firm can ensure at least $\pi_1(\overline{p}, \overline{p})$ social welfare at price $\overline{p}$, even if the private firm sets a higher price, we must have $p_1^* \leq \overline{p}$, which in turn implies that $p_2^* \leq \overline{p}$ since $D_2^r(p_2) = 0$ for all $p_2 > \overline{p}$. We cannot have an equilibrium with $p_2^d < p_1^* = p_2^* \leq \overline{p}$ since otherwise the private firm would gain from slightly undercutting the public firm's price.

Assume that the equilibrium satisfies $p_1^* < p_2^* \leq \overline{p}$. Clearly, if $p_2^d < p_1^* < \overline{p}$, then the private firm would benefit from undercutting price p_1^*; a contradiction. If $p_1^* = p_2^d$, then we must have either $p_2^* = p_2^d$ or $p_2^* = p_2^m$; and thus, we arrived to either an NE_1

[24]In carrying out the verification, $p_2^d < P(k_2)$ and $p_2^d < p_2^m < P(k_1)$ will be helpful.

or NE_2 type profile. If $p_1^* < p_2^d$, then we must have $p_2^* = p_2^m$, which is an NE_2 type equilibrium profile.

Assume that the equilibrium satisfies $p_2^* < p_1^* \leq \overline{p}$. Suppose that $k_1 > k_2$. Then $\overline{p} < P(k_2)$, and therefore, the public firm faces a positive residual demand, which means that it can increase social welfare by reducing its price; a contradiction. Assume that $k_1 \leq k_2$. If $D_1^r(p_1^*) > 0$, then we cannot have an equilibrium since once again the public firm would benefit from decreasing its price. If $D_1^r(p_1^*) = 0$, then the public firm cannot gain from altering its price. However, the private firm could benefit from increasing its price by the concavity of D if $p_2^* < p^M$. For the same reason the private firm would gain from decreasing its price if $p_2^* > p^M$ subject to $p_2^* \geq P(k_2)$. Hence, we just can have an equilibrium if $p_2^* = \max\left\{p^M, P(k_2)\right\}$ and we arrived at an NE_3 type profile. □

We have to remark that NE_3 is a very implausible equilibrium since it requires that the public firm sets very high prices in the market and practically does not want to enter the market. While in the case of NE_3 the public firm cannot increase social welfare it still can increase consumer surplus and its own income by setting a price below p^M, which could be a natural secondary goal for the public firm if it has to select between prices resulting in the same social welfare.

We continue with the case of public leadership.

Proposition 4.3 (Public firm moves first)**.** *Assume that $p_2^m > p^c$ and Assumptions 4.1-4.4 hold. Then the sequential-move game with the public firm as a first mover has the following unique subgame-perfect Nash-equilibrium in pure strategies:*

$$p_1^* = p_2^d, p_2^*(p_1) = \begin{cases} p_2^m & \text{if } p_1 < p_2^d, \\ p_2^d & \text{if } p_1 = p_2^d, \\ p_1 & \text{if } p_2^d < p_1 \leq \max\{P(k_2), p^M\}, \\ \max\{P(k_2), p^M\} & \text{if } \max\{P(k_2), p^M\} < p_1. \end{cases} \qquad (SPNE_1)$$

Proof. First, we determine the reaction function $p_2^*(\cdot)$ of the private firm. Observe that the private firm's best response correspondence can be obtained by altering its best response correspondence (7) derived for the simultaneous-move game in the case of $p_2^d < p_1 \leq \max\{P(k_2), p^M\}$ for which in the public leadership game the private firm matches the public firm's price p_1.

The first period action of the public firm depends on the decision of the private firm when the public firm sets price p_2^d. In other words the private firm's reaction function is either the one given by $SPNE_1$ or

$$p_2^*(p_1) = \begin{cases} p_2^m & \text{if } p_1 \leq p_2^d; \\ p_1 & \text{if } p_2^d < p_1 \leq \max\{P(k_2), p^M\}, \\ \max\{P(k_2), p^M\} & \text{if } \max\{P(k_2), p^M\} < p_1. \end{cases} \tag{8}$$

Concerning the reaction function given by $SPNE_1$, the public firm maximizes social welfare in the first period by choosing price $p_1^* = p_2^d$, and thus, $SPNE_1$ is indeed a subgame-perfect Nash equilibrium.

Turning to the second reaction function given by (8), a first period social welfare maximizing price does not exist if the private firm reacts in the way given by (8) since the public firm wants to set its price as close as possible to p_2^d, but above p_2^d. Hence, (8) does not yield a subgame-perfect Nash equilibrium. □

Finally, we consider the case of private leadership.

Proposition 4.4 (Private firm moves first)**.** *Assume that $p_2^m > p^c$ and Assumptions 4.1-4.4 hold. Then the sequential-move game with the private firm as a first mover has the following two types of subgame-perfect Nash-equilibria:*

$$p_2^* = p_2^d, p_1^*(p_2) = \begin{cases} p_2 & \text{if } p_2 \leq p_2^d, \\ p_1 \in [0, p_2] & \text{if } p_2^d < p_2 \leq P(k_1), \\ p_1 \in [0, P(k_1)] & \text{if } p_2 > P(k_1); \end{cases} \qquad (SPNE_1)$$

and

$$p_2^* = p_2^m, p_1^*(p_2) = \begin{cases} p_1 \in [0, p_2] & \text{if } p_2 \leq P(k_1), \\ p_1 \in [0, P(k_1)] & \text{if } p_2 > P(k_1); \end{cases} \qquad (SPNE_2)$$

where the continuum of $SPNE_1$ as well as the continuum $SPNE_2$ equilibria are payoff equivalent, respectively. Moreover, if $k_1 \leq k_2$ and $k_1 \leq D(p^M)$, then the *sequential-move game has in addition the following set of subgame perfect Nash-equilibria*

$$
\begin{aligned}
p_2^* &= \max\left\{p^M, P(k_2)\right\}, \\
p_1^*(p_2) &= \begin{cases} p_1 \in [0, p_2] & \textit{if } p_2 < P(k_2), \\ p_1 \in [0, b] & \textit{if } P(k_2) \leq p_2 < p^M, \\ p_1 \in \left(\max\left\{p^M, P(k_2)\right\}, b\right] & \textit{if } p_2 = \max\left\{p^M, P(k_2)\right\}, \\ p_1 \in [0, b] & \textit{if } \max\left\{p^M, P(k_2)\right\} < p_2 \leq P(k_1), \\ p_1 \in [0, P(k_1)] & \textit{if } P(k_1) < p_2; \end{cases}
\end{aligned}
\tag{$SPNE_3$}
$$

where the continuum of $SPNE_3$ equilibria are payoff equivalent. Finally, no other equilibrium in pure strategies exists.

Proof. We solve the game by backward induction. Observe that the best response correspondence of the public firm is given by (6) since it remains the same as in the case of simultaneous moves. In addition, if the private firm sets a price not higher than p_2^d, then the public firm should not set a price below the private firm's price since anticipating this behavior the private firm would set definitely price p_2^m in period 1.

However, if the price set by the private firm is high enough so that it can serve the entire market, then the public firm loses its influence on social welfare. It might be beneficial for the private firm to set an extremely high price if its capacity is larger than the public firm's capacity and the public firm cannot do better by matching or undercutting the private firm's price. Note that in this latter case only the reaction of the public firm at price $\max\left\{p^M, P(k_2)\right\}$ matters.

Taking this into account, we can obtain $SPNE_1$, $SPNE_2$ and $SPNE_3$ as possible types of subgame-perfect equilibria. □

We have to remark that $SPNE_3$ is an implausible equilibrium since it requires, for the same reasons as explained after the proof of Proposition 5.1, that the private firm anticipates a very strange reaction of the public firm. Hence, in all three games only type 1 or type 2 equilibria are plausible. But what will decide which of these two equilibria is to be played? We now collect arguments in favor of $(SP)NE_1$. Firstly,

$(SP)NE_1$ is the only Pareto-optimal equilibrium among all the subgame-perfect Nash-equilibria. Secondly, an outcome very close to $(SP)NE_1$ can be enforced by the public firm. Namely, assume that the public company sets $p_1 = p_2^d + \epsilon$ in any of the three cases. Then if the private firm sets its price slightly below this level, it will be strictly better off than in the case of $(SP)NE_1$ or $(SP)NE_2$. Although the social welfare will be a bit lower than in the case of $(SP)NE_1$, but far higher than in the case of $(SP)NE_2$. Therefore, this act is worth for the public firm to avoid the risk of an $(SP)NE_2$ type outcome.

To sum up, we argued that $p_1^* = p_2^* = p_2^d$ is the most plausible outcome that is expected to be played.

4.3.2 The weak private firm case

Our second case to be analyzed occurs when $p_1^m > p^c \geq p_2^m$. We begin the analysis with the following lemma which asserts that the private firm does not intend to set any price below the market clearing price.

Lemma 4.3. *Under Assumptions 4.1-4.3 and $p_1^m > p^c \geq p_2^m$ the private firm's strategies $p_2 < p^c$ are strictly dominated in all three possible orderings.*

Proof. The private firm can only be worse off by selling all its capacity at a lower price than the market clearing price due to the definition of p^c and the fact that $p_2^m \leq p^c$. □

Before solving the game, we have to define how the firms share the market in the case of price ties. In particular, we employ the same tie-breaking rule as Deneckere and Kovenock [1992].

Assumption 4.5. If the two firms set the same price, then we assume for the sequential-move games that the demand is allocated first to the second mover and for the simultaneous-move game that the demand is allocated in proportion of the firms' capacities.

In contrast to the previous section, there is only one (subgame-perfect) Nash-equilibrium in the weak private firm case resulting in a market-clearing outcome. This is pointed out in the following proposition.

Proposition 4.5. *If $p_1^m > p^c \geq p_2^m$ and Assumptions 4.1-4.3, 4.5 are satisfied, then each price-setting game with an exogenously given ordering of moves has the following (subgame-perfect) Nash-equilibria in pure strategies with the following equilibrium prices:*

$$p_1^* \in [0, p^c], p_2^* = p^c \quad (SPNE_4)$$

Moreover, there is no other equilibrium in pure strategies.

Proof. Now we do not consider the different orderings separately, like we did in the previous subsection. It is easy to see that any equilibrium of type $SPNE_4$ specifies (subgame-perfect) Nash-equilibrium prices in all the three possible orderings.

We briefly show that there are no other equilibrium price profiles left. Since by $p_1^m > p_2^m$ we have $k_1 > k_2$, which in turn implies $P(k_1) < P(k_2)$ none of the firms will set a price above $P(k_1)$. Moreover, we cannot have an equilibrium with $P(k_1) \geq p_2^* > p_1^*$ and $p_2^* > p^c$ since this would imply that the private firm has to serve residual demand, which would result in less profits for the private firm than setting p^c by the concavity of the residual demand function and by $p_2^m < p^c$. In addition, there cannot be an equilibrium with $p^c < p_2^* < p_1^* \leq P(k_1)$ either because then the public firm could increase social welfare by unilaterally decreasing its price. These arguments are valid for any ordering of moves.

It remains to be shown that we cannot have $p^c < p_1^* = p_2^* \leq P(k_1)$ in an equilibrium, which we will check separately for the different orderings of moves. Concerning the simultaneous-move case, the private firm would have an incentive to undercut the public firm's price. Turning to private leadership, the private firm has to serve residual demand by Assumption 4.5, and therefore, it would prefer price p^c to p_2^* by the concavity of the residual demand function and by $p_2^m < p^c$. Finally, considering public leadership, the private firm matches the public firm's price in region $[p^c, P(k_1)]$, and thus, the public firm maximizes social welfare by setting its price not larger than p^c. □

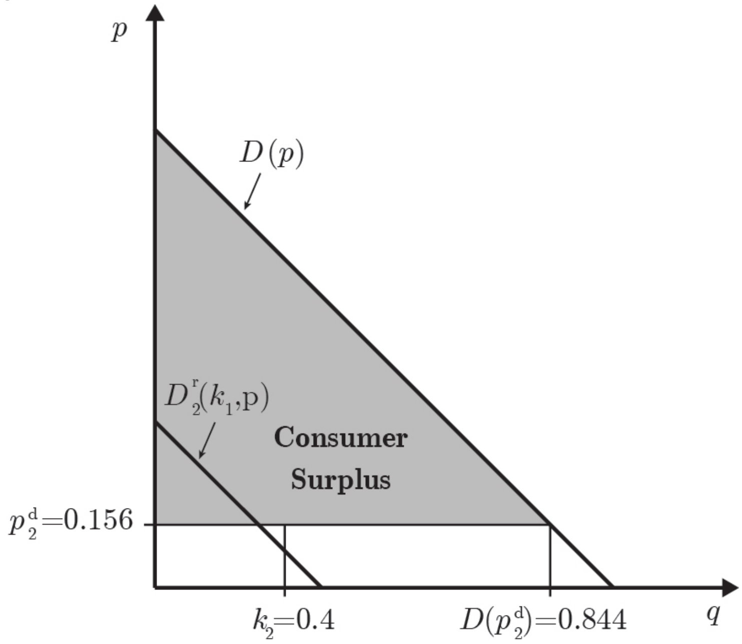
4.4 A numerical example

We illustrate our main results summarized in the previous section by the following example.

Example 4.1. Let the demand function be $D(p) = 1 - p$. We assume that $k_1 = 0.5$, $k_2 = 0.4$ and in the mixed version of the game firm 1 is the public firm, while firm 2 is the private firm. We calculated the equilibrium prices and payoffs both for the standard version of the game and for the mixed version as well. According to our earlier arguments, we assumed in the example that NE_1 or $SPNE_1$-type equilibria are played in each case. The calculated equilibrium prices and quantities are as follows: $p_1^* = p_2^* = p_2^d = 0.156$, $q_1^* = 0.444$, $q_2^* = k_2 = 0.4$.

The $(SP)NE_1$-type equilibrium is illustrated in Figure 6 below.

Figure 6: Equilibrium prices and quantities - Example 4.1



The following tables show the values calculated for all the three possible orderings of moves.

Table 3: Calculated values for the simultaneous-moves case.

	Standard version	Mixed version
p_1^*	0.2254	0.1563
p_2^*	0.2068	0.1563
π_1^*	0.0900	0.4870
π_2^*	0.0720	0.0625
Social welfare	**0.4715**	**0.4870**

Note that in the first column of Table 3 the equilibrium prices, profits and social welfare given for the standard version of the game are expected values as there is no pure-strategy equilibrium in the case of simultaneous moves. We have computed these expected values by employing the explicit solution of the Bertrand-Edgeworth game determined by Kreps and Scheinkman [1983].

Table 4: Calculated values when firm 1 moves first.

	Standard version	Mixed version
p_1^*	0.3000	0.1563
p_2^*	0.3000	0.1563
π_1^*	0.0900	0.4870
π_2^*	0.1200	0.0625
Social welfare	**0.4550**	**0.4870**

Table 5: Calculated values when firm 2 moves first.

	Standard version	Mixed version
p_1^*	0.3000	0.1563
p_2^*	0.1800	0.1563
π_1^*	0.0900	0.4870
π_2^*	0.0720	0.0625
Social welfare	**0.4550**	**0.4870**

The values in Tables 3-5 show the social welfare-increasing effect of the appearance of a public firm, which we emphasized in Section 4.3.

4.5 Corollaries and concluding remarks of the section

In this subsection we collect the corollaries of our analysis carried out in the previous subsection. Our first corollary determines the endogenous order of moves based on a two-period timing game in which both firms can select between two periods for setting their prices. If one accepts our arguments brought forward in favor of a type 1 equilibrium (that is, an NE_1 or an $SPNE_1$) in case of a strong private firm, then by checking Propositions 4.2-4.4 one immediately sees that the three type 1 equilibria result in the same equilibrium price p_2^d and the same equilibrium payoffs. The case of a weak private firm is even simpler by Proposition 4.5.

Corollary 4.1. *Assuming that a type 1 equilibrium is played in the case of a strong private firm, the ordering of price decisions does not matter.*

Now we turn to the question whether replacing a public firm by a private firm (privatization) has a social welfare increasing effect. If one compares the equilibrium payoffs in Proposition 4.1 with the type 1 equilibrium payoffs in Propositions 4.2-4.4 and Proposition 4.5, one can observe that for each ordering of moves switching from the standard PTO Bertrand-Edgeworth game (i.e. when there are only private firms on the market) to its mixed version strictly increases social welfare.

Corollary 4.2. *Assuming that a type 1 equilibrium is played in the case of a strong private firm and capacities are in a range such that the standard simultaneous-move PTO Bertrand-Edgeworth game does not have an equilibrium in pure strategies, then the appearance of a public firm makes the outcome more competitive, i.e. the social welfare of the mixed version of the PTO Bertrand-Edgeworth game is higher than that of the standard PTO Bertrand-Edgeworth duopoly game.*

Earlier research in this field has pointed out that in general the standard simultaneous-move version of the Bertrand-Edgeworth game does not have an equilibrium in pure strategies (see, for example, Proposition 4.1).[25] Considering Propositions 2-6, we see that the simultaneous-move Bertrand-Edgeworth mixed duopoly game always has an equilibrium in pure strategies.

Corollary 4.3. *In contrast with the standard simultaneous-move Bertrand-Edgeworth game its mixed duopoly version always has an equilibrium in pure strategies under Assumptions 4.1-4.3.*

To conclude, we have found that the appearance of a public firm is advantageous from various points of view. First, the timing of decisions does not play a role since all games with an exogenously given order of moves result in the same outcome. Second, the appearance of a public firm increases social welfare. Third, the mixed version of the simultaneous-move PTO Bertrand-Edgeworth game always has an equilibrium in pure strategies.

In our analysis we focused on the interesting case in which the standard PTO Bertrand-Edgeworth game does not have an equilibrium in pure strategies. It should be emphasized that in the other case (i.e. $p^c \geq \max\{p_1^m, p_2^m\}$) there is no real difference concerning the market outcome between the standard and mixed versions

[25] In fact the class of demand curves that admit an equilibrium in pure strategies for arbitrary capacity levels cannot intersect both axes (Tasnádi 1999).

of the Bertrand-Edgeworth game. In particular, sales take place only at the market clearing price and the entire demand is served at that price regardless of the ordering of moves. However, similarly to the strong private firm case the simultaneous-move game and the private leadership game has an additional implausible outcome of type 3, whenever $k_1 \leq k_2$ and $k_1 \leq D(p^M)$, in which the private firm sets price $p_2^* = \max\left\{p^M, P(k_2)\right\}$ and the public firm a higher price $p_1^* > \max\left\{p^M, P(k_2)\right\}$. We consider this latter outcome as implausible since this would require that the public firm does not want to enter the market in order to achieve at least a positive income or a higher consumer surplus though social welfare remains the same. The equilibria can be determined in an analogous way to the strong private firm case.[26]

We have not investigated the situation yet, where more than one private firms exist on the market. However, we should be aware that our knowledge concerning the Bertrand-Edgeworth oligopoly game with only private firms is limited. Even the existence of an equilibrium of multi-period games with exogenously given ordering of moves is not known for the case in which at least pairs of firms move in different time periods. The most recent results on the mixed-strategy equilibria of the simultaneous-move Bertrand-Edgeworth oligopoly game by Hirata [2009] and De Francesco and Salvadori [2010] point to the difficulty of the problem.

In the current section we have investigated the production-to-order version of the Bertrand-Edgeworth game. In the next section we consider the production-in-advance case for which answering the same questions addressed in this section turns out to be much harder. Namely, the PIA case will not reduce to a price-setting game, separate price and production decisions need to be considered.

[26] A detailed proof is available upon request from the authors.

5 Mixed Bertrand-Edgeworth duopolies - Production-in-advance framework

Production-to-order and production-in-advance has been compared in many frameworks. In this section we investigate a mixed production in advance version of the capacity-constrained Bertrand-Edgeworth duopoly game and determine the solution of the respective timing game. We show that a pure-strategy (subgame-perfect) Nash-equilibrium point exists for all possible orderings of moves. It is pointed out that unlike the production-to-order case, the equilibrium of the timing game lies at simultaneous moves. An analysis of the public firm's impact on social welfare is also carried out. All the results are compared to those of the production-to order version of the respective game.

The remainder of the section is organized as follows. In Section 5.1 we present our framework, Sections 5.2 - 5.4. contain the analysis of three cases: the strong private firm case, the weak private firm case and the high unit cost case, respectively. The analysis is carried out for all possible exogenously given orderings of moves, Section 5.5 solves the timing game, and finally, we conclude in Section 5.6.

The results of this section are put down in Balogh and Tasnádi [2014].

5.1 The framework

The demand is given by function D on which we impose the following restrictions:

Assumption 5.1. The demand function D intersects the horizontal axis at quantity a (where $a > 0$) and the vertical axis at price b. D is strictly decreasing, concave and twice continuously differentiable on $(0, b)$; moreover, D is right-continuous at 0, left-continuous at b and $D(p) = 0$ for all $p \geq b$.

Clearly, no price-setting firm will set its price above b. Let us denote by P the inverse demand function. Thus, $P(q) = D^{-1}(q)$ for $0 < q \leq a$, $P(0) = b$, and $P(q) = 0$ for $q > a$.

On the producers' side we have a public firm and a private firm, that is, we consider a so-called mixed duopoly. We label the public firm with 1 and the private firm with 2. Henceforth, we will also label the two firms by i and j, where $i, j \in \{1, 2\}$ and $i \neq j$. Our assumptions imposed on the firms' cost functions are as follows:

Assumption 5.2. The two firms have identical $c \in (0, b)$ unit costs up to the positive capacity constraints k_1, k_2 respectively.

We shall denote by p^c the market clearing price and by p^M the price set by a monopolist without capacity constraints, i.e. $p^c = P(k_1 + k_2)$ and $p^M = \arg\max_{p \in [0,b]} (p - c) D(p)$. In what follows $p_1, p_2 \in [0, b]$ and $q_1, q_2 \in [0, a]$ stand for the prices and quantities set by the firms.

For any firm i and for any quantity q_j set by its opponent j we shall denote by $\overline{p}_i(q_j)$ the profit maximizing price on the firms' residual demand curves $D_i^r(p, q_j) = (D(p) - q_j)^+$, i.e. $\overline{p}_i(q_j) = \arg\max_{p \in [0,b]} (p - c) D_i^r(p, q_j)$, where in the definition of $\overline{p}_i(q_j)$ we do not include the capacity constraint of firm i. For notational convenience we also introduce $p_i^m(q_j)$ as the price maximizing profits with respect to the residual demand curve subject to the capacity constraint of firm i (i.e. the high-price firm produces the quantity given by its residual demand curve). Clearly, p^c, $\overline{p}_i$ and p_i^m are well defined whenever $c < P(q_j)$ and Assumptions 6.1-6.2 are satisfied. If $c \geq P(q_j)$, then $\overline{p}_i(q_j)$ and $p_i^m(q_j)$ are not unique, as any price $p_i > P(q_j)$ together with quantity $q_i = 0$ results in $\pi_i = 0$ and π_i cannot be positive. In other words, any price $p_i > P(q_j)$ is profit-maximizing on the residual demand curve. For notational convenience we define $\overline{p}_i(q_j)$ and $p_i^m(q_j)$ by b in case of $c \geq P(q_j)$.

For a given quantity q_j we shall denote the inverse residual demand curve of firm i by $R_i(\cdot, q_j)$. It can be checked that $R_i(q_i, q_j) = P(q_i + q_j)$ and $p_i^m(q_j) = \max\{\overline{p}_i(q_j), R_i(k_i, q_j)\}$. Let $\overline{q}_i(q_j) = \arg\max_{q_i \in [0,a]} (R_i(q_i, q_j) - c) q_i$ and $q_i^m(q_j) = \min\{\overline{q}_i(q_j), k_i\}$. Clearly, $\overline{q}_i(q_j) = D_i^r(\overline{p}_i(q_j), q_j)$ and $q_i^m(q_j) = D_i^r(p_i^m(q_j), q_j)$.

Let us denote by $p_i^d(q_j)$ the smallest price for which

$$(p_i^d(q_j) - c) \min\left\{k_i, D\left(p_i^d(q_j)\right)\right\} = (p_i^m(q_j) - c) q_i^m(q_j),$$

whenever this equation has a solution.[27] Provided that the private firm has 'sufficient' capacity, that is $\max\{p^c, c\} < p_2^m(k_1)$, then if it is a profit-maximizer, it is indifferent to whether serving residual demand at price level $p_2^m(q_1)$ or selling $\min\{k_2, D\left(p_2^d(q_1)\right)\}$ at the lower price level $p_2^d(q_1)$. Note that if $R_i(k_i, q_j) \geq \overline{p}_i(q_j)$, then $p_i^d(q_j) = p_i^m(q_j)$.[28] We shall denote by $\tilde{q}_j$ the quantity for which $\overline{q}_i(\tilde{q}_j) = k_i$ in case of $p^M < P(k_i)$ and quantity zero otherwise. From Deneckere and Kovenock (1992, Lemma 1) it follows that $p_i^d(\cdot)$ is strictly decreasing on its domain and it can be also verified that $\overline{p}_i(\cdot)$, $\overline{q}_i(\cdot)$, and $p_i^m(\cdot)$ are also strictly decreasing on the region in which the profit maximization problem with respect to the residual demand curve has an interior solution (i.e. not lying on one of the axes). Moreover, $q_i^m(\cdot)$ is strictly decreasing on $[\tilde{q}_j, k_j]$ and constant on $[0, \tilde{q}_j]$.

We assume efficient rationing on the market, and thus, the firms' demands equal

$$\Delta_i\left(D, p_1, q_1, p_2, q_2\right) = \begin{cases} D\left(p_i\right) & \text{if} \quad p_i < p_j, \\ T_i(p, q_1, q_2), & \text{if} \quad p = p_i = p_j \\ \left(D\left(p_i\right) - q_j\right)^+ & \text{if} \quad p_i > p_j, \end{cases}$$

for all $i \in \{1, 2\}$, where T_i stands for a tie-breaking rule.[29] We will consider two sequential-move games (one with the public firm as the first mover and one with the private firm as the first-mover) and a simultaneous-move game. We employ the same tie-breaking rule as Deneckere and Kovenock [1992].

Assumption 5.3. If the two firms set the same price, then we assume for the sequential-move games that the demand is allocated first to the second mover and for the simultaneous-move game that the demand is allocated in proportion of the firms' capacities.

[27] The equation defining $p_i^d(q_j)$ has a solution for any $q_j \in [0, k_j]$ if, for instance, $p_i^m(k_j) \geq \max\{p^c, c\}$, which will be the case in our analysis when we will refer to $p_i^d(q_j)$.

[28] This can be the case if $p^M < P(k_1)$.

[29] The selection of the appropriate tie-breaking rule will ensure the existence of a Nash equilibrium or subgame perfect Nash equilibrium in order to avoid the consideration of ε-equilibria implying a more difficult analysis without substantial gain.

Now we specify the firms' objective functions. The public firm aims at maximizing total surplus, that is,

$$\pi_1(p_1, q_1, p_2, q_2) = \int_0^{\min\left\{(D(p_j)-q_i)^+, q_j\right\}} R_j(q, q_i)dq + \int_0^{\min\{a, q_i\}} P(q)dq - c(q_1+q_2), \quad (9)$$

where $0 \leq p_i \leq p_j \leq b$. The private firm is a profitmaximizer, and therefore,

$$\pi_2(p_1, q_1, p_2, q_2) = p_2 \min\{k_2, \Delta_2(D, p_1, q_1, p_2, q_2)\} - cq_2. \quad (10)$$

We divide our analysis into three cases.

1. The *strong private firm case*, where we assume that $\overline{p}_2(k_1) > \max\{p^c, c\}$. This means that the private firm's capacity is large enough to have strategic influence on the outcome.

2. The *weak private firm case*, where we assume that $\overline{p}_1(k_2) > p^c \geq \overline{p}_2(k_1) > c$. In this case the private firm's capacity is not large enough to have strategic influence on the outcome, but it has a unique profit-maximizing price on the residual demand curve.

3. The *high unit cost case*, where we assume that $c \geq P(k_1)$. In this case if the public firm produces at its capacity level, then there is no incentive for the private firm to enter the market, because the cost level is too high.

The three cases are well defined and disjoint from each other: in the first two cases $P(k_1) > c$ is satisfied (otherwise $c > \overline{p}_2(k_1)$ would have to hold, a contradiction) and these cases are divided by the relation of p^c and $\overline{p}_2(k_1)$. Additionally, in the third case $P(k_1) \leq c$.

We now determine all the equilibrium strategies of both firms for the three possible orderings of moves in each of the three main cases. Within every case we begin with the simultaneous moves subcase, thereafter we focus on the public-firm-moves-first subcase, finally we analyze the private-firm-moves-first subcase. The results are always illustrated with numerical examples. For better visibility, the most interesting equilibria are depicted.

5.2 The strong private firm case

The following two inequalities remain true for all three orderings of moves, therefore we do not discuss them separately in each subsection.

Lemma 5.1. *Under Assumptions 5.1-5.3 and* $\overline{p}_2(k_1) > \max\{p^c, c\}$ *we must have*

$$p_2^* \geq p_2^d(q_1^*) \tag{11}$$

in any equilibrium $(p_1^*, q_1^*, p_2^*, q_2^*)$.

Proof. We obtain the result directly from the definition of $p_2^d(q_1)$. For any $q_1 \in [0, k_1]$, the private firm is better off by setting $p_2 = p_2^m(q_1)$ and $q_2 = q_2^m(q_1)$, than by setting any price $p_2 < p_2^d(q_1)$ and any quantity $q_2 \in [0, k_2]$. □

Lemma 5.2. *Under Assumptions 5.1-5.3 and* $\overline{p}_2(k_1) > \max\{p^c, c\}$ *we have in case of simultaneous moves and public leadership that*

$$p_2^* \leq \max\{P(k_2), p^M\} \tag{12}$$

in any equilibrium $(p_1^*, q_1^*, p_2^*, q_2^*)$.

Proof. Suppose that $p_2^* > \max\{P(k_2), p^M\}$. If $p_2^* \leq p_1^*$, then the private firm would be better off by setting price $\max\{P(k_2), p^M\}$ and quantity $D\left(\max\{P(k_2), p^M\}\right)$. If $p_2^* > p_1^*$, then the private firm serves residual demand, and therefore switching to action $(p_2^m(q_1^*), q_2^m(q_1^*))$, $\left(\max\{P(k_2), p^M\}, D\left(\max\{P(k_2), p^M\}\right)\right)$ or just undercutting the public firm would be beneficial. For both cases we have obtained a contradiction. □

5.2.1 Simultaneous moves

For the case of simultaneous moves we have two possible[30] pure-strategy Nash equilibrium families. The first equilibrium family contains profiles where the private firm maximizes its profit on the residual demand choosing $p_2^* = p_2^m(q_1^*)$ and $q_2^* = q_2^m(q_1^*)$, while the public firm can choose any price level not greater than $p_2^d(q_1^*)$ and produce

[30] Provided that certain conditions hold true.

any non-negative amount up to its capacity. In the second equilibrium the private firm acts as a monopolist up to its capacity limit on the original demand curve, while the public firm is not present on the market. This is put down in the next proposition.

Proposition 5.1 (Simultaneous moves). *Let Assumptions 5.1-5.3 and* $\overline{p}_2(k_1) > \max\{p^c, c\}$ *be satisfied. A strategy profile*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)) \tag{13}$$

is for a quantity $q_1^* \in [0, k_1]$ *and for any price* $p_1^* \in \left[0, p_2^d(q_1^*)\right]$ *a Nash-equilibrium in pure strategies if and only if*

$$\pi_1\left(p_2^d(q_1^*), q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)\right) \geq \pi_1\left(P(k_1), k_1, p_2^m(q_1^*), q_2^m(q_1^*)\right),^{31} \tag{14}$$

where there exists a nonempty closed subset H *of* $[0, k_1]$ *satisfying condition (14).*[32] *Moreover, if* $\pi_1(p_2', 0, p_2', q_2') \geq \pi_1(P(k_1), k_1, p_2', q_2')$*, where* $p_2' = \max\{P(k_2), p^M\}$ *and* $q_2' = D(\max\{P(k_2), p^M\})$*, then for all* $p_1 \in [0, b]$

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0, p_2', q_2') \tag{15}$$

are also equilibrium profiles.[33] *Finally, no other equilibrium in pure strategies exists.*

Proof. Assume that $(p_1^*, q_1^*, p_2^*, q_2^*)$ is an arbitrary equilibrium profile. We divide our analysis into three subcases. In the first case (Case A) we have $p_1^* = p_2^*$, in the second one (Case B) $p_1^* > p_2^*$ holds true, while in the remaining case we have $p_1^* < p_2^*$ (Case C).

Case A: We claim that $p_1^* = p_2^*$ implies $q_1^* + q_2^* = D(p_2^*)$. Suppose that $q_1^* + q_2^* < D(p_2^*)$. Then[34] because of $p_2^* > \max\{p^c, c\}$ by a unilateral increase in output the

[31] Clearly, $P(k_1) < p_2^m(q_1^*)$ is a necessary condition for (14).

[32] In particular, there exists a subset $[\overline{q}, k_1]$ of H.

[33] Observe that those Nash equilibria appearing in (15) for which $p_1 < p_2'$ are also included in (13).

[34] Observe that by Lemma 5.1, the monotonicity of $p_2^d(\cdot)$ and $\overline{p}_2(k_1) > \max\{p^c, c\}$ we have $p_2^* \geq p_2^d(q_1) \geq p_2^d(k_1) > \max\{p^c, c\}$.

public firm could increase social welfare or the private firm could increase its profit; a contradiction. Suppose that $q_1^* + q_2^* > D(p_2^*)$. Then the public firm could increase social welfare by decreasing its output or if $q_1^* = 0$, the private firm could increase its profit by producing only $D(p_2^*)$; a contradiction.

We know that we must have $p_1^* = p_2^* \geq p_2^d(q_1^*)$ by Lemma 5.1. Assume that $q_1^* > 0$. Then we must have $q_2^* = \min\{k_2, D(p_2^*)\}$, since otherwise the private firm could benefit from reducing its price slightly and increasing its output sufficiently (in particular, by setting $p_2 = p_2^* - \varepsilon$ and $q_2^* = \min\{k_2, D(p_2)\}$). Observe that $p_2^m(0) = p_2^d(0) = p_2'$, $p_2^m(q_1) = p_2^d(q_1)$ for all $q_1 \in [0, \tilde{q}_1]$ and $p_2^m(q_1) > p_2^d(q_1)$ for all $q_1 \in (\tilde{q}_1, k_1]$.[35] Moreover, it can be verified by the definitions of $p_2^m(q_1^*)$ and $p_2^d(q_1^*)$ that $q_1^* + k_2 \geq D(p_2^d(q_1^*)) \geq D(p_2^*)$, where the first inequality is strict if $q_1^* > \tilde{q}_1$. Thus, $q_1^* > \tilde{q}_1$ is in contradiction with $q_2^* = \min\{k_2, D(p_2^*)\}$ since we already know that $q_1^* + q_2^* = D(p_2^*)$ in Case A. Hence, an equilibrium in which both firms set the same price and the public firm's output is positive exists if and only if $p_2^m(q_1^*) = p_2^d(q_1^*)$ (i.e., $q_1^* \in (0, \tilde{q}_1)$) and (14) is satisfied. This type of equilibrium appears in (13).

Moreover, it can be verified that $(p_1^*, q_1^*, p_2^*, q_2^*) = (p_2', 0, p_2', q_2')$ is an equilibrium profile in pure strategies if and only if

$$\pi_1(p_2', 0, p_2', q_2') \geq \pi_1(P(k_1), k_1, p_2', q_2'), \tag{16}$$

where $p_2' = \max\{P(k_2), p^M\}$ and $q_2' = D(\max\{P(k_2), p^M\})$.

Case B: Suppose that $p_1^* > p_2^* \geq p_2^d(q_1^*)$ and $D(p_2^*) > q_2^*$. Then the public firm could increase social welfare by setting price $p_1 = p_2^*$ and $q_1 = D(p_2^*) - q_2^*$; a contradiction.

Assume that $p_1^* > p_2^* \geq p_2^d(q_1^*)$ and $D(p_2^*) = q_2^*$. Then in an equilibrium we must have $q_1^* = 0$, $p_2^* = p_2'$ and $q_2^* = q_2'$. Furthermore, it can be checked that these profiles specify equilibrium profiles if and only if equation (16) is satisfied.

Clearly, $p_1^* > p_2^* \geq p_2^d(q_1^*)$ and $D(p_2^*) < q_2^*$ cannot be the case in an equilibrium since the private firm could increase its profit by producing $q_2 = D(p_2^*)$ at price p_2^*.

Case C: In this case $p_2^* = p_2^m(q_1^*)$ and $q_2^* = q_2^m(q_1^*)$ must hold, since otherwise the

[35] We recall that $\tilde{q}_i$ has been defined after $p_i^d(q_j)$.

private firm's payoff would be strictly lower. In particular, if the private firm sets a price not greater than p_1^*, we are not anymore in Case C; if $q_2^* > \min\{D_2^r(p_2^*, q_1^*), k_2\}$, then the private firm either produces a superfluous amount or is capacity constrained; if $q_2^* < \min\{D_2^r(p_2^*, q_1^*), k_2\}$, then the private firm could still sell more than q_2^*; and if $q_2^* = \min\{D_2^r(p_2^*, q_1^*), k_2\}$, then the private firm will choose a price-quantity pair maximizing profits with respect to its residual demand curve $D_2^r(\cdot, q_1^*)$ subject to its capacity constraint. In addition, in order to prevent the private firm from undercutting the public firm's price we must have $p_1^* \leq p_2^d\left(q_1^*\right)$.

Clearly, for the given values p_1^*, p_2^* and q_2^* from our equilibrium profile the public firm has to choose a quantity $q_1' \in [0, k_1]$, which maximizes function $f(q_1) = \pi_1\left(p_1^*, q_1, p_2^*, q_2^*\right)$ on $[0, k_1]$. We show that $q_1' = q_1^*$ must be the case. Obviously, it does not make sense for the public firm to produce less than q_1^* since this would result in unsatisfied consumers. Observe that for all $q_1 \in [q_1^*, \min\left\{D\left(p_2^*\right), k_1\right\}]$

$$\begin{aligned} f(q_1) &= \int_0^{D\left(p_2^*\right)-q_1} \left(R_2(q, q_1) - c\right) dq + \int_0^{q_1} \left(P(q) - c\right) dq - c(q_1 - q_1^*) = \\ &= \int_0^{D\left(p_2^*\right)} P(q) dq - D(p_2^*)c - c(q_1 - q_1^*). \end{aligned} \tag{17}$$

Since $-c(q_1 - q_1^*)$ is a function of only q_1 we see that f is strictly decreasing on $[q_1^*, \min\left\{D\left(p_2^*\right), k_1\right\}]$.

Subase (i): In case of $k_1 \leq D\left(p_2^*\right)$ we have already established that q_1^* maximizes f on $[0, k_1]$. Moreover, (p_1^*, q_1^*) maximizes $\pi_1\left(p_1, q_1, p_2^*, q_2^*\right)$ on $[0, p_2^*) \times [0, k_1]$ since equation (17) is not a function of p_1^*. Hence, for any $p_1 \leq p_2^d\left(q_1^*\right)$ such that $p_1 < p_2^*$ we have that $\left(p_1, q_1^*, p_2^m\left(q_1^*\right), q_2^m\left(q_1^*\right)\right)$ specifies a Nash equilibrium for any $q_1 \in (0, k_1]$ satisfying $k_1 \leq D\left(p_2^m\left(q_1^*\right)\right)$. However, note that in case of $q_1^* \in [0, \tilde{q}_1]$ and $p_1 = p_2^d\left(q_1^*\right)$ we are leaving Case C and obtain a Case A Nash equilibrium.

Observe that $p_2^m\left(k_1\right) > \max\left\{p^c, c\right\}$ implies that $k_1 < D\left(p_2^m\left(k_1\right)\right)$, and therefore we always have Subcase (i) equilibrium profiles. If $k_1 = D\left(p_2^m\left(q_1\right)\right)$ has a solution for $q_1 \in [0, k_1)$, then we shall denote its solution by $\widetilde{q}$, and otherwise let $\widetilde{q} = 0$. Since $D\left(p_2^m\left(\cdot\right)\right)$ is a continuous and strictly increasing function, interval $[\widetilde{q}, k_1] \cap (0, k_1]$ determines the set of quantities yielding an equilibrium for Subcase (i).

Subase (ii): Turning to the more complicated case of $k_1 > D(p_2^*)$, we also have to investigate function f above the interval $[D(p_2^*), k_1]$ in which region the private firm does not sell anything at all at price p_2^* and

$$f(q_1) = \int_0^{\min\{q_1, D(p_1^*)\}} (P(q) - c)\, dq - cq_2^* - c\,(q_1 - D(p_1^*))^+ . \tag{18}$$

Observe that we must have $P(k_1) < p_2^*$. If the public firm is already producing quantity $q_1 = D(p_2^*)$, the private firm does not sell anything at all and contributes to a social loss of cq_2^*. Therefore, $f(q)$ is increasing on $[D(p_2^*), \min\{D(p_1^*), k_1\}]$.

Assume that $k_1 \leq D(p_1^*)$. Then for any $p_1 \leq p_2^d(q_1^*)$ we get that $(p_1, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*))$ is a Nash equilibrium if and only if

$$\begin{aligned} \pi_1\left(p_2^d(q_1^*), q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)\right) &\geq \pi_1\left(p_2^d(q_1^*), k_1, p_2^m(q_1^*), q_2^m(q_1^*)\right) = \\ &= \pi_1\left(P(k_1), k_1, p_2^m(q_1^*), q_2^m(q_1^*)\right), \end{aligned} \tag{19}$$

where the last equality follows from the inequalities $p_1^* \leq P(k_1) \leq p_2^*$ valid for this case and the fact that social welfare is maximized in (p_1, q_1) subject to the constraint that the private firm does not sell anything at all if the public firm sets an arbitrary price not greater than $P(k_1)$ and produces k_1.

Assume that $k_1 > D(p_1^*)$. Therefore, $f(q)$ would be decreasing on $[D(p_1^*), k_1]$. However, it can be checked that the public firm could increase social welfare by switching to strategy $(P(k_1), k_1)$ from strategy $(p_1^*, D(p_1^*))$. In addition, any strategy (p_1, k_1) with $p_1 \leq P(k_1)$ maximizes social welfare subject to the constraint that the private firm does not sell anything at all. Therefore, $\left(p_2^d(q_1^*), q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)\right)$ is a Nash equilibrium if and only if condition (14) is satisfied. Comparing equation (19) with equation (14), we can observe that we have derived the same necessary and sufficient condition for a strategy profile being a Nash equilibrium, which is valid for Subcase (ii).

So far we have established that there exists a function g, which uniquely determines the highest equilibrium price as a function of quantity q produced by the public firm. Clearly, $g(q) = p_2^d(q)$, where the domain of g is not entirely specified. At least we know from Subcase (i) that the domain of g contains $[\widetilde{q}, k_1]$. Observe also that the equilibrium profiles of Subcase (i) satisfy condition (14). Let $u(q_1) = \pi_1\left(p_2^d(q_1), q_1, p_2^m(q_1), q_2^m(q_1)\right)$ and $v(q_1) = \pi_1(P(k_1), k_1, p_2^m(q_1), q_2^m(q_1))$. Hence, q_1 determines a Nash equilibrium profile if and only if $u(q_1) \geq v(q_1)$. It can be verified that u and v are continuous, and therefore, set $H = \{q \in [0, k_1] \mid u(q) \geq v(q)\}$ is a closed set containing $[\widetilde{q}, k_1]$. □

Example 5.1. Let the demand curve take the form of $D(p) = 1 - p$. From the demand curve, the following functions can be directly derived: $p_2^m(q_1) = \frac{1-q_1-c}{2}$, $q_2^m(q_1) = \frac{1-q_1+c}{2}$, while $p_2^d(q_1) = \frac{(p_2^m(q_1)-c)q_2^m(q_1)}{k_2} + c = \frac{(1-q_1-3c)(1-q_1+c)+4ck_2}{4k_2}$.

For the illustration of the first Nash equilibrium profile mentioned in the statement let $k_1 = 0.5$, $k_2 = 0.4$ and $c = 0.1$. Now the following values can be calculated directly from the exogenously given data: $p^c = 0.1$, $p_2^m(k_1) = 0.3$, $q_2^m(k_1) = 0.2$, $p_2^d(k_1) = 0.2$.

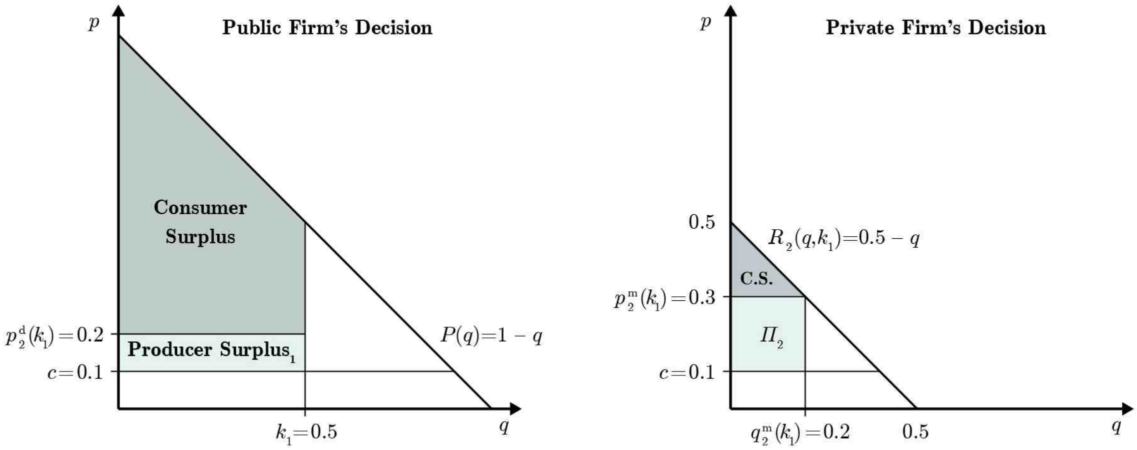
The first Nash equilibrium profile mentioned in the statement is realized with the above values as follows:

$$(p_1^*, q_1^*, p_2^*, q_2^*) = \left(p_1^*, q_1^*, \frac{1 - q_1^* - c}{2}, \frac{1 - q_1 + c}{2}\right)$$

where $q_1^* \in [0, 0.5]$ and $p_1^* \in [0, 0.2]$.

In particular, if $q_1^* = k_1 = 0.5$ and $p_1^* = p_2^d(k_1) = 0.2$, then $p_2^* = 0.3$ and $q_2^* = 0.2$ (see Figure 7 below). Calculating the social welfare (the sum of dark gray and light gray areas in Figure 7) and the private firm's profit (the light gray area indicated by π_2), we obtain $\pi_1 = 0.435$ and $\pi_2 = 0.04$. It is easy to check that for this profile the necessary Condition (14) is satisfied.

Figure 7: Equilibrium - Example 5.1 (both firms have positive output)



Clearly, p_1^* and q_1^* can vary within the given ranges. Reducing p_1^* results in lower producer surplus for the public firm, but equally large increase in consumer surplus. Thus, payoffs remain the same. Altering q_1^* replaces the residual demand curve, and results in varying payoffs. The possible payoff intervals can also be calculated for the example: $\pi_1 \in [0.28, 0.435]$ and $\pi_2 \in [0.04, 0.2]$.

The second Nash equilibrium profile mentioned in the statement can also occur with the given demand function, capacities and unit cost. It can be checked easily that $\pi_1(p_2', 0, p_2', q_2') \geq \pi_1(P(k_1), k_1, p_2', q_2')$. In what follows $(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0, 0.6, 0.4)$ is also a Nash-equilibrium profile, where the private firm sells its entire capacity at $P(k_2)$ price and the public firm does not enter the market (i.e. it can choose any price level to its zero output). The payoffs are $\pi_1 = 0.28$ and $\pi_2 = 0.2$.

In case $p^M > P(k_2)$, the private firm acts as a monopolist in the latter equilibrium. By changing the capacities and the unit cost to $k_1 = 0.1$, $k_2 = 0.8$ and $c = 0.1$ the private firm becomes a monopolist and the equilibrium lies at

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0, 0.55, 0.45)$$

for all $p_1 \in [0, 1]$. The payoffs in this case are as follows: $\pi_1 = 0.304$, while $\pi_2 = 0.203$.

5.2.2 Public firm moves first

We continue with the case of public leadership. Here, we have a unique family of pure-strategy subgame-perfect Nash equilibria, where the public firm produces its capacity limit at a price not greater than $p_2^d(k_1)$. The private firm serves residual demand and acts as a monopolist on the residual demand curve, as presented in the following proposition.

Proposition 5.2 (Public firm moves first)**.** *Let Assumptions 5.1-5.3 and* $\overline{p}_2(k_1) > \max\{p^c, c\}$ *be satisfied. Then the set of SPNE prices and quantities are given by*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, k_1, p_2^m(k_1), q_2^m(k_1)) \tag{20}$$

for any $p_1 \leq p_2^d(k_1)$*.*

Proof. First, we determine the best reply $BR_2 = (p_2^*(\cdot,\cdot), q_2^*(\cdot,\cdot))$ of the private firm. Observe that the private firm's best response correspondence can be obtained from the proof of Proposition 5.1. Assuming that $q_1 > 0$,[36] $BR_2(p_1, q_1) =$

$$\begin{cases} \{(p_2^m(q_1), q_2^m(q_1))\} & \text{if } p_1 < p_2^d(q_1); \\ \{(p_2^m(q_1), q_2^m(q_1)), (p_1, \min\{k_2, D(p_1)\})\} & \text{if } p_1 = p_2^d(q_1); \\ \{(p_1, \min\{k_2, D(p_2)\})\} & \text{if } p_2^d(q_1) < p_1 \leq \max\{P(k_2), p^M\}; \\ \{\left(\max\{P(k_2), p^M\}, D\left(\max\{P(k_2), p^M\}\right)\right)\} & \text{if } \max\{P(k_2), p^M\} < p_1. \end{cases}$$

Though there are two possible best replies for the private firm to the public firm's first-period action $\left(p_2^d(q_1), q_1\right)$, in an SPNE the private firm must respond with $(p_2^m(q_1), q_2^m(q_1))$ because otherwise, there will not be an optimal first-period action for the public firm. Hence, the public firm maximizes social welfare in the first period by choosing price $p_1^* = p_2^d(k_1)$ and quantity k_1. □

Example 5.2. We recall only that simultaneous-move outcome from our example, which matches the SPNE emerging in case of public leadership. Let the demand curve take the form of $D(p) = 1 - p$. the capacities and the unit cost are fixed at $k_1 = 0.5$, $k_2 = 0.4$ and $c = 0.1$.

[36] It is easy to see that if $q_1 = 0$, then $BR_2(p_1, 0) = \left(\max\{P(k_2), p^M\}, D\left(\max\{P(k_2), p^M\}\right)\right)$.

Then the actions associated with the only subgame-perfect Nash equilibrium profile are

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, 0.5, 0.3, 0.2)\,.$$

where $p_1^* \in [0, 0.3]$. The social welfare and the private firm's profit are as follows: $\pi_1 = 0.435$ and $\pi_2 = 0.04$.

5.2.3 Private firm moves first

Now we consider the case of private leadership. In this case, there exist two types of subgame-perfect Nash equilibria. In the first equilibrium profile the private firm becomes a monopolist up to its capacity limit, while the public firm remains outside the market. The second equilibrium family is somewhat more complicated: the private firm produces on the original demand curve at the highest price level for which it is still of the public firm's interest to remain on the residual demand curve and produce less than it would produce on the original demand curve. The equilibrium profiles with their necessary conditions are given formally in the following proposition.

Proposition 5.3 (Private firm moves first)**.** *Let Assumptions 5.1-5.3 and $\overline{p}_2(k_1) > \max\{p^c, c\}$ be satisfied. If $\pi_1(p_2', 0, p_2', q_2') \geq \pi_1(P(k_1), k_1, p_2', q_2')$, where $p_2' = \max\{P(k_2), p^M\}$ and $q_2' = D(\max\{P(k_2), p^M\})$, then the equilibrium actions of the firms associated with an SPNE are the following ones*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0, p_2', q_2')\,, \tag{21}$$

where $p_1 \in [0, b]$ can be an arbitrary price. If $\pi_1(p_2', 0, p_2', q_2') < \pi_1(P(k_1), k_1, p_2', q_2')$, then the equilibrium actions of the firms associated with an SPNE are the following ones:

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, D_1^r(\tilde{p}_2, \min\{D(\tilde{p}_2), k_2\}), \tilde{p}_2, \min\{D(\tilde{p}_2), k_2\}) \tag{22}$$

where $p_1 \in [0, \tilde{p}_2]$ and $\tilde{p}_2 =$

$$\sup\{p_2 \mid \pi_1(p_1, D_1^r(p_2, \min\{D(p_2), k_2\}), p_2, \min\{D(p_2), k_2\}) \geq \pi_1(P(k_1), k_1, p_2, \min\{D(p_2), k_2\})\}\,.$$

In addition, if $\pi_1(p_2', 0, p_2', q_2') = \pi_1(P(k_1), k_1, p_2', q_2')$*, then both (21) and (22) are SPNE.*

Proof. If $\pi_1(p_2', 0, p_2', q_2') \geq \pi_1(P(k_1), k_1, p_2', q_2')$, then the private firm becomes a monopolist on the market or sells its entire capacity since this is the best enforceable outcome for the private firm. Considering the other case $\pi_1(p_2', 0, p_2', q_2') < \pi_1(P(k_1), k_1, p_2', q_2')$, just like in the previous sequential case, we determine the reaction function of the second mover. In particular, $BR_1(p_2, q_2) =$

$$\begin{cases} \{(p_1, D_1^r(p_2, q_2) \mid p_1 \leq p_2)\} & \text{if } \pi_1(p_1, D_1^r(p_2, q_2), p_2, q_2) > \pi_1(P(k_1), k_1, p_2, q_2); \\ \{(p_1, k_1) \mid p_1 \leq P(k_1)\} & \text{if } \pi_1(p_1, D_1^r(p_2, q_2), p_2, q_2) < \pi_1(P(k_1), k_1, p_2, q_2); \\ \{(p_1, D_1^r(p_2, q_2) \mid p_1 \leq p_2)\} \cup & \\ \{(p_1, k_1) \mid p_1 \leq P(k_1)\} & \text{if } \pi_1(p_1, D_1^r(p_2, q_2), p_2, q_2) = \pi_1(P(k_1), k_1, p_2, q_2); \end{cases}$$

Concerning the reaction function given by BR_1, the private firm maximizes its profit in the first period by selling its entire capacity k_2 at the highest price p_2, at which it is still not worth for the public firm to sell its entire capacity k_1. □

Example 5.3. Let the demand function again be $D(p) = 1 - p$. To illustrate the case where $\pi_1(p_2', 0, p_2', q_2') \geq \pi_1(P(k_1), k_1, p_2', q_2')$, let $k_1 = 0.5$, $k_2 = 0.4$ and $c = 0.1$.

The following values can be calculated directly from the exogenously given data: $p^c = 0.1$, $p^M = 0.55$, $q^M = 0.45$, $P(k_2) = 0.6$. In what follows, in the first step the private firm will set $p_2^* = P(k_2) = 0.6$ and $q_2^* = k_2 = 0.4$. It can be checked that for these values the public firm has no incentive to enter the market in the second step.

Thus, the actions associated with the SPNE in this case is for all $p_1 \in [0, 1]$:

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0, 0.6, 0.4)$$

The payoffs in this case are as follows: $\pi_1 = 0.28$ and $\pi_2 = 0.2$.

Turning to the case where $\pi_1(p_2', 0, p_2', q_2') < \pi_1(P(k_1), k_1, p_2', q_2')$, we fix the capacities and the unit cost at $k_1 = 0.6$, $k_2 = 0.3$ and $c = 0.1$. With these values it can be checked that the public firm will enter the market. Being aware of this, the private firm sets the highest price level ($\tilde{p_2}$) at which it can still sell its entire capacity so that the public firm has no incentive to undercut the price level set by the private firm. In this case $\tilde{p_2} = 0.487$. The public firm will then satisfy residual demand at $\tilde{p_2}$ price level, i.e. $q_1^* = 0.213$. The public firm can set its price to any level within $[0, 0.487]$. To sum up, the actions associated with the SPNE in this case are as follows:

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.213, 0.487, 0.4),$$

where $p_1 \in 0, 0.487$. The payoffs are $\pi_1 = 0.36$ and $\pi_2 = 0.116$.

5.3 The weak private firm case

The main assumption of the case: $\overline{p}_1(k_2) > p^c \geq \overline{p}_2(k_1) > c$. We begin the analysis with the following lemma which dictates that the private firm does not intend to set any price below the market clearing price.

Lemma 5.3. *Assume that Assumptions 5.1-5.3 and $\overline{p}_1(k_2) > p^c \geq \overline{p}_2(k_1) > c$ hold true. Given any strategy (p_1, q_1) of the public firm, the private firm's strategies (p_2, q_2) with price level $p_2 < p^c$ and any quantity $q_2 > 0$ are strictly dominated, for instance by a strategy with $p_2 = p^c$ and $q_2 > 0$, in all three possible orderings.*

Proof. If $p_2 \leq p^c$, then the private firm can sell its entire capacity, independently from the public firm's strategy. Clearly, given any (p_1, q_1) and $q_2 > 0$, replacing the private firm's price level by $p_2 = p^c$, π_2 increases, thus, the private firm's strategy with the lower price level becomes strictly dominated. □

5.3.1 Simultaneous moves

Here, we have two main types of subgame perfect Nash equilibria. The first type is similar to that of the strong private firm - private leadership case, that is, the private firm sets the highest price level at which it can still produce on the original demand curve. as a particular case of this equilibrium, clearing the market may emerge. The second type contains profiles for which the private firm is a monopolist on the original demand curve.

Proposition 5.4 (Simultaneous moves). *Assume that $\overline{p}_1(k_2) > p^c \geq \overline{p}_2(k_1) > c$ and Assumptions 5.1-5.3 hold. A strategy profile*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = \left(p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\}\right) \tag{23}$$

where $p^* \in [0, \hat{p}]$, defines a Nash equilibrium family in pure strategies if and only if all of the following conditions hold:

$$\max\{p^M, P(k_2)\} \geq \hat{p} \geq p_2^m(q_1^*) \quad ^{37} \tag{24}$$

and[38]

$$\pi_1(p^c, k_1, \hat{p}, \min\{k_2, D(\hat{p})\}) \leq \pi_1(p^*, \min\{k_1, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\})\}, \hat{p}, \min\{k_2, D(\hat{p})\}). \tag{25}$$

In particular, if $\hat{p} = p^c$, then

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, k_1, p^c, k_2) \tag{26}$$

is a Nash equilibrium. Moreover, if $\pi_1(p_2', 0, p_2', q_2') \geq \pi_1(P(k_1), k_1, p_2', q_2')$, where $p_2' = \max\{P(k_2), p^M\}$ and $q_2' = D(\max\{P(k_2), p^M\})$, then for all $p_1 \in [0, b]$

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0, p_2', q_2') \tag{27}$$

are also equilibrium profiles. Finally, no other equilibrium exists in pure strategies.

Proof. Assume that $(p_1^*, q_1^*, p_2^*, q_2^*)$ is an arbitrary equilibrium profile. It can be verified that $q_1^* + q_2^* = D(p')$, where p' stands for the highest price from p_1^*, p_2^* at which at least one firm sells a positive amount. Like in the analysis of the strong private firm case, we divide our analysis into three subcases. In the first case (Case A) we have $p_1^* = p_2^*$, in the second one (Case B) $p_1^* > p_2^*$ holds, while in the remaining case we have $p_1^* < p_2^*$ (Case C).

Case A: By Lemma 5.3 we have $p_1^* = p_2^* \geq p^c$. First, we verify that the strategy profile given by (23) is a Nash-equilibrium profile for any $\hat{p} \geq p^c$ if

[37] Note that $p_2^m(q_1^*) \geq P(q_1^*) \geq p^c$.

[38] We note that from the definition of $p_2^m(\cdot)$, $\max\{p^M, P(k_2)\} = p_2^m(0)$.

(24) and (25) are satisfied. Hence, firms set quantities $q_2^* = \min\{k_2, D(\hat{p})\}$ and $q_1^* = D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\})$. By the second inequality in (24), the private firm has no incentive to increase its price. If $D(\hat{p}) \geq k_2$, then decreasing p_2 is trivially irrational for the private firm that already sells its entire capacity. In case $k_2 > D(\hat{p})$, we obtain a particular equilibrium $(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, 0, \hat{p}, D(\hat{p}))$, which means that the public firm is not present on the market, and therefore, by the first inequality in (24) the private firm has no incentive to decrease its price.

Now we consider the public firm's actions. Clearly, increasing the public firm's price would not increase, but in fact reduce total surplus if $q_1^* > 0$. Moreover, prices $p_1^* = p_2^* = p^c$ with quantities $q_1^* = D_1^r(\hat{p}, \min\{k_2, D(p^c)\}) = k_1$ and $q_2^* = \min\{k_2, D(p^c)\} = k_2$ would result in the best possible outcome for the public firm. Hence, we still have to investigate the effect of a potential price decrease by the public firm in the case of $p_1^* = p_2^* > p^c$. If the public firm reduces its price without increasing its quantity, obviously total surplus cannot increase. To analyze the case in which the public firm decreases its price and increases its quantity at the same time, observe that the sum of consumer surplus and the two firms' revenues (which equals $\pi_1(p_1, q_1, p_2, q_2) + c(q_1 + q_2)$) is only a function of the highest price at which sales are still positive. Therefore, total surplus is strictly decreasing in q_1 on $(q_1^*, D(\hat{p}))$ and strictly increasing in q_1 on $[D(\hat{p}), k_1]$ for a given $p_1 < p_1^*$. To see the latter statement notice that within $[D(\hat{p}), k_1]$ the superfluous production of the private firm remains the same, that is its entire production. Hence, we have shown that the benchmark action of the public firm in order to determine whether it has an incentive to reduce its price is (p^c, k_1), which is in line with Condition (25).

Turning to the case where Condition (24) is violated, we show that (23) cannot be a Nash-equilibrium profile. If $\hat{p} < p_2^m(q_1^*)$ the private firm will increase its price until $p_2^m(q_1)$ to become a monopolist on the residual demand curve, where we are not in Case A of our analysis any more. Note that any $p_1^* \in [0, \hat{p}]$ results in the same outcome, but if $p_1^* \neq p_2^*$, we are again either in Case B or in Case C. If $\max\{p^M, P(k_2)\} < \hat{p}$, the public firm will switch to price $\max\{p^M, P(k_2)\}$.

As a special case of $\hat{p} = p^c$, clearing the market is always a Nash equilibrium for the following reason: by $p^c \geq \overline{p}_2(k_1)$ the private firm cannot be better off by unilaterally increasing its price even by reducing its quantity, accordingly. Note that the market-clearing equilibrium ensures that an equilibrium in pure strategies always exists in the weak private firm case.

Now we show that no other equilibrium exists given that $p_1^* = p_2^* \geq p^c$. Assume that $q_2^* < \min\{k_2, D(p_1^*)\}$. In such cases the private firm gets better off by slightly undercutting p_1^* and selling $q_2^* = \min\{k_2, D(p_1^* - \varepsilon)\}$. Now assume that $q_1^* \neq D_1^r(p_1^*, \min\{k_2, D(p_1^*)\}$. If the left hand side is larger, then there is superfluous production that results in welfare loss; if the left hand side is smaller, then there is a loss in consumer surplus. Thus, there are no more equlibria, if $p_1^* = p_2^*$.

Case B: By Lemma 5.3 $p_1^* > p_2^* \geq p^c$. By decreasing p_1 to $p_1 = p_2^*$, the public firm can always increase social welfare, unless $q_1^* = 0$. In the extreme case of $q_1^* = 0$, p_1^* can obviously be any nonnegative amount. Besides, if $k_2 \geq D(p_2^*)$ and Condition (27) holds, we arrive at the second Nash equilibrium family mentioned in the statement. If $k_2 < D(p_2^*)$, then the public firm can increase social welfare by setting price $p_1 = p_2^*$ and quantity $q_1^* = D(p_2^*) - k_2$.

Case C: Now we have $p_2^* > p_1^*$. As already shown in Case A, this case emerges in equilibrium if $(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\})$, and $p_1^* < \hat{p}$, that is, we have the Nash equilibrium mentioned in the statement. It remains to show that there is no other possible equilibrium in this case. If $p_2^* > p_1^*$, then $p_2^* = p_2^m(q_1^*)$ and $q_2^* = D_2^r(p_2^*, q_1^*) = q_2^m(q_1^*)$ must hold, since otherwise the private firm's payoff would be strictly lower. The arguments for this are analogous to those mentioned in the strong private firm case.[39] From $p^c > \overline{p}_2(k_1)$ we have that $\overline{q}_2(k_1) > k_2$. Thus, due to the fact that $\overline{q}_2(\cdot)$ is increasing[40] in q_1, for any $q_1 < k_1$, $\overline{q}_2(q_1) > \overline{q}_2(k_1) > k_2$. In what follows, q_2^* must equal k_2. It is easy to see that for this case the

[39]In particular, if the private firm sets a price not greater than p_1^*, we are not anymore in Case C; if $q_2^* > D_2^r(p_2^*, q_1^*)$, then the private firm produces a superfluous amount; if $q_2^* < D_2^r(p_2^*, q_1^*)$, then the private firm could still sell more than q_2^*; and if $q_2^* = D_2^r(p_2^*, q_1^*)$, then the private firm will choose a price-quantity pair maximizing profits with respect to its residual demand curve $D_2^r(\cdot, q_1^*)$.

[40]Because $\overline{p}_2(\cdot)$ is a decreasing function in q_1

only possible type of equilibrium is characterized in the statement. □

Example 5.4. Let the demand curve take the form of $D(p) = 1 - p$. The following capacity and unit cost levels lead to the weak private firm case: $k_1 = 0.9$, $k_2 = 0.02$, $c = 0.01$. From these exogenously given values we can determine $p^c = 0.08$ and $\tilde{p_2} = 0.102$. In this case we have several Nash equilibrium profiles, which are not payoff equivalent. For all $\hat{p} \in [0.08, 0.102]$ and any $p_1 \in [0, \hat{p}]$,

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.98 - \hat{p}, \hat{p}, 0.02)$$

defines the family of Nash equilibrium profiles. In particular, if $\hat{p} = p^c$, then

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.9, 0.08, 0.02)$$

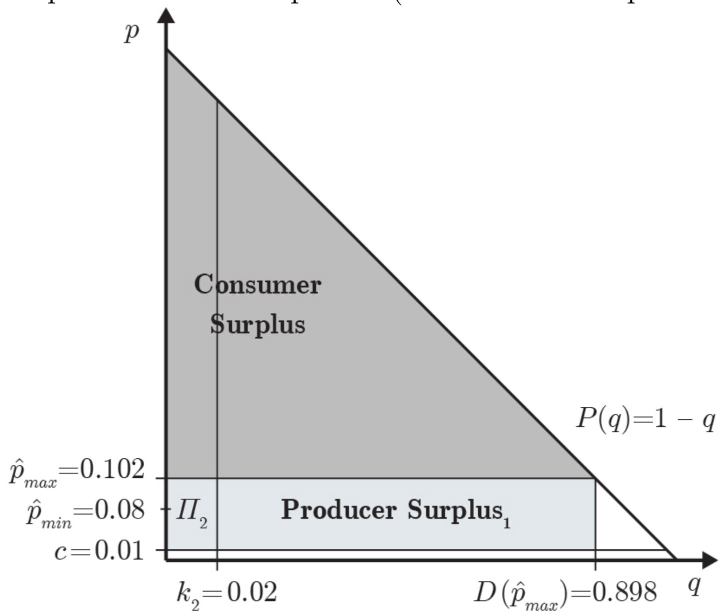
and the social welfare associated to the market clearing equilibrium is $\pi_1 = 0.4876$, while the private firm's profit is $\pi_2 = 0.0014$.

As another equilibrium example, where the firms do not choose the market clearing price, let $\hat{p} = 0.102$ (see Figure 8 below). Then the equilibrium profile is

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.878, 0.102, 0.02),$$

the corresponding payoffs are $\pi_1 = 0.4858$ (the sum of dark and light gray areas) and $\pi_2 = 0.0018$ (the light gray area indicated by π_2).

Figure 8: Equilibrium - Example 5.4 (both firms have positive output)



Clearly, for this equilibrium family $\pi_2(\cdot)$ is increasing in $\hat{p}$, while $\pi_1(\cdot)$ is decreasing in $\hat{p}$. The payoff intervals can also be calculated, in particular, $\pi_1 \in [0.4858, 0.4876]$, $\pi_2 \in [0.0014, 0.0018]$

5.3.2 Public firm moves first

The case of public leadership is somewhat simpler. Namely, the firms clear the market in the only equilibrium family[41]. The results of public leadership are collected in the following proposition.

Proposition 5.5 (Public leadership)**.** *Assume that $\overline{p}_1(k_2) > p^c \geq \overline{p}_2(k_1) > c$ and Assumptions 5.1-5.3 hold. Then the prices and quantities associated with the pure strategy SPNE are*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, k_1, p^c, k_2)$$

where $p^* \in [0, P(k_1)]$.

[41] We speak about family, because the p_1^* can vary within a given range

Proof. We determine the reaction function $BR_2 = (p_2^*(\cdot,\cdot), q_2^*(\cdot,\cdot))$ of the private firm. Like in the strong private firm case, the private firm's best response correspondence can be obtained from the proof of Proposition 5.4, the corresponding simultaneous case.

$$BR_2(p_1, q_1) = \begin{cases} (p_1, \min\{k_2, D_2^r(p_1, q_1)\}) & \text{if } p_2^m(q_1) \leq p_1; \\ (p_2^m(q_1), q_2^m(q_1)) & \text{if } p_2^m(q_1) > p_1. \end{cases} \tag{28}$$

The reaction function dictates that the public firm maximizes social welfare in the first period by choosing any price level $p_1^* \leq p^c$ and quantity k_1. □

Example 5.5. We recall the example outcome from the simultaneous case that matches the actions associated to the only Nash-equilibrium in public leadership. Let the demand curve take the form of $D(p) = 1 - p$. The capacities and the unit cost are fixed at $k_1 = 0.9$, $k_2 = 0.02$ and $c = 0.01$. Then $p^c = 0.08$. We fix the share of the state in the mixed firm at $\alpha = 0.5$. The public firm will sell its entire capacity at a $p_1^* \in [0, p^c]$ market clearing price. The private firm will react with the market clearing price, and will also sell its entire capacity. This ensures the highest possible social welfare in this setting. Thus, for all $p_1 \in [0, 0.08]$ the actions associated with the only SPNE are

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.9, 0.08, 0.02),$$

where the corresponding payoffs are $\pi_1 = 0.4876$ and $\pi_2 = 0.0014$.

5.3.3 Private firm moves first

Finally, we consider the case of private leadership. The only pure-strategy equilibrium family of this case also appears in the simultaneous-moves subcase of the weak private firm case. Namely, the private firm produces on the original demand curve at the highest possible price level for which it is still of the public firm's interest to allow the private firm to do so. The equilibrium family is given formally in the following proposition.

Proposition 5.6 (Private leadership). *Assume that* $\overline{p}_1(k_2) > p^c \geq \overline{p}_2(k_1) > c$ *and Assumptions 5.1-5.3 hold. Then the prices and quantities associated with the pure strategy SPNE are*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\})$$

where $p^* \in [0, \hat{p}]$*, if and only if* $\hat{p} \geq p^c$ *and* $p_2^* = \hat{p}$ *is the highest price level for which*

$$\pi_1(p^c, k_1, \hat{p}, \min\{k_2, D(\hat{p})\}) \leq \pi_1(p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\}) \quad (29)$$

Proof. We determine the reaction function $BR_1 = (p_1^*(\cdot,\cdot), q_1^*(\cdot,\cdot))$ of the public firm. The public firm's best response correspondence can also be obtained from the proof of Proposition 5.4, the corresponding simultaneous-move case.

$$BR_1(p_2, q_2) = \begin{cases} (p^*, k_1) & \text{if Condition (29) does not hold;} \\ (p_2, D_1^r(p_2, q_2)) & \text{if Condition (29) holds.} \end{cases} \quad (30)$$

where $p^* \in [0, \hat{p}]$.

The reaction function prescribes that the private firm maximizes its profit in the first period by choosing the highest possible price level, where the public firm is still better off (i.e. the social surplus is higher) by reacting with the same price and serving residual demand, than by undercutting p_2.[42] A highest price level $\hat{p}$ exists for every demand function, because if both firms choose price level p^c and sell their entire capacities (i.e. they clear the market), then Condition (29) always holds. □

Example 5.6. We recall an example outcome from the simultaneous-move case. Let the demand curve take the form of $D(p) = 1 - p$. the capacities and the unit cost are fixed at $k_1 = 0.9$, $k_2 = 0.02$ and $c = 0.01$. Then $\tilde{p_2} = 0.102$. The private firm will choose $p_2^* = \tilde{p_2}$ and sells its entire capacity. The public firm will serve residual demand as it is not worth undercutting the private firm's price which would cause superfluous production. Thus, for all $p_1 \in [0, 0.102]$ the actions associated with the only SPNE are

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, 0.878, 0.102, 0.02),$$

[42]Depending on the parameters, it can also occur that the public firm has zero output on the residual demand curve.

where the corresponding payoffs are $\pi_1 = 0.4858$ and $\pi_2 = 0.0018$.

5.4 The high unit cost case

The main assumption of this case is $c \geq P(k_1)$. For the proof of this case being disjoint from the previous two cases, we refer the reader to Section 5.1.

5.4.1 Simultaneous moves

In this subcase we have three types of pure-strategy Nash equilibria. The first type consists of profiles where the private firm produces on the original demand curve at the highest possible price level. In the second type, the public firm produces at its capacity limit, while the private firm does not enter the market. Finally, in the third type, the private firm acts as a monopolist on the residual demand curve.

Proposition 5.7 (Simultaneous moves). *Assume that $c \geq P(k_1)$ and Assumptions 5.1-5.3 hold. A strategy profile NE_1*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\})$$

where $p^* \in [0, \hat{p}]$, defines a Nash equilibrium family in pure strategies if and only if all of the following conditions hold:

$$\hat{p} > c \tag{31}$$

$$\max\{p^M, P(k_2)\} \geq \hat{p} \tag{32}$$

$$\hat{p} \geq p_2^m(q_1^*) \quad [43] \tag{33}$$

$$\pi_1(0, k_1, \hat{p}, \min\{k_2, D(\hat{p})\}) \leq \pi_1(p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\}) \tag{34}$$

A strategy profile NE_2

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, k_1, p_2^*, 0)$$

[43] Such a $\hat{p}$ exists because $\max\{p^M, P(k_2)\} = p_2^m(0) > c$, and thus, there exist an r for which $p_2^m(r) = c$. In particular, if $q_1^* \in [0, r]$, Conditions 32 and 33 are satisfied.

where $p^* \in [0, P(k_1)]$, *and* p_2^* *can be any nonnegative amount, also defines a Nash equilibrium family. Moreover, a strategy profile* NE_3

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*))$$

is for a quantity $q_1^* \in [0, k_1]$ *and for any price* $p_1^* \in \left[0, p_2^d(q_1^*)\right]$ *a Nash-equilibrium in pure strategies if and only if all of the following conditions hold:*

$$P(q_1^*) > c \tag{35}$$

$$\pi_1(0, k_1, p_2^m(q_1^*), q_2^m(q_1^*)) \leq \pi_1(p_1^*, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*)) \tag{36}$$

Finally, no other equilibrium exists in pure strategies.

Proof. Assume that $(p_1^*, q_1^*, p_2^*, q_2^*)$ is an arbitrary equilibrium profile. We divide our analysis again into two subcases. In the first case (Case A) we have $p_1^* \geq p_2^*$, while in the second case we have $p_1^* < p_2^*$ (Case C). Separated from the two cases, we show that no other equilibrium profiles exist in pure strategies.

Case A:

Trivially, the private firm will not set its price below the unit cost, unless its output is zero. For $q_2^* = 0$ the only possible equilibrium is characterized in NE_2. For any other case, $p_1^* \geq p_2^* \geq c$ holds in Case A. Together with quantities $q_2^* = \min\{k_2, D(\hat{p})\}$ and $q_1^* = D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\})$ this is a Nash-equilibrium profile for any $\hat{p} \geq c$, if Conditions (31) - (34) hold. By Condition (33), the private firm has no incentive to increase its price. If $D(\hat{p}) \geq k_2$, then decreasing p_2 is trivially irrational for the private firm that sells its entire capacity in this setting. In case $k_2 > D(\hat{p})$, we obtain a particular equilibrium $(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, 0, \hat{p}, D(\hat{p}))$, which means that the public firm is not present on the market. As far as the public firm's actions are concerned, if and only if condition (34) holds, then by reducing p_1 or increasing q_1 the public firm causes less gain in social welfare, than the loss caused by the superfluous production of the private firm. Turning to the case where condition (33) does not hold, the private firm will increase its price until $p_2^m(q_1)$ to become a monopolist on the residual demand curve, where we are not in Case A in our analysis any more. Note that any $p_1^* \in [0, \hat{p}]$ results in the same outcome, but if $p_1^* < p_2^*$, we are again in Case B. Turning to the second equilibrium family, we remain in Case A if $p_1^* \geq p_2^*$ and $(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, k_1, p_2^*, 0)$. It is easy to see that the private firm has no incentive to enter the market and the public firm has no incentive to reduce its production. As far as the third equilibrium type is concerned, if $(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*))$, we cannot have $p_1^* > p_2^*$. If $p_1^* = p_2^m(q_1^*) > c$, then it is of the private firm's interest to undercut the public firm's price. If $p_1^* < p_2^m(q_1^*)$, we are in Case B.

Case B:

Now we have $p_2^* > p_1^*$. This case may emerge in equilibrium if $(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\})$, that is, we have the first Nash equilibrium family (NE_1) mentioned in the statement, and $p_1^* < \hat{p}$. Besides, if $p_1 \in [0, P(k_1)]$ and $q_1 = k_1$, together with $p_2 > p_1$ and $q_2 = 0$ we also obtain a Nash equilibrium profile (NE_2), where the private firm is not present on the market. Here, the private firm cannot realize a positive profit by entering the market. Turning to the case where the private firm is a profit-maximizer on the residual demand curve, if Conditions (35) - (36) are satisfied, and $p_1^* < p_2^m(q_1^*)$, then $(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1^*, q_1^*, p_2^m(q_1^*), q_2^m(q_1^*))$ is an equilibrium profile (NE_3), where $p_1^* < p_2^*$.

Now we show that no other equilibrium exists in pure strategies. Firstly, given $p_1^* = p_2^*$, assume that $q_2^* < \min\{k_2, D(\hat{p})\}$. In such cases the private firm gets better off by slightly undercutting p_1 and selling $q_2^* = \min\{k_2, D(\hat{p})\}$. Now assume that $q_1^* \neq D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}$. If the left hand side is larger, then there is superfluous production that results in welfare loss; if the left hand side is smaller, then there is a loss in consumer surplus. Thus, there are no more equilibria, if $p_1^* = p_2^*$. Given $p_1^* > p_2^*$, the public firm can increase social welfare by decreasing its price, unless the public firm has zero output, which is a particular case of the equilibrium family NE_1. It remains to show that there is no other possible equilibrium in case $p_2^* > p_1^*$. Here, we may arrive at three possible equilibria mentioned in the statement. Provided that the private firm's output is zero ($q_2^* = 0$), we can only arrive at Nash equilibrium family NE_2, since in this case $q_1^* = k_1$ must hold, otherwise the private firm could raise its profit by entering the market. If $q_2^* \neq 0$, then the private firm produces either on the residual demand curve at a residual profit-maximizing price (i.e. $p_2^* = p_2^m(q_1^*)$, that is equilibrium family NE_3),[44], or the private firm determines the highest price level at which it still remains on the original demand curve[45] (that is, $p_2^* = \hat{p}$). Here, we can only have an NE_1 type equilibrium, since the private firm - producing on the original demand curve - will not produce less than $\min\{k_2, D(\hat{p})\}$. Thus, there are no more equilibria given $p_2^* > p_1^*$. □

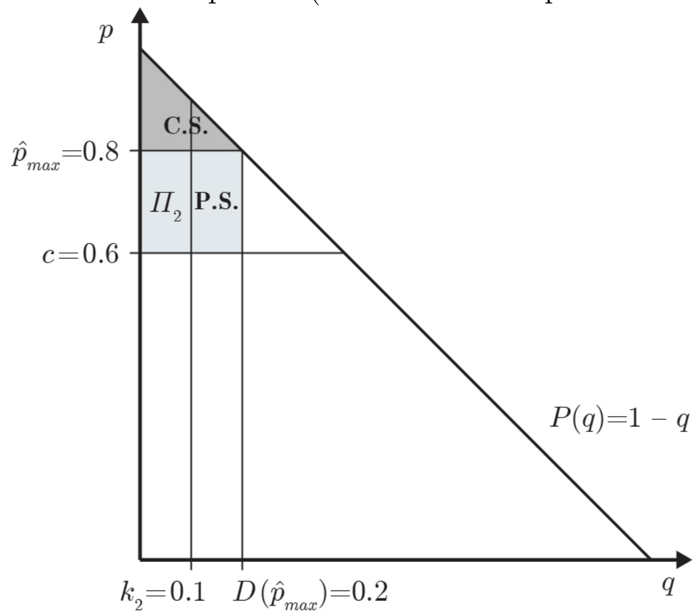
Example 5.7. Let the demand curve take the form of $D(p) = 1 - p$. The following capacity and unit cost levels lead to the high unit cost case: $k_1 = 0.5$, $k_2 = 0.1$, $c = 0.6$.

We give examples to the equilibria in the order they are listed in the statement. Firstly, from these exogenously given values we can calculate the interval where $\hat{p}$ can be taken from, leading to Nash equilibria which are not payoff equivalent: $\hat{p} \in [0.6, 0.8]$. We can choose $\hat{p} = 0.8$ (see Figure 9). This leads to the following values: $p_1^* \in [0, 0.8]$; $q_1^* = 0.1$; $p_2^* = 0.8$; $q_2^* = 0.1$. In this case $\pi_1 = 0.06$ (sum of dark and light gray areas); $\pi_2 = 0.04$ (light gray area indicated by π_2).

[44]Given $p_2^* = p_2^m(q_1^*)$ and $q_2^* = D_2^r(p_2^*, q_1^*) = q_2^m(q_1^*)$ must hold, since otherwise the private firm's payoff would be strictly lower. In particular, if the private firm sets a price not greater than p_1^*, we are not anymore in Case B; if $q_2^* > D_2^r(p_2^*, q_1^*)$, then the private firm produces a superfluous amount; if $q_2^* < D_2^r(p_2^*, q_1^*)$, then the private firm could still sell more than q_2^*; and if $q_2^* = D_2^r(p_2^*, q_1^*)$, then the private firm will chooses a price-quantity pair maximizing profits with respect to its residual demand curve $D_2^r(\cdot, q_1^*)$.

[45]i.e. the public firm has no interest in generating superfluous production at $\hat{p}$ price level

Figure 9: Equilibrium - Example 5.7 (both firms have positive output - case 1)



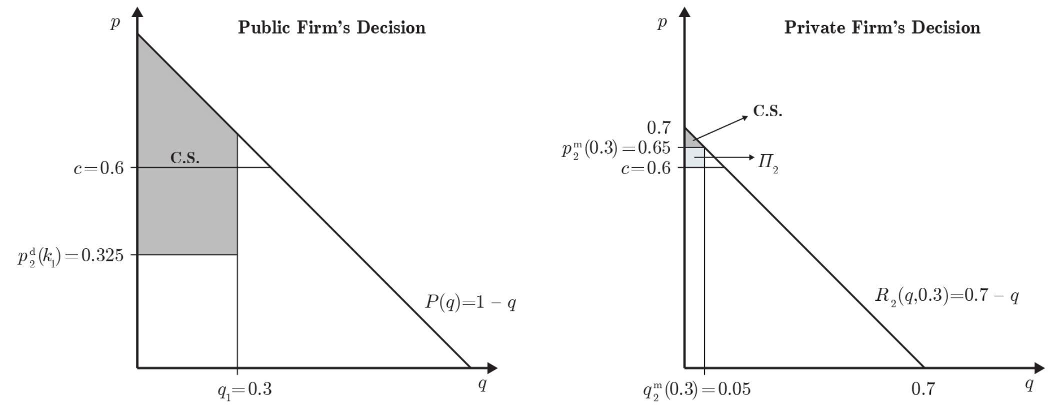
Depending on $\hat{p}$, profit levels can vary in the following intervals: $\pi_1 \in [0.06, 0.08]$ and $\pi_2 \in [0, 0.04]$.

Turning to the second equilibrium type, where the private firm is not present on the market, we obtain $p_1^* \in [0, 0.5]$; $q_1^* = 0.5$; $p_2^* \in \mathbb{R}$; $q_2^* = 0$. Profit levels are as follows: $\pi_1 = 0.08$; $\pi_2 = 0$.

Finally, for the illustration of the third equilibrium we have that any $q_1 \in [0, k_1]$ leads to a NEP. Let us fix $q_1 = 0.3$. Now $p_2^m(0.3) = 0.65$ an $p_2^d(0.3) = 0.325$. Thus, $p_1^* \in [0, 0.325]$; $q_1^* = 0.3$; $p_2^* = 0.65$; $q_2^* = 0.05$. In this case, $\pi_1 = 0.0787$ and $\pi_2 = 0.0013$. Depending on q_1, profit levels can vary in the following intervals: $\pi_1 \in [0.06, 0.08]$ and $\pi_2 \in [0, 0.04]$ (see Figure 10). [46]

[46] We note that here $p_1^* < c$, still, it is of the public firms interest to produce a positive amount, as this action leads to a positive change in consumer surplus. This is the reason why there is no producer surplus indicated on the left-hand-side of Figure 10.

Figure 10: Equilibrium - Example 5.7 (both firms have positive output - case 2)



5.4.2 Public firm moves first

In the high unit cost case with public leadership we obtain that the private firm does not enter the market, while the public firm's output equals its capacity. This result is formalized in the following proposition.

Proposition 5.8 (Public leadership). *Assume that $c > P(k_1)$ and Assumptions 5.1-5.3 hold. Then the prices and quantities associated with the pure strategy SPNE are*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, k_1, p_2^*, 0)$$

where $p^* \in [0, P(k_1)]$ and p_2^* can be any nonnegative amount.

Proof. We determine the reaction function $BR_2 = (p_2^*(\cdot,\cdot), q_2^*(\cdot,\cdot))$ of the private firm. The private firm's best response correspondence can be obtained from the proof of Proposition 5.7, the corresponding simultaneous case.

$$BR_2(p_1, q_1) = \begin{cases} (p, 0) & \text{if } q_1 = k_1; \\ (p_1, \min\{k_2, D_2^r(p_1, q_1)\}) & \text{if } q_1 < k_1 \text{ and } p_2^m(q_1) \leq p_1. \\ (p_2^m(q_1), q_2^m(q_1)) & \text{if } q_1 < k_1 \text{ and } p_2^m(q_1) > p_1. \end{cases} \quad (37)$$

The reaction function dictates that the public firm maximizes social welfare in the first period by choosing any price level $p^* \in [0, P(k_1)]$ and quantity k_1. □

Example 5.8. Let the demand curve take the form of $D(p) = 1 - p$. We recall an outcome from the simultaneous case with the exogenous values $k_1 = 0.5$; $k_2 = 0.1$; $c = 0.6$. The private firm is not present on the market, we obtain $p_1^* \in [0, 0.5]$; $q_1^* = 0.5$; $p_2^* \in \mathbb{R}$; $q_2^* = 0$. Profit levels are as follows: $\pi_1 = 0.08$; $\pi_2 = 0$.

5.4.3 Private firm moves first

Finally, we consider the case of private leadership. Here, in equilibrium, the private firm chooses the highest price level at which it can still produce on the original demand curve. However, it can occur that no such price level exists. In the latter case, the private firm does not enter the market and the public firm produces its capacity.

Proposition 5.9 (Private leadership)**.** *Assume that $c \geq P(k_1)$ and Assumptions 5.1-5.3 hold. Then the prices and quantities associated with the pure strategy SPNE are*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, D_1^r(\hat{p}, \min\{k_2, D(\hat{p})\}), \hat{p}, \min\{k_2, D(\hat{p})\})$$

where $p^* \in [0, \hat{p}]$, if and only if $\max\{p^M; P(k_2)\} > c$ *and $p_2^* = \hat{p}$ is the highest price level, where*

$$\pi_1(0, k_1, p_2, \min\{k_2, D(p_2)\}) \leq \pi_1(p^*, D_1^r(p_2, \min\{k_2, D(p_2)\}), p_2, \min\{k_2, D(p_2)\}) \tag{38}$$

In the case no $\hat{p}$ price level exists that satisfies condition (38), then the prices and quantities associated with the pure strategy SPNE are

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^*, k_1, p_2^*, 0)$$

where $p^* \in [0, P(k_1)]$ and p_2^* can be any nonnegative amount.

Proof. We determine the reaction function $BR_1 = (p_1^*(\cdot,\cdot), q_1^*(\cdot,\cdot))$ of the public firm. The public firm's best response correspondence can also be obtained from the proof of Proposition 5.7, the corresponding simultaneous case.

$$BR_1(p_2, q_2) = \begin{cases} (p^*, k_1) & \text{if Condition (38) does not hold;} \\ (p_2, D_1^r(p_2, q_2)) & \text{if Condition (38) holds.} \end{cases} \tag{39}$$

where $p^* \in [0, \hat{p}]$.

The reaction function dictates that the private firm maximizes its profit in the first period by choosing the highest possible price level, where the public firm is still better off (i.e. the social welfare is higher) by reacting with the same or lower price and serving residual demand, than by undercutting p_2 and selling its entire capacity. A highest price level $\hat{p}$, however, may not exist for every demand function and cost level. In case no such $\hat{p}$ exists, the private firm does not enter the market, because otherwise the public firm would undercut the private firm's price, resulting in a negative private profit. □

Example 5.9. Let the demand curve remain $D(p) = 1 - p$, while the capacities and the unit costs are fixed again at $k_1 = 0.5$, $k_2 = 0.1$, $c = 0.6$.

It can be calculated that $\hat{p} = 0.8$. This leads us to the following values: $p_1^* \in [0, 0.8]$; $q_1^* = 0.1$; $p_2^* = 0.8$; $q_2^* = 0.1$. In this case $\pi_1 = 0.06$; $\pi_2 = 0.04$.

5.5 Solution of the timing game

When the ordering of moves is not exogenously given, we arrive at the timing game. The payoffs of the timing game are the equilibrium payoffs of the firms in each ordering of moves. The equilibrium of the timing game for all the three main cases can be derived from Propositions 5.1-5.9, by comparing the payoffs of both firms for different orderings of moves.

Before we give the solution of the timing game, we provide a summary of the payoffs that were calculated in the numerical examples after Propositions 5.1-5.9, respectively. Table 6 provides numerical evidence of the solution of the timing game for the particular demand function $D(p) = 1 - p$, with exogenously given capacities and cost levels.

Table 6: Example payoff levels for the demand function $D(p) = 1 - p$

Cases	**Strong private firm**	**Weak private firm**	**High unit cost**
$\mathbf{k_1}$	0.5	0.9	0.5
$\mathbf{k_2}$	0.4	0.02	0.1
$\mathbf{c}$	0.1	0.01	0.6
π_1**: Public firm's equilibrium payoff (social welfare)**			
sim. moves	$\in [0.28, 0.435]$	$\in [0.4858, 0.4876]$	$\in [0.06, 0.08]$
as leader	0.435	0.4876	0.08
as follower	0.28	0.4858	0.06
π_2**: Private firm's equilibrium payoff (profit)**			
sim. moves	$\in [0.04, 0.2]$	$\in [0.0014, 0.0018]$	$\in [0, 0.04]$
as leader	0.2	0.0018	0.04
as follower	0.04	0.0014	0

It is easy to see from Table 6 that in all the three main cases any firm has the highest payoff with certainty in the case when it is the first mover. Thus, as every firm wants to become the leader and there cannot be two leaders at the same time, the outcome of the timing game is simultaneous moves. The equilibrium of the timing game for any concave, twice continuously differentiable demand function is precisely stated in the following proposition.

Proposition 5.10. *Assume that Assumptions 5.1-5.3 hold. For any cost and capacity levels, the equilibrium of the timing game lies at simultaneous moves.*

Proof. The result comes directly from Propositions 5.1-5.9. □

5.6 Corollaries and concluding remarks of the section

Our main results are collected in the following corollaries. We focus on the differences between the production-to-order case - which was investigated in earlier work

- and the production-in-advance case from the point of view of equilibrium strategies, social welfare effects and equilibrium analysis of the timing game. The first corollary determines the endogenous order of moves in a two-period timing game of the production-in-advance framework, where both firms can choose between two periods for setting their prices and quantities.

Corollary 5.1. *In the production-in-advance framework both firms want to move first, therefore the equilibrium of the timing game lies at simultaneous moves.*

We turn to the problem of the public firm's influence on social welfare. One can make a comparison with the results for the production-to-order case presented in Balogh and Tasnádi [2012]. In the PIA case the social welfare becomes lower - no matter what pure-strategy Nash equilibria are played - than that of the PTO case. This result is stated in the next corollary.

Corollary 5.2. *When playing the production-in-advance type of the Bertrand-Edgeworth game, the equilibrium strategies lead to a decrease in social welfare compared to the PTO case.*

The third main result of the section is implicitly given: independently from the parameters and the orderings of firms' decisions, the production-in-advance type mixed Bertrand-Edgeworth duopoly always has at least one pure-strategy Nash equilibrium. This result remained the same as that in the mixed PTO case. However, we emphasize that in the case of standard Bertrand-Edgeworth duopolies, there is a lack of pure-strategy equilibria (see e.g. Deneckere and Kovenock [1992]). We state the existence of a pure-strategy equilibrium in the third corollary.

Corollary 5.3. *We have at least one pure-strategy (subgame-perfect) Nash equilibrium in all three analyzed cases and for all three orderings of moves.*

We note that it follows directly from the proofs of the present section that the pure strategy equilibria are not interchangeable, thus, theoretically it is possible that the firms do not navigate to an equilibrium point even if they try to do so.

The results concerning pure-strategy Nash equilibria and the timing game are summarized in the following Table 7.

Table 7: Comparison of the PTO and PIA cases

	Production-to-order	*Production-in-advance*
Equilibrium in pure strategies	Yes	Yes
Timing game equlibrium	All possible orderings	Simultaneous moves

As far as the public firm's social welfare effect is concerned, one can make a direct comparison either between the standard and mixed production-to-order cases, or between the mixed PTO and PIA cases. in both comparisons, it turns out that the mixed PTO case results in higher social welfare in equilibrium. However, unfortunately, we cannot make a direct comparison between the standard and mixed PIA models, as the mixed equilibrium of the standard model has not yet been characterized and is out of our scope.

The results suggest that it really matters whether a public firm has some influence in an oligopoly market. Further research directions may include the application of our model to markets with asymmetric information, partial public ownership, and oligopolies with more than two firms. One can notice that our assumptions are quite general in the present section, as well as throughout the dissertation. However, in order to obtain plausible results in the mentioned topics, more strict assumptions may be needed.

6 Semi-mixed Bertrand-Edgeworth duopolies

This section aims at investigating two further models within the field of mixed Bertrand-Edgeworth duopolies. While in Sections 4 and 5 the share of the state in the public firm was fixed at 100 % - that is, the public firm had pure public ownership -, in this section we will allow for mixed public and private ownership within one firm.

Such oligopolies, where one firm has partial public ownership emerge frequently on the producer side of real-life markets. We provided several examples in the introductory section of the dissertation.

Now we focus on the equilibrium analysis of the semi-mixed Bertrand-Edgeworth duopoly. We will assume that one of the firms is purely private, i.e. it is a profit-maximizer, while in the other one we allow for a certain share of the state. This will modify the mixed-ownership firm's payoff function compared to that of Sections 4 and 5. We aim at finding pure-strategy Nash-equilibrium profiles, solving the timing game, and analyzing the mixed-ownership[47] firm's impact on social welfare compared to the cases of Sections 4 and 5.

We will show later on that for these models the pure-strategy equilibrium analysis becomes simpler, however, the existence of pure-strategy (subgame-perfect) Nash equilibria cannot be guaranteed.

Both the production-to-order (PTO) case and the production-in-advance (PIA) framework are discussed in this section, within separate subsections (6.2 and 6.3, respectively). For the PTO case we wish to emphasize that the results to be presented in detail were introduced in Tasnádi [2013]. However, as the referred contribution fits directly into the scope of the present thesis, we will present its results in detail. As far as the PIA case is concerned, the results are published in Balogh [2014].

As the model assumptions remain the same in Sections 6.2 and 6.3 - apart from the difference that lies in the PTO and PIA frameworks -, we present the assump-

[47]From now on, we will refer to the mixed-ownership firm as mixed firm or firm 1, while the purely private firm remains firm 2.

tions in Section 6.1, separately from the PTO and PIA cases. The assumptions to be presented can be simplified for the PTO case, we will indicate the possible simplifications in Section 6.2.

6.1 Model specification

The demand is given by function D on which we impose the same restrictions as in the previous sections:

Assumption 6.1. The demand function D intersects the horizontal axis at quantity a (where $a > 0$) and the vertical axis at price b. D is strictly decreasing, concave and twice continuously differentiable on $(0, b)$; moreover, D is right-continuous at 0, left-continuous at b and $D(p) = 0$ for all $p \geq b$.

We denote by P the inverse demand function. Thus, $P(q) = D^{-1}(q)$ for $0 < q \leq a$, $P(0) = b$, and $P(q) = 0$ for $q > a$.

On the producer side we now have a mixed firm and a private firm. As mentioned earlier, we label the public firm with 1 and the private firm with 2. Just like before, we will also label the two firms by i and j, where $i, j \in \{1, 2\}$ and $i \neq j$. Let us denote the public share in the mixed firm by α. We will focus on the cases where $\alpha \in (0; 1)$. Trivially, if $\alpha = 1$, then we arrive at a pure public firm, while for $\alpha = 0$ we have a standard duopoly.

Our usual assumptions imposed on the firms' cost functions are as follows:

Assumption 6.2. The two firms have identical $c \in (0, b)$ unit costs up to the positive capacity constraints k_1, k_2 respectively.

We shall again denote by p^c the market clearing price and by p^M the price set by a monopolist without capacity constraints, i.e. $p^c = P(k_1 + k_2)$ and $p^M = \arg\max_{p \in [0,b]} (p - c) D(p)$. In what follows $p_1, p_2 \in [0, b]$ and $q_1, q_2 \in [0, a]$ stand for the prices and quantities set by the firms.

For any firm i and for any quantity q_j set by its opponent j we shall denote by $\overline{p}_i(q_j)$ the profit maximizing price on the firms' residual demand curves $D_i^r(p, q_j) =$

$(D(p)-q_j)^+$, i.e. $\overline{p}_i(q_j) = \arg\max_{p\in[0,b]}(p-c)D_i^r(p,q_j)$, where in the definition of $\overline{p}_i(q_j)$ we do not include the capacity constraint of firm i. For notational convenience we also introduce $p_i^m(q_j)$ as the price level that maximizes profits with respect to the residual demand curve subject to the capacity constraint of firm i (i.e. the high-price firm produces the quantity given by its residual demand curve). Clearly, p^c, $\overline{p}_i$ and p_i^m are well defined whenever $c < P(q_j)$ and Assumptions 6.1-6.2 are satisfied. If $c \geq P(q_j)$, then $\overline{p}_i(q_j)$ and $p_i^m(q_j)$ are not unique, as any price $p_i > P(q_j)$ together with quantity $q_i = 0$ results in $\pi_i = 0$ and π_i cannot be positive. In other words, any price $p_i > P(q_j)$ is profit-maximizing on the residual demand curve. For notational convenience we define $\overline{p}_i(q_j)$ and $p_i^m(q_j)$ by b in case of $c \geq P(q_j)$.

For a given quantity q_j we shall denote the inverse residual demand curve of firm i by $R_i(\cdot,q_j)$. It can be checked that $R_i(q_i,q_j) = P(q_i+q_j)$ and $p_i^m(q_j) = \max\{\overline{p}_i(q_j), R_i(k_i,q_j)\}$. Let $\overline{q}_i(q_j) = \arg\max_{q_i\in[0,a]}\left(R_i\left(q_i,q_j\right)-c\right)q_i$ and $q_i^m(q_j) = \min\{\overline{q}_i(q_j), k_i\}$. Clearly, $\overline{q}_i(q_j) = D_i^r\left(\overline{p}_i(q_j),q_j\right)$ and $q_i^m(q_j) = D_i^r\left(p_i^m(q_j),q_j\right)$.

Let us denote by $p_i^d(q_j)$ the smallest price for which

$$(p_i^d(q_j)-c)\min\left\{k_i, D\left(p_i^d(q_j)\right)\right\} = (p_i^m(q_j)-c)q_i^m(q_j),$$

whenever this equation has a solution. Provided that the private firm has 'sufficient' capacity, that is $\max\{p^c,c\} < p_2^m(k_1)$, then if it is a profit-maximizer, it is indifferent whether serving residual demand at price level $p_2^m(q_1)$ or selling $\min\{k_2, D\left(p_2^d(q_1)\right)\}$ at the lower price level $p_2^d(q_1)$. Note that if $R_i(k_i,q_j) \geq \overline{p}_i(q_j)$, then $p_i^d(q_j) = p_i^m(q_j)$. We have presented in the previous sections that $p_i^d(\cdot)$ is strictly decreasing over the region it is defined on and $\overline{p}_i(\cdot)$, $\overline{q}_i(\cdot)$, and $p_i^m(\cdot)$ are also strictly decreasing over the region in which the profit maximization problem with respect to the residual demand curve has an interior solution (i.e. not lying on one of the axes). Moreover, $q_i^m(\cdot)$ is strictly decreasing on the region where $q_i^m(q_j) < k_i$, otherwise constant.

We assume again efficient rationing on the market, and thus, the firms' demands equal

$$\Delta_i\left(D,p_1,q_1,p_2,q_2\right)=\begin{cases} D\left(p_i\right) & \text{if} \quad p_i<p_j, \\ T_i(p,q_1,q_2), & \text{if} \quad p=p_i=p_j \\ \left(D\left(p_i\right)-q_j\right)^+ & \text{if} \quad p_i>p_j, \end{cases}$$

for all $i \in \{1,2\}$, where T_i stands for a tie-breaking rule. We will consider two sequential-move games (one with the public firm as the first mover and one with the private firm as the first-mover) and a simultaneous-move game.

Assumption 6.3. If the two firms set the same price, then we assume for the sequential-move games that the demand is allocated first to the second mover and for the simultaneous-move game that the demand is allocated in proportion of the firms' capacities.

Now we specify the firms' payoff functions. This is the point where the main difference lies between mixed and semi-mixed models. We will assume that the mixed firm's profit function consists of a weighted sum of total surplus and own profit, where the weights are given by the share of the state in the firm.

The private firm remains a profit-maximizer, and therefore,

$$\pi_2(p_1,q_1,p_2,q_2)=p_2\min\left\{k_2,\Delta_2\left(D,p_1,q_1,p_2,q_2\right)\right\}-cq_2. \tag{40}$$

The mixed firm aims at maximizing a weighted sum of total surplus and its own profit, that is,

$$\begin{aligned}\pi_1(p_1,q_1,p_2,q_2)=&(1-\alpha)p_1\min\{k_1,\Delta_1(D,p_1,k_1,p_2,k_2)\}+\\ &+\alpha\int_0^{\min\left\{(D(p_j)-q_i)^+,q_j\right\}}R_j(q,q_i)dq+\\ &+\alpha\int_0^{\min\{a,q_i\}}P(q)dq-\alpha c(q_1+q_2),\end{aligned} \tag{41}$$

where $0\leq p_i\leq p_j\leq b$.

Finally, we introduce the notion $p_1^s(q_2)$ as the payoff maximizing price of the mixed firm given it serves residual demand[48], where the residual demand curve is

[48]We note that $p_1^s(q_2)\neq p_1^m(q_2)$, since $p_1^m(q_2)$ is a pure producer-surplus-maximizing price on the residual demand curve.

nothing else but the original demand curve shifted to the left-hand-side by q_2 units. Formally,

$$p_1^s(q_2) = \arg\max_{p\in[0,b]} (1-\alpha)p_1 D_1^r(p_1) + \alpha \int_0^{D(p_1)} P(q)dq. \tag{42}$$

Now we turn to presenting the results of the PTO case, where the above model assumptions can be simplified slightly.

6.2 Production-to-order framework

As we have already pointed out in Section 4, the game reduces to a price-setting game in the PTO case. Therefore some of the definitions presented in Section 6.1 can be simplified: $p_i^m(\cdot)$, $p_i^d(\cdot)$ and $p_i^s(\cdot)$ have only one single value, because the firm producing at the lower price level automatically sells its entire capacity.[49] Therefore $p_i^m(\cdot)$, $p_i^d(\cdot)$ and $p_i^s(\cdot)$ can be replaced by $p_i^m(k_j)$, $p_i^d(k_j)$ and $p_i^s(k_j)$, or simply p_i^m, p_i^d and p_i^s.[50]

6.2.1 Characterization of pure-strategy equilibria

In this subsection we present the results of Tasnádi [2013]. We specify the tie-breaking rule: if the firms set the same price, they share the demand in proportion of their capacities, that is

$$\Delta_i\left(D, p_1, q_1, p_2, q_2\right) = \frac{k_i}{k_1+k_2} D\left(p\right) \text{ if } p = p_i = p_j \tag{43}$$

Another important feature of the PTO case is that the unit cost can be normalized to 0, because there are no unsold items. We now state the proposition concerning the existence and the characterization of pure-strategy Nash-equilibria. It is established that a pure-strategy Nash equilibrium exists only if both residual payoff-maximizing price levels are lower than the market clearing price - independently from the orderings of moves.

[49] Except for the extreme case where $k_j > b$, which would mean that the lower-price firm is in fact not restricted by its capacity.

[50] We note that the same distinction can be noticed between Sections 4 and 5.

Proposition 6.1 (Tasnádi, 2013). *Under Assumptions 6.1-6.3, the necessary and sufficient condition for the existence of a pure-strategy Nash equilibrium in the production-to-order type semi-mixed Bertrand-Edgeworth duopoly is that* $\max\{p_1^s; p_2^m\} \leq p^c$. *Provided that this condition is satisfied, the only pure Nash equilibrium is*

$$p_1^* = p_2^* = p^c. \tag{44}$$

Proof. First, we show that if a pure equilibrium exists, it cannot be anything else, but (44). Assume first that $p_1^* < p_2^*$. We consider first the $D(p_1^*) > k_1$ case. If $D_2^r(p_2^*) > 0$, then the mixed firm could raise its producer surplus without altering social welfare, that is, it could increase its payoff. If $D_2^r(p_2^*) \leq 0$, then the private firm could increase its payoff by setting a lower price level. Now let us consider the $D(p_1^*) \leq k_1$ case. Then the private firm could realize a positive profit by altering its price to the lower p_1^*, because $D(0) = b > k_1$. Therefore, there is no equilibrium satisfying $p_1^* < p_2^*$.

We turn to the case of $p_1^* > p_2^*$. First, we note that $p_2^* \neq 0$, because the private firm has an incentive to set a positive price level. If $D(p_2^*) > k_2$, then the private firm could sell its entire capacity at a higher price level, resulting in a higher profit. If $D(p_2^*) \leq k_2$, then it is of the mixed firm's interest to decrease its price below p_2^*, which would result in a higher producer surplus, but no change in social welfare.

Observing the case of $p_1^* = p_2^*$, if $p_1^* = p_2^* > p^c$, then both firms will decrease their price levels. Price levels below p^c are trivially irrational. Given $p_1^* = p_2^* = p^c$, it is of neither of the firms' interest to unilaterally raise its price, due to the facts that $\max\{p_1^s; p_2^m\} \leq p^c$ and that the payoff functions given residual demand are strictly concave.

On the contrary, if $\max\{p_1^s; p_2^m\} > p^c$, at least one of the firms would unilaterally raise its price to the residual payoff-maximizing level. This is why there is no pure equilibrium provided that $\max\{p_1^s; p_2^m\} > p^c$. □

This result is rather negative. As we argued in the introduction, the lack of pure Nash equilibrium makes the applicability of the observed model questionable.

However, in this case the lack arises only for certain parameter settings.

Tasnádi [2013] extents the analysis to the existence of mixed-strategy Nash equilibria and shows - without characterization - that a mixed equilibrium always exists.

The solution of the timing game for the parameter setting where a pure-strategy equilibrium exists, lies at simultaneous moves, as every ordering of moves results in the same outcome.

6.2.2 A numerical example

In this subsection we present a numerical example, then we turn to the production-in-advance framework.

Example 6.1. Let the demand curve take the form of $D(p) = 1 - p$. The capacities and the unit cost are fixed at $k_1 = 0.4$, $k_2 = 0.2$ and $c = 0$. We obtain that $p^c = 0.4$. We fix the share of the state in the mixed firm at $\alpha = 0.5$. Let us check whether a pure Nash equilibrium exists. It is easy to calculate that $p_1^s = 0.4$ [51], while $p_2^m = 0.3$. Neither of these two values exceeds p^c, the existence condition is therefore satisfied.

Thus, the firms' actions associated with the only pure Nash equilibrium are

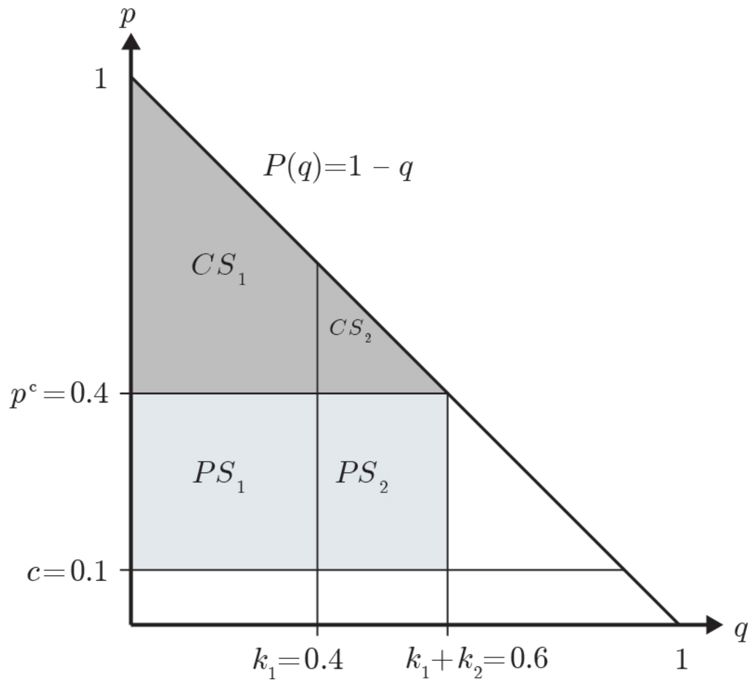
$$(p_1^*, p_2^*) = (0.4, 0.4),$$

where the both firms sell their entire capacities. The calculated payoffs are $\pi_1 = 0.5\text{x}0.16 + 0.5\text{x}0.42 = 0.29$ and $\pi_2 = 0.08$.

The equilibrium price and quantities are illustrated in the following Figure 11.

[51] The value of p_1^s is unique, since the mixed firm's own producer surplus is included in its payoff function. Therefore, decreasing the mixed firm's price would result in less producer surplus, and although the social welfare would remain the same, the overall payoff would turn lower. We note that p_1^s would equal p_1^m provided that firm 1 is also purely private.

Figure 11: Equilibrium price and quantities - Example 6.1



6.3 Production-in-advance framework

The solution of this case is published in Balogh [2014]. The assumptions of Section 6.1 remain valid and cannot be simplified in the production-in-advance case. There might also be superfluous production, therefore the unit cost level cannot be normalized to 0.

6.3.1 Characterization of pure-strategy equilibria

When observing the PIA case, the main difference is that the game does not reduce to a price-setting game. Therefore, both price and production decisions have to be considered when verifying an equilibrium profile.

We will state in the following proposition that there is only one possible pure-strategy Nash equilibrium for this case, namely, clearing the market. A necessary and sufficient condition for the existence is that $\max\{p_1^s(k_2); p_2^m(k_1)\} \leq p^c$. The result matches the one pointed out for the PTO case. In the proof of the proposition we follow the logic of the PTO case, however, it is somewhat different for the PIA case, due to the fact that production decisions are not implicit.

Proposition 6.2. *Under Assumptions 6.1-6.3, the necessary and sufficient condition for the existence of a pure-strategy Nash equilibrium in the production-in-advance type semi-mixed Bertrand-Edgeworth duopoly is that* $\max\{p_1^s(k_2); p_2^m(k_1)\} \leq p^c$. *Provided that this condition is satisfied, the only pure Nash equilibrium is*

$$(p_1^*, q_1^*, p_2^*, q_2^*) = (p^c, k_1, p^c, k_2). \tag{45}$$

Proof. We consider a price profile and a quantity profile for the two firms. An incentive for any firm to unilaterally alter either its price or quantity level means the considered profile is not a pure Nash equilibrium. First, we show that if a pure equilibrium exists, it cannot be anything else, but 45.

Assume that $p_1^* < p_2^*$. We consider first the $D(p_1^*) > k_1$ case. Here, any $q_1 < k_1$ is irrational for the mixed firm, as a lower-than-k_1 production leads to both less social welfare and less production surplus. If $D_2^r(p_2^*) > 0$, then by setting a slightly higher price, the mixed firm could raise its producer surplus without altering social welfare, that is, it could increase its payoff. If $D_2^r(p_2^*) \leq 0$, then the private firm could increase its payoff by setting a lower price level and entering the market. Now let us consider the $D(p_1^*) \leq k_1$ case. Then the private firm could realize a positive profit by altering its price to the lower p_1^*, because $D(0) = b > k_1$. Therefore, there is no equilibrium satisfying $p_1^* < p_2^*$.

We turn to the case of $p_1^* > p_2^*$. First, we note that $p_2^* \neq 0$, because the private firm has an incentive to set a positive price level to have a positive profit. If $D(p_2^*) > k_2$, then the private firm could sell its entire capacity at a higher price level, resulting in a higher profit. If $D(p_2^*) \leq k_2$, then it is of the mixed firm's interest to decrease its price below p_2^*, which would result in a higher (positive) producer surplus, but no change in social welfare.

Observing the case of $p_1^* = p_2^*$, if $p_1^* = p_2^* = p^* > p^c$, then $D(p^*) < k_1 + k_2$, therefore, by applying the tie-breaking rule, for any i, $q_i^* < k_i$. Thus, both firms will undercut the other firm's price level and sell its entire capacity at a slightly lower price. In firm 1's case this step would increase both its producer surplus and social welfare. Turning to price levels below p^c, these are trivially irrational for both firms, as they could sell their entire capacities at a higher price level, namely p^c. Given $p_1^* = p_2^* = p^c$, it is of neither of the firms' interest to unilaterally raise its price, due to the facts that $\max\{p_1^s(k_2); p_2^m(k_1)\} \leq p^c$ and that the payoff functions given residual demand are strictly concave. As far as the quantities are concerned, selling the entire capacities results in the highest possible payoffs for both firms, therefore $q_1^* = k_1$ and $q_2^* = k_2$ must hold.

Contrary to the existence condition, if $\max\{p_1^s(k_2); p_2^m(k_1)\} > p^c$, at least one of the firms would unilaterally raise its price to the residual payoff-maximizing level and set its quantity as dictated by the residual demand function. This is why there is no pure equilibrium provided that $\max\{p_1^s(k_2); p_2^m(k_1)\} > p^c$. □

We obtained a similar result to that of the PTO case. Considering the timing game, the equilibrium lies again at simultaneous moves.

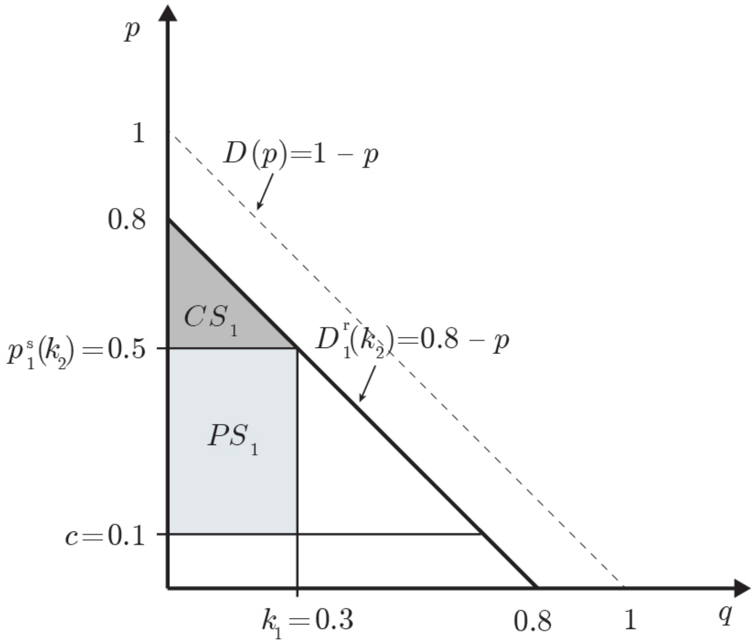
6.3.2 A numerical example

To illustrate the result, we present a numerical example, where the unit cost is strictly positive.

Example 6.2. We will calculate with the following values and demand function: $D(p) = 1 - p$, $k_1 = 0.3$, $k_2 = 0.2$, $c = 0.1$. We obtain that $p^c = 0.5$. We fix the share of the state in the mixed firm at $\alpha = 0.5$. Let us check whether a pure Nash equilibrium exists. It is easy to calculate that $p_1^s(k_2) = 0.5$, while $p_2^m(k_1) = 0.4$.

Determining $p_1^s(k_2)$ is illustrated below in Figure 12.

Figure 12: Determining $p_1^s(k_2)$



Clearly, as the pure producer surplus of the mixed firm has a weight of 0.5 in its payoff function, $p_1^s(k_2)$ will be the highest price level at which firm 1 can still sell its entire capacity.

Neither of $p_1^s(k_2)$ and $p_2^m(k_1)$ exceeds p^c, the existence condition is therefore satisfied.

Thus, the firms' actions associated with the only pure Nash equilibrium are

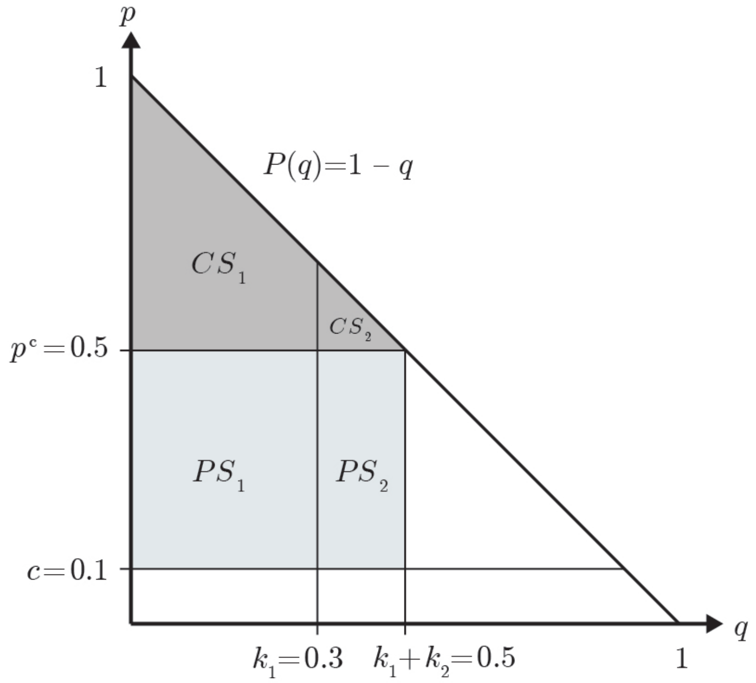
$$(p_1^*, q_1^*, p_2^*, q_2^*) = (0.5, 0.3, 0.5, 0.2),$$

where both firms sell their entire capacities at the market clearing price. The calculated payoffs are $\pi_1 = 0.22$ and $\pi_2 = 0.08$.

We also present a counter-example, where there is a lack of pure-strategy Nash equilibria. It can easily be verified that if ceteris paribus $k_1 > 0.7$, then the example does not have a pure Nash equilibrium point any more.

The equilibrium price and quantities are illustrated below in Figure 13.

Figure 13: Equilibrium price and quantities - Example 6.2



6.4 Corollaries and concluding remarks of the section

We investigated the semi-mixed Bertrand-Edgeworth duopolies with capacity constraints. One of the firms was purely private, therefore profit-maximizer, while the other one was partially owned by the state - we called it the mixed firm. The mixed ownership structure of the mixed firm was captured in its payoff function, which was a weighted sum of total welfare and its own profit.

There are several markets in practical life where the state has some interest - but not entire ownership - in one of the competing firms, providing motivation for the investigation of semi-mixed models. The model framework used in this section can be embedded into the framework we have used throughout the thesis.

Our results for the semi-mixed versions are weaker than what we presented in Sections 4 and 5 where we assumed that one of the competing firms is under pure public ownership. We investigated both the production-to-order and the production-in-advance cases. It has turned out that the different frameworks result in similar outcomes. The results concerning pure equilibria and the timing game are summarized in corollaries. However, due to the lack of pure equilibria for certain parameter settings, we cannot state a general result for the social welfare effect of the appearance of a mixed firm.

As the reader might have noticed, the semi-mixed frameworks provided less complicated results concerning pure-strategy Nash equilibria than the mixed frameworks presented in Sections 4 and 5. A factor lying in the background is as follows. A useful feature of a purely public firm is that given certain conditions, its price can vary within a given range, resulting in the same payoff level. Assume that $D(p_1) = k_1$. Then, by setting a lower $p_1' < p_1$ price level, the consumer surplus generated by firm 1 increases while its producer surplus decreases by the same magnitude. The aggregate effect results in no change in social welfare. This argument has been used several times in the proofs of Sections 4 and 5. However, this technique is not valid for the semi-mixed frameworks, allowing for less pure-strategy equilibria and less complicated analysis at the same time.

The first corollary of the section considers the existence of pure-strategy Nash equilibria. The result we obtained is similar to what one can find in Deneckere and Kovenock [1992], where the purely private firm case is investigated: for certain parameter settings there exists no pure equilibrium.

Corollary 6.1. *The necessary and sufficient condition for the existence of a pure (subgame-perfect) Nash equilibrium point is that* $\max\{p_1^s(k_2); p_2^m(k_1)\} < p^c$*. Whenever the assumption is satisfied, the only pure equilibrium is clearing the market.*

Our second corollary considers the timing game. Assuming endogenous timing, i.e. endogenous decision of the ordering of moves, we have obtained that for the parameter settings where we have a pure (SP)NE, it is all the same whether a firm becomes a leader or a follower.

Corollary 6.2. *If the price- (and quantity-) setting game has an equilibrium in pure strategies, then the equilibria of the timing game lie at all three possible orderings of moves.*

Further research directions in the field of semi-mixed duopolies might include extending the results to the more-than-two-firm case. Assuming that any of the competing firms may have partial public ownership could also be helpful to induce practical applications. However, it is likely that for certain parameter settings, one has to face the lack of pure-strategy Nash equilibria.

7 Conclusion

The dissertation aimed at analyzing firm behavior in the mixed and semi-mixed Bertrand-Edgeworth duopolies by means of game theory. The main concept we used throughout the thesis was the most frequently used equilibrium concept, the Nash equilibrium.

Assuming partial public ownership on the market, we provided a thorough analysis concerning equilibrium prices and quantities, endogenous timing and social welfare effects. The contribution of the thesis to knowledge is broadening the theory of Bertrand-Edgeworth models for a better understanding of duopolistic markets with public ownership.

A mixed duopoly model considers a market situation, where one of the two competing firms is under pure or partial state ownership. Thus, the state does not act as an outside regulator on such markets, but as a market participant, it aims at driving the market to a socially better equilibrium. Therefore, when modelling mixed duopolies by means of game theory, social welfare appears in the public firm's payoff function.

There are several duopoly and oligopoly models with different strengths and focuses. The first duopol models - the Cournot- and Bertrand-duopolies - were born in the 19th century, and since then, many directions in improving them have been investigated in the literature concerning both price-setting and quantity-setting frameworks. A model that tackles several critics that have been addressed to Cournot- and Bertrand-type models is Bertrand-Edgeworth duopoly. The main advantage of this model family is that it can handle the problem of unlimited capacities. On the other hand, if two private firms compete on the market, we have to face a lack of pure-strategy Nash equilibria for certain parameter settings, which makes practical applicability rather difficult.

The theory of duopolies with public ownership was analyzed first by Merrill and Schneider [1966]. Endogenizing the timing of decisions began later. Endogenous timing in mixed Bertrand-Edgeworth duopolies with capacity constraints was investigated in the present dissertation.

We analyzed all together four variants of the mixed Bertrand-Edgeworth model. The common features of the presented models are as follows:

- There are two firms competing on the market of a homogenous good.
- The decision variables of the firms are both price and quantity.
- The consumer side is given by a market demand function, which is monotone, strictly deceasing and twice continuously differentiable.
- The two firms cannot produce a higher amount than their respective capacity constraints.
- Both firms have constant and identical unit costs.
- One of the competing firms has pure private ownership, while there is a certain share of the state in the other one.
- All the parameters are common knowledge.

The four models can be differentiated either according to the share of the state in the public firm, or according to the timing of demand satisfaction. We consider on the one hand, mixed and semi-mixed models, while, on the other hand, we assume both production-to-order (PTO) and production-in-advance (PIA) frameworks. PTO means that production takes place only after sales are realized, while in the PIA framework items are produced before they are sold. In the latter case there might emerge supplies that cannot be sold (think of the markets of perishable goods), while the PTO framework lets the game reduce to a price-setting game, as quantities are obtained by substituting the firms' price levels into the demand curve.

As far as the timing of decisions is concerned, the two decisions can be made simultaneously, or sequentially, where the latter variant consists of public leadership and private leadership. We always took into consideration all three possible orderings of moves.

The following Table 8 can be helpful in positioning the four models.

Table 8: Model variants

Models	**Pure public ownership**	**Limited public ownership**
PTO	mixed PTO	semi-mixed PTO
PIA	mixed PIA	semi-mixed PIA

Now we provide an overview of the contents of each section, afterwards we summarize the results based on the research questions we stated.

The introductory section stated the overall aim of the thesis as well as its contribution to knowledge. The game-theoretic methodology is clarified and we provided motivation by recalling real-life oligopolistic markets where public ownership is present. The key research questions were also presented in Section 1.

Section 2 provided an introduction to the most simple duopolistic models, the Cournot- and the Bertrand-duopolies. With the aim of reducing the shortcomings of these classical models, we introduced the Bertrand-Edgeworth competition. By recalling contributions from the relating literature, we discussed the question of rationing rules and the existence of pure-strategy Nash equilibria.

Section 3 offered a survey on mixed oligopolies. We introduced the production-to-order and production-in-advance frameworks and put down the main assumptions of the four models we discussed.

The formal discussion began in Section 4, where we carried out the analysis of the mixed PTO model. We gave the formal assumptions of the model and characterized pure-strategy Nash equilibria for the so-called strong-private-firm and weak-private-firm cases, making a distinction in the private firm's capability of influencing equilibrial outcome. We also provided a numerical example to illustrate the results. The implicit solution of the timing game and the public firm's social welfare effect are also discussed. The results of Section 4 are published in Balogh and Tasnádi [2012].

Section 5 considered the mixed PIA framework. In this more complicated case we analyzed separately the strong-private-firm case, the weak-private-firm case and the high-unit-cost case for all three possible orderings of moves. Numerical examples are attached to each case. We also presented the solution of the timing game and analyzed the public firm's social welfare effect. The results of Section 5 can be found in Balogh and Tasnádi [2014].

Section 6 observed the two semi-mixed frameworks. The main distinction of this section compared to the previous ones is that we have not allowed for purely public ownership in the public firm. On the contrary, we considered a purely private firm and a so-called mixed firm with an exogenously given (less-than-one) ratio of public ownership. We recalled the results on equilibrial firm behavior for the PTO case and presented similar results for the PIA framework. The analyses in Section 6 have proved to be less complicated than those of Sections 4 and 5, however, the results were rather negative concerning the existence of pure-strategy Nash equilibrium profiles. Some results of Section 6 can also be found in Balogh [2014].

We turn to giving our answers to the research questions and stating the main results of the dissertation.

The first question referred to the existence of pure-strategy Nash equilibria: Under what conditions does a pure-strategy Nash equilbrium in a Bertrand-Edgeworth duopoly with public ownership exist?

Answering this question is quite straightforward. In Sections 4 and 5 we showed that at least one pure-strategy Nash equilibrium exists even in the regions, where the standard version of the game does not have any. In other words: we proved that replacing one of the private firms to a purely public firm results in the appearance of at least one pure Nash equilibrium. For the semi-mixed variants the result is not so positive. We obtained that if the residual payoff-maximizing price levels (which can be directly calculated from the firms' capacity constraints and the market demand function) of any of the firms exceeds the unit cost level, then the game does not have any pure Nash equilibrium. This result matches that of the standard Bertrand-Edgeworth duopoly (with purely private firms), therefore the appearance of partial public ownership gives no remedy for the lack of pure equilibria.

Based on these arguments, we can state the first main result.

Main result 1. *Under quite general conditions for the demand function, there exists at least one pure-strategy Nash equilibrium in the mixed Bertrand-Edgeworth duopoly, given any parameter setting. The semi-mixed Bertrand-Edgeworth duopoly has a pure-strategy Nash equilibrium if and only if an extra condition (see pages 93 and 96) is satisfied for the firms' capacities.*

The solutions of the existence problem were constructive throughout the dissertation, which means that we also characterized the pure-strategy Nash equilibrium points whenever they existed, and thus, answered the following second research question: Given entire or partial public ownership in one of the competitors in a Bertrand-Edgeworth duopoly and provided that a pure Nash equilibrium exists, what are the equilibrium prices and quantities of both firms for the simultaneous and the sequential versions of the game?

We have had several types of pure Nash equilibria in the previous sections. It is desirable to provide a brief and transparent summary of the pure equilibria of the observed models. As there are several model versions and there exist three timing variants to each setting, we present here only the simultaneous-moves-case equilibria. The sequential-moves equilibria can be derived from the simultaneous equilibria in most of the cases. Whenever we obtained multiple equilibria, these were not interchangeable. All the equilibria are given in the propositions of Sections 4, 5 and 6. In the following list, when necessary, we recall the notations having been used throughout the dissertation.

1. **Mixed PTO.** In the *strong-private-firm case*, the price levels $p_1^* \leq p_2^* = p_2^d$ and $p_1^* \leq p_2^d$, $p_2^* = p_2^m$ lead to pure equilibria. This means that the private firm becomes either a monopolist on the residual demand curve, or stays on the original demand curve and sells its entire capacity. Additionally, under certain conditions (see page 42) , the private firm can become a monopolist in equilibrium up to its capacity limit.

In the *weak-private-firm case*, firms set the market clearing price in equilibrium.

2. **Mixed PIA.** In the *strong-private-firm case*, the price and quantity levels $p_1^* \leq p_2^d(q_1^*)$, $q_1^* \in [0, k_1]$, $p_2^* = p_2^m(q_1^*)$, $q_2^* = q_2^m(q_1^*)$ lead to a pure equilibrium under certain conditions, i.e. the private firm is a monopolist, but only on the residual demand curve. Additionally, also under certain extra conditions, the private firm can become a monopolist in equilibrium up to its capacity limit.
In the *weak-private-firm case*, the private firm will choose in equilibrium the highest price level at which it can sell its entire capacity provided that the public firm has no incentive to undercut the private firm's price. A particular case of this equilibrium is clearing the market. Besides, under certain conditions, the private firm might become a monopolist up to its capacity limit.
In the *high-unit-cost-case*, all kinds of equilibria mentioned in the previous two cases are possible under certain conditions.

3. **Semi-mixed PTO.** A pure (SP)NE exists if and only if $\max\{p_1^s; p_2^m\} \leq p^c$. Provided that the condition is satisfied, the only pure equilibrium is clearing the market, i.e. $p_1^* = p_2^* = p^c$.

4. **Semi-mixed PIA.** A pure (SP)NE exists if and only if $\max\{p_1^s(k_2); p_2^m(k_1)\} \leq p^c$. Provided that the condition is satisfied, the only pure equilibrium is clearing the market, i.e. $p_1^* = p_2^* = p^c; q_1^* = k_1; q_2^* = k_2$.

The results on the characterization of pure-strategy Nash equilibria are stated in the second main conclusion.

Main result 2. *The characterization of pure-strategy Nash equilibria depend strongly on the model assumptions. The five main types of Nash equilibria in the mixed PTO, mixed PIA, semi-mixed PTO and semi-mixed PIA Bertrand-Edgeworth duopolies are as follows: (1) the firms clear the market; (2) the private firm is a monopolist on the residual demand curve; (3) the private firm sells its entire capacity and earns as much as if it were a monopolist on the residual demand curve; (4) the* *private firm is a monopolist on the market demand curve up to its capacity limit; (5) the private firm sells its entire capacity at the highest price level, where it is still not worth for the public firm to undercut the private firm's price.*

Timing of decisions is often endogenized in the recent literature. This means that the firms play in fact a two-stage game. In the first stage they decide when to announce their price and production decisions (before the other firm, after the other firm, or at the same time). In the second stage, firms announce their price and production levels. The following, third research question focused on the problem of endogenous timing. Which ordering of decisions emerges if a private and a purely or partially public firm compete on the market in a Bertrand-Edgeworth duopoly provided that timing is endogenous?

We obtained different results in the observed models concerning endogenous timing. In the mixed PTO case all three orderings of moves were timing game equilibria. Considering the mixed PIA case, both firms were better off if they become the leader, therefore, the timing game equilibrium lies at simultaneous moves. Finally, in the semi-mixed cases, we could determine the equilibrium of the timing game only for parameter settings, where the price- and quantity-setting game had a pure equilibrium. For this case all three possible orderings were equilibria of the timing game, as the ordering of price and quantity decisions did not matter concerning payoffs.

The results on endogenous timing are presented in the third main conclusion.

Main result 3. *Timing of decisions does not matter, i.e. the timing game has multiple equilibria in the production-to-order mixed Bertrand-Edgeworth duopoly. The same is true for both the production-to-order and the production-in-advance cases of the semi-mixed Bertrand-Edgeworth duopoly for the parameter settings, where a pure-strategy Nash equilibrium exists in the price- and production-setting game. Finally, the timing game equilibrium of the production-in-advance mixed Bertrand-Edgeworth duopoly lies at simultaneous moves.*

When the state enters a duopoly market by acquiring partial or entire ownership in one of the competing firms, the level of social welfare generated on the market may not remain the same, as there is a modification in one firm's objective function. Our last research question referred to the change in social welfare the presence of a public firm may cause: What is the direction and magnitude of social welfare change the appearance of a purely or partially public firm generates in a Bertrand-Edgeworth duopoly framework?

Based on the most plausible pure-strategy Nash equilibria of the observed model, we could derive the public firm's social welfare effect for the parameter settings, where we had at least one pure Nash equilibrium. We pointed out that the mixed PTO case resulted in higher social welfare than the pure PTO case, i.e. the appearance of the public firm made the outcome more competitive, providing a surprising result. For the mixed PIA case we concluded that the result is less competitive than that of the mixed PTO case, i.e. the social welfare becomes lower in equilibrium. As far as the semi-mixed models are concerned, we could conclude for the favorable parameter-settings, where a pure strategy exists, that the social welfare remains the same as that of the standard (purely private) case. Consequently, socializing a certain proportion of a firm does not result in welfare growth, unless the firm gets under pure public ownership.

The following table provides a summary of the social welfare effects experienced in the different models wherever a direct comparison could be made.

Table 9: Social welfare effects of public ownership

	Public (or mixed) firm's social welfare effect
Mixed PTO	Positive (compared to standard PTO)
Mixed PIA	Negative (compared to mixed PTO)
Semi-mixed PTO	No effect (compared to standard PTO)
Semi-mixed PIA	No effect (compared to standard PIA)

The results concerning social welfare effects are stated in our last main conclusion.

Main result 4. *The production-to-order mixed Bertrand-Edgeworth duopoly environment leads to a higher social welfare in equilibrium than the standard version of the game. In the production-in-advance mixed Bertrand-Edgeworth duopoly the social welfare becomes lower than that of the production-to-order mixed duopoly. Finally, the appearance of a partially public firm generates no change in social welfare in equilibrium provided that there is a pure-strategy Nash equilibrium of the price- and quantity-setting game.*

To sum up the results, the dissertation contributed to the understanding of duopolies, where public ownership is present on the supplier side of the market.

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Appendix: The author's scientific activity

Journal articles

Tamas L. Balogh, Attila Tasnadi (2012): Does Timing of Decisions in a Mixed Duopoly Matter? Journal of Economics 106: 233-249.

Tamas L. Balogh, Janos Kormos (2014): A computational model of outguessing in two-player non-cooperative games. Acta Univ. Sap. Informatica 6: 71-88.

Balogh Tamás László (2014): Állami tulajdonrész duopol piacokon. Competitio 13: 63-74.

Working papers

Tamas L. Balogh, Christian Ewerhart (2013): On the origin of r-concavity. University of Zurich

Tamas L. Balogh, Attila Tasnadi (2014): Mixed duopolies with advance production. HAS "Lendület" Strategic Interactions Research Group

Conference talks

Balogh Tamás László, Bertók Kornél (2011): A gazdaságinformatikus képzés hallgatói megítélése a Debreceni Egyetemen. Conference title: Informatika a felsöoktatásban (Debrecen). Date: 24-27th August 2011

Tamas L. Balogh, Attila Tasnadi (2012): Effects of Advance Production on a Price Setting Mixed Duopoly. Conference title: SING 8 (Budapest). Date: 16-18th July 2012

Tamas L. Balogh, Janos Kormos (2014): A computational model of outguessing. Conference title: ICAI (Eger). Date: 29th January - 1st February 2014

Balogh Tamás László, Bánszki I., Beringer D, Varga L. (2014): A mozgás és gondolkodás együttes élménye: A Medve Szabadtéri Matekverseny. Conference title: Matematikatanárok Rátz László Vándorgyülése (Keszthely). Date: 3-6th July 2014

Other talks

University of Debrecen, Faculty of Economics and Business Administration, Research Forum: six talks between January 2011 and June 2014

University of Zurich: talk at the Microeconomics PhD Seminar, September 2012