Neighbourhood sequences in different grids

PhD thesis

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Köszönetnyilvánítás

Ezúton is szeretnék köszönetet mondani mindazoknak, akik közvetlenül vagy közvetve hozzájárultak a disszertációm elkészítéséhez.

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A Cajun father decided to send his oldest son to Louisiana State University. He would be the first member of the family ever to attend college. The son returned home at Christmas after his first semester. The proud father asked the son:

"Son, tell, yo' daddy what you been learnin' at that college."

"Well, daddy, I been takin' Biology, History, Algebra . . . "

The father interrupts:

"Oh! Algebra you says?! Dat's fine. Speak to yo' daddy some Algebra, so I can hear what it sound like."

"But daddy, Algebra ain't no language, it's . . ."

The father interrupts again in a stern tone:

"Boy, if you know what's good for you, you best speak yo' daddy some Algebra right heah and now!"

The boy sighs:

"Oh, okay daddy. Here goes: 'pi r squared.'"

The father knocks the boy over with a slap.

*"Fool! Everybody know dat pie are round! It's *cornbread* dat are squared!"*

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Chapter 1 : Introduction

In this section we offer a brief survey both of a part of digital geometry and this thesis itself.

The digital geometry is an important part of image processing. In digital geometry the spaces we work in consist of discrete points with integer coordinates. Two points are neighbours if their coordinate difference values are at most 1. In the n dimensional square (cubic) grids there are n kinds of neighbourhood relations according to the number of differences. In the hexagonal grid only 1 neighbourhood criterion is widely used, while in the case of the triangular grid there are three kinds of neighbours. We define the distance of two points as the number of steps in a shortest path (it is possible several shortest paths of the same path-length exist), where by a step we mean a movement from a point to one of its neighbour points. This distance function depends not only on the points, but also the given neighbourhood criterion. Varying the neighbourhood relations in a path, we get the concept of neighbourhood sequences. In the thesis we analyse the distance functions based on neighbourhood sequences in different grids, such as the most usual square- and cubic-, or arbitrary (possibly infinite) dimensional grids as well as the hexagonal and the triangular ones.

In the next subsections we present a short bibliographic introduction as a survey and the structure of the thesis.

1.1 A brief history of digital geometry

In this part we show some references on grids in digital geometry. The history of neighbourhood sequences is also presented.

The birth of digital geometry is connected to [57]. In that paper Rosenfeld and Pfaltz defined two basic neighbourhood relations in the square grid. They gave two types of motions in the two dimensional square grid. The cityblock motion allows horizontal and vertical movements only, while with the chessboard motion the diagonal directions are also permitted. So in this grid we have two kinds of distances, based on these motions. Figure 1 shows a point together with those points, which have distance 1 from it. Both cityblock and chessboard distances are shown. Moreover, the theory of neighbourhood sequences started with this paper. The authors recommended the alternate use of the two possible motions for distance measuring. This so-called octagonal distance obtains a better approximation for the Euclidean distance than the ones using only one kind of steps. We come back to the history of neighbourhood sequences later.

The higher dimensional rectangular grids were also investigated ([4, 60]). There are some survey papers about the digital metrics of square grids. Short summaries of these kinds of examinations can be found in [38, 44] and in [56]. In the case of the square grid, each coordinate of a point is independent of the others. For n dimensions we use n coordinates. In the n dimensional cubic grid, the structure of the nodes is isomorphic to the structure of the n dimensional cubes. In Figure 1, both options are shown in two dimensions, because we will use both kinds of grid.

We note here, that another branch of the Mathematics works on these grids. The field called 'Geometry of Numbers' has nice theorems about these grids. (To read more on the topic: [8, 28, 29, 39].) They use the concept 'lattice' with the same meaning with which we use 'grid' here. Among other eminent mathematicians, P. Erdős ([20, 30]) and

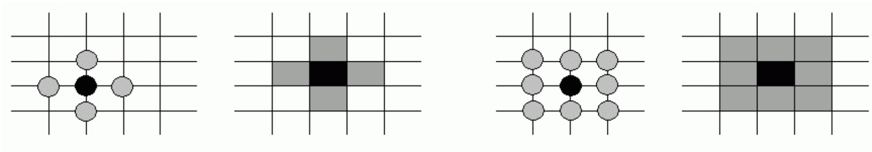


Fig. 1: 'Cityblock' and 'chessboard' neighbourhood relations in the square grid of nodes and of regions

L. Lovász ([27, 41]) also have some results in this field. They also use the hexagonal and the triangular lattices. So let us see where and how these grids are used in the field of digital geometry.

Parallel to the square grid the hexagonal and the triangular grid were also investigated in digital geometry. A connection among the cubic, hexagonal and triangular grids can be found in [37, 52a, 52, 61], and we will explain it.

We can say that the hexagonal grid is popular. There is a natural neighbourhood relation in this grid and it is used almost every time. One of the first papers about the hexagonal plane is [43], in which the name 'rhombic array' was introduced. In pattern recognition the idea of using a non-cartesian coordinate system can be found, for example, in [3, 5, 19] and in [25]. There are some fairly old papers on this topic; hence, the hexagonal grid is also well described.

The hexagonal grid is used in many applications and in practice as well, because it is very natural and simple. The distance function based on the neighbourhood relation of the hexagonal grid can be found in [42].

The triangular grid is the third "basic grid". The 3 kinds of neighbourhood criteria of the triangular grid can be found in [19], where thinning algorithms are shown in the three basic grids.

The triangular grid is a valid concurrent plane to the rectangular one in digital geometry. The triangular grids play an increasing role in geometric modelling; many 3D-scanners produce triangulations. These grids are generally not regular, but at high enough resolution they are close to regular ones. The human retina is often modelled by a Delauney triangulation. Many algorithms of computer graphics are also given for the triangular grid ([24, 58]). Therefore we can say that the triangular grid is also one of the most important grids in digital geometry, in digital image processing and in the theory of cellular neural networks also (see [55]).

Now we will consider the theory of neighbourhood sequences.

Formally, in digital geometry we use a discrete space, i.e. points can have only integer coordinates. In the rectangular cases we do not have more restrictions than this for the values of coordinates. Two different points in $\mathbb{Z}^m$ are k-neighbours, $(k, m \in \mathbb{N}, k \leq m)$, if their corresponding coordinate values are equal up to at most k exceptions, and the difference of the exceptional values are at most 1. After fixing k, we may define the distance of two points as the number of steps of the shortest path between these points, where a step means moving from a point to one of its k-neighbours (see [13, 15]). It is easy to check that by this definition we get a metric on $\mathbb{Z}^m$, for each $k \in \{1, 2, ...m\}$, and that these metrics are different for the separate values of k.

To obtain these metrics we fixed k in the beginning, in other words, we used the same k in each step for walking from a point p to a point q in $\mathbb{Z}^m$. The situation is more complicated if we can change the value of k after every step. For example, by allowing arbitrary mixtures of cityblock and chessboard motions we obtain the concept of neighbourhood sequences in two dimensions. Generally, a sequence $(b(i))_{i=1}^{\infty}$ is called a neighbourhood sequence in $\mathbb{Z}^m$ if $b(i) \in \{1, ..., m\}$ $(i \in \mathbb{N})$. The sequence is periodic if there is some $l \in \mathbb{N}$ such that $b(i + l) = b(i)$ for every $i \in \mathbb{N}$. The concept of periodic neighbourhood sequences was introduced in [12] by Das, Chakrabarti and Chatterji, and in more general way (not connected to any specific neighbourhood relation) in [62, 63] by Yamashita and his co-authors. The general, not necessarily periodic neighbourhood sequences for $\mathbb{Z}^n$ and $\mathbb{Z}^\infty$ were introduced in [23] by Fazekas *at al.* (We mention that the sequences in [12] were called "neighbourhood sequences", while in [23] "generalized neighbourhood sequences", but for simplicity we use the above definition.)

With the help of a neighbourhood sequence $(b(i))_{i=1}^{\infty}$ we may define the distance of $p, q \in \mathbb{Z}^m$ in the following way. We take the length of a shortest path from p to q, but at the i-th step now we may move from a point to another if and only if they are $b(i)$-neighbours. Certainly this notion is a generalization of the original one, as we may choose $b(i) = k$ for each $i \in \mathbb{N}$, with any $k \in \{1, ..., m\}$.
In [12] the authors gave an algorithm which generates a shortest path between any $p, q \in \mathbb{Z}^m$ in case of periodic neighborhood sequences.

As we mentioned, the neighbourhood sequence $(b(i))_{i=1}^{\infty}$ with $b(i) = k$ $(i \in \mathbb{N})$ generates a metric on $\mathbb{Z}^m$ $(m \in \mathbb{N})$ for any $1 \leq k \leq m$. However, it is easy to find neighbourhood sequences, even periodic ones, such that the distances with respect to these sequences do not provide metrics on $\mathbb{Z}^m$. In [12] the authors gave a nice characterization of the periodic neighbourhood sequences, for which the above defined distance functions provide a metric on $\mathbb{Z}^m$. We extended this result to arbitrary neighbourhood sequences in $\mathbb{Z}^m$ and we generalized it also to the infinite dimensional digital plane $\mathbb{Z}^\infty$ in [48a, 48].

The main advantage of neighbourhood sequences over the classical distances, using only one neighbourhood criterion at each step, is that they provide more flexibility in moving in space. Making use of this property, Das and Chatterji [2, 10, 14, 16] were able to determine distance functions that provide a good approximation of the Euclidean distance in the square grid. The authors in [12, 16] analyzed the geometric properties of the octagons occupied by a neighbourhood sequence during "spreading" on the 2D plane. This is another aspect of the analysis of neighbourhood sequences, examining how the occupied areas develop step by step. It is obvious that in the square grid using only one type of neighbourhood in each step, we get squares (in the case of chessboard) or diamonds (in the case of cityblock). Using both kinds of steps we obtain octagons as digital circles. Some results about these wave-front sets were described by Das *et al.* (see [16]). So, the investigation of hyperpheres with neighbourhood sequences started in [16] in two dimensions with periodic neighbourhood sequences. In the higher dimensions, hyperspheres were also analyzed using only constant neighbourhood criteria in [15]. In the three dimensional digital space Danielsson described them by periodic neighbourhood sequences ([9]). In [48a, 48] we used their vertices for a proof in an implicit way with generalized neighbourhood sequences in arbitrary dimensions, while Hajdu analyzed them in an explicit way in the arbitrary finite dimensional spaces ([31]).
We would like to mention here that in [34] we presented some practical examples where we used neighbourhood sequences in the 3 dimensional RGB cube to measure the distances of colours. We showed examples for the following image processing methods: fuzziness,

region growing and clustering.

We introduced the concept of neighbourhood sequences for the triangular grid [49a, 49, 50]. An algorithm to find a shortest path by using arbitrary neighbourhood sequence can be found in [49a, 49] and the detailed analysis of the algorithm in [49]. We showed the strange property that the distance based on neighbourhood sequences in the triangular grid may not be symmetric. In [50] there is a necessary and sufficient condition for neighbourhood sequences generating metric distances. In [51] a characterization and analysis of some properties of digital circles is given while in [33] we approximated the Euclidean circles by using neighbourhood sequences in the triangular grid as well. It turned out that the digital circles of the triangular grid are better approximations to the Euclidean circles than the ones in the square grid.

We will explain most of our results about the triangular grid in Chapter 5.

After this historical overview the structure of the thesis is shown.

1.2 The structure of the thesis

In Chapter 2 we show some basic definitions. We present some formal and informal preliminaries as well.

In Chapter 3 the square grids (in arbitrary dimensions) are described. In Section 3.2 we write about the shortest path problem, and we present an algorithm which generates a minimal path if possible. In Section 3.3 a formula to calculate the distance is presented and in Section 3.4 we give a necessary and sufficient condition for a distance based on a neighbourhood sequence to be a metric.

Chapter 4 is about the hexagonal grid. We present some results, most of them simple and well-known. We show the connection between the hexagonal and the cubic grids as well. In Chapter 5, on the triangular grid, there are several sections. First we give some specific definitions and notation. In Section 5.2 we write about the shortest path problem, and we present an algorithm which generates a minimal path. In Section 5.3 we give a necessary and sufficient condition for a distance based on a neighbourhood sequence to be a metric. In the next section, the embedding of the triangular grid into the cubic grid is shown. In Section 5.5 a formula to calculate the distance is presented and in Section 5.6 we describe the digital circles of the triangular grid. In Section 5.7 some interesting practical examples are drawn about using neighbourhood sequences in triangular networks.

Finally, in Chapter 6 some directions of further research are presented, such as distance of sequences, other grids (especially the "3-plane triangular grid") and some digital curves, for example, parabolas.

Chapter 2 : Basic definitions and notion

First we introduce a notation. In this thesis we will use the function $sgn(x)$ for $x \in \mathbb{R}$:

$$sgn(x) = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}$$

The concept of distance plays an important role in this thesis. We use the name *"distance"* for any function which is defined on the square of the Universe (V). Usually we use points (or regions) with integer coordinates and therefore the elements of the Universe are vectors with integer values.

Definition 2.1 *A function $d : V \times V \to \mathbb{R} \cup \{\infty\}$ is called a metric on V if it satisfies the following conditions:*

1. *$\forall p, q \in V$: $d(p, q) \geq 0$, and $d(p, q) = 0$ if and only if $p = q$ (positive definiteness),*
2. *$\forall p, q \in V$: $d(p, q) = d(q, p)$ (symmetry),*
3. *$\forall p, q, r \in V$: $d(p, q) + d(q, r) \geq d(p, r)$ (triangular inequality).*

We note that usually the metric properties are defined as above, but the non-negative property is a consequence of the other properties, as we show. Assume that $d(p, q) < 0$ for some p, q. Then because of symmetry $d(q, p) < 0$. Let us calculate the sum of them: $d(p, q)+d(q, p) < 0$, which contradicts triangular inequality: it should be $d(p, q)+d(q, p) \geq d(p, p)$, where $d(p, p) = 0$.

Sometimes only those distances are called metrics which always have finite values, and the distances allowing distance ∞ are called generalized metrics. We used these definitions in [53], but in this thesis we will use Definition 2.1.

We can use more types of neighbours in an arbitrary grid. Informally, we assume that two geometrical objects (of the same dimension n as the space where we are) are neighbours of each other if there is at least one point which is on the border of both. Two geometrical objects are neighbours of type m (or shortly, m-neighbours) if we can make at most m steps from one to the other in such a way that in each step we step through an ($n - 1$ dimensional) border line of two objects.

Analogously, we can define the relation *"m-neighbourhood"* among the nodes in (planar) graphs. Two nodes are neighbours if they are on the border of the same region. They are m-neighbours if they are neighbours and the shortest path between them includes at most m edges.

Sometimes we use the concept of strict m-neighbours when two objects are m-neighbours, but they are not $(m - 1)$-neighbours.

It is obvious that these m-neighbourhood relations are reflexive and symmetric relations. Moreover they have the following inclusion properties. All $(m - 1)$-neighbours of an object are its m-neighbours as well.

Using this definition of m-neighbourhood we have to pay attention to the coordinatization of the grid. The aim is to order the coordinate values to the objects such a way that we can redefine the neighbourhood criteria in natural way by using coordinate values.

We adopt some definitions from the literature mentioned earlier.

According to the possible types of neighbours, we can define the so-called neighbourhood sequences.

Definition 2.2 *The infinite sequence* $B = (b(i))_{i=1}^{\infty}$, *– in which the values* $b(i) \in \mathbb{N}$ *are possible types of neighbourhood criteria in the digital space that is used – is called a neighbourhood sequence (or abbreviated n.s.). If for some* $l \in \mathbb{N}$, $b(i) = b(i+l)$ *holds for every* $i \in \mathbb{N}$, *then* B *is called periodic (with period* l*). In periodic cases we will use the abbreviation* $B = (b(1), ..., b(l))$.

Since 1984 the concept of neighbourhood sequences has appeared. In 1987, Das *et al.* [12] used the theory of periodic neighbourhood sequences in arbitrary finite dimensional square grids. The periodic property of such sequences was dropped by Fazekas *et al.* [23] in 2002.

Definition 2.3 *Let* p *and* q *be two points and* $B = (b(i))_{i=1}^{\infty}$ *a n.s. A finite point sequence* $\Pi(p, q; B)$ *of the form* $p = p_0, p_1, \ldots, p_m = q$, *where* p_{i-1}, p_i *are* $b(i)$*-neighbour points for* $1 \leq i \leq m$, *is called a* B*-path from* p *to* q. *We write* $m = |\Pi(p, q; B)|$ *for the length of the path.*

In the case of the finite dimension, there is a B-path between any two points with any neighbourhood sequence B. However, in $\mathbb{Z}^{\infty}$ it is possible that there are no B-paths between two points. For example if the set $\{|p(i) - q(i)| \, : \, i \in \mathbb{N}\}$ is unbounded, then there are no neighbourhood sequences B, for which a B-path would exist between the points $p = (p(i))_{i=1}^{\infty}$ and $q = (q(i))_{i=1}^{\infty}$. The i-th coordinate of the point p is indicated by $p(i)$, which we will use many times.

Definition 2.4 *When* B*-paths exist between the points* p *and* q, *denote by* $\Pi^*(p, q; B)$ *a shortest path between them, and set* $d(p, q; B) = |\Pi^*(p, q; B)|$. *We call* $d(p, q; B)$ *the* B*-distance of* p *and* q. *In those cases when finite* B*-paths do not exist between points* p *and* q *we put* $d(p, q; B) = \infty$.

Note that the shortest path problem in our case looks more like a graph-theoretical problem (see [7, 36]) than the problem in Euclidean space. There can be more shortest paths opposite to the Euclidean case when it is always only 1.

Chapter 3 : Square grids

As we mentioned in the introduction, the square and cube grids are well examined. In this chapter we present some results about the square grids in arbitrary finite and infinite dimensions.

The structure of the nodes of these grids is isomorphic with the structure of the n dimensional cubes. The number and the positions of the neighbours are the same. The grid of nodes, and the grid obtained by representing each (hyper)cube with its central point are identical within a translation. So the dual of the n dimensional square grid is an n dimensional square grid as well. Therefore, we do not care about what grid we use. The coordinate values are assigned to those objects what we use, and we will refer to them as points (however they can be the n-dimensional cubes, as well). In the square grids we use the same number of independent coordinate values as the dimension of the space.

In the n dimension we have n-types of neighbourhood criteria according to the number of different coordinate values. If we use strict neighbourhood relations then in the 2 dimensional plane the points have four 1-neighbours and four 2-neighbours, as a square has 4 sides and 4 corners. In the 3 dimensional space the points have six 1-neighbours, twelve 2-neighbours, and eight 3-neighbours, as a cube has 6 sides 12 edges and 8 corners.

In the next section we recall some specific definitions and known results for these grids.

3.1 Definitions, preliminaries

Throughout the chapter, n will denote an arbitrary natural number and N will denote an arbitrary element of the set $\mathbb{N} \cup \{\infty\}$. Let $\mathbb{Z}^N$ be the N-dimensional digital space, i.e. $\mathbb{Z}^N = \{(z(i))_{i=1}^N : z(i) \in \mathbb{Z}\}$. We shall refer to the elements of $\mathbb{Z}^N$ as points.

Definition 3.1 *Let p and q be two points in $\mathbb{Z}^N$. Let $k \in \mathbb{N} \cup \{\infty\}$ with $1 \leq k \leq N$. The points p and q are k-neighbours if the following two conditions hold:*

1. $|p(i) - q(i)| \leq 1$ *for* $1 \leq i \leq N$,
2. $\sum_{i=1}^N |p(i) - q(i)| \leq k$.

(One can check that the formal definition above gives the same concept as the intuitive definition in the previous chapter.)

In the case of any finite dimension, there is a B-path between any two points with any neighbourhood sequence B. However, in $\mathbb{Z}^\infty$ it is possible that $d(p, q; B) = \infty$ for two points p and q and a neighbourhood sequence B.

Definition 3.2 *Let B_1 and B_2 be two neighbourhood sequences in $\mathbb{Z}^N$. We say that B_1 is faster than B_2 if*

$$d(p, q; B_1) \leq d(p, q; B_2) \quad \textit{for all } p, q \in \mathbb{Z}^N.$$

We denote this relation by $B_1 \sqsupseteq^* B_2$.

The relation $\sqsupseteq^*$ was introduced by Das [11] in the two dimensional space with periodic neighbourhood sequences, and by Fazekas in [22] in three dimensions. Fazekas *et al.* in [23] generalize this relation to higher dimensions with arbitrary neighbourhood sequences.

Definition 3.3 *Let p and q be two points in $\mathbb{Z}^N$. For any $1 \leq i \leq N$ put $w(i) = |p(i) - q(i)|$, and $w_{p,q} = (w(i))_{i=1}^N$. The point $w_{p,q}$ is called the absolute difference of p and q. Moreover let the vector $v_{p,q}$ be the ordered absolute difference of the points in the following way. The multiset of the elements of $v_{p,q}$ is the same as the multiset of $w_{p,q}$, but they are in non-increasing order, i.e. for all $i < j$ implies $v(i) \geq v(j)$. In obvious cases we omit the indices of w and v.*

Definition 3.4 *Let $m \in \mathbb{N}$ and $B = (b(i))_{i=1}^{\infty}$, an ND-n.s. Put $b^{(m)}(i) = \min(b(i), m)$ and $B^{(m)} = (b^{(m)}(i))_{i=1}^{\infty}$. The sequence $B^{(m)}$ is called the m-dimensional limited sequence of B. Denote by $f_i(j)$ the j-th subsums of the i-dimensional limited sequence of B, i.e., put*

$$f_i(j) = \begin{cases} \sum_{k=1}^{j} b^{(i)}(k), & \textit{if } 1 \leq j, \\ 0, & \textit{if } j = 0 \end{cases}$$

The following lemma is from Das and his co-authors.

Lemma 3.1 *Let p and q be two points in a finite dimensional digital space $\mathbb{Z}^n$, and B a periodic n-dimensional n.s. with period l. The length of the minimal path from p to q determined by B, is given by the equation*

$$d(p, q; B) = \max_{i=1}^{n} d_i(p, q),$$

where

$$d_i(p, q) = l \left\lfloor \frac{a_i}{f_i(l)} \right\rfloor + h(z_i; B_i) = \sum_{j=1}^{l} \left\lfloor \frac{a_i + g_i(j)}{f_i(l)} \right\rfloor .$$

They used the following notation:

$$a_i = \sum_{j=1}^{n-i+1} v(j);$$

where v is the sorted absolute difference vector of p and q. The sequences $B_i = (b_i(1), \ldots, b_i(l))$ are given such that $\forall i, 1 \leq i \leq n$,

$$b_i(j) = \min(b(j), n - i + 1).$$

Furthermore, $z_i = a_i \mod f_i(l)$ and

$$h(z_i; B_i) = \min\{k | f_i(k) \geq z_i\},$$

that is

$$f_i(h(z_i; B_i) - 1) < z_i \leq f_i(h(z_i; B_i));$$

and

$$g_i(j) = f_i(l) - f_i(j - 1) - 1, \text{ for } 1 \leq j \leq l.$$

Finally, the function signed as $\lfloor y \rfloor$ is the floor function (i.e. the integer part of y).

For later use we need to introduce a further definition.

Definition 3.5 *Let $B = (b(i))_{i=1}^{\infty}$, an N dimensional neighbourhood sequence. The sequence $B(j) = (b(i))_{i=j}^{\infty}$ is called the j-shifted sequence of B.*

The following lemma (from [23]) is very useful if we would like to decide numerically whether a neighbourhood sequence is faster than another one or not.

Lemma 3.2 *Let B_1 and B_2 be two ND-n.s.-es. Then*

$$d(p, q; B_1) \leq d(p, q; B_2) \quad \text{for all } p, q \in \mathbb{Z}^N$$

if and only if

$$f_k^{(1)}(i) \geq f_k^{(2)}(i) \quad \text{for all } i, k \in \mathbb{N},\ 1 \leq k \leq N,$$

where $f_k^{(1)}(i)$ and $f_k^{(2)}(i)$ correspond to B_1 and B_2, respectively.

Remark 3.1 *As a simple consequence of the previous lemma, we obtain that if B_1 and B_2 are ∞D-n.s.-s with $B_1 \sqsupseteq^* B_2$, then for every $i \in \mathbb{N}$ among the first i elements of B_1 there are at least as many symbols ∞ as there are among the first i elements of B_2.*

Observe that the neighbourhood sequences have the following property.

Lemma 3.3 *For any ND-n.s. B and its limited sequences the following relations hold:*

$$B \sqsupseteq^* B^{(k)} \sqsupseteq^* B^{(j)},\ \text{for all } j \leq k \leq N.$$

Proof. It is trivial, by the definition of limited sequences. □

Remark 3.2 *If the sequence B is periodic, and it contains the element ∞, then B contains ∞ at infinitely many positions.*

3.2 The shortest path problem

The following algorithm provides one of the shortest B-paths between two arbitrary points in $\mathbb{Z}^N$, if such a path exists.

Algorithm 3.1
Input: An ND-n.s. $B = (b(i))_{i=1}^{\infty}$ and $p, q \in \mathbb{Z}^N$, such that $d(p, q; B) < \infty$.
step 1. *Let $w^{(0)}$ be the absolute difference of p and q, $t(i) = sgn(p(i) - q(i))$, $(1 \leq i \leq n)$, and put $j = 0$ and $\Pi = (p)$.*
step 2. *If $w^{(j)}(i) = 0$ for every i with $1 \leq i \leq n$ then goto step 8, else set $j = j + 1$.*
step 3. *Put $w^{(j)} = w^{(j-1)}$.*
step 4. *If $b(j)$ is finite, then select the largest $b(j)$ entries of $w^{(j)}$. If $b(j)$ is infinite, then select all the entries of $w^{(j)}$.*
step 5. *For each selected $w^{(j)}(i)$ with if $w^{(j)}(i) \neq 0$, let $w^{(j)}(i) = w^{(j-1)}(i) - 1$.*
step 6. *Append to the path Π the point x_j defined by $x_j(i) = q(i) + w^{(j)}(i)t(i)$ $(1 \leq i \leq n)$.*
step 7. *Goto 2.*
step 8. *Output Π as a minimal B-path between p and q, and j as the length of this path.*
End.

The following theorem is about the correctness of our algorithm. We use the term step as step of the algorithm, however some steps are complex.

Theorem 3.1 *Algorithm 3.1 terminates after finitely many steps and provides a B-path with minimal length between the points p and q if there is any path between them.*

Proof. Observe that for every $j \geq 0$, x_{j-1} and x_j are $b(j)$-neighbours. Moreover, by its definition, $w^{(j)}$ is the absolute difference of x_j and q. Let $p = y_0, y_1, y_2, ..., y_j, ...$ be any point sequence, where y_{j-1} and y_j are $b(j)$-neighbours for $j \geq 0$, and let $v^{(j)}$ be the absolute difference of y_j and q. We show that if $v^{(j)}$ is identically zero for some $j \geq 0$, then so is $w^{(j)}$. For this purpose, put for $t \geq 0$

$$||v^{(j)}||_t = \sum_{i=1}^{N} \max(v^{(j)}(i) - t, 0), \quad (j \geq 0),$$

and in particular

$$||w^{(j)}||_t = \sum_{i=1}^{N} \max(w^{(j)}(i) - t, 0), \quad (j \geq 0).$$

We even claim that for every $t \geq 0$ and $j \geq 0$ we have $||w^{(j)}||_t \leq ||v^{(j)}||_t$. We proceed by induction. For $j = 0$, as $x_0 = y_0 = p$, we certainly have $w^{(0)} = v^{(0)}$, whence $||w^{(0)}||_t = ||v^{(0)}||_t$ $(t \geq 0)$. Suppose, that for some $j \geq 0$ we have

$$||w^{(j)}||_t \leq ||v^{(j)}||_t \quad (t \geq 0).$$

We should prove that the same is valid with $j + 1$ in place of j. Suppose the contrary and choose an l such that

$$||w^{(j+1)}||_l > ||v^{(j+1)}||_l.$$

Now we distinguish three cases.

First, suppose that $||w^{(j+1)}||_l = \infty$. Then we also have $||w^{(j)}||_l = \infty$. Hence, by $||w^{(j)}||_l \leq ||v^{(j)}||_l$ we get $||v^{(j)}||_l = \infty$. Therefore by $||v^{(j+1)}||_l < \infty$, $b_j = \infty$ and $||v^{(j)}||_{l+1} < \infty$. Combining this with the induction hypothesis for $t = l + 1$, we get $||w^{(j)}||_{l+1} < \infty$. However, as $||w^{(j)}||_{l+1} < \infty$ and $b_j = \infty$, in view of Step 5 of the Algorithm $||w^{(j+1)}||_l < \infty$. This is a contradiction, so this case cannot hold.

Now, we assume that $||w^{(j+1)}||_l < \infty$, and also $||w^{(j)}||_l < \infty$. Then by $||v^{(j+1)}||_l < \infty$ we have $||v^{(j)}||_{l+1} < \infty$. If $||v^{(j)}||_l < \infty$ is also true, then combining this with the induction hypothesis for $t = l + 1$, we obtain

$$||v^{(j)}||_l - ||w^{(j)}||_l \geq (||v^{(j)}||_l - ||v^{(j)}||_{l+1}) - (||w^{(j)}||_l - ||w^{(j)}||_{l+1}).$$

Here, the right hand side equals the difference of the numbers of entries in the sequences v and w, respectively, which are larger than l. In the $(j + 1)$th step we can modify maximum these amounts of coordinate values in v and w, respectively. So, this inequality immediately implies $||w^{(j+1)}||_l \leq ||v^{(j+1)}||_l$, which is a contradiction. On the other hand, if $||v^{(j)}||_l = \infty$, then by $||v^{(j+1)}||_l < \infty$ we get $b_j = \infty$. In this case, in view of Step 5 of the Algorithm, we have $||w^{(j+1)}||_l = ||w^{(j)}||_{l+1}$. Certainly, $||v^{(j+1)}||_l \geq ||v^{(j)}||_{l+1}$ is also valid. Hence, by the induction hypothesis we get a contradiction, and our statement is proved in this case.

Finally, assume that $||w^{(j+1)}||_l < \infty$, but $||w^{(j)}||_l = \infty$. Then again $b_j = \infty$, and by the previous argument we get $||w^{(j+1)}||_l = ||w^{(j)}||_{l+1}$ and $||v^{(j+1)}||_l \geq ||v^{(j)}||_{l+1}$. By using the induction hypothesis with $t = l+1$ we get $||w^{(j+1)}||_l \leq ||v^{(j+1)}||_l$, which is a contradiction.

Thus, we proved that

$$||w^{(j)}||_t \leq ||v^{(j)}||_t, \text{ for every } j \geq 0, t \geq 0,$$

which implies that if $v = (0)_{i=1}^N$ then so is w. By our assumption $d(p,q;B) < \infty$, there is a minimal B-path $p = y_0, y_1, ..., y_m = q$ between p and q. Hence, $p = x_0, x_1, ..., x_j = q$ is also a minimal path, with $j = m$. Thus, the algorithm terminates after finitely many steps, and outputs a minimal B-path between p and q. □

We illustrate our algorithm on the following simple examples for the ∞ dimensional case.

Example 3.1 *Let $B = (3, \infty, 2, ...)$, a ∞D-n.s., and $p = (3, 2, 2, 2, 1, 1, 1, 1, ...)$ ($p(i) = 1$ for $i \geq 5$), and $q = (0)_{i=1}^{\infty}$.*
The algorithm provides the following path:

$$\begin{aligned}\Pi = (&(3,2,2,2,1,1,1,1,1,1,1,...) = p,\\ &(2,1,1,2,1,1,1,1,1,1,1,...),\\ &(1,0,0,1,0,0,0,0,0,0,0,...),\\ &(0,0,0,0,0,0,0,0,0,0,0,...) = q).\end{aligned}$$

Thus the B-distance of p and q is $d(p,q;B) = 3$.

Example 3.2 *Let $B = (3, \infty, 4, 1, \infty, 6, ...)$, a ∞D-n.s., $p = (1)_{i=1}^{\infty}$ and $q = (3, 6, 4, 4, 5, 2, -1, 0, 4, 3, 2, 2, 2...)$ ($q(i) = 2$ for $i \geq 11$).*
The algorithm provides the following path:

$$\begin{aligned}\Pi = (&(1,1,1,1,1,1,\quad 1,1,1,1,1,1,1,...) = p,\\ &(1,2,2,1,2,1,\quad 1,1,1,1,1,1,1,...),\\ &(2,3,3,2,3,2,\quad 0,0,2,2,2,2,2,...),\\ &(2,4,3,3,4,2,\quad 0,0,3,2,2,2,2,...),\\ &(2,5,3,3,4,2,\quad 0,0,3,2,2,2,2,...),\\ &(3,6,4,4,5,2,-1,0,4,3,2,2,2,...) = q).\end{aligned}$$

Thus the B-distance of p and q is $d(p,q;B) = 5$.

3.3 Functional form of the distance

As one can see (Lemma 3.1), the formula presented by Das *et al.* is very complex, and it strongly uses periodicity (l is the period of the neighbourhood sequence).

Our aim is to get a simpler formula which calculates distance with non-periodic neighbourhood sequences also. To get our main result, the hyperspheres of the digital spaces will help. Therefore in this section, first we describe the way a neighbourhood sequence spreads in the digital space starting from a point of $\mathbb{Z}^N$. We also investigate the ∞D hyperspheres in this section.

One can investigate the hyperspheres independently of their center using the absolute difference vector instead of the actual coordinate values of the points. Now we will refer to the point o as the origin of the hypersphere.

Let B be an ND-neighbourhood sequence. For $k \in \mathbb{N}$, let

$$O_k^B = \{p \in \mathbb{Z}^N \mid d(o, p; B) \leq k\}.$$

So O_k^B is the region occupied by B after k steps starting from o. We will call it the (digital) hypersphere with radius k using the ND neighbourhood sequence B. If it is obvious, we will omit B and simply use O_k.

The following remarks are the extensions of the former results in [31], summarizing some simple observations about the geometric behavior of hyperspheres. They are true in the infinite dimensional space as well.

Remark 3.3

- *In N dimensional digital space for any $k \in \mathbb{N}$, the hypershere O_k^B does not depend on the order of the first k elements of B; and*
- *The sequence of regions $(O_k^B)_{k=1}^{\infty}$ is a strictly monotone increasing sequence. That is, $l > k$ implies $O_l^B \supsetneq O_k^B$.*

We present the hypervoxels of the hyperspheres. This procedure gives the points which are at the maximal possible Euclidean distance from the origin among the points which are in the same B-distance from o. (We used this construction in a proof in [48a].)

Theorem 3.2 *Let B be an ND-n.s. and k be a natural number. The vertices of O_k^B are exactly those points whose coordinates are the permutation of the values of*

$$x = \left(\delta_l \sum_{j=1}^{k} \sigma_{b(j),l} \right)_{l=1}^{N}$$

where $\sigma_{a,b} = 1$ if $a \geq b$, and $\sigma_{a,b} = 0$ in other cases, $\delta_l = \pm 1$ for every $l \in \mathbb{N}, 1 \leq l \leq N$, and δ_l can be different in different permutations.

Proof. It is a consequence of the main theorem of [31] in finite dimensions. In the infinite dimensional space the same technical calculation works. $\square$

Remark 3.4 *In the infinite dimensional space, the sequence $|x|$ with the values $\left(x_l = \sum_{j=1}^{k} \sigma_{b(j),l} \right)_{l=1}^{\infty}$ is convergent for each finite value of k. In case the limit $k \to \infty$, the sequence $|x|$ can be divergent, for example when B is given by the form $b(i) = i$.*

The vertices of our digital hyperspheres fully determine the polyhedron. Knowing that the points of the polyhedron have distances from the origin at most k, the set of the points which are exactly at distance k from the origin are the following points: $O_k \setminus O_{k-1}$ (the difference of the sets of two neighbour digital hyperspheres).

In ∞D, it is possible that the distance of two points is not finite, because there is no B-path between them for a given n.s. B. The following lemma (from [48]) gives the necessary and sufficient condition for the B-distance to be finite in ∞D.

Lemma 3.4 *The finite path exists (i.e. the B-distance is finite) between two points if and only if*

- *the difference vector of the points (w) has a largest element (i.e. the multiset of the values $\{w(i)\}$ is bounded), and*
- *the sequence B contains the symbol ∞, at least k times, where k is the largest number in the difference vector (w) of the points, which occurs infinitely many times.*

The previous lemma is useful to decide whether the algorithm given in the previous section can provide a shortest path, or there are no paths. (It recommended to check this property before applying the algorithm, because it cannot terminate if no paths exist.)

And now we are ready to present the functional form of the distance of any two points in nD or in ∞D defined by a given neighbourhood sequence. To formalize our main result we need only the concept of limited neighbourhood sequences (Definition 3.4). This concept has a very natural meaning. If two points p and q differ only in number k coordinate values in N dimensions (for all the other coordinates $p(i) = q(i)$), then to construct a shortest path between them by the algorithm (see the previous section), we do not change more values than k in a step. Therefore their B-distance and $B^{(m)}$-distance are the same if $m \geq k$.

Now we are in a position to state our main proposition about the B-distance when it is finite.

Proposition 3.1 *The B-distance of any two points p and q is given by*

$$d(p, q; B) = \max_{i<N+1} (d_i(w)),$$

where

$$d_i(w) = \max\left\{ h \,\middle|\, \sum_{j=1}^{i} v(j) > \sum_{j=1}^{h-1} b^{(i)}(j) \right\}.$$

For proving that the formula above is correct, we need the following lemmas.

Lemma 3.5 *For the vertices of polyhedron O_k the values $d_i(w)$ in Proposition 3.1 are equal for all i.*

Proof. For a vertex x of O_k the values $v(j)$ are in the next form: $v(j) = \sum_{l=1}^{k} \sigma_{b(l),j}$. Then $\sum_{j=1}^{i} \sum_{l=1}^{k} \sigma_{b(l),j} = \sum_{j=1}^{k} b^{(i)}(j)$, since the lefthandside summarizes exactly the values $b^{(i)}(j)$

up to k. So for all i the values $d_i(w) = k$ and the proof is finished. $\square$

Lemma 3.6 *If $d_i(w_{p,q}) \leq k$ for all i, then q is in O_k with origin p.*

Proof. The lemma is a simple consequence of Lemma 9 in [31] with technical calculations. $\square$

As an obvious consequence of the previous lemma, the B-distance of the points p and q above is not greater than k:

Corollary 3.1 *If $d_i(w; B) \leq k$ for all possible values of i, then $d(w; B) \leq k$.*

Now we are ready to prove that the formula in Proposition 3.1 is valid, as the next theorem states:

Theorem 3.3 *The B-distance of points p and q is*

$$d(p, q; B) = \max_{i<N+1} (d_i(w)),$$

where

$$d_i(w) = \max \left\{ h \left| \sum_{j=1}^{i} v(j) > \sum_{j=1}^{h-1} b^{(i)}(j) \right. \right\}.$$

Proof. Assume that the formula in Proposition 3.1 gives k for the distance. We will show that in this case w is in C_k, but it is not in C_{k-1}. From Lemma 3.6 it is clear that in this case $w \in C_k$. To prove the other part, we assume that for the value m, $d_m(w) = k$. (Such a natural number m must exist.) It is clear with the following definition of q', that the B-distance of p and q' is not greater than the B-distance of p and q:

$$q'(i) = \begin{cases} q(i), & \text{if } w(i) \text{ is among the } m \text{ highest value of the multiset } w; \\ p(i), & \text{in other cases.} \end{cases}$$

Then p and q' differ at most m coordinate values; therefore their B-distance equals their $B^{(m)}$-distance. Using $B^{(m)}$ to go from p to q' in the first $k-1$ steps, we can change $\sum\limits_{j=1}^{k-1} b^{(m)}(j)$ coordinate values. With these steps we cannot reach q' because $d_m(w) = k$; therefore $\sum\limits_{j=1}^{m} v(j) > \sum\limits_{j=1}^{k-1} b^{(m)}(j)$, and $\sum\limits_{j=1}^{m} v(j) = \sum\limits_{j=1}^{N} w_{p,q'}(j)$. So the $B^{(m)}$-distance of p and q' is greater than $k-1$. So $d(p, q; B) = k$ which was to be proved. $\square$

Remark 3.5 *It is only a simple calculation to check that our formula gives the same result as Das' formula (see Lemma 3.1) in the case of periodic sequences in finite dimensions.*

Using the formula given above one can understand the form of the Lemma 3.2.

In practice the 2 and 3 dimensional cases are usual; therefore, we calculate the distances for these special digital spaces.

Corollary 3.2 *In $\mathbb{Z}^2$ the points p and q are in B-distance*

$$d(p,q;B) = \max\left\{v(1), \max\left\{i \,\middle|\, v(1)+v(2) > \sum_{j=1}^{i-1} b(j)\right\}\right\}$$

with a 2-dimensional neighbourhood sequence B.

Corollary 3.3 *Let B be a 3D-n.s. and p, q be two points in the three dimensional space. Then*

$$d(p,q;B) = \max\left\{v(1), d_2, d_3\right\},$$

where

$$d_2 = \max\left\{i \,\middle|\, v(1)+v(2) > \sum_{j=1}^{i-1} b^{(2)}(j)\right\}, \text{ and}$$

$$d_3 = \max\left\{i \,\middle|\, v(1)+v(2)+v(3) > \sum_{j=1}^{i-1} b(j)\right\}.$$

For the sake of interest, let us see the formula in ∞D.

Corollary 3.4 *Let B be an ∞D-n.s. It is easy to check that the B-distance of p and q (two two points in the infinite dimensional digital space) can be written in the following form. Let*

$$d_1(w) = \sup_{i\in\mathbb{N}}\{w(i)\},$$

and $m \geq 0$ be the greatest value among the values $w(i)$ which occurs infinitely many times in the multiset of $\{w(i)\}$. Define k_m as the place of the m-th value ∞ in the n.s. B, i.e.:

$$k_m = \inf\{i| \text{ there are at least number } m \text{ values } \infty \text{ among the first } i \text{ elements of } B\}.$$

Then

$$d(p,q;B) = \max\left\{d_1(w), k_m, d'(w';B')\right\},$$

where $w' = (\max(0, w(i) - m)_{i=1}^{\infty}$ and B' is the n.s. which resulted from deleting the first m occurences of the ∞ symbol of B, and finally, $d'(w';B') = d(w';B') + m$.

3.4 Condition for a distance to be a metric

The distance based on an arbitrary neighbourhood sequence in general does not satisfy the conditions of a metric, as the following example shows.

Example 3.3 *The neighborhood sequence $B = (2,1)$ does not generate a metric. Let $p = (0,0,...)$, $q = (1,1,...)$, $r = (2,2,...)$, and $p(i) = q(i) = r(i)$ for $i \geq 3$. In this case the distances are the following: $d(p,q;B) = 1$, $d(q,r;B) = 1$ and $d(p,r;B) = 3$. Hence part 3) of Definition 2.1 does not hold for this B-distance.*

However, in geometry those distances are more useful which have metric properties. In this section we give a necessary and sufficient condition for a distance based on a neighbourhood sequence to be a metric.

Lemma 3.7 *Let p and q be arbitrary points in $\mathbb{Z}^\infty$, and let the ∞D-n.s. B_1 be faster than the ∞D-n.s. B_2. If there is no B_1-path between p and q, then there is no B_2-path between them either.*

Proof. From Definition 3.2 and Lemma 3.2, if $d(p,q;B_1) = \infty$ and B_1 is faster than B_2, then $d(p,q;B_2) = \infty$. $\square$

The next theorem is the extension of the result of Das *et al.* [12], concerning periodic neighbourhood sequences in the finite dimension, to the general case. To prove their result, the authors in [12] introduced relatively complicated geometric notions, such as the wave-front of a neighbourhood sequence, etc. In contrast, to formulate and prove our result, we need only the simple concepts of the faster relation and the shifted sequence.

Theorem 3.4 *The distance function based on an ND-n.s. B is a metric on $\mathbb{Z}^N$ if and only if $B(i)$ is faster than B for all $i \in \mathbb{N}$.*

Proof. First we prove sufficiency. The validity of properties 1) and 2) of Definition 2.1 is trivial; it can be seen, e.g. following the shortest path algorithm in Section 3.2. Indeed, the distance $d(p,q;B)$ depends only on the absolute difference w of p and q, and on B. As the definition of w is symmetric in p and q, $d(p,q;B) = d(q,p;B)$. It is clear that the distance is zero if and only if the absolute difference of the points has only zero elements, i.e. if the points are the same. Otherwise the distance is a positive integer or infinite. In square grids all distances generated by neighbourhood sequences satisfy these two properties. Hence, it is enough to deal with the triangular inequality, i.e. property 3).

Now we prove that property 3) is true if and only if $B(i)$ is faster than B for every $i \in \mathbb{N}$. Let $p,q,r \in \mathbb{Z}^N$, such that their distances are finite. Then we can find a B-path Π between p and r which is a concatenation of a minimal B-path between p and q, and a minimal $B(i)$-path between q and r, where $i = d(p,q;B) + 1$, and $B(i)$ is the i-shifted sequence of B. Hence

$$|\Pi| = d(p,q;B) + d(q,r;B(i)).$$

The assumption that $B(i)$ is faster than B means that

$$d(p,r;B(i)) \leq d(p,r;B).$$

Thus

$$|\Pi| \leq d(p,q;B) + d(q,r;B).$$

By the definition of the B-distance we have

$$d(p,r;B) \leq |\Pi|,$$

from whence we get

$$d(p,r;B) \leq d(p,q;B) + d(q,r;B).$$

Now, suppose that not all the distances are finite between p, q and r. If $d(p,q;B) = \infty$ or $d(q,r;B) = \infty$ then 3) is trivially valid. Assume that $d(p,r;B) = \infty$, but $d(p,q;B) = s < \infty$. If there were a $B(s)$-path between q and r, then there would also be a B-path between p and r. (We could concatenate a shortest B-path between p and q, with length s, and a $B(s)$-path between q and r.) As the shifted sequence $B(s)$ is faster than B, by Lemma 3.7 there is no B-path between q and r. So $d(q,r;B) = \infty$, and 3) is valid in this case, too.

Now we prove necessity. Assume that for some $j \in \mathbb{N}$, $B(j)$ is not faster than B, but $d(p,q;B)$ has property 3). In this case by Definition 3.2 there exist $p, q \in \mathbb{Z}^N$ such that $d(p,q;B(j)) = k$, and $d(p,q;B) < k$. The B-distance of two points depends only on their absolute difference, so we may assume that the coordinate values of p are non-negative, and q is the origin: $p(i) \geq 0$ and $q(i) = 0$, for all $1 \leq i \leq n$. We need a corner of a hypersphere with radius j, so we define the point $r \in \mathbb{Z}^N$ in the following way:

$$r(i) = -|\{b(l) : l \leq j \text{ and } b(l) \geq i\}| \quad \text{for all } i \ (1 \leq i \leq N).$$

By our algorithm it is easy to see that $d(q,r;B) = j$ and q is an element of one of the shortest paths between p and r. Then

$$d(p,r;B) = d(q,r;B) + d(p,q;B(j)) = j + k,$$

as a shortest B-path between p and r can be obtained as a concatenation of a shortest $B(j)$-path from p to q and a shortest B-path from q to r. Thus

$$d(p,q;B) + d(q,r;B) < k + j = d(p,r;B).$$

But we assumed that $d(p,q;B)$ has property 3). This is a contradiction, and the proof is complete. □

Remark 3.6 *By Lemma 3.2 one can decide, at least in principle, whether an ND-n.s. defines a metric or not.*

Note, that in 2-dimensional digital space those n.s.-es define metrics which correspond to the Lyndon-words. All words above the alphabet $\{1, 2\}$ which is not a real power of a shorter word has exactly 1 cyclic permutation which is a Lyndon word [40]. So in our concept each periodic n.s. has a shifted n.s. which generates a metric, moreover this shifted n.s. is a minimum in a certain sense. A periodic neighbourhood sequence defines a metric distance if it is the minimal (by lexical order) among its cyclic permutations (i.e. shifted sequences). The definition of Lyndon-words can be extended to the non-periodic case as well.

Triangular grids

The square- and the cube-grids as well as their higher dimensional forms are widely used. In the plane there are two other regular grids: the hexagonal and the triangular ones.

The dual of the hexagonal grid is the triangular grid (see Fig. 2). The grid of triangular nodes is isomorphic to the grid of hexagonal areas. We investigate these – so-called hexagonal – grids in Chapter 4. The so-called triangular grid is the triangular grid of areas which isomorphic to the grid of nodes of the hexagonal grid. The triangular grid is discussed in Chapter 5.

We will use three coordinate values to refer to the points of the hexagonal and triangular grid. We also will present the relationship between $\mathbb{Z}^3$ in which we have three independent coordinate values and hexagonal or triangular grids (in which we use three coordinates, but they are not independent). As we will show the hexagonal grid is a subplane of $\mathbb{Z}^3$. Opposite to the hexagonal the triangular grid is two parallel planes in $\mathbb{Z}^3$. We can call these grids as one and two-plane triangular grids. Extending this definition we get the n-plane triangular grids, we will return to them in Section 6.2.1.

These grids are more exotic from the view of practice. As we will describe later, the hexagonal grid is simple and in many cases it has better properties than the square grid. The triangular grid is more complex. It has some new properties which are not usual in the digital geometry. These interesting properties seem to be useful in practice. Let us look at first the hexagonal grid, and after this at the triangular grid. They are treated in the next two chapters.

Chapter 4 : The hexagonal grid

In this section we investigate the hexagonal grid. We overview some previous results, mainly the techniques that we will use for the triangular grid.

In [42] the distance based on this neighbourhood relation was calculated with two independent coordinate values. Because of the symmetric properties of the grid, in [37] the description uses three coordinates which have zero sum. Following Her's way we use 3 coordinate values to preserve the symmetry of the grid. In many applications the hexagonal grid seems to be more useful than the square grid (see for example [19]).

As we mentioned earlier, the grids of hexagonal regions and triangular nodes are dual of each other. We usually use the term hexagonal grid instead of triangular grid of nodes. This grid is one kind of triangular grid (because its symmetric properties, for example rotating by $\frac{2\pi}{3}$).

4.1 Basic definitions, coordinates

In digital image processing the neighbourhood criterion, illustrated in Figure 3, is used almost every time, since this is the most natural for a human observer. The figure shows the neighbourhood relations in both interpretations of the hexagonal grid.

By the formal definition, two objects are neighbours
- in the grid of triangular nodes: if there is a direct connection between these nodes,
- in the grid of hexagonal areas: if these hexagons have a common side.

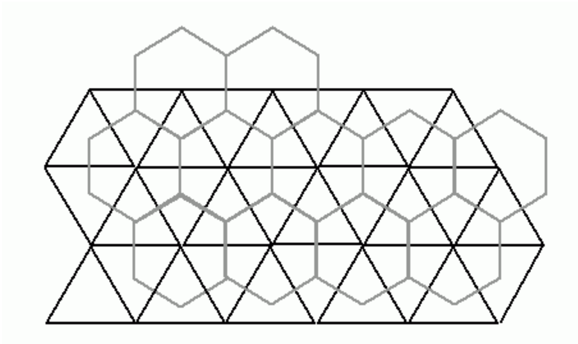


Fig. 2: The connection between the triangular and the hexagonal grids

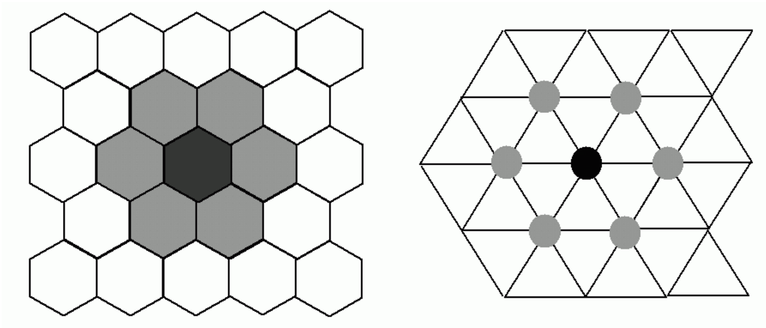


Fig. 3: Neighbours in the hexagonal grid of areas and in the triangular grid of nodes

One can easily check that every object has 6 neighbours. The usually used hexagonal distance (in some literature referred as 'honeycomb distance') is based on this neighbourhood criterion at each step.

We introduce the coordinate system in the same way, as it was given in [37] for nodes of the triangular grid. We assign 3 coordinate values to the objects. The necessity of this procedure is to be able to handle mathematically this hexagonal structure, for example we can calculate numerically the distance of two objects.

The following procedure helps us to make this assigning in the grid of hexagonal areas.

Procedure 4.1 *Choose a point for the origin, whose coordinate values are* $(0, 0, 0)$*. Then fix three coordinate axes as lines crossing the centre of the origin which are orthogonal to two sides of it. The directions of the axes* x, y *and* z *are taken as* $0, \frac{2\pi}{3}$*, and* $\frac{4\pi}{3}$*, respectively. We assign the coordinate values to the points inductively. If the coordinates of a point are* (a_1, a_2, a_3)*, then let the coordinates of its neighbour in the direction* x *be* $(a_1 + 1, a_2, a_3 - 1)$*, and in the opposite direction* $(a_1 - 1, a_2, a_3 + 1)$*. Similarly in the direction of* y $(a_1 - 1, a_2 + 1, a_3)$*, and in the opposite direction of* y*, it is* $(a_1 + 1, a_2 - 1, a_3)$*. According to the way the coordinates are introduced, moving in the direction* z*, the coordinate values are* $(a_1, a_2 - 1, a_3 + 1)$*, and in the opposite direction* $(a_1, a_2 + 1, a_3 - 1)$*.*

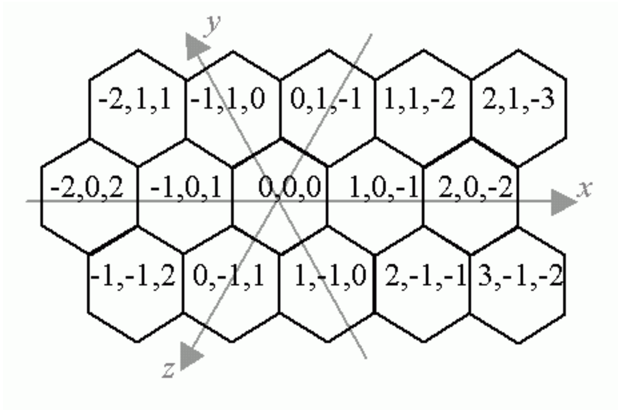


Fig. 4: Coordinate values on the hexagonal grid

Figure 4 shows a part of the result of this procedure. We use 3 coordinate values to reflect the geometrical symmetry of the grid.

Remark 4.1 *The sum of the coordinate values of every point is zero. So each point has only two independent coordinate values. Two points are neighbours if one of their coordinate value is the same and the differences of the other two corresponding coordinate values are* 1 *and* -1*, respectively.*

Remark 4.2 *In [42] the authors used only 2 coordinates which are independent. We use three values, with zero sum. So our third coordinate value can be easily calculated from the other two values.*

Since we consider only one type of neighbourhood relation, the concept of neighbourhood sequences is identical. We introduce only one hexagonal distance, based on the neighbourhood criterion at each step.

Definition 4.1 *A sequence of objects for which a coordinate value remains constant forms a lane.*

Figure 5 shows some lanes. Observe that a line including hexagons only of a lane is parallel to one of the coordinate axes.

4.2 Shortest path and distance

In [42] a formula was given to calculate the distance of two points of the hexagonal grid represented by two independent coordinates. Now we give an algorithm which produces a shortest path between two arbitrary points of the hexagonal grid. Our procedure is natural, and uses our definition of lanes.

Procedure 4.2 *Let us consider two points, and take the absolute values of the differences between their corresponding coordinates. If they have equal corresponding coordinates, then we go along the lane for which this coordinate value remains constant.*

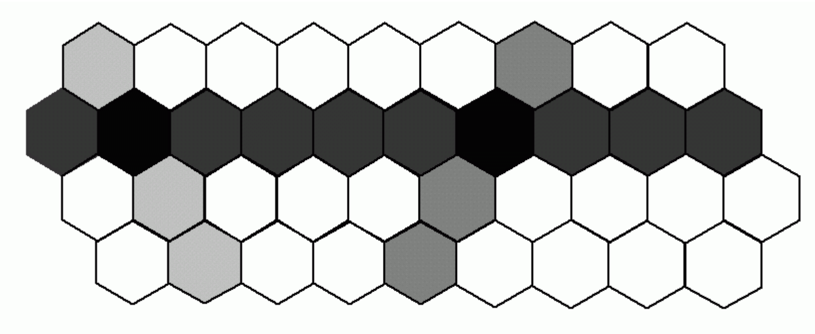


Fig. 5: Examples for lanes on the hexagonal grid

If all corresponding coordinate values are different, then there are two, whose absolute differences are smaller, than the third.
(If the third, the greatest, is equal to another, then since the differences have zero sum, there must be coinciding corresponding coordinate values.)
So in this case we go along those two lanes, where the coordinates, which have smaller absolute differences, remain constant:
First, we go along the lane where one of the two smaller values is constant, until the other coordinate reaches its destination value. Then we take the lane, where the already fixed coordinate remains constant, until we reach the destination point.

The Procedure 4.2 shows an algorithm which can solve the minimal path problem in case of hexagonal grid. Using this algorithm we can easily calculate the length of the shortest path. The path-length is the largest value, from the absolute values of the difference of the corresponding coordinates. Formally, we have the following theorem.

Theorem 4.1 *Let $p = (p(1), p(2), p(3))$ and $q = (q(1), q(2), q(3))$ be two points. Their distance is:*

$$d(p,q) = max(|p(1) - q(1)|, |p(2) - q(2)|, |p(3) - q(3)|).$$

Proof. This theorem is equivalent to the main proposition of [42]. $\square$

Moreover, this distance has nice properties.

Theorem 4.2 *The distance defined above is a metric.*

Proof. By Theorem 4.1, $d(p,q) \geqslant 0$, and $d(p,q) = 0$ if and only if $p = q$. The fact $d(p,q) = d(q,p)$ is trivial. The triangular inequality follows from:

$$\max(|p(1)-q(1)|, |p(2)-q(2)|, |p(3)-q(3)|)+\max(|q(1)-r(1)|, |q(2)-r(2)|, |q(3)-r(3)|) \geqslant$$

$$\geqslant \max(|p(1)-q(1)|+|q(1)-r(1)|, |p(2)-q(2)|+|q(2)-r(2)|, |p(3)-q(3)|+|q(3)-r(3)|) \geqslant$$

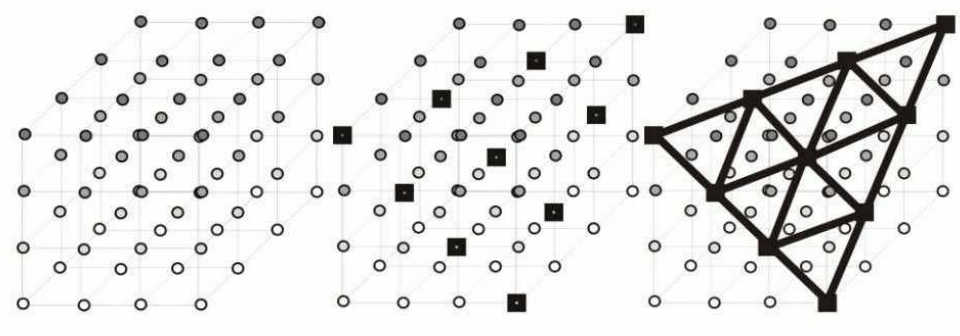


Fig. 6: The hexagonal plane as part of $\mathbb{Z}^3$

$$\geqslant \max(|p(1) - r(1)|, |p(2) - r(2)|, |p(3) - r(3)|). \qquad \square$$

The distance introduced above is close to the Euclidean distance. If the Euclidean length of a side of the hexagon is 1, and the hexagonal distance of two points is k, then the Euclidean distance of the center of these hexagons is between $1.5k$ and $\sqrt{3}k$.

4.3 Connection between the hexagonal and the cubic grids

Let us see which points of $\mathbb{Z}^3$ have coordinate values like the hexagonal grid. We use the triangular grid of nodes (which is equivalent to the hexagonal grid of regions.)
Figure 6 shows the points of $\mathbb{Z}^3$, and sign those ones which have zero-sum coordinate values. As we can see the triangular plane of nodes is simply an oblique plane in $\mathbb{Z}^3$. Due to this fact in [37] Her transferred some geometric properties to hexagonal plane from $\mathbb{Z}^3$.

Since, the points having zero-sum coordinate values are in a plane in $\mathbb{Z}^3$, we can rename the hexagonal grid to one-plane triangular grid.

It is easy to show that a lane is a line in $\mathbb{Z}^3$ which is the meeting of a plane in which a coordinate value is constant and the hexagonal plane.

Remark 4.3 *The neighbourhood criterion in the hexagonal plane is according to the 2-neighbours in $\mathbb{Z}^3$.*

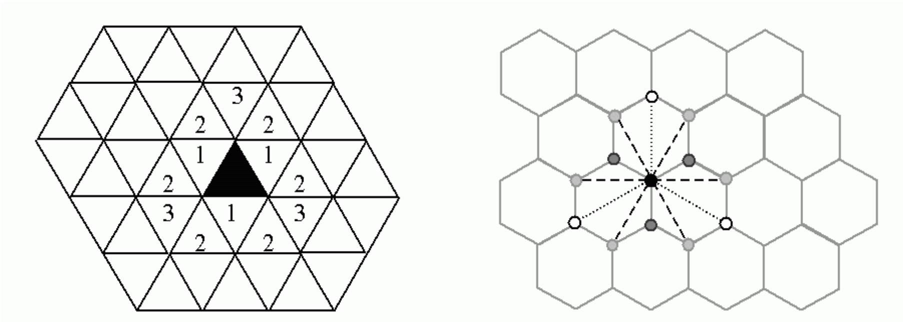


Fig. 7: Types of neighbours in the triangular grid of areas and in the hexagonal grid of nodes

Chapter 5 : The triangular grid

The grid of the triangular areas (the so-called triangular grid) is isomorphic to the grid of hexagonal nodes. They are also well known. In this chapter, we mostly use the triangular regions. This grid is completely different from the previous hexagonal grid; they have very different properties as we show in the next part.

We introduced the concept of neighbourhood sequences to this grid as we will explain it in this chapter. The triangular grid also has triangular symmetry, therefore three coordinates are recommended to analyze this system. There are three kinds of neighbours of each point in this grid. We present a suitable method to formulate the concept of neighbourhoods and neighbourhood sequences in the triangular grid. We use some analogies between the hexagonal and the triangular grids. In Section 5.2 we give an algorithm which finds a shortest path between two arbitrary points. The distances based on neighbourhood sequences can have some interesting properties. We analyse their properties in Section 5.3. In the next two sections we embed the points of the triangular grid in the cube grid and we present a formula to calculate the distance of points. Section 5.6 is about the digital circles, some properties are shown in which the circles of the triangular grid differ from the circles of the square grid. Finally, a possible application is shown, in which we can use some strange properties of these distances.

5.1 Neighbourhoods, coordinates and basic definitions

In this part we present most of the formal definitions we need to describe distances in the triangular grid.

The neighbourhood relations in triangular grid is based on the widely used relations (see [19]), we use three types of neighbours as Fig. 7 shows.

Each triangle (not considering the original one) has three 1-neighbours, nine 2-neighbours (the 1-neighbours, and six more 2-neighbours), and twelve 3-neighbours (nine 2-neighbours, and three more 3-neighbours). In Figure 7, for hexagonal grid of nodes we use the dark grey points to represent the 1-neighbours. With these points the light grey ones are the 2-neighbours, and with them the white points are the 3-neighbours. (Only the 1-neighbours are directly connected by a side, the 2 and 3-neighbours are at the positions of diagonals, respectively.)

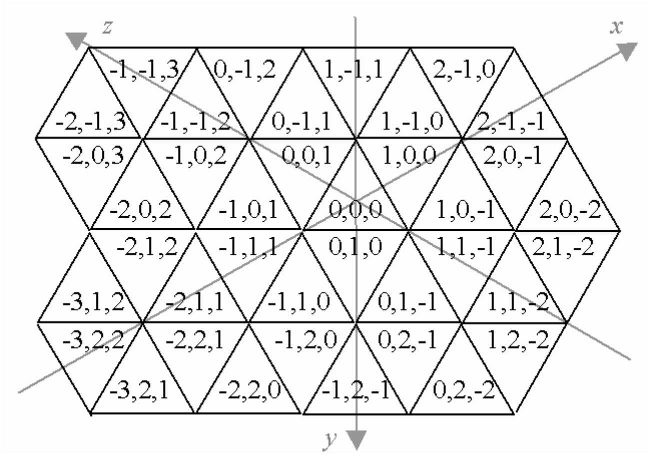


Fig. 8: Coordinate values on the triangular grid

These relations are reflexive (i.e., the pixel marked dark triangle is a 1-, 2-, and 3-neighbour of itself). In addition, all 1-neighbours of a pixel are its 2-neighbours and all 2-neighbours are 3-neighbours, as well (i.e., increasing and inclusion properties).

To describe mathematically the triangular grid we need an appropriate coordinatization. For similar reasons as in the case of the hexagonal grid, we use three coordinate values to represent the triangles. The next procedure shows how we will order the coordinate values to the triangles.

Procedure 5.1 *Choose a point for the origin, whose coordinate values are* $(0, 0, 0)$*. Take the three lines through the centre of the origin triangle, which are orthogonal to its sides. Fix these lines as the coordinate axes* x, y *and* z*, as Figure 8 shows. We assign the coordinate values to the points inductively. Let the coordinate values of a triangle* p *be known. Consider a triangle* q *which has not coordinate values yet and has a common side with* p*. This common side is orthogonal to one of the coordinate axes. According to the direction of this axis, we increase or decrease the corresponding coordinate value of* p *by 1 to get the corresponding coordinate of* q*. The other two values of* p *and* q *are equal.*

Fig. 8 shows a part of the triangular grid with the associated coordinate values.

Due to the coordinate values we can redefine the neighbourhood relations.

Definition 5.1 *The points* p *and* q *of the triangular grid are* m*-neighbours* ($m = 1, 2, 3$)*, if the following two conditions hold:*

1. $|p(i) - q(i)| \leqslant 1$*, for* $i = 1, 2, 3$*,*
2. $|p(1) - q(1)| + |p(2) - q(2)| + |p(3) - q(3)| \leqslant m$*.*

Remember that we use the term strict m*-neighbours if the second condition is equality.*

Remark 5.1 *It is easy to check that the formal definition above with the presented coordinate values (Fig. 8) gives the neighbourhood relations shown in Fig. 7.*

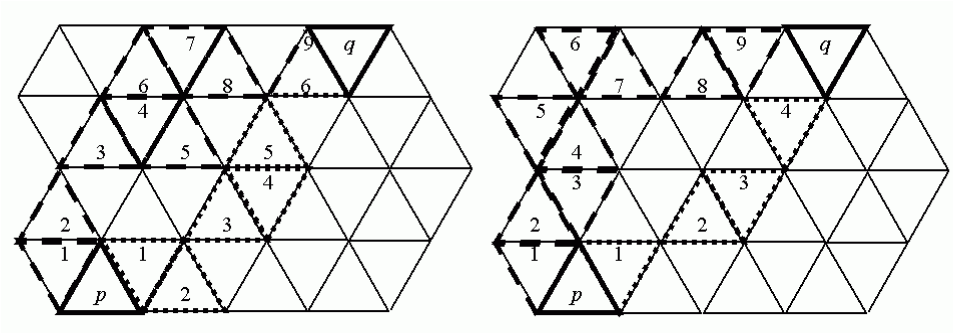


Fig. 9: Paths of different lengths from p to q using B_1 and B_2

In this grid we can define several types of distances according to the neighbourhood criteria used. Since there are 3 types of neighbourhoods in this grid, we will use the neighbourhood sequences containing numbers of the set $\{1, 2, 3\}$.

Now, we show some paths defined by neighbourhood sequences in the triangular grid.

In Figure 9 there are some paths given between the points $p = (-3, 2, 1)$ and $q = (2, -1, 0)$ by the help of $B_1 = (1, 1, 2)$ and $B_2 = (1, 3, 1, 2, 2, 2, 2, 2, 2, 2, 2...)$. As we can see, there are paths with different lengths between the points. On the left-hand side of Figure 9 we show two paths with the neighbourhood sequence B_1, one of them has length 10 and the other has length 7. The points of the paths are represented as dashed and dotted shapes. The numbers in the triangles refer to the steps of the given path. Similarly, in the right-hand side of the figure, we show other paths, using B_2, of length 5 (it is the shortest path) and 10 between the same points.

We can define the lanes similarly as in the hexagonal case.

Definition 5.2 *The points having the same value as x, y, or z-coordinate, form a lane.*

Remark 5.2 *Each lane is orthogonal to one of the coordinate axis. For the points of a lane a coordinate value is fixed. The other two values changes by ± 1.*

Remark 5.3 *The points and their coordinate values are assigned by a one-to-one mapping. We can see this in the following way using the concept of lanes. Let us fix two coordinate values. They define two non-parallel lanes, whose intersection contains two points. The third coordinate of each of them should have a value such that the sum of the coordinates are* 0 *and* 1*, respectively. Moreover the points of the triangular plane are exactly the points with three integer coordinate values with sum of coordinate value* 0 *or* 1.

There are two types of points according to the values of the sum of its coordinates. To distinguish them we define the parity of the points.

Definition 5.3 *If the sum of the coordinate values of the point p is* 0*, then we call the point p even. If the sum is* 1*, then the point p has odd parity.*

In our figures the even and odd triangles are of the shape $\triangle$ and ∇, respectively. The points of the grid have the following properties.

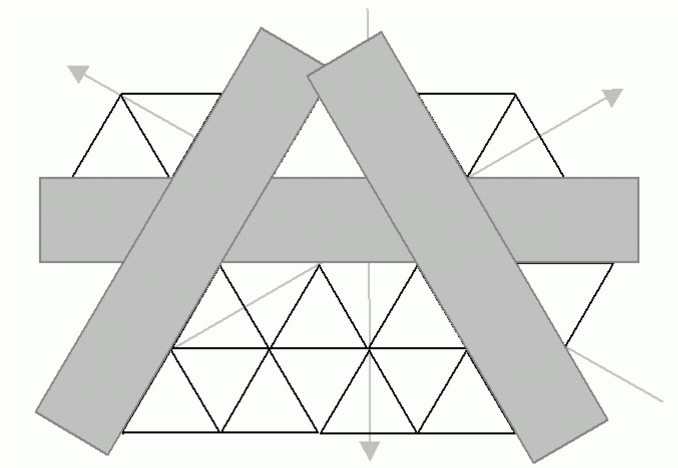


Fig. 10: Examples for lanes in the triangular grid

Remark 5.4 *If two points are* 1*-neighbours, then there exist two lanes containing both of them, and their parities are different. If two points are strict* 2*-neighbours then only one lane contains both of them and their parities are the same. If two points are strict* 3*-neighbours then their parities are different and no lane contains both of them.*

We can define distance between points and lanes, which depends only on these objects and independent of the neighbourhood sequences. The distance of a point and a lane is the minimal number of lanes which we have to go through to reach the point from the lane. We can calculate this distance with iteration.

Procedure 5.2 *The distance between the lane A and the point p is zero if p is on the lane A. The next parallel lanes of the lane which contains the point p have distance* 1 *from p. Let the distance of the lane A' and p be d. Then the next parallel lane with A', – which does not have distance $d-1$ from the point p – has distance $d+1$ from p.*

The distance between a lane and a point, with respect to the distance of points, has the following property.

Remark 5.5 *If the distance between the lane A and the point p is d, then starting from p we can reach some points of A in the d-th step by using the constant neighbourhood sequence* (2)*.*

Definition 5.4 *The difference $w_{p,q} = (w(1), w(2), w(3))$ of two points p and q is defined by: $w(i) = q(i) - p(i)$. If $w(1) + w(2) + w(3) = 0$, then the parity of w is even, else it is odd. Let $v_{p,q}$ be the sorted difference of the points, i.e. we order the values of $w_{p,q}$ by non-decreasing way by their absolute values. Formally:*

$$v(1) = \left\{ w(i) \,\middle|\, \max_i (|w(i)|) \right\}, v(3) = \left\{ w(i) \,\middle|\, \min_i (|w(i)|) \right\}$$

and $v(2)$ is the third value among $w(i)$. In obvious cases we omit the indices of $w_{p,q}$ and $v_{p,q}$.

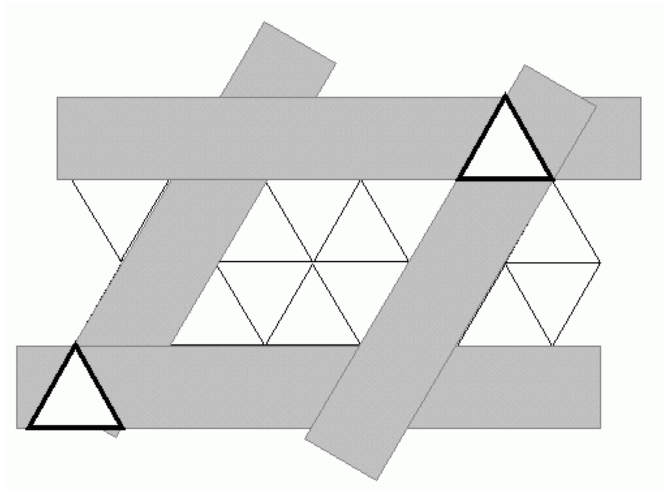


Fig. 11: An example for a parallelogram between two points

The distance using the neighbourhood sequence (1) is a special one, for using it we introduce the abbreviation as follows.

Notion 5.1 *The distance of any two points with the neighbourhood sequence* (1) *(i.e. in which every element is* 1*) is called* 1*-distance.*

As in Procedure 4.2 in the hexagonal case, we can get from a triangle to any other one by the neighbourhood sequence (1), using one or two lanes. In the latter case, the angle of the directions of the motion on these lanes can be chosen to be $\frac{2\pi}{3}$. (In one of these lanes that coordinate value remains constant, which has the smallest absolute value in the difference w of the points. In the other one that coordinate value is fixed which has the second largest absolute value in w.) Generally, we obtain a parallelogram, see Figure 11.

5.2 The shortest paths

In the triangular grid, we have some difficulties in changing the coordinate values. Such difficulties do not occur in case of the hexagonal and square grid. In the triangular grid, when moving from a point to one of its neighbours, we have to take care of the parity of these points. Namely, we have to change the coordinates of a point in such a way that the sum of the coordinate values must be 0 or 1.

Now we give an algorithm which solves the problem of constructing a shortest path between two given points. We prove that the algorithm is correct, i.e. it finds a shortest path from the first point to the other one.

Algorithm 5.1
Input: two points p, q; a neighbourhood sequence B.
step 1. *Let w be the difference of p and q, and let $x_0 = p$, $\Pi = (x_0)$ and $j = 0$.*
step 2. *If $w(i) = 0$ $(i = 1, 2, 3)$, then go to step 11.*

step 3. *Let $j = j + 1$. Let h_i $(i = 1, 2, 3)$ be a permutation of $(1, 2, 3)$, such that $|w(h_1)| \geqslant |w(h_2)| \geqslant |w(h_3)|$, and $sgn(w(h_1)) \neq sgn(w(h_2))$.*
step 4. *If $b(j) = 1$, then if x_{j-1} is even/odd, change by 1 the positive/negative one from $w(h_1)$ and $w(h_2)$, respectively:*
$w(h_i) = sgn(w(h_i))|w(h_i) - 1|$, where $i = 1$ or 2; go to step 8.
step 5. *If $b(j) = 2$, then let $w(h_1) = sgn(w(h_1))|w(h_1) - 1|$ and*
$w(h_2) = sgn(w(h_2))|w(h_2) - 1|$; go to step 8.
step 6. *If the parity of x_{j-1} is even, and w has two coordinates with positive values, then let $w(i) = sgn(w(i))|w(i) - 1|$ $(i = 1, 2, 3)$, else let $w(h_1) = sgn(w(h_1))|w(h_1) - 1|$ and $w(h_2) = sgn(w(h_2))|w(h_2) - 1|$.*
step 7. *If the parity of x_{j-1} is odd, and w has two negative coordinate values, then let $w(i) = sgn(w(i))|w(i) - 1|$ $(i = 1, 2, 3)$, else let*
$w(h_1) = sgn(w(h_1))|w(h_1) - 1|$ and $w(h_2) = sgn(w(h_2))|w(h_2) - 1|$.
step 8. *Let $x_j(i) = q(i) - w(i)$ $(i = 1, 2, 3)$.*
step 9. *Concatenate x_j to the path Π.*
step 10. *Go to step 2.*
step 11. *Output: one of the shortest paths Π from p to q, using B and the length of the path is j. End.*

Now we give a detailed description of the algorithm.

In the first step we initialize the algorithm: p is the starting point, $p = x_0$ is the first element of the path Π which contains x_0 only, w is the difference between the last point of Π and q. The length of the path j starts from 0.

In step 2 we check whether we have finished or not. If yes, we go to Step 11, where the output values are given, and the algorithm terminates.

In step 3 we increase the length of the path j, and order the elements of the difference w. First, suppose that among these elements, there exists one, with largest absolute value. In this case, this element has opposite sign from the others, or some of the others are equal to zero. If two elements have the same absolute value, and the third one has smaller absolute value, then the two elements with larger absolute value have opposite signs. Then the third element must be 0 or ± 1, because of the restriction of the sum of the coordinates of w. Hence the permutation satisfies our conditions. If all three elements have the same absolute value, then this value must be 1, and their sum is ± 1. Hence the permutation can be made in every case.

As we mentioned and showed in Figure 11, we can connect points x_j and q by two lanes and we have a parallelogram. If we can move to a 1- or a 2-neighbour of x_j then we make this step on the lane which goes through x_j and is closer to q.

We use step 4 if we move to a 1-neighbour. In this case we can move from x_{j-1} to a point of different parity. We decrease by 1 one of the absolute values of the first two elements of the permutation of w. If x_{j-1} is even, then the sum of the elements of w is 1, if q is odd, and the sum of the elements of w is 0, if q is even. Since w has a non-zero element, $w(h_1)$ or $w(h_2)$ must be positive, as well. So we can change this positive value. If x_{j-1} is odd, than the sum of the elements of w is -1, if q is even, and the sum of the elements of w is 0, if q is odd. Since w has a non-zero element, $w(h_1)$ or $w(h_2)$ must be negative, as well. So we can change this negative value. Thus, at this step we get closer to q in a coordinate by 1.

Let us consider step 5. If we can move to a 2-neighbour of x_{j-1}, we have two cases. First, if there is only one non-zero element of w, then it must be ± 1, so we change this element to 0. In the other case, the first two elements of the permutation have opposite signs, hence we move from x_{j-1} to a point of the same parity, by changing these two values.

In step 4 and 5 we step in the lane which has smaller distance from the destination point. We can decrease the higher absolute values in w, but if we can move to a 3-neighbour then we step in the lane in which we can decrease the higher absolute values in w, and if possible we step to the next parallel lane which is closer to the end-point. So if we move to a 3-neighbour of x_j, then if possible we step to the point which is in the intersection of two lanes which are closer to q than the previous ones. (This point is on the next lanes as x_j.) If this is impossible, then we move to a 2-neighbour of x_j. We describe these cases below.

We use step 6 or 7 if $b(j) = 3$, so we move to a 3-neighbour of x_{j-1}. If the parity of x_{j-1} is even (Step 6), then we may step to an odd point, and change all of its coordinates by 1, if w contains two positive and one negative values. If x_{j-1} has odd parity, and w contains two negative and one positive values, then we step to an even point, by changing every coordinate value by 1. If w does not let us to do so, we can step only to a 2-neighbour, like at Step 5.

In step 8 and 9 the algorithm calculates the coordinates of x_j, by using the values of w, and adds x_j to the path Π (w is the difference of q and x_j).

Step 10 guarantees the repetition of the procedure, starting from step 2.

Before we present some examples how the algorithm works in practice, we prove that it is correct.

Theorem 5.1 *Algorithm 5.1 provides a shortest path.*

Proof. Let $p = y_0, y_1, \dots y_m = q$ be a B-path, and for $i = 0, \dots, m$ put $v_i = (q(1) - y_i(1), q(2) - y_i(2), q(3) - y_i(3))$ and $h_i = |y_i(1) - q(1)| + |y_i(2) - q(2)| + |y_i(3) - q(3)|$. Similarly, for the path provided by the algorithm ($p = x_0, x_1, \dots x_n = q$), let $w_i = (q(1)-x_i(1), q(2)-x_i(2), q(3)-x_i(3))$ and $g_i = |x_i(1)-q(1)|+|x_i(2)-q(2)|+|x_i(3)-q(3)|$ $(i = 1, \dots, n)$.

We show that $g_i \leqslant h_i$ for all $i \leqslant \min(m, n)$. We use induction. For $i = 0$ we have $g_0 = h_0$, so our assumption holds. Suppose that $g_i \leqslant h_i$. We prove that $g_{i+1} \leqslant h_{i+1}$. We distinguish three cases according to the values of $b(i)$.

If $b(i) = 1$, then $g_{i+1} = g_i - 1$, and $h_{i+1} \geqslant h_i - 1$. Hence, by $g_i \leqslant h_i$, we have $g_{i+1} \leqslant h_{i+1}$.

If $b(i) = 2$, then $h_{i+1} \geqslant h_i - 2$, and if $g_i \geqslant 2$ then $g_{i+1} = g_i - 2$. Hence, using $g_i \leqslant h_i$, we get $g_{i+1} \leqslant h_{i+1}$. If $g_i = 1$, then $g_{i+1} = 0$, and $g_{i+1} \leqslant h_{i+1}$ holds.

Finally, let $b(i) = 3$. Then $g_{i+1} = g_i - 3$ yields $g_{i+1} \leqslant h_{i+1}$, as $h_{i+1} \geqslant h_i - 3$. If $g_{i+1} = 0$, then we also have $g_{i+1} \leqslant h_{i+1}$. Otherwise, $g_{i+1} = g_i - 2$, and we have two possibilities: the parity of x_j is even, and w_j contains two negative values, or the parity of x_j is odd, and w_j contains two positive values. If $h_i > g_i$, then $g_{i+1} \leqslant h_{i+1}$, since $h_{i+1} \geqslant h_i - 3$. If $h_i = g_i$, then the parity of w_i is the same, as the parity of v_i. We have the following cases. If v_i has the same number of positive and negative elements as w_i, then y_{i+1} can differ from y_i in at most two coordinates, and we still have the inequality $g_{i+1} \leqslant h_{i+1}$. Otherwise y_{i+1} differs from y_i in all coordinates, and we have $g_{i+1} \leqslant h_{i+1}$ again, since the difference of some coordinate of y_{i+1} and q must grow. If $h_i = g_i$ then v_i cannot contain the same number of positive elements as the number of

w	j	$w(h_1)$	$w(h_2)$	$w(h_3)$	$b(j)$	x_j
$(-3,4,-2)$	0	–	–	–	–	$(3,-4,2)$
$(-3,4,-2)$	1	y:4	$x:-3$	$z:-2$	2	$(2,-3,2)$
$(-2,3,-2)$	2	y:3	$x:-2$	$z:-2$	3	$(1,-2,1)$
$(-1,2,-1)$	3	y:2	$x:-1$	$z:-1$	1	$(1,-1,1)$
$(-1,1,-1)$	4	$x:-1$	y:1	$z:-1$	3	$(0,0,0)$

Table 1: Construction of a shortest path to Example 5.1

negative elements of w_i, and vice versa. This is because the number of the positive and negative elements in w_i is the same as in the difference between q and p (until one or more of them become zero). If v_i contains an element c, which has opposite sign in w_i, then h_i must be greater than the sum, where c is replaced by 0. Since we assumed that $g_i = h_i$, it is a contradiction.

We have $g_i \leqslant h_i$ for all $i \leqslant \min(m, n)$, and the sequence g_i strictly monotonously tends to zero. This implies that $n \leqslant m$. So the algorithm stops after finitely many steps, and it provides a shortest path from p to q. □

The following remark is a consequence of the work of the algorithm.

Remark 5.6 *The distance of any two points p and q, with respect to a neighbourhood sequence B, depends on the difference and the parity of the points, and on the neighbourhood sequence only.*

Algorithm 5.1 is a greedy algorithm, since at every step it changes as many coordinate values as possible to get closer to the end point.

Let us analyze the complexity of Algorithm 5.1. It is clear that we need only memory to store the point where we are (x_i), and what is the following element of the neighbourhood sequence. So if we can write the output path and we can read the sequence while the algorithm is running, then we need only constant memory.

What is the time-complexity of our algorithm?

It is easy to show that there is a constant upper bound for the time that an iteration takes. (In Steps 3-9 there is ordering of three elements, evaluation of conditions and changing values.) So an iteration takes maximum c time. In the worst case (with neighbourhood sequence (1)) we must make $|w(1)| + |w(2)| + |w(3)|$ steps, where w is the difference of the start and end points.

So our algorithm terminates at most after $c(|w(1)| + |w(2)| + |w(3)|)$ time, which is linear in the different of the coordinate values of the starting and ending points. So we can say that our algorithm is efficient.

The next examples show how the algorithm works in practice.

Example 5.1 *Let $p = (3, -4, 2)$ and $q = (0, 0, 0)$ be two points, and $B = (2, 3, 1, 3)$ a neighbourhood sequence. Table 1 shows how the values change during the algorithm. The notation used in Table 1 is the same as used at Algorithm 5.1. The first row of the table contains the initial values of the algorithm. Every row contains values obtained after moving to the next point. The presented shortest path is in Figure 12.*

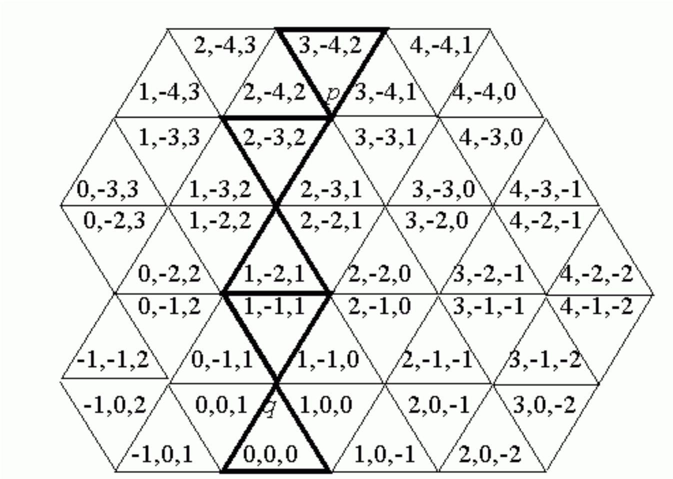


Fig. 12: The shortest path in Example 5.1

w	j	$w(h_1)$	$w(h_2)$	$w(h_3)$	$b(j)$	x_j
$(-1,2,-1)$	0	$-$	$-$	$-$	$-$	$(1,-2,1)$
$(-1,2,-1)$	1	y:2	x:-1	z:-1	1	$(1,-1,1)$
$(0,1,-1)$	2	x:-1	y:1	z:-1	3	$(0,0,0)$

Table 2: The shortest path from r to s in Example 5.2

Further in this subsection, we study distance functions with strange properties. We illustrate that our algorithm works also in the case, when the distance based on a neighbourhood sequence is not symmetric and/or does not meet the triangular inequality.

Example 5.2 *Let* $r = (1, -2, 1)$ *and* $s = (0, 0, 0)$ *be two points, and* $B = (1, 3, 2)$ *a neighbourhood sequence. First, we calculate* $d(r, s; B)$ *by using Algorithm 5.1. By Table 2 we get* $d(r, s; B) = 2$. *Now let us calculate* $d(s, r; B)$. *The result is in Table 3:* $d(s, r; B) = 3$. *As* $d(r, s; B) \neq d(s, r; B)$, *this distance function is not symmetric.*

Example 5.3 *Let* $r = (0, 0, 0)$, $s = (0, 1, -1)$ *and* $t = (0, 2, -2)$ *be three points, and* $B = (2, 1, 1)$ *a neighbourhood sequence. The calculation of* $d(r, s; B)$ *is in Table 4. So* $d(r, s; B) = 1$. *In Table 5 we present the determination of* $d(s, t; B) = 1$. *The calculation of* $d(r, t; B)$ *is presented in Table 6. As we can see,* $d(r, t; B) = 3$, *and* $d(r, t; B) > d(r, s; B) + d(s, t; B)$. *Thus the triangular inequality does not hold.*

w	j	$w(h_1)$	$w(h_2)$	$w(h_3)$	$b(j)$	x_j
$(1,-2,1)$	0	$-$	$-$	$-$	$-$	$(0,0,0)$
$(1,-2,1)$	1	y:-2	x:1	z:1	1	$(1,0,0)$
$(0,-1,0)$	2	y:-2	z:1	x:0	3	$(1,-1,1)$
$(0,0,-1)$	3	y:-1	x:0	z:0	2	$(1,-2,1)$

Table 3: The shortest path from s to r in Example 5.2

w	j	$w(h_1)$	$w(h_2)$	$w(h_3)$	$b(j)$	x_j
(0,1, −1)	0	–	–	–	–	(0,0,0)
(0,1, −1)	1	y:1	z:−1	x:0	2	(0,1, −1)

Table 4: Calculating a shortest path from $(0, 0, 0)$ to $(0, 1, -1)$ by $B = (2, 1, 1)$

w	j	$w(h_1)$	$w(h_2)$	$w(h_3)$	$b(j)$	x_j
(0,1, −1)	0	–	–	–	–	(0,1, −1)
(0,1, −1)	1	y:1	z:−1	x:0	2	(0,2, −2)

Table 5: Calculating a shortest path from $(0, 1, -1)$ to $(0, 2, -2)$ by $B = (2, 1, 1)$

As we can see in the previous examples, we can find neighbourhood sequences which do not generate metrics.

Example 5.4 *The distance function generated by the neighbourhood sequence $B = (3, 1)$ is non-symmetric and non-triangular.*

In the hexagonal and rectangular cases the distance functions must be symmetric, because by the symmetry of the grid we can interchange the start- and endpoints. However in the triangular plane this does not work in every case because of the parity of the points. In Theorem 4.2 we showed that the hexagonal distance is a metric. In Chapter 3 we showed that in rectangular case the triangular inequality may not hold, in Theorem 3.4 we presented a necessary and sufficient condition for metric B-distances. The non-symmetric distances are a new phenomenon in the digital geometry, they appear in the triangular grid with neighbourhood sequences. In the next section we are going to give a condition for the neighbourhood sequence to generate a metric distance in the triangular grid.

5.3 Condition for metric distances

In view of the previous examples, we have the following natural question: knowing B, how can we decide whether the distance function defined by B is a metric on the triangular plane. We give the answer in this section.

Lemma 5.1 *The* 1*-distance of p and q is given by $d(p, q; (1)) = |w(1)| + |w(2)| + |w(3)|$.*

w	j	$w(h_1)$	$w(h_2)$	$w(h_3)$	$b(j)$	x_j
(0,2, −2)	0	–	–	–	–	(0,0,0)
(0,2, −2)	1	y:2	z:−2	x:0	2	(0,1, −1)
(0,1, −1)	2	y:1	z:−1	x:0	1	(0,2, −1)
(0,0, −1)	3	z:−1	x:0	y:0	1	(0,2, −2)

Table 6: Calculating a shortest path from $(0, 0, 0)$ to $(0, 2, -2)$ by $B = (2, 1, 1)$

Proof. The statement is an easy consequence of the definitions. □

We have seen that it is possible to use a 3-step only like a 2-step (Algorithm 5.1). As is shown in the previous section, we can step by 1 or 2 in all possible cases in the shortest path (expect if we need only a 1-step to reach the endpoint and we have a 2 in the neighbourhood sequence).

Now we introduce the concept of *minimal equivalent neighbourhood sequence*, which will be very helpful.

Definition 5.5 *Let B and B' be two neighbourhood sequences. B' is the minimal equivalent neighbourhood sequence of B, if the following conditions hold:*

1. *$d(p, q; B) = d(p, q; B')$ for all points p, q; and*
2. *for each neighbourhood sequence B_1, if $d(p, q; B) = d(p, q; B_1)$ for all points p, q, then $b'(i) \leqslant b_1(i)$ for all i.*

Lemma 5.2 *The minimal equivalent neighbourhood sequence B' of B is uniquely determined, and it is given by*

- *$b'(i) = b(i)$, if $b(i) < 3$,*
- *$b'(i) = 3$, if $b(i) = 3$ and there is no $j < i$ such that $b'(j) = 3$,*
- *$b'(i) = 3$, if $b(i) = 3$ and there is some $b'(l) = 3$ with $l < i$, and $\sum_{k=j+1}^{i-1} b'(k)$ is odd, where $j = \max\{l \mid l < i, b'(l) = 3\}$,*
- *$b'(i) = 2$, otherwise.*

Proof. We start from the beginning of the neighbourhood sequence B, and we give the value $b'(i)$ using $b(i)$ and the previous part of sequence B'.

It is obvious, that if $b(i) < 3$ then $b'(i) = b(i)$, because we can use these steps as strict 1- or 2-steps.

The question is when we may use the strict 3-steps and when it is impossible (to move farther by a strict 3-step than a 2-step).

Let $b(i) = 3$, and let $p = p_0, p_1, \dots p_n = q$ be a minimal path from p to q.

We can change all coordinate values in the i-th step in the following two cases: (according to the parity and the direction)

- p_{i-1} is odd and we need to decrease two coordinates, and increase only one value to go to q the difference of q and p_{i-1} has two negative and a positive value,
- p_{i-1} is even and we need to increase two coordinate values, and decrease only one of them: the difference of q and p_{i-1} has two positive and a negative value.

It is easy to show that these two possibilities can happen if and only if:
this $b(i)$ =3 is the first 3 value in the neighbourhood sequence B; or
$b(i)$ =3 is not the first 3, but $\sum_{k=j+1}^{i-1} b'(k)$ is odd, where $j = \max\{l|l < i, b'(l) = 3\}$.

Indeed, if this subsum is even then we are in a point which has the same parity as the point after the previous 3-step, hence we cannot use a strict 3-step in the i-th step.

By the above argument one can easily see that B' is uniquely determined. Thus the lemma is proved. $\square$

Corollary 5.1 *Let B' be the minimal equivalent neighbourhood sequence of an arbitrary neighbourhood sequence B. Let $j \in \mathbb{N}$ and let $r = \sum_{i=1}^{j} b'(i)$. There are points p and q, such that their 1-distance is r and B'-distance is j, i.e. we can use all elements 3 of the neighbourhood sequence B' to strict 3-steps.*

Lemma 5.3 *The distance based on a neighbourhood sequence $B = (b(i))_{i\in\mathbb{N}}$, is non-symmetric if and only if $b(i) = 3$ for an $i \in \mathbb{N}$, and one of the following conditions is true with $i = \min\{\, l|b(l) = 3\}$:*

- $\sum_{k=1}^{i-1} b(k)$ *is odd; or*
- *there is a j such that $b(j) = 1$, and $i < j$.*

Proof. First assume that the first condition holds. We give an example when this distance is not symmetric.

Let i be the index of the first 3 value of B, and l is the index of the previous 1 value, i.e. $l < i, b(l) = 1$, and there is no h, such that $l < h < i$ and $b(h) = 1$. There must be such an index l, because $\sum_{k=1}^{i-1} b(k)$ is odd, and this part of B does not contain 3 values, only 1 and 2 values.

Let $x = \frac{\sum_{k=1}^{l-1} b(k)}{2}$ and $y = \frac{\sum_{k=l+1}^{i-1} b(k)}{2} = i - l - 1$. These sums must be even by our assumptions and the choices of the indexes i and l. Let $p = (0, 0, 0)$ and $q = (-1, -x-y-1, x+y+2)$. We can calculate the distances between these points by using Algorithm 5.1. We get $d(p, q; B) = i$, because a shortest path goes through the following points: before the l-th step we are in $(0, -x, x)$, after this we are in $(0, -x, x+1)$. Before the i-th step we are in $(0, -x-y, x+y+1)$ (we arrive here from p on the lane where the first coordinate value is 0), and after the i-th step we are in $(-1, -x-y-1, x+y+2) = q$ (we take a strict 3-step). But $d(q, p; B) = i + 1$, because a shortest path goes through the following points: before the l-th step we are in $(-1, -y-1, y+2)$, after this we are in $(-1, -y, y+2)$. Before the i-th step we are in $(-1, 0, 2)$ (we go from q to $(-1, 0, 2)$ on the lane where the first coordinate value is -1), and after the i-th step we are in $(0, 0, 1)$, so we need one more step to reach p (because we cannot use a strict 3-step).

Now we assume that the first condition is false, but the second is true. We provide a counterexample again.

Let i be the same as above and $b(j) = 1$, such that $i < j$, and j is minimal with these properties. Then let $x = \frac{\sum_{k=1}^{i-1} b(k)}{2}$ (it is an integer because the first condition is false). Let B' be the minimal equivalent sequence of B. It follows from Lemma 5.2, that

$b'(k) = 2$ for all $i < k < j$, hence let $y = \frac{\sum_{k=i+1}^{j-1} c(k)}{2} = j - i - 1$. Let $p = (0,0,0)$ and $q = (1, -x-y-2, x+y+1)$. Then a shortest path from p to q: We start from $p = (0,0,0)$, before the i-th step we go to $(0, -x, x)$ on the lane where the first coordinate value is zero, after this step we are in $(1, -x-1, x+1)$ (in this step we can go to another lane with a strict 3-step). After the i-th step we move on the lane where the first coordinate is still 1, and before the j-th step we arrive the point $(1, -x-y-1, x+y+1)$, and after the j-th step $(1, -x-y-2, x+y+1) = q$, hence this distance is j. From q to p with B: we started from $q = (1, -x-y-2, x+y+1)$, before the i-th step $(1, -y-2, y+1)$, after this $(1, -y-1, y)$ (in this step we cannot go to another lane because the parity and the direction, hence we cannot move a strict 3-step, so we use the lane where the first coordinate is 1 to go away). Before the j-th step we are in $(1, -1, 0)$, and after the j-th step $(1, 0, 0)$, and we need one more step to reach p, hence this distance is $j + 1$.

Finally we show that if both conditions are false then the B-distance is symmetric. Let p and q be two arbitrary points. We prove that $d(p, q; B) = d(q, p; B)$.

If B does not contain the value 3, then we use Algorithm 5.1 to construct the shortest paths. It is easy to see that we have no problem if we use only 1- and 2-neighbours (we can move strict 2-steps and strict 1-steps from each point to any direction), hence $d(p, q; B) = d(q, p; B)$.

If there is a 3 value in B, then we must use the minimal equivalent sequence B' instead of B. We know from the false conditions and from Lemma 5.2 that the subsum before the first 3 value is even, and after this we have only 2 values in B'. Let i be the index of the 3 in B'. Put $x = \frac{\sum_{k=1}^{i-1} b(k)}{2}$. This sum is even because the first condition of the theorem is false. If $d(p, q; (1)) < 2x + 1$, then we have the same case as if B does not contain the value 3. If $d(p, q; (1)) = 2x + 1$ or $d(p, q; (1)) = 2x + 2$, then we step only to a 2-neighbour at the last step, so both distances are i. If $d(p, q; (1)) > 2x + 2$ then we either can use the strict 3 in the i-th step, or not. Let us see first the case when we can use this strict 3-step. Before the i-th step we are in a point with the same parity as our starting point. As we can see in the proof of Lemma 5.2, we can use the strict 3-step if

- we go from an odd point and we need to decrease two coordinates, and increase only one value to go to q
- we go from an even point and we need to increase two coordinate values, and decrease only one of them.

It is obvious that if p and q have different parities then we can use this 3-step in the same way (strict 3-steps in both paths or in none of them). If the parity of p and q are the same, then we can use this strict 3-step only in one way. After this strict 3-step we are in a point of opposite parity than p. Hence in the last step we must step only to a 1-neighbour to reach the endpoint, but we may step to a 2-neighbour, so now we lose what we won at the strict 3-step. So in this case we have $d(p, q; B) = d(q, p; B)$ again. □

Lemma 5.4 *Let B be a neighbourhood sequence, which does not contain the element 3. For the distance based on this B, the triangular inequality does not hold if and only if there are such i and j that $\sum_{k=1}^{i} b(k) > \sum_{k=j+1}^{j+i} b(k)$.*

Proof. If the neighbourhood sequence B does not contain 3, then our case is similar to the case of the square grid, because we can move strict 2-steps and strict 1-steps from each point to any direction like in the rectangular case. Thus the same arguments work as in the rectangular case (Theorem 3.4). $\square$

Remark 5.7 *If the neighbourhood sequence B contains the element 3, then in general we have difficulties. However, if B generates a symmetric distance function, then in the minimal equivalent neighbourhood sequence of B there is at most one 3 value. In this case one can easily prove that we need to check the subsums in Lemma 5.4 only before this 3 value to decide that this B-distance satisfies the triangular inequality or not.*

Now we are in the position to answer the question when a neighbourhood sequence generates a metric.

Theorem 5.2 *Let B be a neighbourhood sequence. The distance function based on B is a metric if and only if the following conditions hold:*

- *if $b(j) = 3$ and $b(i) = 1$ then $i < j$,*
- *if B contains 3, then $\sum\limits_{b(k)=1} b(k)$ is even,*
- $\sum\limits_{k=1}^{i} b(k) \leqslant \sum\limits_{k=j+1}^{j+i} b(k)$, *when $i + j < l$, where l is the index of the first 3 value in B, (if 3 is not in B then this condition must hold for all i, j).*

Proof. By Remark 5.7, the theorem is a simple consequence of Lemmas 5.3 and 5.4. $\square$

According to the previous theorem we can exactly give the neighbourhood sequences, which provide metrics.

Corollary 5.2 *B generates a metric if and only if B has one of the following forms:*

a) $b(1) = 1$, B does not contain 3, and $\sum\limits_{k=1}^{i} b(k) \leqslant \sum\limits_{k=j+1}^{j+i} b(k)$ for all $i, j \in \mathbb{N}$.

b) $b(i) > 1$ for all $i \in \mathbb{N}$.

c) $b(1) = 1$, if l is the minimal index such that $b(l) = 3$ then $\sum\limits_{k=1}^{l-1} b(k)$ is even, $\sum\limits_{k=1}^{i} b(k) \leqslant \sum\limits_{k=j+1}^{j+i} b(k)$ if $i + j < l$ and $b(k) > 1$ for $k > l$.

The condition c) is a mixture of a) and b). The first part of B has the same property as in case a), and the other part beginning by $b(l)$ has the same property as in case b).

Neighbourhood sequences, which are both periodic and give metric distances, are of special importance and interest.

Corollary 5.3 *A periodic B generates a metric if and only if one of the following conditions holds:*

there are only 1 and 2 values, and $\sum\limits_{k=1}^{i} b(k) \leqslant \sum\limits_{k=j+1}^{j+i} b(k)$ for all $i, j \in \mathbb{N}$, or

there are only 2 and 3 values in B.

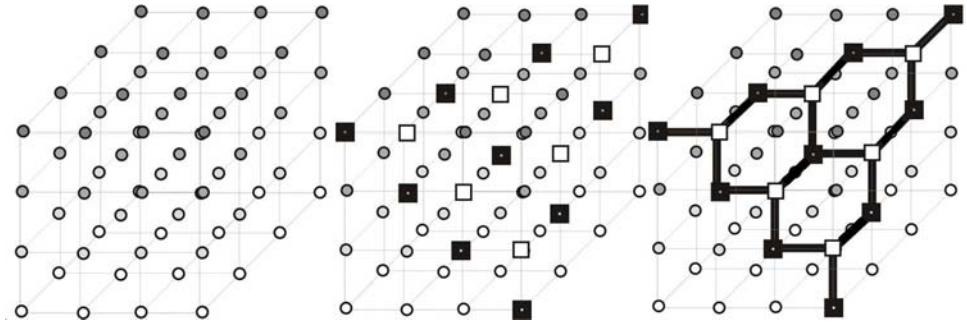


Fig. 13: The hexagonal grid of nodes as two parallel planes in $\mathbb{Z}^3$

We will calculate the B-distance of any two points with a given neighbourhood sequence in Section 5.5.

5.4 Relationship between the triangular and the cubic grids

Let us see how the triangular plane can embed into $\mathbb{Z}^3$, i.e. what points in $\mathbb{Z}^3$ are used to represent the points of the triangular grid. Figure 13 shows the points of the grid of hexagonal nodes in the cubic grid. They form two parallel planes, according to the parities of points.

The lanes of this grid are also parallel planes with a plane including two coordinate axes in $\mathbb{Z}^3$ joined with the two planes representing the triangular grid.

As we can see the considered points of $\mathbb{Z}^3$ are in the planes in which the sum of coordinate values are 0 (black boxes) or 1 (white boxes) hence we can call the triangular grid as the two-plane triangular grid. In Figure 13 we connect the nodes which are 1-neighbours. As we can see they form a hexagonal grid of nodes which is the triangular grid.

The concept of minimal equivalent neighbourhood sequences gets an easily understandable explanation. To build a shortest path between two points of the cubic grid we do not care about the 'parity' of the points. We can use the points in our path with any kind of coordinate sum. Unlike in the previous case in triangular grid we cannot step to a point which is not included in the two given planes of the grid. Using the minimal equivalent neighbourhood sequence instead of the original one we reduce the number of the possible wrong steps to 1. (We have to care only the first occurrence of the element 3, as we will show in the next section.)

In the next section we will use this mapping from the triangular grid into the cubic grid to determine the distance of points.

5.5 Formulae for distance

The concept of the minimal equivalent neighborhood sequences shows that it is possible that both theoretically and practically we cannot change all the three coordinate values to step closer to the endpoint even if there is an element 3 in the neighbourhood sequence.

In this section we use the previous mapping and modify the formula of Proposition 3.1 to our case. We will use the minimal equivalent neighbourhood sequence B' instead of the original neighbourhood sequence B. Remember the following from Section 5.3. As we used in the proof of Lemma 5.3 building a shortest path in some cases it is worth modifying all the three coordinate values at 3-steps, and in some cases not.

Corollary 5.4 *Going from p to q it is better to use a 3-step as a strict 3-step by the first element 3 of the neighbourhood sequence B (let its index be k), if and only if one of the following holds:*

1. *p is even, $\sum_{i=1}^{k-1} b(i)$ is even and we need to decrease two coordinates, and increase only one value to go to q;*
2. *p is odd, $\sum_{i=1}^{k-1} b(i)$ is odd and we need to decrease two coordinates, and increase only one value to go to q;*
3. *p is odd, $\sum_{i=1}^{k-1} b(i)$ is even and we need to increase two coordinate values, and decrease only one of them to direction to q;*
4. *p is even, $\sum_{i=1}^{k-1} b(i)$ is odd and we need to increase two coordinate values, and decrease only one of them.*

From the concept of minimal equivalent neighbourhood sequence it is obvious that by building a shortest path from the point p to q, it is worth modifying all coordinate values only when there is an element 3 in the minimal equivalent neighbourhood sequence of B. Therefore we must use it instead of B in our calculation. Moreover it is possible (depending the coordinate values of p and q), to use the first element 3 as value 2 (in the cases that are not listed above in Corollary 5.4). For this case we will use the concept of *reduced minimal equivalent neighbourhood sequence.*

Definition 5.6 *The reduced minimal equivalent neighbourhood sequence B" of B is given by:*

- *$b"(k) = 2$, where k is the index of the first element 3 of B;*
- *$b"(i) = b'(i)$, for all other value of i, where $b'(i)$ are the correspondent elements of the minimal equivalent neighbourhood sequence B' of B.*

We introduce some notion using the sorted difference v of the points (Definition 5.4):

$$d'_2 = \max\left(i \,\middle|\, |v(1)| + |v(2)| > \sum_{j=1}^{i-1} b'^{(2)}(j)\right),$$

$$d'_3 = \max\left(i \,\middle|\, |v(1)| + |v(2)| + |v(3)| > \sum_{j=1}^{i-1} b'(j)\right),$$

$$d"_2 = \max\left(i \left| |v(1)| + |v(2)| > \sum_{j=1}^{i-1} b"^{(2)}(j) \right.\right),$$

$$d"_3 = \max\left(i \left| |v(1)| + |v(2)| + |v(3)| > \sum_{j=1}^{i-1} b"(j) \right.\right).$$

Finally we can state our theorem using the abbreviations above.

Theorem 5.3 *The distance from a point p to q with a given neighbourhood sequence B can be calculated in the following way:*

$$d' = \max\left(|v(1)|, d'_2, d'_3\right)$$

and

$$d" = \max\left(|v(1)|, d"_2, d"_3\right).$$

The first equation gives the result ($d(p,q;B) = d'$) if we can modify all the 3 coordinate value by step of the first element 3 ($= b(k)$). We must use the second equation in other cases (and get $d(p,q;B) = d"$).

So we must use the value $d"$ in the following case: if the distance $d' \geq k$ and one of the followings hold:

- *p is even, $\sum_{i=1}^{k-1} b(i)$ is even, and the difference $w_{p,q}$ has two negative and a positive value;*
- *p is odd, $\sum_{i=1}^{k-1} b(i)$ is odd and the difference of q and p has two negative and a positive value;*
- *p is odd, $\sum_{i=1}^{k-1} b(i)$ is even and the difference of q and p has two positive and a negative value;*
- *p is even, $\sum_{i=1}^{k-1} b(i)$ is odd and the difference of q and p has two positive and a negative value.*

Proof. Using our previous notations and lemmas, with the shortest path algorithm (Algorithm 5.1) and Corollary 5.4, it is consequence of Theorem 3.1 (and Corollary 3.3). □

The previous theorem is one of the important results. In the triangular grid the different of any two points is 'balanced', i.e. $w(1) + w(2) + w(3) \in \{-1, 0, 1\}$. The differences of the coordinate values are distributed among the three values. If the two points are in a common lane, then we use the 2-limited neighbourhood sequence. Elsewhere the minimal or the restricted minimal equivalent neighbourhood sequence can be used according to the parity and direction of the points.

Now we show an example.

Example 5.5 *Let $r = (1, -2, 1)$ and $s = (0, 0, 0)$ be two points, and $B = (3, 1, 1)$ a neighbourhood sequence. The minimal equivalent neighbourhood sequence of B is: $B' = (3, 1, 1, 2, 1, 1, 2, 1, 1, ...)$ with replaying part $(2, 1, 1)$.*

First we will calculate the value of $d(r, s; B)$.
The value of v is $(2, -1, -1)$ in this case. Try to use the first form of Theorem 5.3. $d'_2 = \max\left(i \,\middle|\, |2| + |-1| > \sum_{j=1}^{i-1} b'^{(2)}(j)\right) = 2$ and $d'_3 = \max\left(i \,\middle|\, 2 + 1 + 1 > \sum_{j=1}^{i-1} b'(j)\right) = 2$, therefore $d' = \max(|2|, 2, 2) = 2$. This is greater then $k = 1$, and the parity of r is even, $\sum_{i=1}^{k-1} b(i) = 0$ is even and the difference w is even, moreover it has two negative and a positive value, therefore we must use the second formula, i.e. the n.s. $B" = (2, 1, 1)$: $d"_2 = 2$, $(3 > 2)$ and $d"_3 = 3$, $(4 > 2 + 1)$. Therefore $d" = 3$ and so $d(r, s; B) = 3$.

Now let us calculate $d(s, r; B)$. Then $v_{s,r} = (-2, 1, 1)$. Using the first form: $d' = \max(2, 2, 2) = 2$. It is greater than $k = 1$, but the other conditions fail, so the result: $d(s, r; B) = 2$. Thus this distance function is not symmetric.

As we mentioned before, the non-symmetric distance functions are exotic in the field of digital geometry, since they are not in the square and hexagonal cases.

Watching the formulae for calculating distance we state the following important property.

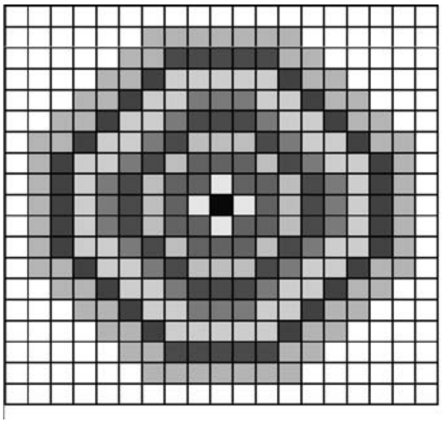
Theorem 5.4 *A distance $d(p, q; B) > k$ depends on the order of the first k elements of B if and only if there is a permutation of these elements such that using it as the initial part of the neighbourhood sequence the distance function is not symmetric.*

Proof. Let B be a neighbourhood sequence for which the distance is not symmetric. So let p and q be two points that $d(p, q; B) \neq (q, p; B)$. We can assume that $d(p, q; B) = k < d(q, p; B) = l$. If we reorder the first k elements of B in inverse order then we get that $d(q, p; B) = k$. (We have a shortest path from q to p which is the symmetric pair of the original shortest path between p and q.) Let us prove the other direction. Let us assume that the distance depends on the ordering. One can check (and it can be found in [33]) that interchanging an element $b(i) = 2$ with any other element $b(j)$ (, where $i, j < k$) does not occur dependency. Dependency occurs only if there are $b(i) = 3$ and $b(j) = 1$. But in these cases using Lemma 5.3 we can have a non-symmetric distance with these elements. $\square$

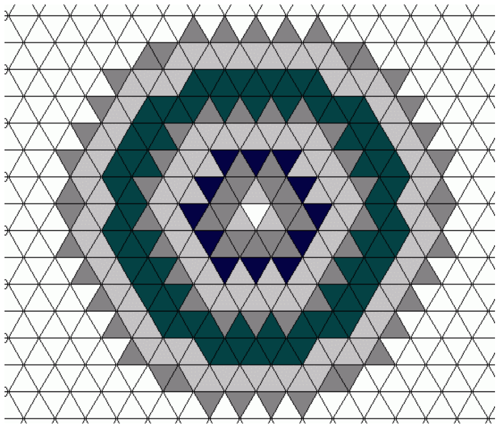
5.6 Digital circles

In this section we present some other properties in which the triangular grid differs from the square grid. We analyze the changing and development of wave-fronts, and give an illustrated description of the digital circles with neighbourhood sequences in triangular grid. At the end of the section we show a characterization of digital circles.

In this section we investigate the way a neighbourhood sequence spreads in the digital space starting from a point of the triangular grid. This spreading is translation-invariant among the points of the same parity and it is central-symmetric concerning points with different parities. So, for simplicity we may choose the origin o as the starting point.



(a) Circles in square grid



(b) Circles in triangular grid

Fig. 14: Examples for growing digital circles up to radius 8

Definition 5.7 *Let B be a neighbourhood sequence in the triangular grid. For $k \in \mathbb{N}$, let*

$$C_k^B = \{p \mid d(o, p; B) \leq k\}.$$

C_k^B is the region (digital circle) occupied by B after k steps.

Remember that in square grid we use the

$$O_k^B = \{q \mid d(o, q; B) \leq k\}$$

notations for the occupied regions where q is a point in the square grid ($\mathbb{Z}^2$), and in this case $1 \leq b(i) \leq 2$ for all i, and k is a natural number. (In obvious cases we reduce the notion to C_k and O_k.)

In Fig. 14 there are some examples of growing digital circles.

In the following we summarize some simple observations about the digital circles. We underline some properties which are different for the digital circles in square grid and in triangular grid.

In the square grid the region O_k occupied by k steps of a neighbourhood sequence B is independent of the ordering of the first k element of B (see Remark 3.3).

Contrary to the case of square grid, it is possible for a n.s. B and for a $k \in \mathbb{N}$, that the region C_k does depend on the order of the first k elements of B. This property is based on the fact shown in Theorem 5.4. We will show an example. The regions $C_2^{(1,3)}$ and $C_2^{(3,1)}$ differ as Fig. 15 shows in the margins of row 3.

The inclusion property of the digital circles generated by a neighbourhood sequence (see Remark 3.3 for case of square grids) stands in the triangular grid as well.

Remark 5.8 *For any neighbourhood sequence B, the sequence of regions $(C_k)_{k=1}^{\infty}$ is a strictly monotone increasing sequence. That is, $k > l$ implies $C_k \supsetneq C_l$.*

In the square grid it is impossible for $k \neq l$ and any two neighbourhood sequence B_1 and B_2 that $O_k^{B_1} = O_l^{B_2}$. This statement follows from the fact, that for any n.s. B the point $p(0, k) \in O_l$ if and only if $l \geq k$ (when $o = (0, 0)$).

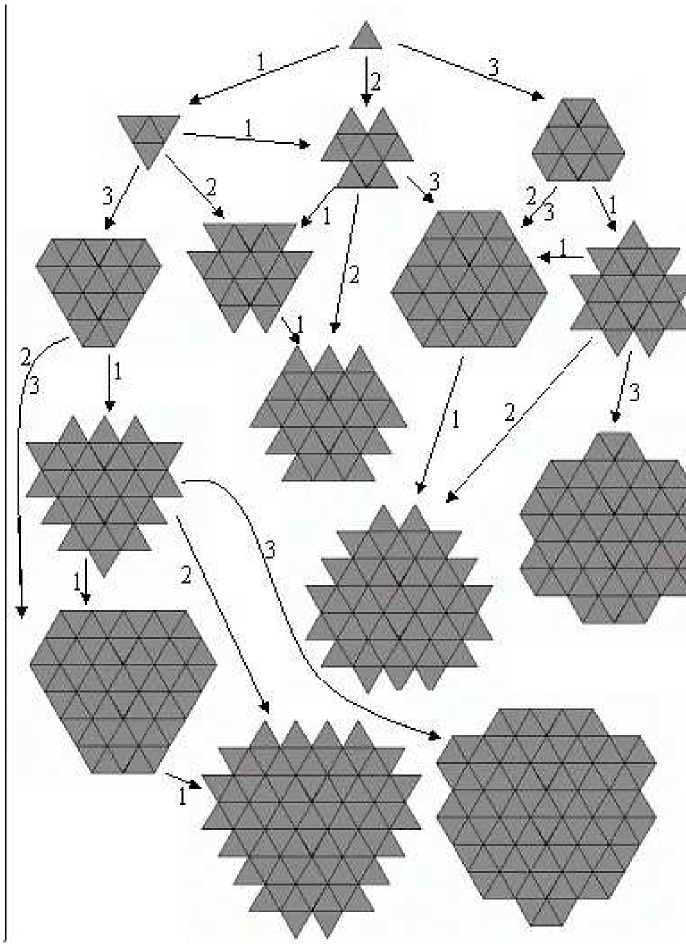


Fig. 15: Basic digital circles and their hierarchy

Lemma 5.5 *Contrary to the square grid, in the triangular grid there are neighbourhood sequences B_1, B_2 and $k, l \in \mathbb{N}$ such that $C_k^{B_1} = C_l^{B_2}$ with $k \neq l$.*

Proof. We present an example. Let $B_1 = (1)$ and $B_2 = (2)$ then $C_2^{(1)} = C_1^{(2)}$ (see Fig. 15). $\square$

In the square grid the digital circles with the same radius form a well ordered set. Formally, for all pairs of B_1, B_2 one of the following relations hold:

$$O_r^{B_1} \subseteq O_r^{B_2} \text{ or } O_r^{B_1} \supset O_r^{B_2}.$$

Moreover

$$O_r^{(1)} \subseteq O_r^{B_1} \subseteq O_r^{(2)}.$$

(The circle $O_r^{B_1}$ is 'bigger' than $O_r^{B_2}$ if and only if the number of 2's are greater among the first r elements in B_1 than B_2.)

original edge type	after a 1-step	after a 2-step	after a 3-step
'sawtooth'	'smooth'	'sawtooth'	'smooth'
'hilly'	'hilly'	'hilly'	'hilly'
'smooth'	'sawtooth'	'smooth'	'smooth'

Table 7: State transition table of edges by taking a step

Contrary, in the triangular grid the following property holds.

Lemma 5.6 *In the triangular grid there are neighbourhood sequences B_1, B_2 and $r \in \mathbb{N}$ such that $C_r^{B_1} \nsubseteq C_r^{B_2}$ and $C_r^{B_1} \nsupseteq C_r^{B_2}$.*

Proof. We present an example. Let $B_1 = (1,3)$ and $B_2 = (3,1)$. The $p = (2,-1,-1) \in C_2^{(1,3)}$, but $p \notin C_2^{(3,1)}$. The $q = (1,-2,1) \notin C_2^{(1,3)}$, but $q \in C_2^{(3,1)}$. $\square$

We will use the concept of minimal equivalent neighbourhood sequences in this section as well. In the triangular grid, for certain neighbourhood sequences, it can happen that a 3-step is equivalent to a 2-step for our investigations (, i.e. the digital circles in triangular grid have the following property).

Remark 5.9 *According to the definition of the minimal equivalent n.s. (Definition 5.5) we obtain the same digital circles using them with the original ones, i.e. ($C_k^{B'} = C_k^B$) for any $k \in \mathbb{N}$ and minimal equivalent n.s. B' of B.*

In Fig. 15 we present some simple digital circles obtained in a few steps (small radii).

In the next part we explore the wavefronts.

We present the types and the development of wavefronts in the triangular plane. In [16] Das and Chatterji showed that for every 2D periodic neighbourhood sequence B, (O_k) is always an octagon. In rectangular grid we have two kinds of sides: 'smooth' and 'stair'-types as shown in Figure 14. The vertical and horizontal edges are 'smooth', the other four edges are 'stair'-type. In the next cases the octagon is degenerate, i.e. it is a square with only one type edges. With the n.s. $B_1 = (1)$ we get only four 'stair'-type edge, while using the n.s. $B_2 = (2)$ we get a square with only 'smooth' edges. In case we use both 1-step and 2-step our result is a non-degenerated octagon.

In the triangular plane we have three kinds of possible "limit lines" (edges). These are the 'smooth', the 'hilly' and the 'sawtooth'. They change to each other by using a step from C_k to C_{k+1}. In Fig. 16, there is the diagram of changing them by a step using different neighbourhood criteria. (The used growing direction is bottom-up.) In the first rows (1) we can see how to modify the 'sawtooth' with various length; in middle rows (2) the 'hilly' and in the last rows (3) the 'smooth' edges after different type of steps. (We used steps by neighbourhood criteria 1, 2 and 3 at columns (b) (c) and (d) respectively.)

The diagram of the changing of types of edges is given by the Table 7 and by Fig. 17.

In the next statements we summarize our experiences.

Proposition 5.1

- *After a 3-step our smooth and sawtooth edges go to smooth lines.*

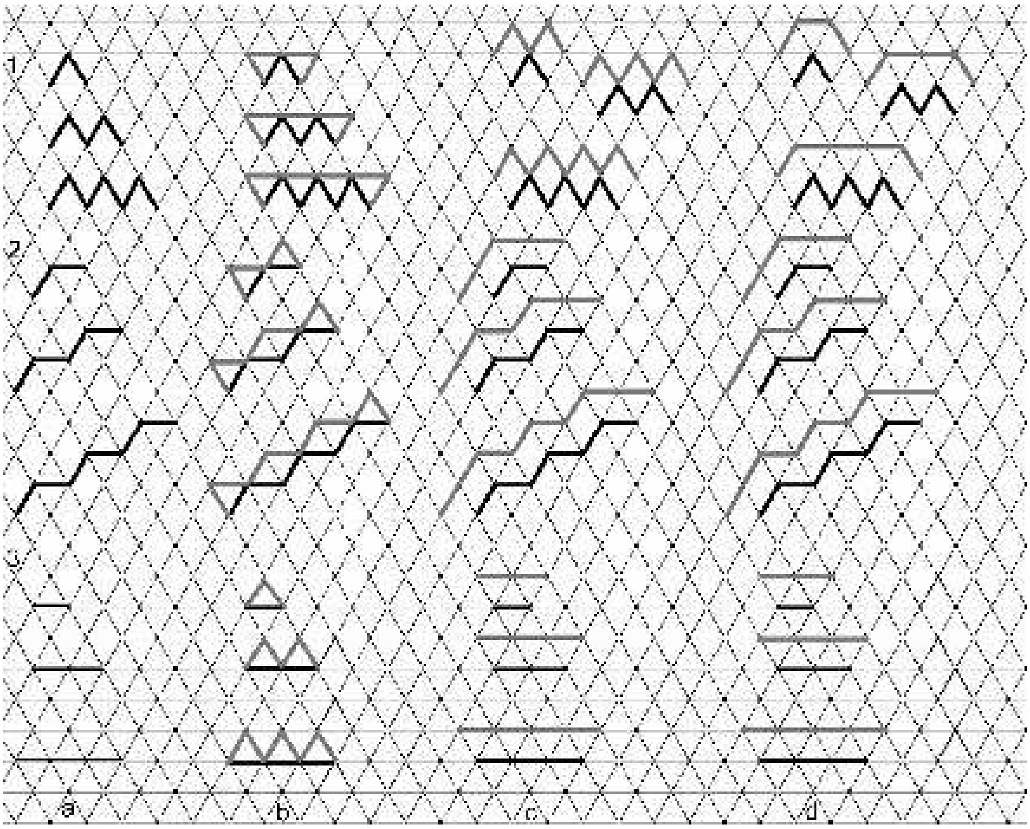


Fig. 16: Changing the edges after a step in upward direction (a) original edges (b) after a 1-step (c) after a 2-step (d) after a 3-step

- *The 2-steps do not change the type of the edges.*
- *The 'hilly' edge cannot change into another type of edge.*

In Fig. 15 we showed the basic digital circles and here we analyzed the edges. In the next part we analyze how the possible vertices i.e. the connections of the possible type of edges change in growing steps. In Table 8 we show what kind of vertices occur in different digital circles. Fig. 18 shows all of the cases of changing corners by a step in upward direction.

Table 9 gathers the types of corners which occur in basic digital circles (we refer to the basic circles of Fig. 15 as the name of possible steps (signs on the edges of the graph) to get them starting with the origin-triangle). We use here all digital circles occuring in Fig. 15, for which the figure does not contain all the three possible growing steps.

Based on Table 8 we summarize how the vertices change via the growing procedure.

Proposition 5.2

- *There are two types of vertices between 'hilly' edges, as we used type 6 and 7. (Their difference can be seen in Fig. 18.)*

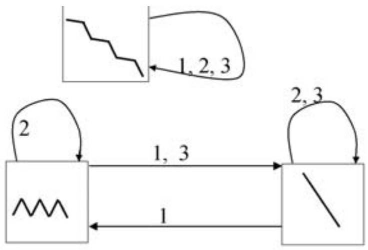


Fig. 17: State transition diagram of types of the edges of digital circles

original corner type	edges after a 1-step	edges after a 2-step	edges after a 3-step
'smooth'-'smooth' (1)	'sawtooth'-'sawtooth' (3)	'smooth'-'smooth' (1)	'smooth'-'smooth' (1)
'smooth'-'sawtooth' (2)	'smooth'-'sawtooth' (2)	'smooth'-'sawtooth' (2)	'smooth'-'smooth' (1)
'sawtooth'-'sawtooth' (3)	'smooth'-'smooth' (1)	'sawtooth'-'sawtooth' (3)	'smooth'-'hilly'-'smooth' (4,4)
'smooth'-'hilly' (4)	'sawtooth'-'hilly' (5)	'smooth'-'hilly' (4)	'smooth'-'hilly' (4)
'sawtooth'-'hilly' (5)	'smooth'-'hilly' (4)	'sawtooth'-'hilly' (5)	'smooth'-'hilly' (4)
'hilly'-'hilly' (type 6)	'hilly'-'hilly' (type 7)	'smooth'-'hilly'-'smooth' (4,4)	'smooth'-'hilly'-'smooth' (4,4)
'hilly'-'hilly' (type 7)	'smooth'-'hilly'-'smooth' (4,4)	'hilly'-'sawtooth'-'hilly' (5,5)	'hilly'-'hilly' (type 6)

Table 8: State transition table of vertices by taking a step

- *The following vertices can start a new type of edge, as we denote in the table by two values: between two 'sawtooth' edges with a 3-step we get a new 'hilly' edge between the 'smooth' ones; between two 'hilly' edges a new 'smooth' one appears (in case 6 with 2-step or 3-step) and we get a 'smooth' and a 'sawtooth' one using 1-step and 2-step, respectively, in case 7.*

We draw the state transition diagram of the vertices in Fig. 19. The double arrows mean that using these transitions, we get more vertices (two of the same type).

As we can see, our edge-types and vertex-types are in a closed set, i.e. we cannot step out from the above used sets by the growing steps. One can check also, that all kinds of vertices and edges occur in digital circles.

Using the basic digital circles and our growing tables we get all possible digital circles of the triangular grid.

In the remaining part of this section - based on our previous experience - we characterize the digital circles with neighbourhood sequences in the triangular plane.

Since neighbourhood sequences spread in an "isotropic" way, the occupied regions are somehow symmetric objects. More precisely, we have the following lemma.

Lemma 5.7 *Let B be a n.s. and $k \in \mathbb{N}$. If a point p with coordinates (p_1, p_2, p_3) belongs*

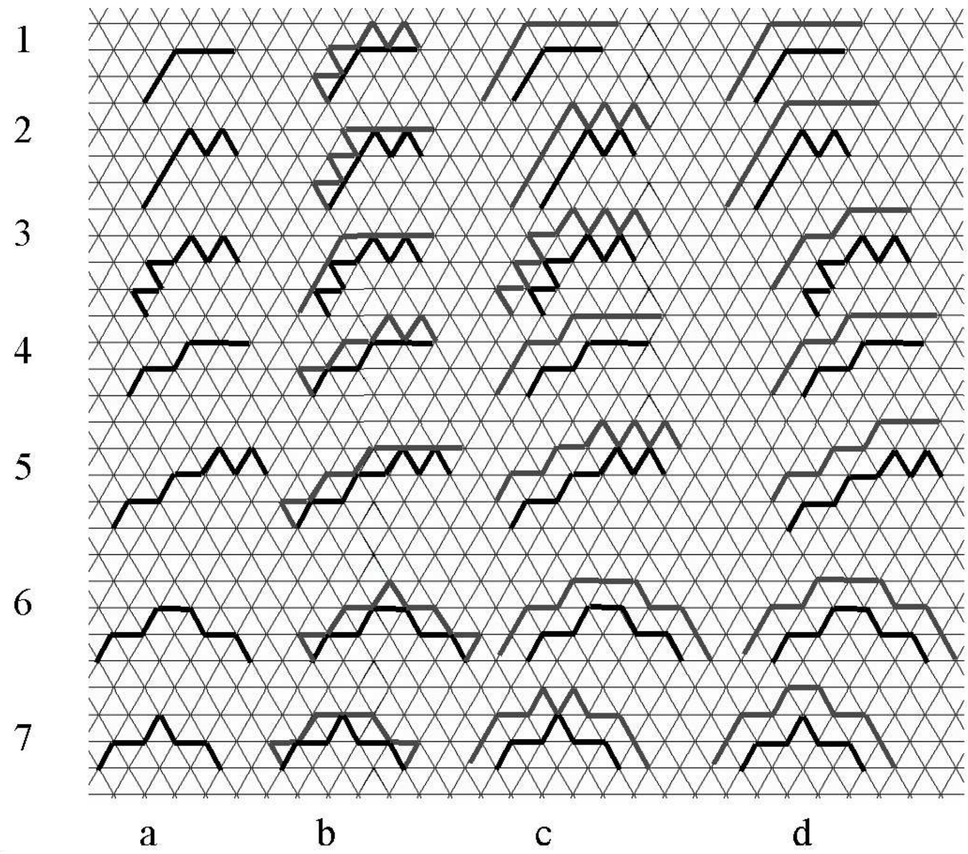


Fig. 18: Changing the corners after a step (a) original edges (b) after a 1-step (c) after a 2-step (d) after a 3-step

to C_k, then the points with coordinates $(p_{i_1}, p_{i_2}, p_{i_3})$ also belong to C_k. Here (i_1, i_2, i_3) is an arbitrary permutation of $(1, 2, 3)$.

Proof. There is no special coordinate; each of them plays equal role. Therefore permutating them, we also get points with the same distance from the Origin. □

Using the above results we know that the lines, – for which the regions occupied by neighbourhood sequences are symmetric – are the coordinate axes. Moreover the digital circles are invariant for the rotation with $2k\pi/3$ for all $k \in \mathbb{Z}$. (In general, these 6 points have the same coordinate values with permutation.)

In the following, based on Lemma 5.2 and Remark 5.9 we will use the minimal equivalent neighbourhood sequence B' instead of the original sequence B. In Table 10 and in Fig. 20 we show the types of the possible digital circles C_k. In the figure we can see the state transition diagram for them.

Theorem 5.5 *Table 10 contains all possible digital circles, and they are in the correct places, respectively.*

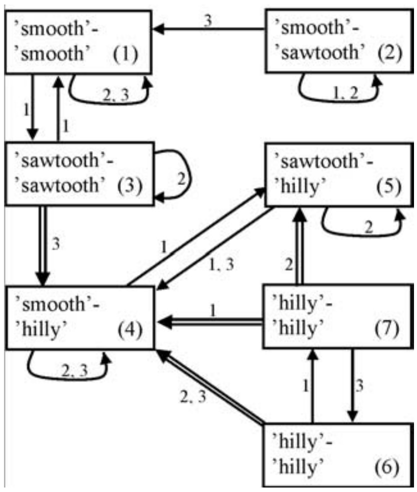


Fig. 19: State transition diagram of corners by growing digital circles

name and sign of vertex-type	'smooth'-'smooth' (1)	'smooth'-'sawtooth' (2)	'sawtooth'-'sawtooth' (3)	'smooth'-'hilly' (4)	'saw-tooth'-'hilly' (5)	'hilly'-'hilly' (type 6)	'hilly'-'hilly' (type 7)
occurrence in basic circles generated by steps ...	23=32 132=1311	21=12 22=112= =121=211 =1111	231=321= 312=3111= =1131 1321= 1312=13111	313 1313	-	313 1313	-

Table 9: Vertex-types of basic digital circles in triangular grid

In proving this theorem we will use the following facts.

Lemma 5.8 *Table 10 contains the digital circles for all possible neighbourhood sequences.*

Proof. It is evident, that – using for steps the equivalent neighbourhood sequence B' instead of B (using Remark 5.9) – all initial parts of all neighbourhood sequences occur in the second column. □

Proposition 5.3 *All basic digital circles in Fig. 15 occur in Table 10, and their types are correct.*

Now in proving Theorem 5.5, we will use induction.

Proof. By using Lemma 5.8 we know that all possible initial parts for the possible neighbourhood sequences are in Table 10. Therefore we need to prove only the statement that,

A) triangle	basic: 0 (only the starting triangle) or by a 1-step
B) hexagon – six 'smooth' edges	only one 3-step and the others are 1-steps and 2-steps; the sum after the 3-step is even
C) hexagon – three 'smooth' and three 'sawtooth' edges	with only 1-steps and 2-steps (without any 3-step)
D) hexagon – six 'sawtooth' edges	only one 3-step and the others are 1-steps and 2-steps; the sum after the 3-step is odd
E) enneagon – six 'hilly' and three 'smooth'	only odd steps: 1-steps and 3-steps by turns (with minimum 2 3-steps); without any 2-step or repetition (double 1-step) with 3-step at last
F) enneagon – six 'hilly' and three 'sawtooth'	only 1-steps and 3-steps by turns (minimum 2 3-steps) with 1-step at last
G) dodecagon – six 'hilly' and six 'smooth'	at least (a 2-step or repeated 1-steps) and at least two 3-steps and after the last 3-step the sum is even
H) dodecagon – six 'hilly' and six 'sawtooth'	at least (a 2-step or repeated 1-steps) and at least two 3-steps and after the last 3-step the sum is odd

Table 10: The possible types of digital circles

for all rows of the table, the given digital circles are correct. Our proof is by induction. From Proposition 5.3 we know this fact for the basic digital circles. Now we suppose for each digital circle that it is in the correct row in Table 10. Our induction steps are based on state transitions of the wave-fronts (Table 8 and 7 for the vertices and edges respectively). Using these facts we get the state transition diagram that we show in Fig. 20. Thus Theorem 5.5 is proved. □

In Fig. 20 we used the minimal equivalent neighbourhood sequences to represent all neighbourhood sequences. For this reason, one cannot see arrows representing 3-steps from the types of digital circles for which the sum of the elements after the last used 3-step must be even. For example types B, E and G are in this position. In that case, if the next element of the n.s. B is 3 then we get the same result as we get with element 2. (Therefore we would have used both values 2 and 3 on the arrows representing 2-steps if we had used the original n.s. B.)

We can use the state transition diagram in Fig. 20 as an automaton with, as a starting state, the starting triangle and alphabet $\{1, 2, 3\}$. The terminal state(s) will be X, where X is the type of the desired polygon ($X \subseteq$ {A,B,C,D,E,F,G,H}).

Now we analyze the convexity of the digital circles. In a strict sense there are many concave occupied regions among the digital circles. It is evident that in the square grid the occupied area O_k is convex if and only if $b(i) = 2$ for all $i < k$ in the neighbourhood sequence B. Hence we can say that only the square with 'smooth' edges is convex. In the triangular grid we have the next theorem.

Theorem 5.6 *In the triangular grid the digital circle C_k is convex if and only if it is one of the following types: A (triangle) or B (hexagon with 'smooth' edges).*

Proof. It is evident that a region is not complex if it has a 'hilly' or 'sawtooth' edge. Therefore the statement follows. □

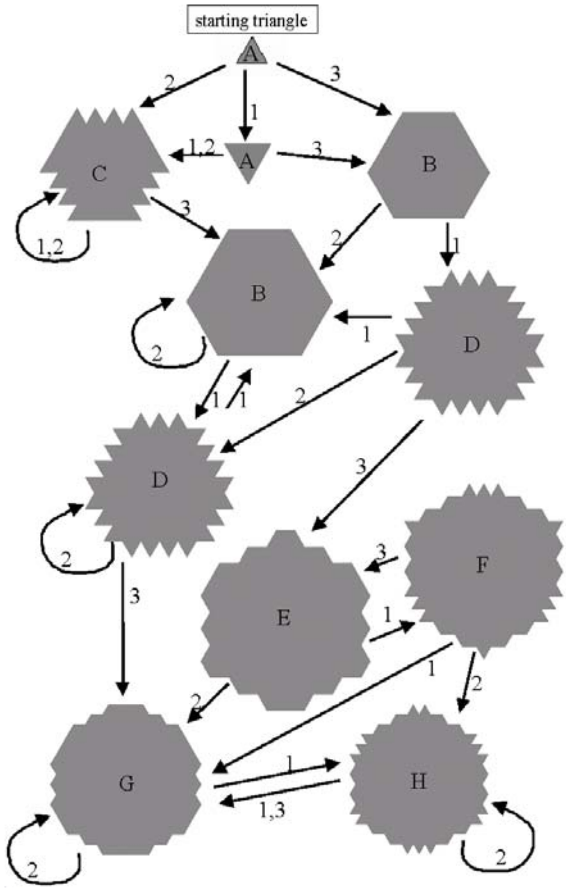


Fig. 20: State transition of types of digital circles

In this section we were growing regions in the triangular grid using 3 kinds of neighbouring relations in various n.s. In image processing, region growing is often used method for analyzing pictures (find a connected region etc., see [1, 26, 6, 21, 34, 59]). In our method, with growing digital circles, we did not care about other properties of the picture. In practice, starting from a point of an image, a variation of our method can be used. We unite only those new points of the wave-front set to our region which satisfy another desired property. We can finish the method when our region does not change, getting the result, which may depend on the n.s. used. (Using a n.s. B, our result is B-connected, i.e. the definition of connectedness and therefore the resulting picture depends on B.)

In CNN-UM structures, in some image processes there are effective algorithms which are based on morphological procedures of waves on binary pictures. It could be interesting further research to analyze these algorithms in the triangular grid. In practice, it would also be interesting to examine the development of the wave-front sets in the case of "barrels". Another possible direction of future research is the analysis of meeting waves, etc. It would be interesting if one mixed our method of region growing with the methods used in practice.

5.7 Application for networks

In this part we draw a possible application of the neighbourhood sequences in the triangular grid. Assume that we have a triangular network. Each signal going in the network uses a neighbourhood sequence in the following way. The sender orders a neighbourhood sequence for the signal, and the signal occupies the digital circle according to the ordered neighbourhood sequence in each step. (We use a discrete time-scale.)

When a sender sends a question to its environment, it waits for the answer. If an another station catches the signal of the question and knows the answer, then it sends the answer.

It is a natural condition to use metrical distances. If the same neighbourhood sequence is used in both ways, then the following statements are true.

Proposition 5.4 *Using neighbourhood sequences which generate metrics we have the following properties.*

- *The question signal is going at the same time to an answerer as the time of its answer arriving back.*
- *An answer signal can never overtake its question signal. It means that if a node gets an answer signal, then it cannot happen that it gets the question signal later on.*

Proof. It is evident that the first statement comes from the symmetric property. The second one is a consequence of the triangular inequality. □

In this case we can use finite neighborhood sequences, which means that after the step by the last element, the signal does not go further. With this finite cutting we can avoid the overload of the system; i.e. it is possible to kill signals, therefore only finitely many signals occur in the system.

Other possibility uses a special property of distances based on triangular neighbourhood sequences (see Lemma 5.5 and 5.6). In this case we use 'faster' neighbourhood sequences for the answer signals than the question signals spread. If an answerer is found, then its signal will reach the question signal and kill it. If a node gets the answer signal for a question, then it will not distribute the question signal any more. We can use infinite neighbourhood sequences in this case.

The model above using triangular neighbourhood sequences cannot work in the square-grid with neighbourhood sequences or in the hexagonal grid (using the natural neighbourhood relations).

This section was about a possible application, and it leads us to the next chapter 'Further research'.

Chapter 6 : Further research

In this part we show some possible directions for further research. In [11, 22, 23] the lattice properties of the neighbourhood sequences were analyzed in square grids. In [32] the authors analyzed the structure of the space of neighbourhood sequences defining the velocity and the distance of the neighbourhood sequences for square grids. We also have some results for triangular grids. We would like to continue the research of these topics. Now, in this chapter we are going to mention some other directions.

6.1 Distances of sequences

Using the neighbourhood sequences, we have many distances in digital geometry. But are the neighbourhood sequences useful only in digital spaces?

In the field of mathematical analysis, the theory of sequences is well described. In Chapter 3 we used the space $\mathbb{Z}^\infty$. We can imagine the points of this space as integer sequences. Analogously, the points of the space $\mathbb{Z}^n$ are finite sequences. Why do we restrict our analysis to only sequences of integers? We can allow sequences over reals: $\mathbb{R}^n$ or $\mathbb{R}^\infty$. We can define neighbourhood relations among sequences. The weight sequences allow us to make this relation in a very fine way. Using weight-sequences and neighbourhood sequences we defined a very wide scale of distances in [53]. We analyzed some properties of the sequences, such as convergence, related to their weighted B-distances. Using appropriate weight functions, we can have not only generalized metrics, but metrical distances getting only finite distance values.

Moreover it is possible to define distances not only for sequences, but also for functions. We can use $\mathbb{R} \to \mathbb{R}$ functions instead of $\mathbb{N} \to \mathbb{R}$. It is a topic of further research to investigate these distances and analyze them with respect to aspects of well-known properties of functions.

6.2 Other grids

As we mentioned in Chapter 2, using the intuitive definitions of m-neighbourhoods we can extend our theory to arbitrary graphs. From our viewpoint the planar graphs are more interesting than non-planar ones. It can be a further direction of research to extend our theory to more general grids. In this chapter we investigate some other triangular grids in detail. We will show the structure of the 3-plane triangular grid in two dimensions. After this we can define the n-plane grid and we show that for $n > 3$ these grids are non-planar. These results can be found in [52, 52a].

6.2.1 Triangular grids with more planes

As we showed in Figures 6 and 13, the points of the hexagonal and of the triangular grids form oblique planes (1 and 2 planes, respectively) in $\mathbb{Z}^3$. Let us continue the sequences of triangular grids with the three-plane grid, in which the points are from three parallel planes from $\mathbb{Z}^3$.

We will call top-plane the plane which has the highest value in the sum of coordinate values, similarly we use the terms bottom- and middle-planes. According to these three planes, the points of this grid have three possible 'parities' in contrast to the traditional triangular grid in which we have only two.

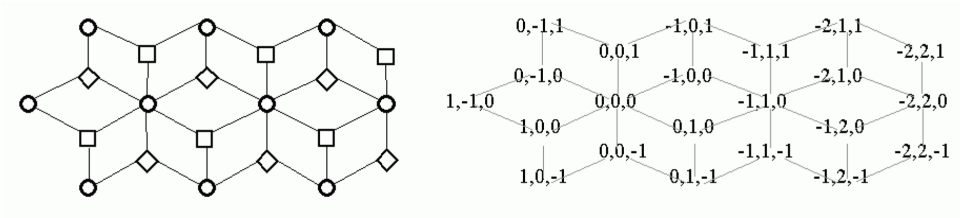


Fig. 21: The points of the 3-plane triangular grid and their coordinate values

Place of the points	num. and	types of neighb.	coordinate difference		place of neighbours
A point in the top plane:	3	1-type	1 different coordinate:	+1	in middle plane
	6	2-type	2	+1, -1	in bottom
	3	2-type	2	+1, +1	in top
	3	3-type	3	+1, +1, -1	in middle
A point in the middle plane:	3	1-type	1 different coordinate:	+1	in top plane
	3	1-type	1	-1	in bottom plane
	6	2-type	2	+1, -1	in middle
	3	3-type	3	+1, +1, -1	in top plane
	3	3-type	3	+1, -1, -1	in bottom plane
A point in the bottom plane:	6	1-type	1 different coordinate:	-1	in middle
	3	2-type	2	+1, -1	in top plane
	3	2-type	2	-1, -1	in bottom plane
	3	3-type	3	+1, -1, -1	in middle

Table 11: Numbers and types of neighbours in the three-plane triangular grid

The points have the number and types of neighbours as shown in Table 11.

As we can see, the points of the top-plane are in the same situation as the points of bottom-plane because of symmetry.

We can draw this grid to the plane using symmetry. The middle plane with six 2-neighbours means the classical hexagonal grids. Considering the 1-neighbours we have 3-3 points both in the top and bottom plane. It is easy to check that these conditions are satisfied by the grid in Figure 21. We use 3 types of points according to the 3 planes (the signs 'o' mean the points of the middle plane; the squares are the points of the top-plane and the diamonds form the bottom-plane), and we marked them by coordinate values on the right hand side of Figure 21. (Because of symmetry we use the points with 0 and ±1 coordinate sums.)

Now we make the dual of this graph. (We connect the centres of neighbour regions.) The result can be seen in Fig. 22. As we can see, the three-plane triangular grid of regions is planar.

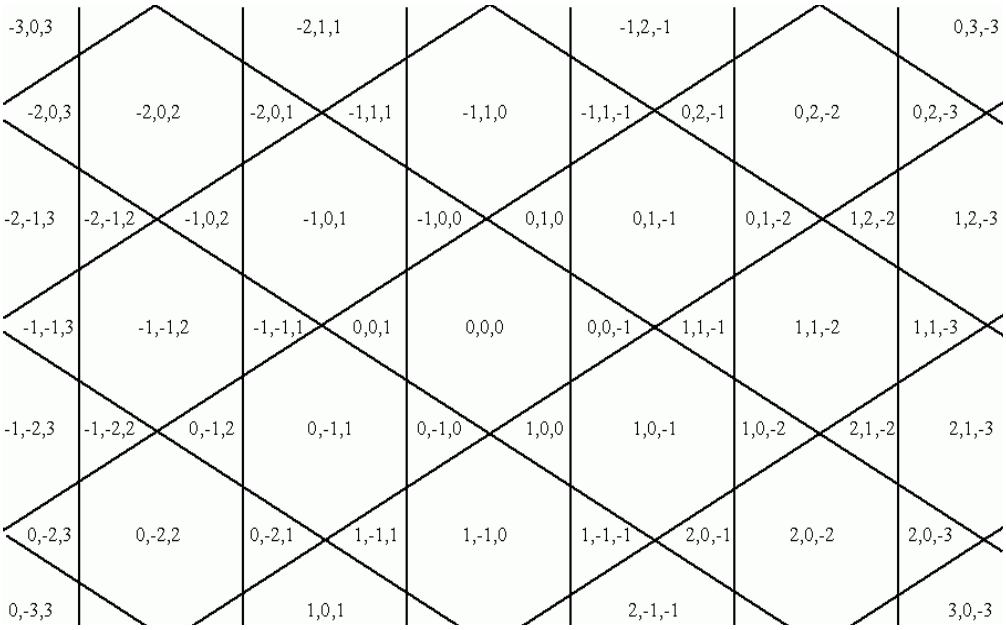


Fig. 22: The coordinatization of the 3-plane triangular grid of regions as planar graph

So this grid looks like a mix of hexagonal and triangular (actually it is the union of a one- and a two-plane grid). This grid is also known; in [55] Radványi examines it as regular planar graph $T(6,3,6,3)$. This grid is not examined from our viewpoint yet; we can introduce coordinate values (they can be seen in Fig. 22) and neighbourhood criteria in this grid.

Unfortunately, in this grid the relation of neighbourhood does not look very natural, but the dual (the grid of nodes) is nice (Fig. 21). As we can see, there are 3 types of nodes and, accordingly, in Figure 22 there are 3 types of regions; they represent the points of the 3 different planes, respectively.

Remark 6.1 *Our coordinatization and the neighbourhood criteria in 3 dimensions match.*

Fig. 23 shows how we can embed this grid to the cubic grid.

We define the lanes as in previous triangular grids. The objects where a coordinate value is fixed form a lane. Figure 24 shows the lanes in both interpretations of the grid. The lanes in the grid of regions seem to be straight lanes, and in the grid of nodes they seem to be section planes of the 3 dimensional cube-grid.

For more planes triangular grids, we use the planes with points which have coordinate sums in the set $\{0, 1, ...n-1\}$. First, look at the four-plane grid.

Theorem 6.1 *The four-plane triangular grid is non-planar.*

Proof. We will shown that the four-plane grid has a topological subgraph $K_{3,3}$, i.e., the Kuratowski graph is the homeomorphic part of the grid. (It is a sufficient condition for

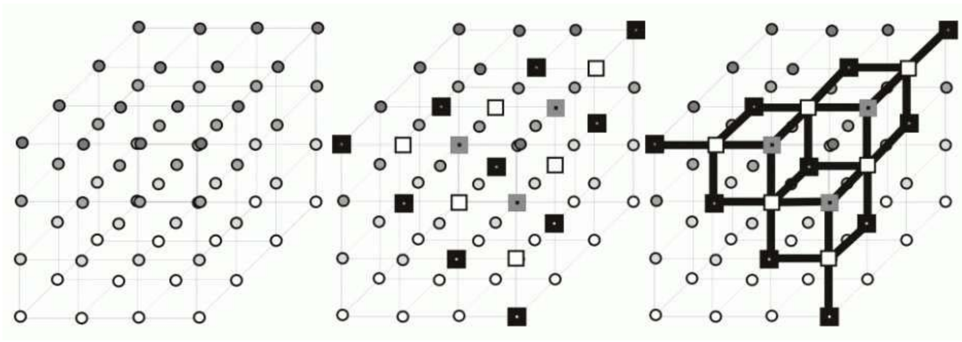


Fig. 23: The 3-plane triangular grid of nodes as three parallel planes in $\mathbb{Z}^3$

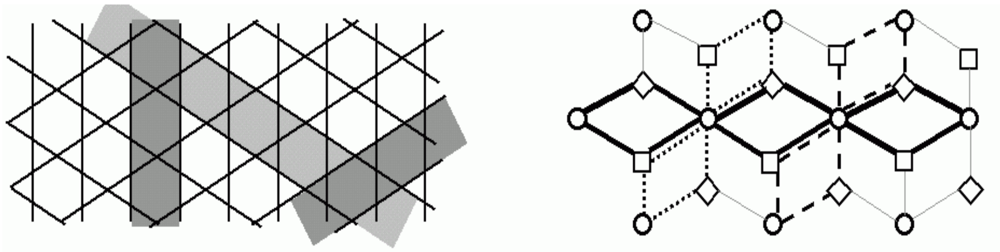
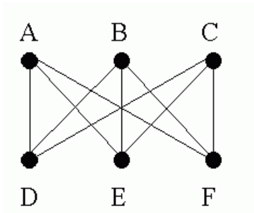


Fig. 24: Examples for lanes in the 3-plane triangular grid of regions and in the grid of nodes



A	B	C
(0,0,0)	(1,1,1)	(1,1,-1),(1,2,-1),(0,2,-1) (2,1,-1),(2,1,0),(2,0,0)

(1,0,0),(1,1,0)	(0,2,0),(0,1,0),(0,1,1)	(0,0,1),(1,0,1),(2,0,1)
D	E	F

Fig. 25: The Kuratowski graph ($K_{3,3}$) and the coordinate values of nodes of the grid, respectively

our statement, see Theorem 10.5.1. in [35], pp. 243.) In Figure 25 we can see an example satisfying this property. As we can see, the nodes are 1-neighbours of each-other, respectively. □

We also can define triangular grids using by more planes than four. So we take a general definition for n-plane triangular grids in $\mathbb{Z}^3$.

Corollary 6.1 *The n-plane triangular grids for $n > 4$ are non-planar.*

Proof. Since the four-plane triangular grid is a proper part of the n-plane triangular grid for $n > 4$, the statement follows. □

6.3 Approximating curves

As we mentioned before there are some papers on approximating Euclidean circles in digital grids with neighbourhood sequences. In ([33]) we made the approximation in the square-, in the hexagonal and in the triangular grids as well. We plan an approximation for other curves also. The program drawing digital parabolas and ellipses in the triangular grid with respect to neighbourhood sequences already exists.

Note that the best approximations for circles and parabolas are given by non-periodic neighbourhood sequences. If one can use only one neighbourhood sequence for all sizes of circles in the square grid, then the ratio of densities of 1's and 2's is non-rational (related to $\sqrt{2}$).

About parabolas, we have the following definition and facts.

Definition 6.1 *Let A be a fixed lane, i.e. the set of those points of the digital plane for which a coordinate value is fixed. Let p be a point which is not in A, and let B be a neighbourhood sequence. Let the set P of the points q for which there exist a point $r \in A$ such that $d(p, q; B) = d(r, q; B)$, and there are not any points $s \in A$ such that $d(s, q; B) < d(r, q; B)$, be the parabola with parameters p, A and B.*

First we use the definition in the square-grid.

Remark 6.2 *The distance between a lane A and a point p is independent of the neighbourhood sequence B in case of square-grid.*

Figure 26 shows digital parabolas with neighbourhood sequences $(1), (2), (1, 2)$ and $B_\square = (2, 1, 1, 2, 1, 1, 1, 1, 2, 1, ...)$. The last non-periodic case is a better approximation for the parabola in the Euclidean space with the respective parameters. The neighbourhood sequences for the best approximation have the property that the values 2 occur in decreasing number in them. (For example at $B_\square$ we use the values $b(i) = 2$ iff i is a square.)

At first sight one can see that most of the parabolas are not connected.

In the triangular grid we defined a distance of points and lanes, which is independent of the neighbourhood sequences (Procedure 5.2). But in this grid, contrary to the case of the square grid, this distance can depend on the given neighbourhood sequence, therefore we use the same definition of parabolas as in the square grid (Definition 6.1). In Figure 27 some parabolas can be seen in the triangular grid.

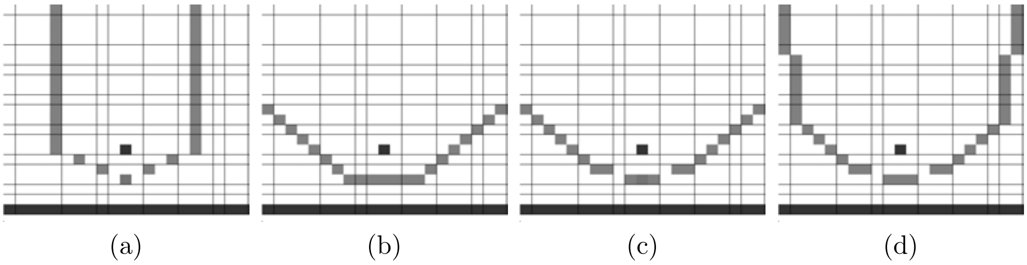


Fig. 26: Digital parabolas with neighbourhood sequences (a) (1), (b) (2), (c) (1, 2), (d) $B_\square$

As we mentioned at the end of Section 5.6, the meeting wave-fronts can also be interesting in practice. The points of a parabola are exactly the points of two meeting wave-fronts: one wavefront is a spreading digital circle from point p and the other one is a wave-front starting from lane A (, where p and A are the parameters of the parabola). In the triangular grid, both wavefronts depend on the neighbourhood sequence.

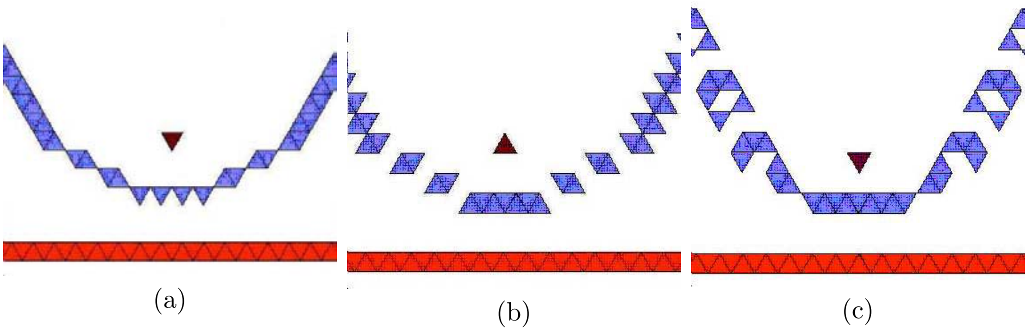


Fig. 27: Digital parabolas in the triangular grid with neighbourhood sequences (a) (1), (b) (2), (c) (1, 2, 3)

Investigating other digital curves and analyzing their properties (with respect to the digital plane and the properties of the curves in the Euclidean plane) can be an interesting job.

6.4 Random walk

We note here that the author wrote his first master thesis in the field of random walks in square grids ([47].) Several kinds of random walks can be in several kinds of grids. In many cases the neighbourhood sequences can be used instead of the frequently used constant neighbourhood criteria. Using neighbourhood sequences in a random walk, we can get some properties of Lévy-flights [18]:

Remark 6.3 *Allowing changes in many coordinate values in a step, we can go very far from the point in the sense that many 1-steps are needed to go this distance.*

It can be an interesting topic of future work to analyze some kinds of random walk in grids using n.s.-es.

6.5 Other applications

There are some other applications in the literature for periodic neighbourhood sequences in square and cube grids, mostly in the area of image processing. Some of them can be applicable with triangular grids as well.

We just want to mention other possible applications, for example in the fields of cellular automata. The birth of the theory of cellular automata is connected with John von Neumann. In this field, the term 'Neumann-neighbourhood' is used for 1-neighbours (in our terminology), and the 'Moore-neighbourhood' is used for n-neighbourhoods in the n dimensional square grid. As we showed in some figures, the meeting wave-fronts can draw figures which look like Conway's life-games.

Another possible field of applications are the picture languages and some simulations in physical systems.

Chapter **7** : Summary

The dissertation consists of six chapters. After a brief historical overview of the digital geometry (first chapter) the basic notion and notations are given in the second chapter. Some preliminaries and some of our new results are presented in the third chapter about the square grids. In the fourth chapter some well-known results are shown using our terminology about the hexagonal grid. In the fifth chapter several new results are presented on the triangular grid. In the sixth chapter there are some further directions of the research, in some of them we already have some results.

In the **first chapter** the introduction and a brief bibliographic history of digital geometry is given.

In the **second chapter** the concept of neighbours and neighbourhood sequences ($B = (b(i))_{i=1}^{\infty}$) are presented. In different grids we use different types of neighbours. The definitions of the B-paths and B-distances are also given: with the help of a neighbourhood sequence $(b(i))_{i=1}^{\infty}$ we may define the distance of points p, q in the following way. We take the length of a shortest path from p to q, but at the i-th step now we may move from a point to another if and only if they are $b(i)$-neighbours.

In the **third chapter** we use the concept of neighbourhood sequences in square-grids ($\mathbb{Z}^N, N \in \mathbb{N} \cup \{\infty\}$). The values $b(i)$ can be positive integers up to the dimension of the space. We analyze the infinite dimensional digital space as well, in this case the symbol ∞ also may appear in the neighbourhood sequence. Our first result is the following one. We present an algorithm to solve the shortest path problem with arbitrary neighbourhood sequences with arbitrary dimensions. The proof of the correctness of the algorithm is given as well.
The concept of limited neighbourhood sequences is introduced, as $B^{(m)} = \left(b^{(m)}(i)\right)_{i=1}^{\infty}$, where $b^{(m)}(i) = \min(b(i), m)$ for all i. The sorted (absolute) difference vector v of two points is also defined. By their help we derive a fairly simple formula to calculate B-distances of points. The B-distance of the points p and q is $d(p, q; B) = \max\limits_{i<N+1}(d_i(w))$, where $d_i(w) = \max\left\{ h \,\middle|\, \sum\limits_{j=1}^{i} v(j) > \sum\limits_{j=1}^{h-1} b^{(i)}(j) \right\}$. In the infinite dimensional space this distance can be infinite.
The faster relation among the neighbourhood sequences is recalled. We introduce the concept of shifted sequences, which is useful to give a necessary and sufficient condition for a neighbourhood sequence to generate a metric. A B-distance is a metric in a square grid if and only if each shifted sequence of B is faster than B. It is one of our main results.

The **fourth chapter** is about the hexagonal grid. On the hexagonal grid we use one natural distance function, based on the natural neighbourhood relation. We use three coordinate values to describe the grid. The sum of the coordinate values is zero for each point. A procedure is presented to solve the shortest path problem in this grid using our concept of lanes. By the help of the presented shortest path we calculate the distance of two objects repeating the known result in symmetric form using three coordinates. We show that this distance is a metric. The advantage of this grid over the triangular and rectangular grids is that the methods used for approximating the Euclidean distance are simpler.

In the **fifth chapter** we present many interesting new results. We analyze three types of neighbourhood relations in the triangular grid. We introduce 3 coordinates and the concept of neighbourhood sequences ($B = (b(i))_{i=1}^{\infty}$, $1 \leq b(i) \leq 3$ for all values of i), and with their help we are able to define distance functions on the triangular grid. The concept of lanes is investigated as well. One of our results is that we present an algorithm, which constructs a shortest path from one point to another. We also examine the complexity of our greedy algorithm. In some cases a value $b(i) = 3$ does not give more chances to step closer to the endpoint as a value 2. Therefore we introduce the concept of minimal equivalent neighbourhood sequence $B'(b'(i))_{i=1}^{\infty}$ which is very helpful. Examining the distance functions generated by neighbourhood sequences, we show that these distances do not generate a metric for every sequence. Since we have two types of points (according to the sum of their coordinate values we call them even and odd points, respectively), we have to face some additional problems that are not present in the square and the hexagonal grids. For example, we can have non-symmetric distance functions. One of our main results about the triangular grid is the presented sufficient and necessary condition for neighbourhood sequences to determine metric spaces. In the triangular grid a neighbourhood sequence B defines a metric if and only if the following conditions hold: if $b(j) = 3$ and $b(i) = 1$ then $i < j$,
if B contains 3, then $\sum\limits_{b(k)=1} b(k)$ is even, and
$\sum\limits_{k=1}^{i} b(k) \leqslant \sum\limits_{k=j+1}^{j+i} b(k)$, when $i + j < l$, where l is the index of the first 3 value in B, (if 3 is not in B then this condition must hold for all $i, j \in \mathbb{N}$).
We showed how the points of the triangular grid can be embedded into the cubic grid. Using this mapping we modified the formula of the cubic grid for calculating distances in the triangular case. We use the minimal equivalent n.s.-es and the reduced minimal equivalent n.s.-es (B"). The sorted difference vector v of the points is used as well. One of the most important results is the following calculation of the B-distance from a point p to a point q: Let $d'_2 = \max\left(i \,\middle|\, |v(1)| + |v(2)| > \sum\limits_{j=1}^{i-1} b'^{(2)}(j)\right)$, $d'_3 = \max\left(i \,\middle|\, \sum\limits_{l=1}^{3} |v(l)| > \sum\limits_{j=1}^{i-1} b'(j)\right)$, $d"_2 = \max\left(i \,\middle|\, |v(1)| + |v(2)| > \sum\limits_{j=1}^{i-1} b"^{(2)}(j)\right)$, $d"_3 = \max\left(i \,\middle|\, \sum\limits_{l=1}^{3} |v(l)| > \sum\limits_{j=1}^{i-1} b"(j)\right)$, $d' = \max(|v(1)|, d'_2, d'_3)$ and $d" = \max(|v(1)|, d"_2, d"_3)$. The distance from a point p to q with a given neighbourhood sequence B equals to d" if and only if if the distance $d' \geq k$ (where k is the minimal value such that $b(k) = 3$, maybe infinite) and one of the followings hold:
p is even, $\sum\limits_{i=1}^{k-1} b(i)$ is even, and the difference $w_{p,q}$ has two negative and a positive value;
p is odd, $\sum\limits_{i=1}^{k-1} b(i)$ is odd and the difference of q and p has two negative and a positive value;

p is odd, $\sum_{i=1}^{k-1} b(i)$ is even and the difference of q and p has two positive and a negative value;
p is even, $\sum_{i=1}^{k-1} b(i)$ is odd and the difference of q and p has two positive and a negative value.
The distance has the value d' in other cases.
We also characterize the digital circles. We give the possible types of edges and vertices of these digital polygons and studied the development of the possible wave-fronts step by step. The digital circles in the triangular grid can have three kinds of edges such as 'smooth', 'sawtooth' and 'hilly'. The possible seven kinds of vertices are also presented. We list the types of the digital circles occupied by neighbourhood sequences, and performed the symmetry and convexity analysis of these regions. A digital circle can be a triangle, a hexagon, an enneagon or a dodecagon. Some interesting properties of these digital circles are presented. For example the same digital circle can have several radii depending on the used neighbourhood sequences.
We present possible applications for communication networks.

Finally, in the **sixth chapter**, some directions for further research are shown. We have achieved some results extending the concept of B-distances from integer-sequences to real-sequences, as well as some additional results about the generalization of triangular grids. The so-called 3-plane triangular grid is shown in the plane. We prove that the n-plane triangular grids are non-planar grids for $n > 3$. It might be interesting to continue these lines of research. Furthermore, the concept of neighbourhood sequences could be used for research in areas like digital curves, random walks or cellular automata. Some examples for digital parabolas are presented both in the square grid and in the triangular grid.

Összefoglaló

A dolgozat hat fejezetből és azon belül több alfejezetből áll.

Az **első fejezet** rövid (történeti) bevezetést tartalmaz a digitális geometria, illetve a szomszédsági sorozatok témakörébe. Itt mutatjuk be a témánkkal kapcsolatos irodalmat is.

Ezután a **második fejezetben** ismertetjük az alapdefiníciókat, úgymint a szomszédsági sorozatokat ($B = (b(i))_{i=1}^{\infty}$, ahol a $b(i)$ értékek lehetséges szomszédságokat jelentenek), illetve a segítségükkel értelmezett utakat, távolságokat. A p, q pontok B-távolsága egyenlő a p-ből a q-ba vezető egyik legrövidebb B-út hosszával, amelyben az i-edik lépésben egy pontról valamelyik $b(i)$-szomszédjára léphetünk tovább.

A **harmadik fejezetben** a négyzetrácsot, illetve magasabb dimenziós formáit tárgyaljuk. A szomszédsági sorozat elemei pozitív egészek, nem nagyobbak a tér dimenziójánál. A végtelen dimenziós digitális teret is vizsgáljuk, ezesetben a ∞ szimbólum is szerepelhet a szomszédsági sorozatokban. Megjegyezzük, hogy a végtelen dimenziós digitális térben végtelen távolságok is előfordulnak, ekkor nem létezik B-út a két pont között. Első eredményeinkként algoritmust adunk a legrövidebb B-út probléma megoldására (arra az esetre, ha van út a két pont között), és bizonyítjuk annak helyességét. Ezután egy viszonylag egyszerű képlettel határozzuk meg a B-távolságot két pont között. A formulát a B által generált hipergömbök csúcsainak koordinátái segítségével határozzuk meg. A szomszédsági sorozatokon alapuló távolságokra a háromszög-egyenlőtlenség nem mindig áll fenn. Az általunk bevezetett korlátozott-, illetve eltolt szomszédsági sorozatok segítségével adunk egy szükséges és elégséges feltételt a B szomszédsági sorozatokra, ahhoz hogy a megfelelő B-távolság metrika legyen: A digitális térben egy B szomszédsági sorozaton alapuló távolságfüggvény pontosan akkor metrika, ha a $B(i)$ eltolt sorozat 'gyorsabb', mint B bármely i-re. Ez az egyik legfontosabb eredményünk a négyzetes rácsokra vonatkozóan, ráadásul ezen eredményeink nagy része kiterjeszthető a digitális térről a valósszám-sorozatokra is, ahogy utalunk rá a hatodik fejezetben.

Ezután, a **negyedik fejezetben** áttérünk a hatszögrács tanulmányozására, ahol egyféle szomszédság az általánosan használt. Itt három koordinátát használunk a rács leírására, valamint eljárást adunk a legrövidebb út előállítására az általunk bevezetett 'sávok' segítségével. A három koordináta segítségével szimmetrikus alakban határozzuk meg két pont távolságát. Bebizonyítjuk, hogy ez a távolság metrika.

Az **ötödik fejezet** a háromszögrácsról szól. Ebben a fejezetben sok új eredményt ismertetünk. A háromszögrácsot, a szimmetriát megőrizve ugyancsak három koordinátával írjuk le. A pontok koordináta-összege nulla vagy egy, így megkülönböztetünk páros, illetve páratlan pontokat. A bevezetett koordinátarendszer és a három lehetséges szomszédsági viszony összeillik, vagyis a koordináták segítségével ugyanazokat a szomszédsági relációkat definiálhatjuk. Bevezetjük a szomszédsági sorozatokat (amiknek most az elemei az $\{1, 2, 3\}$ halmazból vehetnek fel értékeket), értelmezzük az általuk generált utakat és távolságokat. Fontos eredményünk, hogy algoritmust adunk egy legrövidebb B-út előállítására bármely adott ponttól egy adott másik pontig. Előfordulhat, hogy

hiába van hármas érték a szomszédsági sorozat egy helyén, azt nem tudjuk igazán kihasználni a legrövidebb útban, a pontok paritása miatt csak egy 2-szomszédra érdemes továbblépni. Ennek a tulajdonságnak a kezelésére bevezetjük a minimális ekvivalens és a redukált minimális ekvivalens szomszédsági sorozatokat minden szomszédsági sorozathoz.
A háromszögrácson a háromszögegyenlőtlenség esetleges nem teljesülésétől függetlenül előfordulnak nem-szimmetrikus B-távolságok is. A következő fontos eredményünk egy szükséges és elégséges feltétel a B szomszédsági sorozatokra ahhoz, hogy az általa generált B-távolság metrika legyen: A B-távolság pontosan akkor metrikus, ha teljesülnek a következők:
ha $b(j) = 3$ és $b(i) = 1$ akkor $i < j$,
ha a B-ben szerepel a 3, akkor $\sum\limits_{b(k)=1} b(k)$ páros,
$\sum\limits_{k=1}^{i} b(k) \leqslant \sum\limits_{k=j+1}^{j+i} b(k)$, ahol $i + j < l$, arra az l-re ami a B-ben levő első 3 helye, (ha a 3 nincs a B-ben akkor a feltételnek minden $i, j \in \mathbb{N}$ párra fenn kell állnia).
Ugyancsak levezetünk képletet, amellyel bármely pontból bármely pontba a B-távolság meghatározható. A formulát egyébként a kockarácsra működő formula alapján szerkesztjük meg, felhasználva a háromszögrács pontjainak a kockarácsba való beágyazhatóságát. A szomszédsági sorozatokkal terjedő jelek hullámfrontjait is vizsgáljuk, hogyan változnak lépésről lépésre. Összehasonlítjuk a négyzet- és a háromszögrács digitális köreit. A háromszögrácson néhány érdekes, nem szokványos tulajdonságot is bemutatunk, pl. azt hogy egy adott körnek különböző sugarai lehetnek a felhasznált szomszédsági sorozattól függően. (Különböző sorozatok generálhatják ugyanazt a kört különböző sugárral.) Leírjuk a nyolcféle kialakuló kört a háromszögrácson. Ezeknek a köröknek háromféle oldala ('fűrészfog','dombos', illetve 'sima'), illetve hétféle csúcsa lehet. A digitális körök tulajdonképpen három-, hat-, kilenc-, illetve tizenkétszögek a háromszögrácson. A digitális körök szimmetria és a konvexitás vizsgálata ugyancsak megtalálható a dolgozatban. Egy lehetséges hálózatos alkalmazást is felvázoltunk, ahol a háromszögrács éppen a nem szokványos tulajdonságai miatt tűnik használhatónak.

A **hatodik fejezetben** néhány további kutatási irányt mutatunk be. Az első ezek közül a szomszédsági sorozatok alkalmazása véges, illetve végtelen valósszám-sorozatok távolságmérésére. Itt megjegyezzük, hogy alkalmas súlysorozat és szomszédsági sorozat kombinációjával elérhető, hogy a távolság mindig véges legyen még végtelen sorozatok esetében is.

A hatszögrács egy, a háromszögrács pontjai pedig két ferde sík pontjainak felelnek meg a kockarácsban. Felvázoltuk az 'több-síkos' háromszögrács családot. A következő tag pontjai három darab ferde sík pontjainak felelnek meg a kockarácsban. A hatodik fejezet második részében megmutatjuk e rácsot síkbarajzolva, és bebizonyítjuk, hogy a 4-, illetve még több-sík pontjait tartalmazó háromszögrácsok nem síkbarajzolhatóak.

Egyéb érdekes kutatási irány lehet pl. görbék közelítése, illetve digitális megfelelőik vizsgálata. A disszertációban definiáljuk a digitális parabolákat a négyzet- és a háromszögrácson, néhány példával illusztráljuk őket.

Ugyancsak érdekes lehet a bolyongások vagy a sejtautomaták vizsgálata, a szomszédsági sorozatok segítségével különböző rácsokon.

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