

IMPLICATIONS BETWEEN GENERALIZED CONVEXITY PROPERTIES OF REAL FUNCTIONS

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ABSTRACT. Motivated by the well-known implications among t -convexity properties of real functions, analogous relations among the upper and lower M -convexity properties of real functions are established. More precisely, having an n -tuple (M_1, \dots, M_n) of continuous two-variable means, the notion of the descendant of these means (which is also an n -tuple (N_1, \dots, N_n) of two-variable means) is introduced. In particular, when all the means M_i are weighted arithmetic, then the components of their descendants are also weighted arithmetic means. More general statements are obtained in terms of the generalized quasi-arithmetic or Matkowski means. The main results then state that if a function f is M_i -convex for all $i \in \{1, \dots, n\}$, then it is also N_i -convex for all $i \in \{1, \dots, n\}$. Several consequences are discussed.

1. INTRODUCTION

In the theory of convex functions the notion of t -convexity plays an important role. For $t \in]0, 1[$ a real function $f : I \rightarrow \mathbb{R}$ (where I is a nonempty real interval) is termed t -convex (cf. Kuhn [4], Nikodem–Páles [7]) if, for all $x, y \in I$, the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds. The $\frac{1}{2}$ -convex functions are usually called *Jensen convex*. If a function is t -convex for all $t \in]0, 1[$ then it is simply called *convex*. Among the many implications related to t -convexity properties we mention the following ones:

- (1) If f is Jensen convex then it is \mathbb{Q} -convex, i.e., t -convex for all $t \in [0, 1] \cap \mathbb{Q}$ (Kuczma [3]);
- (2) If f is t -convex for some $t \in]0, 1[$, then it is Jensen convex (Daróczy–Páles [1]);
- (3) If f is t -convex for some $t \in]0, 1[$, then, by a result of Kuhn [4], there exists a subfield K of \mathbb{R} such that

$$\{s \in]0, 1[\mid f \text{ is } s\text{-convex}\} =]0, 1[\cap K.$$

For more general results related to higher-order convexity notions refer to the paper by Gilányi and Páles [2].

We recall now the notion of *second-order divided difference* defined for $f : I \rightarrow \mathbb{R}$ and pairwise distinct elements x, y, z of I by

$$[x, y, z; f] := \frac{f(x)}{(y-x)(z-x)} + \frac{f(y)}{(x-y)(z-y)} + \frac{f(z)}{(x-z)(y-z)}.$$

In terms of this concept, the t -convex functions have the following easy-to-see characterization: *A function $f : I \rightarrow \mathbb{R}$ is t -convex if and only if, for all $x, y \in I$ with $x \neq y$, we have $[x, tx + (1-t)y, y; f] \geq 0$.*

In the paper Nikodem–Páles [7], t -convex functions were also characterized by the nonnegativity of a certain second-order derivative which is analogous to the standard characterization of twice differentiable

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convex functions. In this paper, an inequality related to the second-order divided differences was also established which turned out to be a key tool for the proofs of the main results therein.

Proposition 1.1. (Chain Inequality) *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$. Then, for all $n \in \mathbb{N}$, $x_0 < x_1 < \dots < x_{n+1}$ in I , and for all $i \in \{1, \dots, n\}$, the following inequalities hold:*

$$\min_{1 \leq j \leq n} [x_{j-1}, x_j, x_{j+1}; f] \leq [x_0, x_i, x_{n+1}; f] \leq \max_{1 \leq j \leq n} [x_{j-1}, x_j, x_{j+1}; f].$$

To demonstrate the use of this inequality, we show that t -convexity implies Jensen convexity for every real function $f : I \rightarrow \mathbb{R}$. Assume that $f : I \rightarrow \mathbb{R}$ is t -convex for some $t \in]0, \frac{1}{2}[$ and let $x, y \in I$ with $x < y$ be arbitrary points. Set

$$x_0 := x, \quad x_1 := tx + (1-t)\frac{x+y}{2}, \quad x_2 := \frac{x+y}{2}, \quad x_3 := t\frac{x+y}{2} + (1-t)y, \quad x_4 := y.$$

Then

$$x_1 = tx_0 + (1-t)x_2, \quad x_2 = tx_3 + (1-t)x_1, \quad x_3 = tx_2 + (1-t)x_4,$$

whence, by the t -convexity of f , we have

$$[x_0, x_1, x_2; f] \geq 0, \quad [x_1, x_2, x_3; f] \geq 0, \quad [x_2, x_3, x_4; f] \geq 0.$$

In view of the Chain Inequality, this implies that $[x_0, x_2, x_4; f] \geq 0$ also holds, which is equivalent to the Jensen convexity of f .

The Jensen convexity property of a function is equivalent to the restricted condition

$$f(\frac{1}{2}x + \frac{1}{2}y) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) \quad (x, y \in I, x < y).$$

On the other hand, for $t \in]0, 1[\setminus \{\frac{1}{2}\}$, the t -convexity property is equivalent to the condition

$$\begin{cases} f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \\ f((1-t)x + ty) \leq (1-t)f(x) + tf(y) \end{cases} \quad (x, y \in I, x < y),$$

that is, the t -convexity property can be expressed in terms of two inequalities over the triangle $\{(x, y) \in I^2 \mid x < y\}$. It turns out that these two inequalities are not consequences of each other for every $t \in]0, 1[\setminus \{\frac{1}{2}\}$. In 2014, for every transcendental number t , Lewicki and Olbryś [5] constructed a function $f : I \rightarrow \mathbb{R}$ such that

$$\begin{cases} f(tx + (1-t)y) < tf(x) + (1-t)f(y) \\ f((1-t)x + ty) > (1-t)f(x) + tf(y) \end{cases} \quad (x, y \in I, x < y).$$

It is, however, unknown if these two inequalities are equivalent to each other for rational, or more generally, for algebraic t . (Moreover, the particular case $t = \frac{1}{3}$ also has not been answered yet.)

In this paper, for a given two-variable mean $M : \{(x, y) \in I^2 \mid x \leq y\} \rightarrow \mathbb{R}$, we consider the class of functions $f : I \rightarrow \mathbb{R}$ satisfying the inequality

$$f(M(x, y)) \leq \frac{y - M(x, y)}{y - x} f(x) + \frac{M(x, y) - x}{y - x} f(y)$$

for all $x, y \in I$ with $x < y$. Such functions will be called M -convex. In this terminology, the t -convexity of a function $f : I \rightarrow \mathbb{R}$ is equivalent to its convexity with respect to the means \mathbb{A}_t and \mathbb{A}_{1-t} , where, for $s \in [0, 1]$, the *weighted arithmetic mean* $\mathbb{A}_s : \{(x, y) \in \mathbb{R}^2 \mid x \leq y\} \rightarrow \mathbb{R}$ defined by

$$\mathbb{A}_s(x, y) = sx + (1-s)y.$$

Observe that, for $x < y$ and $0 < s < t < 1$, we have

$$x = \min(x, y) = \mathbb{A}_1(x, y) < \mathbb{A}_t(x, y) < \mathbb{A}_s(x, y) < \mathbb{A}_0(x, y) = \max(x, y) = y.$$

Motivated by the above-described implications among t -convexity properties, we are going to establish analogous relations among the \mathbb{A}_t -convexity properties of real functions. More generally, we will introduce and investigate the notions of upper and lower M -convexity for extended real valued functions. The main results of the paper then establish several implications between these convexity properties.

More precisely, having an n -tuple (M_1, \dots, M_n) of continuous means, we introduce the notion of the descendant of these means which is also an n -tuple (N_1, \dots, N_n) of means. In several cases, we explicitly construct the descendant of a given n tuple of means. In particular, when all the means M_i are weighted arithmetic then the components of their descendants are also weighted arithmetic means. More general statements are also obtained in terms of the generalized quasi-arithmetic or Matkowski means. In our main results we then prove that if a function f is M_i -convex for all $i \in \{1, \dots, n\}$, then it is also N_i -convex for all $i \in \{1, \dots, n\}$.

2. NOTATIONS AND TERMINOLOGY

If $n, m \in \mathbb{Z}$ then set $\{k \in \mathbb{Z} \mid n \leq k \text{ and } k \leq m\}$ will be denoted by $\{n, \dots, m\}$. According to this convention $\{n, \dots, m\} = \emptyset$ if $m < n$ and $\{n, \dots, m\}$ is the singleton $\{n\}$ if $n = m$.

Given a subset $S \subseteq \mathbb{R}$ and $n \in \mathbb{N}$, we denote the set of increasingly and strictly increasingly ordered n -tuples of S by S_{\leq}^n and $S_{<}^n$, i.e.,

$$S_{\leq}^n := \{(t_1, \dots, t_n) \in S^n \mid t_1 \leq \dots \leq t_n\} \quad \text{and} \quad S_{<}^n := \{(t_1, \dots, t_n) \in S^n \mid t_1 < \dots < t_n\},$$

respectively.

A function $M : S_{\leq}^2 \rightarrow \mathbb{R}$ is called a *two-variable mean on S* and a *two-variable strict mean on S* if

$$x \leq M(x, y) \leq y \quad ((x, y) \in S_{\leq}^2) \quad \text{and} \quad x < M(x, y) < y \quad ((x, y) \in S_{<}^2),$$

respectively. We note that, two-variable means are usually defined on the Cartesian product S^2 , however, in our approach the values of means on $S_{>}^2 := S^2 \setminus S_{\leq}^2$ are irrelevant. Obviously, if $T \subseteq S$, then the restriction $M|_{T_{\leq}^2}$ is also a mean on T .

In the subsequent sections I always denotes a nonempty interval of \mathbb{R} .

The most important class of two-variable means that appears in the consequences of our results is the class of generalized quasi-arithmetic means introduced by J. Matkowski [6] in 2010: We say that a function $M : I_{\leq}^2 \rightarrow \mathbb{R}$ is a *generalized quasi-arithmetic mean* or a *Matkowski mean* if there exist continuous, strictly increasing functions $f, g : I \rightarrow \mathbb{R}$ such that

$$M(x, y) = \mathbb{M}_{f,g}(x, y) := (f + g)^{-1}(f(x) + g(y)) \quad ((x, y) \in I_{\leq}^2).$$

Under the conditions of this definition, it is obvious that $\mathbb{M}_{f,g}$ is a continuous strict mean on I which is also strictly increasing in each of its variables.

If $s \in]0, 1[$ and $f : I \rightarrow \mathbb{R}$ is a continuous strictly increasing function then the Matkowski mean $\mathbb{M}_{sf, (1-s)f}$ is called a *weighted quasi-arithmetic mean*. We can see that

$$\mathbb{M}_{sf, (1-s)f}(x, y) = f^{-1}(sf(x) + (1-s)f(y)) \quad ((x, y) \in I_{\leq}^2).$$

For $s = 1/2$ this function is termed a (*symmetric*) *quasi-arithmetic mean*. Finally, observe that the mean $\mathbb{M}_{s \text{id}, (1-s) \text{id}}$ equals the weighted arithmetic mean \mathbb{A}_s .

3. AUXILIARY RESULTS

Theorem 3.1. *For $n \in \mathbb{N}$ and for the vectors $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{R}_+^n$, define the two-diagonal matrix*

$$A(u, v) := \begin{pmatrix} 0 & u_1 & \dots & 0 & 0 \\ v_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & u_n \\ 0 & 0 & \dots & v_n & 0 \end{pmatrix} \in \mathbb{R}_+^{(n+1) \times (n+1)}. \quad (1)$$

Then all the eigenvalues of $A(u, v)$ are real numbers. Furthermore, the eigenvalues of $A(u, v)$ are smaller than 1 if and only if $w_1, \dots, w_n > 0$, where $w_{-1} := w_0 := 1$, and

$$w_k := w_{k-1} - u_k v_k w_{k-2} \quad (k \in \{1, \dots, n\}). \quad (2)$$

Proof. In the sequel, denote by I_k the unit matrix of the matrix algebra $\mathbb{R}^{k \times k}$, and for a square matrix $S \in \mathbb{R}^{k \times k}$, denote by P_S the characteristic polynomial of S defined for $\lambda \in C$ by $P_S(\lambda) := \det(\lambda I_k - S)$.

Let $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{R}_+^n$ be fixed. Define $A_0(u, v) := 0$ and

$$A_k(u, v) := \begin{pmatrix} 0 & u_1 & \dots & 0 & 0 \\ v_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & u_k \\ 0 & 0 & \dots & v_k & 0 \end{pmatrix} \in \mathbb{R}_+^{(k+1) \times (k+1)} \quad (k \in \{1, \dots, n\}). \quad (3)$$

Then $A_n(u, v) = A(u, v)$. Observe that $P_{A_0(u, v)}(\lambda) = \lambda$ and $P_{A_1(u, v)}(\lambda) = \lambda^2 - u_1 v_1$. Expanding the determinant of the characteristic polynomial by its last row, we can easily deduce the following recursive formula for $P_{A_{k+1}(u, v)}$:

$$P_{A_{k+1}(u, v)}(\lambda) = \lambda P_{A_k(u, v)}(\lambda) - u_{k+1} v_{k+1} P_{A_{k-1}(u, v)}(\lambda) \quad (k \in \{1, \dots, n-1\}). \quad (4)$$

Now, we are going to prove that, for all $k \in \{1, \dots, n\}$, the characteristic polynomials of $A_k(u, v)$ and $A_k(\sqrt{uv}, \sqrt{uv})$ are identical, where $\sqrt{uv} := (\sqrt{u_1 v_1}, \dots, \sqrt{u_n v_n})$.

We prove this statement by induction on k . For $k = 0$, the statement is trivial. For $k = 1$, we have that

$$P_{A_1(u, v)}(\lambda) = \lambda^2 - u_1 v_1 = \lambda^2 - \sqrt{u_1 v_1} \sqrt{u_1 v_1} = P_{A_1(\sqrt{uv}, \sqrt{uv})}(\lambda).$$

Assume that we have established the identity $P_{A_j(u, v)} = P_{A_j(\sqrt{uv}, \sqrt{uv})}$ for $j \leq k$. By using the recursive formula (4) twice and the inductive assumption, for $k \in \{1, \dots, n-1\}$, we get

$$\begin{aligned} P_{A_{k+1}(u, v)}(\lambda) &= \lambda P_{A_k(u, v)}(\lambda) - \sqrt{u_{k+1} v_{k+1}} \sqrt{u_{k+1} v_{k+1}} P_{A_{k-1}(u, v)}(\lambda) \\ &= P_{A_{k+1}(\sqrt{uv}, \sqrt{uv})}(\lambda). \end{aligned}$$

This completes the proof of the identities $P_{A_k(u, v)} = P_{A_k(\sqrt{uv}, \sqrt{uv})}$.

The matrix $A_n(\sqrt{uv}, \sqrt{uv})$ is symmetric with real entries, therefore its characteristic polynomial has only real roots, whence it follows that the eigenvalues of $A_n(u, v) = A(u, v)$ are also real. The eigenvalues of $A_n(\sqrt{uv}, \sqrt{uv})$ are smaller than one if and only if the eigenvalues of the symmetric matrix $I_{n+1} - A_n(\sqrt{uv}, \sqrt{uv})$ are positive, which is equivalent to the positive definiteness of $I_{n+1} - A_n(\sqrt{uv}, \sqrt{uv})$. In view of the Sylvester test, this holds if and only if all the leading principal minor determinants of $I_{n+1} - A_n(\sqrt{uv}, \sqrt{uv})$ are positive, i.e., if

$$P_{A_k(u, v)}(1) = P_{A_k(\sqrt{uv}, \sqrt{uv})}(1) > 0 \quad (k \in \{0, \dots, n\}). \quad (5)$$

By the recursive formula (4) applied for $\lambda = 1$, it results that $P_{A_k(\sqrt{uv}, \sqrt{uv})}(1) = w_k$ for all $k \in \{0, \dots, n\}$, therefore, (5) is equivalent to the inequalities $w_1, \dots, w_n > 0$. \square

The next result offers a sufficient condition in order that the eigenvalues of the matrix $A(u, v)$ be smaller than 1.

Lemma 3.2. *Let $n \in \mathbb{N}$ and $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{R}^n$ be with positive components. Assume that*

$$v_1 \leq 1, \quad \max\{u_1 + v_2, \dots, u_{n-1} + v_n\} \leq 1, \quad u_n < 1. \quad (6)$$

Then the system of inequalities $w_1, \dots, w_n > 0$ holds, where w_1, \dots, w_n are defined as in Theorem 3.1. Consequently, all the eigenvalues of the two-diagonal matrix $A(u, v)$ defined by (1) are smaller than 1.

Proof. Observe that the positivity of v_2, \dots, v_n and (6) yield that $u_1, \dots, u_n < 1$.

To show that w_k is positive for all $k \in \{1, \dots, n\}$, we shall prove that

$$w_k > 0 \quad \text{and} \quad (1 - u_k)w_{k-1} \leq w_k < w_{k-1} \quad (k \in \{1, \dots, n-1\}). \quad (7)$$

For $k = 1$, the second chain of inequalities is equivalent to $1 - u_1 \leq 1 - u_1 v_1 \leq 1$, which easily follows from $0 < v_1 \leq 1$ and $0 < u_1$. Hence $w_1 > 0$ also holds.

Assume that we have proved (7) for some $k \in \{1, \dots, n-1\}$. Then, using the recursion (2) and using the right hand side inequality in (7), we get

$$w_{k+1} = w_k - u_{k+1}v_{k+1}w_{k-1} < w_k - u_{k+1}v_{k+1}w_k = w_k(1 - u_{k+1}v_{k+1}) < w_k.$$

On the other hand, using the upper estimate for w_{k-1} obtained from (7), we get

$$\begin{aligned} w_{k+1} &= w_k - u_{k+1}v_{k+1}w_{k-1} \geq w_k - u_{k+1}v_{k+1} \frac{w_k}{1 - u_k} = w_k \frac{1 - u_k - u_{k+1}v_{k+1}}{1 - u_k} \\ &\geq w_k \frac{1 - u_k - u_{k+1}(1 - u_k)}{1 - u_k} = w_k(1 - u_{k+1}) > 0, \end{aligned}$$

which completes the proof of (7). \square

Lemma 3.3. *For $n \in \mathbb{N}$ and for the vectors $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{R}^n$ with positive components, define the two-diagonal matrix $A(u, v)$ by (1). Then there exists an eigenvector of $A(u, v)$ with positive components whose eigenvalue is also positive.*

Proof. We follow the argument of the standard proof of the Perron–Frobenius Theorem. Consider the set

$$S_{n+1} := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0, \dots, x_n \geq 0, x_0 + \dots + x_n = 1\}.$$

Then S_{n+1} is a compact convex set in \mathbb{R}^{n+1} . Let $u, v \in \mathbb{R}^n$ be fixed vectors with positive components and let A_0, \dots, A_n be the row vectors of the matrix $A(u, v)$. Observe that

$$A(u, v)x = (\langle A_0, x \rangle, \dots, \langle A_n, x \rangle) \quad (x \in \mathbb{R}^{n+1}), \quad (8)$$

furthermore the sum $\langle A_0, x \rangle + \dots + \langle A_n, x \rangle$ does not vanish on S_{n+1} . Indeed, if for some $x \in S_{n+1}$ we have $\langle A_0, x \rangle + \dots + \langle A_n, x \rangle = 0$, then, by the nonnegativity of the terms on the left hand side of this equation, it follows that $\langle A_i, x \rangle = 0$ for all $i \in \{0, \dots, n\}$. Using the positivity of the parameters u_i and v_i , these equalities imply $x = 0$, which contradicts $x \in S_{n+1}$. Now consider the mapping $F : S_{n+1} \rightarrow \mathbb{R}^{n+1}$ defined by

$$F(x) := \frac{A(u, v)x}{\langle A_0, x \rangle + \dots + \langle A_n, x \rangle} \quad (x \in S_{n+1}).$$

By (8), we have that $F(S_{n+1}) \subseteq S_{n+1}$, and F is trivially continuous on S_{n+1} , hence, by the Brouwer Fixed Point Theorem, there exists a fixed point $c \in S_{n+1}$ of the function F . Then we have

$$A(u, v)c = (\langle A_0, c \rangle + \dots + \langle A_n, c \rangle)F(c) = (\langle A_0, c \rangle + \dots + \langle A_n, c \rangle)c,$$

which shows that c is an eigenvector of $A(u, v)$ with eigenvalue $\lambda := \langle A_0, c \rangle + \dots + \langle A_n, c \rangle > 0$. Therefore, by $A(u, v)c = \lambda c$, the following system of equations hold for the coordinates (c_0, \dots, c_n) :

$$\begin{aligned} u_1 c_1 &= \lambda c_0, \\ u_{i+1} c_{i+1} + v_i c_{i-1} &= \lambda c_i \quad (i \in \{1, \dots, n-1\}), \\ v_n c_{n-1} &= \lambda c_n. \end{aligned} \quad (9)$$

If $c_i = 0$ for some $i \in \{0, \dots, n\}$, then the nonnegativity of the terms on the left hand side of the i th equation yields that $c_j = 0$ for $j \in \{i-1, i+1\} \cap \{1, \dots, n-1\}$. This results that c has to be zero, which contradicts $c \in S_{n+1}$. \square

4. AUXILIARY RESULTS FROM FIXED POINT THEORY

For our purposes, we recall some notions and results related to fixed point theorems.

Definition. We say that the function $d : X \times X \rightarrow \mathbb{R}$ is a *semimetric* on the set X if, for all $x, y \in X$, it possesses the following properties:

- (1) d is *positive definite*, i.e., for all $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
- (2) and d is *symmetric*, i.e., for all $x, y \in X$, $d(x, y) = d(y, x)$.

The pair (X, d) is called *semimetric space*.

If (X, d_X) and (Y, d_Y) are semimetric spaces then a function $f : X \rightarrow Y$ is called *L -Lipschitzian* with respect to the pair of semimetrics (d_X, d_Y) if there exists $0 \leq L$ such that

$$d_Y(f(x), f(y)) \leq L d_X(x, y) \quad (x, y \in X). \quad (10)$$

The function f is said to have the *Lipschitz property* if there exists $L \geq 0$ such that (10) holds. The *Lipschitz modulus* of f is defined by

$$\text{Lip } f := \sup_{\substack{x, y \in X \\ x \neq y}} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

Obviously, f has the Lipschitz property if and only if $\text{Lip } f$ is finite. If $\lambda := \text{Lip } f < 1$ then f is called a *λ -contraction*.

It is an immediate consequence of these definitions, that for a subset $D \subseteq X$ and a contraction $f : D \rightarrow X$ with respect to the semimetric d_X , the map f can have at most one fixed point in D . Indeed, if x and y are both fixed points of f in D , then

$$d_X(x, y) = d_X(f(x), f(y)) \leq \lambda d_X(x, y),$$

which implies $d_X(x, y) \leq 0$, whence $x = y$ follows.

The following lemma is useful to compute the Lipschitz modulus of real valued functions.

Lemma 4.1. *Let $f, g : I \rightarrow \mathbb{R}$ be differentiable functions such that the derivative of g does not vanish on the interval I . Then, for the Lipschitz modulus of the function $f \circ g^{-1} : g(I) \rightarrow \mathbb{R}$, the following formula holds:*

$$\text{Lip } f \circ g^{-1} = \sup_{t \in I} \left| \frac{f'(t)}{g'(t)} \right|.$$

Proof. Due to the assumptions of the lemma, $g : I \rightarrow g(I)$ is a continuous and strictly monotone function. Therefore $g^{-1} : g(I) \rightarrow \mathbb{R}$ is well-defined. Thus, applying the Cauchy Mean Value Theorem, we have that

$$\text{Lip } f \circ g^{-1} = \sup_{\substack{x, y \in g(I) \\ x \neq y}} \frac{|f \circ g^{-1}(x) - f \circ g^{-1}(y)|}{|x - y|} = \sup_{\substack{u, v \in I \\ u \neq v}} \frac{|f(u) - f(v)|}{|g(u) - g(v)|} = \sup_{t \in I} \left| \frac{f'(t)}{g'(t)} \right|.$$

□

In what follows, we recall first the following generalization of the Tychonov Fixed Point Theorem established by Halpern and Bergman. For the formulation of this result, we define the notion of the *inward set of a convex subset K* of a locally convex space X by

$$\text{inw}_K(x) := x + \mathbb{R}_+(K - x) = \{x + t(y - x) \mid y \in K, t \geq 0\} \quad (x \in K).$$

Observe that $K \subseteq \text{inw}_K(x)$ holds for all $x \in K$. On the other hand, for an interior point x of K , we have $\text{inw}_K(x) = X$, therefore the inclusion $y \in \text{inw}_K(x)$ is always trivial if $x \in K \setminus \partial K$, where the notation ∂K stands for the set of boundary points of K .

Theorem 4.2. (Halpern–Bergman) *Let X be Hausdorff locally convex space and let $K \subseteq X$ be a compact convex set. Let $f : K \rightarrow X$ be a continuous weakly inward map, i.e., assume that $f(x) \in \overline{\text{inw}_K(x)}$ holds for all $x \in \partial K$. Then the set of fixed points of f is a nonempty compact subset of K .*

If $f(K) \subseteq K$, then $f(x) \in \overline{\text{inw}_K(x)}$ trivially holds for all $x \in \partial K$, therefore, in this case, the above result reduces to the Tychonov Fixed Point Theorem.

The fixed point theorem stated below that we are going to use for the existence proofs in our main results is consequence of the Halpern–Bergman Fixed Point Theorem. It establishes the existence of the fixed point for continuous maps defined over a convex polyhedron.

Theorem 4.3. Let $c_1, \dots, c_m \in \mathbb{R}^n$ and $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ and assume that the polyhedron $K \subseteq \mathbb{R}^n$ defined by

$$K := \{x \in \mathbb{R}^n \mid \langle c_k, x \rangle \leq \gamma_k, k \in \{1, \dots, m\}\} \quad (11)$$

is bounded. Let $f : K \rightarrow \mathbb{R}^n$ be a continuous function with the following property

$$\langle c_k, f(x) \rangle \leq \gamma_k \quad \text{for all } x \in K \text{ and for all } k \in \{1, \dots, m\} \text{ such that } \langle c_k, x \rangle = \gamma_k. \quad (12)$$

Then the set of fixed points of f is a nonempty compact subset of K .

Proof. By our assumption, K is a compact convex set. It suffices to show that, for all $x \in K$,

$$\text{inw}_K(x) = \{u \in \mathbb{R}^n \mid \langle c_k, u \rangle \leq \gamma_k \text{ for all } k \in \{1, \dots, m\} \text{ such that } \langle c_k, x \rangle = \gamma_k\} \quad (13)$$

because condition (12) then implies that $f(x) \in \text{inw}_K(x)$ for all $x \in K$, whence the Halpern–Bergman Fixed Point Theorem yields the existence the fixed point of f .

Let $x \in K$ be fixed. If $u \in \text{inw}_K(x)$, then there exists $y \in K$ and $t \geq 0$ such that $u = (1-t)x + ty$. Then, for $k \in \{1, \dots, m\}$ such that $\langle c_k, x \rangle = \gamma_k$, we have

$$\langle c_k, u \rangle = \langle c_k, (1-t)x + ty \rangle = (1-t)\langle c_k, x \rangle + t\langle c_k, y \rangle = (1-t)\gamma_k + t\langle c_k, y \rangle \leq (1-t)\gamma_k + t\gamma_k = \gamma_k,$$

which proves the inclusion \subseteq in (13).

For the reversed inclusion, let $u \in \mathbb{R}^n$ be an element such that $\langle c_k, u \rangle \leq \gamma_k$ for all $k \in \{1, \dots, m\}$ such that $\langle c_k, x \rangle = \gamma_k$. Choose $t > 0$ such that

$$t \geq \frac{\langle c_k, u - x \rangle}{\gamma_k - \langle c_k, x \rangle} \quad \text{for all } k \in \{1, \dots, m\} \text{ such that } \langle c_k, x \rangle < \gamma_k$$

and define $y \in \mathbb{R}^n$ by $y := \frac{1}{t}(u - x) + x$. Then, distinguishing the cases whether $\langle c_k, x \rangle = \gamma_k$ or not, for every $k \in \{1, \dots, m\}$, we get that

$$\langle c_k, u - x \rangle \leq t(\gamma_k - \langle c_k, x \rangle).$$

Therefore, for every $k \in \{1, \dots, m\}$,

$$\langle c_k, y \rangle = \langle c_k, \frac{1}{t}(u - x) + x \rangle \leq (\gamma_k - \langle c_k, x \rangle) + \langle c_k, x \rangle = \gamma_k.$$

This proves that $y \in K$. On the other hand, from the definition of y , we have that $u = (1-t)x + ty$, consequently, $u \in \text{inw}_K(x)$. \square

5. THE DESCENDANTS OF MEANS

Assume that we are given an $n \geq 2$ member sequence of means $M_1, \dots, M_n : I_{\leq}^2 \rightarrow \mathbb{R}$. In this section, we are going to deal with existence and uniqueness of an increasing sequence of means $N_1, \dots, N_n : I_{\leq}^2 \rightarrow \mathbb{R}$ such that, for all $(x, y) \in I_{\leq}^2$, the identities

$$\begin{aligned} N_1(x, y) &= M_1(x, N_2(x, y)), \\ N_i(x, y) &= M_i(N_{i-1}(x, y), N_{i+1}(x, y)) \quad (i \in \{2, \dots, n-1\}), \\ N_n(x, y) &= M_n(N_{n-1}(x, y), y) \end{aligned} \quad (14)$$

hold. The i^{th} element of the sequence N_1, \dots, N_n will be called the i^{th} descendant of the n -tuple (M_1, \dots, M_n) of means. Observe that (14) states that, for $(x, y) \in I_{\leq}^2$, the vector $(N_1(x, y), \dots, N_n(x, y)) \in [x, y]_{\leq}^n$ is a fixed point of the mapping $\varphi_{(x,y)} : [x, y]_{\leq}^n \rightarrow \mathbb{R}^n$ defined by

$$\varphi_{(x,y)}(t_1, \dots, t_n) := (M_1(x, t_2), \dots, M_i(t_{i-1}, t_{i+1}), \dots, M_n(t_{n-1}, y)). \quad (15)$$

The first main result of this section establishes the existence and uniqueness of the fixed points of $\varphi_{(x,y)}$, i.e., the nonemptiness and singletonness of the set

$$\Phi_{(x,y)} := \{\xi \in [x, y]_{\leq}^n \mid \varphi_{(x,y)}(\xi) = \xi\}. \quad (16)$$

The existence and uniqueness of the fixed point is obvious if $x = y$, therefore, we restrict our attention to the case $x < y$.

Theorem 5.1. *Let $n \geq 2$ and $M_1, \dots, M_n : I_{\leq}^2 \rightarrow I$ be means. For $(x, y) \in I_{\leq}^2$, define the mapping $\varphi_{(x,y)}$ and the fixed point set $\Phi_{(x,y)}$ by (15) and (16), respectively. Then, for all $(x, y) \in I_{\leq}^2$, the following statements hold:*

- (1) *If all the means M_1, \dots, M_n are continuous, then the fixed point set $\Phi_{(x,y)}$ is nonempty and compact. If the means M_1, \dots, M_n are strict, then $\Phi_{(x,y)} \subseteq]x, y[_{\leq}^n$.*
- (2) *The set $\Phi_{(x,y)}$ is a singleton if there exist semimetrics $d_1, \dots, d_n : [x, y]^2 \rightarrow \mathbb{R}_+$ such that the estimates*

$$\begin{aligned} d_1(M_1(x, s), M_1(x, v)) &\leq b_1 d_2(s, v), \\ d_i(M_i(t, s), M_i(u, v)) &\leq a_i d_{i-1}(t, u) + b_i d_{i+1}(s, v) \quad (i \in \{2, \dots, n-1\}), \\ d_n(M_n(t, y), M_n(u, y)) &\leq a_n d_{n-1}(t, u) \end{aligned} \quad (17)$$

hold for all $t, s, u, v \in [x, y]$ with some positive real numbers a_2, \dots, a_n and b_1, \dots, b_{n-1} such that $w_1, \dots, w_{n-1} > 0$, where $w_{-1} := w_0 := 1$ and

$$w_i := w_{i-1} - a_{i+1} b_i w_{i-2} \quad (i \in \{1, \dots, n-1\}). \quad (18)$$

Proof. Let $(x, y) \in I_{\leq}^2$ be arbitrarily fixed. Then the set $K := [x, y]_{\leq}^n$ is a compact convex set which is characterized by the following $(n+1)$ inequalities: $(t_1, \dots, t_n) \in K$ holds if and only if

$$-t_1 \leq -x, \quad t_1 - t_2 \leq 0, \quad \dots, \quad t_{n-1} - t_n \leq 0, \quad t_n \leq y. \quad (19)$$

Therefore, K is a polyhedron of the form (11) with $m = n+1$, suitably chosen vectors $c_1, \dots, c_{n+1} \in \mathbb{R}^n$ and scalars $\gamma_1, \dots, \gamma_{n+1} \in \mathbb{R}$. Thus, in order to show that the fixed point set of the continuous function $f := \varphi_{x,y}$ is a nonempty compact subset of $K = [x, y]_{\leq}^n$, we need to verify that condition (12) is satisfied.

For the sake of brevity, denote $t_0 := x$ and $t_{n+1} := y$. If, for some $k \in \{2, \dots, n\}$, the k th inequality holds with equality in (19), then $t_{k-1} = t_k$. Therefore, by the mean value property of the means M_{k-1} and M_k , we get

$$s_{k-1} = M_{k-1}(t_{k-2}, t_k) \leq t_k = t_{k-1} \leq M_k(t_{k-1}, t_{k+1}) = s_k,$$

which proves that the vector s satisfies the k th inequality in (19).

On the other hand, by the mean value properties of M_1 and M_n , we have $x \leq M_1(x, t_2) = s_1$ and $s_n = M_n(t_{n-1}, y) \leq y$, therefore, s also satisfies the first and last inequality in (19) and thus the verification of condition (12) is complete.

To prove the second part of the statement (1), assume that all the means M_1, \dots, M_n are strict and let $(\xi_1, \dots, \xi_n) \in \Phi_{(x,y)}$. Then

$$M_1(x, \xi_2) = \xi_1, \quad M_2(\xi_1, \xi_3) = \xi_2, \quad \dots, \quad M_n(\xi_{n-1}, y) = \xi_n. \quad (20)$$

If $x = \xi_1$ then the strict mean property of M_1 and the identity $M_1(x, \xi_2) = \xi_1$ imply that $\xi_1 = \xi_2$. Now the strict mean property of M_2 and the identity $M_2(\xi_1, \xi_3) = \xi_2$ yield that $\xi_2 = \xi_3$. Continuing this argument, it follows that $\xi_{n-1} = \xi_n$. Finally, the strict mean property of M_n and $M_n(\xi_{n-1}, y) = \xi_n$ imply that $\xi_n = y$. This leads to the contradiction $x = y$. Hence, we may assume that $x < \xi_1$. Applying the strict mean property of M_1, \dots, M_n and the equalities in (20), we get $\xi_i < \xi_{i+1}$ recursively for $i \in \{1, \dots, n-1\}$ and finally $\xi_n < y$, which proves that $(\xi_1, \dots, \xi_n) \in]x, y[_{\leq}^n$.

To prove (2), assume that there exist semimetrics $d_1, \dots, d_n : [x, y]^2 \rightarrow \mathbb{R}_+$ such that the estimates in (17) hold and let $a := (a_2, \dots, a_n)$ and $b := (b_1, \dots, b_{n-1})$ such that each members of the sequence w_1, \dots, w_{n-1} , defined by (18) with $w_{-1} := w_0 := 1$, is positive. According to the previous lemmas, the matrix $A(a, b)$ has an eigenvector $c := (c_1, \dots, c_n)$ with positive components and with eigenvalue $0 < \lambda < 1$. This means that c and λ satisfy the following system of linear equations:

$$\begin{aligned} a_2 c_2 &= \lambda c_1, \\ a_{i+1} c_{i+1} + b_{i-1} c_{i-1} &= \lambda c_i \quad (i \in \{2, \dots, n-1\}), \\ b_{n-1} c_{n-1} &= \lambda c_n. \end{aligned} \quad (21)$$

We show that $\varphi_{(x,y)}$ is a λ -contraction with respect to the semimetric $D_c : [x, y]^n \times [x, y]^n \rightarrow \mathbb{R}_+$ defined by

$$D_c((u_1, \dots, u_n), (v_1, \dots, v_n)) := c_1 d_1(u_1, v_1) + \dots + c_n d_n(u_n, v_n)$$

for all $(u_1, \dots, u_n), (v_1, \dots, v_n) \in [x, y]^n$. To prove this, let (t_1, \dots, t_n) and (s_1, \dots, s_n) be arbitrary elements of $[x, y]_{\leq}^n$. For the sake of brevity, set $t_0 = s_0 = x$ and $t_n = s_n = y$. Using the estimates in (17) and then the identities in (21), we obtain that

$$\begin{aligned} D_c(\varphi_{(x,y)}(t_1, \dots, t_n), \varphi_{(x,y)}(s_1, \dots, s_n)) &= \sum_{i=1}^n c_i d_i(M_i(t_{i-1}, t_{i+1}), M_i(s_{i-1}, s_{i+1})) \\ &\leq c_1 b_1 d_2(t_2, s_2) + \left(\sum_{i=2}^{n-1} c_i a_i d_{i-1}(t_{i-1}, s_{i-1}) + c_i b_{i+1} d_i(t_{i+1}, s_{i+1}) \right) + c_n a_n d_n(t_{n-1}, s_{n-1}) \\ &= \lambda(c_1 d_1(t_1, s_1) + \dots + c_n d_n(t_n, s_n)) = \lambda D_c((t_1, \dots, t_n), (s_1, \dots, s_n)). \end{aligned}$$

This results the uniqueness of the fixed point of $\varphi_{(x,y)}$. \square

Definition 5.2. Let $n \geq 2$ and $M_1, \dots, M_n : I_{\leq}^2 \rightarrow \mathbb{R}$ be continuous means. For $i \in \{1, \dots, n\}$, we say that $N : I_{\leq}^2 \rightarrow \mathbb{R}$ is an i^{th} descendant of the n -tuple of means (M_1, \dots, M_n) if, for all $(x, y) \in I_{\leq}^2$, we have

$$N(x, y) \in \bigcup \{ \xi_i \mid (\xi_1, \dots, \xi_n) \in \Phi_{(x,y)} \} \quad \text{if } x < y \quad \text{and} \quad N(x, y) = x \quad \text{if } x = y, \quad (22)$$

where $\Phi_{(x,y)}$ is the fixed point set of the mapping $\varphi_{(x,y)} : [x, y]_{\leq}^n \rightarrow \mathbb{R}^n$ defined by (15). The class of all such functions is denoted by $\mathcal{D}_i(M_1, \dots, M_n)$.

Note that, in view of Theorem 5.1, the continuity of the means M_1, \dots, M_n implies that the descendant functions are well-defined. As a direct consequence of the compactness of the fixed point set $\Phi_{(x,y)}$, we obtain that the family $\mathcal{D}_i(M_1, \dots, M_n)$ has a minimal and a maximal element in the following sense: there exist $N_i^-, N_i^+ \in \mathcal{D}_i(M_1, \dots, M_n)$ such that $N_i^-(x, y) \leq N(x, y) \leq N_i^+(x, y)$ for all $N \in \mathcal{D}_i(M_1, \dots, M_n)$ and for all $x, y \in I$. It is also obvious that each element of $\mathcal{D}_i(M_1, \dots, M_n)$ is a strict mean provided that all the means M_1, \dots, M_n are strict.

Remark 5.3. The uniqueness of the fixed point of the map $\varphi_{(x,y)}$ cannot be stated in general. For instance, let $n \geq 2$, and let $M_1 := \max$, $M_n := \min$ and $M_i := \mathbb{A}_{\frac{1}{2}}$ for $i \in \{2, \dots, n-1\}$ over the interval \mathbb{R} . Then, for $(x, y) \in \mathbb{R}_{<}^2$, the fixed point equation $(t_1, \dots, t_n) = \varphi_{(x,y)}(t_1, \dots, t_n)$ is equivalent to

$$(t_1, \dots, t_n) = \left(t_2, \frac{t_1 + t_3}{2}, \dots, \frac{t_{n-2} + t_n}{2}, t_{n-1} \right).$$

An easy computation shows that this equality is satisfied if and only if $t_1 = \dots = t_n$. Therefore, $\Phi_{(x,y)} = \{(t_1, \dots, t_n) \mid t_1 = \dots = t_n \in [x, y]\}$.

Considering Matkowski means, we obtain useful corollaries of Theorem 5.1.

Theorem 5.4. Let $n \geq 2$ and $f_1, \dots, f_n, g_1, \dots, g_n : I \rightarrow \mathbb{R}$ be continuous, strictly increasing functions. For $(x, y) \in I_{<}^2$, define the function $\varphi_{(x,y)} : [x, y]_{\leq}^n \rightarrow \mathbb{R}^n$ as

$$\varphi_{(x,y)}(t_1, \dots, t_n) := (\mathbb{M}_{f_1, g_1}(x, t_2), \dots, \mathbb{M}_{f_i, g_i}(t_{i-1}, t_{i+1}), \dots, \mathbb{M}_{f_n, g_n}(t_{n-1}, y)). \quad (23)$$

Then, for $(x, y) \in I_{<}^2$, the fixed point set $\Phi_{(x,y)}$ defined by (16) is nonempty and compact. Furthermore, $\Phi_{(x,y)}$ is a singleton if

$$\begin{aligned} a_i &:= \text{Lip} [f_i \circ (f_{i-1} + g_{i-1})^{-1}] < +\infty & (i \in \{2, \dots, n\}), \\ b_i &:= \text{Lip} [g_i \circ (f_{i+1} + g_{i+1})^{-1}] < +\infty & (i \in \{1, \dots, n-1\}), \end{aligned} \quad (24)$$

and if the constants w_1, \dots, w_{n-1} defined by (18) are positive.

Proof. Because of the definition of the means $\mathbb{M}_{f_1, g_1}, \dots, \mathbb{M}_{f_n, g_n}$, for all $(x, y) \in I_{<}^2$, the mapping $\varphi_{(x, y)}$ is continuous, thus, based on the Theorem 5.1, the corresponding fixed point set $\Phi_{(x, y)}$ is a nonempty compact subset of $[x, y]_{\leq}^n$. Due to the strictness of generalized quasi-arithmetic means it also follows that $\Phi_{(x, y)} \subseteq]x, y[_{<}^n$.

Now assume that (24) and $w_1, \dots, w_{n-1} > 0$ hold and fix a point $(x, y) \in I_{<}^2$. To show that $\Phi_{(x, y)}$ is a singleton, for $i \in \{1, \dots, n\}$, define the semimetrics $d_i : I \times I \rightarrow \mathbb{R}_+$ as

$$d_i(s, t) := |(f_i + g_i)(s) - (f_i + g_i)(t)| \quad (s, t \in I).$$

Note that in our case, for all $i \in \{1, \dots, n\}$, the function d_i is a *metric*, i.e., in addition of the properties (1) and (2) of semimetrics, d_i also satisfies the *triangle inequality*, namely, for all $i \in \{1, \dots, n\}$, we have

$$d_i(s, t) \leq d_i(s, r) + d_i(r, t) \quad (r, s, t \in I).$$

Let $t, s, u, v \in [x, y]$ be arbitrary. Then, for all $i \in \{2, \dots, n-1\}$, we have the following estimation:

$$\begin{aligned} d_i(M_i(t, s), M_i(u, v)) &= |(f_i + g_i)(\mathbb{M}_{f_i, g_i}(t, s)) - (f_i + g_i)(\mathbb{M}_{f_i, g_i}(u, v))| \\ &= |f_i(t) + g_i(s) - f_i(u) - g_i(v)| \leq |f_i(t) - f_i(u)| + |g_i(s) - g_i(v)| \\ &\leq \text{Lip}[f_i \circ (f_{i-1} + g_{i-1})^{-1}]d_{i-1}(t, u) + \text{Lip}[g_i \circ (f_{i+1} + g_{i+1})^{-1}]d_{i+1}(s, v) \\ &= a_i d_{i-1}(t, u) + b_i d_{i+1}(s, v). \end{aligned}$$

On the other hand, for $i = 1$ and $i = n$, we get

$$d_1(M_1(x, s), M_1(x, v)) \leq b_1 d_2(s, v) \quad \text{and} \quad d_n(M_n(t, y), M_n(u, y)) \leq a_n d_{n-1}(t, u).$$

Therefore, all the estimates in (17) are satisfied. Thus, in view of the Theorem 5.1, for all $(x, y) \in I_{<}^2$, the fixed point set $\Phi_{(x, y)}$ is indeed a singleton. \square

Corollary 5.5. *Let $n \geq 2$ and $f_1, \dots, f_n, g_1, \dots, g_n : I \rightarrow \mathbb{R}$ be differentiable, strictly increasing functions such that $(f_i + g_i)'$ does not vanish on I for all $i \in \{1, \dots, n\}$. Assume further that*

$$\begin{aligned} a_i &:= \sup_{t \in I} [f'_i \cdot (f'_{i-1} + g'_{i-1})^{-1}](t) < +\infty \quad (i \in \{2, \dots, n\}), \\ b_i &:= \sup_{t \in I} [g'_i \cdot (f'_{i+1} + g'_{i+1})^{-1}](t) < +\infty \quad (i \in \{1, \dots, n-1\}). \end{aligned} \quad (25)$$

Finally, for $(x, y) \in I_{<}^2$, define the function $\varphi_{(x, y)} : [x, y]_{\leq}^n \rightarrow \mathbb{R}^n$ as in (23). Then, for all $(x, y) \in I_{<}^2$, the fixed point set $\Phi_{(x, y)}$ defined by (16) is a nonempty compact subset of $[x, y]_{\leq}^n$, and, it is a singleton if the constants w_1, \dots, w_{n-1} defined by (18) are positive.

Proof. In view of Theorem 5.4, we only need to verify that $\Phi_{(x, y)}$ is a singleton, which in turn is obvious. Using Lemma 4.1 and the conditions in (25), one can easily see that the estimations in (24) of Theorem 5.4 hold, i.e., the constants a_2, \dots, a_n and b_1, \dots, b_{n-1} are real numbers. Thus the proof is complete. \square

Theorem 5.6. *Let $n \geq 2$, $s_1, \dots, s_n \in]0, 1[$, and $h : I \rightarrow \mathbb{R}$ be a continuous, strictly increasing function. Then, for all $(x, y) \in I_{<}^2$, the fixed point set $\Phi_{(x, y)}$ (defined by (16)) of the mapping $\varphi : [x, y]_{\leq}^n \rightarrow \mathbb{R}^n$ defined by*

$$\varphi_{(x, y)}(t_1, \dots, t_n) := (\mathbb{M}_{s_1 h, (1-s_1)h}(x, t_2), \dots, \mathbb{M}_{s_i h, (1-s_i)h}(t_{i-1}, t_{i+1}), \dots, \mathbb{M}_{s_n h, (1-s_n)h}(t_{n-1}, y))$$

is the singleton $\{(\mathbb{M}_{\sigma_1 h, (1-\sigma_1)h}(x, y), \dots, \mathbb{M}_{\sigma_n h, (1-\sigma_n)h}(x, y))\}$, where

$$\sigma_i := \left(\sum_{j=i}^n \prod_{k=1}^j \frac{s_k}{1-s_k} \right) \left(\sum_{j=0}^n \prod_{k=1}^j \frac{s_k}{1-s_k} \right)^{-1} \quad (i \in \{1, \dots, n\}). \quad (26)$$

Proof. In order to apply Theorem 5.4, let $f_i := s_i \cdot h$ and $g_i := (1-s_i) \cdot h$ for $i \in \{1, \dots, n\}$. Then it immediately follows that the fixed point set $\Phi_{(x, y)}$ is nonempty and compact for all $(x, y) \in I_{<}^2$.

To show that $\Phi_{(x,y)}$ is a singleton define the constants a_2, \dots, a_n , b_1, \dots, b_{n-1} , and w_1, \dots, w_{n-1} as in Theorem 5.4. We need to show that conditions (24) and $w_1, \dots, w_{n-1} > 0$ hold. Observe that, for $i \in \{1, \dots, n\}$, we have $f_i + g_i = h$ and

$$\begin{aligned} a_i &= \text{Lip} [f_i \circ (f_{i-1} + g_{i-1})^{-1}] = \text{Lip}[s_i \cdot h \circ h^{-1}] = s_i \quad (i \in \{2, \dots, n\}), \\ b_i &= \text{Lip} [g_i \circ (f_{i+1} + g_{i+1})^{-1}] = \text{Lip}[(1 - s_i) \cdot h \circ h^{-1}] = 1 - s_i \quad (i \in \{1, \dots, n-1\}). \end{aligned}$$

Thus each of the constants a_2, \dots, a_n and b_1, \dots, b_{n-1} are finite, on the other hand, under the notation $(u_1, \dots, u_{n-1}) := (a_2, \dots, a_n)$ and $(v_1, \dots, v_{n-1}) := (b_1, \dots, b_{n-1})$, they satisfy the condition (6) of Lemma 3.2. Therefore, the inequalities $w_1, \dots, w_{n-1} > 0$ hold and hence $\Phi_{(x,y)}$ is a singleton.

Finally, we verify that, for all $(x, y) \in I_{<}^2$, the vector $(\mathbb{M}_{\sigma_1 h, (1-\sigma_1)h}(x, y), \dots, \mathbb{M}_{\sigma_n h, (1-\sigma_n)h}(x, y))$ is a fixed point of $\varphi_{(x,y)}$. For this purpose, we show first that $\sigma_1, \dots, \sigma_n$ fulfill the following system of linear equations:

$$\begin{aligned} \sigma_1 &= s_1 + (1 - s_1)\sigma_2, \\ \sigma_i &= s_i\sigma_{i-1} + (1 - s_i)\sigma_{i+1} \quad (i \in \{2, \dots, n-1\}), \\ \sigma_n &= s_n\sigma_{n-1}. \end{aligned} \tag{27}$$

We prove the above equality for $i \in \{2, \dots, n-1\}$. First observe that

$$\prod_{k=1}^i \frac{s_k}{1 - s_k} = \frac{s_i}{1 - s_i} \prod_{k=1}^{i-1} \frac{s_k}{1 - s_k} = s_i \left(1 + \frac{s_i}{1 - s_i}\right) \prod_{k=1}^{i-1} \frac{s_k}{1 - s_k} = s_i \left(\prod_{k=1}^{i-1} \frac{s_k}{1 - s_k} + \prod_{k=1}^i \frac{s_k}{1 - s_k} \right).$$

Adding this identity to the equality

$$\sum_{j=i+1}^n \prod_{k=1}^j \frac{s_k}{1 - s_k} = s_i \sum_{j=i+1}^n \prod_{k=1}^j \frac{s_k}{1 - s_k} + (1 - s_i) \sum_{j=i+1}^n \prod_{k=1}^j \frac{s_k}{1 - s_k}$$

side by side, we get the desired identity $\sigma_i = s_i\sigma_{i-1} + (1 - s_i)\sigma_{i+1}$. In the cases $i = 1$ and $i = n$ the proof of (27) is completely analogous.

Using (27), after some calculation we easily get that

$$\begin{aligned} \mathbb{M}_{\sigma_1 h, (1-\sigma_1)h}(x, y) &= \mathbb{M}_{s_1 h, (1-s_1)h}(x, \mathbb{M}_{\sigma_2 h, (1-\sigma_2)h}(x, y)), \\ \mathbb{M}_{\sigma_i h, (1-\sigma_i)h}(x, y) &= \mathbb{M}_{s_i h, (1-s_i)h}(\mathbb{M}_{\sigma_{i-1} h, (1-\sigma_{i-1})h}(x, y), \mathbb{M}_{\sigma_{i+1} h, (1-\sigma_{i+1})h}(x, y)) \quad (i \in \{2, \dots, n-1\}), \\ \mathbb{M}_{\sigma_n h, (1-\sigma_n)h}(x, y) &= \mathbb{M}_{s_n h, (1-s_n)h}(\mathbb{M}_{\sigma_{n-1} h, (1-\sigma_{n-1})h}(x, y), y), \end{aligned}$$

which proves that $(\mathbb{M}_{\sigma_1 h, (1-\sigma_1)h}(x, y), \dots, \mathbb{M}_{\sigma_n h, (1-\sigma_n)h}(x, y))$ is indeed a fixed point of $\varphi_{(x,y)}$. \square

Theorem 5.7. Let $n \geq 2$, $j \in \{1, \dots, n\}$ and $p, q, h_1, \dots, h_{n-1} : I \rightarrow \mathbb{R}$ be continuous, strictly increasing functions, and set $h_0 := h_n := 0$. For $(x, y) \in I_{<}^2$, define the mapping $\varphi_{(x,y)} : [x, y]_{\leq}^n \rightarrow \mathbb{R}^n$ by (15), where

$$M_i := \begin{cases} \mathbb{M}_{p+h_{i-1}, h_i}, & \text{if } i \in \{1, \dots, j-1\}, \\ \mathbb{M}_{p+h_{i-1}, h_i+q} & \text{if } i = j, \\ \mathbb{M}_{h_{i-1}, h_i+q}, & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

Then, for $(x, y) \in I_{<}^2$, the fixed point set $\Phi_{(x,y)}$ defined by (16) is the singleton $\{(\xi_1, \dots, \xi_n)\}$, where the coordinates are defined by the following two-way recursion:

$$\xi_j := \mathbb{M}_{p,q}(x, y) \quad \text{and} \quad \xi_i := \begin{cases} \mathbb{M}_{p, h_i}(x, \xi_{i+1}) & \text{if } i \in \{1, \dots, j-1\}, \\ \mathbb{M}_{h_{i-1}, q}(\xi_{i-1}, y) & \text{if } i \in \{j+1, \dots, n\}. \end{cases} \tag{28}$$

Proof. Let $(x, y) \in I_{\leq}^2$ be fixed. By Theorem 5.1, the set $\Phi_{(x,y)}$ is nonempty. Let $(\xi_1, \dots, \xi_n) \in \Phi_{(x,y)}$ be arbitrary and denote $\xi_0 := x$ and $\xi_{n+1} := y$. Then, by the definition of Matkowski means, we have

$$\begin{aligned} (p + h_{i-1} + h_i)(\xi_i) &= (p + h_{i-1})(\xi_{i-1}) + h_i(\xi_{i+1}), & \text{if } i \in \{1, \dots, j-1\}, \\ (p + h_{i-1} + h_i + q)(\xi_i) &= (p + h_{i-1})(\xi_{i-1}) + (h_i + q)(\xi_{i+1}), & \text{if } i = j, \\ (h_{i-1} + h_i + q)(\xi_i) &= h_{i-1}(\xi_{i-1}) + (h_i + q)(\xi_{i+1}), & \text{if } i \in \{j+1, \dots, n\}. \end{aligned} \quad (29)$$

Adding up these inequalities for $i \in \{1, \dots, n\}$ side by side, it follows that

$$p(\xi_j) + h_0(\xi_1) + h_n(\xi_n) + q(\xi_j) = p(\xi_0) + h_0(\xi_0) + h_n(\xi_{n+1}) + q(\xi_{n+1}).$$

This simplifies to

$$(p + q)(\xi_j) = p(x) + q(y),$$

which is equivalent to the equality on the left hand side of (28). This computation also shows that ξ_j is uniquely determined.

To prove the first equality on the right hand side of (28), assume that $1 \leq j-1$ and let $k \in \{1, \dots, j-1\}$ be fixed. Adding up the equalities in (29) for $i \in \{1, \dots, k\}$, we arrive at

$$p(\xi_k) + h_0(\xi_1) + h_k(\xi_k) = p(\xi_0) + h_0(\xi_0) + h_k(\xi_{k+1}),$$

which reduces to $(p + h_k)(\xi_k) = p(x) + h_k(\xi_{k+1})$ proving the first equality on the right hand side of (28) for $i = k$.

Analogously, to verify the second equality on the right hand side of (28), assume that $j+1 \leq n$ and let $k \in \{j+1, \dots, n\}$ be fixed. Adding up the equalities in (29) for $i \in \{k, \dots, n\}$, we obtain

$$h_{k-1}(\xi_k) + h_n(\xi_n) + q(\xi_k) = h_{k-1}(\xi_{k-1}) + h_n(\xi_{n+1}) + q(\xi_{n+1}).$$

This yields $(h_{k-1} + q)(\xi_k) = h_{k-1}(\xi_{k-1}) + q(y)$, which validates the second equality on the right hand side of (28) for $i = k$.

In view of the uniqueness of ξ_j and the recursive system of equalities on the right hand side of (28), we can see that, for $i \neq j$, the value of ξ_i is also uniquely determined. \square

6. UPPER AND LOWER SECOND-ORDER DIVIDED DIFFERENCES

Consider the following binary operations on the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$: for two extended real numbers x, y , their *upper* and *lower sums* are defined by

$$x \dot{+} y := \begin{cases} x + y, & \text{if } \max\{x, y\} < +\infty, \\ +\infty, & \text{if } \max\{x, y\} = +\infty, \end{cases} \quad x \dagger y := \begin{cases} x + y, & \text{if } \min\{x, y\} > -\infty, \\ -\infty, & \text{if } \min\{x, y\} = -\infty, \end{cases}$$

respectively. It is easy to see, that the pairs $(\overline{\mathbb{R}}, \dot{+})$ and $(\overline{\mathbb{R}}, \dagger)$ are commutative semigroups. Apart from the standard cases, the only difference between these operations is that

$$(-\infty) \dot{+} (+\infty) = (+\infty) \dot{+} (-\infty) = +\infty \quad \text{and} \quad (-\infty) \dagger (+\infty) = (+\infty) \dagger (-\infty) = -\infty.$$

Furthermore, the both of the operations $\dot{+}$ and \dagger restricted to pairs of real numbers are the same as the standard addition of the reals. As direct consequences of the definitions, for all $x, y \in \overline{\mathbb{R}}$, we have the following easy-to-see properties:

$$x \dagger y \leq x \dot{+} y \quad \text{and} \quad -(x \dagger y) = (-x) \dot{+} (-y), \quad (30)$$

furthermore, we have the following equivalences:

$$\begin{aligned} 0 \leq x \dot{+} y &\Leftrightarrow -x \leq y & \text{and} & \quad 0 \leq x \dagger y &\Leftrightarrow (-\infty < \min\{x, y\} \text{ and } -x \leq y), \\ x \dagger y \leq 0 &\Leftrightarrow x \leq -y & \text{and} & \quad x \dot{+} y \leq 0 &\Leftrightarrow (\max\{x, y\} < +\infty \text{ and } x \leq -y). \end{aligned} \quad (31)$$

Definition 6.1. Let $D \subseteq \mathbb{R}$ and $f : D \rightarrow \overline{\mathbb{R}}$. The *upper second-order divided difference* of f at three distinct points x, y, z of D is an extended real number defined by

$$[x, y, z; f] := \frac{f(x)}{(y-x)(z-x)} \dot{+} \frac{f(y)}{(x-y)(z-y)} \dot{+} \frac{f(z)}{(x-z)(y-z)}.$$

Similarly, the *lower second-order divided difference* of f at the distinct points x, y, z of D is

$$[x, y, z; f] := \frac{f(x)}{(y-x)(z-x)} \dot{+} \frac{f(y)}{(x-y)(z-y)} \dot{+} \frac{f(z)}{(x-z)(y-z)}.$$

Obviously, the above second-order divided differences are symmetric functions of (x, y, z) . Observe that if the inequalities $x < y < z$ hold, then the coefficients of $f(x)$ and $f(z)$ are positive and the coefficient to $f(y)$ is negative.

As a direct consequence of the above definition and (30) we obtain

Proposition 6.2. Let $D \subseteq \mathbb{R}$ and $f : D \rightarrow \overline{\mathbb{R}}$. Then, for all distinct points $x < y < z$ of D ,

$$[x, y, z; f] \leq [x, y, z; f] \quad \text{and} \quad -[x, y, z; f] = [x, y, z; -f].$$

Proposition 6.3. (Extended Chain Inequality) Let $D \subseteq \mathbb{R}$ and $f : D \rightarrow \overline{\mathbb{R}}$. Then, for all $n \in \mathbb{N}$ and $x_0 < x_1 < \dots < x_{n+1}$ in D and for all $i \in \{1, \dots, n\}$ the following inequalities hold:

$$\min_{1 \leq j \leq n} [x_{j-1}, x_j, x_{j+1}; f] \leq [x_0, x_i, x_{n+1}; f] \leq [x_0, x_i, x_{n+1}; f] \leq \max_{1 \leq j \leq n} [x_{j-1}, x_j, x_{j+1}; f].$$

Proof. We only need to prove the first inequality, because the second one is trivial and the last one is the consequence of the first and Proposition 6.2.

The statement is trivial for $n = 1$, therefore we may assume that $n \geq 2$. Let $x_0 < x_1 < \dots < x_{n+1}$ be arbitrary elements of D and $i \in \{1, \dots, n\}$. If either the left hand side of the first inequality equals $-\infty$ or the right hand side equals $+\infty$, then there is nothing to prove. In the remaining case, for all $j \in \{1, \dots, n\}$, we have that $[x_{j-1}, x_j, x_{j+1}; f] > -\infty$ and $[x_0, x_i, x_{n+1}; f] < +\infty$. The first inequality implies, for all $j \in \{1, \dots, n\}$ that

$$\min\{f(x_{j-1}), -f(x_j), f(x_{j+1})\} > -\infty.$$

In view of $n \geq 2$, the set $\{1, \dots, n\}$ contains at least two elements, therefore, for all $j \in \{1, \dots, n\}$, we obtain that $f(x_j) \in \mathbb{R}$ and $\min\{f(x_0), f(x_{n+1})\} > -\infty$. Thus, $f(x_i) \in \mathbb{R}$ and hence the inequality $[x_0, x_i, x_{n+1}; f] < +\infty$ yields $\max\{f(x_0), f(x_{n+1})\} < +\infty$, which proves that, for all $j \in \{0, \dots, n+1\}$, we have $f(x_j) \in \mathbb{R}$. In this case, the first inequality is a consequence of [7, Corollary 1]. \square

7. UPPER AND LOWER M -CONVEXITY

Definition 7.1. For a fixed strict mean $M : I_{\leq}^2 \rightarrow \mathbb{R}$, we say that the function $f : I \rightarrow \overline{\mathbb{R}}$ is *lower M -convex* if

$$[x, M(x, y), y; f] \geq 0 \quad ((x, y) \in I_{<}^2) \quad (32)$$

holds. On the other hand, the function f is called *upper M -convex* if

$$[x, M(x, y), y; f] \geq 0 \quad (33)$$

holds on the same domain.

Note that, due to Proposition 6.2, if f is lower M -convex, then it is also upper M -convex.

The *lower* and *upper M -concavity* of functions can be also interpreted, namely we may consider (32) and (33) with the reverse inequality. It is easy to check, that these definitions are equivalent to the upper and lower M -convexity of the function $-f$, respectively.

Lemma 7.2. Let $M : I_{\leq}^2 \rightarrow \mathbb{R}$ be a strict mean and $f : I \rightarrow \overline{\mathbb{R}}$. Then the following statements hold.

(a) The function f is lower M -convex if and only if $f(u) > -\infty$ for all $u \in I$ and, for all $(x, y) \in I_{<}^2$, the inequalities $f(M(x, y)) < +\infty$ and

$$f(M(x, y)) \leq \frac{y - M(x, y)}{y - x} f(x) + \frac{M(x, y) - x}{y - x} f(y) \quad (34)$$

hold.

(b) The function f is upper M -convex if and only if, for all $(x, y) \in I_{<}^2$, the inequality

$$f(M(x, y)) \leq \frac{y - M(x, y)}{y - x} f(x) + \frac{M(x, y) - x}{y - x} f(y) \quad (35)$$

holds.

Proof. First we prove the statement (b). Suppose that f is an upper M -convex function meaning that $[x, M(x, y), y; f] \geq 0$ for all $(x, y) \in I_{<}^2$. Due to the first property of upper addition in (31), this inequality is equivalent to

$$\frac{f(M(x, y))}{(M(x, y) - x)(y - M(x, y))} \leq \frac{f(x)}{(M(x, y) - x)(y - x)} + \frac{f(y)}{(x - y)(M(x, y) - y)}, \quad (36)$$

where $(x, y) \in I_{<}^2$. Using that $(M(x, y) - x)(y - M(x, y))$ is positive, we obtain, for all $(x, y) \in I_{<}^2$, that (35) is valid.

To prove the reverse implication of (b), suppose that (35) holds on the domain indicated. Then (36) is also valid and, in view of the first property of upper addition in (31), this implies (35).

In the second step we prove the statement (a). Suppose that f is lower M -convex, i.e., we have $[x, M(x, y), y; f] \geq 0$ for all $(x, y) \in I_{<}^2$. Due to the first property of lower addition in (31), it follows that, $(x, y) \in I_{<}^2$, we have $-\infty < \min\{f(x), -f(M(x, y)), f(y)\}$ and

$$\frac{f(M(x, y))}{(M(x, y) - x)(y - M(x, y))} \leq \frac{f(x)}{(M(x, y) - x)(y - x)} + \frac{f(y)}{(x - y)(M(x, y) - y)}. \quad (37)$$

Thus, for all $u \in I$, we get $-\infty < f(u)$ and, by the positivity of $(M(x, y) - x)(y - M(x, y))$, (37) is equivalent to (34) and $f(M(x, y)) < +\infty$ on the domain indicated.

To prove the reversed implication of the statement (a), suppose that $f(M(x, y)) < +\infty$ and (34) hold for all $(x, y) \in I_{<}^2$ and we have $-\infty < f(u)$ for all $u \in I$. Then (37) is also valid and, in view of the first property of lower addition in (31), this implies (34). \square

In the following proposition we show that, for certain rational numbers t , there exists an upper \mathbb{A}_t -convex extended real valued function f , which is not upper \mathbb{A}_{1-t} -convex. Therefore, f is not t -convex. It is an open problem if there exists a *real-valued* function f with these properties. This result is analogous to that of Lewicki and Olbryś [5] (which works for transcendental values of t).

Proposition 7.3. Denote by \mathbb{Q}_0 and \mathbb{Q}_1 the following subsets of the rationals:

$$\mathbb{Q}_0 := \left\{ \frac{2k}{2n-1} \mid k \in \mathbb{Z}, n \in \mathbb{N} \right\} \quad \text{and} \quad \mathbb{Q}_1 := \left\{ \frac{2k-1}{2n-1} \mid k \in \mathbb{Z}, n \in \mathbb{N} \right\}.$$

Then \mathbb{Q}_0 and \mathbb{Q}_1 are disjoint subsets of \mathbb{Q} and

$$\begin{aligned} \mathbb{Q}_0 + \mathbb{Q}_0 &\subseteq \mathbb{Q}_0, & \mathbb{Q}_0 + \mathbb{Q}_1 &\subseteq \mathbb{Q}_1, & \mathbb{Q}_1 + \mathbb{Q}_1 &\subseteq \mathbb{Q}_0, \\ \mathbb{Q}_0 \mathbb{Q}_0 &\subseteq \mathbb{Q}_0, & \mathbb{Q}_0 \mathbb{Q}_1 &\subseteq \mathbb{Q}_0, & \mathbb{Q}_1 \mathbb{Q}_1 &\subseteq \mathbb{Q}_1. \end{aligned} \quad (38)$$

Let $I \subseteq \mathbb{R}$ be an interval such that $a := \sup I \in I \cap \mathbb{Q}_1$. Let $h : I \rightarrow \mathbb{R}$ be an arbitrary convex function and define the function $f : I \rightarrow \overline{\mathbb{R}}$ by

$$f(x) := \begin{cases} h(x) & \text{if } x \in (I \cap \mathbb{Q}_0) \cup \{a\}, \\ +\infty & \text{if } x \in I \setminus (\mathbb{Q}_0 \cup \{a\}). \end{cases}$$

Then, for all $t \in]0, 1[\cap \mathbb{Q}_1$, the function f is upper \mathbb{A}_t -convex and is not upper \mathbb{A}_{1-t} -convex.

Proof. The inclusions in (38) follow from elementary calculation with rational fractions.

Let $x, y \in I$ with $x < y$ and $t \in]0, 1[\cap \mathbb{Q}_1$ be arbitrarily fixed. Then $1 - t \in \mathbb{Q}_0$. We need to check that (33) is satisfied with \mathbb{A}_t for the function f , which is equivalent to the validity of the inequality

$$f(tx + (1 - t)y) \leq tf(x) \dot{+} (1 - t)f(y). \quad (39)$$

If $\max\{f(x), f(y)\} = +\infty$, then the right hand side of (39) is equal to $+\infty$, thus, we can suppose that the right hand side is finite, that is $f(x) = h(x)$ and $f(y) = h(y)$. Now we have that $x \in \mathbb{Q}_0$ and $y \in \mathbb{Q}_0 \cup \mathbb{Q}_1$. Then, using (38), it follows that $tx + (1 - t)y \in \mathbb{Q}_0$. Therefore, applying the convexity of h , we get

$$f(tx + (1 - t)y) = h(tx + (1 - t)y) \leq th(x) + (1 - t)h(y) = tf(x) \dot{+} (1 - t)f(y).$$

This proves that f is upper \mathbb{A}_t -convex for all $t \in]0, 1[\cap \mathbb{Q}_1$.

To show that f is not upper \mathbb{A}_{1-t} -convex, let $y := a \in \mathbb{Q}_1$ and let $x \in I \cap \mathbb{Q}_0$ be an arbitrary point. It follows from (38) that the convex combination $(1 - t)x + ty$ belongs to \mathbb{Q}_1 and it is also different from a . Therefore we have $f((1 - t)x + ty) = +\infty$ and $(1 - t)f(x) \dot{+} tf(y) = (1 - t)h(x) + th(y) \in \mathbb{R}$, which means that (39) cannot be satisfied. \square

Definition 7.4. For a function $f : I \rightarrow \overline{\mathbb{R}}$, define the following two classes of means:

$$\begin{aligned} \underline{\mathcal{M}}_f &:= \{M : I_{\leq}^2 \rightarrow \mathbb{R} \mid M \text{ is a strict mean and } f \text{ is lower } M\text{-convex}\}, \\ \overline{\mathcal{M}}_f &:= \{M : I_{\leq}^2 \rightarrow \mathbb{R} \mid M \text{ is a strict mean and } f \text{ is upper } M\text{-convex}\}. \end{aligned}$$

Note, that, due to the strictness of the means in the definition, the above sets can be also empty. The following proposition shows a certain algebraic closedness property of the classes $\underline{\mathcal{M}}_f$ and $\overline{\mathcal{M}}_f$.

Proposition 7.5. For a function $f : I \rightarrow \overline{\mathbb{R}}$, the following statements hold:

- (a) if $M, N_1, N_2 \in \underline{\mathcal{M}}_f$ (resp. $M, N_1, N_2 \in \overline{\mathcal{M}}_f$) and $N_1 < N_2$ on the set I_{\leq}^2 , then $M \circ (N_1, N_2) \in \underline{\mathcal{M}}_f$ (resp. $M \circ (N_1, N_2) \in \overline{\mathcal{M}}_f$), and
- (b) if $M, N \in \underline{\mathcal{M}}_f$ (resp. $M, N \in \overline{\mathcal{M}}_f$), then $M \circ (\min, N)$ and $M \circ (N, \max)$ also belong to $\underline{\mathcal{M}}_f$ (resp. to $\overline{\mathcal{M}}_f$).

Proof. We verify the statements for the class $\overline{\mathcal{M}}_f$ only. The proof in the other case is completely analogous and also based on Lemma 7.2.

Let $(x, y) \in I_{\leq}^2$ be arbitrarily fixed, furthermore consider the points $p_1 := N_1(x, y)$ and $p_2 := N_2(x, y)$. (Obviously, under the conditions of (a), it follows that $p_1 < p_2$.) Using these notations, in view of Lemma 7.2, we need to show, that

$$f(M(p_1, p_2)) \leq \frac{y - M(p_1, p_2)}{y - x} f(x) \dot{+} \frac{M(p_1, p_2) - x}{y - x} f(y) \quad (40)$$

holds. By applying the M - and then the N_1 - and N_2 -convexity of f , we have the following calculation:

$$\begin{aligned} & f(M(p_1, p_2)) \\ & \leq \frac{p_2 - M(p_1, p_2)}{p_2 - p_1} f(p_1) \dot{+} \frac{M(p_1, p_2) - p_1}{p_2 - p_1} f(p_2) \\ & = \frac{p_2 - M(p_1, p_2)}{p_2 - p_1} f(N_1(x, y)) \dot{+} \frac{M(p_1, p_2) - p_1}{p_2 - p_1} f(N_2(x, y)) \\ & \leq \frac{p_2 - M(p_1, p_2)}{p_2 - p_1} \left(\frac{y - p_1}{y - x} f(x) \dot{+} \frac{p_1 - x}{y - x} f(y) \right) \dot{+} \frac{M(p_1, p_2) - p_1}{p_2 - p_1} \left(\frac{y - p_2}{y - x} f(x) \dot{+} \frac{p_2 - x}{y - x} f(y) \right) \\ & = \frac{y - M(p_1, p_2)}{y - x} f(x) \dot{+} \frac{M(p_1, p_2) - x}{y - x} f(y). \end{aligned}$$

Thus the inequality (40) is satisfied, which means the statement (a) is true.

A completely similar calculation shows that the statement (b) is also valid. \square

Corollary 7.6. *For a function $f : I \rightarrow \overline{\mathbb{R}}$, the classes*

$$\underline{\mathcal{M}}_f^* := \{M \in \underline{\mathcal{M}}_f \mid M \text{ is separately continuous in both variables}\},$$

$$\overline{\mathcal{M}}_f^* := \{M \in \overline{\mathcal{M}}_f \mid M \text{ is separately continuous in both variables}\}$$

have no isolated points with respect to the pointwise convergence, namely for all $M \in \underline{\mathcal{M}}_f^$ (resp. $M \in \overline{\mathcal{M}}_f^*$) there exist sequences of means $(L_n), (U_n) \subseteq \underline{\mathcal{M}}_f^*$ (resp. $(L_n), (U_n) \subseteq \overline{\mathcal{M}}_f^*$), such that $L_n < M < U_n$ for all $n \in \mathbb{N}$, furthermore $L_n \rightarrow M$ and $U_n \rightarrow M$ pointwise on $I_{<}^2$ as $n \rightarrow \infty$.*

Proof. We prove the statement only for the class $\underline{\mathcal{M}}_f^*$.

Let $M \in \underline{\mathcal{M}}_f^*$ be an arbitrarily fixed mean. We show only that the sequence (U_n) exists, because the existence of (L_n) can be proved similarly.

Let $U_0 = \max$ and, for $n \geq 1$, let $U_n := M \circ (M, U_{n-1})$. In the first step we show that the sequence (U_n) belongs to $\underline{\mathcal{M}}_f^*$. To see this, we prove, by induction, that $M < U_n < U_{n-1}$ for all $n \in \mathbb{N}$ on $I_{<}^2$. Let $(x, y) \in I_{<}^2$ be fixed. For $n = 1$, using that M is a strict mean, we get

$$U_1(x, y) = M(M(x, y), U_0(x, y)) = M(M(x, y), y) \in]M(x, y), y[=]M(x, y), U_0(x, y)[.$$

Assume that $M < U_n < U_{n-1}$ hold on $I_{<}^2$ for some $n \geq 2$. Using this assumption, for $n + 1$, we obtain that

$$U_{n+1}(x, y) = M(M(x, y), U_n(x, y)) \in]M(x, y), U_n(x, y)[.$$

Hence $M(x, y) < U_{n+1}(x, y) < U_n(x, y)$ follows for all $(x, y) \in I_{<}^2$, which completes the proof of the induction. Thus, due to Proposition 7.5, it follows that $(U_n) \subseteq \underline{\mathcal{M}}_f$. Moreover, by the definition, U_n is a strict mean and separately continuous in both variables for all $n \in \mathbb{N}$, hence $(U_n) \subseteq \underline{\mathcal{M}}_f^*$.

In the second step we show, that $U_n \downarrow M$ pointwise on $I_{<}^2$ as $n \rightarrow \infty$. Let $(x, y) \in I_{<}^2$ be arbitrarily fixed again. Obviously, the sequence $(U_n(x, y)) \subseteq]x, y[$ is convergent, because it is monotone decreasing and bounded from below by $M(x, y)$. Denote $\lim_{n \rightarrow \infty} U_n(x, y)$ by $U^*(x, y)$ which, of course, cannot be smaller than $M(x, y)$. Upon taking the limit $n \rightarrow \infty$ in the identity

$$U_n(x, y) = M(M(x, y), U_{n-1}(x, y)),$$

we get that

$$U^*(x, y) = M(M(x, y), U^*(x, y)).$$

The inequality $M(x, y) < U^*(x, y)$ would contradict the strictness of M , therefore, $U^*(x, y) = M(x, y)$ must be valid. \square

The following theorem is one of the main results of this paper. Roughly speaking, it states that the lower M -convexity property is inherited by the descendants.

Theorem 7.7. *Let $f : I \rightarrow \overline{\mathbb{R}}$, $n \geq 2$ and $M_1, \dots, M_n \in \underline{\mathcal{M}}_f$ be continuous strict means. Then, for all $i \in \{1, \dots, n\}$, we have $\mathcal{D}_i(M_1, \dots, M_n) \subseteq \underline{\mathcal{M}}_f$.*

Proof. Let $i \in \{1, \dots, n\}$ and $N \in \mathcal{D}_i(M_1, \dots, M_n)$ be arbitrarily fixed. We have already seen that, under our conditions, N is a strict mean. If $(x, y) \in I_{<}^2$, then there exists $k \in \{1, \dots, n\}$ and $(\xi_1, \dots, \xi_n) \in \Phi_{(x, y)}$ such that $N(x, y) = \xi_k$, furthermore, with $\xi_0 := x$ and $\xi_{n+1} := y$, we have

$$M_j(\xi_{j-1}, \xi_{j+1}) = \xi_j \quad (j \in \{1, \dots, n\}).$$

Using this and, for all $j \in \{1, \dots, n\}$, the lower M_j -convexity of the function f , we obtain

$$0 \leq \lfloor \xi_{j-1}, \xi_j, \xi_{j+1}; f \rfloor \quad (j \in \{1, \dots, n\}).$$

Now, applying the Extended Chain Inequality, we get that

$$0 \leq \min_{1 \leq j \leq n} \lfloor \xi_{j-1}, \xi_j, \xi_{j+1}; f \rfloor \leq \lfloor x, \xi_k, y; f \rfloor = \lfloor x, N(x, y), y; f \rfloor.$$

This means, by the definition, that f is lower N -convex, that is $N \in \underline{\mathcal{M}}_f$. \square

Corollary 7.8. *Let $f : I \rightarrow \overline{\mathbb{R}}$, $n \geq 2$, $s_1, \dots, s_n \in]0, 1[$, and let $h : I \rightarrow \mathbb{R}$ be a continuous, strictly increasing function. Assume that $\mathbb{M}_{s_i h, (1-s_i)h} \in \underline{\mathcal{M}}_f$ for all $i \in \{1, \dots, n\}$. Then, for all $i \in \{1, \dots, n\}$, the Matkowski mean $\mathbb{M}_{\sigma_i h, (1-\sigma_i)h}$ also belongs to $\underline{\mathcal{M}}_f$, where*

$$\sigma_i := \left(\sum_{j=i}^n \prod_{k=1}^j \frac{s_k}{1-s_k} \right) \left(\sum_{j=0}^n \prod_{k=1}^j \frac{s_k}{1-s_k} \right)^{-1} \quad (i \in \{1, \dots, n\}). \quad (41)$$

Proof. For $(x, y) \in I_{<}^2$, define the mapping $\varphi_{(x,y)} : [x, y]_{\leq}^n \rightarrow \mathbb{R}^n$ as in Theorem 5.6. In view of this theorem, it follows that, for all $(x, y) \in I_{<}^2$, the fixed point set $\Phi_{(x,y)}$ is the singleton $\{(\xi_1, \dots, \xi_n)\}$, where $\xi_i = \mathbb{M}_{\sigma_i h, (1-\sigma_i)h}(x, y)$. Thus, for $i \in \{1, \dots, n\}$, the function $\mathbb{M}_{\sigma_i h, (1-\sigma_i)h}$ is the i^{th} descendant of the n -tuple of means $(\mathbb{M}_{s_1 h, (1-s_1)h}, \dots, \mathbb{M}_{s_n h, (1-s_n)h})$. Therefore, due to Theorem 7.7, we obtain that $\mathbb{M}_{\sigma_i h, (1-\sigma_i)h} \in \underline{\mathcal{M}}_f$ for all $i \in \{1, \dots, n\}$. \square

Corollary 7.9. *Let $n \geq 2$, $p, q, h_1, \dots, h_{n-1} : I \rightarrow \mathbb{R}$ be continuous, strictly increasing functions and $f : I \rightarrow \overline{\mathbb{R}}$. Set further $h_0 := h_n := 0$ and assume that there exists $j \in \{1, \dots, n\}$ such that*

$$\{\mathbb{M}_{p+h_{i-1}, h_i} \mid 1 \leq i \leq j-1\} \cup \{\mathbb{M}_{p+h_{j-1}, q+h_j}\} \cup \{\mathbb{M}_{h_{i-1}, q+h_i} \mid j+1 \leq i \leq n\} \subseteq \underline{\mathcal{M}}_f.$$

Then $N_1, \dots, N_n \in \underline{\mathcal{M}}_f$, where, for all $(x, y) \in I_{\leq}^2$,

$$N_j(x, y) = \mathbb{M}_{p,q}(x, y) \quad \text{and} \quad N_i(x, y) = \begin{cases} \mathbb{M}_{p, h_i}(x, N_{i+1}(x, y)) & \text{if } i \in \{1, \dots, j-1\}, \\ \mathbb{M}_{h_{i-1}, q}(N_{i-1}(x, y), y) & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

Proof. The method of the proof is the same as that of Corollary 7.8. For $(x, y) \in I_{\leq}^2$, define the mapping $\varphi_{(x,y)}$ as in (15) by the using the means M_1, \dots, M_n , where

$$M_i := \begin{cases} \mathbb{M}_{p+h_{i-1}, h_i} & \text{if } i \in \{1, \dots, j-1\}, \\ \mathbb{M}_{p+h_{i-1}, h_i+q} & \text{if } i = j, \\ \mathbb{M}_{h_{i-1}, h_i+q} & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

Due to Theorem 5.7, it follows that, for all $(x, y) \in I_{\leq}^2$, the fixed point set $\Phi_{(x,y)}$ is the singleton $\{(\xi_1, \dots, \xi_n)\}$, where we have

$$\xi_j := \mathbb{M}_{p,q}(x, y) \quad \text{and} \quad \xi_i := \begin{cases} \mathbb{M}_{p, h_i}(x, \xi_{i+1}) & \text{if } i \in \{1, \dots, j-1\}, \\ \mathbb{M}_{h_{i-1}, q}(\xi_{i-1}, y) & \text{if } i \in \{j+1, \dots, n\}. \end{cases}$$

Thus, for $i \in \{1, \dots, n\}$, the function $N_i : I_{\leq}^2 \rightarrow \mathbb{R}$, $N_i(x, y) := \xi_i$ is the i^{th} descendant of the n -tuple of means $(\mathbb{M}_{p+h_0, h_1}, \dots, \mathbb{M}_{p+h_{j-1}, h_j+q}, \dots, \mathbb{M}_{h_{n-1}, h_n+q})$. Hence, by Theorem 7.7, it follows that $N_i \in \underline{\mathcal{M}}_f$ for all $i \in \{1, \dots, n\}$. \square

8. \mathbb{A}_t -CONVEXITY OF EXTENDED REAL VALUED FUNCTIONS

In this section we investigate a special subclass of $\underline{\mathcal{M}}_f$ and $\overline{\mathcal{M}}_f$, respectively. For an extended real valued function $f : I \rightarrow \overline{\mathbb{R}}$ consider the sets $\underline{\mathcal{C}}_f$ and $\overline{\mathcal{C}}_f$ defined by

$$\underline{\mathcal{C}}_f := \{0 < t < 1 \mid \text{for all } (x, y) \in I_{<}^2 : [x, \mathbb{A}_t(x, y), y; f] \geq 0\},$$

and

$$\overline{\mathcal{C}}_f := \{0 < t < 1 \mid \text{for all } (x, y) \in I_{<}^2 : [x, \mathbb{A}_t(x, y), y; f] \geq 0\}.$$

If f is real-valued, then clearly these two sets are the same, therefore, we will simply denote them by \mathcal{C}_f . Note that, by the definitions, both sets can also be empty. On the other hand, these sets can be easily identified with the subclass of weighted arithmetic means in $\underline{\mathcal{M}}_f$ and $\overline{\mathcal{M}}_f$ respectively, more precisely we have the following identifications

$$t \in \underline{\mathcal{C}}_f \iff \mathbb{A}_t|_{I_{\leq}^2} \in \underline{\mathcal{M}}_f \quad \text{and} \quad t \in \overline{\mathcal{C}}_f \iff \mathbb{A}_t|_{I_{\leq}^2} \in \overline{\mathcal{M}}_f.$$

The motivation for our investigations is a well known result, which is due to N. Kuhn [4], and which is about the structure of the set of parameters for which a given real valued function is convex. The theorem says that if $f : I \rightarrow \mathbb{R}$ is an arbitrary function and

$$\mathcal{C}_f^\circ := \mathcal{C}_f \cap (1 - \mathcal{C}_f) = \{0 < t < 1 \mid f \text{ is simultaneously } \mathbb{A}_t\text{-convex and } \mathbb{A}_{1-t}\text{-convex on } I\},$$

then we have that either $\mathcal{C}_f^\circ = \emptyset$ or $\mathcal{C}_f^\circ = K \cap]0, 1[$, where K is a subfield of \mathbb{R} . Moreover, the reverse of this statement is also valid: if $K \subseteq \mathbb{R}$ is a given subfield, then there exists a function $f : I \rightarrow \mathbb{R}$ such that \mathcal{C}_f° equals to the intersection $K \cap]0, 1[$.

The following results are about such algebraical closedness properties of the sets $\underline{\mathcal{C}}_f$ and $\overline{\mathcal{C}}_f$.

Theorem 8.1. *Given a function $f : I \rightarrow \overline{\mathbb{R}}$, the following statements hold for $S \in \{\underline{\mathcal{C}}_f, \overline{\mathcal{C}}_f\}$:*

- (1) *if $t, s_1, s_2 \in S$ with $s_1 < s_2$, then $ts_2 + (1 - t)s_1 \in S$,*
- (2) *if $t, s \in S$, then ts and $1 - (1 - t)(1 - s)$ belong to S , and*
- (3) *S is dense in the open unit interval, provided that it is not empty.*

Proof. We verify only the statements about $S = \underline{\mathcal{C}}_f$. The proof for $S = \overline{\mathcal{C}}_f$ is analogous.

Let $t, s_1, s_2 \in \underline{\mathcal{C}}_f$ with $s_1 < s_2$. Then the means $\mathbb{A}_t, \mathbb{A}_{s_1}$ and \mathbb{A}_{s_2} belong to $\underline{\mathcal{M}}_f$ and, because of $s_1 < s_2$, we have $\mathbb{A}_{s_2} < \mathbb{A}_{s_1}$ on $I_{<}^2$. Using Proposition 7.5 for $M := \mathbb{A}_t$, $N_1 := \mathbb{A}_{s_2}$ and $N_2 := \mathbb{A}_{s_1}$, we obtain that $\mathbb{A}_t \circ (\mathbb{A}_{s_2}, \mathbb{A}_{s_1}) \in \underline{\mathcal{M}}_f$. On the other hand, for $(x, y) \in I_{<}^2$, we have

$$\begin{aligned} \mathbb{A}_t \circ (\mathbb{A}_{s_2}, \mathbb{A}_{s_1})(x, y) &= \mathbb{A}_t(\mathbb{A}_{s_2}(x, y), \mathbb{A}_{s_1}(x, y)) = \mathbb{A}_t(s_2x + (1 - s_2)y, s_1x + (1 - s_1)y) \\ &= (ts_2 + (1 - t)s_1)x + (1 - (ts_2 + (1 - t)s_1))y = \mathbb{A}_{ts_2 + (1 - t)s_1}(x, y), \end{aligned}$$

consequently $ts_2 + (1 - t)s_1 \in \underline{\mathcal{C}}_f$, which proves (1).

To prove (2), observe that, under our notation, $\min = \mathbb{A}_1$ and $\max = \mathbb{A}_0$ on $I_{<}^2$. Thus, according to the second statement of Proposition 7.5, the means $\mathbb{A}_t \circ (\mathbb{A}_1, \mathbb{A}_s)$ and $\mathbb{A}_t \circ (\mathbb{A}_s, \mathbb{A}_0)$ belong to $\underline{\mathcal{M}}_f$. Then the same calculation yields that $1 - (1 - t)(1 - s)$ and ts belong to $\underline{\mathcal{C}}_f$, respectively.

To verify (3) assume that $\underline{\mathcal{C}}_f$ is not empty and indirectly suppose that $\underline{\mathcal{C}}_f$ is not dense in $]0, 1[$, that is there exist $\alpha < \beta$ in $[0, 1]$ such that $\underline{\mathcal{C}}_f \cap]\alpha, \beta[$ is empty. We may assume that the interval $]\alpha, \beta[$ is maximal, or equivalently, for all $\varepsilon > 0$, the intersection $\underline{\mathcal{C}}_f \cap]\alpha - \varepsilon, \beta + \varepsilon[$ is not empty. Observe that, due to the second assertion of the theorem, it easily follows that $0 < \alpha$ and $\beta < 1$. Indeed, if $t \in \underline{\mathcal{C}}_f$ is arbitrary, then, due to the fact that $\underline{\mathcal{C}}_f$ is closed under the multiplication, for all $k \in \mathbb{N}$, the value t^k belongs again to $\underline{\mathcal{C}}_f$. Thus any open neighborhood of zero contains an element from $\underline{\mathcal{C}}_f$, which means $0 < \alpha$. Similarly, using the closedness of $\underline{\mathcal{C}}_f$ under the operation $(t, s) \mapsto 1 - (1 - t)(1 - s)$, we get that $\beta < 1$. Thus we obtained that $[\alpha, \beta] \subseteq]0, 1[$. Now, let $t \in \underline{\mathcal{C}}_f$ be arbitrarily fixed and $(r_n), (s_n) \subseteq \underline{\mathcal{C}}_f$ be sequences such that $r_n \nearrow \alpha$ and $s_n \searrow \beta$ as $n \rightarrow \infty$. Then, in view of the first assertion of the theorem, $ts_n + (1 - t)r_n \in \underline{\mathcal{C}}_f$ for all $n \in \mathbb{N}$ and $ts_n + (1 - t)r_n \rightarrow t\beta + (1 - t)\alpha \in]\alpha, \beta[$ as $n \rightarrow \infty$. Therefore, for sufficiently large n , we get that $ts_n + (1 - t)r_n \in]\alpha, \beta[$, which contradicts the emptiness of $\underline{\mathcal{C}}_f \cap]\alpha, \beta[$ and hence $\underline{\mathcal{C}}_f$ must be dense in $]0, 1[$. \square

Remark 8.2. The result stated in Theorem 8.1 is not analogous to that of Kuhn [4]. In general, the set $\overline{\mathcal{C}}_f$ is not of the form $]0, 1[\cap K$, where K is a subfield of \mathbb{R} . To see this, it is sufficient to construct a function $f : I \rightarrow \overline{\mathbb{R}}$ such that the set $\overline{\mathcal{C}}_f$ is not closed under the addition of their elements.

Indeed, let $f : I \rightarrow \mathbb{R} \cup \{+\infty\}$ be the function defined in Proposition 7.3. For arbitrarily fixed parameters $s, t \in]0, 1[\cap \mathbb{Q}_1 \subseteq \overline{\mathcal{C}}_f$ with $s + t < 1$, in view of (38), the sum $s + t$ belongs to \mathbb{Q}_0 . To prove that $s + t \notin \overline{\mathcal{C}}_f$, we construct $x < y$ in I such that

$$f((s + t)x + (1 - (s + t))y) > (s + t)f(x) + (1 - (s + t))f(y). \quad (42)$$

Let $x \in I \cap \mathbb{Q}_0$ be arbitrarily fixed and set $y := a$. Then, using again (38), the convex combination $u := (s + t)x + (1 - (s + t))y$ belongs to $I \cap \mathbb{Q}_1$ and it is also different from a . Consequently $f(u) = +\infty$, on the other hand

$$(s + t)f(x) + (1 - (s + t))f(y) = (s + t)h(x) + (1 - (s + t))h(y) \in \mathbb{R},$$

thus (42) is satisfied.

Corollary 8.3. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \rightarrow \overline{\mathbb{R}}$, $n \geq 2$ and $s_1, \dots, s_n \in \mathcal{C}_f$. Then $\sigma_i \in \mathcal{C}_f$ for all $i \in \{1, \dots, n\}$, where*

$$\sigma_i := \left(\sum_{j=i}^n \prod_{k=1}^j \frac{s_k}{1-s_k} \right) \left(\sum_{j=0}^n \prod_{k=1}^j \frac{s_k}{1-s_k} \right)^{-1}. \quad (43)$$

Proof. Apply Corollary 7.8 under $h := \text{id}$. □

Corollary 8.4. *For a function $f : I \rightarrow \overline{\mathbb{R}}$ the following statements hold:*

- (1) *if $1/2 \in \mathcal{C}_f$ then $r \in \mathcal{C}_f$ for all $r \in \mathbb{Q} \cap]0, 1[$,*
- (2) *if $\ell/m \in \mathcal{C}_f$ for some $\ell, m \in \mathbb{N}$ with $\ell < m$ and $\ell \neq m/2$, then, for all $n \geq 2$ and for all $i \in \{1, \dots, n\}$, the fraction*

$$r_i := \frac{\ell^{n+1} - \ell^i(m - \ell)^{n+1-i}}{\ell^{n+1} - (m - \ell)^{n+1}}$$

belongs to \mathcal{C}_f .

Proof. To prove (1), assume that $1/2 \in \mathcal{C}_f$ and let $p, q \in \mathbb{N}$ be arbitrarily fixed numbers such that $q > 1$ and $p < q$. For $q = 2$, the statement (1) is trivial, thus we may assume that $q > 2$. Now set $n := q - 1$ and $i_0 := q - p$. Then $n \geq 2$ and $i_0 \in \{1, \dots, n\}$, thus, using Corollary 8.3 for $s_1 := \dots = s_n := 1/2$, we get that

$$\sigma_{i_0} = \frac{n - i_0 + 1}{n + 1} = \frac{q - 1 - (q - p) + 1}{q - 1 + 1} = \frac{p}{q}.$$

This means that $\mathbb{Q} \cap]0, 1[\subseteq \mathcal{C}_f$.

To prove (2), assume that $\ell/m \in \mathcal{C}_f$ for some $\ell, m \in \mathbb{N}$, where $\ell < m$ and $2\ell \neq m$. Let further $n \geq 2$ be arbitrarily fixed and set $s_1 := \dots = s_n := \ell/m$. Then a simple calculation shows that $\sigma_i = r_i$ for all $i \in \{1, \dots, n\}$. Due to Corollary 8.3, we get that $r_i \in \mathcal{C}_f$ for all $i \in \{1, \dots, n\}$. □

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