



Hölder- and Minkowski-type inequalities for generalized quasi-arithmetic means

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Abstract

The purpose of this paper is to establish several necessary and sufficient conditions to ensure the validity of a general functional inequality in terms of generalized quasi-arithmetic means. In particular cases, we consider Hölder-, Minkowski-, and Jensen-type inequalities. Generalized quasi-arithmetic means are defined by taking strictly monotone generating functions instead of strictly monotone and continuous ones.

Keywords Generalized quasi-arithmetic mean · Hölder-type inequality · Minkowski-type inequality · Jensen convexity

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1 Introduction

This paper aims to study the functional inequality of the form

$$M_0(\varphi(x_{1,1}, \dots, x_{1,k}), \dots, \varphi(x_{n,1}, \dots, x_{n,k})) \leq \Phi(M_1(x_{1,1}, \dots, x_{n,1}), \dots, M_k(x_{1,k}, \dots, x_{n,k})),$$

for all $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{n,1}, \dots, x_{n,k}) \in I_1 \times \dots \times I_k,$
(1.1)

where $n, k \in \mathbb{N}$, $M_j: I_j^n \rightarrow I_j$ are n -variable means on I_j ($j \in \{0, \dots, k\}$), $\varphi: I_1 \times \dots \times I_k \rightarrow I_0$, $\Phi: I_1 \times \dots \times I_k \rightarrow \mathbb{R}$ ($I_0, \dots, I_k$ are subintervals of $\mathbb{R}$) in the case when all M_i -s are generalized quasi-arithmetic means. More precisely, we characterize, for a given pair (φ, Φ) , all sequences of generalized quasi-arithmetic means $(M_0, \dots, M_k)$ satisfying this inequality.

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Furthermore, we will study its weighted counterpart, that is we involve the space of weights $\Lambda_n := [0, \infty)^n \setminus \{(0, \dots, 0)\}$ in the inequality (1.1) and we obtain

$$\begin{aligned} & \tilde{M}_0(\varphi(x_{1,1}, \dots, x_{1,k}), \dots, \varphi(x_{n,1}, \dots, x_{n,k}); \lambda_1, \dots, \lambda_n) \\ & \leq \Phi(\tilde{M}_1(x_{1,1}, \dots, x_{n,1}; \lambda_1, \dots, \lambda_n), \dots, \tilde{M}_k(x_{1,k}, \dots, x_{n,k}; \lambda_1, \dots, \lambda_n)), \quad (1.2) \\ & \text{for all } \lambda \in \Lambda_n, \text{ and } (x_{1,1}, \dots, x_{1,k}), \dots, (x_{n,1}, \dots, x_{n,k}) \in I_1 \times \dots \times I_k, \end{aligned}$$

where all remaining parameters are the same as above except $\tilde{M}_j: I_j^n \times \Lambda_n \rightarrow I_j$ ($j \in \{0, \dots, k\}$) which, in this case, are n -variable weighted means on suitable intervals.

Inequality (1.2) seems to be more general than (1.1) as it contains (1.1) as a special case (when $\lambda \equiv 1$). Nevertheless, these two approaches are equivalent in view of our main result. These inequalities generalize several known problems. For example, if $\varphi = \Phi$ is the arithmetic mean, then we get the convexity-type inequalities; if $\varphi = \Phi$ is the sum, then our problem reduces to Minkowski-type inequalities, and so on. It will turn out that $\varphi = \Phi$ is generally the important subclass of these inequalities providing meaningful simplifications. The proofs of our main results are elementary, for the verification of one of the implications among the several equivalent conditions, we use Helly's Theorem and also Farkas' Lemma.

Let us also emphasize that converse inequalities can be obtained using the standard reflection-type techniques. Roughly speaking, we consider reflected means M_j^* defined on $-I_j$ as $M_j^*(x) := -M_j(-x)$ instead of the original ones and apply natural substitutions; see for example [15, sect. 2.2]. We skip the details for the sake of compactness.

Finally, we refer to the papers [1–3, 5–11, 13], where the reader can find results related to the standard quasi-arithmetic setting.

2 The concept of generalized left inverse and generalized quasi-arithmetic means

The concept of generalized quasi-arithmetic means was mentioned for the first time in [14, Example 4.5(3)] as an important subclass of semideviation means (a large class of means defined for the first time in [12]). Furthermore, these means are the particular cases of the generalized Bajraktarević means introduced in [4]. The first comprehensive study of this family was presented in the recent paper [16].

Here we recall some relevant concepts and results from the past. In the sequel, the set of points in I , where f is continuous, will be denoted by C_f .

Lemma 2.1 ([4, Lemma 1]) *Let $f: I \rightarrow \mathbb{R}$ be a strictly increasing function. Then there exists a uniquely determined increasing function $f^{(-1)}: \text{conv}(f(I)) \rightarrow I$ which is the left inverse of f , i.e., for all $x \in I$,*

$$(f^{(-1)} \circ f)(x) = x. \quad (2.1)$$

Furthermore, $f^{(-1)}$ is continuous and $(f \circ f^{(-1)})(y) = y$ for all $y \in f(I)$.

For instance, if $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) := x + \text{sign}(x)$, then f is strictly increasing and discontinuous at 0, furthermore $f(\mathbb{R}) = (-\infty, -1) \cup \{0\} \cup (1, \infty)$. Thus $\text{conv}(f(\mathbb{R})) = \mathbb{R}$ and

$$f^{(-1)}(y) = \begin{cases} y + 1 & \text{if } y \in (-\infty, -1), \\ 0 & \text{if } y \in [-1, 1], \\ y - 1 & \text{if } y \in (1, \infty). \end{cases}$$

Then (2.1) holds for all $x \in \mathbb{R}$, the equality $(f \circ f^{(-1)})(y) = y$ is satisfied for all $y \in f(\mathbb{R})$, however for $y \in \mathbb{R} \setminus f(\mathbb{R}) = [-1, 0) \cup (0, 1]$, we have that $(f \circ f^{(-1)})(y) = 0$, which shows that $f^{(-1)}$ is not a right inverse for f .

The function $f^{(-1)}$ described in the above lemma is called the *generalized left inverse of the strictly increasing function* $f : I \rightarrow \mathbb{R}$. It is clear that the restriction of $f^{(-1)}$ to $f(I)$ equals the inverse of f in the standard sense. Therefore, $f^{(-1)}$ is the continuous and increasing extension of the inverse of f to the smallest interval containing the range of f . It turns out that the generalized left inverse follows the composition principle known for the common inverse. This property, however, requires a simple proof.

Lemma 2.2 *Let I, J be nonempty open subintervals of $\mathbb{R}$, and let $f : I \rightarrow J, g : J \rightarrow \mathbb{R}$ be strictly increasing functions. Then $(f \circ g)^{(-1)} = g^{(-1)} \circ f^{(-1)}$.*

Proof Note that $f \circ g$ is strictly increasing. Take any $y_0 \in \text{conv}(f \circ g(I))$, and set $x_0 := (f \circ g)^{(-1)}(y_0)$. Then for all $x, x' \in I$ with $x < x_0 < x'$ we have $f \circ g(x) < y_0 < f \circ g(x')$.

Applying, side-by-side, first $f^{(-1)}$, then $g^{(-1)}$, and using that these functions are increasing, we get $x \leq g^{(-1)} \circ f^{(-1)}(y_0) \leq x'$. Finally, upon taking the limit $x \uparrow x_0$ and $x' \downarrow x_0$ we get $g^{(-1)} \circ f^{(-1)}(y_0) = x_0 = (f \circ g)^{(-1)}(y_0)$, which completes the proof. $\square$

Let us now recall the easy but useful lemma.

Lemma 2.3 ([16, Lemma 2.2]) *Let $f : I \rightarrow \mathbb{R}$ be a strictly increasing function and let $x \in I$ and $u \in \text{conv}(f(I))$. Then we have the following equivalences:*

- (i) $f^{(-1)}(u) < x$ holds if and only if $u < f_-(x)$.
- (ii) $f^{(-1)}(u) \leq x$ holds if and only if $u \leq f_+(x)$.
- (iii) $x < f^{(-1)}(u)$ holds if and only if $f_+(x) < u$.
- (iv) $x \leq f^{(-1)}(u)$ holds if and only if $f_-(x) \leq u$.

Given a strictly increasing function $f : I \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$, the n -variable generalized quasi-arithmetic mean $A_f^{[n]} : I^n \rightarrow I$ and the n -variable weighted generalized quasi-arithmetic mean $\tilde{A}_f^{[n]} : I^n \times \Lambda_n \rightarrow I$ are defined by the following formulas:

$$A_f^{[n]}(x) := f^{(-1)}\left(\frac{f(x_1) + \cdots + f(x_n)}{n}\right) \quad (x \in I^n), \quad (2.2)$$

$$\tilde{A}_f^{[n]}(x; \lambda) := f^{(-1)}\left(\frac{\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)}{\lambda_1 + \cdots + \lambda_n}\right) \quad (x \in I^n, \lambda \in \Lambda_n). \quad (2.3)$$

Clearly, for any $x \in I^n$, we have that $A_f^{[n]}(x) = \tilde{A}_f^{[n]}(x; \lambda)$ if $\lambda_1 = \cdots = \lambda_n$.

Lemma 2.4 ([16, Proposition 3.3]) *Let $n \in \mathbb{N} \setminus \{1\}$, let $f, g : I \rightarrow \mathbb{R}$ be strictly increasing functions, and let $t \in I$. Assume that $A_f^{[n]}(x) \leq A_g^{[n]}(x)$ holds for all $x \in I^n$. Then, we have the following assertions:*

- (i) If g is lower semicontinuous at t , then so is f .
- (ii) If f is upper semicontinuous at t , then so is g .
- (iii) The function f is continuous at t if and only if g is continuous at t .

3 Main results

Our main result reads as follows.

Theorem 3.1 *Let $k \in \mathbb{N}$, let $I_0, I_1, \dots, I_k$ be nonempty open intervals and denote $I := I_1 \times \dots \times I_k$. For all $j \in \{0, 1, \dots, k\}$, let $f_j : I_j \rightarrow \mathbb{R}$ be strictly increasing, let $\varphi : I \rightarrow I_0$ be arbitrary and let $\Phi : I \rightarrow \mathbb{R}$ be separately increasing and upper semicontinuous. Assume that the set $\Gamma := \Phi^{-1}(C_{f_0}) = \{t \in I : \Phi(t) \in C_{f_0}\}$ is a dense subset of I . Then the following assertions are equivalent.*

(a) *For all $n \in \mathbb{N}$ and for all $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{n,1}, \dots, x_{n,k}) \in I$,*

$$\begin{aligned} & A_{f_0}^{[n]}(\varphi(x_{1,1}, \dots, x_{1,k}), \dots, \varphi(x_{n,1}, \dots, x_{n,k})) \\ & \leq \Phi\left(A_{f_1}^{[n]}(x_{1,1}, \dots, x_{n,1}), \dots, A_{f_k}^{[n]}(x_{1,k}, \dots, x_{n,k})\right). \end{aligned}$$

(b) *For all $n \in \mathbb{N}$, for all $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{n,1}, \dots, x_{n,k}) \in I$, and for all $(\lambda_1, \dots, \lambda_n) \in \Lambda_n$,*

$$\begin{aligned} & \tilde{A}_{f_0}^{[n]}(\varphi(x_{1,1}, \dots, x_{1,k}), \dots, \varphi(x_{n,1}, \dots, x_{n,k}); \lambda_1, \dots, \lambda_n) \\ & \leq \Phi\left(\tilde{A}_{f_1}^{[n]}(x_{1,1}, \dots, x_{n,1}; \lambda_1, \dots, \lambda_n), \dots, \right. \\ & \quad \left. \tilde{A}_{f_k}^{[n]}(x_{1,k}, \dots, x_{n,k}; \lambda_1, \dots, \lambda_n)\right). \end{aligned}$$

(c) *For all $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{k+1,1}, \dots, x_{k+1,k}) \in I$ and for all $(\lambda_1, \dots, \lambda_{k+1}) \in \Lambda_{k+1}$,*

$$\begin{aligned} & \tilde{A}_{f_0}^{[k+1]}(\varphi(x_{1,1}, \dots, x_{1,k}), \dots, \varphi(x_{k+1,1}, \dots, x_{k+1,k}); \lambda_1, \dots, \lambda_{k+1}) \\ & \leq \Phi\left(\tilde{A}_{f_1}^{[k+1]}(x_{1,1}, \dots, x_{k+1,1}; \lambda_1, \dots, \lambda_{k+1}), \dots, \right. \\ & \quad \left. \tilde{A}_{f_k}^{[k+1]}(x_{1,k}, \dots, x_{k+1,k}; \lambda_1, \dots, \lambda_{k+1})\right). \end{aligned}$$

(d) *For all $j \in \{1, \dots, k\}$, there exists a nonnegative function $a_j : \Gamma \rightarrow \mathbb{R}$ such that for all $(x_1, \dots, x_k) \in I$ and $(t_1, \dots, t_k) \in \Gamma$,*

$$f_0(\varphi(x_1, \dots, x_k)) - f_0(\Phi(t_1, \dots, t_k)) \leq \sum_{j=1}^k a_j(t_1, \dots, t_k)(f_j(x_j) - f_j(t_j)).$$

(e) *There exists a separately increasing concave function $\Psi : \text{conv}(f_1(I_1)) \times \dots \times \text{conv}(f_k(I_k)) \rightarrow \mathbb{R}$ such that, for all $(x_1, \dots, x_k) \in I$,*

$$f_0(\varphi(x_1, \dots, x_k)) \leq \Psi(f_1(x_1), \dots, f_k(x_k)) \quad (3.1)$$

and, for all $(t_1, \dots, t_k) \in \Gamma$,

$$f_0(\Phi(t_1, \dots, t_k)) \geq \Psi(f_1(t_1), \dots, f_k(t_k)). \quad (3.2)$$

Remark In general, we cannot remove the density assumption about the set Γ defined in the above theorem. However, we show that this condition is satisfied under very natural (and less technical) assumptions. For instance, if we suppose that the function $\Phi : I \rightarrow \mathbb{R}$ is strictly increasing in one of its variables, then the set $\Gamma = \Phi^{-1}(C_{f_0})$ is dense in I . Indeed, assume that Φ is strictly increasing in its k th variable. Let $(t_1, \dots, t_{k-1}) \in I_1 \times \dots \times I_{k-1}$

be arbitrary. Then the map $t_k \mapsto \phi(t_1, \dots, t_{k-1}, t_k)$ is an injective mapping of I_k into I_0 . Therefore, the inverse image by this map of the set of discontinuity points of f_0 which is countable is also countable. Therefore, the inverse image by this map of the set of continuity points of f_0 is co-countable and hence it is dense in I_k . This proves that Γ is dense in I .

Proof of Theorem 3.1 The proof of implication $(a) \Rightarrow (b)$ is standard, it is based on the continuity of generalized quasi-arithmetic means with respect to weights and the upper semi-continuity of Φ .

Applying assertion (b) for $n = k + 1$, we can see that (b) implies (c) .

To verify the implication $(c) \Rightarrow (d)$, assume that the assertion (c) is valid. To prove the existence of nonnegative functions described in assertion (d) it is sufficient to show that, for all fixed $t := (t_1, \dots, t_k) \in \Gamma$, there exists $(a_1, \dots, a_k) \in [0, \infty)^k$ such that

$$f_0(\varphi(x)) - f_0(\Phi(t)) \leq \sum_{j=1}^k a_j (f_j(x_j) - f_j(t_j))$$

holds for all $x = (x_1, \dots, x_k) \in I$. To see this, we prove that the intersection of the family of closed convex sets $\{H(x) : x \in I\}$ is not empty, where

$$H(x) := \left\{ (a_1, \dots, a_k) \in [0, \infty)^k : f_0(\varphi(x)) - f_0(\Phi(t)) \leq \sum_{j=1}^k a_j (f_j(x_j) - f_j(t_j)) \right\}, \quad x \in I.$$

First we show that there exists a finite (in fact a k -element) subset of I over which the intersection is bounded and hence it is compact.

Let $e_1, \dots, e_k$ denote the standard basis for $\mathbb{R}^k$. Choose $\varepsilon > 0$ so that $t - \varepsilon e_j \in I$ be valid for all $j \in \{1, \dots, k\}$. Then

$$H(t - \varepsilon e_j) = \left\{ (a_1, \dots, a_k) \in [0, \infty)^k : f_0(\varphi(t - \varepsilon e_j)) - f_0(\Phi(t)) \leq a_j (f_j(t_j - \varepsilon) - f_j(t_j)) \right\},$$

which, by the strict monotonicity of f_j , shows that, for $(a_1, \dots, a_k) \in H(t - \varepsilon e_j)$, we have

$$a_j \in \left[0, \frac{f_0(\varphi(t - \varepsilon e_j)) - f_0(\Phi(t))}{f_j(t_j - \varepsilon) - f_j(t_j)} \right].$$

Therefore,

$$\bigcap_{j=1}^k H(t - \varepsilon e_j) \subseteq \prod_{j=1}^k \left[0, \frac{f_0(\varphi(t - \varepsilon e_j)) - f_0(\Phi(t))}{f_j(t_j - \varepsilon) - f_j(t_j)} \right].$$

According to an infinite variant of Helly's Theorem, if the intersection $\bigcap_{x \in I} H(x)$ were empty, then would there exists a $(k + 1)$ -member subfamily whose intersection is also empty. That is, there exist $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{k+1,1}, \dots, x_{k+1,k}) \in I$ such that the following system of linear inequalities

$$f_0(\varphi(x_{i,1}, \dots, x_{i,k})) - f_0(\Phi(t)) \leq \sum_{j=1}^k a_j (f_j(x_{i,j}) - f_j(t_j)) \quad (i \in \{1, \dots, k + 1\})$$

has no solution for $(a_1, \dots, a_k) \in [0, \infty)^k$. Using a standard version of the celebrated Farkas Lemma, it follows that there exists $\lambda = (\lambda_1, \dots, \lambda_{k+1}) \in \Lambda_{k+1}$ such that

$$\sum_{i=1}^{k+1} \lambda_i (f_j(x_{i,j}) - f_j(t_j)) \leq 0 \quad (j \in \{1, \dots, k\}) \quad \text{and} \quad (3.3)$$

$$\sum_{i=1}^{k+1} \lambda_i (f_0(\varphi(x_{i,1}, \dots, x_{i,k})) - f_0(\Phi(t))) > 0. \quad (3.4)$$

For $j \in \{1, \dots, k\}$, the inequality (3.3) implies that

$$\frac{\sum_{i=1}^{k+1} \lambda_i f_j(x_{i,j})}{\sum_{i=1}^{k+1} \lambda_i} \leq f_j(t_j),$$

which then yields that

$$\tilde{A}_{f_j}^{[k+1]}(x_{1,j}, \dots, x_{k+1,j}; \lambda_1, \dots, \lambda_{k+1}) = f_j^{(-1)}\left(\frac{\sum_{i=1}^{k+1} \lambda_i f_j(x_{i,j})}{\sum_{i=1}^{k+1} \lambda_i}\right) \leq f_j^{(-1)}(f_j(t_j)) = t_j.$$

From the inequality (3.4) it follows that

$$\frac{\sum_{i=1}^{k+1} \lambda_i f_0(\varphi(x_{i,1}, \dots, x_{i,k}))}{\sum_{i=1}^{k+1} \lambda_i} > f_0(\Phi(t)).$$

Therefore, using that $\Phi(t) \in C_{f_0}$, we get

$$\begin{aligned} & \tilde{A}_{f_0}^{[k+1]}(\varphi(x_{1,1}, \dots, x_{1,k}), \dots, \varphi(x_{k+1,1}, \dots, x_{k+1,k}); \lambda_1, \dots, \lambda_{k+1}) \\ &= f_0^{(-1)}\left(\frac{\sum_{i=1}^{k+1} \lambda_i f_0(\varphi(x_{i,1}, \dots, x_{i,k}))}{\sum_{i=1}^{k+1} \lambda_i}\right) > \Phi(t) \\ &\geq \Phi\left(\tilde{A}_{f_1}^{[k+1]}(x_{1,1}, \dots, x_{k+1,1}; \lambda_1, \dots, \lambda_{k+1}), \dots, \tilde{A}_{f_k}^{[k+1]}(x_{1,k}, \dots, x_{k+1,k}; \lambda_1, \dots, \lambda_{k+1})\right), \end{aligned}$$

which contradicts assumption (c). The contradiction so obtained shows that the intersection $\bigcap_{x \in I} H(x)$ is not empty.

In the next step, we show that assertion (d) implies (e). Assume that (d) holds and, for $(t_1, \dots, t_k) \in \Gamma$, define the affine function $\Psi_{t_1, \dots, t_k} : \mathbb{R}^k \rightarrow \mathbb{R}$ by

$$\Psi_{t_1, \dots, t_k}(u_1, \dots, u_k) := f_0(\Phi(t_1, \dots, t_k)) + \sum_{j=1}^k a_j(t_1, \dots, t_k)(u_j - f_j(t_j)).$$

Due to the nonnegativity of $a_1(t_1, \dots, t_k), \dots, a_k(t_1, \dots, t_k)$, it follows that $\Psi_{t_1, \dots, t_k}$ is separately increasing. According to condition (d), for all $(x_1, \dots, x_k) \in I$, we also have that

$$f_0(\varphi(x_1, \dots, x_k)) \leq \Psi_{t_1, \dots, t_k}(f_1(x_1), \dots, f_k(x_k)) \quad (3.5)$$

and

$$f_0(\Phi(t_1, \dots, t_k)) = \Psi_{t_1, \dots, t_k}(f_1(t_1), \dots, f_k(t_k)). \quad (3.6)$$

Now we define $\Psi : \text{conv}(f_1(I_1)) \times \dots \times \text{conv}(f_k(I_k)) \rightarrow [-\infty, \infty]$ as follows

$$\Psi(u_1, \dots, u_k) := \inf_{(t_1, \dots, t_k) \in \Gamma} \Psi_{t_1, \dots, t_k}(u_1, \dots, u_k).$$

Then Ψ is concave and separately increasing because it is the pointwise infimum of affine (and hence also concave) separately increasing functions. For any fixed $(x_1, \dots, x_k) \in I$, taking the infimum of the right-hand side in inequality (3.5) with respect to $(t_1, \dots, t_k) \in \Gamma$, we can see that (3.1) holds. It is obvious that $\Psi(u) < \infty$ for all $u \in \text{conv}(f_1(I_1)) \times \dots \times \text{conv}(f_k(I_k))$. By the inequality (3.5), we have that $\Psi(u) > -\infty$ for all $u \in f_1(I_1) \times \dots \times f_k(I_k)$. Using the concavity of Ψ , it follows that the inequality $\Psi(u) > -\infty$ is true for all u belonging to the convex hull of the set $f_1(I_1) \times \dots \times f_k(I_k)$, which equals $\text{conv}(f_1(I_1)) \times \dots \times \text{conv}(f_k(I_k))$. Hence, the function Ψ takes only finite values over its domain. The equality (3.6) and the inequality $\Psi_{t_1, \dots, t_k} \geq \Psi$ show that the inequality (3.2) is valid for all $(t_1, \dots, t_k) \in \Gamma$.

Finally, assume that (e) holds. To show that (a) is valid, let $n \in \mathbb{N}$, and let $(x_{i,1}, \dots, x_{i,k}) \in I$ for $i \in \{1, \dots, n\}$. Let $(t_1, \dots, t_k) \in \Gamma$ be arbitrary such that the inequality

$$A_{f_j}^{[n]}(x_{1,j}, \dots, x_{n,j}) < t_j \quad (j \in \{1, \dots, k\}) \quad (3.7)$$

be valid. This implies that

$$\frac{1}{n} \sum_{i=1}^n f_j(x_{i,j}) < f_j(t_j) \quad (j \in \{1, \dots, k\}). \quad (3.8)$$

On the other hand, by inequality (3.1), we have

$$f_0(\varphi(x_{i,1}, \dots, x_{i,k})) \leq \Psi(f_1(x_{i,1}), \dots, f_k(x_{i,k})) \quad i \in \{1, \dots, n\}.$$

Therefore, using the concavity first, then the separate increasingness of Ψ together with the inequalities in (3.8), finally applying the inequality (3.2), we conclude that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n f_0(\varphi(x_{i,1}, \dots, x_{i,k})) &\leq \frac{1}{n} \sum_{i=1}^n \Psi(f_1(x_{i,1}), \dots, f_k(x_{i,k})) \\ &\leq \Psi\left(\frac{1}{n} \sum_{i=1}^n f_1(x_{i,1}), \dots, \frac{1}{n} \sum_{i=1}^n f_k(x_{i,k})\right) \\ &\leq \Psi(f_1(t_1), \dots, f_k(t_k)) \leq f_0(\Phi(t_1, \dots, t_k)). \end{aligned}$$

Thus, we have proved that, for all $(t_1, \dots, t_k) \in \Gamma$ which satisfies the inequalities in (3.7),

$$\frac{1}{n} \sum_{i=1}^n f_0(\varphi(x_{i,1}, \dots, x_{i,k})) \leq f_0(\Phi(t_1, \dots, t_k)).$$

Hence, it follows that

$$\begin{aligned} A_{f_0}^{[n]}(\varphi(x_{1,1}, \dots, x_{1,k}), \dots, \varphi(x_{n,1}, \dots, x_{n,k})) &= f_0^{(-1)}\left(\frac{1}{n} \sum_{i=1}^n f_0(\varphi(x_{i,1}, \dots, x_{i,k}))\right) \\ &\leq \Phi(t_1, \dots, t_k). \end{aligned}$$

Finally, we apply that the set Γ is dense in I and Φ is upper semicontinuous on this product set. Hence, we can take the upper limit as

$$\Gamma \ni (t_1, \dots, t_k) \rightarrow \left(A_{f_1}^{[n]}(x_{1,1}, \dots, x_{n,1}), \dots, A_{f_k}^{[n]}(x_{1,k}, \dots, x_{n,k})\right)$$

in the above inequality, we obtain that assertion (a) is valid. $\square$

Theorem 3.2 Let $k \in \mathbb{N}$, let $I_0, I_1, \dots, I_k$ be nonempty open intervals and denote $I := I_1 \times \dots \times I_k$. For all $j \in \{0, 1, \dots, k\}$, let $f_j : I_j \rightarrow \mathbb{R}$ be strictly increasing and let $\varphi : I \rightarrow I_0$ be separately strictly increasing. Assume, for some $n \in \mathbb{N} \setminus \{1\}$, that for all $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{n,1}, \dots, x_{n,k}) \in I$, the inequality

$$\begin{aligned} & A_{f_0}^{[n]}(\varphi(x_{1,1}, \dots, x_{1,k}), \dots, \varphi(x_{n,1}, \dots, x_{n,k})) \\ & \leq \varphi\left(A_{f_1}^{[n]}(x_{1,1}, \dots, x_{n,1}), \dots, A_{f_k}^{[n]}(x_{1,k}, \dots, x_{n,k})\right) \end{aligned} \quad (3.9)$$

holds. Then, for all $j \in \{1, \dots, k\}$, the function f_j is continuous on I_j and f_0 is continuous on $\varphi(I)$. Furthermore, φ is also continuous on I .

Proof Fix $\ell \in \{1, \dots, k\}$ arbitrarily. We show that $f_\ell : I_\ell \rightarrow \mathbb{R}$ is continuous. For a given vector $z = (z_1, \dots, z_k) \in I$ let us define a function $\psi_{z,\ell} : I_\ell \rightarrow I_0$ by

$$\psi_{z,\ell}(x) := \varphi(z_1, \dots, z_{\ell-1}, x, z_{\ell+1}, \dots, z_k),$$

that is, x appears at the ℓ -th coordinate of the argument of φ . Since φ is separately strictly increasing, the function $\psi_{z,\ell}$ is strictly increasing on I_ℓ (for all $z \in I$). In particular, $\psi_{z,\ell}$ possesses an increasing left inverse $\psi_{z,\ell}^{(-1)} : \text{conv}(\psi_{z,\ell}(I_\ell)) \rightarrow I_\ell$. For an arbitrary $y \in I_\ell^n$ let us substitute

$$x_{i,j} = \begin{cases} y_i & \text{for } (i, j) \in \{1, \dots, n\} \times \{\ell\} \\ z_j & \text{for } (i, j) \in \{1, \dots, n\} \times (\{1, \dots, k\} \setminus \{\ell\}) \end{cases}$$

into (3.9). Then, for $j \in \{1, \dots, k\} \setminus \{\ell\}$, we have that $A_{f_j}^{[n]}(x_{1,j}, \dots, x_{n,j}) = A_{f_j}^{[n]}(z_j, \dots, z_j) = z_j$, hence (3.9) simplifies to

$$A_{f_0}^{[n]}(\psi_{z,\ell}(y_1), \dots, \psi_{z,\ell}(y_n)) \leq \psi_{z,\ell}(A_{f_\ell}^{[n]}(y_1, \dots, y_n)) \quad (y_1, \dots, y_n) \in I_\ell^n. \quad (3.10)$$

Therefore, since $\psi_{z,\ell}^{(-1)}$ is an increasing function with $\psi_{z,\ell}^{(-1)} \circ \psi_{z,\ell} = \text{id}$, we get

$$\psi_{z,\ell}^{(-1)} \circ A_{f_0}^{[n]}(\psi_{z,\ell}(y_1), \dots, \psi_{z,\ell}(y_n)) \leq A_{f_\ell}^{[n]}(y_1, \dots, y_n) \quad (y_1, \dots, y_n) \in I_\ell^n.$$

In view of Lemma 2.2 we get know that $\psi_{z,\ell}^{(-1)} \circ f_0^{(-1)} = (f_0 \circ \psi_{z,\ell})^{(-1)}$, and therefore

$$A_{f_0 \circ \psi_{z,\ell}}^{[n]}(y_1, \dots, y_n) \leq A_{f_\ell}^{[n]}(y_1, \dots, y_n) \quad (y_1, \dots, y_n) \in I_\ell^n.$$

Therefore, in view of Lemma 2.4, we get that $C_{f_0 \circ \psi_{z,\ell}} = C_{f_\ell}$ holds for all $z \in I$.

Now assume, to the contrary, that f_ℓ is not continuous at a certain point $x_0 \in I_\ell$. Then f_0 is not continuous at all points belonging to the set

$$S := \{\psi_{z,\ell}(x_0) : z \in I\} = \{\varphi(z_1, \dots, z_{\ell-1}, x_0, z_{\ell+1}, \dots, z_k) : z \in I\}.$$

Since $z \mapsto \psi_{z,\ell}(x_0)$ is separately strictly increasing on I except with respect to the ℓ th variable, we find that S is an uncountable set, i.e. f_0 is not continuous at uncountably many points, which is impossible as f_0 was assumed to be strictly increasing. This contradiction shows that f_ℓ must be continuous on I_ℓ and hence $C_{f_\ell} = I_\ell$. Consequently, $C_{f_0 \circ \psi_{z,\ell}} = I_\ell$, which shows that $f_0 \circ \psi_{z,\ell}$ is continuous on I_ℓ for all $z \in I$. Having in mind that $f_0, \psi_{z,\ell}$ are strictly increasing on their domains, it follows that $\psi_{z,\ell}$ is continuous on I_ℓ and f_0 is continuous on the open interval $\psi_{z,\ell}(I_\ell)$ for all $z \in I$. Therefore, f_0 is continuous over the union

$$\bigcup_{z \in I} \psi_{z,\ell}(I_\ell) = \varphi(I).$$

Finally, using that $\psi_{z,\ell}$ is continuous on I_ℓ for all $\ell \in \{1, \dots, k\}$ and $z \in I$, we show that φ is continuous on I . Let $(p_1, \dots, p_k) \in I$ be a point at which we want to verify the continuity of φ . Let $\varepsilon > 0$ be arbitrary. Let $e_1, \dots, e_k$ be the standard basis in $\mathbb{R}^k$. We will recursively construct the positive numbers $\delta_1, \dots, \delta_k > 0$ as follows: Let $\ell \in \{1, \dots, k\}$ and assume that δ_j has been constructed for $j \in \{1, \dots, \ell - 1\}$. Then the map

$$I_\ell - p_\ell \ni x \mapsto \psi_{z,\ell}(p_\ell + x) = \varphi\left(p + \sum_{j=1}^{\ell-1} \delta_j e_j + x e_\ell\right)$$

(where $z = p + \sum_{j=1}^{\ell-1} \delta_j e_j$) is continuous and strictly increasing on $I_\ell - p_\ell$, which is a neighbourhood of 0. Therefore, there exists $\delta_\ell > 0$ such that

$$\varphi\left(p + \sum_{j=1}^{\ell} \delta_j e_j\right) - \varphi\left(p + \sum_{j=1}^{\ell-1} \delta_j e_j\right) < \frac{\varepsilon}{k}.$$

Adding up these inequalities side by side for $\ell \in \{1, \dots, k\}$, it follows that

$$\varphi\left(p + \sum_{j=1}^k \delta_j e_j\right) - \varphi(p) < \varepsilon.$$

Analogously, we can construct $\delta'_1, \dots, \delta'_k > 0$ such that

$$\varphi(p) - \varphi\left(p - \sum_{j=1}^k \delta'_j e_j\right) < \varepsilon.$$

Therefore, since φ is separately strictly increasing, for all $x = (x_1, \dots, x_k) \in \prod_{\ell=1}^k (p_\ell - \delta'_\ell, p_\ell + \delta_\ell)$, we have

$$\varphi(p) - \varepsilon < \varphi\left(p - \sum_{j=1}^k \delta'_j e_j\right) \leq \varphi(x) \leq \varphi\left(p + \sum_{j=1}^k \delta_j e_j\right) < \varphi(p) + \varepsilon,$$

and hence

$$|\varphi(x) - \varphi(p)| < \varepsilon,$$

which shows that φ is continuous at p . $\square$

Corollary 3.3 *Let $k \in \mathbb{N}$, let $f_j : \mathbb{R}_+ \rightarrow \mathbb{R}$ be strictly increasing for $j \in \{0, 1, \dots, k\}$. Then the following conditions are equivalent.*

(a) *For all $n \in \mathbb{N}$ and for all $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{n,1}, \dots, x_{n,k}) \in \mathbb{R}_+^k$,*

$$\begin{aligned} & A_{f_0}^{[n]}(x_{1,1} + \dots + x_{1,k}, \dots, x_{n,1} + \dots + x_{n,k}) \\ & \leq A_{f_1}^{[n]}(x_{1,1}, \dots, x_{n,1}) + \dots + A_{f_k}^{[n]}(x_{1,k}, \dots, x_{n,k}). \end{aligned}$$

(b) *For all $n \in \mathbb{N}$, for all $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{n,1}, \dots, x_{n,k}) \in \mathbb{R}_+^k$, and for all $(\lambda_1, \dots, \lambda_n) \in \Lambda_n$,*

$$\begin{aligned} & \widetilde{A}_{f_0}^{[n]}(x_{1,1} + \dots + x_{1,k}, \dots, x_{n,1} + \dots + x_{n,k}; \lambda_1, \dots, \lambda_n) \\ & \leq \widetilde{A}_{f_1}^{[n]}(x_{1,1}, \dots, x_{n,1}; \lambda_1, \dots, \lambda_n) + \dots \\ & \quad + \widetilde{A}_{f_k}^{[n]}(x_{1,k}, \dots, x_{n,k}; \lambda_1, \dots, \lambda_n). \end{aligned}$$

(c) For all $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \mathbb{R}_+^k$,

$$A_{f_0}^{[2]}(x_1 + \dots + x_k, y_1 + \dots + y_k) \leq A_{f_1}^{[2]}(x_1, y_1) + \dots + A_{f_k}^{[2]}(x_k, y_k).$$

(d) The functions $f_0, f_1, \dots, f_k$ are continuous and the function $\Psi : f_1(\mathbb{R}_+) \times \dots \times f_k(\mathbb{R}_+) \rightarrow \mathbb{R}$ defined by

$$\Psi(u_1, \dots, u_k) := f_0(f_1^{-1}(u_1) + \dots + f_k^{-1}(u_k))$$

is concave.

Proof Applying Theorem 3.1 with

$$\varphi(x_1, \dots, x_k) := \Phi(x_1, \dots, x_k) := x_1 + \dots + x_k \quad (x_1, \dots, x_k \in \mathbb{R}_+), \quad (3.11)$$

we can see that assertions (a) and (b) are equivalent to each other. In view of this theorem we also have that (d) implies (a).

By taking $\lambda_1 := \lambda_2 := 1$ and $\lambda_j := 0$ for $j \geq 2$ in assertion (b), we get that assertion (c) holds (with an obvious substitution).

Now assume that assertion (c) is valid. Then, applying Theorem 3.2 with $n = 2$ and φ defined by (3.11), it follows that f_j is continuous for all $j \in \{0, 1, \dots, k\}$. Then the generalized left inverse of each of these functions is just their standard inverse and their range is convex. Using this, the inequality in (d), with the substitutions $x_i = f_i^{-1}(u_i)$, $y_i = f_i^{-1}(v_i)$ yields that the function Ψ defined in assertion (d) is Jensen concave. Due to its continuity, it follows that it is concave. Therefore assertion (d) follows. $\square$

In a completely analogous manner, we can obtain the a result for Hölder-type inequalities.

Corollary 3.4 Let $k \in \mathbb{N}$, let $f_j : \mathbb{R}_+ \rightarrow \mathbb{R}$ be strictly increasing for $j \in \{0, 1, \dots, k\}$. Then the following conditions are equivalent.

(a) For all $n \in \mathbb{N}$ and for all $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{n,1}, \dots, x_{n,k}) \in \mathbb{R}_+^k$,

$$A_{f_0}^{[n]}(x_{1,1} \dots x_{1,k}, \dots, x_{n,1} \dots x_{n,k}) \leq A_{f_1}^{[n]}(x_{1,1}, \dots, x_{n,1}) \dots A_{f_k}^{[n]}(x_{1,k}, \dots, x_{n,k}).$$

(b) For all $n \in \mathbb{N}$, for all $(x_{1,1}, \dots, x_{1,k}), \dots, (x_{n,1}, \dots, x_{n,k}) \in \mathbb{R}_+^k$, and for all $(\lambda_1, \dots, \lambda_n) \in \Lambda_n$,

$$\begin{aligned} & \tilde{A}_{f_0}^{[n]}(x_{1,1} \dots x_{1,k}, \dots, x_{n,1} \dots x_{n,k}; \lambda_1, \dots, \lambda_n) \\ & \leq \tilde{A}_{f_1}^{[n]}(x_{1,1}, \dots, x_{n,1}; \lambda_1, \dots, \lambda_n) \dots \tilde{A}_{f_k}^{[n]}(x_{1,k}, \dots, x_{n,k}; \lambda_1, \dots, \lambda_n). \end{aligned}$$

(c) For all $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \mathbb{R}_+^k$,

$$A_{f_0}^{[2]}(x_1 \dots x_k, y_1 \dots y_k) \leq A_{f_1}^{[2]}(x_1, y_1) \dots A_{f_k}^{[2]}(x_k, y_k).$$

(d) The functions $f_0, f_1, \dots, f_k$ are continuous and the function $\Psi : f_1(\mathbb{R}_+) \times \dots \times f_k(\mathbb{R}_+) \rightarrow \mathbb{R}$ defined by

$$\Psi(u_1, \dots, u_k) := f_0(f_1^{-1}(u_1) \dots f_k^{-1}(u_k))$$

is concave.

In our final corollary we characterize the Jensen convexity of generalized quasi-arithmetic means.

Corollary 3.5 *Let $f : I \rightarrow \mathbb{R}$ be strictly increasing. Then the following conditions are equivalent.*

(a) *For all $n \in \mathbb{N}$ and for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in I^n$,*

$$A_f^{[n]}\left(\frac{x_1 + y_1}{2}, \dots, \frac{x_n + y_n}{2}\right) \leq \frac{1}{2}\left(A_f^{[n]}(x_1, \dots, x_n) + A_f^{[n]}(y_1, \dots, y_n)\right).$$

(b) *For all $n \in \mathbb{N}$, for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in I^n$, and for all $(\lambda_1, \dots, \lambda_n) \in \Lambda_n$,*

$$\begin{aligned} \tilde{A}_f^{[n]}\left(\frac{x_1 + y_1}{2}, \dots, \frac{x_n + y_n}{2}; \lambda_1, \dots, \lambda_n\right) \\ \leq \frac{1}{2}\left(\tilde{A}_f^{[n]}(x_1, \dots, x_n; \lambda_1, \dots, \lambda_n) + \tilde{A}_f^{[n]}(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n)\right). \end{aligned}$$

(c) *For all $(x_1, x_2), (y_1, y_2) \in I^2$,*

$$A_f^{[2]}\left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}\right) \leq \frac{1}{2}\left(A_f^{[2]}(x_1, x_2) + A_f^{[2]}(y_1, y_2)\right).$$

(d) *The function f is continuous and the function $\Psi : f(I)^2 \rightarrow \mathbb{R}$ defined by*

$$\Psi(u, v) := f\left(\frac{f^{-1}(u) + f^{-1}(v)}{2}\right)$$

is concave.

(e) *f is twice continuously differentiable with a positive first derivative and either $f'' = 0$ on I or f'' is positive and $\frac{f'}{f''}$ is concave.*

Proof By taking $k = 2$ and functions $\varphi, \Phi : I^2 \rightarrow I$ defined as

$$\varphi(x, y) := \Phi(x, y) := \frac{x + y}{2}$$

the equivalence of the first four assertions follows from Theorem 3.1 and Theorem 3.2 in the same manner as for the equivalence of the analogous conditions for Minkowski-type inequalities.

The equivalence of assertions (a) and (e) was obtained by the authors in [15]. □

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