Generalizations of some results from Riemannian geometry to Finsler geometry

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Preface

In the last decades Finsler geometry produced remarkable development. Many papers and books on this topic have been published. Specially, a lot of results from Riemannian geometry have been extended for Finsler manifolds.

Probably the first work in Finsler geometry was the PhD thesis of Paul Finsler (1918). But more one half of a century before Riemann (in 1854) pointed the difference between the case of what is known as Riemannian geometry and the general case (see [**Spi75**] for an English translation). He state in his address : "The study of the metric which is the fourth root of a quartic differential form is quite time-consuming (zeitraubend) and does not throw new light to the problem."

After Einstein's formulation of general relativity, Riemannian geometry became widely used and the Levi-Civita connection came to the forefront. This connection is both torsion free and metric-compatible.

Though Finsler geometry was originated in calculus of variations, geometrically a Finsler manifold means that at each tangent space a norm, varying smoothly, is given, not necessarily induced by an inner product. In the first half on the 20-th century the tools and techniques appropriate for treatment of Finsler geometry were developed.

On a Finsler manifold there does not exist, in general, a linear metrical connection. The generalizations of the Levi-Civita connection induced by a Riemannian metric live just in the vertical bundle π^*TM or TTM, however, there are several ones. The differences between these connections are in the level of the metric compatibility and the torsion. The first of these generalizations were proposed by J.L. Synge (1925), J.H. Taylor (1925), L. Berwald (1928) [**Ber28**] and, most important, Elie Cartan(1934) [**Car34**] — the last one is metric compatible, but has the largest number of non-vanishing torsion tensors —; after a short time, S.S. Chern [**Che43, Che48, Che96**] proposed a different generalization, which is identical with the connection proposed later by Rund (see[**Ana96**])— it is not fully metric compatible but it has less number of non-vanishing torsion tensors. These connections can be used to prove many results from Riemannian geometry in Finslerian context (see [**AP94, BCS00**]). Another useful connection in Finsler geometry is the Berwald connection ([**Ber28, BCS00, Mat86**])— it has no torsion but it has a great deviance from the compatibility with the metric. In [**Aba96**] and [**MA94**] one can find nice characterizations of these connections, illustrating there similarities and differences.

In the last decades important generalizations of Finsler spaces have been proposed. These generalizations have applications in Mechanics, Physics, Variational Calculus and many other fields. Some of the generalized Finsler spaces are Lagrange spaces, Hamilton spaces, generalized Lagrange spaces and others. The Romanian school initiated by R. Miron has important contributions in the field (see [**Mir89, Mir85, Mir86, MA94**]). Though S. S. Chern says [**Che96**] that Finsler geometry is more natural than Riemannian geometry as a concept, the computational part of the subject requires much more effort.

Like in Riemannian geometry the Finsler spaces of constant curvature (constant flag curvature) constitute an important class of Finsler spaces. Finsler spaces of constant negative curvature are studied by Akbar-Zadeh [**AZ88**]. The structure of that kind of spaces is well clarified however Finsler manifolds of positive curvature have not been completely understood yet. Recently, results on Finsler spaces of positive (constant) curvature are obtained by Shen (see [**She96**]) and by Bryant (see [**Bry96**]). The latter gave examples of non-Riemannian Finsler structures with constant positive curvature on the 2-sphere.

In this thesis, first (Chapter 2), we prove some properties of real and complex Finsler manifolds of positive bisectional curvature (see [**KP00**] , [**Pet02**]). Here results concerning intersections of submanifolds in real and complex (Kähler) Finsler manifolds, and also results concerning coincidence of correspondences in Kähler Finsler manifolds are proved. Among these we prove that for two compact, totally geodesic submanifolds of a real, complete, connected Finsler manifold with positive sectional curvature have non-void intersection, if the sum of their dimensions is greater than the dimension of the manifold.

The last decades have meant a great development of global Riemannian geometry. It is an important project to try to generalize these to Finsler settings. It is a remarkable fact that the Jacobi equation, the second variation formula and the index form for Finsler manifolds look exactly like their counterparts in Riemannian case. These enable one to prove in Finslerian context the Cartan-Hadamard theorem,the Bonnet-Myers theorem and the Synge theorem [**AP94, BCS00**]. The Morse Index Theorem was also generalized to Finsler manifolds. That was due to Lehmann [**Leh64**]; see Matsumoto for an exposition [**Mat86**]. On the other, in the Riemannian and semi-Riemannian case, the Morse Index Theorem where the ends are submanifolds is also proved by many authors (Ambrose [**Amb61**], Bolton [**Bol77**], Kalish [**Kal88**], Piccione and Tausk [**PT99**]).

In Chapter 3 we prove the Morse Index Theorem for variable endpoints in the case of Finsler manifolds (published in [**Pet**]). We show that, despite the fact that the second fundamental form is not symmetric, the Morse Index Form is symmetric and this fact is crucial in the proofs.

During the last years several generalizations of Finsler spaces have been proposed and studied (see [**AM95**], [**MA94**]). Warped product of manifolds is an important tool in applications of Riemannian and semi-Riemannian geometry to relativity (for example Robertson-Walker space-time and Schwarzschild geometry, see [**O'N83**]).

The last chapter (Chapter 4) is devoted to constructing the warped product of Finsler manifolds [**KPV01**]. The constructed warped metric has almost all properties of a Finsler metric. The only exception is that the warped metric is not of class C^2 on the zero section of the product. But it is smooth on $\widetilde{M} \times \widetilde{N}$ (where $\widetilde{M} = TM \setminus zerosection$), so we can use the Cartan connections of the factors. We show some relations between the Cartan connections of the factors and the warped product manifold. These properties enable to construct Cartan connection of the warped product manifold from the Cartan connections of the factors. The notions of umbilical point of a Finsler manifold and the umbilical submanifold are defined. The geodesics with respect to this connection are characterized. It is proved that the leaves of the product manifold are totally geodesic and the fibers are umbilical. Finally we give explicit relations in order to compute the curvature of warped product from the curvatures of the factors.

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CHAPTER 1

Preliminaries

1. Fundamentals of real Finsler geometry

Let M be a real manifold M of dimension n, (TM, π, M) the tangent bundle of M. The vertical bundle of the manifold M is the vector bundle $\overline{\pi}\colon \mathcal{V} \to TM$ given by $\mathcal{V} = \ker d\pi \subset T(TM)$. (x^i) will denote local coordinates on an open subset U of M, and (x^i, y^i) the induced coordinates on $\pi^{-1}(U) \subset TM$. The radial vertical vector field ι is locally given by $\iota(u^a \frac{\partial}{\partial x^a}) = u^a \frac{\partial}{\partial y^a}|_u$.

A Finsler metric on M is a a function $F\colon TM \to \mathbb{R}_+$ satisfying the following properties:

(1) F^2 is smooth on $\widetilde{M}$, where $\widetilde{M} = TM \setminus \{0\}$,
(2) $F(u) > 0$ for all $u \in \widetilde{M}$,
(3) $F(\lambda u) = |\lambda| F(u)$ for all $u \in TM$, $\lambda \in \mathbb{R}$,
(4) For any $p \in M$ the indicatrix $I_x(p) = \{u \in T_pM \mid F(u) < 1\}$ is strongly convex.

A manifold endowed with a Finsler metric F is called a Finsler manifold.

From the condition 4 it follows that the quantities $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial x^i \partial x^j}$ means positive definite matrix, so a Riemannian metric $\langle\ ,\ \rangle$ can be introduced in the vertical bundle $(\mathcal{V}, \overline{\pi}, TM)$.

In this thesis we use the Cartan connection, which is a good vertical connection in $\mathcal{V}$, i.e. a $\mathbb{R}$-linear map

$$\nabla\colon \mathfrak{X}\widetilde{M} \times \mathfrak{X}(\mathcal{V}) \to \mathfrak{X}(\mathcal{V})$$

having the usual properties of a covariant derivations, metrical with respect to g, and 'good' in the sense that the bundle map $\Lambda\colon T\widetilde{M} \to \mathcal{V}$ defined by $\Lambda(X) = \nabla_X \iota$ is a bundle isomorphism when restricted to $\mathcal{V}$. The latter property induces the horizontal subspaces $H_u = \ker \Lambda$ for all $u \in \widetilde{M}$, which is direct summand of the vertical subspaces $V_u = \mathrm{Ker}\,(d\pi)_u$:

$$T\widetilde{M} = \mathcal{H} \oplus \mathcal{V}$$

$\Theta\colon \mathcal{V} \to \mathcal{H}$ denotes the horizontal map associated to the horizontal bundle $\mathcal{H}$. For a tangent vector field X on M we have its vertical lift X^V and its horizontal lift X^H to $\widetilde{M}$.

Using Θ first we get the radial horizontal vector field $\chi = \Theta \circ \iota$. Secondly we can extend the covariant derivation ∇ of the vertical bundle to the whole tangent bunlde of $\widetilde{M}$. Denoting it with the same letter, for horizontal vector fields H we have

$$\nabla_X H = \Theta(\nabla_X(\Theta^{-1}(H))) \quad \forall X \in \mathfrak{X}\widetilde{M},$$

and then, an arbitrary vector field $Y \in \mathfrak{X}\widetilde{M}$ is decomposed into vertical and horizontal parts, so

$$\nabla_X Y = \nabla_X Y^V + \nabla_X Y^H.$$

Thus $\nabla : \mathfrak{X}(T\widetilde{M}) \times \mathfrak{X}(T\widetilde{M}) \to \mathfrak{X}(T\widetilde{M})$ is a linear connection on $\widetilde{M}$ induced by a good vertical connection. Its torsion θ and curvature R are defined as usual:

$$\nabla_X Y - \nabla_Y X = [X, Y] + \theta(X, Y) \ \ \forall X, Y \in \mathfrak{X}T\widetilde{M}$$
$$R_Z(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ \ \forall X, Y, Z \in \mathfrak{X}T\widetilde{M}$$

and the torsion has the property that for horizontal vectors $\theta(X, Y)$ is a vertical vector [**AP94**]. The curvature operator Ω is a global $T^*\tilde{M} \otimes T\tilde{M}$-valued 2-form. That means that $\Omega(X, Y)$ is a global $T\tilde{M}$-valued 1-form for any $X, Y \in T\tilde{M}$ by the relation $\Omega(X, Y)Z = R_Z(X, Y)$ for any $X, Y, Z \in \mathfrak{X}(T\tilde{M})$, and Ω is well defined. Specially the sectional curvature of ∇ along a curve σ is given as follows:

$$R_{\dot\sigma}(U^H, U^H) = \langle \Omega(\dot\sigma^H, U^H)U^H, \dot\sigma^H \rangle$$

for any $U \in \mathfrak{X}(M)$. This is called the horizontal flag curvature in [**AP94**]. The horizontal flag curvature is the most important contraction of the curvature operator because it appears in the second variation formula.

We often use that the torsion of two horizontal vectors is a vertical one, that is $\theta(X,Y) \in \mathcal{V}$ for all $X, Y \in \mathcal{H}$ [**AP94**].

The metrical property of the Cartan connection is also important [**AP94**]:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

In the following we shall present the first and second variation of the length, as in [**AP94**].

DEFINITION 1.1. *A regular curve* $\sigma : [a,b] \to M$ *is a* C^1 *curve such that*

$$\forall t \in [a,b] \qquad \dot\sigma(t) = d\sigma_t(\frac{d}{dt}) \neq 0.$$

The *length* with respect to the Finsler metric $F : TM \to \mathbb{R}^+$, of the regular curve σ is given by

$$L(\sigma) = \int_a^b F(\dot\sigma(t))dt$$

A geodesic for the Finsler metric F is a curve which is a critical point of the energy functional. We present now the one parameter variation of a curve:

DEFINITION 1.2. *Let* $\sigma_0 : [a,b] \to M$ *be a curve with* $F(\dot\sigma_0) = c_0$. *A regular variation of* σ_0 *is a* C^1*-map*

$$\Sigma : (-\varepsilon, \varepsilon) \times [a,b] \to M$$

such that

(1) $\sigma_0(t) = \Sigma(0,t), \forall t \in [a,b]$
(2) $\forall s \in (-\varepsilon,\varepsilon)$ *the curve* $\sigma_s(t) = \Sigma(s,t)$ *is a regular curve in* M*;*
(3) $F(\dot\sigma_s) = c_s > 0, \forall s \in (-\varepsilon,\varepsilon)$.

A regular variation Σ *is fixed if it moreover satisfies*

(4) $\sigma_s(a) = \sigma_0(a)$ *and* $\sigma_s(b) = \sigma_0(b)$ *for all* $s \in (-\varepsilon,\varepsilon)$.

For a regular variation Σ of σ_0 we define the function $l_\Sigma : (-\epsilon,\epsilon) \to \mathbb{R}^+$ by

$$l_\Sigma(s) = L(\sigma_s).$$

DEFINITION 1.3. *A regular curve* σ_0 *is a geodesic for* F *iff*

$$\frac{dl_\Sigma}{ds}(0) = 0$$

for all fixed regular variations Σ *of* σ_0.

In [**AP94**] there is derived the first and the second variation of the length functional. It is also derived the differential equation of geodesics and it is shown that every geodesic for F is also a geodesic for the Cartan connection, and conversely, the geodesics of the Cartan connection are geodesics of the Finsler metric.

It is used there the pulled-back of the Cartan connection along a curve. The pulled-back bundle does not live on TM, but on $\widetilde{TM}$. Anyway the construction is not very complicated and it is clear. We briefly present it here.

Let $\Sigma : (-\epsilon, \epsilon) \times [a, b] \to M$ be a regular variation of a curve $\sigma_0 : [a, b] \to M$. Let

$$p : \Sigma^*(TM) \to (-\epsilon, \epsilon) \times [a, b]$$

be the pull back bundle, and $\gamma : \Sigma^*(TM) \to TM$ be the fiber map which identifies each $\Sigma^*(TM)_{(s,t)}$ with $T_{\Sigma(s,t)}M$ for all $(s, t) \in (-\epsilon, \epsilon) \times [a, b]$. A local frame for $\Sigma^*(TM)$ is given by the local fields

$$\frac{\partial}{\partial x^i}|_{(s,t)} = \gamma^{-1}(\frac{\partial}{\partial x^i}|_{\Sigma(s,t)})$$

for $i = 1, \ldots n$. An element $\xi \in \mathfrak{X}(\Sigma^*(TM))$ can be written locally by

$$\xi(s, t) = u^i(s, t)\frac{\partial}{\partial x^i}|_{(s,t)},$$

and a local frame on $T(\Sigma^*(TM))$ is given by $\partial_s, \partial_t, \dot{\partial}_i$, where $\partial_s = \frac{\partial}{\partial s}, \partial_t = \frac{\partial}{\partial t}$ and $\dot{\partial}_i = \frac{\partial}{\partial u^i}$.

There are two particularly important sections of $\Sigma^*(TM)$:

$$T = \gamma^{-1}(d\Sigma\ (\frac{\partial}{\partial t})) = \frac{\partial \Sigma^i}{\partial t}\frac{\partial}{\partial x^i}$$

and

$$U = \gamma^{-1}(d\Sigma\ (\frac{\partial}{\partial s})) = \frac{\partial \Sigma^i}{\partial s}\frac{\partial}{\partial x^i}$$

Definition 1.4. *The section* U *is the transversal vector of* Σ.

By setting $\Sigma^*\tilde{M} = \gamma^{-1}(\tilde{M})$, we have that $T \in \mathfrak{X}(\Sigma^*\tilde{M})$ and $T(s, t) = \gamma^{-1}(\dot{\sigma}_s(t))$.

We may pull-back $T\tilde{M}$ over $\Sigma^*\tilde{M}$ by using γ, obtaining the map $\tilde{\gamma}:\gamma^*(T\tilde{M})\rightarrow T\tilde{M}$ which identifies, for any $u\in\Sigma^*\tilde{M}_{(s,t)}=\gamma^{-1}(\tilde{M}_{\Sigma(s,t)})$, $\gamma^*(t\tilde{M})_u$ with $T_{\gamma(u)}\tilde{M}$.

We shall enounce now the first and the second variation of the length for Finsler metric.

Theorem 1.5. [**AP94**] *Let $F:TM\longrightarrow\mathbb{R}^+$be a Finsler metric on a manifold M. Take a regular curve $\sigma_0:[a,b]\longrightarrow M$, with $F(\dot{\sigma}_0)\equiv c_0\geq 0$, and let $\Sigma:(-\varepsilon,\varepsilon)\times[a,b]\rightarrow M$ be a regular variation of σ_0. Then*

$$\frac{dl_\Sigma}{ds}(0)=\frac{1}{c_0}\{\langle U^H,T^H\rangle_{\dot{\sigma}_0}|_a^b-\int_a^b\langle U^H,\nabla_{T^H}T^H\rangle_{\dot{\sigma}_0}dt\}.$$

In paricular if the variation is fixed we have

$$\frac{dl_\Sigma}{ds}(0)=-\frac{1}{c_0}\int_a^b\langle U^H,\nabla_{T^H}T^H\rangle_{\dot{\sigma}_0}dt.$$

The equation of geodesics is obtained as a corollary:

Corollary 1.6. *Let $F:TM\longrightarrow\mathbb{R}^+$be a Finsler metric on a manifold M and $\sigma_0:[a,b]\longrightarrow M$ a regular curve. Then σ is a geodesic for F iff $\nabla_{T^H}T^H\equiv 0$ where $T^H(u)=\chi_u(\dot{\sigma}(t))\in\mathcal{H}_u$ for all $u\in\tilde{M}_{\sigma(t)}$.*

Now it follows the second variation of arc-length.

Theorem 1.7. [**AP94**] *Let $F:TM\longrightarrow\mathbb{R}_+$ be a Finsler metric on a manifold M. Take a geodesic $\sigma_0:[a,b]\longrightarrow M$, with $F(\dot{\sigma}_0)\equiv 1$, and let $\Sigma:(-\varepsilon,\varepsilon)\times[a,b]\rightarrow M$ be a regular variation of σ_0. Then*

$$\begin{aligned}\frac{d^2l_\Sigma}{ds^2}(0) &= \langle\nabla_{U^H}U^H,T^H\rangle_{\dot{\sigma}_0}|_a^b\\ &\quad+\int_a^b[\|\nabla_{T^H}U^H\|^2_{\dot{\sigma}_0}-\langle\Omega(T^H,U^h),T^H\rangle_{\dot{\sigma}_0}-|\frac{\partial}{\partial t}\langle U^H,T^H\rangle_{\dot{\sigma}_0}|^2]dt\end{aligned}$$

where $\|H\|^2_u=\langle H,H\rangle_u$ for all $u\in\tilde{M}$ and $H\in\mathcal{H}_u$. In particular, if the variation Σ is fixed we have

$$\frac{d^2l_\Sigma}{ds^2}(0)=\int_a^b[\|\nabla_{T^H}U^H\|^2_{\dot{\sigma}_0}-\langle\Omega(T^H,U^h),T^H\rangle_{\dot{\sigma}_0}-|\frac{\partial}{\partial t}\langle U^H,T^H\rangle_{\dot{\sigma}_0}|^2]dt$$

2. Some notions in complex Finsler geometry

We recall some facts about Kähler-Finsler manifolds (see [**AP94**]).

Let M be a complex manifold of complex dimension. The complexification $T_{\mathbb{C}}M$ of the real tangent bundle is decomposed as

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

where $T^{1,0}M$ is the holomorphic tangent bundle over M and $T^{0,1}M$ is the conjugate of $T^{1,0}M$. $T^{1,0}M$ is also a complex manifold of $\dim_{\mathbb{C}} T^{1,0}M = n$. $T^{1,0}M$ and $T^{0,1}M$ are the eigenspaces of the complex structure J belonging to the eigenvalues i and $-i$, respectively.

A complex Finsler metric on a complex manifold is a continuous function $F : T^{1,0}M \to \mathbb{R}$ satisfying

i) $G := F^2$ is smooth on $\widetilde{M} = TM \setminus \{\text{zero section}\}$,
ii) $F(v) > 0,\ \forall\ v \in \widetilde{M}$,
iii) $F(\mu_\xi(v)) = |\xi| F(v)$ for all $v \in T^{1,0}M$ and $\xi \in \mathbb{C}$.

Recall that $\mu_\varepsilon : T^{1,0}M \to T^{1,0}M$ is given by $\mu_\xi(p,v) = (p, \xi v),\ \forall\ (p,v) \in T^{1,0}M$ and $\xi \in \mathbb{C}$. F is called strongly pseudoconvex if the Levi matrix $(G_{\alpha\overline{\beta}})$ is positive definite on $\widetilde{M}$, where

$$G_{\alpha\overline{\beta}} = \frac{\partial G^2}{\partial v^\alpha \partial \overline{v}^\beta}.$$

The complex vertical bundle is

$$\mathcal{V}_{\mathbb{C}} = \ker d\pi \subset T_{\mathbb{C}}\widetilde{M}$$

There is a canonical isomorphism $\iota_v : T^{1,0}_{\pi(v)} \to \mathcal{V}_v$ The complex radial vertical vector field $\iota : \widetilde{M} \to \mathcal{V}$ is defined by $\iota(v) = \iota_v(v) \quad \forall\ v \in T^{1,0}\widetilde{M}$. The projection $d\pi$ commutes with J. It follows that we have the splitting $\mathcal{V}_{\mathbb{C}} = \mathcal{V}^{1,0} + \mathcal{V}^{0,1}$. The complex vertical bundle is $\mathcal{V} = \mathcal{V}^{0,1} = \ker d\pi \subset T^{1,0}\widetilde{M}$. The complex horizontal bundle is a complex subbundle of $T_{\mathbb{C}}\widetilde{M}$ which is a direct summand of $\mathcal{V}$ and it is J-invariant. We have also the splitting $\mathcal{H}_{\mathbb{C}} = \mathcal{H}^{1,0} + \mathcal{H}^{0,1}$.

The complex horizontal map is a complex bundle map $\Theta : \mathcal{V}_{\mathbb{C}} \to T_{\mathbb{C}}$ which commutes with J and the conjugation and which satisfies the relation

$(d\pi \circ \Theta)_v|_{\mathcal{V}^{1,0}} = \iota_v^{-1}|_{\mathcal{V}^{1,0}}$. The complex radial (Liouville) horizontal vector field is given by $\chi = \Theta \circ \iota$.

Then there exists a unique good vertical connection which makes the Hermitian structure $(G_{\alpha\overline{\beta}})$ in the vertical bundle $\mathcal{V}$ parallel. It can be extended via the horizontal map to a complex linear connection on $\widetilde{M}$. This is called the complex Chern Finsler connection ∇.

The geodesics are characterized by the equation:

$$\nabla_{T^H+\overline{T^H}}T^H = 0.$$

The torsion θ, and τ of ∇ are defined as follows:

$$\theta(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad \forall X,Y \in \mathfrak{X}(T^{1,0}\widetilde{M})$$

$$\tau(X,\overline{Y}) = \nabla_X \overline{Y} - \nabla_{\overline{Y}} X - [X,\overline{Y}], \quad \forall X,Y \in \mathfrak{X}(T^{1,0}\widetilde{M})$$

the curvature Ω are defined as usual. The holomorphic bisectional curvature is given as follows

$$R(T,U) = \langle \Omega(T^H + \overline{T^H}, U^H + \overline{U^H}), U^H, T^H \rangle \qquad \forall\, T, H \in T^{1,0}M.$$

It is to derive that in the case of the Chern-Finsler connection this takes the form

$$R(T,U) = \langle \Omega(T^H, \overline{U^H})U^H, T^H \rangle$$

A strongly pseudoconvex Finsler metric F is called Kähler if its (2,0)-torsion θ satisfies

$$\forall\, H \in \mathcal{H} \qquad \theta(H,\chi) = 0$$

and it is called strongly Káhler if its torsion satisfies

$$\forall H, K \in \mathcal{H} \qquad \theta(H,K) = 0.$$

The horizontal (1,1) torsion is defined by

$$\tau^{\mathcal{H}}(X,\overline{Y}) = \Theta(\tau(X,\overline{Y}))$$

where Θ is the horizontal map. The symmetric product $\langle\!\langle \cdot, \cdot \rangle\!\rangle \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is locally given by

$$\langle\!\langle H, K \rangle\!\rangle_v = G_{\alpha\beta}(v) H^\alpha H^\beta \qquad \forall\, H, K \in \mathcal{H}_v,\, v \in \widetilde{M}.$$

It is clearly globally well defined and satisfies $\langle\!\langle H, \chi \rangle\!\rangle = 0$ for all $H \in \mathcal{H}$.

In the proof of Theorem 2.2 the second variation formula will play a crucial role: Consider $F\colon T^{1,0}M \to \mathbb{R}$ be a Kähler Finsler metric on a complex manifold M. Take a geodesic $\sigma_0\colon [a,b] \to M$ with $F(\dot{\sigma}_o) = 1$, and a regular variation $\Sigma\colon (-\varepsilon, \varepsilon) \times [a,b] \to M$ of σ_0. Then it is known [**AP94**]

$$\begin{aligned}\frac{d^2\ell_\Sigma}{ds^2}(0) \quad = \quad & \mathrm{Re}\langle \nabla_{U^H+\overline{U^H}} U^H, T^H \rangle_{\dot{\sigma}_0} |_a^b + \\ & + \int_a^b \{ \| \nabla_{T^H+\overline{T^H}} U^H \|^2_{\dot{\sigma}_0} - | \frac{\partial}{\partial t} \mathrm{Re}\langle U^H, T^H \rangle_{\dot{\sigma}_0} |^2 - \\ & - \mathrm{Re} \Big[\langle \Omega(T^H, \overline{U^H}) U^H, T^H \rangle_{\dot{\sigma}_0} - \langle \Omega(U^H, \overline{T^H}) U^H, T^H \rangle_{\dot{\sigma}_0} \\ & + \langle\!\langle \tau^{\mathcal{H}}(U^H, \overline{T^H}), U^H \rangle\!\rangle_{\dot{\sigma}_0} - \langle\!\langle \tau^{\mathcal{H}}(T^H, \overline{U^H}), U^H \rangle\!\rangle_{\dot{\sigma}_0} \Big] \} \, dt.\end{aligned}$$

CHAPTER 2

Frankel Type Theorems for Finsler Manifolds

1. Introduction

J. L. Synge [**Syn26, Syn36**] proved in 1936 that an even dimensional orientable compact manifold with positive sectional curvature is simply connected. He used the formula of the second variation of the arc-length, derived by him in an earlier paper [**Syn26**].

In 1970 T. J. Frankel [**Fra61**] continued the study of positively curved manifolds using the Synge's techniques and applying them to a different situation, namely "the position" of certain submanifolds of a manifold.

He proved that two compact totally geodesic submanifolds V and W of dimensions r and s, respectively, of an n-dimensional manifold complete connected Riemannian manifold M with positive sectional curvature always have a nonempty intersection provided $r + s \geq n$. Unfortunately totally geodesic submanifolds are not common occurrence. If M is a Kähler manifold the situation is much more satisfactory. In this case instead of requiring that V and W are totally geodesic, he needed only to assume that they are complex analytic submanifolds.

These results are extended by Gray [**Gra70**] to the case of nearly Kähler manifolds, by S. Machiafava [**Mar90**] to the case of quaternionic Kähler spaces, by L. Ornea [**Orn92**] to the case of locally conformal Kähler manifolds and by T. Q. Binh, L. Ornea and L. Tamássy [**BOT99**] to the case of of Sasakian manifolds with positive bisectional curvature.

Holomorphic correspondences are generalizations of holomorphic mappings as multivalued maps of a complex manifold [**BB84**], [**KP91**]. Fixed points of correspondences of complex Kähler manifolds have been studied by T. Frankel [**Fra61**]. He proved that for a Kähler manifold of positive sectional curvature a correspondence always has a fixed point (i.e. it intersects the diagonal of $N \times N$). The method of its proof, based upon the second variation formula of geodesics, proved effective in different situations [**AP94**],[**Fra61**].

L. Kozma and the present author generalized Frankel's results on intersections of submanifolds for the case of Finsler manifolds [**KP00**](Theorems 2.1, 2.2 in this work). The result of Frankel concerning correspondences are extended to the case of Kähler Finsler by the present author [**Pet02**]. We mention that we deduce results on coincidence of correspondences (Theorem 2.3), while Frankel's theorem refers only to fixed points of a correspondence. Some consequences regarding coincidence of mappings and fixed point properties for classes of mappings defined on Kähler Finsler manifolds are obtained(Theorems 2.6, 2.7 and Corollaries 2.4 and 2.5). The proof follows the line of the original version of Frankel, however, at some points more elaborated arguments are needed due, to the Finslerian context.

2. Frankel Type Theorems

We begin to present the theorems on intersection of submanifolds of a Finsler and a Kähler Finsler manifold.

THEOREM 2.1. [**KP00**] *If V and W are two compact totally geodesic submanifolds of a real complete connected Finsler manifold (M, F) of positive sectional curvature, and $dim\,V + dim\,W \geq dim\,M$, then $V \cap W \neq \emptyset$.*

PROOF. We assume that V and W do not intersect each other. Then there is a shortest geodesic $\sigma(t)$ from V to W with the endpoints $\sigma(a) \in V, \sigma(b) \in W$.

All quantities from the tangent level are now horizontally lifted to the second tangent level along the tangent curve $\dot{\sigma}$ of the geodesic σ. Its reason is that the Cartan connection lives there and we want to use the parallel translation of this linear connection. The horizontal lift from $T_{\sigma(a)}M$ and $T_{\sigma(b)}M$ to $H_{\dot{\sigma}(a)}$ and $H_{\dot{\sigma}(b)}$, resp. will be simply denoted by the superscript H.

Since σ is the shortest geodesic from V to W it strikes V and W orthogonally by the Gauss lemma: $\dot\sigma^H(a) \perp T^H_{\sigma(a)}V$ and $\dot\sigma^H(b) \perp T^H_{\sigma(b)}W$.

Let $P \subset H_{\dot\sigma(b)}\widetilde{M}$ be the parallel translated of $T^H_{\sigma(a)}V$ with respect to the Cartan connection along $\dot\sigma$ to the point $\dot\sigma(b)$. The parallel translation of the Cartan connection is angle-preserving, therefore $P \perp \dot\sigma^H(b)$ as well, so $\dim(P + T^H_{\sigma(b)}(W)) \leq \dim M - 1$. Then

$$\begin{aligned}\dim(P \cap T^H_{\sigma(b)}W) &= \\ &= \dim P + \dim T^H_{\sigma(b)}W - \dim(P + T^H_{\sigma(b)}W) \geq \\ &\geq \dim V + \dim W - (\dim M - 1) \geq 1.\end{aligned}$$

Thus there is $w^H \in P \cap T^H_{\sigma(b)}W$ with $\langle w^H, w^H\rangle = 1$. Clearly w^H must be the parallel translated along $\dot\sigma$ of some $v^H \in T^H_pV$ with $\langle v^H, v^H\rangle = 1$. Let U^H be the unit tangent horizontal vector field along $\dot\sigma$ obtained by parallel translation of v^H. Consider the variation Σ of σ with transversal vector field $X = d\pi(U^H)$. Then, by the second variation formula (cf. Theorem 1.7)(cf. [**AP94**], p. 38) we have

$$\begin{aligned}\frac{d^2\ell_\Sigma}{ds^2}(0) &= \langle \nabla_{U^H}U^H, T^H\rangle_{\dot\sigma}\big|_a^b \\ &+ \int_a^b \left[\|\nabla_{T^H}U^H\|^2_{\dot\sigma} - \langle \Omega(T^H, U^H)U^H, T^H\rangle_{\dot\sigma} - |\frac{\partial}{\partial t}\langle U^H, T^H\rangle_{\dot\sigma}|^2\right] dt,\end{aligned}$$

where T and U are the tangential and transversal vector fields, resp., of the variation Σ. U^H is parallel along $\dot\sigma$ and $T^H \circ \dot\sigma = \dot\sigma^H$, so $\nabla_{T^H}U^H|_{\dot\sigma} = \nabla_{\dot\sigma^H}U^H = 0$. Thus the first term of the integral vanishes. So does the last term, for $U^H \perp T^H$ holds along $\dot\sigma$. The end terms can be omitted, since we have chosen such variation where all transversal curves are geodesics, therefore $\nabla_{U^H}U^H = 0$. Summarizing we have

$$\frac{d^2\ell_\Sigma}{ds^2}(0) = -\int_a^b \langle \Omega(T^H, U^H)U^H, T^H\rangle_{\dot\sigma}\, dt = -\int_a^b R_{\dot\sigma}(U^H, U^H)\, dt < 0,$$

thus contradicting the minimality of σ. □

THEOREM 2.2. [**KP00**] *If V and W are two compact complex analytic submanifolds of a strongly Kähler Finsler manifold (M, F) of positive holomorphic bisectional curvature, and $\dim_{\mathbb{C}} V + \dim_{\mathbb{C}} W \geq \dim_{\mathbb{C}} M$, then $V \cap W \neq \emptyset$.*

PROOF. We use here Frankel's method again. Suppose that $V \cap W = \emptyset$. Then, there exists a minimazing geodesic $\sigma\colon [a,b] \to M$. Let $\sigma(a) \in V$, $\sigma(b) \in W$, σ is orthogonal to V and W in $\sigma(a)$ and $\sigma(b)$, resp.

We construct a regular variation $\Sigma\colon (-\varepsilon, \varepsilon) \times [a,b] \to M$ of σ such that $\nabla_{T^H + \overline{T^H}} U^H = 0$. Let $P \subset H_{\dot\sigma(b)} T^{1,0}M$ be the parallel translated of $T^H_{\sigma(a)}(V)$ with respect to the Chern-Finsler connection along $\dot\sigma$ to the point $\dot\sigma(b)$. Considering the horizontal lifts to $\widetilde{M}$ along $\dot\sigma$, analogously to the real case we get

$$\begin{aligned}\dim_{\mathbb{C}}\Big(P \cap (T^H_{\sigma(b)}W)\Big) &= \dim_{\mathbb{C}} P + \dim_{\mathbb{C}}(T^H_{\sigma(b)}W) - \dim_{\mathbb{C}}\Big(P + (T^H_{\sigma(b)}W)\Big)\\ &\geq \dim_{\mathbb{C}} V + \dim_{\mathbb{C}} W - (\dim_{\mathbb{C}} M - 1) \geq 1.\end{aligned}$$

So we can choose a vector U^H in the intersection. Its parallel translated along $\dot\sigma$ will be denoted by U^H, too. Since U^H is orthogonal to $\dot\sigma$ at the endpoint, it remains orthogonal along the entire curve by the metrical property of the Chern-Finsler connection. We consider the regular variation of σ with transversal vector field U.

In this case the second variation formula reduces to the following form:

$$\begin{aligned}&\frac{d^2\ell_\Sigma}{ds^2}(0) = \mathrm{Re}\langle \nabla_{U^H + \overline{U^H}} U^H, T^H\rangle_{\dot\sigma}\big|_a^b +\\ &+ \int_a^b \left\{ \|\nabla_{T^H+\overline{T^H}} U^H\|^2_{\dot\sigma} - \Big|\frac{\partial}{\partial t}\mathrm{Re}\langle U^H, T^H\rangle_{\dot\sigma}\Big|^2 - \mathrm{Re}\,[R_{\dot\sigma}(T,U)]\right\} dt,\end{aligned}$$

because of Proposition 2.6.7 in [**AP94**, p. 120].

The first term of the integral vanishes, for U^H is parallel along σ, and therefore, by the hypothesis on the holomorphic bisectional curvature all the remaining terms here will be negative except the first one at most. We consider also the variation belonging to the transversal vector field JU^H, and prove that the initial terms belonging to U^H, and JU^H cannot be positive at the same time. This will give the contradiction.

Therefore we calculate $\nabla_{JU^H + \overline{JU^H}} JU^H$.

$$\nabla_{JU^H + \overline{JU^H}} JU^H = J\nabla_{JU^H + \overline{JU^H}} U^H = J(\nabla_{JU^H} U^H + \nabla_{\overline{JU^H}} U^H)$$

Using the torsion we have

$$\nabla_{JU^H} U^H = \nabla_{U^H} JU^H + [JU^H, U^H] + \theta(JU^H, U^H)$$

The last term $\theta(JU^H, U^H)$ vanishes because F is strongly Kähler Finsler metric. Because of Proposition 2.6.7 in [**AP94**, p. 120],

$$\begin{aligned}
\nabla_{\overline{JU^H}} U^H &= \nabla_{U^H} \overline{JU^H} - [U^H, \overline{JU^H}] \\
&= J\left[\nabla_{\overline{U^H}} U^H + [U^H, \overline{U^H}]\right] - [U^H, \overline{JU^H}] \\
&= J\nabla_{\overline{U^H}} U^H + J[U^H, \overline{U^H}] - [U^H, \overline{JU^H}].
\end{aligned}$$

It follows now

$$\begin{aligned}
\nabla_{JU^H + \overline{JU^H}} JU^H &= J\left(\nabla_{U^H} JU^H + [JU^H, U^H] + J\nabla_{\overline{U^H}} U^H\right) \\
&\quad + J\left(J[U^H, \overline{U^H}] - [U^H, \overline{JU^H}]\right) \\
&= -\nabla_{U^H + \overline{U^H}} U^H + J[JU^H, U^H] - J[U^H, \overline{U^H}] - [U^H, \overline{JU^H}].
\end{aligned}$$

Now V and W are complex submanifolds, U^H is a horizontal lift, and tangent to $T^H_{\sigma(a)}V$ and $T^H_{\sigma(b)}W$ at $\dot\sigma(a)$ and $\dot\sigma(b)$, respectively. Since the horizontal space is a complex linear space, and we use the Chern Finsler connection, all Lie brackets above are horizontal vectors, and are orthogonal to T^H at $\sigma(a)$ and $\sigma(b)$. So

$$\mathrm{Re}\langle \nabla_{JU^H + \overline{JU^H}} JU^H, T^H\rangle = -\mathrm{Re}\langle \nabla_{U^H + \overline{U^H}} U^H, T^H\rangle.$$

□

3. Product of Kähler Finsler manifolds

In this section we construct the product of strongly Kähler Finsler manifolds.

Let (M_1, F_1), (M_2, F_2) be two strongly Kähler Finsler manifolds with the Chern-Finsler connection. Consider the product manifold $M_1 \times M_2$ with the metric

$$F(v_1, v_2) = \sqrt{F_1^2(v_1) + F_2^2(v_2)} \;\; \forall \; (v_1, v_2) \in TM_1 \times TM_2.$$

This is homogeneous, smooth and positive definite on $\widetilde{M}_1 \times \widetilde{M}_2$ because F_1, F_2 have these properties on $\widetilde{M}_1, \widetilde{M}_2$. The Levi matrix of F is positive definite on $\widetilde{M_1} \times \widetilde{M_2}$ because it is of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where A, B are the Levi matrix of F_1, F_2.

Let $\mathcal{H}_1, \mathcal{H}_2$ the horizontal bundle of the manifolds $(M_1, F_1), (M_2, F_2)$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

The metrics F_1, F_2 induce the Hermitian structures $\langle , \rangle_1$ and $\langle , \rangle_2$ on the horizontal bundles. It follows that on the bundle $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ we have the Hermitian metric

$$\langle X + U, Y + V \rangle = \langle X, Y \rangle_1 + \langle U, V \rangle_2.$$

The Chern-Finsler connection of the product manifold is related to the Chern-Finsler connections of M_1 and M_2 as follows:

$$\nabla_{X+U}(Y + V) = \nabla_X Y + \nabla_U V, \ \forall \ X, Y \in \mathfrak{X}(\mathcal{H}_1), \ U, V \in \mathfrak{X}(\mathcal{H}_2).$$

From these relation follows that the product manifold is strongly Kähler if the M_1 and M_2. The bisectional curvature of $M_1 \times M_2$ satisfies the relation:

$$R(X+U, Y+V) = R(X, Y) + R(U, V) \ \ \forall X, Y \in T^{1,0}M_1, \text{and } U, V \in T^{1,0}M_1.$$

We have the isomorphism $_o : T_{\mathbb{R}}M_1 \to T^{1,0}M_1$

$$\forall u \in T_{\mathbb{R}}M_1 \quad u_o = \frac{1}{2}(u - iJu).$$

Using the above isomorphism we can associate to F a function F^o : $T_{\mathbb{R}}M_1 \to \mathbb{R}^+$ by setting

$$\forall u \in T_{\mathbb{R}}M_1 \quad F^o(u) = F(u_o).$$

It is shown in [**AP94**, p.114] that the geodesics of F and F^o are the same if F is Kähler.

Applying these facts we show that if $\sigma = (\alpha, \beta)$ is a geodesic for F, then α and β are geodesic for F_1 and F_2, resp. In fact, α is also a geodesic for F^o, therefore, applying our result about geodesic on real warped product in [**KPV01**] for $f \equiv 1$, α and β are geodesic for F_1^o and F_2^o, resp. It follows by [**AP94**, p.114] again that α and β are geodesics for F_1 and F_2 resp.

It follows that $\dot{\alpha}(t) \neq 0$ and $\dot{\beta}(t) \neq 0$. That means that F is of smooth along the curve.

4. Coincidence of correspondences in Kähler-Finsler Manifolds

In the next part of the chapter we present some results on coincidence of correspondences of Kähler Finsler manifold, and some results on coincidence of mappings and fixed point theorems in Kähler Finsler manifolds (see [**Pet02**]).

A *holomorphic correspondence* of a complex manifold N with itself is a complex analytic submanifold of $N \times N$. Two (holomorphic) correspondences V, W are said to have a *coincidence* iff $V \cap W \neq \emptyset$. A holomorphic correspondence $V \subset N \times N$ is called *transversal* if $T_{(p,q)}V \oplus T_{(p,q)}(\{p\} \times N) = T_{(p,q)}(N \times N)$ and $T_{(p,q)}V \oplus T_{(p,q)}(N \times \{q\}) = T_{(p,q)}(N \times N)$ hold for all $(p, q) \in V$. Since $T_{(p,q)}(\{p\} \times N)$ and $T_{(p,q)}(N \times \{q\})$ are orthogonal, it follows that any vector orthogonal to V at (p, q) cannot be tangent to $\{p\} \times N$ or $N \times \{q\}$.

A holomorphic map $f : N \to N$ gives rise to a correspondence, the graph $G(f)$ of f; $G(f) = \{(p, f(p))|\ p \in N\}$. $G(f)$ is a special type of correspondence since f is single valued. Let $\Delta = \{(p, p)|\ p \in N\}$ be the diagonal of $N \times N$. It is clear that a map f has a fixed point whenever $G(f)$ intersects the diagonal Δ. A correspondence will be said to have a fixed point if it intersects the diagonal.

The main result is the following:

THEOREM 2.3. [**Pet02**] *Two holomorphic compact correspondences V, W, — one of them is tranversal, — of a connected strongly Kähler Finsler manifold N with positive holomorphic bisectional curvature have a coincidence, if $dim_{\mathbb{C}}V + dim_{\mathbb{C}}W \geq 2dim_{\mathbb{C}}N$.*

PROOF. The correspondences are complex analytic submanifolds V, W of $N \times N$. On the product manifold $N \times N$ we consider the metric $F : T^{1,0}N \times T^{1,0}N \to \mathbb{R}^+$ given by

$$F(v_1, v_2) = \sqrt{F_1^2(v_1) + F_1^2(v_2)} \text{ for } (v_1, v_2) \in T^{1,0}N \times T^{1,0}N.$$

We use the notations used in [**AP94**] and [**KP00**]. We take $M = N \times N$ and V, W are submanifolds of M.

We need only to show that V and W intersect. Suppose $V \cap W = \emptyset$. Then there exists a minimal geodesic $\sigma : [a,b] \to M$. Let $\sigma(a) \in V$, $\sigma(b) \in W$. σ is orthogonal to V and W in $\sigma(a)$ and $\sigma(b)$, respectively i. e. $\dot{\sigma}^H(a) \perp T^{(1,0)H}_{\sigma(a)} V$ and $\dot{\sigma}^H(b) \perp T^{(1,0)H}_{\sigma(b)} W$. According to the last argument of the previous section the geodesic has the form $\sigma = (\alpha, \beta) \in N \times N$ where both α and β geodesics. By the assumption of transversality of V or W we have $\dot{\alpha} \neq 0$ and $\dot{\beta} \neq 0$. Then it follows that F is smooth along σ.

We construct a regular variation $\Sigma : (-\varepsilon, \varepsilon) \times [a,b] \to M$ such that $\nabla_{T^H + \overline{T}^H} U^H = 0$. Denoting by the horizontal lift of $T^{1,0}M$ to horizontal space in $\dot{\sigma}(b)$, let $P \subset H_{\dot{\sigma}(b)} T^{1,0}M$ be the parallel translation of $T^H_{\sigma(a)}(V)$ with respect to the Chern-Finsler connection along $\dot{\sigma}$ to the point $\dot{\sigma}(b)$. Considering the horizontal lifts to $\widetilde{M}$ along $\dot{\sigma}$ we get

$$\dim_{\mathbb{C}}(P \cap (T^H_{\sigma(b)}(W)) = \dim_{\mathbb{C}} P + \dim_{\mathbb{C}}(T^H_{\sigma(b)} W) - \dim_{\mathbb{C}}(P + (T^H_{\sigma(b)} W)) \geq 1,$$

for $\dim_{\mathbb{C}}(P + T^H_{\sigma(b)} W) \leq 2\dim_{\mathbb{C}} N - 1$.

So we can choose a vector U^H in the intersection. Its parallel translation along $\dot{\sigma}$ will be denoted by U^H, too. Since U^H is orthogonal to $\dot{\sigma}$ at the end point, it remains orthogonal along the entire curve by the metrical property of the Chern-Finsler connection. We consider the regular variation of σ with transversal vector field U.

In this case the second variation formula reduces to the following form

$$\frac{d^2 l_\Sigma}{ds^2}(0) = \mathrm{Re}\ \langle \nabla_{U^H + \overline{U}^H} U^H, T^H \rangle_{\dot{\sigma}} \Big|_a^b +$$

$$+ \int_a^b \left\{ \|\nabla_{T^H + \overline{T}^H} U^H\|^2_{\dot{\sigma}} - \left| \frac{\partial}{\partial t} \mathrm{Re}\ \langle U^H, T^H \rangle_{\dot{\sigma}} \right|^2 - \mathrm{Re}\ [R_{\dot{\sigma}}(T,U)] \right\} dt$$

because of Proposition 2.6.7 in [**AP94**, p. 120].

The first term of the integral is zero for U is parallel along $\dot{\sigma}$. Furthermore, U^H and T^H are orthogonal.

By the hypothesis on the holomorphic sectional curvature all the terms will be negative or zero except the first one at most.

In fact we have

$$\frac{d^2 l_\Sigma}{ds^2}(0) = \mathrm{Re}\ \langle \nabla_{U^H + \overline{U}^H} U^H, T^H \rangle_{\dot{\sigma}} \Big|_a^b - \int_a^b \mathrm{Re}\ [R_{\dot{\sigma}}(T,U)] dt.$$

The integral is positive because $R_{\dot{\sigma}}(T,U) = R_{\dot{\sigma}}(T_1,U_1) + R_{\dot{\sigma}}(T_2,U_2)$ where $T_1 = \dot{\alpha} \neq 0$ and $T_2 = \dot{\beta} \neq 0$ and U_1, U_2 are orthogonal to T_1, T_2 resp.

By the minimality of σ it follows that $\frac{d^2 l_\Sigma}{ds^2}(0) \geq 0$ for any transversal vector field U.

If we consider the variation belonging to the transversal vector JU^H, we show that the initial terms in the second variation cannot be positive in the same time (for the variations corresponding to U^H and JU^H respectively). This will give the contradiction.

Therefore we calculate $\nabla_{JU^H+\overline{JU}^H} JU^H$.

$$\nabla_{JU^H+\overline{JU}^H} JU^H = J(\nabla_{JU^H} U^H + \nabla_{\overline{JU}^H} U^H).$$

Using the torsion we have

$$\nabla_{JU^H} U^H = \nabla_{U^H} JU^H + [JU^H, U^H] + \theta(JU^H, U^H).$$

The last term $\theta(JU^H, U^H)$ vanishes because F is strongly Kähler Finsler metric and using again the Proposition 2.6.7 in [**AP94**, p. 120] it follows :

$$\nabla_{\overline{JU}^H} U^H = \nabla_{U^H} \overline{JU}^H - [U^H, \overline{JU}^H] =$$

$$= J[\nabla_{\overline{U}^H} U^H + [U^H, \overline{U}^H]] - [U^H, \overline{JU}^H] =$$

$$= J\nabla_{U^H} U^H + J[U^H, \overline{U}^H] - [U^H, \overline{JU}^H].$$

It follows now

$$\nabla_{JU^H+\overline{JU}^H} JU^H = J(\nabla_{U^H} JU^H + [JU^H, U^H] + J\nabla_{\overline{U}^H} U^H + J[U^H, \overline{U}^H] - [U^H, \overline{JU}^H]) =$$

$$= -\nabla_{U^H+\overline{U}^H} U^H + J[JU^H, U^H] - J[U^H, \overline{U}^H] - [U^H, \overline{JU}^H].$$

Now V and W are complex manifolds, U^H is a horizontal lift, and tangent to $\widetilde{V}$ and $\widetilde{W}$ in $\dot{\sigma}(a)$ and $\dot{\sigma}(b)$ respectively. Since the horizontal space is a complex linear space, and we use the Chern-Finsler connection, all the brackets above are horizontal vectors, and are orthogonal to T^H in $\dot{\sigma}(a)$ and $\dot{\sigma}(b)$. So

$$\text{Re } \langle \nabla_{JU^H+\overline{JU}^H} JU^H, T^H \rangle = -\text{Re } \langle \nabla_{U^H+\overline{U}^H} U^H, T^H \rangle.$$

This means that $\frac{d^2 l_\Sigma}{ds^2}(0)$ cannot be non-negative for U and JU at the same time, wich gives the contradiction. □

We can easily formulate some consequences concerning coincidence of mappings and fixed point properties in Kähler Finsler manifold using the above theorem.

Let us consider a Kähler Finsler manifold M and two holomorphic maps $f, g : M \to M$.

Corollary 2.4. [**Pet02**] *Let M be a compact Kähler Finsler manifold of positive holomorphic bisectional curvature and $f, g : M \to M$ holomorphic maps. There exists at least one point $p \in M$ such that $f(p) = g(p)$.*

Corollary 2.5. [**Pet02**]*Let M be a compact Kähler Finsler manifold of positive holomorphic bisectional curvature and $f : M \to M$ holomorphic map. The map f has at least one fixed point.*

Theorem 2.6. *Let M be a Kähler Finsler manifold of positive holomorphic bisectional curvature and N be a compact complex analytic submanifold of M with $dim_{\mathbb{C}} N \geq \left[\frac{dim_{\mathbb{C}} M}{2}\right] + 1$. If $f, g : N \to M$ are holomorphic embeddings then they have at least one coincidence.*

Proof. If $f, g : N \to M$ are holomorphic embeddings, the images $f(N)$ and $g(N))$ are compact complex analytic submanifolds of M. Now we consider V and W to be $N \times f(N)$ and $N \times f(M)$, respectively as submanifolds of the product manifold $M \times M$.

The condition in the theorem means exactly that

$$\dim_{\mathbb{C}} V + \dim_{\mathbb{C}} W \geq \dim_{\mathbb{C}} (M \times M).$$

Now the results follows from Theorem 1, because V and W are compact submanifolds of $M \times M$. f □

Theorem 2.7. [**Pet02**] *Let M be a Kähler Finsler manifold of positive holomorphic bisectional curvature and N be a compact complex analytic submanifold of M with $dim_{\mathbb{C}} N \geq \left[\frac{dim_{\mathbb{C}} M}{2}\right] + 1$. If $f : N \to M$ is holomorphic embedding then it has at least one fixed point.*

CHAPTER 3

Morse Index Theorems in Finsler Geometry

1. Introduction

It is a remarkable fact that the Jacobi equation the second variation formula of the arc-length and the index form in Finsler spaces look exactly like their counterparts in Riemannian geometry (see [**AP94**], [**BC93**], [**Che96**], [**BCS00**]). Many global results are obtained in Finslerian context (for example Cartan Hadamard theorem, Bonnet-Myers theorem and Synge's theorem, see [**AP94**], [**Aus55**], [**BC93**], [**BCS00**]).

The Morse Index Theorem also generalizes in Finsler case. That was due to Lehmann [**Leh64**], see Matsumoto [**Mat86**] for an exposition and Milnor [**Mil63**] for background.

In the Riemann and semi-Riemann cases the Morse Index Theorem is also generalized where the ends are submanifolds by Ambrose [**Amb61**], Bolton [**Bol77**], Kalish [**Kal88**], Piccione and Tausk [**PT99**].

In this chapter we prove the second variation formula for the energy functional in Finsler geometry. First we discuss the Morse Index Theorem in the classical case, where the ends are fixed points and then the case where the ends are submanifolds of a Finsler manifold.

The main difference between the Riemannian and Finsler case is that the second fundamental form of a submanifold is not symmetric. We show (Section 6, p. 37) that the Morse Index form is symmetric and this allows us to prove the Morse Index Theorem in the case of variable end points.

In Section 2 variation formulas for the energy functional are proved. We consider a regular two parameter variation and the pulled back of the Cartan connection along the curve. Then we derive formulas for the first and the second variation of the energy functional.

In the next sections (Sections 3 and 4) we introduce the Jacobi fields and Morse Index Form, and we recall some properties, mainly from [**AP94**].

The following section (Section 5) is devoted to prove the Morse Index Theorem for fixed endpoints of the geodesic. The proof follows the line from [**Mil63**]. The results are the same as results obtained by [**Leh64**] (presented in [**Mat86**]).

Section 6 deals with the Morse Index form where the ends are submanifolds. The results of this section are from the author's paper [**Pet**]. First we prove the symmetry of the Morse Index Form. Despite the fact that the second fundamental form of a submanifold is not symmetric, the Morse Index form is symmetric. The Morse Index theorem where the ends are submanifolds is proved in two steps: first we consider the case where one end point is in a submanifold and the other is fixed (Section 7, Theorem 3.33), and after that we prove the general case (Section 8, Theorem 3.34). The index is computed using P-Jacobi fields (Definition 3.29). The proof follows the line of Morse [**Mil63**] and Piccione and Tausk [**PT99**].

2. Variation Formulae

In order to prove the Morse index theorem in the case where the ends are submanifolds we prove the first and the second variation of the energy functional [**Mat86**].

Definition 3.1. [**AP94**]. *A regular curve $\sigma : [a, b] \to M$ is a C^1-curve such that*

$$\forall\ t \in [a, b] \quad \dot{\sigma}(t) = d\sigma_t \left(\frac{d}{dt}\right) \neq 0.$$

The length, with respect to the Finsler metric $F : TM \to \mathbb{R}^+$ of the regular curve is given by

$$L(\sigma) = \int_a^b F(\dot{\sigma}(t))dt\ ,$$

and the energy is given by

$$E(\sigma) = \int_a^b F^2(\dot{\sigma}(t))dt\ .$$

Definition 3.2. [**Mat86**] *Let* $\sigma_0 : [a,b] \to M$ *be a regular curve with* $F(\dot\sigma_0) \equiv c_0$*. A regular two parameter variation of* σ_0 *is a* C^1*-map* $\Sigma : U \times [a,b] \to M$ *where* $U \in \mathbb{R}^2$ *is a neighborhood of* $0 \in \mathbb{R}^2$*such that*

(1) $\sigma_0(t) = \Sigma(0,t), \ \forall\, t \in [a,b]$,
(2) *for every* $(x,y) \in U$ *the curve* $\sigma_{(x,y)}(t) = \Sigma(x,y)(t)$ *is a regular curve in* M*,*
(3) $F(\dot\sigma_{(x,y)}) \equiv c_{(x,y)} > 0$ *for every* $(x,y) \in U$*.*

A regular variation is fixed iff it moreover satisfies:

(4) $\sigma_{(x,y)}(a) = \sigma_0(a)$ *and* $\sigma_{(x,y)}(b) = \sigma_0(b)$ *for all* $(x,y) \in U$*.*

A regular variation is a geodesic variation iff it moreover satisfies

(5) *for every* $(x,y) \in U$ *the curve* $\sigma_{(x,y)}(t) = \Sigma(x,y)(t)$ *is a geodesic curve in* M

For a regular variation of σ_0 *we defined the function* $E_\Sigma : U \to \mathbb{R}^*$ *given by*

$$E_\Sigma(x,y) = E(\sigma_{(x,y)})$$

We use again the pulled-back of the Cartan connection along a curve. Again the pulled-back bundle does not live on TM, but on $T\tilde{M}$. We briefly present it here.

Let $\Sigma : U \times [a,b] \to M$ be a regular variation of a curve $\sigma_0 : [a,b] \to M$. Let

$$p : \Sigma^*(TM) \to U \times [a,b]$$

be the pull back bundle, and $\gamma : \Sigma^*(TM) \to TM$ be the fiber map which identifies each $\Sigma^*(TM)_{(x,y,t)}$ with $T_{\Sigma(x,y,t)}M$ for all $(x,y,t) \in U \times [a,b]$. A local frame for $\Sigma^*(TM)$ is given by the local fields

$$\frac{\partial}{\partial x^i}|_{(x,y,t)} = \gamma^{-1}(\frac{\partial}{\partial x^i}|_{\Sigma(x,y,t)})$$

for $i = 1, \dots n$. An element $\xi \in \mathfrak{X}(\Sigma^*(TM))$ can be written locally by

$$\xi(x,y,t) = u^i(x,y,t)\frac{\partial}{\partial x^i}|_{(x,y,t)},$$

and a local frame on $T(\Sigma^*(TM))$ is given by $\partial_x, \partial_y \partial_t, \dot\partial_i$, where $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}\partial_t = \frac{\partial}{\partial t}$ and $\dot\partial_i = \frac{\partial}{\partial u^i}$.

There are three particularly important sections of $\Sigma^*(TM)$:

$$\begin{aligned} T &= \gamma^{-1}(d\Sigma\ (\frac{\partial}{\partial t})) = \frac{\partial \Sigma^i}{\partial t}\frac{\partial}{\partial x^i}, \\ X &= \gamma^{-1}(d\Sigma\ (\frac{\partial}{\partial x})) = \frac{\partial \Sigma^i}{\partial x}\frac{\partial}{\partial x^i} \text{ and} \\ Y &= \gamma^{-1}(d\Sigma\ (\frac{\partial}{\partial y})) = \frac{\partial \Sigma^i}{\partial y}\frac{\partial}{\partial x^i} \end{aligned}$$

Definition 3.3. *The sections X and Y are the transversal vectors of Σ.*

By setting $\Sigma^*\tilde{M} = \gamma^{-1}(\tilde{M})$, we have that $T \in \mathfrak{X}(\Sigma^*\tilde{M})$ and $T(x,y,t) = \gamma^{-1}(\dot{\sigma}_{(x,y)}(t))$.

We may pull-back $T\tilde{M}$ over $\Sigma^*\tilde{M}$ by using γ, obtaining the map $\tilde{\gamma}$: $\gamma^*(T\tilde{M}) \to T\tilde{M}$ which identifies, for any $u \in \Sigma^*\tilde{M}_{(x,y,t)} = \gamma^{-1}(\tilde{M}_{\Sigma(x,y,t)})$, $\gamma^*(t\tilde{M})_u$ with $T_{\gamma(u)}\tilde{M}$.

We shall state now the first and the second variation of energy .

Theorem 3.4. *Let $F : TM \to \mathbb{R}^+$ be a Finsler metric on a manifold M. We consider the regular two parameter variation of $\sigma_0 : [a,b] \to M$ with $F(\dot{\sigma}_0) = c_0 > 0$ and let $\Sigma : U \times [a,b] \to M$ be a regular variation of σ_0. Then*

$$\frac{1}{2}\frac{\partial E_\Sigma}{\partial x}(0,0) = \left\{ \langle X^H, Y^H \rangle_{\dot{\sigma}_0}\Big|_a^b - \int_a^b \langle X^H, \nabla_{T^H} T^H \rangle_{\dot{\sigma}_0} \right\} dt\ .$$

If the variation is fixed we have

$$\frac{1}{2}\frac{\partial E_\Sigma}{\partial x}(0) = -\int_a^b \langle X^H, \nabla_{T^H} T^H \rangle_{\dot{\sigma}_0} dt.$$

Proof.

$$E_\Sigma(s) = \int_a^b G(\dot{\sigma}_s)dt, \text{where}\ \ G = F^2\ .$$

Now T^H, X^H denote the horizontal liftings in this bundle of the tangent vector to the curve and to the transversal vector.

$$\frac{\partial E_\Sigma}{\partial x} = \frac{\partial}{\partial x}\int_a^b G(\dot{\sigma}_s)dt = \int_a^b \frac{\partial}{\partial x} G(\dot{\sigma}_s)dt =$$

$$= \int_a^b \frac{\partial}{\partial x}\langle \chi(\dot{\sigma}_s), \chi(\dot{\sigma}_s)\rangle_{\dot{\sigma}_0} dt\ .$$

Now

$$\frac{\partial}{\partial x}\langle \chi(\dot\sigma_s),\chi(\dot\sigma_s)\rangle_{\dot\sigma_0} = 2\langle \nabla_{X^H}T^H, T^H\rangle_{\dot\sigma_0} =$$

$$= 2\{\langle \nabla_{T^H}T^H, T^H\rangle_{\dot\sigma_0} - \langle [T^H, X^H], T^H\rangle_{\dot\sigma_0} - \langle \theta(T^H, X^H), T^H\rangle_{\dot\sigma_0}\}\ .$$

But $\theta(T^H, X^H)$ is a vertical vector and so is $[T^H, X^H]$. This means that

$$\begin{aligned}\frac{\partial}{\partial x}\langle T^H, T^H\rangle &= 2\langle \nabla_{T^H}X^H, T^H\rangle_{\dot\sigma_0}\\ &= 2\{\frac{\partial}{\partial t}\langle X^H, T^H\rangle_{\dot\sigma_0} - \langle X^H, \nabla_{T^H}T^H\rangle_{\dot\sigma_0}\}\ .\end{aligned}$$

Finally

$$\frac{\partial E_\Sigma}{\partial x}(0) = 2\{\langle X^H, T^H\rangle_{\dot\sigma_0}\Big|_a^b - \int_a^b \langle X^H, \nabla_{T^H}T^H\rangle_{\dot\sigma_0} dt\}\ .$$

□

THEOREM 3.5. *Let $F : TM \to \mathbb{R}^+$ be a Finsler metric on a manifold M. Let $\sigma_0 : [a,b] \to M$ with $F(\sigma_0) \equiv 1$ and let $\Sigma : U \times [a,b] \to M$ be a geodesic two parameter variation of σ_0. Then*

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 E_\Sigma}{\partial x \partial y}(0,0)(X,Y) &= \int_a^b (\langle \nabla_{T^H}X^H, \nabla_{T^H}Y^H\rangle - \langle \Omega(T^H, X^H)Y^H, T^H\rangle_T) dt\\ &= \langle \nabla_{T^H}X^H, Y^H\rangle\Big|_a^b - \int_a^b \langle \nabla_{T^H}\nabla_{T^H}X^H - \Omega(T^H, X^H)T^H, Y^H\rangle_T dt.\end{aligned}$$

If the variation is fixed we have

$$\frac{1}{2}\frac{\partial^2 E_\Sigma}{\partial x \partial y}(0,0)(X,Y) = \int_a^b \langle \nabla_{T^H}\nabla_{T^H}X^H - \Omega(T^H, X^H)T^H, Y^H\rangle_T dt.$$

PROOF. In the proof of the first variation formula we saw that

$$\frac{\partial E_\Sigma}{\partial x}(0)(X,Y) = 2\int_a^b \langle \nabla_{T^H}X^H, T^H\rangle_{\dot\sigma_0} dt\ .$$

The integrand is a continuous function on $U \times [a,b]$, we lift it over $\Sigma^* M$ (the pulled back bundle of the variation) and compute

$$\frac{\partial}{\partial y}\langle \nabla_{T^H}X^H, T^H\rangle_{\dot\sigma_0} = \langle \nabla_{Y^H}\nabla_{T^H}X^H, T^H\rangle_{\dot\sigma_0} + \langle \nabla_{T^H}U^H, \nabla_{Y^H}\nabla_{T^H}\rangle_{\dot\sigma_0} =$$

$$= \langle \nabla_{T^H}\nabla_{Y^H}X^H, T^H\rangle_{\dot\sigma_0} - \langle \nabla_{[T^H,Y^H]}X^H, T^H\rangle_{\dot\sigma_0} - (\Omega(T^H, X^H)Y^H, T^H\rangle_{\dot\sigma_0} +$$

$$+ \langle \nabla_{T^H}X^H, \nabla_{T^H}Y^H\rangle_{\dot\sigma_0} - \langle \nabla_{T^H}X^H, [T^H, Y^H]\rangle_{\dot\sigma_0} - \langle \nabla_{T^H}X^H, \theta(T^H, Y^H)\rangle_{\dot\sigma_0}\ .$$

Now $\nabla_{T^H} X^H$ is horizontal vector, $\theta(T^H, Y^H)$ and $[T^H, Y^H]$ are vertical and for every V vertical vector

$$\langle \nabla_V X^H, T^H \rangle_{\dot\sigma_0} = 0$$

Using the fact that $\nabla_{T^H} T^H = 0$ because σ_0 is a geodesic and by the same arguments as in the proof of the first variation we obtain the result. □

3. Jacobi Fields

Next we will define the Jacobi fields [**AP94**].

DEFINITION 3.6. *A geodesic variation $\Sigma : (-\varepsilon, \varepsilon) \times [0, a] \to M$ of a geodesic $\sigma_0 : [0, a] \to M$ is a regular variation of σ_0 such that $\sigma_s = \Sigma(s, \cdot)$ is a geodesic $\forall s \in (-\varepsilon, \varepsilon)$.*

That means that if we consider

$$\forall u \in \widetilde{M}_{\sigma_s(t)}, \quad T^H(u) = \chi_u(\dot\sigma_s(t))$$

it follows that

$$\forall s \in (-\varepsilon, \varepsilon), \quad \nabla_{T^H} T^H \Big|_{\dot\sigma_s} = 0 \ .$$

We consider as above

$$U^H(u) = \chi_u \left(\frac{\partial \Sigma^a}{\partial s} \frac{\partial}{\partial x^a} \Bigg|_{\sigma_s(t)} \right) \ .$$

$$0 = \nabla_{U^H} \nabla_{T^H} T^H = \nabla_{T^H} \nabla_{U^H} T^H + \nabla_{[U^H, T^H]} T^H + \Omega(U^H, T^H) T^H =$$

$$= \nabla_{T^H} \nabla_{T^H} U^H + \nabla_{T^H}([U^H, T^H)] + \theta(U^H, T^H)) + \nabla_{[U^H, T^H]} T^H - \Omega(T^H, U^H) T^H.$$

Now

$$[U^H, T^H] = -\theta(U^H, T^H) \quad ,$$

$[U^H, T^H]$ is a vertical vector, but $(\nabla_V T^H)(\sigma_s) = 0$.

Finally we have

Let $\Sigma : (-\varepsilon, \varepsilon) \times [0, a] \to M$ be a geodesic variation of the geodesic $\sigma_0 : [0, a] \to M$ in a Finsler manifold M. We consider

$$J(t) = \frac{\partial \Sigma^a}{\partial s}(0, t) \frac{\partial}{\partial x^a} \Bigg|_{\sigma_0(t)} \in T_{\sigma_0(t)} M$$

and

$$J^H(t) = \chi_{\dot{\sigma}_0(t)}(J(t)) \in \mathcal{H}_{\dot{\sigma}_0(t)}, \text{for} t \in [0, a].$$

Then

$$\nabla_{T^H} \nabla_{T^H} J^H - \Omega(T^H, J^H) T^H = 0.$$

Because $T^H(\dot{\sigma}_0(t)) = \chi(\dot{\sigma}_0(t))$ the above equation can be written as

$$\nabla_\chi \nabla_\chi J^H - \Omega(\chi, J^H)\chi \equiv 0$$

along of $\dot{\sigma}_0$.

DEFINITION 3.7. *Let $\sigma_0 : [0, a] \to M$ be a geodesic. A vector field J along σ is a Jacobi field iff it satisfies the Jacobi equation*

$$\nabla_{T^H} \nabla_{T^H} J^H - \Omega(T^H J^H) T^H = 0 \ ,$$

for $t \in [0, a]$ where $J^H = \chi_{\dot{\sigma}(t)}(J(t))$.

It follows that $\dot{\sigma}$ and $t\dot{\sigma}$ are Jacobi fields; the first one is never zero, the second vanish in $t = 0$. We note the set of all Jacobi fields along σ by $\mathcal{J}(\sigma)$.

In local coordinates, the Jacobi equation is a second order differential equation system. Given $J(0)$ and $(\nabla_{T^H} J^H)(0)$ there is a unique solution of the system defined on $[0, a]$. The set of the solutions is a vector space of dimension n.

DEFINITION 3.8. *Let $\sigma : [0, a] \to M$ be a geodesic. The point $\sigma(t_0)$ is conjugate with $\sigma(0)$ along σ, where $t_0 \in (0, a]$ if there exists a non-zero Jacobi field J, along σ such that $J(0) = 0 = J(t_0)$.*

It is important that the zeroes of a Jacobi field J are discrete; indeed if it is not true, we have that $J(t_0) = 0$ and $\nabla_{T^H} J^H(t_0) = 0$ for $t_0 \in [0, a]$ and from the property of uniqueness of the solution of a Cauchy problem follows that $J \equiv 0$.

Next we shall prove two results regarding the behavior of a Jacobi field along a geodesic.

PROPOSITION 3.9. *Let $J \in \mathcal{J}(\sigma)$ be a Jacobi field along a geodesic $\sigma : [0,a] \to M$ in a Finsler manifold M. Then*

$$\langle J^H, T^H \rangle_{\dot\sigma(t)} = t \langle \nabla_{T^H} J^H, T^H \rangle_{\dot\sigma(0)} + \langle J^H, T^H \rangle_{\dot\sigma(0)}.$$

PROOF. We have

$$\frac{d}{dt} \langle \nabla_{T^H} J^H, T^H \rangle_{\dot\sigma} = T^H \langle \nabla_{T^H} J^H, T^H \rangle_{\dot\sigma} =$$

$$= \langle \nabla_{T^H} \nabla_{T^H} J^H, T^H \rangle_{\dot\sigma} = \langle \Omega(T^H, J^H) T^H, T^H \rangle_{\dot\sigma} = 0.$$

Then

$$\langle \nabla_{T^H} J^H, T^H \rangle_{\dot\sigma} = \langle \nabla_{T^H} J^H, T^H \rangle_{\dot\sigma(0)} \ .$$

Moreover

$$\frac{d}{dt} \langle J^H, T^H \rangle = T^H \langle J^H, T^H \rangle = \langle \nabla_{T^H} J^H, T^H \rangle_{\dot\sigma} \equiv$$

$$\equiv \langle \nabla_{T^H} J^H, T^H \rangle_{\dot\sigma(0)} \ .$$

□

COROLLARY 3.10. *Let $J \in \mathcal{J}(\sigma)$ be a Jacobi field along a geodesic $\sigma : [0,a] \to M$ in a Finsler manifold M. Suppose that*

$$\langle J^H, T^H \rangle_{\dot\sigma(0)} = \langle J^H, T^H \rangle_{\dot\sigma(a)}.$$

Then

$$\langle J^H, T^H \rangle_{\dot\sigma} \equiv \langle J^H, T^H \rangle_{\dot\sigma(0)}$$

and

$$\langle \nabla_{T^H} J^H, T^H \rangle_{\dot\sigma} \equiv 0.$$

DEFINITION 3.11. *Let $\sigma : [0,a] \to M$ be a geodesic in a Finsler manifold M. A proper Jacobi field along σ is a Jacobi field $J \in \mathcal{J}(\sigma)$ such that*

$$\langle J^H, T^H \rangle_{\dot\sigma} \equiv 0.$$

We shall denote by $\mathcal{J}_0(\sigma)$ the set of all Jacobi fields along σ .

4. The Morse Index Form

In this section we shall investigate the Morse Index from which results from the second variation of the energy [**AP94**] .

DEFINITION 3.12. *Let $\sigma : [a,b] \to M$ a geodesic in a Finsler manifold M; we say that σ is a normal geodesic if it is parameterized by arc-length, that is $F(\dot{\sigma}) \equiv 1$. Particularly $T(\sigma) = \dot{\sigma}$.*

Let $\sigma : [a,b] \to M$ be a normal geodesic in a Finsler manifold M, we note by $\mathfrak{X}[a,b]$ the space of vector fields ξ along σ such that

$$\langle \xi^H, T^H \rangle_T \equiv 0.$$

Moreover, we note by $\mathfrak{X}_0[a,b]$ the subspace of the vector fields $\xi \in \mathfrak{X}[a,b]$ such that $\xi(a) = \xi(b) = 0$.

DEFINITION 3.13. *The Morse Index Form $I = I_a^b : \mathfrak{X}[a,b] \times \mathfrak{X}[a,b] \to \mathbb{R}$ of a normal geodesic $\sigma : [a,b] \to M$ is the bilinear symmetric form*

$$I(\xi,\eta) = \int_a^b \langle \nabla_{T^H} \xi^H, \nabla_{T^H} \eta^H \rangle_T - \langle \Omega(T^H, \xi^H)\eta^H, T^H \rangle_T dt,$$

for $\xi, \eta \in \mathfrak{X}[a,b]$.

LEMMA 3.14. *Let $\sigma : [a,b] \to M$ be a normal geodesic in a Finsler manifold M and let $\xi \in \mathfrak{X}[a,b]$ be smooth. Then*

$$I(\xi,\eta) = \langle \nabla_{T^H} \xi^H, \eta^H \rangle_T \Big|_a^b - \int_a^b \langle \nabla_{T^H} \nabla_{T^H} \xi^H - \Omega(T^H, \xi^H) T^H, \eta^H \rangle dt$$

for $\eta \in \mathfrak{X}[a,b]$.

PROOF. Suppose that η is smooth (if not we broke the geodesic in a finite number of pieces on which η is smooth). Then we have

$$\frac{d}{dt} \langle \nabla_{T^H} \xi^H, \eta^H \rangle_T = T^H \langle \nabla_{T^H} \xi^H, \eta^H \rangle_T =$$
$$= \langle \nabla_{T^H} \nabla_{T^H} \xi^H, \eta^H \rangle_T + \langle \nabla_T \xi^H, \nabla_T \eta^H \rangle_T$$
$$\langle \Omega(T^H, \xi^H)\eta^H, T^H \rangle = -\langle \Omega(T^H, \xi^H) T^H, \eta^H \rangle_T$$

Substituting these into the expression of the Morse Index Form we obtain the above formula. □

The kernel of the Morse Index Form consists of proper Jacobi fields.

COROLLARY 3.15. *Let $\sigma : [a,b] \to M$ be a normal geodesic in a Finsler manifold M, and $\xi \in \mathfrak{X}[a,b]$. Then $I(\xi, \mathfrak{X}_0[a,b]) = \{0\}$ if and only if ξ is a proper Jacobi field. Particularly*

$$\ker I\Big|_{\mathfrak{X}_0[a,b]} = \mathfrak{X}_0[a,b] \cap \mathcal{J}_0(\sigma)$$

PROOF. Suppose that $I(\xi, \mathfrak{X}_0[a,b]) = \{0\}$. Then $\forall \eta \in \mathfrak{X}_0[a,b]$,

$$0 = I(\xi, \eta) = -\int_a^b \langle \nabla_{T^H} \nabla_{T^H} \xi^H - \Omega(T^H, \xi^H) T^H, \eta^H \rangle_T dt \ ,$$

and it follows that $\xi \in \mathcal{J}_0(\sigma)$. The converse is obvious. □

There is a relationship between Jacobi fields and conjugate points. We try to exploit this.

DEFINITION 3.16. *Let $\sigma : [a,b] \to M$ be a normal geodesic in a Finsler manifold M. We say that σ does not contain conjugate points if $\sigma(t)$ and $\sigma(a)$ are not conjugate along σ for $t \in [a,b]$. We said that $\sigma(b)$ is the first conjugate point with $\sigma(a)$ along σ if $\sigma(b)$ is conjugate with $\sigma(a)$ and all points $\sigma(t)$, $t \in (a,b)$ are not conjugate with $\sigma(a)$.*

PROPOSITION 3.17. *Let $\sigma : [a,b] \to M$ be a normal geodesic in a Finsler manifold M which does not contain conjugate points. The Morse Index form I_a^b is positive definite on $\mathfrak{X}_0[a,b]$.*

PROOF. In fact we suppose that $\exp_{\sigma(a)}$ is a local diffeomorphism . σ is local minimizing for the arc-length. Then I is positive definite on $\mathfrak{X}_0[a,b]$.

We suppose that $\xi \in \mathfrak{X}_0[a,b]$ has the property that $I(\xi, \eta) = 0$. We will show that $\xi \in ker\, I$. Let $\eta \in \mathfrak{X}_0[a,b]$,then $\forall \varepsilon \in \mathbb{R}^+$ we have

$$0 \leq I(\xi + \varepsilon\eta, \xi + \varepsilon\eta) = \varepsilon(I(\xi, \eta) + \varepsilon I(\eta, \eta))$$

Divide now by ε and let $\varepsilon \to 0^+$ ($\varepsilon \to 0^-$ respectively). We obtain that $I(\xi, \eta) \geq 0 \quad (I(\xi, \eta) \leq 0$ resp.) and it follows that $I(\xi, \eta) = 0$.

It follows that ξ is a Jacobi field such that $J(a) = J(b) = 0$. But $\sigma(b)$ is not conjugate with $\sigma(a)$ along $\sigma \ \Rightarrow\ \xi \equiv 0$. □

COROLLARY 3.18. *Let $\sigma : [a,b] \to M$ be a normal geodesic in a Finsler manifold M which does not contain conjugate points. Let $\xi \in \mathfrak{X}[a,b]$ and $J \in \mathcal{J}_0(\sigma)$ such that $\xi(a) = J(a)$ and $\xi(b) = J(b)$. Then*

$$I(J,J) \leq I(\xi,\xi) \ ,$$

and equality holds if and only if $J \equiv \xi$.

PROOF.

$$I(J,\xi) = \langle \nabla_T J^H, \xi^H \rangle_T \Big|_a^b = \langle \nabla_{T^H} J^H, J^H \rangle_T \Big|_a^b = I(J,J) \ .$$

For $J \not\equiv \xi$

$$0 \leq I(\xi - J, \xi - J) = I(\xi,\xi) - 2I(\xi,J) + I(J,J) = I(\xi,\xi) - I(J,J) \ .$$

□

The above result shows that the Jacobi fields minimize the Morse Index form between the vector fields with same beginning and end points.

The Morse Index form becomes positive semi-definite in the first conjugate point.

PROPOSITION 3.19. *Let $\sigma : [a,b] \to M$ be a normal geodesic on a Finsler manifold M such that $\sigma(b)$ is the first conjugate point with $\sigma(a)$ along σ. Then I_a^b is positive semi-definite on $\mathfrak{X}_0[a,b]$ and*

$$\ker I_a^b|_{\mathfrak{X}_0[a,b]} = \mathfrak{X}_0[a,b] \cap \mathcal{J}_0(\sigma) \neq \{0\}.$$

PROOF. We only show that $I_a^b \geq 0$. The second assertion is obvious. Let $b' \in (a,b)$ and define $T_{b'} : \mathfrak{X}_0[a,b] \to \mathfrak{X}_0[a,b']$ by

$$T_{b'}(\xi)(t) = \xi(bt/b') \ .$$

It is clear that the application $T_{b'}$is an isomorphism; we can define a bilinear symmetric form by $I_{b'} : \mathfrak{X}_0[a,b] \times \mathfrak{X}_0[a,b] \to \mathbb{R}$ by

$$I_{b'}(\xi,\eta) = I_a^{b'}(T_{b'}(\xi), T_{b'}(\eta))$$

Then

$$I_a^b(\xi,\xi) = \lim_{b' \to b} I_{b'}(\xi,\xi) \geq 0 \ ,$$

for all $\xi \in \mathfrak{X}_0[a,b]$. □

Next we shall prove the following

PROPOSITION 3.20. *Let* $\sigma : [a, b] \to M$ *be a normal geodesic in a Finsler manifold* M *. Then exists* $t_0 \in (a, b)$ *such that* $\sigma(t_0)$ *is conjugate with* $\sigma(a)$ *along* σ *if and only if exists* $\xi \in \mathfrak{X}_0[a, b]$ *such that* $I_a^b(\xi, \xi) < 0$.

PROOF. If there exists such a field ξ it follows that exists $t_0 \in (a, b)$ such that $\sigma(t_0)$ and $\sigma(a)$ are conjugate along σ.

Conversely let $t_0 \in (a, b)$ such that $\sigma(t_0)$ and $\sigma(a)$ are conjugate points along σ. Then exists a non-zero Jacobi field $J \in \mathfrak{X}[a, t_0]$.

Let $t' \in (a, t_0)$ and $t'' \in (t_0, b)$ such that $J(t') \neq 0$ and

$$d_F(\sigma(t'), \sigma(t'')) < ir(\sigma(t'')).$$

Particularly $\sigma|_{[t',t'']}$ does not contain conjugate points to $\sigma(t'')$.

Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a path with $\gamma(0) = \sigma(t')$ and $\gamma'(0) = J(t')$. If $\gamma = \exp_{\sigma(t'')}(\widetilde{\gamma})$ let Σ be the geodesic variation such that

$$\Sigma(s, t) = \exp_{\sigma(t'')}(t\widetilde{\gamma}(s)).$$

The vector U which is transversal of Σ is a proper Jacobi field which is in $\mathfrak{X}[t', t'']$ such that $U(t') = J(t')$ and $U(t'') = 0$.

We define now $\xi \in \mathfrak{X}_0[a, b]$ by

$$\xi(t) = \begin{cases} J(t) & \text{for } t \in [a, t'] \\ U(t) & \text{for } t \in [t', t''] \\ 0 & \text{for } t \in [t'', b]. \end{cases}$$

We also note by $t' \in \mathfrak{X}[a, t'']$ the extension of J obtained by considering $J'(t) = 0$ for $t \in [t_0, t'']$. It is clear that J' is not smooth in t_0, so it is not a Jacobi field on $[t', t'']$.

It follows that

$$\begin{aligned} I_a^b(\xi, \xi) &= I_a^{t'}(\xi, \xi) + I_{t'}^{t''}(\xi, \xi') \\ &= I_a^{t'}(J, J) + I_{t'}^{t''}(U, U) < I_a^{t'}(J, J) + I_{t'}^{t''}(J', J') \\ &= I_a^{t'}(J, J) + I_{t'}^{t_0}(J, J) = I_a^{t_0}(J, J) = 0. \end{aligned}$$

□

Particularly a geodesic which contains conjugate points cannot realize the minima of the distance between his end-points.

COROLLARY 3.21. *Let $\sigma : [a,b] \to M$ be a normal geodesic in a Finsler manifold M. Suppose that there exists $t_0 \in [a,b]$ such that $\sigma(t_0)$ and $\sigma(a)$ are conjugate along σ.Then σ does not minimize the distance, that is $d_F(\sigma(a), \sigma(b)) < L(\sigma)$.*

PROOF. If σ is distance minimizing , then the Morse Index form I_a^b along σ must be positive semi-definite according to the above Proposition. □

COROLLARY 3.22. *Let $\sigma : [a,b] \to M$ be a normal geodesic $\hat{in}$ a Finsler manifold M. Suppose that the Morse Index I_a^b is positive definite on $\mathfrak{X}_0[a,b]$. Then σ contains no conjugate points .*

5. Morse Index Theorem for Finsler manifolds

Next we will introduce some notions which we will need in order to prove the Morse Index Theorem. Let M a Finsler manifold and $p, q \in M$.

We note by $\Omega(M, p, q)$ the space of piecewise smooth vector fields which has the beginning point in p and the end point in q.

So for $\sigma : [0,1] \to M$, $\sigma \in \Omega(M,p,q)$ if and only if:

(1) exists a sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ in $[0,1]$ such that $\sigma|_{[t_{i-1}, t_i]}$ is smooth for $i = \overline{1,k}$.
(2) $\sigma(0) = p$, $\sigma(1) = q$.

By the tangent space to Ω in a curve $\omega \in \Omega$ we will understand the vector space of the vector fields piecewise smooth,with $W(0) = W(1) = 0$. We shall note this space with $T\Omega_\sigma$.

DEFINITION 3.23. *Let $\sigma : [0,1] \to M$ in a Finsler manifold M. The points p and q are conjugate along σ if there exists a non-zero Jacobi field J alongσ with $J(p) = J(q) = 0$.*

The multiplicity of p and q as conjugate points along σ is equal with the dimension of the vector space of such kind of Jacobi fields.

We recall that the nullity (the null space) of the Morse Index form consists by the vectors $\xi_q \in T\Omega_\sigma$

$$I_0^1(\xi, \eta) = 0, \forall \eta \in T\Omega_\sigma.$$

The nullity of I_0^1 is the ν-dimension of the null space. I_0^1 is degenerate if $\nu > 0$.

We saw that a vector field $W_1 \in T\Omega_\sigma$ is in the null space of the Morse Index form if and only if W_1 is a Jacobi field. We can state the following proposition.

PROPOSITION 3.24. *I_0^1 is degenerate if and only if $p = \sigma(0)$ and $q = \sigma(1)$ are conjugate along σ. The nullity of I_0^1 is equal with the multiplicity of p and q as conjugate points.*

PROOF. The proof is obvious. □

It follows that the nullity of I_0^1 is finite. It also follows that there exists only a finite number of Jacobi fields linear independent along σ.

Observation. The nullity ν satisfies $0 \leq \nu < n$.

The index λ of the Morse Index form

$$I_0^1 : T\Omega_\sigma \times T\Omega_\sigma \to \mathbb{R}$$

is the maximum dimension of the subspace of $T\Omega_\sigma$ on which I_0^1 is negative definite.

THEOREM 3.25. *(The Morse Index Theorem for Finsler manifolds) The Index λ of the Morse Index form I_0^1 is equal with the number of points $\sigma(t)$, with $0 < t < 1$ with the property that $\sigma(t)$ and $\sigma(0)$ are conjugate points along σ, every such a point being counted with its multiplicity. That number is always finite.*

PROOF. Every point $\sigma(t)$ is contained in an open set U such that every two points from U are joined by a minimal geodesic which depends differentiable of its endpoints. We choose a division of $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\sigma|_{[t_{i-1}, t_i]}$ is in a such kind of set U; it follows that every $\sigma|_{[t_{i-1}, t_i]}$ is minimal geodesic.

Let $T\Omega_\sigma(t_0, t_1, \dots, t_k) \in T\Omega_\sigma$ be the vector space of the vector fields V along σ such that:

(1) $V|_{[t_{i-1},t_i]}$ is a Jacobi field along $\sigma|_{[t_{i-1},t_i]}$ for every i;
(2) V is zero at the ends of the interval $t = 0$, $t = 1$.

$T\Omega_\sigma(t_0, \dots, t_k)$ subspace finite dimensional of the space of Jacobi fields along σ.

Let $\mathcal{T}' \in T\Omega_\gamma$ be the vector space consisting of the vector fields $V \in T\Omega_\sigma$ such that $V(t_0) = 0$, $V(t_1) = 0, \dots, V(t_n) = 0$.

Lemma 3.26. *The vector space $T\Omega_\sigma$ can be written as a direct sum $T\Omega_\sigma(t_0, \dots, t_k) \oplus \mathcal{T}'$. This subspaces are mutually orthogonal with respect to the scalar product defined by I_0^1. Moreover, the restriction of I_0^1 to $\mathcal{T}'$ is positive definite.*

Proof. For a vector field $W \in T\Omega_\gamma$ let J_1 the unique Jacobi field with the property that $J_1(t_i) = W(t_i)$, $i = \overline{0,k}$. It is clear that $J_1 - W \in \mathcal{T}'$.

Thus these two spaces $T\Omega_\sigma(t_0, \dots, t_k)$ and $\mathcal{T}'$ generate $T\Omega_\sigma$ and have in their intersection only the null vector field.

For $J_1 \in T\Omega_\sigma(t_0, \dots, t_k)$ şi $W \in \mathcal{T}'$ the Morse Index form is

$$I_0^1(J_1, W) = \langle \nabla_{T^H} J_1^H | W \rangle - \int_0^1 \langle W | 0 \rangle = 0,$$

i.e. these two spaces are orthogonal.

It remains to proof that $I_0^1(W, W) \geq 0$ for $W \in \mathcal{T}'$

$$I_0^1(W, W) \geq I_0^1(J_1, J_1) = 0$$

We prove that $I_0^1(W, W) > 0$, $W \in \mathcal{T}'$, $W \neq 0$. Suppose that $I_0^1(W, W) = 0$.

Then W is in the null space of I_0^1.

But the null space of I_0^1 consists of Jacobi fields only. Because $\mathcal{T}'$ contains only the null Jacobi fields it follows that $W = 0$.

Finally it follows that $I_0^1|_{\mathcal{T}' \times \mathcal{T}'} > 0$. □

From these relations follows the following lemma:

LEMMA 3.27. *The index (nullity) of I_0^1 is equal to the index (nullity) of the restriction of I_0^1 to the space $T\Omega_\sigma(t_0, \ldots, t_k)$ of broken Jacobi fields. Particularly, the index λ is always finite because $T\Omega_\sigma(t_0, \ldots, t_k)$ is a finite dimensional vector space.*

PROOF. Let σ_τ be the restriction of σ to the interval $[0, \tau]$. Then $\sigma_\tau : [0, \tau] \to M$ is a geodesic from $\sigma(0)$ to $\sigma(\tau)$. Let $\lambda(\tau)$ the index of the Morse Index form I_0^τ associated to this geodesic. We are interested in $\lambda(1)$.

I. $\lambda(\tau)$ is a monotone function.

For $\tau < \tau'$ there exists a space of dimension $\lambda(\tau)$ $\mathcal{V}$ of vector fields along σ_τ which are zero in $\sigma(0)$ and $\sigma(\tau)$ such that the Morse Index form I_0^τ is negative definite on ν. Any vector field from $\mathcal{V}$ can be extended to a vector field along $\sigma_{\tau'}$ which is constant null between $\sigma(\tau)$ and $\sigma(\tau')$. In that way we obtained a vector space $\lambda(\tau)$-dimensional of vector fields along $\sigma_{\tau'}$ and $I_0^{\tau'}$ is negative definite on it. It follows that $\lambda(\tau) \leq \lambda(\tau')$.

II. $\lambda(\tau) = 0$ for t small enough.

For τ small enough σ_τ is a minimal geodesic and $\lambda(\tau) = 0$ (it does not contain conjugate points).

Next we shall study the discontinuities of $\lambda(\tau)$. First we will show that $\lambda(\tau)$ is left-continuous.

III. For ε small enough $\lambda(\tau - \varepsilon) = \lambda(\tau)$.

$\lambda(1)$ can be interpreted as the index of a quadratic form defined on the finite dimensional vector space $T\Omega_\sigma(t_0, \ldots, t_k)$. Suppose that $t_i < \tau < t_{i+1}$. The index $\lambda(\tau)$ is in fact the index of the form I_0^τ on the corresponding vector space of broken Jacobi fields along σ_τ. This is constructed using the subdivision $0 = t_0 < t_1 < \cdots < t_i < \tau$ of $[0, \tau]$. Because a broken Jacobi field is unique determined by its value in its broken points $\sigma(t_i)$ this vector space is isomorphic to the direct sum

$$\Sigma = TM_{\sigma(t_1)} \oplus \cdots \oplus TM_{\sigma(t_i)}.$$

Σ does not depend of τ. the quadratic form I_0^τ depends continuously of τ on Σ.

Now I_0^τ is negative definite on a subspace $V \leq \Sigma$ of dimension $\lambda(\tau)$. For τ' closely enough to τ, $I_0^{\tau'}$ is negative definite on $V \;\Rightarrow \Longrightarrow \lambda(\tau') \geq \lambda(\tau)$. But for $\tau' = \tau - \varepsilon < \tau \;\Rightarrow\; \lambda(\tau-\varepsilon) < \lambda(\tau) \;\Rightarrow\; \lambda(\tau-\varepsilon) = \lambda(\tau)$.

IV. Let ν be the nullity of the Morse Index form I_0^τ. For $\varepsilon > 0$ small enough we have

$$\lambda(\tau+\varepsilon) = \lambda(\tau) + \nu.$$

□

The function $\lambda(t)$ "jumps" with ν when the variable t goes through a discontinuity point with multiplicity ν and in the other points is continuous. These completes the assertion in the Index Theorem.

LEMMA 3.28. *Let $\lambda(t+\varepsilon) \leq \lambda(\tau) + \nu$.*

PROOF. Let I_0^τ and Σ as in the proof of the assertion III in the previous lemma.

$$\dim \Sigma = n_i$$

I_0^τ is positive definite on a subspace $V' \subset \Sigma$ of dimension $n_i - \lambda(\tau) - \nu$. For τ' close enough to τ, $I_0^{\tau'}$ is positive definite on V'. It follows that

$$\lambda(\tau') \leq \dim \Sigma - \dim V' \leq \lambda(\tau) + \nu.$$

We first prove that $\lambda(\tau+\varepsilon) > \lambda(\tau) + \nu$.

Let $W_1, \ldots, W_{\lambda(\tau)}$ $\lambda(\tau)$-vector fields along σ_τ which are zero at the end-points such that the matrix $(I_0^\tau(W_i, W_j))$ is negative definite. Let $J_1, \ldots, J_\nu$ be ν-linear independent Jacobi fields along σ_τ, which are zero at the end-points. The vectors $\nabla_{T^H} J_n^H \in T\widetilde{M}_{\sigma(t)}$ are linear independent. We can choose $X_1, \ldots, X_\nu$ ν- vector fields such that the matrix $(\langle \nabla_{T^H} J_n^H | X_k(\tau) \rangle)$ is the identity matrix $\nu \times \nu$.

We extend these vector fields J_h and W_i to $\sigma_{\tau+\varepsilon}$ by the condition to be zero for $\tau \leq t \leq \tau + \varepsilon$.

Using the second variation formula we have:

$$I_0^{\tau+\varepsilon}(J_h, W_i) = 0 \ ,$$

$$I_0^{I+\varepsilon}(J_h, X_k) = 2\delta_{hk} \ ,$$

where δ_{hk} is the Kronecker symbol.

Let now c small enough and consider $\lambda(\tau)+\nu$ vector fields

$$W_1, \dots, W_{\lambda(\tau)}, c^{-1}J_1 - cX_1, \dots, c^{-1}J_\nu - cX_\nu$$

along $\gamma_{\tau+\varepsilon}$. We show that these vector fields generate a vector space of dimension $\lambda(\tau)+\nu$ on which the quadratic form $I_0^{\tau+\varepsilon}$ is negative definite.

The matrix of $I_0^{\tau+\varepsilon}$ with respect to this base is:

$$\begin{pmatrix} I_0^\tau(W_i, W_j) & cA \\ cA^t & -4I + i^2 B \end{pmatrix},$$

with A and B matrix which not depends of c. For c small enough this matrix is negative definite.

This proves the assertion IV. □

The Morse Index theorem follows clearly now from the assertions II, III and IV. □

6. Morse Index Form where the ends are submanifolds

The results of this section are from [**Pet**].

Now let $P \subset M$ be a submanifold of M of dimension k and consider $\sigma : [a,b] \to M$ be a normal geodesic in M with $\sigma(a) \in P$ and $\dot\sigma^H(a)$ be in the normal bundle of P (i.e. $\dot\sigma^H(a) \perp (T_{\dot\sigma(a)}P)^H$).

Let $\widetilde{\mathfrak{X}}^P = \mathfrak{X}^P[a,b]$ be the vector space of all piecewise smooth vector fields X along σ such that $X^H(a) \in T_{\dot\sigma(a)}\widetilde{P}$ and let $\mathfrak{X}^P$ be the subspace of $\widetilde{\mathfrak{X}}^P$ consisting of these X such that X^H is orthogonal to $\dot\sigma^H$ along the curve and $X(b)=0$.

In this case the Morse index form becomes $I^P : \widetilde{\mathfrak{X}}^P \times \widetilde{\mathfrak{X}}^P \longrightarrow \mathbb{R}$,

(2) $$I^P(X,Y) = \langle \nabla_{T^H} X^H, Y^H \rangle_T \Big|_a^b - \langle \mathbb{I}_T(X^H, Y^H), T^H \rangle_T \Big|_a$$

$$- \int_a^b \langle \nabla_{T^H}\nabla_{T^H} X^H - \Omega(T^H, X^H)T^H, Y^H \rangle_T dt.$$

We need to prove that I^P is symmetric. Because of (2) we have only to prove that

$$\langle \mathbb{I}_T(X^H, Y^H), T^H \rangle \Big|_a = \langle \mathbb{I}_T(Y^H, X^H), T^H \rangle \Big|_a.$$

But

$$\mathbb{I}_T(X^H,Y^H) = \nabla_{X^H}Y^H - \nabla^*_{X^H}Y^H$$

$$\mathbb{I}_T(Y^H,X^H) = \nabla_{Y^H}X^H - \nabla^*_{Y^H}X^H.$$

Now

$$\begin{aligned}\mathbb{I}_T(X^H,Y^H) - \mathbb{I}_T(Y^H,X^H) &= \nabla_{X^H}Y^H - \nabla_{Y^H}X^H - (\nabla^*_{X^H}Y^H - \nabla^*_{Y^H}X^H)\\ &= [X^H,Y^H] + \theta(X^H,Y^H) - ([X^H,Y^H]^* - \theta^*(X^H,Y^H)).\end{aligned}$$

But this Lie brackets and torsions are all vertical vectors, and so orthogonal to T^H. This implies that

$$\langle \mathbb{I}_T(X^H,Y^H), T^H\rangle_T = \langle \mathbb{I}_T(Y^H,X^H), T^H\rangle_T \ ,$$

and follows that the Morse index form is symmetric.

Here is the first main difference from the Riemannian case, because the second fundamental form in the Finslerian case is not symmetric (only for totally geodesic submanifolds, see [**Dra86**]), but the Morse index form is symmetric.

If we consider a piecewise smooth curve $\sigma : [a,b] \to M$ we obtain the following expression for the Morse index form:

$$I^P(X,Y) = \int_a^b \langle \Omega(T^H,X^H)T^H - \nabla_{T^H}\nabla_{T^H}X^H, Y^H\rangle_T dt \tag{3}$$

$$+\langle \nabla_{T^H}X^H, Y^H\rangle_T\Big|_a^b - \langle \mathbb{I}_T(X^H,Y^H), T^H\rangle_T\Big|_a +$$

$$+\sum_{i=1}^{N-1}\langle (\nabla_{T^H}X^H)^+\Big|_{t_i} - (\nabla_{T^H}X^H)^-\Big|_{t_i}, Y^H\rangle_T \ ,$$

where $a = t_0 < \cdots < t_N = b$ is a partition of $[a,b]$ such that σ is smooth on each interval $[t_i,t_{i+1}]$, $i = \overline{0,N-1}$.

It is easy to see that σ is a stationary point for the energy functional defined on the set $\Omega_{P,\sigma(b)}$ of all piecewise smooth curves $\sigma : [a,b] \to M$ joining P and $\sigma(b)$. The vector space $\widetilde{\mathfrak{X}}^P$ is a subspace of the tangent space of $\Omega_{P,\sigma(b)}$ and $I^P|_{\mathfrak{X}^P}$ is a symmetric bilinear form given by the second variation of the energy at the stationary point σ. We want to describe the index of I^P in $\mathfrak{X}^P$ defined as follows. If $\mathcal{E}$ is a subspace of $\widetilde{\mathfrak{X}}^P$, then the index of I^P in $\mathcal{E}$ is the number

$$\text{ind}\,(I^P, \mathcal{E}) = \sup\{\dim(\mathcal{B}) : \ \mathcal{B} \text{ is a subspace of } \mathcal{A} \text{ with } I^P|_{\mathcal{B}} < 0\}$$

and we set

$$\text{ind}\,(I^P) = \text{ind}\,(I^P, \mathfrak{X}^P).$$

The number ind (I^P) will be called the Morse index of σ.

Remember (Definition 3.7) Jacobi field along a geodesic $\sigma : [a, b] \to M$ is a vector field J which satisfies the Jacobi equation

$$\nabla_{T^H} \nabla_{T^H} J^H - \Omega(T^H, J^H) T^H \equiv 0 \tag{4}$$

where $J^H(t) = \chi_{\dot\sigma(t)}(J(t))$.

$\dot\sigma$ and $t\dot\sigma$ are Jacobi fields; the first one never vanishes, the second one vanishes only at $t = 0$.

Definition 3.29. [**Pet**] *A P-Jacobi field J is a Jacobi field which satisfies in addition*

$$J(a) \in T_{\sigma(a)} P$$

and

$$\langle \nabla_{T^H} J^H + A_{T^H} J^H, Y^H \rangle_T \Big|_a = 0 \tag{5}$$

for all $Y \in (T_{\sigma(a)} P)^H$, where A_{T^H} is the operator defined by

$$\langle A_{T^H} X^H, Y^H \rangle_T = \langle \mathbb{I}_T(X^H, Y^H), T^H \rangle_T.$$

The last condition means in fact that

$$\nabla_{T^H} J^H + A_{T^H} J^H \in ((T_{\sigma(a)} P)^H)^\perp \ .$$

The dimension of the vector space of all P-Jacobi fields along σ is equal to n and the dimension of the vector space of the Jacobi fields satisfying

$$\langle J^H, T^H \rangle = 0$$

is equal to $n - 1$.

If P is a point, then a P-Jacobi field is a Jacobi field J along σ such that $J(a) = 0$.

Two points $\sigma(t_0)$ and $\sigma(t_1)$, $t_0, t_1 \in [a, b]$ are said to be conjugate along σ if there exists a nonzero Jacobi field J along σ with $J(t_0) = 0$ and $J(t_1) = 0$. A point $\sigma(t_0)$, $t_0 \in [a, b]$ is said to be a P-focal point along σ if there exists a non-null P-Jacobi field J along σ with $J(t_0) = 0$. The geometrical multiplicity $\mu^P(t_0)$ of a P-focal point $\sigma(t_0)$ is the dimension of the vector space of all P-Jacobi field along σ that vanish in t_0. If $\sigma(t_0)$ is not P-focal point we set $\mu^P(t_0) = 0$.

Analogously with the classical case the set of all P-focal points along σ is discrete, hence finite.

If $J_1 \dots J_n$ is a basis for the space of P-Jacobi fields along σ and $l_1 \dots l_n$ is a parallelly transported orthogonal basis in $(T_{\sigma(t)}M)^H$ along $\dot\sigma$ then the smooth function $f(t) = \det(\langle J_i, l_j\rangle)$ has only simple zeroes in $[a, b]$, i.e. zeroes of finite multiplicity exactly at those points $t_0 \in [a, b]$ such that $\sigma(t_0)$ is a P-focal point along σ. Analogously for all $\sigma(t_0)$ the set of points which are conjugate to $\sigma(t_0)$ is finite.

We describe now the kernel of $I^P|_{\mathfrak{X}^P}$. Let

$$\mathcal{J}_0 = \{P\text{-Jacobi field } J \text{ along } \sigma : \ J(b) = 0\}.$$

Lemma 3.30. [**Pet**] *Let (M, F) be a Finsler manifold and $P \subset M$ be a submanifold of M. The kernel of the restriction of the bilinear form I^P to $\mathfrak{X}^P$ is equal to $\mathcal{J}_0$.*

Proof. A P-Jacobi field that vanishes at a point on $[a, b]$ has the property that J^H is orthogonal to T^H and so $\mathcal{J}_0 \subset \mathfrak{X}^P$.

If $X \in \mathfrak{X}^P$ is in the $\operatorname{Ker} I^P|_{\mathfrak{X}^P}$ it follows that $\nabla_{T^H}\nabla_{T^H}X^H - \Omega(T^H, X^H)T^H$ is parallel to T^H and that X satisfies equation (5).

Since $\nabla_{T^H}\nabla_{T^H}X^H - \Omega(T^H, X^H)T^H$ is also orthogonal to T^H it follows that X is a Jacobi field.

This means that $\operatorname{Ker} I^P|_{\mathfrak{X}^P} = \mathcal{J}_0$. □

Lemma 3.31. [**Pet**] *Let (M, F) be a Finsler manifold and $\sigma : [a, b] \to M$ be a geodesic, and $P \subset M$ be a submanifold of M. Suppose that there are no P-focal points along σ. Let $X, J \in \widetilde{\mathfrak{X}}^P$ be vector fields orthogonal to σ*

with X a P-Jacobi field such that $X(b) = J(b)$. Then

$$I^P(X,X) \geq I^P(J,J)$$

with equality iff $X = J$.

PROOF. Set $k = \dim P$. For $i = \overline{1,k}$ we choose Jacobi fields J_i such that $J_i^H(a)$ are a basis for $(T^H_{\sigma(a)}P)^H$ and such

$$\nabla_{T^H} J^H|_a = -A^P_{T^H} J^H|_a.$$

For $i = k+1, \ldots, n-1$ choose Jacobi fields such that $J_i(a) = 0$ and the vectors $\nabla_{T^H} J^H|_a$ form a basis in $((T_{\sigma(a)}P)^H)^\perp \cap (T^H(a))^\perp$.

Then J_i's form a basis of the space of P-Jacobi fields orthogonal to σ.

Define now $\overline{J}_i = J_i$ for $i = \overline{1,k}$ and $\overline{J}_i(t) = J_i(t)/(t-a)$, $J_i^H(a) = (\nabla_{T^H} J^H)|_a$, $i = \overline{k+1, n-1}$. Because there are no P-focal points along σ and because

$$\dot{\sigma}^H(a) \perp (T_{\sigma(a)}P)^\perp, (T_{\sigma(a)}M)^H = (T_{\sigma(a)}P)^H \oplus ((T_{\sigma(a)}P)^H)^\perp$$

it follows that the vectors $\overline{J}_i(t)$ form a basis for $(\dot{\sigma}^H(t))^\perp$ for $t \in [a,b]$.

$$I^P(J,X) = \langle \nabla_{T^H} J^H, X^H \rangle \Big|_a^b = \langle \nabla_{T^H} J^H, J^H \rangle \Big|_a^b = I^P(J,J) \tag{6}$$

The Morse index form I^P is positive definite if the normal geodesic $\sigma : [a,b] \to M$ contains no P-focal points σ and it is length minimizing among nearby curves. Then I^P is positive semidefinite on $\widetilde{\mathfrak{X}}^P \setminus \mathcal{J}_0$.

Assume that $X \in \widetilde{\mathfrak{X}}^P$ such that $I^P(X,X) = 0$. Take $Y \in \widetilde{\mathfrak{X}}^P$, then for any $\varepsilon \in \mathbb{R}^+$

$$0 \leq I(X + \varepsilon Y, X + \varepsilon Y) = \varepsilon[2I(X,Y) + \varepsilon I(Y,Y)]$$

Dividing by ε and letting $\varepsilon \to 0^+$ (respectively $\varepsilon \to 0^-$) we get $I^P(X,Y) \geq 0$ (respectively $I^P(X,Y) \leq 0$) and so $I^P(X,Y) = 0$. That means that $X \in \operatorname{Ker} I^P = \mathcal{J}_0$, that means that $J(b) = 0$ in contradiction with the fact that σ contains no P-focal points.

For $X \neq J$ we have now

$$0 < I^P(X-J, X-J) = I(X,X) - 2I(X,J) + I(J,J) = I(X,X) - I(J,J)$$

□

We need the following definition.

DEFINITION 3.32. [**Pet**] *A partition* $a = t_0 < t_1 < \cdots < t_N = b$ *of* $[a, b]$ *is said to be normal if the following conditions are satisfied*

(a) for all $i \geq 1$ *and all* $t \in (t_i, t_{i+1}]$, *the point* $\sigma(t)$ *is no conjugate to* $\sigma(t_i)$ *along* σ

(b) for all $t \in (t_0, t_1]$ *the point* $\sigma(t)$ *is not* P*-focal along* σ.

Since the set of all P-focal points along σ is finite it is easy to see that exists $\delta > 0$ such that every partition $t_0, \ldots, t_N$ of $[a, b]$ with $t_{i+1} - t_i \leq \delta$ for all i is finite.

Given a normal partition we define the subspaces of $\mathfrak{X}^P$

$$\mathfrak{X}_0^P = \{X \in \mathfrak{X}^P : \mathfrak{X}(t_i) = 0, \ \forall \ i \geq 1\} \tag{7}$$

$$\mathfrak{X}_J^P = \{X \in \mathfrak{X}^P : \mathfrak{X}|_{[t_i, t_{i+1}]} \text{ is Jacobi } \forall \ i \geq 1 \text{ and } X|_{[t_0, t_1]} \text{ is } P\text{-Jacobi}\}.$$

We define

$$\phi : \mathfrak{X}_J^P \to \bigoplus_{i=1}^{N-1} (\dot{\sigma}^H(t_i))^{\perp} \tag{8}$$

given by setting $\phi(X) = (X(t_1), X(t_2), \ldots, X(t_{N-1}))$. Since $\sigma(t_i)$ and $\sigma(t_{i+1})$ are non-conjugate for $i \geq 1$ then $X|_{[t_i, t_{i+1}]}$ is unique determined by the values $X(t_i), X(t_{i+1})$; since $\sigma(t_1)$ is not P-focal $X|_{[t_0, t_1]}$ is uniquely determined by the value $X(t_1)$. It follows that ϕ is an isomorphism.

This shows that $\mathfrak{X}_0^P \cap \mathfrak{X}_J^P = \{0\}$ and that $\mathfrak{X}_0^P + \mathfrak{X}_J^P = \mathfrak{X}^P$, hence we have

$$\mathfrak{X}_0^P \oplus \mathfrak{X}_J^P = \mathfrak{X}^P. \tag{9}$$

7. Morse Index Theorem with one variable endpoint

Now we prove the Morse Index theorem with one variable end point.

THEOREM 3.33. [**Pet**] *Let* (M, F) *be a Finsler manifold,* P *a submanifold of* M *and* $\sigma : [a, b] \to M$ *a geodesic with* $\sigma(a) \in P$ *and* $\dot{\sigma}^H(a) \in ((T_{\sigma(a)}P)^H)^{\perp}$. *Then*

$$ind\ I^P = \sum_{t_0 \in (a,b)} \mu^P(t_0) < \infty.$$

PROOF. For $[\alpha,\beta] \subset [a,b]$ let $I_{[\alpha,\beta]}$ be the bilinear form of (2), the restricted Morse index form (2) for the restricted geodesic $\sigma|_{[\alpha,\beta]}$. For $t \in (a,b)$ we write $i(t) = \text{ind } (I^P_{[a,t]})$, $i(b) = \text{ind } (I^P)$. The function $i : [a,b] \to \mathbb{N}$ is non-decreasing.

We show that $i(t)$ is piecewise constant left-continuous on $[a,b]$ and that $i(t^+) - i(t^-) = \mu^p(t)$ for all $t \in (a,b)$.

Let $t \in (a,b)$ be fixed and choose a normal partition $t_0, \dots, t_N$ on $[a,b]$ such that $t \in (t_i, t_{i+1})$ for some $i \geq 1$ (we allow $t = t_{i+1}$ if $t = b$ and we set $i = N-1$).

Let us denote $\mathfrak{X}^P_J([a,t])$ and $\mathfrak{X}^P_0([a,t])$ the spaces defined in (7), replacing the interval $[a,b]$ by $[a,t]$ (and using the normal partition $t_0, \dots, t_i$ of $[a,t]$).

The direct sum (9) is orthogonal with respect to the inner product $I^P_{[a,t]}$ i.e. $I^P_{[a,t]}(X_0, X_J) = 0$ for all $X_0 \in \mathfrak{X}^P_0([a,t])$ and $X_J \in \mathfrak{X}^P_J([a,t])$ which follows from (2).

For $X \in \mathfrak{X}^P_0([a,t])$.

$$I^P_{[a,t]}(X,X) = I^P_{[t_0,t_1]}(X,X) + \sum_{j=1}^{i-1} I_{[t_j,t_{j+1}]}(X,X) + I_{[t_i,t]}(X,X).$$

In the inequality $I^P(X,X) > I^P(J,J)$ we take the Jacobi field $J \equiv 0$ and it follows that $I^P(X,X) > 0$ i.e.

$$I^P_{[a,t]}\Big|_{\mathfrak{X}^P_0[a,t]} \geq 0.$$

It follows that

$$i(t) = \text{ind } (I^P_{[a,t]}) = \text{ind } (I^P_{[a,t]}, \mathfrak{X}^P_J([a,t]).$$

As in (8) the space $\mathfrak{X}^P_J([a,t])$ is isomorphic to the space $\mathfrak{X}_*$ defined by

$$\mathfrak{X}_* = \bigoplus_{j=1}^{i} (\dot{\sigma}^H(t_j))^\perp.$$

We denote this isomorphism by

$$\phi_t : \mathfrak{X}^P_J([a,t]) \to \mathfrak{X}_* \ .$$

If $s \in (t_i, t_{i+1}]$ the arguments above can be repeated by replacing t with s (the space $\mathfrak{X}_*$ will be the same). We can use the isomorphism ϕ_s between

$\mathfrak{X}_J^P([a,s])$ and $\mathfrak{X}_*$ to define a symmetric bilinear from I_s on $\mathfrak{X}_*$ corresponding to $I^P_{[a,s]}$. Clearly $i(s) = \text{ind }(I_s)$.

We have a one parameter family of symmetric bilinear forms on the fixed finite dimensional space $\mathfrak{X}_*$ and it is not difficult to see that I_s depends continuously on s.

We decompose $\mathfrak{X}_* = \mathfrak{X}_*^- \oplus \mathfrak{X}_*^0 \oplus \mathfrak{X}_*^+$ where I_t is positive (negative) definite on $\mathfrak{X}_*^+(\mathfrak{X}_*^-)$ and $\mathfrak{X}_*^0 = \operatorname{Ker} I_t$. We assume that the decomposition is I_t orthogonal.

$$i(t) = \dim \mathfrak{X}_* \ .$$

Because of the orthogonality of the decomposition $\mathfrak{X}_0^P([a,t]) \oplus \mathfrak{X}_J^P([a,t])$ with respect to $I^P_{[a,t]}$ it follows that the kernel of the restriction of $I^P_{[a,t]}$ to $\mathfrak{X}_J^P([a,t])$ is the intersection of $\mathfrak{X}_J^P([a,t])$ and $\operatorname{Ker} I^P_{[a,t]}$, the last one being computed by Lemma 1. $\mathcal{J}_0 \subset \mathfrak{X}_J^P([a,t])$ and denote $\mathcal{J}_*$ the subspace of $\mathfrak{X}_*$ which corresponds to $\mathcal{J}_0$ (i.e. $\mathcal{J}_* = \phi_t(\mathcal{J}_0)$). It is clear that $\mathfrak{X}_*^0 = \mathcal{J}_*$ and $\dim \mathcal{J}_*$ is just the multiplicity $\mu^P(t)$ of $\sigma(t)$ as a P-focal point.

By the continuous dependence of I_s on s we see that for $\varepsilon > 0$ sufficiently small and $s \in [t-\varepsilon, t+\varepsilon]$, I_s is negative definite on $\mathfrak{X}_*^-$ so that is $i(s) \geq i(t)$. For $s \in [t-\varepsilon, t]$ we have also $i(s) \leq i(t)$ so it follows that $i(s) = i(t)$, i.e. i is constant on $[t-\varepsilon, t]$. This means that i is left continuous.

Suppose now that $t < b$. The same continuity argument shows that there exists $\varepsilon > 0$ such that I_s is positive definite on $\mathfrak{X}_*^+$ for $s \in [t, t+\varepsilon]$, so that $i(s)$ is bounded above by the codim $\mathfrak{X}_*^+$. For $\sigma(t)$ not P-focal point this is equal to $i(t)$ so $i(s) = i(t)$ for $s \in [t-\varepsilon, t+\varepsilon]$.

If $\sigma(t)$ is a P-focal point we only obtain, using the same arguments, that $i(s) \leq i(t) + \mu_\sigma^P(t)$. Let $s \in [t, t_{i+1}]$ and $X = (x_1, \dots, x_i) \in \mathfrak{X}_*$.

Let $X_1 \in \mathfrak{X}_J^P([a,t])$ and $X_2 \in \mathfrak{X}_J^P([a,b])$ be the vector fields corresponding to $X \in \mathfrak{X}_*$ i.e. $X_1 = \phi_t^{-1}(X)$, $X_2 = \phi_s^{-1}(X)$. Extend X_1 to zero on $[t,s]$. It follows then $I_t(X,X) = I^P_{[a,s]}(X_1,X_1)$ and $I_s(X,X) = I^P_{[a,s]}(X_2,X_2)$. The vector fields X_1, X_2 differ at most in the interval $[t_i, s]$. The restriction of X_1 to $[t_i, t]$ is the unique Jacobi field such that $X_1(t_i) = v_i$ and $X_1(t) = 0$ while the restriction of X_2 to $[t_i, s]$ is the unique Jacobi field such that $X_2(t_i) = v_i$

and $X_2(s) = 0$. We have

$$I_t(X,X) - I_s(X,X) = I_{[t_i,s]}(X_1,X_1) - I_{[t_i,s]}(X_2,X_2).$$

Apply now the Lemma 4 to the geodesic $\sigma|_{[t_i,s]}$ (with starting and ending point interchanged) for the Jacobi X_2, vector field X_1 and submanifold equal to the point $\sigma(s)$. It follows that

$$I_t(X,X) \geq I_s(X,X).$$

The inequality is strict if $X_i \neq 0$. But this holds for $X \in \mathcal{J}_*$ and $X \neq 0$ because the corresponding vector field $\phi_t^{-1}(X)$ on $\mathfrak{X}_J^P([a,t])$ is an unbroken Jacobi vector field. We conclude that $I_s(X,X) < 0$ for $X \in \mathcal{J}_*$, $X \neq 0$ and hence for all nonzero $X \in \mathfrak{X}_*^- \oplus \mathcal{J}_*$ which implies that I_s is negative definite on this space and $i(s) \geq i(t) + \mu^P(t)$. □

8. Morse Index Theorem with two variable endpoints

We extend now the Morse Index Theorem to the case of two variable endpoints. For this we now assume that P and Q are submanifolds of $M, \sigma : [a,b] \to M$ is a geodesic with $\sigma(a) \in P$, $\dot\sigma^H(a) \in ((T_{\sigma(a)}, P)^H)^\perp$, $\sigma(b) \in Q$, $\dot\sigma^H(b) \in ((T_{\sigma(b)}Q)^H)^\perp$.

Let us denote by $\mathfrak{X}^{(P,Q)}$ the vector space of all piecewise smooth vector fields X along σ such that X^H is orthogonal to $\dot\sigma^H$, $X(a) \in T_{\sigma(a)}P$, $X(b) \in T_{\sigma(b)Q}$. We consider the following symmetric bilinear form

$$(10)\ . \qquad I^{(P,Q)}(X,Y) = I^P(X,Y) + \langle \mathbb{I}_T^Q(X,Y), T^H\rangle_T\Big|_b$$

Let $\mathcal{J}^Q$ denote the subspace of $\mathfrak{X}^{(P,Q)}$ consisting of all P-Jacobi fields and $\mathcal{A}$ be the symmetric bilinear form on $\mathcal{J}^Q$ obtained by the restriction of $I^{(P,Q)}$. It follows that

$$\mathcal{A}(J_1,J_2) = \langle \mathbb{I}_T^Q(J_1,J_2), T^H\rangle_T\Big|_b + \langle \nabla_{T^H} J_1^H, J_2^H\rangle\Big|_b, \quad J_1, J_2 \in \mathcal{J}^Q\ .$$

For $t \in [a,b]$ we introduce

$$\mathcal{J}[t] = \{J(t) :\ J \text{ is } P\text{-Jacobi}\} \subset T_{\sigma(t)}M.$$

For $t \in (a,b]$, $\sigma(t)$ is not P-focal if

$$\mathcal{J}[t] = T_{\sigma(t)}M.$$

Now we can prove the extension of Morse Index Theorem for geodesics between submanifolds.

THEOREM 3.34. [**Pet**] *Let (M, F) be a Finsler manifold, P, Q be submanifolds of M and $\sigma : [a, b] \to M$ be a geodesic such that $\sigma(a) \in P$, $\dot\sigma^H(a) \in ((T_{\sigma(a)}P)^H)^\perp$, $\sigma(b) \in Q$, $\dot\sigma(b) \in ((T_{\sigma(b)}Q)^H)^\perp$. Suppose that $\mathcal{J}[b] \supset T_{\sigma(b)}Q$. Let $\mathcal{U}$ be a subspace of $\mathfrak{X}^{(P,Q)}$ which contains the space of P-Jacobi fields along σ in $\mathfrak{X}^{(P,Q)}$. Then*

$$ind\ (I^{(P,Q)}, \mathcal{U}) = ind\ (I^P, \mathfrak{X}^P \cap \mathcal{U}) + ind\ (\mathcal{A}, \mathcal{J}).$$

PROOF. $\mathfrak{X}^P$ is the subspace of $\mathfrak{X}^{(P,Q)}$ consisting of those vectors V such that $V(b) = 0$ moreover the restriction of $I^{(P,Q)}$ to $\mathfrak{X}^P$ is precisely I^P. Defining $\mathcal{J}_0$ as above, let $\mathcal{J}_1$ be a subspace of $\mathcal{J}^Q$ such that $\mathcal{J}^Q = \mathcal{J}_0 \oplus \mathcal{J}_1$. It is clear that $\mathfrak{X}^{(P,Q)} = \mathfrak{X}^P + \mathcal{J}_1$ $(T_{\sigma(b)}Q \subset \mathcal{J}(b))$. From (10) it follows that this decomposition is $I^{(P,Q)}$ orthogonal i.e. $I^{(P,Q)}(X, J) = 0$ for all $X \in \mathfrak{X}^P$ and $J \in \mathcal{J}_1$. Since $\mathcal{J}_1 \subset \mathcal{U}$ we have that $\mathcal{U} = (\mathcal{U} \cap \mathfrak{X}^P) \oplus \mathcal{J}_1$. It follows that

$$\text{ind}\ (I^{(P,Q)}, \mathcal{U}) = \text{ind}\ (I^P, \mathfrak{X}^P \cap \mathcal{U}) + \text{ind}\ (\mathcal{A}, \mathcal{J}_1).$$

To finish the proof we simply observe that ind $(\mathcal{A}, \mathcal{J}_1)$ = ind $(\mathcal{A}, \mathcal{J})$ because $\mathcal{J}_0 \subset \text{Ker}\,(\mathcal{A})$. ◻

CHAPTER 4

Warped Product of Finsler Manifolds

1. Introduction

In Riemannian (semi-Riemannian) geometry the warped product of Riemannian (semi-Riemannian) manifolds is an important tool which helps to construct geometrical models of theoretical physics. It is the case, for example of Robertson-Walker space-time, which is a relativistic model of the flow of a perfect fluid and for Schwarzschild geometry, which is the simplest relativistic model of a universe with a single star — it gives a model for the solar system better than any Newtonian model, and it also gives the simplest model for the black hole (see [**O'N83**]).

In this chapter we construct the warped product of Finsler manifolds. Let M and N be two Finsler manifolds with Finsler metrics F_1, F_2 resp., $M \times N$ be the product manifold and let $f : M \longrightarrow \mathbb{R}^+$ be a smooth function, called the warped function. The function $F : \widetilde{M} \times \widetilde{N} \longrightarrow \mathbb{R}$, defined by

$$F(v_1, v_2) = \sqrt{F_1^2(v_1) + f^2(\pi_1(v_1))F_2^2(v_2)}$$

is a Finsler metric on the product manifold $M \times N$, except the property that it is not smooth on the vectors of the form $(v_1, 0)$ and $(0, v_2) \in TM \times TN$. It is smooth on $\widetilde{M} \times \widetilde{N}$, not on $TM \times TN$, because F is is not smooth on the vectors of the form $(v_1, 0)$ and $(0, v_2) \in TM \times TN$. We construct, by using the Cartan connections of the manifolds M and N, a linear connection on the direct sum of horizontal bundles of M, N, resp. By using the geometry of M, N resp. their Cartan connections, and the properties of the warping function we describe the geometry of the warped Finsler manifold $(M \times_f N, F)$. Then the covariant derivatives are computed (Theorem 4.7), and the geodesics of warped product are characterized (Theorem 4.9). We introduce the notion of umbilical point and the umbilical submanifold (Definition 4.3) in Finsler geometry and we show that the leaves of a warped product are totally geodesic submanifolds, and the fibers are totally umbilical submanifolds (Corollary 4.8). Also the curvatures are computed in this chapter (Theorem 4.10). The results here are from our work [**KPV01**].

2. Preliminaries

In this Chapter we use again the Cartan connection. First we prove some special properties of the Cartan connection.

The Cartan connection does not verify the Koszul formula for all vectors, but this formula is true for the horizontal ones, as is shown in the next Lemma:

LEMMA 4.1. [**KPV01**] *Let (M, F) be a Finsler manifold with Cartan connection ∇. For $X, Y, Z \in \mathcal{H}$ the following relation holds:*

$$2\langle\nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle.$$

PROOF. For the first three terms we use the metrical property of the Cartan connection, and for the last three terms we use the relation satisfied by the torsion as follows:

$$\begin{aligned}
X\langle Y, Z\rangle &= \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle;\\
Y\langle Z, X\rangle &= \langle\nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle;\\
Z\langle X, Y\rangle &= \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle;\\
[Y, Z] &= \nabla_Y Z - \nabla_Z Y - \theta(Y, Z);\\
[Z, X] &= \nabla_Z X - \nabla_X Z - \theta(Z, X);\\
[X, Y] &= \nabla_X Y - \nabla_Y X - \theta(X, Y).
\end{aligned}$$

Summing up and using the fact that for horizontal vectors $\langle X, \theta(Y, Z)\rangle$ is zero because $\theta(Y, Z)$ is vertical for horizontal vectors Y, Z we obtain the Koszul formula. □

We are interested in some properties of the curvature of Cartan connection listed below.

LEMMA 4.2. *Let (M, F) be a Finsler manifold. The curvature of the Cartan connection satisfies the following properties for horizontal vectors X, Y, Z, V, W:*

(1) $R(X,Y) = -R(Y,X)$;
(2) $\langle R_V(X,Y), W\rangle = -\langle R_W(X,Y), V\rangle$;
(3) $R_Z(X,Y) + R_X(Y,Z) + R_Y(Z,X) = 0$;
(4) $\langle R_V(X,Y), W\rangle = \langle R_X(V,W)X, Y\rangle$.

The proof of the previous Lemma can be found in [**AP94**, p. 31], and [**Mat86**, p. 72].

Let P be a submanifold of M of dimension $p < n$ and let us consider $F^* = F|_{TP}$; it is a Finsler metric and thus P becomes a Finsler space. Let $\widetilde{x} \in \widetilde{P}$ and let $P^*_{\widetilde{x}}$ be the $\langle\cdot,\cdot\rangle_{\widetilde{x}}$ orthogonal complement of $T_{\widetilde{x}}TP$ in $T_{\widetilde{x}}TM$. Let $P^\perp$ be the disjoint union of all $P^\perp_{\widetilde{x}}$, $\widetilde{x} \in \widetilde{P}$ and let $\pi^\perp : \widetilde{P}^\perp \to \widetilde{P}$ the natural projection. Then $(P^\perp, \pi^\perp, \widetilde{P})$ admits a natural structure of real differentiable vector bundle, $\operatorname{rank} P^\perp = n - p$. It is the normal bundle of the submanifold P.

Let $\widetilde{X}^*, \overline{Y}$ be respectively a tangent vector field on $\widetilde{P}$ and a cross section in $T\widetilde{P}$ and $\widetilde{X}^*, \overline{Y}^*$ prolongations to $T\widetilde{M}$. Then the restriction of $\nabla_{\widetilde{X}^*}\overline{Y}$ to $T\widetilde{P}$ does not depend upon the choice of prolongations and is denoted by $\nabla^*_{\widetilde{X}}\overline{Y}$. The bundle direct sum decomposition

$$T\widetilde{M} = T\widetilde{P} \oplus P^\perp$$

leads to the Gauss–Weingarten formulae:

$$\nabla_{\widetilde{X}}\overline{Y} = \nabla^*_{\widetilde{X}}\overline{Y} + \mathbb{I}(\widetilde{X}, \overline{Y})$$
$$\nabla_{\widetilde{X}}\overline{\xi} = -\widetilde{A}_{\overline{\xi}}\widetilde{X} + \nabla^\perp_{\widetilde{X}}\overline{\xi}$$

Here $\xi \in \operatorname{Sec}(\widetilde{P}, P^\perp)$ and a similar argument (independence of extensions of $\widetilde{X}, \overline{\xi}$ to $T\widetilde{P}$) leads to the notation $\nabla_{\widetilde{X}}\overline{\xi}$. Then ∇^* is the induced connection, $\mathbb{I}$ the second fundamental form, $\widetilde{A}_{\overline{\xi}}$ the operators of Weingarten and $\nabla^\perp$ is the normal connection ([**Bej99, ADiH88, Dra86**]). Next we define the umbilical point of a Finsler submanifold and the umbilical submanifold.

Definition 4.3. [**KPV01**] *A point* $q \in P$ *is an umbilical point if there exists a vector* $Z \in \mathcal{H}^{\perp}(P)$ *such that* $\mathbb{I}(X,Y) = \langle X, Y\rangle Z$. *The submanifold* P *is said to be totally umbilical if every point of* P *is an umbilical point.*

3. Construction of the warped product

The following results are from [**KPV01**]. Let (M, F_1) and (N, F_2) be Finsler manifolds with Cartan connections ∇^1 and ∇^2, and let $f : M \longrightarrow \mathbb{R}_+$ be a smooth function. Let $p_1 : M \times N \longrightarrow M$, and $p_2 : M \times N \longrightarrow N$. We consider the product manifold $M \times N$ endowed with the metric $F : \widetilde{M} \times \widetilde{N} \longrightarrow \mathbb{R}$,

$$F(v_1, v_2) = \sqrt{F_1^2(v_1) + f^2(\pi_1(v_1))F_2^2(v_2)}.$$

We show that the metric defined above is really a Finsler metric. First it is clear that F is smooth on $\widetilde{M} \times \widetilde{N}$, because F_1 and F_2 are. F is not necessary smooth on the vectors of the form $(v_1, 0)$and$(0, v_2) \in TM \times TN$. This means that F is not a really Finsler metric on the product manifold $M \times N$, therefore the study should be restricted to the domain $\widetilde{M} \times \widetilde{N}$. Secondly F is homogeneous with respect to the vector variables because F_1 and F_2 are. Third, the Hessian of F with respect to the vector variables is of the form:
$\begin{pmatrix} A & 0 \\ 0 & f^2B \end{pmatrix}$ where A and B are the Hessians of the Finsler metrics F_1 and F_2. So the Hessian of F is positive because the Hessians of F_1 and F_2 are. It means that the indicatrix of F is strongly convex. The difference between this metric and a classical Finsler metric is that it not smooth on the vectors of the form $(v_1, 0)$ and$(0, v_2)$.

The product manifold $M \times N$ with the metric $F(v) = F(v_1, v_2)$, for $v = (v_1, v_2) \in \widetilde{M} \times \widetilde{N}$ defined above will be called warped product of the manifolds M, N, and f will be called the warping function. We denote this warped product by $M \times_f N$. We just showed that $(M \times_f N, F)$ is really a Finsler manifold.

Our goal is to express the geometry of warped product by the geometries of M, N and the warping function f. The study follows the line adopted in Riemannian and semi-Riemannian cases [**O'N83**], with the specific situation due to the Finslerian context. In the Finsler case we have no a natural splitting property as in the Riemannian case [**BCS00**, p. 361] but we work on the liftings of the horizontal spaces of M and N. On that spaces we construct the connection.

The manifold M will be called base and the manifold N will be called fiber as in [**O'N83**].

4. The gradient of a function in Finsler geometry

In this section we define the gradient of the smooth function $f : M \longrightarrow \mathbb{R}_+$ with $df_x \neq 0$. We follow the line of Shen [**She01**, p. 37]. Define ∇f_x by

$$\nabla f_x := L_x^{-1}(df_x)$$

where $L_x : T_xM \longrightarrow T_x^*M$ is the Legendre transformation. Shen proves that

$$\nabla f^H = \widehat{\nabla} f$$

where $\widehat{\nabla} f$ is the gradient of f with respect to Riemannian metric induced by the Finsler metric, and

$$F(\nabla f) = \sqrt{\langle \widehat{\nabla} f, \widehat{\nabla} f \rangle_{\nabla f}}.$$

We work with ∇f^H, the horizontal lifting of ∇f which has the property that $F^2(\nabla f) = \langle \nabla f^H, \nabla f^H \rangle_{\nabla f^H}$.

Next we define the Hessian of a function.

DEFINITION 4.4. *The Hessian of a function $f \in \mathcal{F}(M)$ is its second covariant differential $\mathcal{H}^f = \nabla(\nabla f)$.*

LEMMA 4.5. [**KPV01**] *The Hessian $\mathcal{H}^f$ satisfy the following relation:*

$$\mathcal{H}^f(X, Y) = XYf - (\nabla_X Y)f = \langle \nabla_X(\nabla f^H), Y \rangle$$

for $X, Y \in \mathcal{H}$.

PROOF.

$$\mathcal{H}^f(X, Y) = \nabla(df^H)(X, Y) = \langle \nabla_X \nabla f^H, Y \rangle$$

since $Yf = \langle \nabla f^H, Y\rangle$ and it follows that

$$\begin{aligned} XYf &= X\langle \nabla f^H, Y\rangle = \langle \nabla_X \nabla f^H, Y\rangle + \langle \nabla f^H, \nabla_X Y\rangle \\ &= \langle \nabla_X(\nabla f^H), Y\rangle + (\nabla_X Y)f \end{aligned}$$

which implies the assertion. □

If f is smooth on M (i.e. $f : M \longrightarrow \mathbb{R}$ is smooth), the lift of f to $M \times N$ is the map $\widehat{f} := f \circ p_1 : M \times N \longrightarrow \mathbb{R}$. If $a \in T_pM$ and $q \in N$ then the lift $\widehat{a}$ of a to (p, q) is the unique vector in $T_{(p,q)}(M \times q)$ such that $dp_1(\widehat{a}) = a$. If $X \in \mathfrak{X}(M)$ the lift of X to $M \times N$ is the vector field $\widehat{X}$ whose value at each (p, q) is the lift of X_p to (p, q). Because of the product coordinate systems it is clear that $\widehat{X}$ is smooth. It follows that the lift of $X \in \mathfrak{X}(M)$ is the unique element of $\mathfrak{X}(M \times N)$ that is p_1-related to X and p_2-related to the zero vector field on N. The same method could be used to lift objects defined on N to $M \times N$.

Now we prove a Lemma needed in what follows:

Lemma 4.6. [**KPV01**] *If h is a smooth function on M, then the gradient of the lift $h \circ p_1$ of h to $M \times_f N$ is the lift to $M \times_f N$ of the gradient of h on M.*

Proof. Let $v \in TN$. Now $\langle \nabla(h \circ p_1), v^H\rangle = v^H(h \circ p_1) = 0$.

Next for $x \in TM$ we have that

$$\langle d\widetilde{p}_1((\nabla(h\circ p_1))^H), d\widetilde{p}_1(x)\rangle = \langle(\nabla(h\circ p_1))^H, x^H\rangle = (x(h\circ p_1))^H = \langle(\nabla h)^H, dp_1(x)^H\rangle.$$

From these two properties it follows the assertion in the theorem. □

Due to this theorem there will be no confusion if we denote h and ∇h instead of for $h \circ p_1$ and $\nabla(h \circ p_1)$, resp.

5. Properties of warped metrics

Let (M, F_1) and (N, F_2) be two Finsler manifolds, with Finsler metrics F_1, F_2 resp. We consider the product manifold $M \times N$ and the warped metric defined above. We consider the projections $p_1 : M \times N \longrightarrow M$ and $p_2 : M \times N \longrightarrow N$ and the canonical projections $\pi_1 : TM \longrightarrow M$

and $\pi_2 : TN \longrightarrow N$. The projections p_1, p_2 resp. generate the projections $dp_1 : TM \times TN \longrightarrow TM$ and $dp_2 : TM \times TN \longrightarrow TN$, for $v = (v_1, v_2) \in TM \times TN, dp_i(v_1, v_2) = v_i$, $i = 1, 2$.

It is obvious that the fibers $p \times N = p_1^{-1}(p), p \in M$ and the leaves $M \times q = p_2^{-1}(q), q \in N$ are Finsler submanifolds of $M \times_F N$ and the warped metric has the properties:

(1) for each $q \in N$ the map $p_1|_{(M\times q)}$ is an isometry onto M.
(2) for each $p \in M$ the map $p_2|_{(p\times N)}$ is a positive homothety onto N with scale factor $\frac{1}{f}$.
(3) for each $(p, q) \in M \times N$ the leaf $M \times q$ and the fiber $p \times N$ are orthogonal with respect to the Riemannian metrics induced by the Finsler metrics.

The canonical projection π_1 gives rise to the vertical bundle $(\mathcal{V}_1, \widetilde{\pi_1}, TM)$, where $\mathcal{V}_1 = \ker(d\pi_1)$ and $\widetilde{\pi_1} = d\pi_1 : TTM \longrightarrow TM$.The same is true for the manifold N. Now we have that

$$d\pi_1 \times d\pi_2 = d(\pi_1 \times \pi_2) : TTM \times TTN = T(TM \times TN) \longrightarrow TM \times TN$$

and $\ker d(\pi_1 \times \pi_2) = \ker d\pi_1 \oplus \ker d\pi_2$. It follows that the vertical space of the manifold $M \times N$, $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, so the Riemannian metrics $\langle \cdot, \cdot \rangle^1$ and $\langle \cdot, \cdot \rangle^2$, defined on $\mathcal{V}_1$ and $\mathcal{V}_2$ as in the introduction give rise to a Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathcal{V}$ as follows: $\langle \cdot, \cdot \rangle_v = \langle \cdot, \cdot \rangle^1_{v_1} + f^2(\pi_1(v_1)) \langle \cdot, \cdot \rangle^2_{v_2}$. Now let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the horizontal spaces with respect to the Cartan connections ∇^1 and ∇^2 on the Finsler manifolds (M, F_1) and (N, F_2), resp.

We have the direct sum decomposition

$$TT(M \times N) = TTM \oplus TTN = \mathcal{V}_1 \oplus \mathcal{H}_1 \oplus \mathcal{V}_2 \oplus \mathcal{H}_2.$$

Next the Finsler metrics F_1, F_2 on the manifolds M and N resp. generate the Riemannian metrics $\langle , \rangle^1$ and $\langle , \rangle^2$ on the vertical spaces $\mathcal{V}_1$ and $\mathcal{V}_2$, resp. By the horizontal maps these Riemannian metrics are mapped onto horizontal spaces $\mathcal{H}_1, \mathcal{H}_2$ resp. Finally these Riemannian metrics generates a Riemannian metric on $T(TM \times TN)$. In what it follows we work mostly on the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ the direct sum of the liftings of $\mathcal{H}_1$ and $\mathcal{H}_2$ to the $TTM \times TTN$.

The following theorem relates the Cartan connections of M and N to the Cartan connection of $M \times_f N$.

THEOREM 4.7. [**KPV01**] *On* $B = M \times_f N$ *if* $X, Y \in \mathfrak{X}(\mathcal{H}_1)$ *and* $V, W \in \mathfrak{X}(H_2)$ *the following relations are true:*

(1) $\nabla_X Y$ *on* $\mathcal{H}_1 \oplus \mathcal{H}_2$ *is the lift of* $\nabla_X Y$ *on* $\mathcal{H}_1$.
(2) $\nabla_X V = \nabla_V X = (Xf/f)V$.
(3) *nor* $\nabla_V W = \mathbb{I}(V, W) = -(\langle V, W\rangle/f)\nabla f^H$.
(4) $\theta(X, V) = \theta(V, X) = 0$.
(5) *tan* $\nabla_V W \in \mathfrak{X}(N)$ *is the lift of* $\nabla_V W$ *on* N.

PROOF. We apply the Koszul formula (see Lemma 4.1) for $2\langle\nabla_X Y, V\rangle$ and we obtain that it is equal to $-V\langle X, Y\rangle + \langle V, [X, Y]\rangle$ because $[X, V] = [Y, V] = 0$. Because X, Y are lifts from M, $\langle X, Y\rangle$ is constant on fibers (liftings on N), and because $V \in T\widetilde{N}$ follows that $V\langle X, Y\rangle = 0$. Analogously $\langle V, [X, Y]\rangle = 0$. Thus $\langle\nabla_X Y, V\rangle = 0$ for all $V \in \mathfrak{X}(N)$ and it follows formula 1.

First we prove the first equality from 2. The second one will be proved after 3. We have that $X\langle V, Y\rangle = \langle\nabla_X V, Y\rangle + \langle V, \nabla_X Y\rangle = 0$, so $\langle\nabla_X V, Y\rangle = -\langle V, \nabla_X Y\rangle$. We apply the Koszul formula for $2\langle\nabla_X V, W\rangle$, and we observe that all the terms vanish except $X\langle V, W\rangle$.

It follows from the expression of the Riemannian metric induced by the warped metric that $\langle V, W\rangle(v, w) = f^2(\pi_1(v))\langle V_w, W_w\rangle$. This term is constant on leaves. Thus $X\langle V, W\rangle = X(f^2(\pi_1(v))\langle V_w, W_w\rangle) = 2fX(f(\pi_1(v)))\langle V_w, W_w\rangle = 2(\frac{Xf}{f})\langle V, W\rangle$. From these relations we have that $\nabla_X V = (\frac{Xf}{f})V$. Now $\nabla_X V - \nabla_V X = [X, V] + \theta(X, V)$. We can assume that $[X, V] = 0$.

It is obvious that $V\langle W, X\rangle = 0$. But this means that

$$\langle\nabla_V W, X\rangle = -\langle W, \nabla_V X\rangle = -\langle W, (Xf/f)V + \theta(X, V)\rangle = -(Xf/f)\langle V, W\rangle$$

because $\theta(X, V)$ is vertical. Now $\langle\nabla f^H, X\rangle = Xf$. Thus

$$\langle\nabla_V W, X\rangle = -\langle((\langle V, W\rangle/f)\nabla f^H, X\rangle.$$

This yields 3.

$$\begin{aligned}\langle \nabla_V X, W\rangle &= -\langle X, \nabla_V W\rangle = -\langle X, \langle V, W\rangle / f \nabla f^H\rangle \\ &= \frac{1}{f}\langle X, \nabla f^H\rangle\langle V, W\rangle = \langle\langle X, \nabla f^H\rangle / f V, W\rangle.\end{aligned}$$

The above gives the second part of *2* and it follows that

$$\nabla_V X = \nabla_X V = (\frac{Xf}{f})V,$$

and the mixed part of the torsion vanishes $\theta(X, V) = \theta(V, X) = 0$. The last assertion 5 is trivial. □

It is a remarkable fact that the torsion vanishes on the mixed part. This will let us to compute the curvature of warped product.

Now the next Corollary easily follows:

COROLLARY 4.8. [**KPV01**] *The leaves $M \times q$ of a warped product are totally geodesic; the fibers $p \times M$ are totally umbilical.*

PROOF. By the claim 1 in the Theorem 4.7 in the theorem it follows that for a geodesic α in M its lifting on $M \times_f N$ is also a geodesic. The second assertion comes from 3 of Theorem 4.7. □

6. Geodesics of warped product manifolds

In a warped product manifold a curve γ can be written as $\gamma(s) = (\alpha(s), \beta(s))$ where the curves α and β are the projections of γ into M and N, resp. Now we give conditions for a curve in the warped product to be geodesic with respect to the warped metric.

THEOREM 4.9. [**KPV01**] *A curve $\gamma = (\alpha, \beta)$ in $M \times_f N$ is a geodesic if and only if*

(1) $\nabla_{\alpha'^H}\alpha'^H = \frac{||\beta'^H||^2}{f}\nabla f^H$,
(2) $\nabla_{\beta'^H}\beta'^H = \frac{-2}{f\circ\alpha}\frac{(d(f\circ\alpha))^H}{ds}\beta'^H$

PROOF. We work in an interval around $s = 0$.

Case 1. $\gamma'(0)$ is neither in $T_{\alpha(0)}M$ nor in $T_{\beta(0)}N$. Then $\alpha'(0) \neq 0$ and $\beta'(0) \neq 0$. So we can suppose that α is an integral curve for X in M and β is an integral curve for V in N. Also we denote by X and V the lifts on $M \times_f N$.

It follows that γ is a geodesic curve if and only if $\nabla_{X^H+V^H}(X^H+V^H)=0$. But this means that

$$\nabla_{X^H}X^H+\nabla_{X^H}V^H+\nabla_{V^H}X^H+\nabla_{V^H}X^H=0.$$

Now we use Theorem 4.7 from the previous section and we have that

$$\nabla_{X^H}X^H-\frac{||V^H||^2}{f}\nabla f^H=0$$

and

$$2\frac{X^Hf}{f}V+\nabla_{V^H}V^H=0.$$

Case 2. Suppose that $\gamma'(0)\in T_{\alpha(0)}M$. If γ is a geodesic, because $M\times\beta(0)$ is totally geodesic, it follows that γ remains in $M\times\beta(0)$. Thus β is constant and the assertions of the theorem are trivial. Conversely if condition (2) from Theorem 4.7 holds, since $\beta'(0)=0$ it follows that β is constant. Then condition (1) in Theorem 4.7 implies that α is a geodesic, and so is γ.

Case 3. Suppose that $\gamma'(0)\in T_{\beta(0)}N$ and nonzero. Suppose that ∇f is not zero, because otherwise $\alpha(0)\times N$ is totally geodesic and the conclusion follows as in the *Case 1.* Now if γ is a geodesic, it follows that on no interval around 0 γ remains in the totally umbilical fiber $p\times N$. It follows that there is a sequence $\{s_i\}\to 0$ such that for all i, $\gamma'(s_i)$ is neither in $T_{\alpha(s_i)}M$ or in $T_{\beta(s_i)}N$. The assertions in the theorem follows by continuity from the first case. Conversely, if (1) in the theorem is true it follows that $\nabla_{\alpha'(0)^H}\alpha'(0)^H\neq 0$ hence there exists a sequence $\{s_i\}$ as above, and using again the first case it follows that γ is a geodesic. □

7. Curvature of warped product manifolds

Now we express the curvature of the warped product. The curvature tensor is defined by the relation

$$R_Z(X,Y)=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z.$$

Because the projection p_1 is an isometry it follows that the lift of the curvature on M is equal to the curvature of the warped product when is computed for vectors from on $\mathcal{H}_1$.

THEOREM 4.10. [**KPV01**] *Let $M \times_f N$ be a warped product of Finsler manifolds with curvature tensor R and let $X, Y, Z \in \mathcal{H}_1$ and $U, V, W \in \mathcal{H}_2$. Let$R^M_Z$ and R^N_U denote the curvature tensors of the manifolds (M, F_1) and (N, F_2) resp. The following relations are true:*

(1) $R_Z(X, Y) \in \mathfrak{X}(\mathcal{H}_1)$ *is the lift of* $R^M_Z(X, Y)$ *on* M.
(2) $R_Y(V, X) = -(\frac{H^f(X,Y)}{f})V$, *where* H^f *is the Hessian of* f.
(3) $R_X(V, W) = (Xf/f)\theta(V, W)$.
(4) $R_W(X, V) = (\frac{\langle V,W\rangle}{f})\nabla_X(\nabla f)$.
(5) $R_U(V, W) = R^N_U(V, W) - (\frac{\langle \nabla f, \nabla f\rangle}{f^2})\{\langle V, U\rangle W - \langle W, U\rangle V\}$.

PROOF. 1. This is true because the projection p_1 is an isometry and the leaves are totally geodesic.

2. Because $[V, X] = 0$ it follows that $\nabla_V \nabla_X Y - \nabla_X \nabla_V Y = R_Y(V, X)$. By Theorem 4.7 we have that $\nabla_V \nabla_X Y = (\frac{(\nabla_X Y)f}{f})V$ because $\nabla_X Y \in \mathfrak{X}(\mathcal{H}_1)$. The second term

$$\begin{aligned}\nabla_X \nabla_V Y &= \nabla_X(\frac{Yf}{f}V) = X(Yf/f)V + (Yf/f)\nabla_X V \\ &= [(XY)f/f + YfX(1/f)]V + (Yf/f)(Xf/f)V.\end{aligned}$$

Because $X(1/f) = -Xf/f^2$ the last expression reduces to $(XYf/f)V$. Thus

$$R_Y(V, X) = -[(XYf - (\nabla_X Y)f)/f]V = -(H^f(X, Y)/f)V.$$

3. We can assume that $[V, W] = 0$. It follows that

$$R_X(V, W) = \nabla_V \nabla_W X - \nabla_W \nabla_V X.$$

But

$$\nabla_V \nabla_W X = \nabla_V((Xf/f)W) = V(Xf/f)W + (Xf/f)\nabla_V W.$$

Now $V(Xf/f) = 0$ because Xf/f is constant on the fibers. This implies that

$$R_X(V, W) = (Xf/f)[\nabla_V W - \nabla_W V] = (Xf/f)\theta(V, W).$$

We note that $R_X(V, W) \in \mathcal{V}_2$ by the properties of the Cartan connection.

By the symmetry of curvature $\langle R_V(X,Y),W\rangle = \langle R_X(V,W),Y\rangle = 0$ because $R_X(V,W)$ is vertical. Now we use 2, the curvature symmetries, and then we obtain that relation 3 is true.

4. We have that $\langle R_W(X,V),U\rangle = \langle R_X(W,U),W\rangle = 0$ because of the point above. We use here the properties from Lemma 4.2. Now $R_X(V,W)$ is vertical and it follows that

$$\begin{aligned}\langle R_W(V,X),Y\rangle &= \langle R_Y(V,X),W\rangle = H^f(X,Y)\langle V,W\rangle \\ &= (\langle V,W\rangle/f)\langle \nabla_X(\nabla f),Y\rangle,\end{aligned}$$

which gives assertion 4.

5. Again we can assume that $[U,V]$ is zero.

$$\begin{aligned}R(V,W)U &= \nabla_V\nabla_W U - \nabla_W\nabla_V U = \nabla_V\{-(\langle W,U\rangle/f)\nabla f^H + \nabla_V^N U\} \\ &\quad -\nabla_W\{-(\langle V,U\rangle/f)\nabla f^H + \nabla_V^N U\} = -(\langle \nabla_V W,U\rangle \\ &\quad +\langle W,\nabla_V U\rangle)(\nabla f^H/f) - (\langle W,U\rangle/f)\nabla_V(\nabla f^H) \\ &\quad +\nabla_V\nabla_W^N U + (\langle \nabla_W V,U\rangle + \langle V,\nabla_W U\rangle)(\nabla f^H/f) \\ &\quad +(\langle V,U\rangle/f)\nabla_W(\nabla f^H) - \nabla_W\nabla_V^N U = (\langle \nabla_W V - \nabla_V W,U\rangle \\ &\quad -\langle W,\nabla_V U\rangle - \langle V,\nabla_W U\rangle)(\nabla f^H/f) + \nabla_V^N\nabla_W^N U - \nabla_W^N\nabla_V^N U \\ &\quad -(\langle V,\nabla_W^N U\rangle/f)\nabla f^H + (\langle W,\nabla_V^N U\rangle_)(\nabla f^H) \\ &\quad +(\langle V,U\rangle/f)(\langle \nabla f^H,\nabla f^H\rangle/f) - (\langle W,U\rangle/f)(\langle \nabla f^H,\nabla f^H\rangle/f)V \\ &= R^N(V,W)U + \frac{\langle \nabla f^H,\nabla f^h\rangle}{f^2}(\langle V,U\rangle W - \langle W,U\rangle V)\ .\end{aligned}$$

We use that $\langle V,\nabla_W U\rangle = \langle V,\nabla_W^N U\rangle$, and the properties from Theorem 4.7. Thus we have

$$R_U(V,W) = R_U^N(V,W) + (\frac{\langle \nabla f^H,\nabla f^h\rangle}{f^2})(\langle V,U\rangle W - \langle W,U\rangle V).$$

$\square$

Összefoglaló

Néhány Riemann geometriai eredmény általánosítása a Finsler geometriai esetre

Az utóbbi évtizedekben a Finsler geometriában számos figyelemre méltó eredmény született. Rengeteg dolgozat és több könyv látott napvilágot, és sok Riemann geometriai összefüggést sikerült általánosítani a Finsler geometriában.

Talán Paul Finsler doktori értekezése számít az első Finsler geometriai munkának (1918). Több mint egy fél évszázaddal korábban Riemann (1854) rámutatott már a Riemann geometria és a nála általánosabb, ma Finsler geometriának nevezett geometria különbségére, de az általánosabb esetet — bonyolultságára hivatkozva — elvetette. Bár a Finsler geometria a variációszámításból ered, legegyszerűbben úgy gondolhatjuk el, hogy minden egyes érintőtérben meg van adva egy norma, amely simán változik, de nem szükségképpen származik belső szorzatból. Egy Finsler sokaságon általában nem létezik lineáris és metrikus konnexió. A Riemann geometriai Levi-Civita konnexiónak az általánosításai többféleképpen képezhetők, pl. a vertikális nyalábon, vagy a második érintőnyalábon. A különféle általánosítások között a különbséget a metrikusságra, illetve a torziómentességre vonatkozó feltételek eltérő volta adja.

Az első ilyen, konnexiókra vonatkozó általánosítást J. L Synge (1925) adta, majd J. H.Taylor (1925), L. Berwald (1928)[**Ber28**], E. Cartan (1934) [**Car34**] vezetett be konnexiót Finsler térben. Ez utóbbi kompatibilis a metrikával, de a legtöbb el nem tűnő torzió tenzora van. Később S.S. Chern (1948) [**Che43, Che48, Che96**] is javasolt egy ezektől különböző konnexiót (ezt definiálta H. Rund is (lásd [**Ana96, Run59**]), amely nem teljesen kompatibilis a metrikával, de kevesebb el nem tűnő torzió tenzora van. A különféle konnexiók más-más szituációban bizonyulnak hasznosnak [**Aba96, MA94**]. Csak jóval később sikerült tisztázni e konnexióknak egymáshoz való viszonyát.

Az utóbbi időben a Finsler geometriának több fontos általánosítása született, mint a Lagrange terek, Hamilton terek, általánosított Lagrange, stb. terek [**AIM93, MA94**]. Ezek hasznosnak bizonyulnak a fizikában, mechanikában, biológiában, és több más területen. Az ilyen irányú általánosításokat elsősorban a román Finsler geometria iskola vizsgálja R. Miron vezetésével [**MA87, MA94, Mir85, Mir86, Mir89**].

Éppúgy mint a Riemann geometriában, a konstans görbületű terek a Finsler terek egy igen fontos osztályát alkotják. A negatív konstans görbületű Finsler tereket Akbar-Zadeh tanulmányozta [**AZ88**]. Ezen terek szerkezete kellőképpen tisztázott, viszont a pozitív görbületű tereké még nem. Nemrégiben Z. Shen [**She96**] és R. Bryant [**Bry02, Bry96, Bry97**] ért el az utóbbival kapcsolatban eredményeket. Bryant példákat adott a kétdimenziós gömbön pozitív konstans görbületű Finsler terekre.

Az értekezés második fejezetében pozitív biszekcionális görbületű Finsler terekre bizonyítunk néhány tulajdonságot a valós és a komplex esetben. Valós és komplex (Kaehler) Finsler sokaságok részsokaságai metszésére igazolunk tételeket pozitív biszekcionális görbület esetén, és Kaehler-Finsler sokaságok megfeleltetéseinek egybeesését vizsgáljuk. Többek közt bebizonyítjuk, hogy két kompakt, totálisan geodetikus részsokaságnak mindig van nemüres metszete, feltéve, hogy a valós, teljes összefüggő Finsler sokaság pozitív szekcionális görbületű, és a részsokaságok dimenzióinak összege eléri a sokaság dimenzióját.

Az elmúlt félévszázadban a globális Riemann geometria hatalmas fejlődésen esett keresztül. Ezért fontos, hogy ezeket mielőbb próbáljuk általánosítani a Finsler geometriai esetre, amennyiben lehetséges. Az egyik ezt lehetővé tevő figyelemre méltó tény az, hogy a Jacobi egyenlet, a második variációs formula, és az indexforma formálisan ugyanúgy néz ki, mint a Riemann geometriai megfelelője. Ez teszi lehetővé a Cartan-Hadamard tétel, a Bonnet-Myers tétel, és a Synge tétel bebizonyítását a Finsler geometriában [**AP94, Aus55, BCS00**]. A Morse index tételt is általánosították Finsler sokaságokra. (lásd [**Leh64**]). Másrészt a Riemann és szemi-Riemann geometriában igazolást nyert a Morse index tétel azon formája is, amikor a geodetikusok végpontjai előírt részsokaságokban mozoghatnak. A 3. fejezetben célunk ennek Finsler geometriai vizsgálata. Megmutatjuk, hogy bár a részsokaságok második alapformája nem szimmetrikus, a Morse indexforma mégis az, s ez kulcsfontosságúnak bizonyul a részsokaságban mozgó végpontú geodetikusra vonatkozó Morse index tétel igazolásában.

A 'warped' szorzat igen jelentős szerepet játszik a Riemann geometria relativitáselméleti alkalmazásaiban, például a Robertson-Walker tér-idő, és a Schwarzschild metrika konstrukciójában [**BO69, O'N83**]. A 4. fejezet Finsler sokaságok 'warped' szorzatának konstrukciójára vonatkozik [**KPV01**]. A konstruált metrika majdnem Finsler metrika, az egyetlen eltérés az, hogy nem minden irányban definiált, speciálisan a komponensekkel párhuzamos irányokban nem. Eredményeink megadják a komponens-sokságok Cartan konnexiói és a szorzat Cartan konnexiója közti kapcsolatot, tovább a görbületek és a geodetikus kapcsolatát. Következményként adódik, hogy az egyik komponens sokaság totálgeodetikus, míg a másik umbilikus.

Az eredmények

Frankel típusú tételek Finsler sokaságokra

J.L.Synge [**Syn36**] 1936-ban bizonyította, hogy a pozitív szekcionális görbületű páros dimenziós irányítható kompakt sokaságok egyszeresen összefüggőek. Bizonyításában az általa korábban levezetett, az ívhosszra vonatkozó második variációs formulát használta. Synge technikáját használva J. Frankel [**Fra61**] 1970-ben kezdete tanulmányozni a pozitív görbületű sokaságokat, különféle szituációkban alkalmazta, különösen a részsokaságok pozícióit vizsgálva. Többek közt azt igazolta, hogy pozitív görbületű teljes összefüggő Riemann sokaság két kompakt totálgeodetikus részsokasága mindig metszi egymást, amennyiben dimenzióik összege nagyobb, vagy egyenlő, mint a teljes sokaság dimenziója. A totálgeodetikus részsokaságok meglehetősen speciálisak, viszont a komplex esetben sokkal gyengébb feltételek mellett is sikerült levezetni a konklúziót, nevezetesen totálgeodetikus részsokaságok helyett elegendő komplex analitikus részsokaságokat tekinteni.

Ezeket az eredményeket számos esetre kiterjesztették: A. Gray [**Gra70**] a majdnem Kaehler sokaságok esetére, S. Marchiafava [**Mar90**] a kvaternionikus Kaehler sokaságokra, L. Ornea [**Orn92**] a lokálisan konform Kaehler sokaságokra, s végül T.Q. Binh, L. Ornea és L. Tamássy [**BOT99**] a pozitív szekcionális görbületű Sasaki sokaságokra.

A holomorf megfeleltetések a holomorf leképezések általánosításait jelenti, mint a komplex sokaságok többértékű leképezései. T. Frankel vizsgálta a komplex Kaehler sokaságok megfeleltetéseink fixpontjait [**Fra61**]. Azt igazolta, hogy pozitív szekcionális görbületű Kaehler sokaság tetszőleges megfeleltetésének mindig van fixpontja, azaz metszi $N \times N$ diagonálisát. Módszere szintén a második variációs formulán alapult.

A disszertációban Frankel említett eredményeit terjesztjük ki a Finsler sokaságok esetére, a részsokaságok metszésére vonatkozóan (Kozma Lászlóval közös) [**KP00**] dolgozatban publikáltuk az eredményeket, a megfeleltetésekre vonatkozóan pedig a szerző [**Pet02**] dolgozatban. Megjegyezzük, hogy míg Frankel eredménye a megfeleltetések fixpontjaira vonatkozott, itt a megfeleltetések egybeesésére sikerült igazolni állításokat. A bizonyítás menete követi a Riemann geometriai esetet, viszont több helyen bonyolultabb érvelések szükségesek a Finsler geometriai szituációnak köszönhetően.

TÉTEL. [**KP00**] *Ha V és W két totálisan geodetikus részsokasága egy valós, teljes, összefüggő, pozitív szekcionális görbülettel rendelkező (M, F) Finsler térnek, és $dim\, V + dim\, W \geq dim\, M$, akkor $V \cap W \neq \emptyset$.*

TÉTEL. [**KP00**] *Amennyiben V és W két komplex analitikus részsokasága egy pozitív holomorf biszekcionális görbülettel rendelkező (M, F) erősen Kähler Finsler sokaságnak, és $dim_{\mathbb{C}} V + dim_{\mathbb{C}} W \geq dim_{\mathbb{C}}\, M$, akkor $V \cap W \neq \emptyset$.*

Egy komplex N sokaság *holomorf megfeleltetése* nem más, mint $N \times N$ komplex analitikus részsokasága. Két (holomorf) megfeleltetésről, V és W-ről azt mondjuk, hogy egybeesésük van, ha $V \cap W \neq \emptyset$. Egy $V \subset N \times N$ holomorf megfeleltetést *transzverzálisnak* mondunk, ha $T_{(p,q)}V \oplus T_{(p,q)}(\{p\} \times N) = T_{(p,q)}(N \times N)$ és $T_{(p,q)}V \oplus T_{(p,q)}(N \times \{q\}) = T_{(p,q)}(N \times N)$ teljesül minden $(p,q) \in V$–re. Mivel $T_{(p,q)}(\{p\} \times N)$ és $T_{(p,q)}(N \times \{q\})$ ortogonálisak, azonnal következik, hogy egyik (p,q) –beli V-re ortogonális vektor sem lehet érintő $\{p\} \times N$ vagy $N \times \{q\}$-höz.

TÉTEL. [**Pet02**] *Egy pozitív holomorf biszekcionális görbülettel rendelkező, erősen Kähler Finsler N sokaság két holomorf kompakt — legalább egyikük transzverzális, — V, W megfeleltetése egybeeső, amennyiben $dim_{\mathbb{C}}V + dim_{\mathbb{C}}W \geq 2dim_{\mathbb{C}}N$.*

TÉTEL. [**Pet02**] *Legyen N egy pozitív holomorf biszekcionális görbülettel rendelkező, erősen Kähler Finsler sokaság, és $f, g : N \to N$ biholomorf leképezések. Ekkor legalább egy olyan $p \in N$ létezik, melyre $f(p) = g(p)$.*

KÖVETKEZMÉNY. [**Pet02**] *Legyen N egy pozitív holomorf biszekcionális görbülettel rendelkező, erősen Kähler Finsler sokaság, és $f : N \to N$ egy biholomorf leképezés. Ekkor f–nek legalább egy fixpontja van.*

Morse-index tételek a Finsler geometriában

Figyelemreméltó, hogy az ívhosszra vonatkozó második variációs formula és az indexforma pontosan úgy néz ki a Finsler geometriában, mint a Riemann geometriában. Segítségükkel több globális eredményt vezettek le (pl. Cartan-Hadamard tétel, Bonnet-Myers tétel, Synge tétel, stb.)[**AP94**], [**Aus55**], [**BC93**], [**BCS00**].

A Morse-index tételt is általánosította a Finsler esetre D. Lehmann [**Leh64**], lásd még Matsumoto [**Mat86**] könyvét, s a háttért illetően Milnor [**Mil63**] művét. A Riemann és szemi-Riemann geometriában a Morse-index tételt abban az esetben is vizsgálták, amikor a geodetikusok végpontjai egy részsokaságban mozognak [**Amb61**], Bolton [**Bol77**], Kalish [**Kal88**], Piccione és Tausk [**PT99**].

A disszertáció 3. fejezetében igazoljuk a Morse-index tételt, előbb a klasszikus esetben, majd amikor a végpontok megadott részsokaságokban mozoghatnak. A Riemann és a Finsler eset közötti fő különbség abban áll, hogy a részsokaságok második alapformája nem szimmetrikus. Megmutatjuk azonban, hogy a Morse indexforma mégis szimmetrikus, s ez teszi lehetővé, hogy igazoljuk a Morse féle indextételt változó végpontok esetében. Definiáljuk az energiafunkcionál variációs formuláit, majd bevezetjük a Jacobi mezőket, és a Morse indexformát, megmutatjuk alapvető tulajdonságait. A részsokaságokban mozgó végpontú geodetikusokra vonatkozó Morse-index tételt két lépésben igazoljuk, előbb az egyik végpont rögzített. Az indexet a P-Jacobi mezők felhasználásával számítjuk ki. A bizonyítás Morse eredeti [**Mil63**] és Piccione-Tausk [**PT99**] gondolatmenetét követi.

TÉTEL. *(A klasszikus Morse Index tétel Finsler sokaságokra) A I_0^1 Morse indexforma λ indexe megegyezik azon $\sigma(t)$, $(0 < t < 1)$ pontok számával, amelyekre $\sigma(t)$ és $\sigma(0)$ konjugáltak σ mentén. Minden ilyen pontot multiplicitással kell számolni. Az index véges.*

DEFINÍCIÓ. [**Pet**] *J-t P-Jacobi mezőnek nevezzük, ha olyan Jacobi mező, amely kielégíti*

$$J(a) \in T_{\sigma(a)}P$$

és

$$\langle \nabla_{T^H} J^H + A_{T^H} J^H, Y^H \rangle_T \Big|_a = 0 \tag{5}$$

feltételeket minden $Y \in (T_{\sigma(a)}P)^H$–ra, ahol az A_{T^H} operátort

$$\langle A_{T^H} X^H, Y^H \rangle_T = \langle \mathbb{I}_T(X^H, Y^H), T^H \rangle_T$$

adja meg.

TÉTEL. [**Pet**] *Legyen (M, F) egy Finsler sokaság, P pedig M-nek egy részsokasága, továbbá $\sigma : [a, b] \to M$ egy geodetikus, $\sigma(a) \in P$ and $\dot\sigma^H(a) \in ((T_{\sigma(a)}P)^H)^\perp$. Ekkor*

$$ind\ I^P = \sum_{t_0 \in (a,b)} \mu^P(t_0) < \infty.$$

TÉTEL. [**Pet**] *Legyen* (M, F) *egy Finsler sokaság,* P, Q *részsokaságai* M*–nek, és* $\sigma : [a, b] \to M$ *geodetikus, melyre* $\sigma(a) \in P$*,* $\dot{\sigma}^H(a) \in ((T_{\sigma(a)}P)^H)^\perp$*,* $\sigma(b) \in Q$*,* $\dot{\sigma}(b) \in ((T_{\sigma(b)}Q)^H)^\perp$*. Tegyük fel, hogy* $\mathcal{J}[b] \supset T_{\sigma(b)}Q$*. Legyen* $\mathcal{U}$ *egy altere* $\mathfrak{X}^{(P,Q)}$*-nek, mely tartalmazza a* σ *menti,* $\mathfrak{X}^{(P,Q)}$*–beli* P*-Jacobi mezőket. Ekkor*

$$ind\ (I^{(P,Q)}, \mathcal{U}) = ind\ (I^P, \mathfrak{X}^P \cap \mathcal{U}) + ind\ (\mathcal{A}, \mathcal{J}).$$

Finsler sokaságok 'warped' szorzata

A 'warped' szorzat fogalma a Riemann geometriában igen fontos szerepet játszik (lásd [**AB98, Che01, Che99, Che96, Kim95, N$\ddot{9}$6, Ula99**]). Segítségével elméleti fizikai példákat lehet megkonstruálni, például a Robertson-Walker tér-időt, amely a tökéletes folyadék áramlásának relativisztikus modelljét adja, továbbá a Schwarzschild geometriát, amely az egy középpontú univerzum legegyszerűbb relativisztikus modellje - jobb modell a naprendszerre, mint a newtoni (lásd [**O'N83**]).

Ezt a konstrukciót kisebb megszorításokkal ki lehet terjeszteni a Finsler sokaságok esetére. A kiterjesztést Asanov dolgozatai [**Asa98, Asa92**] is motiválják, amelyekben a relativitáselmélet bizonyos modelljei Finsler metrikák 'warped' szorzatával vannak leírva. Például, [**Asa92**]-ban az $\mathbb{R} \times M$-en adott általánosított Schwarzschild metrika tulajdonságai vannak megadva.

A 4. fejezetben két Finsler sokaság 'warped' szorzatát definiáljuk és vizsgáljuk. Célunk az, hogy a szorzat geometriáját a képzésben részvevő faktorok geometriájával írjuk le. Először a Cartan konnexiók kapcsolatát adjuk meg, majd a szorzatban haladó geodetikusokat jellemezzük. Végül a görbületi tenzorok közötti kapcsolatot vezetjük le.

Legyen (M, F_1) és (N, F_2) két Finsler sokaság, Cartan konnexióit jelölje ∇^1 és ∇^2. Legyen továbbá $f : M \longrightarrow \mathbb{R}_+$ egy sima függvény. $p_1 : M \times N \longrightarrow M$, és $p_2 : M \times N \longrightarrow N$ jelöli a projekciókat. Tekintsük az $M \times N$ szorzat–sokaságot, ellátva a $F : \widetilde{M} \times \widetilde{N} \longrightarrow \mathbb{R}$,

$$F(v_1, v_2) = \sqrt{F_1^2(v_1) + f^2(\pi_1(v_1))F_2^2(v_2)}$$

metrikával.

Könnyen látható, hogy a $p \times N = p_1^{-1}(p), p \in M$ fibrumok, illetve az $M \times q = p_2^{-1}(q), q \in N$ levelek $M \times_F N$–nek Finsler részsokaságai, és a 'warped' metrika rendelkezik a következő tulajdonságokkal:

(1) minden egyes $q \in N$–re a $p_1|_{(M\times q)}$ leképezés izometria M–re.
(2) minden egyes $p \in M$–re a $p_2|_{(p\times N)}$ leképezés pozitív homotécia N-re $\frac{1}{f}$ skálafaktorral.
(3) minden egyes $(p, q) \in M \times N$–re az $M \times q$ levél és a $p \times N$ fibrum ortogonálisak a Finsler metrika által indukált Riemann metrikára nézve.

Tétel. [**KPV01**] *$M \times_f N$–en, $X, Y \in \mathfrak{X}(\mathcal{H}_1)$ és $V, W \in \mathfrak{X}(H_2)$ esetén a következők érvényesek:*

(1) *$\nabla_X Y$ a $\mathcal{H}_1 \oplus \mathcal{H}_2$ nyalábon éppen $\nabla_X Y$ on $\mathcal{H}_1$–nek a liftje.*
(2) *$\nabla_X V = \nabla_V X = (Xf/f)V$.*
(3) *$nor \nabla_V W = \mathbb{I}(V, W) = -(\langle V, W\rangle/f)\nabla f^H$.*
(4) *$\theta(X, V) = \theta(V, X) = 0$.*
(5) *$tan \nabla_V W \in \mathfrak{X}(N)$ éppen $\nabla_V W$–nek a liftje.*

Következmény. [**KPV01**] *A 'warped' szorzat $M \times q$ levelei totálgeodetikusak; a $p \times M$ fibrumok pedig totálisan umbilikusak.*

Tétel. [**KPV01**] *Egy $M \times_f N$–beli $\gamma = (\alpha, \beta)$ görbe pontosan akkor geodetikus, ha*

(1) $\nabla_{\alpha'^H} \alpha'^H = \frac{||\beta'^H||^2}{f} \nabla f^H$,
(2) $\nabla_{\beta'^H} \beta'^H = \frac{-2}{f\circ\alpha} \frac{(d(f\circ\alpha))^H}{ds} \beta'^H$.

Tétel. [**KPV01**] *Tekintsük $M \times_f N$-en a Finsler sokaságok 'warped' szorzatát, R görbületi tenzorral. Legyen továbbá $X, Y, Z \in \mathcal{H}_1$ és $U, V, W \in \mathcal{H}_2$. Jelölje R_Z^M és R_U^N az (M, F_1), illetve (N, F_2) sokaságok görbületi tenzorait. A következő összefüggések érvényesek:*

(1) *$R_Z(X, Y) \in \mathfrak{X}(\mathcal{H}_1)$ éppen $R_Z^M(X, Y)$-nek liftje.*
(2) *$R_Y(V, X) = -(\frac{H^f(X,Y)}{f})V$, ahol H^f f–nek a Hessianja.*
(3) *$R_X(V, W) = (Xf/f)\theta(V, W)$.*
(4) *$R_W(X, V) = (\frac{\langle V,W\rangle}{f})\nabla_X(\nabla f)$.*

(5) $R_U(V,W) = R^N_U(V,W) - (\frac{\langle \nabla f, \nabla f \rangle}{f^2})\{\langle V,U\rangle W - \langle W,U\rangle V\}.$

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