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Parallelisms and Finsler structures

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Parallelisms and Finsler structures

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Contents

Introduction

1. Beginning with Euclid, the concept of a *parallelism* has played a key role in the history of geometry, and, later, differential geometry. More or less intuitively, in the Euclidean n-space $\mathbb{R}^n$ two tangent vectors $u \in T_p\mathbb{R}^n$ and $v \in T_q\mathbb{R}^n$ are parallel if u, v and the line segment between p and q 'form three sides of a parallelogram'. More precisely, u and v are parallel if there is a constant vector field X on $\mathbb{R}^n$ such that $X(p) = u$ and $X(q) = v$. 'Constant' means here that X can be linearly combined from the standard frame field

$$(E_i)_{i=1}^n, \quad E_i(p) := (p, e_i) \quad ((e_i)_{i=1}^n \text{ is the canonical basis of } \mathbb{R}^n)$$

such that the coefficients of the linear combination are real numbers.

Given any two points p, q in $\mathbb{R}^n$, define the mapping

$$\mathsf{P}_0(p, q)\colon T_p\mathbb{R}^n \to T_q\mathbb{R}^n,\ v \mapsto \mathsf{P}_0(p, q)(v) := X(q),$$

where X is the constant vector field on $\mathbb{R}^n$ specified by $X(p) = v$. Then:

(P_1) $\mathsf{P}_0(p, q) \in L(T_p\mathbb{R}^n, T_q\mathbb{R}^n)$, i.e., $\mathsf{P}_0(p, q)$ is a linear mapping from $T_p\mathbb{R}^n$ to $T_q\mathbb{R}^n$.

(P_2) $\mathsf{P}_0(a, a) = 1_{T_a\mathbb{R}^n}$, $\mathsf{P}_0(a, q) \circ \mathsf{P}_0(p, a) = \mathsf{P}_0(p, q)$ for all $a, p, q \in \mathbb{R}^n$ (*consistence*).

(P_3) Given a tangent vector $v \in T_p\mathbb{R}^n$, the mapping

$$\mathbb{R}^n \to T\mathbb{R}^n,\ q \mapsto \mathsf{P}_0(p, q)(v)$$

is smooth, and hence a vector field on $\mathbb{R}^n$ (*smoothness*).

We say that the mapping

$$\mathsf{P}_0\colon (p, q) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathsf{P}_0(p, q) \in L(T_p\mathbb{R}^n, T_q\mathbb{R}^n)$$

is the *natural parallelism* of the Euclidean n-space $\mathbb{R}^n$.

Obviously, this kind of direct comparison of two distant tangent vectors to a manifold (or, in the simplest case, to a surface in $\mathbb{R}^3$) is impossible in general. A reasonable extension of the concept of parallelism from the Euclidean plane $\mathbb{R}^2$ to a (smooth) surface in $\mathbb{R}^3$ was discovered only in 1917 by Levi-Civita. However, contrary to the natural parallelism on $\mathbb{R}^n$, Levi-Civita's parallelism is a local concept, operates along curves and is path-dependent.

In the decades after Levi-Civita's breakthrough, the theory of parallelism was quickly extended to arbitrary (finite-dimensional) smooth manifolds, mainly under the name 'affine connection' or simply 'connection'. Having this general concept (mainly due to Élie Cartan), it became also clear that the reason for the existence of an 'absolute parallelism' (that is, a sort of parallelism which satisfies the properties (P_1)–(P_3)) is, at least locally, its vanishing curvature.

2. The simple example of the Euclidean parallelism P_0 suggests also an axiomatic approach in the more general setting of smooth manifolds. Let us *define* an *absolute parallelism* (or simply *parallelism*) on a manifold M as a family

$$\mathsf{P}(p,q)\colon T_pM \to T_qM,\ (p,q) \in M \times M$$

of linear mappings between tangent spaces of M satisfying the consistency condition (P_2) and the smoothness condition (P_3) above.

Our earliest source for a definition of this sort for the notion 'absolute parallelism' ('Fernparallelismus' in German) is Werner Greub's fundamental paper entitled 'Liesche Gruppen und affin zusammenhängende Mannigfaltigkeiten' from 1961 [19]. In a similarly important and inspiring 1972 paper of Joseph A. Wolf [54, 55] we also find conditions (P_1)–(P_3) above as the axioms for 'absolute parallelism'. The definition of parallelism, taken as a base in our Thesis, is borrowed from the monograph of Greub, Halperin and Vanstone [20, 21]. Here the smoothness condition is expressed in a different, but equivalent manner. We note that in the Problems sections of this monograph most of the definitions and results from Greub's earlier paper [19] are translated to the language of vector bundles.

The axiomatic approach to parallelisms based on the requirements (P_1)–(P_3) has a serious drawback. Most manifolds (for instance, even-dimensional spheres) have a non-trivial tangent bundle, and hence have no global frame field – while ‘absolute parallelism’ and ‘global frame field’ are just two sides of the same coin. However, the class of manifolds with absolute parallelism is still quite large. All Lie groups, for example, belong to this class. It is also known that *every* 3*-dimensional, orientable smooth manifold is parallelizable* (see, e.g., Steenrod’s classic book [44] or, for a modern proof, [43]).

3. The first paper in this theme was probably written by É. Cartan in 1923 [8]. There he proved that the vanishing of the curvature tensor of a connection is the necessary condition for the existence of a (complete) absolute parallelism. The next important steps towards understanding the fine structure of manifolds with parallelism were made by Cartan and Schouten [10, 11]. In reference [10] they defined flat connections on Lie groups, thus exhibiting their absolute parallelisms. In reference [11], a local description of parallelized manifolds with compatible Riemannian metric is presented, with some gaps. (‘Compatibility’ means here that the parallel vector fields of the parallelism have constant norm, and their integral curves are Riemannian geodesics.) In his above-mentioned paper, J. A. Wolf extended the work of Cartan and Schouten to compatible pseudo-Riemannian metrics, and completed their classification theorem. In this Thesis we take one step further and investigate parallelized manifolds endowed with compatible (and strongly compatible) Finsler functions.

Before concluding this brief historical overview, we have to mention still one important moment. In 1928, Einstein proposed a unified theory of gravitation and electromagnetism based upon a parallelizable manifold admitting a compatible Riemannian metric [15, 16]. Now we quote a paragraph of Eisenhart’s paper [17]:

‘He was unaware of the existence of the requisite mathematical knowledge and developed it anew. He said: “The new unitary field theory is based on the following mathematical discovery: There are continua with a Riemannian metric and distant parallelism which nevertheless are not euclidean.” Later he gave up hope of founding a satisfactory theory on such a basis. . . .’

Nevertheless, life has not stopped. Field theories based upon manifolds with absolute parallelism form a very active area of present day research. It turned out, among others, that classical general relativity can be recast into absolute parallelism or ‘teleparallelism’ language, see, e.g., [5, 25, 33].

What is maybe more interesting, there is some hope to understand such a mystery as 'dark energy' in this framework [51].

4. There is an extensive literature on manifolds with absolute parallelisms, even if one disregards (as we do) the delicate differential topological questions. However, until now, there has been no unified treatment of the subject. For this reason, in Chapter 2 of the Thesis we have tried to give a systematic introduction to the *geometry* of parallelized manifolds, and to present complete proofs of most results. So, although this chapter lays no claim to deep originality, it is no without novelties. For example, we associate to a parallelism an Ehresmann connection, and this leads immediately to the spray of the parallelism. As a natural weakening of the concept of parallelism, we speak of covering parallelisms. Their usefulness will become clear in Chapter 3 (and just below, in the next paragraph). We would like to emphasize that a covering parallelism exists on every manifold. A quite sophisticated new concept in our Thesis is the concept of conformally conjugate parallelisms. Using that, we obtain a sufficient condition for a Finsler manifold to be Wagnerian in Chapter 3.

5. The detailed contents of the Thesis can be found in the Summary (Chapter 4) in English and in the 'Összefoglaló' (Chapter 5) in Hungarian, so we do not specify them here. In the following overview, we focus on our main results which connect parallelisms, covering parallelisms and Finsler functions.

The most natural relationship between a parallelism P on a manifold M and a Finsler function F on TM is their *compatibility*:

$$F_q \circ \mathsf{P}(p, q) = F_p, \quad \text{for all } p, q \in M$$

('Finsler norms are preserved by parallel translations'). We have the following nice property: *if a Finsler function is compatible with two conjugate parallelisms, then their (necessarily common) generated spray is the canonical spray of the Finsler manifold* (Theorem 3.7). A remarkable consequence of this result is that *a bi-invariant Finsler function on a Lie group is necessarily of Berwald type* (theorem of Latifi and Razavi [34]). We note that in their paper [13], Deng and Hou investigate bi-invariant Finsler functions on a Lie group. They gave a classification of them, they did not notice, however, that the studied structure is actually a Berwald manifold. It would be interesting to compare their result with Szabó's famous classification theorem on (positive definite) Berwald manifolds [45].

More generally, the compatibility of a Finsler function and a *covering parallelism* leads to a characterization of generalized Berwald manifolds. Namely, *a Finsler manifold belongs to the class of generalized Berwald manifolds if, and only if, the base manifold has a covering parallelism which is compatible with the Finsler function* (Theorem 3.13). This result implies that *a Lie group equipped with a left invariant Finsler function is a generalized Berwald manifold* (Theorem 3.14). If, in particular, the Finsler function is compatible with *two conjugate* covering parallelisms, then we obtain the more special class of Berwald manifolds (Theorem 3.17). By changing the conjugacy in the latter theorem to the more general condition *conformal conjugacy*, we get that the Finsler manifold is necessarily a Wagner manifold (Theorem 3.19).

Finally, we investigate the case when a Finsler function F is *strongly compatible* with a parallelism P on the base manifold. Then, by definition, F and P are compatible and have the same pregeodesics. (Actually, by Theorem 3.21, F and P have common *geodesics*.) It turns out that *a Finsler manifold admitting such a parallelism is a Berwald manifold* (Theorem 3.22). As a consequence we obtain that a left invariant Finsler function for a Lie group whose geodesics are the one-parameter subgroups and their translations must be of Berwald type (Corollary 3.23).

Our concluding result is a structure theorem for parallelized Finsler manifolds (Theorem 3.24). Suppose that (M, F) is a connected Finsler manifold, and let P be a parallelism on M, strongly compatible with F. If the geodesics of P are complete and the P-parallel vector fields form a Lie algebra, then there exists a connected and simply connected Lie group G such that

(i) M *is diffeomorphic to* G *up to a factorization by one of its discrete subgroups;*

(ii) P *is 'essentially' the left parallelism of* G*;*

(iii) F *is induced by a bi-invariant (and hence of Berwald type) Finsler function on* TG.

Observe that (i) and (ii) do not involve any Finsler function. This part is a nice result concerning parallelized manifolds discovered probably by É. Cartan [9]. Following the ideas sketched in the proof of Proposition 2.5 in [54], we gave a detailed and complete proof also of properties (i) and (ii).

Chapter 1

Preliminaries

This chapter is devoted to fix our notation and terminology, and to review some basic facts that will be needed in the Thesis. In most cases the conventions of the book [49] are followed.

1.1. Basic conventions.

(1) The identity transformation of a set S will be denoted by 1_S.

(2) The set of natural numbers (including 0) is denoted by $\mathbb{N}$. The set of positive integers is $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

(3) The symbol $\mathbb{R}$ stands for the set of real numbers.

(4) Functions are mappings whose range is a subset of $\mathbb{R}$.

(5) We write $\mathrm{M}_n(\mathbb{R})$ for the ring of all $n \times n$ matrices with entries in $\mathbb{R}$ and 1_n for the identity matrix in $\mathrm{M}_n(\mathbb{R})$. The general linear group $\mathrm{GL}_n(\mathbb{R})$ is the group of invertible matrices in $\mathrm{M}_n(\mathbb{R})$.

(6) The ring of endomorphisms of an R-module V is denoted by $\mathrm{End}(V)$.

(7) The term 'tensor' is used in a restricted sense, to refer to covariant tensors and type $(1, k)$-tensors. If V is an R-module, then $T_k(V)$ (resp. $T_k^1(V)$) denotes the R-module of covariant tensors (resp. vector-valued covariant tensors) of order k. The elements of $T_k^1(V)$ will be interpreted as k-linear mappings $V^k \to V$.

(8) By a neighbourhood of a point in a topological space we mean an open subset containing the point.

1.1 Manifolds, bundles and flows

1.2. By a *manifold* we mean a finite-dimensional smooth manifold, whose underlying topological space is Hausdorff and second countable; the letter M will always stand for an unspecified manifold of dimension n.

The simplest manifold is the Euclidean n-space $\mathbb{R}^n$, where we assume that $n \leq 1$. We shall denote the canonical basis of $\mathbb{R}^n$ and its dual by $(e_i)_{i=1}^n$ and $(e^i)_{i=1}^n$, respectively. Then $(\mathbb{R}^n, (e^i)_{i=1}^n)$ is a global chart for $\mathbb{R}^n$, which defines its canonical smooth structure. If $n = 1$, we write for this chart $(\mathbb{R}, r) := (\mathbb{R}, 1_{\mathbb{R}})$.

A smooth mapping $\varphi\colon M \to N$ is a diffeomorphism if it is bijective and its inverse is smooth as well. The set of all diffeomorphisms from a manifold M onto itself forms a group under composition, which we denote by $\mathrm{Diff}(M)$. For the real algebra of real-valued smooth functions on M we use the notation $C^\infty(M)$. A *curve* in M is a smooth mapping $\gamma\colon I \to M$, where I is an open interval of $\mathbb{R}$, regarded as an open submanifold. If it is necessary, we tacitly assume that $0 \in I$.

1.3. The *tangent space* T_pM to M at a point $p \in M$ is the real n-dimensional vector space consisting of the derivations of the real algebra $C^\infty(M)$ at p; the elements of T_pM are the *tangent vectors* at p to M. The disjoint union TM of all tangent spaces to M can be uniquely endowed with a topology and smooth structure such that the 'foot mapping'

$$\tau\colon TM \to M,\ v \mapsto \tau(v) := p \ \text{ if } \ v \in T_pM$$

is smooth. We say that TM is the *tangent manifold* of M and $\tau\colon TM \to M$ is the *tangent bundle* of M. The *slit tangent bundle* of M is $\mathring{\tau} := \tau \restriction \mathring{T}M$, where $\mathring{T}M := \{v \in TM \mid v \neq 0 \in T_{\tau(v)}M\}$.

The tangent space to $\mathbb{R}^n$ at a point p can be naturally identified with the vector space $\{p\} \times \mathbb{R}^n$; then the tangent manifold of $\mathbb{R}^n$ is simply $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.

If $\varphi\colon M \to N$ is a smooth mapping between manifolds, then its *derivative* is the bundle map $\varphi_*\colon TM \to TN$ whose restriction to a tangent space T_pM is the linear mapping $(\varphi_*)_p\colon T_pM \to T_{\varphi(p)}N$ given by

$$(\varphi_*)_p(v)(h) := v(h \circ \varphi); \quad v \in T_pM,\ h \in C^\infty(N).$$

Instead of $(\varphi_*)_p(v)$ we will often write $\varphi_*(v)$.

1.4. Let $(\mathcal{U}, u) = (\mathcal{U}, (u^i)_{i=1}^n)$ be a chart of M ($u^i := e^i \circ u$), and let D_i denote the standard ith partial derivative in $\mathbb{R}^n$. Given a point $p \in \mathcal{U}$, the functions

$$\left(\frac{\partial}{\partial u^i}\right)_p : f \in C^\infty(M) \mapsto \frac{\partial f}{\partial u^i}(p) := D_i(f \circ u^{-1})(u(p)) \in \mathbb{R}$$

are tangent vectors to M at p, and the family $\left(\left(\frac{\partial}{\partial u^i}\right)_p\right)_{i=1}^n$ is a basis of T_pM. Then

$$v = \sum_{i=1}^n v(u^i)\left(\frac{\partial}{\partial u^i}\right)_p =: v(u^i)\left(\frac{\partial}{\partial u^i}\right)_p \quad \text{for all } v \in T_pM.$$

The summation convention used here will be in force throughout the Thesis.

In particular, the single tangent vector

$$\left(\frac{d}{dr}\right)_t : f \in C^\infty(M) \mapsto \left(\frac{d}{dr}\right)_t(f) := f'(t) \in \mathbb{R}$$

forms a basis of $T_t\mathbb{R}$. The *velocity vector* of a curve $\gamma\colon I \to M$ at $t \in I$ is

$$\dot{\gamma}(t) := (\gamma_*)_t\left(\frac{d}{dr}\right)_t \in T_{\gamma(t)}M.$$

The *velocity field* of γ is the curve $\dot{\gamma}\colon I \to TM,\ t \mapsto \dot{\gamma}(t)$ in TM. If $\varphi\colon M \to N$ is a smooth mapping between manifolds, then

$$(\varphi_*)_{\gamma(t)}(\dot{\gamma}(t)) = \dot{\overline{\varphi \circ \gamma}}(t), \quad t \in I. \tag{1.1}$$

If $x^i := u^i \circ \tau$ and

$$y^i\colon TM \to \mathbb{R},\ v \mapsto y^i(v) := v(u^i),$$

then $\left(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n)\right)$ is a chart of TM called the *chart induced by* $(\mathcal{U}, u)$ and denoted simply by $(\tau^{-1}(\mathcal{U}), (x, y))$.

1.5. A *vector field* on M is a smooth section of the tangent bundle of M, i.e., a smooth mapping $X\colon M \to TM$ which satisfies $\tau \circ X = 1_M$. The $C^\infty(M)$-module of vector fields is denoted by $\mathfrak{X}(M)$. If $X \in \mathfrak{X}(M)$ and $v \in C^\infty(M)$, we define the function $Xf \in C^\infty(M)$ by

$$(Xf)(p) := X_p(f), \quad p \in M.$$

The *Lie bracket* of two vector fields $X, Y \in \mathfrak{X}(M)$ is the unique vector field $[X, Y] \in \mathfrak{X}(M)$ satisfying

$$[X, Y](f) = X(Yf) - Y(Xf), \quad f \in C^\infty(M).$$

This bracket operation makes the real vector space $\mathfrak{X}(M)$ into a Lie algebra.

1.6. A *tensor field* (or simply a *tensor*) on M is a tensor over the $C^\infty(M)$-module $\mathfrak{X}(M)$. We use the notation

$$\mathcal{T}_k(M) := T_k(\mathfrak{X}(M)),\ \mathcal{T}_k^1(M) := T_k^1(\mathfrak{X}(M)); \quad k \in \mathbb{N}$$

(cf. **1.1**(7)). Any tensor field on M can indeed be regarded as a 'field' on M, i.e., as a smooth section of a suitable vector bundle.

In particular, $\mathcal{A}_k(M) \subset \mathcal{T}_k(M)$ is the module of *alternating k-tensor fields* on M. Elements of $\mathcal{A}_k(M)$ are mentioned as *differential forms* of degree k, or *k-forms* for short. The vector space

$$\mathcal{A}(M) := \bigoplus_{k=0}^{n} \mathcal{A}_k(M)$$

is a graded algebra with the wedge product. Its classical graded derivations are the *Lie derivative* $\mathcal{L}_X$, the *substitution operator* i_X ($X \in \mathfrak{X}(M)$) and the *exterior derivative* d.

1.7. Let V be an n-dimensional real vector space. A V-valued 1-form θ on M is a smooth mapping

$$\theta\colon p \in M \mapsto \theta_p \in L(T_pM, V),$$

where $L(T_pM, V)$ is the real vector space of linear mappings from T_pM to V. If $(b_i)_{i=1}^n$ is a basis of V, then θ can be written in the form $\theta = \theta^i \otimes b_i$ (tensor product) for some 1-forms θ^i on M. The exterior derivative of θ is the V-valued 2-form $d\theta = d\theta^i \otimes b_i$ on M. For details, see [36, Section 8.4].

1.8. Let $\varphi\colon M \to N$ be a smooth mapping between manifolds. Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called *φ-related*, written $X \underset{\varphi}{\sim} Y$ if

$$\varphi_* \circ X = Y \circ \varphi. \tag{1.2}$$

If $X, Y \in \mathfrak{X}(M)$ and $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(N)$, then

$$(X \underset{\varphi}{\sim} \widetilde{X} \text{ and } Y \underset{\varphi}{\sim} \widetilde{Y}) \Rightarrow [X, Y] \underset{\varphi}{\sim} [\widetilde{X}, \widetilde{Y}].$$

This result will be quoted as the *related vector field lemma.*

If φ is a diffeomorphism, then

$$\varphi_\# X := \varphi_* \circ X \circ \varphi^{-1}$$

is the unique vector field on N which is φ-related to X. It is called the *push-forward* of X by φ. The mapping

$$\varphi_\# \colon \mathfrak{X}(M) \to \mathfrak{X}(N),\ X \mapsto \varphi_\# X$$

is an isomorphism of Lie algebras, thus

$$\varphi_\#[X, Y] = [\varphi_\# X, \varphi_\# Y]; \quad X, Y \in \mathfrak{X}(M). \tag{1.3}$$

1.9. Consider a smooth function $f \in C^\infty(M)$. The *vertical lift* of f is $f^{\mathsf{v}} := f \circ \tau \in C^\infty(TM)$, its *complete lift* is the smooth function

$$f^{\mathsf{c}} \colon v \in TM \mapsto f^{\mathsf{c}}(v) := v(f) \in \mathbb{R}.$$

Given a vector field X on M, there is a unique vector field X^{v} on TM such that

$$X^{\mathsf{v}} \underset{\tau}{\sim} X \quad \text{and} \quad X^{\mathsf{v}} f^{\mathsf{c}} = (Xf)^{\mathsf{v}} \quad \text{for all } f \in C^\infty(M).$$

This vector field is called the *vertical lift* of X. The *complete lift* of X is the unique vector field X^{c} on TM satisfying

$$X^{\mathsf{c}} f^{\mathsf{v}} = (Xf)^{\mathsf{v}} \quad \text{and} \quad X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}} \quad \text{for all } f \in C^\infty(M).$$

The vertical and complete lifts of smooth vector fields on M generate the $C^\infty(TM)$-module $\mathfrak{X}(TM)$. We define a type $(1,1)$ tensor field

$$\mathbf{J} \in \mathcal{T}^1_1(TM) \cong \operatorname{End}(\mathfrak{X}(TM))$$

by

$$\mathbf{J}(X^{\mathsf{v}}) = 0, \quad \mathbf{J}(X^{\mathsf{c}}) = X^{\mathsf{v}}; \quad X \in \mathfrak{X}(M);$$

it is called the *vertical endomorphism* on TM.

1.10. Define the vector bundle $\pi\colon TM \times_M TM \to TM$ by

$$TM \times_M TM := \{(u,v) \in TM \times TM \mid \tau(u) = \tau(v)\}$$

and by $\pi(u,v) := u$ if $(u,v) \in TM \times_M TM$. Its fibre over u is the n-dimensional real vector space

$$\{u\} \times T_{\tau(u)}M \cong T_{\tau(u)}M.$$

We denote by $\Gamma(\pi)$ the $C^\infty(TM)$-module of smooth sections of π. A distinguished element in $\Gamma(\pi)$ is the *canonical section*

$$\widetilde{\delta}\colon v \in TM \mapsto \widetilde{\delta}(v) := (v,v) \in TM \times_M TM.$$

Every vector field X on M determines the *basic section* $\widehat{X}$ in $\Gamma(\pi)$ given by

$$\widehat{X}(v) := (v, X \circ \tau(v)), \quad v \in TM.$$

Notice that it can be identified with its *principal part* $X \circ \tau$. The module $\Gamma(\pi)$ is locally generated by basic sections. We have a canonical $C^\infty(TM)$-linear injection $\mathbf{i}\colon \Gamma(\pi) \to \mathfrak{X}(TM)$ given by $\mathbf{i}(\widehat{X}) = X^{\mathsf{v}}$, and a canonical $C^\infty(TM)$-linear surjection $\mathbf{j}\colon \mathfrak{X}(TM) \to \Gamma(\pi)$ defined by

$$\mathbf{j}(X^{\mathsf{v}}) = 0,\ \mathbf{j}(X^{\mathsf{c}}) = \widehat{X};\quad X \in \mathfrak{X}(M).$$

The latter can be naturally identified with the (strong) bundle map

$$(\tau_{TM}, \tau_*)\colon TTM \to TM \times_M TM,\ w \mapsto (v, (\tau_*)_v(w)) \quad \text{if } w \in T_vTM.$$

When dealing with Finsler structures, we also need the slit bundle $\mathring{\pi}\colon \mathring{T}M \times_M TM \to \mathring{T}M$, where

$$\mathring{T}M \times_M TM := \{(u,v) \in TM \times_M TM \mid u \in T_{\tau(u)}M \setminus \{0\}\}$$

and $\mathring{\pi} := \pi \restriction \mathring{T}M \times_M TM$.

By a *Finsler tensor field* we mean a tensor on the module $\Gamma(\pi)$ or $\Gamma(\mathring{\pi})$.

1.11. Let X be a vector field on M. A curve $\gamma\colon I \to M$ is an *integral curve* of X if $X \circ \gamma = \dot{\gamma}$. If $0 \in I$, the point $\gamma(0)$ is called the *starting point* of γ. We say that γ is a *maximal integral curve* of X if it does not admit a proper extension as an integral curve of X. Given a point $p \in M$, there exists a unique maximal integral curve $\gamma_p\colon I_p \to M$ of X such that $\gamma_p(0) = p$. Furthermore, there exist an open subset $\mathcal{D}_X$ of $\mathbb{R} \times M$, the so-called *flow domain*, and a smooth mapping $\varphi^X : \mathcal{D}_X \to M$ such that for all $p \in M$, the curve

$$t \in I_p \mapsto \varphi^X(t,p) \in M$$

is the maximal integral curve of X starting at p. The mapping φ^X is called the *(local) flow* of X. Define $M_t := \{p \in M \mid (t,p) \in \mathcal{D}_X\}$. We say that the mappings (in fact diffeomorphisms)

$$\varphi_t^X : M_t \to M_{-t},\ p \mapsto \varphi_t^X(p) := \varphi^X(t,p), \quad t \in \mathbb{R}$$

are the *stages* of the flow. A vector field is *complete* if all of its integral curves are defined on all $t \in \mathbb{R}$. Then the flow domain is $\mathbb{R} \times M$, and the flow is *global*. In this case, $\varphi_t^X \in \mathrm{Diff}(M)$ for all $t \in \mathbb{R}$.

1.2 Sprays and Ehresmann connections

1.12. A *spray* for M is a (not necessarily smooth) section S of the double tangent bundle $\tau_{TM}\colon TTM \to TM$ such that

(S_1) it is of class C^1 and smooth on the slit tangent bundle $\mathring{T}M$;
(S_2) $\tau_* \circ S = 1_{TM}$;
(S_3) $[C,S] = S$, where $C = \mathbf{i}\,\widetilde{\delta}$ is the *Liouville vector field*.

If a spray is smooth on the whole tangent manifold, then we say that it is an *affine spray*. A diffeomorphism φ of M is an *automorphism* of a spray S if $\varphi_{**} \circ S = S \circ \varphi_*$. These form a group under composition denoted by $\mathrm{Aut}(S)$ and called the *automorphism group* of S.

We say that a curve $\gamma\colon I \to M$ is a *geodesic* of S if $S \circ \dot{\gamma} = \ddot{\gamma}$; it is a *pregeodesic* of S if it has a positive reparametrization as a geodesic. The automorphisms of S send geodesics to geodesics, i.e., if $\varphi \in \mathrm{Aut}(S)$ and γ is a geodesic of S, then $\varphi \circ \gamma$ is also a geodesic [49, Proposition 5.1.32].

1.13. We recall that a *covariant derivative* on M is a mapping

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),\ (X,Y) \mapsto \nabla_X Y,$$

such that it is $C^\infty(M)$-linear in its first variable and satisfies

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y \quad \text{for all } X,Y \in \mathfrak{X}(M),\ f \in C^\infty(M).$$

The *torsion* T of ∇ is the type $(1, 2)$ tensor field on M given by

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y], \quad X, Y \in \mathfrak{X}(M),$$

while its *curvature* $R \in \mathcal{T}^1_3(M)$ is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad \text{for } X, Y, Z \in \mathfrak{X}(M).$$

We say that ∇ is *flat* if its curvature vanishes.

A diffeomorphism φ of M is an *automorphism* of ∇ if

$$\nabla_{\varphi_\# X}(\varphi_\# Y) = \varphi_\#(\nabla_X Y) \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

The automorphisms of ∇ form a group under composition denoted by $\mathrm{Aut}(\nabla)$.

1.14. An *Ehresmann connection* in TM is a mapping

$$\mathcal{H} \colon TM \times_M TM \to TTM,\ (u, v) \mapsto \mathcal{H}(u, v)$$

such that

($\mathcal{H}_1$) $\mathcal{H}$ is fibre-preserving and fibrewise linear: $\mathcal{H}(u, v) \in T_u TM$ and, for tangent vectors v_1, v_2 at $\tau(u)$ and real numbers λ_1, λ_2,

$$\mathcal{H}(u, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \mathcal{H}(u, v_1) + \lambda_2 \mathcal{H}(u, v_2);$$

($\mathcal{H}_2$) $\mathbf{j} \circ \mathcal{H} = 1_{TM \times_M TM}$, where $\mathbf{j} = (\tau_{TM}, \tau_*)$;
($\mathcal{H}_3$) $\mathcal{H}$ is smooth over $\mathring{T}M \times_M TM$.

Notice that $\mathcal{H}$ can be considered as a mapping $\mathcal{H} \colon \Gamma(\mathring{\pi}) \to \mathfrak{X}(\mathring{T}M)$ (cf. [49, Remark 7.2.4(e)]), then $X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) \in \mathfrak{X}(\mathring{T}M)$ is the *horizontal lift* of the vector field X with respect to $\mathcal{H}$. To an Ehresmann connection $\mathcal{H}$ we associate the *horizontal projection* $\mathbf{h} := \mathcal{H} \circ \mathbf{j} \in \mathrm{End}(\mathfrak{X}(\mathring{T}M))$ and the $(1, 2)$ tensor field

$$\mathbf{t} := [\mathbf{J}, \mathbf{h}] \in \mathcal{T}^1_2(\mathring{T}M), \tag{1.4}$$

called the *Grifone-torsion* (or *weak torsion*, cf. [22], Definition I.12) of $\mathcal{H}$. In (1.4) the bracket means Frölicher–Nijenhuis bracket, which is discussed in detail, e.g., in [49]. The Grifone-torsion is semibasic in the sense that

$$\mathbf{t}(\mathbf{J}\,\xi, \eta) = 0 \quad \text{for all } \xi, \eta \in \mathfrak{X}(\mathring{T}M).$$

We define the *potential of* $\mathbf{t}$ by

$$\mathbf{t}^\circ := i_S\,\mathbf{t}, \quad S \text{ is a spray}.$$

It is easy to check that $\mathbf{t}^\circ$ does not depend on the choice of the spray S, so $\mathbf{t}^\circ$ is a well-defined $(1,1)$ tensor field on $\mathring{T}M$.

Choose a chart $(\mathcal{U}, u)$ for M, and consider the induced chart $(\tau^{-1}(\mathcal{U}), (x,y))$. If $X \in \mathfrak{X}(M)$ and $X = X^j \frac{\partial}{\partial u^j}$ over $\mathcal{U}$, then the action of $\mathcal{H}$ on the basic section $\widehat{X}$ is given locally by

$$\mathcal{H}(\widehat{X}) = X^{\mathsf{h}} = (X^j)^{\mathsf{v}} \left(\frac{\partial}{\partial x^j} - N^i_j \frac{\partial}{\partial y^i} \right)$$

with some functions N^i_j, smooth on $\tau^{-1}(\mathcal{U}) \cap \mathring{T}M$. These functions are called the *Christoffel symbols* of $\mathcal{H}$ with respect to the chart $(\mathcal{U}, u)$.

1.15. An Ehresmann connection $\mathcal{H}$ is *homogeneous* (resp. positive-homogeneous) if for any $(v,w) \in TM \times_M TM$ and $\lambda \in \mathbb{R}$ (resp. positive λ) we have

$$(\mu_\lambda)_*(\mathcal{H}(v,w)) = \mathcal{H}(\lambda v, w),$$

where μ_λ is the multiplication by λ on TM. If $\mathcal{H}$ is homogeneous and smooth over $TM \times_M TM$, then it is *linear*.

1.16 ([49], Proposition 7.5.11 and [47], (3.3a))**.** *If ∇ is a covariant derivative on M, then there exists a unique linear Ehresmann connection $\mathcal{H}$ in TM such that*

$$[X^{\mathsf{h}}, Y^{\mathsf{v}}] = (\nabla_X Y)^{\mathsf{v}} \quad \textit{for all } X, Y \in \mathfrak{X}(M).$$

The Grifone torsion of $\mathcal{H}$ and the torsion tensor of ∇ are related by

$$\mathbf{t}(X^{\mathsf{c}}, Y^{\mathsf{c}}) = (T(X,Y))^{\mathsf{v}}; \quad X, Y \in \mathfrak{X}(M). \tag{1.5}$$

If $(\mathcal{U}, u)$ is a chart of M and $\Gamma^i_{jk} \in C^\infty(M)$ are the Christoffel symbols of ∇ with respect to this chart, then the Christoffel symbols of $\mathcal{H}$ are the functions $N^i_j = y^k(\Gamma^i_{jk} \circ \tau)$, $i, j \in \{1, \dots, n\}$.

1.17 ([49], Lemma and definition 7.2.13 and Corollary 7.5.6)**.** *If $\mathcal{H}$ is a positive-homogeneous Ehresmann connection in TM, then $S := \mathcal{H}\widetilde{\delta}$ is a spray for M, called the* spray associated to $\mathcal{H}$. *Furthermore, if $\mathcal{H}$ is linear then S is an affine spray.*

Let N^i_j be the Christoffel symbols of $\mathcal{H}$ with respect to a chart $(\mathcal{U}, u)$ of M. Then the coefficients of the associated spray S with respect to this chart are the functions $G^i = \frac{1}{2} y^j N^i_j$ over $\tau^{-1}(\mathcal{U})$.

1.3 Lie groups and actions

Lie groups

1.18. A group G is a *Lie group* if it is endowed with a smooth structure such that the multiplication map $\mu\colon G\times G\to G$, $(g,h)\mapsto \mu(g,h):=gh$ and the inverse map $\nu\colon G\to G$, $g\mapsto \nu(g):=g^{-1}$ are smooth. Unless otherwise stated, we use multiplicative notation for the group operation. The unit element of an abstract Lie group will be denoted by e. The smooth mappings

$$\lambda_a\colon G\to G,\ g\mapsto \lambda_a(g):=ag,\quad \rho_a\colon G\to G,\ g\mapsto \rho_a(g):=ga$$

and

$$\mathbf{c}_a:=\lambda_a\circ\rho_{a^{-1}}\colon g\in G\mapsto aga^{-1}\in G$$

are the left translations, the right translations and the conjugation by $a\in G$, respectively. Then

$$\lambda_a\circ\lambda_b=\lambda_{ab},\quad \rho_a\circ\rho_b=\rho_{ba},\quad \lambda_a\circ\rho_b=\rho_b\circ\lambda_a, \tag{1.6}$$

for all $a,b\in G$.

A *(Lie group) homomorphism* is a smooth mapping between Lie groups which preserves the multiplication. If it is also a diffeomorphism, then we talk about an *isomorphism* of Lie groups. An *automorphism* of a Lie group G is an isomorphism of G onto itself. The automorphisms of G form a group denoted by $\mathrm{Aut}(G)$. The mappings ν, λ_a and ρ_a are diffeomorphisms of G, while $\mathbf{c}_a\in\mathrm{Aut}(G)$ ($a\in G$).

1.19. A vector field X on a Lie group G is *left invariant* if it is invariant under left translations, that is, $(\lambda_a)_\# X=X$ for all $a\in G$. In a less condensed form,

$$(\lambda_a)_*\circ X=X\circ\lambda_a. \tag{1.7}$$

Since each $(\lambda_a)_\#$ preserves Lie brackets by (1.3), the left invariant vector fields of G form a subalgebra of the Lie algebra $\mathfrak{X}(G)$ denoted by $\mathfrak{X}_L(G)$.

A left invariant vector field X is uniquely determined by its value at the identity: for every $a\in G$,

$$X_a=X(\lambda_a(e))\overset{(1.7)}{=}(\lambda_a)_*(X_e). \tag{1.8}$$

Let $v \in T_eG$. The unique left invariant vector field v_L such that $(v_L)_e = v$ is called *the left invariant vector field generated by* v. The mapping

$$\mathfrak{X}_L(G) \to T_eG,\ X \mapsto X_e$$

is a linear isomorphism between the real vector spaces $\mathfrak{X}_L(G)$ and T_eG. In this way the Lie algebra structure of $\mathfrak{X}_L(G)$ induces a Lie algebra structure on T_eG: if $u, v \in T_eG$, then

$$[u, v] := [u_L, v_L]_e. \tag{1.9}$$

The vector space T_eG endowed with this Lie bracket is the *Lie algebra* of G, denoted by $\mathsf{Lie}(G)$. That is, $\mathsf{Lie}(G) := (T_eG, [\,,\,])$.

1.20. Similarly to the previous paragraph, $Y \in \mathfrak{X}(G)$ is *right invariant* if $(\rho_a)_\# Y = Y$ for each $a \in G$. The vector space $\mathfrak{X}_R(G)$ of right invariant vector fields is also a subalgebra of $\mathfrak{X}(G)$. If $v \in T_eG$, then the vector field v_R, defined by $(v_R)_a := (\rho_a)_*(v)$, is the unique right invariant vector field such that $(v_R)_e = v$.

With the help of the inverse map ν we have a natural *isomorphism*

$$\mathfrak{X}_L(G) \to \mathfrak{X}_R(G),\ X \mapsto \nu_\# X$$

between the Lie algebras of left and right invariant vector fields. If $v \in T_eG$, then $\nu_\#(v_L) = -v_R$.

We also have

$$[X, Y] = 0 \quad \text{if } X \in \mathfrak{X}_L(G), Y \in \mathfrak{X}_R(G). \tag{1.10}$$

1.21. A mapping $\alpha\colon \mathbb{R} \to G$ is a *one-parameter subgroup* of G if it is a Lie group homomorphism from the additive group $\mathbb{R}$ to G. In other words, a one-parameter subgroup is a smooth curve $\alpha\colon \mathbb{R} \to G$ such that

$$\alpha(s + t) = \alpha(s)\alpha(t) \quad \text{for all } s, t \in \mathbb{R}.$$

In particular,

$$\alpha(0) = e, \quad \alpha(t)^{-1} = \alpha(-t).$$

The left and right invariant vector fields are complete, that is, their flows are defined on $\mathbb{R} \times G$. In particular, if $X \in \mathfrak{X}_L(G)$ then its flow is given by

$$\varphi^X\colon \mathbb{R} \times G \to G,\ (t, g) \mapsto \varphi^X(t, g) = g \cdot \alpha(t) = \rho_{\alpha(t)}(g), \tag{1.11}$$

where α is the unique one-parameter subgroup of G satisfying $\dot{\alpha}(0) = X_e$. Thus the stages $\varphi_t^X : G \to G,\ g \mapsto \varphi_t^X(g) := \varphi^X(t, g)$ of the flow act as right translations of G.

Analogously, if $Y \in \mathfrak{X}_R(G)$, then its flow φ^Y is

$$\varphi^Y : \mathbb{R} \times G \to G,\ (t, g) \mapsto \varphi_t^Y(g) := \varphi^Y(t, g) = \beta(t) \cdot g = \lambda_{\beta(t)}(g),$$

where β is the unique one-parameter subgroup with $\dot{\beta}(0) = Y_e$. In this case the stages φ_t^Y are left translations of G.

It follows that the integral curves of left and right invariant vector fields through e are one-parameter subgroups.

1.22. The exponential map of a Lie group G is the smooth mapping

$$\exp\colon T_eG \to G,\ v \mapsto \exp(v) := \alpha_v(1),$$

where α_v is the unique one-parameter subgroup of G satisfying $\dot{\alpha}_v(0) = v$. The exponential map has the properties

$$\exp(tv) = \alpha_v(t) \quad \text{and} \quad \exp((s + t)v) = \exp(sv) \exp(tv), \tag{1.12}$$

where $s, t \in \mathbb{R}$, $v \in T_eG$. Another nice feature is the existence of neighbourhoods $\mathcal{V}$ of $0 \in T_eG$ and $\mathcal{U}$ of $e \in G$ such that $\exp \restriction \mathcal{U}$ is a diffeomorphism of $\mathcal{U}$ onto $\mathcal{V}$.

The next two assertions are needed to prove one of our main results, namely Theorem 3.13. The first observation can be found in a paper of Ichijyō [30].

1.23. Fact. Let (V, f) be a finite-dimensional normed space. Then the isometry group

$$\operatorname{iso}(f) := \{A \in \operatorname{End}(V) \mid f \circ A = f\}$$

of f is a Lie subgroup of $\mathrm{GL}(V)$.

The following result can be found in [53] as an exercise; for the readers' convenience we present it with a proof.

1.24. Lemma. Let G be a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$, and let $A\colon I \to \mathsf{Lie}(G)$ be a curve in its Lie algebra. If $\gamma\colon I \to \mathrm{GL}_n(\mathbb{R})$ is a solution of the initial value problem

$$\gamma'(t) = A(t) \cdot \gamma(t), \quad \gamma(0) = 1_n, \tag{1.13}$$

then it takes values only in G.

Conversely, if γ is a curve in G, then $\gamma'(t) = A(t) \cdot \gamma(t)$ for some curve A in $\mathsf{Lie}(G)$.

Proof. (Lemma 3.3 in [4]) We show that (1.13) implies that the curve

$$t \in I \mapsto (t, \gamma(t)) \in \mathbb{R} \times \mathrm{GL}_n(\mathbb{R})$$

is an integral curve of a vector field on $\mathbb{R} \times G$, thus γ must run in G.

Since $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $\mathrm{M}_n(\mathbb{R})$, we can identify its tangent manifold with $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{M}_n(\mathbb{R})$. Consider the right invariant vector field $(A(t))_R$ on $\mathrm{GL}_n(\mathbb{R})$. Then $(A(t))_R(1_n) = (1_n, A(t))$, and we obtain

$$\begin{aligned}\dot{\gamma}(t) &= (\gamma(t), \gamma'(t)) \overset{(1.13)}{=} (\gamma(t), A(t) \cdot \gamma(t)) = \big(\rho_{\gamma(t)}(1_n), \rho_{\gamma(t)}(A(t))\big) \\ &= (\rho_{\gamma(t)})_*(1_n, A(t)) = (\rho_{\gamma(t)})_*((A(t))_R(1_n)) = (A(t))_R(\gamma(t)).\end{aligned}$$

Thus $t \mapsto (t, \gamma(t))$ is an integral curve of the vector field

$$\begin{aligned}&\mathbb{R} \times \mathrm{GL}_n(\mathbb{R}) \to T(\mathbb{R} \times \mathrm{GL}_n(\mathbb{R})) \cong (\mathbb{R} \times \mathbb{R}) \times (\mathrm{GL}_n(\mathbb{R}) \times \mathrm{M}_n(\mathbb{R})), \\ &(t, g) \mapsto ((t, 1), (A(t))_R(g)) = \Big((t, 1), \big(g, (\rho_g)_*(A(t))\big)\Big).\end{aligned} \tag{1.14}$$

However, $(A(t))_R$ is tangent to the submanifold G of $\mathrm{GL}_n(\mathbb{R})$, and, obviously, (1.14) is tangent to $\mathbb{R} \times G$, so the restriction of (1.14) to $\mathbb{R} \times G$ is a vector field. The converse is immediate.

Lie group and Lie algebra actions

1.25. A *right action* of G on M is a smooth mapping

$$A\colon M \times G \to M, \ (p, a) \mapsto A(p, a) =: p \cdot a$$

satisfying

$$p \cdot (ab) = (p \cdot a) \cdot b \quad \text{and} \quad p \cdot e = p; \quad a, b \in G, \ p \in M. \tag{1.15}$$

The action A determines the diffeomorphisms

$$T_a\colon M \to M,\ p \mapsto T_a(p) := T(p,a) = p \cdot a, \quad a \in G,$$

mentioned also as *right translations*. For $p \in M$, the mapping

$$A_p\colon G \to M,\ a \mapsto A_p(a) := A(p,a) = p \cdot a$$

is called the *orbit map* of p. The group action is *transitive* if for all $p \in M$ the orbit map A_p is surjective, i.e., for any two points $p, q \in M$, there exists an element $a \in G$ such that $q = p \cdot a$.

1.26. Suppose that A is a right action of G on M, as above. The *orbit of* $p \in M$ is the subset $p \cdot G := \{p \cdot a \in M \mid a \in G\} = \mathrm{Im}(A_p)$ of M. The orbits of A form a partition of the manifold M. It follows that A is transitive if, and only if, there is only one orbit.

For $p \in M$ the set $G_p := \{a \in G \mid p \cdot a = p\} = A_p^{-1}(p)$ is a subgroup of G, called the *isotropy subgroup at* p. The continuity of A implies that G_p is actually a closed subset of G and hence, by É. Cartan's closed subgroup theorem (see, e.g., [37, Theorem 20.12]), it is a Lie subgroup.

We say that the group action is *free* if for all $p \in M$ the isotropy subgroup at p is trivial: $G_p = \{e\}$. This condition is equivalent to A_p being injective for every $p \in M$. The group action is *locally free* (or *almost free*) if G_p is a discrete subgroup of G, for each $p \in M$.

1.27. Let $\mathfrak{g}$ be an arbitrary finite-dimensional Lie algebra over $\mathbb{R}$. A Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, $v \mapsto \widehat{v}$ is a *(right) action of* $\mathfrak{g}$ *on* M if the evaluation map

$$M \times \mathfrak{g} \mapsto TM,\ (p,v) \mapsto \widehat{v}(p) = \widehat{v}_p$$

is smooth. The Lie algebra action is *complete* if the vector field $\widehat{v}$ is complete for all $v \in \mathfrak{g}$.

1.28. A (right) group action $A\colon M \times G \to M$ naturally induces a right action

$$\widehat{A}\colon \mathsf{Lie}(G) \to \mathfrak{X}(M),\ v \mapsto \widehat{A}(v)$$

of $\mathsf{Lie}(G)$ on M defined by

$$\widehat{A}(v)(p) := ((A_p)_*)_e(v),\ p \in M.$$

It can be easily seen that $\widehat{A}$ satisfies the desired smoothness property, and $\widehat{A}(v)$ is a vector field on M. To show that $\widehat{A}$ preserves the Lie bracket, let $p \in M$, $v \in \mathsf{Lie}(G)$, and consider the left invariant vector field v_L on G generated by v. First we show that v_L and $\widehat{A}(v)$ are A_p-related. For every $g \in G$ we have

$$\begin{aligned}((A_p)_*)_g \circ v_L(g) &\overset{(1.8)}{=} ((A_p)_*)_g \circ ((\lambda_g)_*)_e(v) = ((A_p \circ \lambda_g)_*)_e(v) \\ &\overset{(1.15)}{=} ((A_{p\cdot g})_*)_e(v) =: \widehat{A}(v)(p \cdot g) = \widehat{A}(v) \circ A_p(g),\end{aligned}$$

therefore $v_L \underset{A_p}{\sim} \widehat{A}(v)$ by (1.2). Now let $v, w \in \mathsf{Lie}(G)$. Since

$$v_L \underset{A_p}{\sim} \widehat{A}(v) \quad \text{and} \quad w_L \underset{A_p}{\sim} \widehat{A}(w),$$

by the related vector field lemma (see **1.8**) we have

$$[v_L, w_L] \underset{A_p}{\sim} \big[\widehat{A}(v), \widehat{A}(w)\big] \overset{(1.2)}{\iff} (A_p)_* \circ [v_L, w_L] = \big[\widehat{A}(v), \widehat{A}(w)\big] \circ A_p.$$

We evaluate both sides of the equality at the unit element. Then, on the one hand,

$$\begin{aligned}(A_p)_* \circ [v_L, w_L](e) &= ((A_p)_*)_e[v_L, w_L]_e \\ &\overset{(1.9)}{=} ((A_p)_*)_e([v, w]) =: \widehat{A}([v, w])(p),\end{aligned}$$

while on the other hand

$$\big[\widehat{A}(v), \widehat{A}(w)\big] \circ A_p(e) \overset{(1.15)}{=} \big[\widehat{A}(v), \widehat{A}(w)\big](p),$$

so it follows that $\widehat{A}([v, w])(p) = \big[\widehat{A}(v), \widehat{A}(w)\big](p)$. By the arbitrariness of p, this implies the desired relation $\widehat{A}([v, w]) = \big[\widehat{A}(v), \widehat{A}(w)\big]$.

The $\mathsf{Lie}(G)$-action $\widehat{A}$ on M so described is also called the *infinitesimal generator* of the G-action A on M.

1.29. Let $A\colon M \times G \to M$ be a group action as above, and consider its infinitesimal generator $\widehat{A}\colon \mathsf{Lie}(G) \to \mathfrak{X}(M)$. For every $v \in T_eG$, the vector field $\widehat{A}(v)$ can also be obtained as the velocity field of a global flow on M. (For the definitions of the latter concepts, consult, e.g., with [49, **3.2.3**]). To see this, consider the exponential map of G (**1.22**), choose a tangent vector $v \in T_eG$, and define the mapping

$$\psi^v\colon (t, p) \in \mathbb{R} \times M \mapsto \psi^v(t, p) := \psi^v_t(p) := p \cdot \exp(tv).$$

Using the notation of **1.22** and the identities (1.12), on the one hand, we have

$$\psi^v(0,p) := p\cdot\alpha_v(0) = p\cdot e \overset{(1.15)}{=} p, \quad p\in M.$$

On the other hand, for any $s,t\in\mathbb{R}$,

$$\begin{aligned}\psi^v_t\circ\psi^v_s(p) &:= \psi^v_t(p\cdot\exp(sv)) \overset{(1.15)}{=} p\cdot\exp(sv)\exp(tv)\\ &\overset{(1.12)}{=} p\cdot\exp((s+t)v) =: \psi^v_{t+s}(p),\end{aligned}$$

thus ψ^v is a global flow on M. Denote the velocity field of ψ^v, guaranteed by Proposition 3.2.24 in [49], by $\widehat{v}$. Then the integral curve of $\widehat{v}$ through a point $p\in M$ is the flow line $t\mapsto\psi^v(t,p)$. Therefore, if $p\in M$, then

$$\begin{aligned}\widehat{A}(v)(p) &:= ((A_p)_*)_e(v) = ((A_p)_*)_e((\dot\alpha_v)(0)) \overset{(1.1)}{=} \dot{\overbrace{A_p\circ\alpha_v}}(0)\\ &\overset{(1.12)}{=} \dot{\overbrace{t\mapsto p\cdot\exp(tv)}}(0) = \dot{\overbrace{t\mapsto\psi^v(t,p)}}(0) = \widehat{v}(\psi^v(0,p)) = \widehat{v}(p),\end{aligned}$$

so $\widehat{v}=\widehat{A}(v)$, as indicated.

1.30. Lemma. *Keeping the notation of **1.25** and **1.26**, consider a right Lie group action $A\colon M\times G\to M$. The Lie algebra of the isotropy subgroup G_p of $p\in M$ is*

$$\mathsf{Lie}(G_p) = \operatorname{Ker}((A_p)_*)_e. \tag{1.16}$$

Proof. Indeed, if $v\in\mathsf{Lie}(G_p)$, then $tv\in\mathsf{Lie}(G_p)$ for all $t\in\mathbb{R}$, hence $\exp(tv)\in G_p$, and so $p\cdot\exp(tv)=p$. Thus we obtain

$$((A_p)_*)_e(v) =: \widehat{A}(v)(p) \overset{\mathbf{1.29}}{=} \dot{\overbrace{t\mapsto p\cdot\exp(tv)}}(0) = \dot{\overbrace{t\mapsto p}}(0) = 0,$$

whence $v\in\operatorname{Ker}((A_p)_*)_e$.

Conversely, assume that $v\in\operatorname{Ker}((A_p)_*)_e$. Since $t\mapsto p\cdot\exp(tv)$ is an integral curve of $\widehat{A}(v)$ through p, and $\widehat{A}(v)(p) = ((A_p)_*)_e(v) = 0$, the uniqueness of integral curves implies that the curve $t\mapsto p\cdot\exp(tv)$ is constant. Thus

$$p\cdot\exp(v) = p\cdot\exp(1v) = p\cdot\exp(0v) = p\cdot e = p,$$

whence $\exp(v)\in G_p$, and consequently, $v\in\mathsf{Lie}(G_p)$.

1.31. Lemma. *We continue to keep the notation of **1.25**. If at a point p in M the mapping $((A_p)_*)_e \colon T_eG \to T_pM$ is surjective, then the orbit $p \cdot G$ is open. If $((A_p)_*)_e$ is injective, then the isotropy subgroup G_p is discrete.*

Proof. First we assume that $((A_p)_*)_e$ is surjective, and follow the reasoning of [29, (ii) of Proposition 10.1.6]. The implicit function theorem implies that $p \cdot G = A_p(G)$ is a neighbourhood of p. Since every right translation T_g $(g \in G)$ is a diffeomorphism (and hence homeomorphism), and

$$T_g(p \cdot G) = (p \cdot G) \cdot g \overset{(1.15)}{=} p \cdot (Gg) = p \cdot G,$$

it follows that $p \cdot G$ is also a neighbourhood of $p \cdot g$, therefore the orbit $p \cdot G$ is an open set.

The second assertion is immediate: by the injectivity of $(A_p)_*$ we have $\mathrm{Ker}((A_p)_*)_e = 0$, so (1.16) implies that $\mathsf{Lie}(G_p)$ is trivial. Thus G_p must be a zero-dimensional submanifold of G, hence it is a discrete subgroup.

1.32. The question arises naturally whether any action of a Lie algebra $\mathfrak{g}$ on M is induced by a group action in the sense above; the answer is given by the following important result, due to Palais [42].

Fundamental theorem on Lie algebra actions. [37, Theorem 20.16] *Let M be a manifold, G a simply connected Lie group and let $\mathfrak{g} := \mathsf{Lie}(G)$. If $\widehat{A} \colon \mathfrak{g} \to \mathfrak{X}(M)$ is a complete Lie algebra action of $\mathfrak{g}$ on M, then there exists a unique right Lie group action $M \times G \to M$ of G on M whose infinitesimal generator is $\widehat{A}$.*

1.4 Finsler manifolds

1.33. A *Finsler function* for a manifold M (or on TM) is a function $F \colon TM \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is smooth on $\mathring{T}M$;

(F_2) F is positive-homogeneous, i.e., $F(\lambda v) = \lambda F(v)$ for $v \in TM$, $\lambda > 0$;

(F_3) $F(v) > 0$ if $v \in \mathring{T}M$ and $F(v) = 0$ if $v = 0$;

(F_4) the type $(0, 2)$ Finsler tensor field g defined on basic sections by

$$g(\widehat{X}, \widehat{Y}) := \frac{1}{2} X^{\mathsf{v}}(Y^{\mathsf{v}} F^2); \quad X, Y \in \mathfrak{X}(M)$$

is fibrewise positive definite.

Then the pair (M, F) is a *Finsler manifold*, g is its *metric tensor*, and the function $E := \frac{1}{2}F^2$ is the *energy* associated to F. For the restrictions of F and E we use the notation $F_p := F \restriction T_pM$ and $E_p := E \restriction T_pM$ ($p \in M$). The conditions on a Finsler function imply that it is continuous on TM. Evidently, the energy E shares this property and also that it is smooth on $\mathring{T}M$. It is shown in [52], that E is actually of class C^1 on TM.

The pointwise action of the metric tensor g is given by

$$g_u((u,v),(u,w)) = (E_p)''(u)(v,w) = \frac{1}{2}(F_p)''(u)(v,w), \tag{1.17}$$

where $u \in \mathring{T}_pM$, $v, w \in T_pM$. Using the natural isomorphism between $T_{\tau(u)}M$ and $\{u\} \times T_{\tau(u)}M$, sometimes we write $g_u(v,w)$ instead of $g_u((u,v),(u,w))$. Thus, we can regard g_u as a scalar product on $T_{\tau(u)}M$. In particular, we have

$$g_v(v,v) = F^2(v) \quad \text{for all } v \in \mathring{T}M. \tag{1.18}$$

1.34. Fact (the Finslerian Cauchy–Schwarz inequality [49])**.** Let (M, F) be a Finsler manifold with metric tensor g. Then for every $p \in M$, $u \in \mathring{T}_pM$ and $v \in T_pM$,

$$g_u((u,u),(u,v)) \leq F(u) \cdot F(v).$$

Equality holds if, and only if, v is a non-negative multiple of u.

1.35. There exists a unique spray S for M satisfying

$$i_S d(dE \circ \mathbf{J}) = -dE,$$

called the *canonical spray* of the Finsler manifold. According to [49, Proposition 7.3.4], S determines an Ehresmann connection $\mathcal{H}$ on TM such that

$$X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) = \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]) \quad \text{for all } X \in \mathfrak{X}(M).$$

We say that $\mathcal{H}$ is the *canonical connection* of the Finsler manifold. (For details, we refer to [49, Theorem 9.3.5].)

1.36. The *geodesics* (resp. *pregeodesics*) of a Finsler manifold (M, F) (or simply F) are the geodesics (resp. *pregeodesics*) of its canonical spray.

Given a tangent vector v in TM, there exists a unique maximal geodesic $\gamma_v \colon I \to M$ of (M, F) with initial velocity v. An important property of

Finslerian geodesics is that they have constant speed, that is, the function $F \circ \dot{\gamma}\colon I \to \mathbb{R}$ is constant if $\gamma\colon I \to M$ is a geodesic.

A diffeomorphism φ of M is an *isometry* of a Finsler manifold (M, F) if $F \circ \varphi_* = F$. It can be shown that isometries map geodesics to geodesics. For a proof of this seemingly obvious result, see [3, Proposition 4].

1.37. Lemma. *A spray S over M is the canonical spray of a Finsler manifold (M, F) if, and only if, $SF = 0$ and the pregeodesics of F and S coincide.*

This follows essentially from Theorem 9.6.1 of [49] ('Rapcsák equations'), taking into account Lemma 9.2.20 and Theorem 9.2.22 in this book.

1.38. Fact ([38])**.** If (M, F) is a Finsler manifold and $X \in \mathfrak{X}(M)$, then the following are equivalent:

(i) The stages $\varphi_t^X : M_t \subset M \to M_{-t} \subset M, p \mapsto \varphi_t^X(p) := \varphi^X(t, p)$ of the local flow $\varphi^X : \mathcal{D}_X \subset \mathbb{R} \times M \to M$ of X are isometries between $(M_t, F \restriction TM_t)$ and $(M_{-t}, F \restriction TM_{-t})$.

(ii) $X^{\mathsf{c}}F = 0$.

(iii) If $\gamma\colon I \to M$ is a non-constant geodesic of (M, F), then the function $g_{\dot{\gamma}}(\dot{\gamma}, X \circ \gamma)\colon I \to \mathbb{R}$ is constant.

If one (and hence all) of the conditions above is satisfied, then we say that X is a *Killing vector field* of (M, F).

1.39. Let (M, F) be a Finsler manifold and ∇ a covariant derivative on M. We say that the Finsler function F is *holonomy invariant with respect to* ∇ if the parallel translations induced by ∇ preserve F, that is, for any curve $\gamma\colon I \to M$ and parameters $t_1, t_2 \in I$ we have $F_{\gamma(t_2)} \circ (P_\gamma)_{t_1}^{t_2} = F_{\gamma(t_1)}$. (For the definition of the parallel translations

$$(P_\gamma)_{t_1}^{t_2}\colon T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M; \quad t_1, t_2 \in I$$

with respect to ∇ along γ we refer to [49, Lemma and definition 6.1.58].)

1.40. A triple (M, F, ∇) is a *generalized Berwald manifold* if (M, F) is a Finsler manifold and ∇ is a covariant derivative on M such that F is holonomy invariant with respect to ∇. If, in particular, ∇ is torsion-free, then (M, F, ∇) is a *Berwald manifold*. If (M, F, ∇) is a generalized Berwald manifold and the torsion of ∇ is of the form

$$T = 1_{\mathfrak{X}(M)} \wedge d\sigma := 1_{\mathfrak{X}(M)} \otimes d\sigma - d\sigma \otimes 1_{\mathfrak{X}(M)} \text{ for some } \sigma \in C^\infty(M),$$

then (M, F, ∇) is called a *Wagner manifold*.

In the cases of Berwald and Wagner manifolds the covariant derivative ∇ is unique, so we can denote them simply by (M, F). For generalized Berwald manifolds sometimes we also omit the symbol ∇.

A Finsler manifold (M, F) is a *locally Minkowski manifold* if M admits a flat covariant derivative ∇ such that (M, F, ∇) is a Berwald manifold.

1.41. Fact ([47], Corollary 3.2)**.** Suppose that (M, F, ∇) is a generalized Berwald manifold and denote by $\mathbf{h}_0$ the horizontal projection determined by its canonical connection. Let $\mathcal{H}$ be the linear Ehresmann connection induced by ∇ according to **1.16** with horizontal projection $\mathbf{h}$ and Grifone-torsion $\mathbf{t}$. Then

$$\mathbf{h} = \mathbf{h}_0 + \frac{1}{2}\mathbf{t}^\circ + \frac{1}{2}[\mathbf{J}, (dE \circ \mathbf{t}^\circ)^\#],$$

where E is the energy function, the sharp operator $\#$ is taken with respect to the fundamental 2-form $d(dE \circ \mathbf{J})$ of (M, F), and $\mathbf{t}^\circ$ is the potential of $\mathbf{t}$.

1.42. Fact ([48], Proposition 7)**.** Let (M, F) be a Finsler manifold. The following conditions are equivalent:

(i) (M, F) is a Berwald manifold;
(ii) the canonical spray of (M, F) is an affine spray;
(iii) the canonical connection of (M, F) is a linear connection.

Chapter 2

Parallelisms

The central notion of this chapter, and of the dissertation, is that of a parallelism. In the literature we find many definitions of this concept highlighting different aspects of it (see the overview 2.10). Here we choose the definition below, which can be found in [20] (Problem 14 in Chapter IV). In a slightly different, but equivalent formulation, we find this definition also in the fundamental papers of W. Greub [19] and J. A. Wolf [54].

2.1 The concept of a parallelism

Definition 2.1. Let M be a manifold and consider the vector bundle $\mathcal{P} \to M \times M$ over $M \times M$ whose fibre at a point (p, q) is $L(T_pM, T_qM)$, that is, the real vector space of linear mappings between the tangent spaces at p and q to M. A smooth section P of this vector bundle is called a *parallelism* on M, if

$$\mathsf{P}(r, q) \circ \mathsf{P}(p, r) = \mathsf{P}(p, q) \quad \text{and} \quad \mathsf{P}(p, p) = 1_{T_pM} \tag{2.1}$$

for all $p, q, r \in M$. If M admits a parallelism, we say that M is *parallelizable*; if a parallelism P is given on M, then the pair (M, P) is called a *parallelized manifold* or a *manifold with parallelism*. By a *covering parallelism* of M we mean a family $(\mathcal{U}_\alpha, \mathsf{P}^\alpha)_{\alpha \in \mathcal{A}}$, where $(\mathcal{U}_\alpha)_{\alpha \in \mathcal{A}}$ is an open covering of M and P^α is a parallelism on $\mathcal{U}_\alpha$ for each $\alpha \in \mathcal{A}$.

Notice that (2.1) implies immediately that the mappings $\mathsf{P}(p, q)$ are actually linear *isomorphisms* between the corresponding tangent spaces.

Definition 2.2. A tensor field $A \in \mathcal{T}_k(M)$ (where $k \in \mathbb{N}^*$) is called P-*parallel* or *parallel* (with respect to P) if for any points p, q in M and any tangent vectors $v_1, \dots, v_k \in T_pM$ we have

$$A_q(\mathsf{P}(p,q)(v_1), \dots, \mathsf{P}(p,q)(v_k)) = A_p(v_1, \dots, v_k).$$

Similarly, $B \in \mathcal{T}_k^1(M)$ (for some $k \in \mathbb{N}$) is P-*parallel*, or *parallel* (with respect to P) if

$$B_q(\mathsf{P}(p,q)(v_1), \dots, \mathsf{P}(p,q)(v_k)) = \mathsf{P}(p,q)(B_p(v_1, \dots, v_k)).$$

Remark 2.3. We mention two important special cases of parallel tensor fields.

(1) A vector field X in $\mathfrak{X}(M) \cong \mathcal{T}_0^1(M)$ is P-parallel if $\mathsf{P}(p,q)(X_p) = X_q$ for all $p, q \in M$. By the linearity of the mappings $\mathsf{P}(p,q)$, the P-parallel vector fields form a vector subspace of the real vector space $\mathfrak{X}(M)$, denoted by $\mathfrak{X}_{\mathsf{P}}(M)$.

Due to the relation $\mathsf{P}(p,q)(X_p) = X_q$, a P-parallel vector field is uniquely determined by its value at an arbitrarily chosen point of M. Another consequence is that for any tangent vector $v \in T_pM$ there is a unique P-parallel vector field having v as its value at p. If we want to emphasize that X is P-parallel and satisfies $X_p = v$, we use the notation $v_{\mathsf{P}} :=$; cf. **1.19** and **1.20**.

(2) A Riemannian metric g is parallel with respect to a parallelism P if

$$g_q(\mathsf{P}(p,q)(v), \mathsf{P}(p,q)(w)) = g_p(v,w),$$

for any $p, q \in M$ and $v, w \in T_pM$. In other words, g is P-parallel if for any P-parallel vector fields X and Y on M the function

$$g(X,Y)\colon p \in M \mapsto g_p(X_p, Y_p) \in \mathbb{R}$$

is constant.

Example 2.4. *The natural parallelism of the Euclidean n-space.* Let p, q be arbitrary points in $\mathbb{R}^n$, and consider a tangent vector $v \in T_p\mathbb{R}^n$. Then the mapping P_0 given by

$$\mathsf{P}_0(p,q)(v) = \mathsf{P}_0(p,q)(p,(v^1, \dots, v^n)) := (q,(v^1, \dots, v^n))$$

is a parallelism on $\mathbb{R}^n$, called its natural parallelism. In this case, the P_0-parallel vector fields have the form $\nu^i \frac{\partial}{\partial e^i}$ with some real numbers ν^i (for $i \in \{1, \dots, n\}$), that is, the P_0-parallel vector fields are the 'constant vector fields' of $\mathbb{R}^n$.

Lemma and definition 2.5. *Suppose that* P *is a parallelism on a manifold* M. *Given a point* $p \in M$ *and a basis* $(b_i)_{i=1}^n$ *of* T_pM, *the vector fields* $E_i := (b_i)_{\mathsf{P}}$ *form a frame field of* M. *We say that* $(E_i)_{i=1}^n$ *is a* frame field associated to P.

Conversely, if M *admits a global frame field* $(E_i)_{i=1}^n$, *then the mapping*

$$\begin{aligned}\mathsf{P}\colon (p,q) \in M \times M &\mapsto \mathsf{P}(p,q) \in L(T_pM, T_qM)\\ \mathsf{P}(p,q)(v) &= v^i E_i(q) \text{ if } v = v^i E_i(p) \in T_pM\end{aligned}$$

is a parallelism on M, *and the members of the given frame field are* P-*parallel.*

Proof. The proof is immediate. □

The result above enables us to define a parallelism with the help of a frame field on the manifold. This definition is used, for example, in the book [7], see Section 7.3.

Remark 2.6. (1) From the assertion above it follows that P-parallel vector fields generate the module $\mathfrak{X}(M)$.

(2) If (M, P) is a parallelized manifold, $(E_i)_{i=1}^n$ is a frame field associated to P and $X \in \mathfrak{X}(M)$, then for any $p, q \in M$ we have $X_q = X^i(q)E_i(q)$ and

$$\mathsf{P}(p,q)(X_p) = X^i(p)\big(\mathsf{P}(p,q)(E_i(p))\big) = X^i(p)E_i(q),$$

where the functions X^i are the component functions of X with respect to $(E_i)_{i=1}^n$. Thus, a vector field is P-parallel if, and only if, its component functions with respect to a frame field associated to P are constant. So, with the notation of Remark 2.3(1),

$$\text{if} \quad v = v^i E_i(p) \in T_pM, \quad \text{then} \quad v_{\mathsf{P}} = v^i E_i. \tag{2.2}$$

(3) More generally, a tensor field of type $(0,k)$ or $(1,l)$ ($k \in \mathbb{N}^*$, $l \in \mathbb{N}$) is P-parallel if, and only if, its components with respect to a frame field

$(E_i)_{i=1}^n$ associated to P are constant. Indeed, if $A \in \mathcal{T}_k(M)$ and $p, q \in M$, then, for $i_j \in \{1, \ldots, n\}$, where $j \in \{1, \ldots, k\}$,

$$\begin{aligned} A_{i_1 \ldots i_k}(p) &:= (A(E_{i_1}, \ldots, E_{i_k}))(p) = A_p(E_{i_1}(p), \ldots, E_{i_k}(p)) \quad \text{and} \\ A_{i_1 \ldots i_k}(q) &:= (A(E_{i_1}, \ldots, E_{i_k}))(q) \\ &= A_q\big(\mathsf{P}(p,q)(E_{i_1}(p)), \ldots, \mathsf{P}(p,q)(E_{i_1}(p))\big), \end{aligned}$$

from which, taking into account (1), our claim follows.

If $B \in \mathcal{T}_l^1(M)$, then its components with respect to $(E_i)_{i=1}^n$ are defined by

$$B(E_{i_1}, \ldots, E_{i_l}) = B^r_{i_1 \ldots i_l} E_r,$$

So if $p, q \in M$, then on the one hand

$$\begin{aligned} B^r_{i_1 \ldots i_l}(q) E_r(q) &= (B(E_{i_1}, \ldots, E_{i_l}))(q) \\ &= B_q\big(\mathsf{P}(p,q)(E_{i_1}(p)), \ldots, \mathsf{P}(p,q)(E_{i_l}(p))\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} B^r_{i_1 \ldots i_l}(p) E_r(q) &= \mathsf{P}(p,q)(B^r_{i_1 \ldots i_l}(p) E_r(p)) \\ &= \mathsf{P}(p,q)\big((B(E_{i_1}, \ldots, E_{i_l}))(p)\big), \end{aligned}$$

as wanted.

However, two frame fields can lead to the same parallelism:

Lemma 2.7. *Given a parallelism P on M, any frame field $(\bar{E}_i)_{i=1}^n$ consisting of P-parallel vector fields is associated to P. Thus, two frame fields $(E_i)_{i=1}^n$ and $(\bar{E}_i)_{i=1}^n$ on M define the same parallelism P if, and only if, the transition matrix from $(E_i)_{i=1}^n$ to $(\bar{E}_i)_{i=1}^n$ is a real matrix.*

Proof. Fix a point p in M. Then $(\bar{E}_i(p))_{i=1}^n$ is a basis of T_pM, and thus, by Lemma and definition 2.5, $\big((\bar{E}_i(p))_{\mathsf{P}}\big)_{i=1}^n$ is a frame field associated to P. Since $\bar{E}_i = (\bar{E}_i(p))_{\mathsf{P}}$, our claim follows. Part (2) of the previous remark implies the second assertion of the lemma. □

Remark 2.8. Let (M, P) be a manifold with parallelism and $(E_i)_{i=1}^n$ a frame field associated to P. If $(\mathcal{U}, u)$ is a chart of M then we can define the functions $P^i_j, Q^i_j \in C^\infty(\mathcal{U})$ $(i, j \in \{1, \ldots, n\})$ given by

$$E_j = P^i_j \frac{\partial}{\partial u^i} \quad \text{and} \quad (Q^i_j) = (P^i_j)^{-1}.$$

By the previous lemma the matrices (P^i_j) and (Q^i_j) (with entries from $C^\infty(\mathcal{U})$) differ only by a multiple of a matrix of real entries if we choose another P-parallel frame field.

If M is a manifold such that there exists a strong bundle map

$$\varphi\colon M \times T_a M \to TM,$$

where $a \in M$ is fixed, and $\varphi_p\colon v \in T_a M \mapsto \varphi_p(v) := \varphi(p, v) \in TM$, then the mapping P given on $p, q \in M$ by $\mathsf{P}(p, q) := \varphi_q \circ \varphi_p^{-1}$ is a parallelism on M. Vice versa: given a parallelism P on M if we choose an arbitrary point a, then the mapping

$$M \times T_a M \to TM,\ (p, v) \mapsto (v_{\mathsf{P}})_a = \mathsf{P}(a, p)(v)$$

is a strong bundle map, and hence a trivialization of M.

However, in our investigations the correspondence between frame fields and global trivializations will be more useful, so we highlight this relation in the following.

Remark 2.9. If M is a parallelizable manifold, then the frame fields of M are in a natural bijective correspondence with the global trivializations of M. Indeed, if we fix a frame field $(E_i)_{i=1}^n$ of M, then the mapping

$$\varphi\colon M \times \mathbb{R}^n \to TM,\ (p, (\nu^1, \dots, \nu^n)) \mapsto \nu^i E_i(p)$$

is a trivialization of M. In this situation we say that the trivialization φ (which depends on the chosen frame field) is *associated to* P.

Conversely, given a trivializing map $\varphi\colon M \times \mathbb{R}^n \to TM$, let, for any $i \in \{1, \dots, n\}$,

$$E_i\colon M \to TM,\ E_i(p) := \varphi(p, e_i),$$

where $(e_i)_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$. Then $(E_i)_{i=1}^n$ is a global frame field of M.

To conclude this section, we collect the possible ways (or at least some of them) to define a parallelism on a parallelizable manifold M.

2.10. Overview of the alternative definitions of a parallelism. The following objects differ only by change of viewpoint:

(i) A parallelism

$$\mathsf{P}\colon (p,q) \in M \times M \mapsto \mathsf{P}(p,q) \in L(T_pM, T_qM).$$

(ii) A strong bundle map (i.e., a trivialization)

$$\varphi\colon M \times \mathbb{R}^n \to TM, \quad \text{where } n = \dim M.$$

(iii) A strong bundle map $\varphi\colon M \times T_aM \to TM$, where $a \in M$ is an arbitrarily fixed point of M.

(iv) A linear subspace $\mathfrak{X}_{\mathsf{P}}(M) \subset \mathfrak{X}(M)$ such that the evaluation mapping

$$M \times \mathfrak{X}_{\mathsf{P}}(M) \to TM,\ (p,X) \mapsto X_p$$

is a strong bundle map.

(v) A strong bundle map

$$\varphi\colon M \times V \to TM,$$

where V is a $\dim M$-dimensional real vector space.

(vi) A frame field over M.

Here (v) contains (ii), (iii) and (iv) as special cases.

2.2 Associated structures: Ehresmann connection and geodesics

Throughout this section (M, P) is a parallelized manifold and $(E_i)_{i=1}^n$ is a frame field on M associated to P. Recall that the vector fields E_i are P-parallel.

Lemma and definition 2.11. *For tangent vectors $v, w \in T_pM$, let*

$$\mathcal{H}(v,w) := (v_{\mathsf{P}})_*(w), \tag{2.3}$$

where v_{P} denotes the unique P-parallel vector field such that $v_{\mathsf{P}}(p) = v$, as in (1) of Remark 2.3. Then $\mathcal{H}$ is a linear Ehresmann connection in TM

called the Ehresmann connection associated to (or generated by) P. *The coordinate expression of* $\mathcal{H}$ *with respect to an induced chart* $(\tau^{-1}(\mathcal{U}), (x, y))$ *for* TM *is*

$$\mathcal{H}\left(\widehat{\frac{\partial}{\partial u^j}}\right) = \frac{\partial}{\partial x^j} + y^k (Q_k^l \circ \tau)\Big(\frac{\partial P_l^i}{\partial u^j} \circ \tau\Big)\frac{\partial}{\partial y^i}, \tag{2.4}$$

where the functions P_i^j *and* Q_i^j *are defined as in Remark 2.8.*

Proof. By the properties of the derivative $(v_{\mathsf{P}})_*$ it follows immediately that $\mathcal{H}(v, w) \in T_v TM$ and that $\mathcal{H}$ is fibrewise linear. We have

$$\tau_*(\mathcal{H}(v, w)) = \tau_*((v_{\mathsf{P}})_*(w)) = (\tau \circ v_{\mathsf{P}})_*(w) = (1_M)_*(w) = w,$$

thus $\mathbf{j} \circ \mathcal{H} = (\tau_{TM}, \tau_*) \circ \mathcal{H} = 1_{TM \times_M TM}$ is also satisfied. Finally, $\mathcal{H}$ is smooth on its whole domain: $(v_{\mathsf{P}})_* \colon TM \to TTM$ is evidently smooth, and the vector field v_{P} depends smoothly on the tangent vector $v \in TM$. Hence $\mathcal{H}$ is indeed an Ehresmann connection in TM.

To prove that $\mathcal{H}$ is linear, we only need to show the homogeneity. Let λ be a real number. Then by (2.2) we have $\mu_\lambda \circ v_{\mathsf{P}} = (\lambda v)_{\mathsf{P}}$, so

$$(\mu_\lambda)_*(\mathcal{H}(v, w)) = (\mu_\lambda)_*(v_{\mathsf{P}})_*(w) = ((\lambda v)_{\mathsf{P}})_*(w) =: \mathcal{H}(\lambda v, w).$$

Finally, we derive the coordinate expression (2.4). For $v = v^l E_l(p)$ and $w = w^j E_j(p) = w^j P_j^k(p)\left(\frac{\partial}{\partial u^k}\right)_p$ we have

$$v_{\mathsf{P}} \overset{(2.2)}{=} v^l E_l = v^l P_l^i \frac{\partial}{\partial u^i}, \quad v^l = y^j(v) Q_j^l(p) \quad \text{and} \quad y^k(w) = w^j P_j^k(p).$$

Thus

$$\begin{aligned}
\mathcal{H}(v,w) &= ((v_{\mathsf{P}})_*)_p(w) = w^j P_j^k(p)((v_{\mathsf{P}})_*)_p \left(\frac{\partial}{\partial u^k}\right)_p \\
&= w^j P_j^k(p)\left(\frac{\partial(x^i \circ v_{\mathsf{P}})}{\partial u^k}(p)\left(\frac{\partial}{\partial x^i}\right)_v + \frac{\partial(y^i \circ v_{\mathsf{P}})}{\partial u^k}(p)\left(\frac{\partial}{\partial y^i}\right)_v\right) \\
&= w^j P_j^k(p)\left(\left(\frac{\partial}{\partial x^k}\right)_v + \frac{\partial(v^l P_l^i)}{\partial u^k}(p)\left(\frac{\partial}{\partial y^i}\right)_v\right) \\
&= y^k(w)\left(\left(\frac{\partial}{\partial x^k}\right)_v + y^j(v) Q_j^l(p)\frac{\partial P_l^i}{\partial u^k}(p)\left(\frac{\partial}{\partial y^i}\right)_v\right).
\end{aligned}$$

Hence, for $v \in T_pM$ we have

$$\mathcal{H}\left(\widehat{\frac{\partial}{\partial u^j}}(v)\right) = \mathcal{H}\left(v, \left(\frac{\partial}{\partial u^j}\right)_p\right)$$
$$= \left(\frac{\partial}{\partial x^j}\right)_v + y^k(v)\Big(Q_k^l \frac{\partial P_l^i}{\partial u^j}\Big)(p)\left(\frac{\partial}{\partial y^i}\right)_v,$$

so (2.4) is satisfied. Notice that the Christoffel symbols of $\mathcal{H}$ with respect to $(\mathcal{U}, u)$ are the functions $N_j^i = -y^k(Q_k^l \circ \tau)\Big(\frac{\partial P_l^i}{\partial u^j} \circ \tau\Big)$. □

Corollary and definition 2.12. *The spray S associated to the linear Ehresmann connection $\mathcal{H}$ generated by* P *is given by $S(v) := (v_{\mathsf{P}})_*(v)$. This spray is affine, called the* spray generated by *(or* associated to*)* P*. Its coordinate expression over an induced chart is*

$$S = y^i \frac{\partial}{\partial x^i} + y^k y^j (Q_j^l \circ \tau)\Big(\frac{\partial P_l^i}{\partial u^k} \circ \tau\Big)\frac{\partial}{\partial y^i},$$

where we used the notation introduced above.

Proof. If $\mathcal{H}$ is the canonical Ehresmann connection of (M, P), then the action of its associated spray S on a tangent vector $v \in TM$ is indeed

$$S(v) := \mathcal{H} \circ \widetilde{\delta}(v) = \mathcal{H}(v, v) := (v_{\mathsf{P}})_*(v). \tag{2.5}$$

It is evidently smooth on TM, thus S is an affine spray. By **1.17** the coefficients of S with respect to the given chart are

$$G^i = \frac{1}{2} y^j N_j^i \overset{(2.4)}{=} -\frac{1}{2} y^j y^k (Q_k^l \circ \tau)\Big(\frac{\partial P_l^i}{\partial u^j} \circ \tau\Big),$$

whence the coordinate expression of S. □

It can be seen from (2.5), that our definition of the spray associated to P is equivalent to Brickell and Clark's definition [7, Section 10.3].

Definition 2.13. A curve $\gamma\colon I \to M$ is a *geodesic* of a parallelized manifold (M, P) (or simply of P) if for $t_1, t_2 \in I$ we have

$$\mathsf{P}(\gamma(t_1), \gamma(t_2))\dot{\gamma}(t_1) = \dot{\gamma}(t_2).$$

By a *pregeodesic* of (M, P) we mean a curve in M which has a reparametrization as a geodesic of the parallelized manifold.

We say that (M, P) is *complete* if all of its geodesics are defined on the entire real line.

Lemma 2.14. *The geodesics of a parallelized manifold* (M, P) *are just the integral curves of the* P*-parallel vector fields.*

Proof. First, let $\gamma\colon I \to M$ be an integral curve of a P-parallel vector field X. Then, for any $t_1, t_2 \in I$,

$$\mathsf{P}(\gamma(t_1), \gamma(t_2))\dot\gamma(t_1) = \mathsf{P}(\gamma(t_1), \gamma(t_2))X_{\gamma(t_1)} = X_{\gamma(t_2)} = \dot\gamma(t_2),$$

so γ is a geodesic of (M, P).

Conversely, assume that $\gamma\colon I \to M$ is a geodesic of (M, P). Choose a parameter $t_0 \in I$ and let X be the unique P-parallel vector field such that $X(\gamma(t_0)) = \dot\gamma(t_0)$ (cf. Remark 2.3(1)). Then for any $t \in I$ we have

$$X_{\gamma(t)} = \mathsf{P}(\gamma(t_0), \gamma(t))X_{\gamma(t_0)} = \mathsf{P}(\gamma(t_0), \gamma(t))\dot\gamma(t_0) = \dot\gamma(t),$$

from which our assertion follows. □

Thus in the case of the Euclidean n-space the geodesics of the natural parallelism defined in Example 2.4 are the affinely parametrized straight lines of $\mathbb{R}^n$, as expected.

Corollary 2.15. *A parallelism* P *is complete if, and only if, the* P*-parallel vector fields are complete.*

Proposition 2.16. *If* S *is the spray associated to a parallelism* P*, then the geodesics of* S *and* P *coincide.*

Proof. (cf. Proposition 10.3.1 in [7]) Let X be a P-parallel vector field and consider an integral curve $\gamma\colon I \to M$ of X; by Lemma 2.14, all geodesics of P are of this form. Then $X(\gamma(t)) = \dot\gamma(t)$ for all $t \in I$, thus

$$S_{\dot\gamma(t)} := (X_*)_{\gamma(t)}(\dot\gamma(t)) \overset{(1.1)}{=} \dot{\overline{X \circ \gamma}}(t) = \ddot\gamma(t),$$

which means that $\gamma\colon I \to \mathbb{R}$ is a geodesic of S.

Conversely, suppose that a curve $\gamma\colon I \to M$ is a geodesic of S, that is, $S \circ \dot\gamma = \ddot\gamma$. Let $v := \dot\gamma(0)$ and let X be the unique P-parallel vector field such that $X(\gamma(0)) = v = \dot{\gamma}(0)$. We show that for any $t \in I$ relation $X(\gamma(t)) = \dot{\gamma}(t)$ is satisfied.

After fixing a frame field $(E_i)_{i=1}^n$ associated to P, the mapping

$$w = w^i E_i(p) \in TM \mapsto (p, (w^1, \dots, w^n)) \in M \times \mathbb{R}^n$$

is a diffeomorphism of the form (τ, ψ) for some $\psi \colon TM \to \mathbb{R}^n$. (Actually, it is the inverse of the trivializing map φ associated to P according to Remark 2.9.) Thus, it remains to show that

$$(\tau, \psi) \circ X \circ \gamma = (\tau, \psi) \circ \ddot{\gamma}. \tag{2.6}$$

Equality $\tau \circ X \circ \gamma = \tau \circ \ddot{\gamma}$ is evident. Since $\psi \circ X$ is the constant mapping $q \in M \mapsto (v^1, \dots, v^n) \in \mathbb{R}^n$, we have

$$(\psi_*)_v(S_v) = (\psi_*)_v((X_*)_p(v)) = ((\psi \circ X)_*)_p(v) = 0_{(v^1,\dots,v^n)}. \tag{2.7}$$

Also, for any $t \in I$,

$$\dot{\overline{\psi \circ \dot{\gamma}}}(t) \overset{(1.1)}{=} (\psi_*)_{\dot{\gamma}(t)}(\ddot{\gamma}(t)) \overset{\text{cond.}}{=} (\psi_*)_{\dot{\gamma}(t)}(S_{\dot{\gamma}(t)}) \overset{(2.7)}{=} 0_{\psi(\dot{\gamma}(t))}.$$

It follows that the curve $\psi \circ \dot{\gamma}$ in $\mathbb{R}^n$ is an integral curve of the zero vector field on $\mathbb{R}^n$, thus, by the uniqueness of integral curves, it must be constant. But then

$$\psi(\dot{\gamma}(t)) = \psi(\dot{\gamma}(0)) = \psi(v) = (v^1, \dots, v^n) = \psi(X(\gamma(t))),$$

so (2.6) holds, which concludes the proof. □

2.3 The torsion of a parallelism

In this section we associate an alternating $(1, 2)$ tensor field to a parallelism P, called its torsion. In our definition we again follow [20], but later we will see that other definitions are also possible, and the term 'torsion' will be justified.

Lemma and definition 2.17. *Let (M, P) be a parallelized manifold and fix a point $p \in M$. Define a T_pM-valued 1-form θ on M by*

$$\theta_q(w) := \mathsf{P}(q, p)(w) \quad \textit{if } q \in M, w \in T_qM$$

*(cf. **1.7**). Then the mapping*

$$T^{\mathsf{P}}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),\ (X,Y) \mapsto T^{\mathsf{P}}(X,Y)$$

given by

$$(T^{\mathsf{P}}(X,Y))_q := \mathsf{P}(p,q)(d\theta)_q(X_q, Y_q)$$

is a type $(1,2)$ *tensor field on* M *which is independent of the choice of* p*. It is called the* torsion *of the parallelism* P*. If* X *and* Y *are* P*-parallel vector fields, then*

$$T^{\mathsf{P}}(X,Y) = -[X,Y] = [Y,X]. \tag{2.8}$$

Proof. Let $(E_i)_{i=1}^n$ be a frame field associated to P and consider its dual frame $(E^i)_{i=1}^n$. Then $\theta = E^i \otimes E_i(p)$. Indeed, for $q \in M$ and $w \in T_qM$ we have $\theta_q(w) := \mathsf{P}(q,p)(w) = \mathsf{P}(q,p)(w^i E_i(q))$, where $w^i = (E^i)_q(w)$ $(i = 1, \dots, n)$. Using the linearity of $\mathsf{P}(q,p)$ and the fact that the vector fields E_i are P-parallel, we obtain that

$$\theta_q(w) = (E^i)_q(w)E_i(p) = (E^i \otimes E_i(p))_q(w),$$

as claimed.

It follows that $d\theta = dE^i \otimes E_i(p)$, which is a T_pM-valued 2-form on M. So for any $X, Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned}(T^{\mathsf{P}}(X,Y))_q :=& \mathsf{P}(p,q)(d\theta(X,Y))_q \\ =& \mathsf{P}(p,q)\big((dE^i(X,Y))_q E_i(p)\big) = (dE^i(X,Y))_q E_i(q).\end{aligned} \tag{2.9}$$

It is clear that T^{P} is a type $(1,2)$ tensor field on M, and it can be seen from (2.9) that T^{P} is independent of the choice of $p \in M$.

To prove the second assertion, let $X = X^i E_i$, $Y = Y^i E_i$. Then

$$\begin{aligned}dE^i(X,Y) &= X(E^i(Y)) - Y(E^i(X)) - E^i([X,Y]) \\ &= X(Y^i) - Y(X^i) - [X,Y]^i.\end{aligned}$$

If, in particular, X and Y are P-parallel, then their component functions X^i, Y^i are constant, so in this case $dE^i(X,Y) = -[X,Y]^i$, and

$$T^{\mathsf{P}}(X,Y) \overset{(2.9)}{=} dE^i(X,Y) \otimes E_i = -[X,Y]^i E_i = -[X,Y],$$

thus (2.8) holds as well. □

It turns out from the proof that the torsion of a parallelism can be represented as

$$T^{\mathsf{P}} = dE^i \otimes E_i,$$

where $(E_i)_{i=1}^n$ is a P-parallel frame field and $(E^i)_{i=1}^n$ is its dual frame. We could have used this formula as a definition of T^{P}, but in this case one has to check that $dE^i \otimes E_i$ is independent of the choice of the frame field associated to P.

Remark 2.18. Consider a parallelized manifold (M, P) and a chart $(\mathcal{U}, u)$ for M. With the notation of Remark 2.8, the tensor components of the torsion of P with respect to the given chart are

$$(T^{\mathsf{P}})^i_{jk} = Q^l_j \frac{\partial P^i_l}{\partial u^k} - Q^l_k \frac{\partial P^i_l}{\partial u^j}.$$

Indeed,

$$\begin{aligned} T^{\mathsf{P}}\left(\frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}\right) &= T^{\mathsf{P}}(Q^l_j E_l, Q^s_k E_s) \overset{(2.8)}{=} Q^l_j Q^s_k [E_s, E_l] \\ &= Q^l_j Q^s_k \left[P^i_s \frac{\partial}{\partial u^i}, P^r_l \frac{\partial}{\partial u^r}\right] = Q^l_j Q^s_k P^i_s \frac{\partial P^r_l}{\partial u^i} \frac{\partial}{\partial u^r} - Q^l_j Q^s_k P^r_l \frac{\partial P^i_s}{\partial u^r} \frac{\partial}{\partial u^i} \\ &= Q^l_j \frac{\partial P^r_l}{\partial u^k} \frac{\partial}{\partial u^r} - Q^s_k \frac{\partial P^i_s}{\partial u^j} \frac{\partial}{\partial u^i} \overset{s \rightsquigarrow l, r \rightsquigarrow i}{=} \left(Q^l_j \frac{\partial P^i_l}{\partial u^k} - Q^l_k \frac{\partial P^i_l}{\partial u^j}\right) \frac{\partial}{\partial u^i}. \end{aligned}$$

We can apply Definition 2.2 for the torsion of a parallelism P on M to obtain that the torsion is parallel (more precisely, P-parallel): this holds if for any $p, q \in M$ and $X, Y \in \mathfrak{X}(M)$ we have

$$\mathsf{P}(p,q)(T^{\mathsf{P}}(X,Y))_p = T^{\mathsf{P}}_q(\mathsf{P}(p,q)(X_p), \mathsf{P}(p,q)(Y_p)). \tag{2.10}$$

Lemma 2.19. *The torsion of a parallelism* P *is parallel if, and only if, the set of* P*-parallel vector fields forms a subalgebra of the Lie algebra* $\mathfrak{X}(M)$*: if* X, Y *are* P*-parallel, then* $[X, Y]$ *is* P*-parallel as well.*

Proof. If X and Y are P-parallel vector fields, then $T^{\mathsf{P}}(X, Y) = [Y, X]$, so in this case (2.10) is equivalent to

$$\mathsf{P}(p,q)[Y,X]_p = [Y,X]_q.$$

But this is just the condition on $[Y, X]$ to be P-parallel.

If $X, Y \in \mathfrak{X}(M)$ are arbitrary, and the P-parallel vector fields form a subalgebra of $\mathfrak{X}(M)$, then (2.10) still holds for X and Y since T^{P} is tensorial and $\mathsf{P}(p, q)$ is a linear mapping for any $p, q \in M$. ∎

2.4 P-invariant covariant derivatives

Throughout this section, (M, P) denotes a parallelized manifold. On the analogy of ‘left-invariant connections’ on Lie groups ([26, Section 1 of Chapter II], [18, 17.6]) we introduce P-invariant covariant derivatives. Proposition 2.21 is also a generalization of the corresponding results found in the cited books.

Definition 2.20. We say that a covariant derivative ∇ on M is P-*invariant* if

$$\nabla_X Y \in \mathfrak{X}_{\mathsf{P}}(M) \quad \text{for all} \quad X, Y \in \mathfrak{X}_{\mathsf{P}}(M).$$

Proposition 2.21. *There is a canonical one-to-one correspondence between the set of* P*-invariant covariant derivatives* ∇ *on* M *and the set of* $\mathbb{R}$*-bilinear mappings* α *on* $\mathfrak{X}_{\mathsf{P}}(M) \times \mathfrak{X}_{\mathsf{P}}(M)$ *with values in* $\mathfrak{X}_{\mathsf{P}}(M)$ *such that*

$$\alpha(X, Y) = \nabla_X Y \quad \textit{for all } X, Y \in \mathfrak{X}_{\mathsf{P}}(M).$$

Proof. The mapping

$$\alpha\colon \mathfrak{X}_{\mathsf{P}}(M) \times \mathfrak{X}_{\mathsf{P}}(M) \to \mathfrak{X}(M),\ (X, Y) \mapsto \alpha(X, Y) := \nabla_X Y$$

is evidently an $\mathbb{R}$-bilinear mapping, and the P-invariance of ∇ implies that $\alpha(X, Y) \in \mathfrak{X}_{\mathsf{P}}(M)$.

Conversely, let α be an $\mathbb{R}$-bilinear mapping from $\mathfrak{X}_{\mathsf{P}}(M) \times \mathfrak{X}_{\mathsf{P}}(M)$ to $\mathfrak{X}_{\mathsf{P}}(M)$, and consider a frame field $(E_i)_{i=1}^n$ associated to P. Define a map

$$\begin{aligned} \nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M), \\ (X, Y) &\mapsto \nabla_X Y := X^i Y^j (\alpha(E_i, E_j)) + X^i (E_i Y^j) E_j, \end{aligned} \tag{2.11}$$

where the functions X^i, Y^j on M are given by the relations $X = X^i E_i$ and $Y = Y^i E_i$. Then a simple calculation shows that ∇ is a covariant derivative on M. Furthermore, ∇ is P-invariant, since if X and Y are P-parallel, then $X = a^i E_i$ and $Y = b^i E_i$ for some $a^i, b^i \in \mathbb{R}$, and so

$$\nabla_X Y = a^i b^j (\alpha(E_i, E_j)) + a^i (E_i b^j) E_j = a^i b^j (\alpha(E_i, E_j)).$$

Thus $\nabla_X Y$ is a P-parallel vector field by (2) of Remark 2.6. □

Proposition 2.22. *Consider the bijection between* P*-invariant covariant derivatives on* M *and* $\mathbb{R}$*-bilinear mappings of* $\mathfrak{X}_{\mathsf{P}}(M)$ *to* $\mathfrak{X}_{\mathsf{P}}(M)$ *described in Proposition 2.21.*

(A) The covariant derivative ∇ *corresponding to* $\alpha = 0$ *is a flat covariant derivative, and its torsion* T *is just the torsion of* P.

(B) Assume that the P*-parallel vector fields* $\mathfrak{X}_{\mathsf{P}}(M)$ *form a Lie subalgebra of* $\mathfrak{X}(M)$*. Then*

(i) if $\alpha(X,Y) := [X,Y]$ *for* $X,Y \in \mathfrak{X}_{\mathsf{P}}(M)$*, then the covariant derivative* ∇^+ *corresponding to* α *is also flat and its torsion is* $-T$*;*

(ii) if $\alpha(X,Y) := \frac{1}{2}[X,Y]$ *for* $X,Y \in \mathfrak{X}_{\mathsf{P}}(M)$*, then the covariant derivative* ∇^0 *corresponding to* α *is torsion-free and its curvature* R^0 *is given on* P*-parallel vector fields* X, Y, Z *by*

$$R^0(X,Y)Z = -\frac{1}{4}[[X,Y],Z]. \tag{2.12}$$

Proof. (A) Let ∇ be the covariant derivative corresponding to $\alpha = 0$. First notice that if $Y \in \mathfrak{X}_{\mathsf{P}}(M)$, then $\nabla Y = 0$. Indeed, in this case the component functions of Y with respect to a frame field associated to P are constant, so for any $X \in \mathfrak{X}(M)$ we have that $\nabla_X Y = 0$ by (2.11). Thus the torsion T of ∇ is given on P-parallel vector fields X and Y by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = -[X,Y] \overset{(2.8)}{=} T^{\mathsf{P}}(X,Y), \tag{2.13}$$

whence $T = T^{\mathsf{P}}$. The vanishing of the curvature R of ∇ follows immediately from the fact that all P-parallel vector fields are parallel with respect to ∇ and from (1) of Remark 2.6.

(B) Observe first, that our assumption about $\mathfrak{X}_{\mathsf{P}}(M)$ implies that in the cases (i) and (ii) α indeed maps $\mathfrak{X}_{\mathsf{P}}(M) \times \mathfrak{X}_{\mathsf{P}}(M)$ into $\mathfrak{X}_{\mathsf{P}}(M)$.

(i) Let first $\alpha(X,Y) := [X,Y]$ for X, Y in $\mathfrak{X}_{\mathsf{P}}(M)$. The torsion T^+ of the covariant derivative ∇^+ is given on P-parallel vector fields X, Y by

$$\begin{aligned} T^+(X,Y) &= \nabla^+_X Y - \nabla^+_Y X - [X,Y] \\ &= [X,Y] - [Y,X] - [X,Y] = [X,Y] \overset{(2.13)}{=} -T(X,Y). \end{aligned}$$

To prove that ∇^+ is flat, let $X, Y, Z \in \mathfrak{X}_{\mathsf{P}}(M)$. Then, applying the Jacobi identity, for the curvature R^+ of ∇^+ we have

$$\begin{aligned} R^+(X,Y)Z &= \nabla^+_X \nabla^+_Y Z - \nabla^+_Y \nabla^+_X Z - \nabla^+_{[X,Y]} Z \\ &= [X,[Y,Z]] - [Y,[X,Z]] - [[X,Y],Z] \\ &= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \end{aligned}$$

(ii) Now let α be defined by $\alpha(X,Y) := \frac{1}{2}[X,Y]$. Then the torsion T^0 of ∇^0 is

$$\begin{aligned} T^0(X,Y) &= \nabla^0_X Y - \nabla^0_Y X - [X,Y] \\ &= \frac{1}{2}[X,Y] - \frac{1}{2}[Y,X] - [X,Y] = 0, \end{aligned}$$

so ∇^0 is indeed torsion-free. To prove the expression for the curvature R^0 consider P-parallel vector fields X, Y, Z. Calculating as above, we find

$$\begin{aligned} R^0(X,Y)Z &= \nabla^0_X \nabla^0_Y Z - \nabla^0_Y \nabla^0_X Z - \nabla^0_{[X,Y]} Z \\ &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}\big([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]\big) + \frac{1}{4}[Z,[X,Y]] \\ &= -\frac{1}{4}[[X,Y],Z], \end{aligned}$$

as claimed. □

Proposition 2.23. *The geodesics of a* P*-invariant covariant derivative on* (M,P) *coincide with the geodesics of* (M,P) *if, and only if, the corresponding* $\mathbb{R}$*-bilinear mapping is skew-symmetric.*

Proof. Let $\widehat{\nabla}$ be a P-invariant covariant derivative on M.

First, assume that α is skew-symmetric, or, equivalently, that $\alpha(X,X) = 0$ for every P-parallel vector field X. Fix an $X \in \mathfrak{X}_{\mathsf{P}}(M)$ and let $\gamma\colon I \to M$ be an integral curve of X. Then for any $t \in I$ we have

$$(\widehat{\nabla}^\gamma \dot\gamma)(t) = \widehat{\nabla}_{\dot\gamma(t)} X = \widehat{\nabla}_{X(\gamma(t))} X = \alpha(X,X)(\gamma(t)) = 0,$$

thus γ is a geodesic of $\widehat{\nabla}$.

To prove the converse statement, suppose that the integral curves of P-parallel vector fields are geodesics of $\widehat{\nabla}$. We show that $\alpha(X,X) = 0$ for each $X \in \mathfrak{X}_{\mathsf{P}}(M)$. Given a point $p \in M$, there exists an integral curve $\gamma\colon I \to M$ of X such that $\gamma(0) = p$. Then γ is a geodesic by assumption, so we obtain

$$\alpha(X,X)_p = \alpha(X,X)_{\gamma(0)} = (\widehat{\nabla}_X X)(\gamma(0)) = \widehat{\nabla}_{\dot\gamma(0)} X = (\widehat{\nabla}^\gamma \dot\gamma)(0) = 0,$$

which implies the skew-symmetry of α. □

Corollary 2.24. *The covariant derivatives* ∇, ∇^{+} *and* ∇^{0} *defined in Proposition 2.22 have the same geodesics as* (M, P).

Proof. In all three cases the corresponding $\mathbb{R}$-bilinear mapping α is skew-symmetric, thus the assertion is a consequence of Proposition 2.23 and Lemma 2.14. □

Corollary 2.25. *Let* (M, P) *be a manifold with parallelism. Given a point* p *in* M *and a tangent vector* $v \in T_pM$*, there exists a unique maximal geodesic* $\gamma\colon I \to M$ *of* P *such that* $\dot{\gamma}(0) = v$.

Remark 2.26. The covariant derivative ∇ in (A) of Proposition 2.22 is called the *covariant derivative induced by* P. From now on, unless otherwise stated, if (M, P) is a manifold with parallelism, then ∇ stands for its induced covariant derivative. It is also mentioned as 'Weitzenböck connection' in the literature, mainly among physicists (see, e.g., [31]).

It was highlighted in the first part of the proof of Proposition 2.22, that ∇ is a covariant derivative on M such that all P-parallel vector fields are parallel with respect to ∇. (Actually, ∇ is uniquely determined by this property due to (1) of Remark 2.6.) The exact relation is the following:

Lemma 2.27. *Let* P *be a parallelism on a connected manifold* M. *The necessary and sufficient condition for a vector field* X *on* M *to be* P*-parallel is that* $\nabla X = 0$. *If* M *is not connected, then it is only a necessary condition.*

Proof. It is clear from the definition of ∇ that if X is a P-parallel vector field, then $\nabla X = 0$.

Now suppose that for $X \in \mathfrak{X}(M)$ we have $\nabla X = 0$. Given a P-parallel frame field $(E_i)_{i=1}^n$, the vector field X has the form $X = X^i E_i$, thus if $Y \in \mathfrak{X}(M)$, then

$$0 = \nabla_Y X = \nabla_Y (X^i E_i) = (Y X^i) E_i + X^i \nabla_Y E_i = (Y X^i) E_i,$$

thus $Y X^i = 0$ ($i = \{1, \dots, n\}$). These relations hold for any $Y \in \mathfrak{X}(M)$, thus the component functions X^i of X with respect to $(E_i)_{i=1}^n$ must be constant functions on the components of M. Hence, if M is connected, then X is P-parallel by (2) of Remark 2.6. □

By (2.13), the torsion of the parallelism coincides with the torsion of its induced covariant derivative. Thus in what follows, if there is no danger of confusion, we will use the notation T instead of T^{P}.

Remark 2.28. (1) Let ∇ be the induced covariant derivative of (M, P), and let $\gamma\colon I \to M$ be a curve in M. The parallel translations $(P_\gamma)_{t_1}^{t_2}$ with respect to ∇ along γ (for $t_1, t_2 \in I$) can be simply expressed in terms of P. Namely, if $v \in T_{\gamma(t_1)}M$, then the unique parallel vector field (with respect to ∇) along γ is $v_{\mathsf{P}} \circ \gamma$, hence

$$(P_\gamma)_{t_1}^{t_2}(v) = (v_{\mathsf{P}})_{\gamma(t_2)} = \mathsf{P}(\gamma(t_1), \gamma(t_2))(v_{\mathsf{P}})_{\gamma(t_1)} = \mathsf{P}(\gamma(t_1), \gamma(t_2))(v).$$

(2) We calculate the Christoffel symbols Γ^i_{jk} of ∇ with respect to a chart $(\mathcal{U}, u)$ of M. To do this, let $(E_i)_{i=1}^n$ be a P-parallel frame field, and use the notation of Remark 2.8. Then

$$0 = \nabla_{\frac{\partial}{\partial u^j}} E_l = \nabla_{\frac{\partial}{\partial u^j}}\Big(P_l^k \frac{\partial}{\partial u^k}\Big) = \frac{\partial P_l^k}{\partial u^j}\frac{\partial}{\partial u^k} + P_l^k \Gamma^i_{jk}\frac{\partial}{\partial u^i}.$$

From this we get $\Gamma^i_{jk} = -Q^l_k \frac{\partial P^i_l}{\partial u^j}$, and we can see again that ∇ is well-defined, cf. Lemma 2.7.

Proposition 2.29. *Let (M, P) be a manifold with parallelism, and consider its induced covariant derivative ∇. Then the linear Ehresmann connection associated to (M, ∇) according to **1.16** is just the Ehresmann connection generated by P, which was introduced in Lemma and definition 2.11.*

Proof. Consider two tangent vectors $v, w \in T_pM$. Then $\nabla v_{\mathsf{P}} = 0$ and, of course, $v_{\mathsf{P}}(p) = v$, thus the defining equality (7.5.12) of the Ehresmann connection $\mathcal{H}$ associated to ∇ in [49, Proposition 7.5.11] turns into

$$\mathcal{H}(v, w) := (v_{\mathsf{P}})_*(w) - (\nabla_w v_{\mathsf{P}})^{\uparrow}(v) = (v_{\mathsf{P}})_*(w),$$

because $u \in T_pM \mapsto u^{\uparrow}(v) \in T_vTM$ is linear, see [49, Lemma 4.1.16]. This is just the canonical Ehresmann connection of (M, P) defined in Lemma and definition 2.11. $\square$

The proposition could have been proved using coordinate expressions: by comparing **1.16** and (2) of Remark 2.28 with (2.4). In this case the coincidence of the Ehresmann connections is immediate.

2.5 Automorphisms of a parallelized manifold

Definition 2.30. Let (M, P) and $(\bar{M}, \bar{\mathsf{P}})$ be manifolds with parallelisms. We say that a diffeomorphism $\varphi\colon M \to \bar{M}$ is an *isomorphism of parallelized* *manifolds*, if for any $p, q \in M$

$$(\varphi_*)_q \circ \mathsf{P}(p,q) = \bar{\mathsf{P}}(\varphi(p), \varphi(q)) \circ (\varphi_*)_p. \tag{2.14}$$

An isomorphism of (M, P) onto itself is called an *automorphism* of the parallelized manifold or of P.

Remark 2.31. (1) The automorphisms of (M, P) form a group under composition called the *automorphism group* of (M, P) and denoted by $\mathrm{Aut}(\mathsf{P})$.
(2) With the notation of Remark 2.3(1), equality (2.14) is equivalent to

$$\varphi_* \circ v_{\mathsf{P}} = (\varphi_*(v))_{\bar{\mathsf{P}}} \circ \varphi, \quad \text{for any } v \in TM. \tag{2.15}$$

Indeed, if we evaluate both sides of (2.15) at an arbitrary $q \in M$, and take into account that v_{P} is P-parallel, $(\varphi_*(v))_{\bar{\mathsf{P}}}$ is $\bar{\mathsf{P}}$-parallel, then

$$\begin{aligned} (\varphi_*)_q(v_{\mathsf{P}}(q)) &= (\varphi_*)_q(\mathsf{P}(p,q)(v_{\mathsf{P}})_p), \\ (\varphi_*(v))_{\bar{\mathsf{P}}}(\varphi(q)) &= \bar{\mathsf{P}}(\varphi(p), \varphi(q))\big((\varphi_*(v))_{\bar{\mathsf{P}}}(\varphi(p))\big) \\ &= \bar{\mathsf{P}}(\varphi(p), \varphi(q))(\varphi_*(v)), \end{aligned}$$

in this way we obtain the two sides of (2.14) evaluated at v. Equality (2.15) expresses that for any P-parallel vector field X the vector field $\varphi_\# X$ on $\bar{M}$ is $\bar{\mathsf{P}}$-parallel.
(3) In the book [7] we find some stronger notion of an automorphism: a diffeomorphism φ is called an automorphism of (M, P) if $\varphi_\# E_i = E_i$ for $i \in \{1, \dots, n\}$, where $(E_i)_{i=1}^n$ is a frame field associated to P. This condition is equivalent to the fact that for each P-parallel vector field X we have $\varphi_\# X = X$. However, if $\varphi \in \mathrm{Aut}(\mathsf{P})$, then φ not necessarily leaves all the P-parallel vector fields invariant. We show two examples for this phenomenon, but before, we introduce a new concept to distinguish between the different definitions.

Definition 2.32. Let (M, P) be a manifold with parallelism. We say that a mapping $\varphi \in \mathrm{Diff}(M)$ is a *symmetry* or *translation* of (M, P) (or of P), if $\varphi_\# X = X$ for every P-parallel vector field X.

The symmetries of (M, P) form a group; we denote this group by $\mathrm{Sym}(\mathsf{P})$. Evidently, $\mathrm{Sym}(\mathsf{P})$ is a subgroup of $\mathrm{Aut}(\mathsf{P})$. This subgroup is a proper one, as the following examples illustrate.

Example 2.33. (1) Consider the Euclidean n-space endowed with its natural parallelism P_0 (see Example 2.4). Then $\mathrm{Aut}(\mathsf{P}_0)$ coincides with the group of affine transformations of $\mathbb{R}^n$, however, $\mathrm{Sym}(\mathsf{P}_0)$ consists of the translations of $\mathbb{R}^n$. This example justifies the name 'translation' for this special kind of automorphism.

(2) Let G be a Lie group. Then there exists a unique parallelism P_L such that the left invariant vector fields are the P_L-parallel vector fields; this parallelism will be investigated in Section 2.6. If X is a left invariant vector field and $g \in G$, then $(\rho_g)_\# X$ is again left invariant; to see this, we only have to look at (1.6). However, generally, $(\rho_g)_\# X \neq X$, because it would mean that X is also right invariant.

Proposition 2.34. *If (M, P) is a manifold with parallelism, then the automorphisms of P are automorphisms of its induced covariant derivative ∇ as well. If M is connected then the converse also holds, thus in this case* $\mathrm{Aut}(\mathsf{P}) = \mathrm{Aut}(\nabla)$.

Proof. Let ∇ be the covariant derivative induced by P and consider a diffeomorphism φ of M.

First, suppose that $\varphi \in \mathrm{Aut}(\mathsf{P})$, and let $X, Y \in \mathfrak{X}(M)$. Consider a frame field $(E_i)_{i=1}^n$ associated to P. Then $X = X^i E_i$, $Y = Y^i E_i$ for some smooth functions X^i, Y^i on M and so $\nabla_X Y = X^i(E_i Y^j)E_j$. At a point $p \in M$ we obtain

$$\begin{aligned}(\varphi_\#(\nabla_X Y))_p &= (\varphi_*)_{\varphi^{-1}(p)}\big((\nabla_X Y)_{\varphi^{-1}(p)}\big)\\ &= (\varphi_*)_{\varphi^{-1}(p)}\big(X^i(\varphi^{-1}(p))(E_iY^j)(\varphi^{-1}(p))E_j(\varphi^{-1}(p))\big)\\ &= X^i(\varphi^{-1}(p))(E_iY^j)(\varphi^{-1}(p))(\varphi_*)_{\varphi^{-1}(p)}\big(E_j(\varphi^{-1}(p))\big)\\ &= ((X^i\circ\varphi^{-1})(E_iY^j\circ\varphi^{-1})(\varphi_\# E_j))_p.\end{aligned} \tag{2.16}$$

Since, for example, $\varphi_\# X = (X^i \circ \varphi^{-1})(\varphi_\# E_i)$, we also have

$$\begin{aligned}\nabla_{\varphi_\# X}(\varphi_\# Y) = (X^i\circ\varphi^{-1})((\varphi_\# E_i)(Y^j\circ\varphi^{-1}))(\varphi_\# E_j)\\ + (X^i\circ\varphi^{-1})(Y^j\circ\varphi^{-1})\nabla_{\varphi_\# E_i}(\varphi_\# E_j).\end{aligned} \tag{2.17}$$

Here the second term is zero, because $\varphi_\# E_j$ is P-parallel. Finally,

$$\begin{aligned}(\varphi_\# E_i)(Y^j\circ\varphi^{-1})(p) &= (\varphi_*)_{\varphi^{-1}(p)}\big(E_i(\varphi^{-1}(p))\big)(Y^j\circ\varphi^{-1})\\ &= (E_i)_{\varphi^{-1}(p)}(Y^j\circ\varphi^{-1}\circ\varphi) = (E_i)_{\varphi^{-1}(p)}(Y^j) = (E_iY^j\circ\varphi^{-1})(p),\end{aligned}$$

thus (2.17) at p is equal to (2.16), and it follows that $\varphi \in \mathrm{Aut}(\nabla)$ (see **1.13**).

Conversely, assume that M is connected and $\varphi \in \mathrm{Aut}(\nabla)$. Consider an arbitrary tangent vector $v \in T_pM$; we show that (2.15) holds. Notice that by Lemma 2.27 the vector field $\varphi_\# v_\mathsf{P}$ is P-parallel, since for any $X \in \mathfrak{X}(M)$,

$$\nabla_X(\varphi_\# v_\mathsf{P}) \overset{\text{cond.}}{=} \varphi_\#(\nabla_{(\varphi^{-1})_\# X} v_\mathsf{P}) = 0.$$

Furthermore, at a point p we have

$$\begin{aligned}(\varphi_\# v_\mathsf{P})(p) &= \big((\varphi_\# v_\mathsf{P})(\varphi(p))\big)_\mathsf{P}(p) \\ &= \big((\varphi_*)_p(v_\mathsf{P}(p))\big)_\mathsf{P}(p) = ((\varphi_*)_p(v))_\mathsf{P}(p),\end{aligned}$$

which means that the two P-parallel vector fields $\varphi_\# v_\mathsf{P}$ and $((\varphi_*)_p(v))_\mathsf{P}$ are equal at p, thus they must coincide by Remark 2.3(1). Therefore we have $\varphi_* \circ v_\mathsf{P} \circ \varphi^{-1} = ((\varphi_*)_p(v))_\mathsf{P}$, and the proof is concluded. □

Proposition 2.35. *The automorphisms of* (M, P) *are also automorphisms of its generated spray* S*, that is,* $\mathrm{Aut}(\mathsf{P}) \subset \mathrm{Aut}(S)$.

Proof. Let φ be an automorphism of (M, P) and choose a tangent vector v. Then

$$\begin{aligned}\varphi_{**}(S(v)) &= \varphi_{**}((v_\mathsf{P})_*(v)) = (\varphi_* \circ v_\mathsf{P})_*(v) \\ &\overset{(2.15)}{=} ((\varphi_*(v))_\mathsf{P} \circ \varphi)_*(v) = ((\varphi_*(v))_\mathsf{P})_*(\varphi_*(v)) = S(\varphi_*(v)),\end{aligned}$$

therefore φ is an automorphism of S, cf. **1.12**. □

Corollary 2.36. *The automorphisms of a parallelized manifold map geodesics to geodesics.*

Proof. Let γ be a geodesic of a parallelized manifold (M, P), and consider an automorphism φ of P. Lemma 2.16 implies that γ is a geodesic of the induced spray S as well, and, by the previous lemma, $\varphi \in \mathrm{Aut}(S)$. We conclude from **1.12** that $\varphi \circ \gamma$ is a geodesic of S, and hence also of P. □

Lemma 2.37. *If* φ *is an isomorphism of parallelized manifolds* (M, P) *and* $(\bar{M}, \bar{\mathsf{P}})$*, then for any* $X, Y \in \mathfrak{X}(M)$ *we have*

$$(T^{\bar{\mathsf{P}}} \circ \varphi)(\varphi_* \circ X, \varphi_* \circ Y) = \varphi_* \circ T^\mathsf{P}(X, Y), \tag{2.18}$$

that is, the torsion is preserved by isomorphisms.

Proof. Let $(E_i)_{i=1}^n$ be a frame field on M consisting of P-parallel vector fields, as above. Then the family $(\bar{E}_i)_{i=1}^n$, where

$$\bar{E}_i := \varphi_{\#} E_i := \varphi_* \circ E_i \circ \varphi^{-1}$$

(that is, $\bar{E}_i$ is the push-forward of E_i by φ), is a frame field on $\bar{M}$ associated to $\bar{\mathsf{P}}$. Since

$$\begin{aligned}(T^{\bar{\mathsf{P}}} \circ \varphi)(\varphi_* \circ E_j, \varphi_* \circ E_k) &= (T^{\bar{\mathsf{P}}} \circ \varphi)(\bar{E}_j \circ \varphi, \bar{E}_k \circ \varphi) \\ &\overset{(2.8)}{=} -[\bar{E}_j, \bar{E}_k] \circ \varphi = -[\varphi_{\#} E_j, \varphi_{\#} E_k] \circ \varphi \\ &= -\varphi_* \circ [E_j, E_k] = \varphi_* \circ T^{\mathsf{P}}(E_j, E_k),\end{aligned}$$

(2.18) holds with the choice $(X, Y) := (E_j, E_k)$, and thus for all pair of vector fields on M. $\square$

Lemma 2.38. *Let (M, P) be a parallelized manifold with vanishing torsion. Then every point $p \in M$ has a neighbourhood $\mathcal{U}$ such that there exists an isomorphism from $\mathcal{U}$ onto an open subset $\mathcal{U}_0$ of $\mathbb{R}^n$, where $\mathcal{U}_0$ is endowed with the restriction of the canonical parallelism of $\mathbb{R}^n$.*

Proof. Let $(E_i)_{i=1}^n$ be a frame field on M associated to P. By (2.8), we have

$$[E_i, E_j] = T^{\mathsf{P}}(E_j, E_i) = 0 \quad (i, j \in \{1, \ldots, n\}).$$

Theorem 9.46 in [37] shows that this property is equivalent to having a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ around every point $p \in M$ such that $E_i = \frac{\partial}{\partial u^i}$ $(i \in \{1, \ldots, n\})$ on the set $\mathcal{U}$. In this case the coordinate map $u = (u^1, \ldots, u^n)$ is the sought-for isomorphism between $\mathcal{U}$ and $\mathcal{U}_0 := u(\mathcal{U})$, that is,

$$(u_*)_q \circ \mathsf{P}(p, q) = \mathsf{P}_0(u(p), u(q)) \circ (u_*)_p \tag{2.19}$$

holds for all $p, q \in \mathcal{U}$, where P_0 is the natural parallelism on $\mathcal{U}_0 \subset \mathbb{R}^n$.

To prove (2.19), let $v := v^i E_i(p) = v^i \left(\frac{\partial}{\partial u^i}\right)_p \in T_pM$, and notice that

$(u_*)_p \left(\frac{\partial}{\partial u^i}\right)_p = \left(\frac{\partial}{\partial e^i}\right)_{u(p)}$. Then

$$\begin{aligned}(u_*)_q(\mathsf{P}(p,q)(v)) &= v^i(u_*)_q(\mathsf{P}(p,q)(E_i)_p) = v^i(u_*)_q\left(\frac{\partial}{\partial u^i}\right)_q \\ &= v^i\left(\frac{\partial}{\partial e^i}\right)_{u(q)} = v^i\,\mathsf{P}_0(u(p),u(q))\left(\frac{\partial}{\partial e^i}\right)_{u(p)} \\ &= v^i\,\mathsf{P}_0(u(p),u(q))\left((u_*)_p\left(\frac{\partial}{\partial u^i}\right)_p\right) \\ &= \mathsf{P}_0(u(p),u(q))((u_*)_p(v)),\end{aligned}$$

as we claimed. □

2.6 Parallelisms and Lie groups

The natural parallelisms of Lie groups

We have two natural parallelisms on every Lie group induced by the left and right translations. Now we have a look at these parallelisms from the point of view of the general theory. Throughout this section G is a Lie group and the notation of Section 1.3 is in effect.

Lemma and definition 2.39. *The mapping*

$$\mathsf{P}_L\colon (p,q)\in G\times G\mapsto \mathsf{P}_L(p,q) := ((\lambda_{qp^{-1}})_*)_p \in L(T_pG, T_qG) \quad (2.20)$$

is a parallelism on G, called the left parallelism *of the Lie group. The* right parallelism *of G is*

$$\mathsf{P}_R\colon (p,q)\in G\times G\mapsto \mathsf{P}_R(p,q) := ((\rho_{p^{-1}q})_*)_p \in L(T_pG, T_qG).$$

Then the P_L-parallel vector fields are the left invariant, and the P_R-parallel vector fields are the right invariant vector fields of G.

Proof. By the properties of the derivative, $\mathsf{P}_L(p,q)$ is a linear mapping between the appropriate tangent spaces and $\mathsf{P}_L(p,p) = ((1_G)_*)_p = 1_{T_pG}$. The first condition in (2.1) is a consequence of the chain rule, and the smoothness of the multiplication map of G implies that P_L is smooth.

If $X \in \mathfrak{X}_L(G)$, then for any $p, q \in G$,

$$\mathsf{P}_L(p,q)X_p = ((\lambda_{qp^{-1}})_* \circ X)_p \overset{(1.7)}{=} (X \circ \lambda_{qp^{-1}})_p = X_q.$$

Conversely, if X is P_L-parallel, then

$$\begin{aligned}(X \circ \lambda_p)(q) = X_{pg} &= \mathsf{P}_L(p,pq)X_p = ((\lambda_{pqp^{-1}})_*)_p X_p \\ &= ((\lambda_p)_*)_q((\lambda_{qp^{-1}})_*)_p X_p = ((\lambda_p)_*)_q(\mathsf{P}_L(p,q)X_p) \\ &= ((\lambda_p)_*)_q X_q = ((\lambda_p)_* \circ X)(q),\end{aligned}$$

thus the vector space of P_L-parallel vector fields and the vector space $\mathfrak{X}_L(G)$ are the same.

The proof is similar for P_R. □

In the case of a connected Lie group, the symmetry group of P_L is $\mathrm{Sym}(\mathsf{P}_L) = \{\lambda_g \in \mathrm{Diff}(G) \mid g \in G\}$. Indeed, it is obvious that the left translations are symmetries of P_L. Now assume that $\varphi \in \mathrm{Sym}(\mathsf{P}_L)$, and let $g := \varphi(e)$. Then $\lambda_{g^{-1}}(\varphi(e)) = \lambda_{g^{-1}}(g) = e$, so e is a fixed point of the symmetry $\lambda_{g^{-1}} \circ \varphi$. From this we conclude by [7, Proposition 13.6.1], that $\lambda_g = \varphi$. (Remember that the cited book uses different terminology, see (3) of Remark 2.31 and the definition afterwards).

Besides the left translations, the automorphism group $\mathrm{Aut}(\mathsf{P}_L)$ contains the right translations and the conjugations by the elements of G, since if $X \in \mathfrak{X}_L(G)$ and $g \in G$, then the vector fields $(\rho_g)_\# X$ and $(\mathbf{c}_g)_\# X$ are also left invariant.

Lemma 2.40. *The geodesics of P_L and P_R coincide, and these are the one-parameter subgroups of G and their left and right translations.*

Proof. It follows from **1.21** and Lemma 2.14 that the one-parameter subgroups of G are geodesics through $e \in G$ both for P_L and P_R. We know that $\lambda_g \in \mathrm{Aut}(\mathsf{P}_L)$ for every $g \in G$, thus Corollary 2.36 implies that if α is a one-parameter subgroup, then $\lambda_g \circ \alpha$ is a geodesic of P_L. If $t_1, t_2 \in \mathbb{R}$, then

$$\begin{aligned}&\mathsf{P}_L\big(\rho_g(\alpha(t_1)), \rho_g(\alpha(t_2))\big)\dot{\overline{\rho_g \circ \alpha}}(t_1) \overset{(1.1)}{=} ((\lambda_{\alpha(t_2)\alpha(t_1)^{-1}})_* \circ (\rho_g)_*)(\dot\alpha(t_1)) \\ &\overset{(1.6)}{=} ((\rho_g)_* \circ (\lambda_{\alpha(t_2)\alpha(t_1)^{-1}})_*)(\dot\alpha(t_1)) \\ &= \big((\rho_g)_* \circ \mathsf{P}_L(\alpha(t_1), \alpha(t_2))\big)(\dot\alpha(t_1)) = (\rho_g)_*(\dot\alpha(t_2)) \overset{(1.1)}{=} \dot{\overline{\rho_g \circ \alpha}}(t_2),\end{aligned}$$

so the right translations of one-parameter subgroups are geodesics of P_L as well.

An analogous argument applies to P_R. Thus the one-parameter subgroups and their left and right translations are geodesics of both P_L and P_R. Since every tangent vector v in T_gG can be written in the form $v = (\lambda_g)_*(\dot\alpha(0)) = (\rho_g)_*(\dot\beta(0))$ for some one-parameter subgroups α and β, these curves exhaust the geodesics of P_L and P_R by Corollary 2.25. $\square$

We know from Lemma 2.27, that the induced covariant derivative ∇ of P_L is the one that satisfies $\nabla X = 0$ for every left invariant vector field X on G. The next observation shows that we have already met with the induced covariant derivative of P_R as well.

Lemma 2.41. *The covariant derivative ∇^+ associated to P_L by Proposition 2.22, i.e., the covariant derivative defined by*

$$\nabla^+_X Z := [X, Z], \quad \textit{for } X, Z \in \mathfrak{X}_L(G), \tag{2.21}$$

is the induced covariant derivative of the right parallelism P_R.

Proof. Define a covariant derivative ∇^+ on G by (2.21), and consider a left invariant vector field X on G. If $Y \in \mathfrak{X}(G)$ is right invariant and it is expressed as $Y = Y^i E_i$ in terms of a left invariant frame field $(E_i)_{i=1}^n$, then we have

$$\nabla^+_X Y = \nabla^+_X (Y^i E_i) = (XY^i)E_i + Y^i[X, E_i] = [X, Y] \overset{(1.10)}{=} 0.$$

Thus $\nabla^+ Y = 0$ for any right invariant vector field Y on G, therefore ∇^+ is indeed the covariant derivative induced by P_R. $\square$

Consider the parallelized manifold (G, P_L). The three distinguished covariant derivatives attached to P_L according to Proposition 2.22 are the notable covariant derivatives of a Lie group known from the literature as the ∇^-, ∇^0 and ∇^+ covariant derivatives on G, see, e.g., [40]. This statement motivates our notation in the just cited proposition, however, we shall rather denote the frequently used induced covariant derivative on a parallelized manifold simply by ∇ instead of ∇^-. (Notice that in the case of Lie groups the assumption on $\mathfrak{X}_L(G)$ being a Lie subalgebra is automatically satisfied by **1.19**, so ∇^0 and ∇^+ exist.)

Thus, the torsions $T^- := T^{\mathsf{P}_L}$ and $T^+ := T^{\mathsf{P}_R}$ of the canonical parallelisms of a Lie group act on left invariant vector fields X and Z by

$$T^-(X,Z) = [Z,X] \quad \text{and} \quad T^+(X,Z) = [X,Z].$$

Parallelized manifolds as Lie groups

We have already seen that every Lie group is parallelizable. One may ask: under what conditions is a parallelized manifold (M,P) diffeomorphic to a Lie group such that the given parallelism is essentially the left parallelism of the Lie group?

From the theory of Lie groups we know that the left parallelism P_L is complete, and the left invariant vector fields form a subalgebra of $\mathfrak{X}(G)$ (or, equivalently, the torsion of P_L is parallel; see Lemma 2.19). Thus the completeness of (M,P) and the parallelism of the torsion of P provide necessary conditions. It turns out that in the connected case these conditions are also sufficient for M to be diffeomorphic to a Lie group up to a factorization by a discrete subgroup.

The following important result can be found in Joseph A. Wolf's fundamental paper [54]. For an equivalent formulation, we refer to [21, Problem 17 in Chapter III].

Proposition 2.42. *If (M,P) is a connected parallelized manifold such that P is complete and the torsion of P is parallel, then M is diffeomorphic to $H\backslash G$, where H is a discrete subgroup of a simply connected Lie group G, and the parallelism P is induced by the left parallelism P_L on G.*

Proof. The assumption that the torsion of P is parallel implies by Lemma 2.19 that $\mathfrak{X}_{\mathsf{P}}(M)$, the real vector space of P-parallel vector fields, is a Lie algebra. Choose and fix a point $p \in M$, and for $u, v \in T_pM$ let

$$[u,v] := [u_{\mathsf{P}}, v_{\mathsf{P}}]_p.$$

With this Lie bracket T_pM becomes a (finite-dimensional) Lie algebra (cf. (1.9)). By Lie's third theorem [29, Theorem 9.4.11], there exists a connected and simply connected Lie group G such that T_pM is the Lie algebra $\mathsf{Lie}(G) = T_eG$ of G. Then the mapping

$$T_pM = T_eG \to \mathfrak{X}(M),\ v \mapsto v_{\mathsf{P}} \tag{2.22}$$

is obviously an action of a Lie algebra: it is a Lie algebra homomorphism such that the evaluation mapping is smooth (cf. **1.27**). This Lie algebra action is complete, since the P-parallel vector fields are complete by assumption. Thus, by the fundamental theorem on Lie algebra actions (see **1.32**) there exists a unique right action A of the Lie group G on M whose infinitesimal generator is the mapping given by (2.22). Owing to this property, we use the notation $\widehat{A}$ for this map.

Next we show that the Lie group action A is transitive. More precisely, we prove that the orbit of the point $p \in M$ is open, then the connectedness of M implies the equivalent property that there is only one orbit. Since for $v \in T_eG$ we have

$$((A_p)_*)_e(v) =: \widehat{A}(v)(p) := v_{\mathsf{P}}(p) = v,$$

it follows that $((A_p)_*)_e$ is the identity map of $T_pM = T_eG$, and hence it is surjective. This ensures the openness of the orbit $p \cdot G$ by **1.31**.

Thus, there is a Lie group G acting transitively on M. Now, by Proposition 13.3.3 in [7], M is diffeomorphic to the quotient manifold $H \backslash G$, where H is the isotropy subgroup of an arbitrarily chosen point of M. Let this point be the already fixed point $p \in M$, that is, let $H = G_p$. By the argument above $((A_p)_*)_e$ is also injective, so the isotropy subgroup G_p is discrete, see again **1.31**. A natural diffeomorphism of $G_p \backslash G$ onto M is given by the mapping

$$\psi_p \colon G_p \backslash G \to M, \; G_p g \mapsto \psi_p(G_p g) := p \cdot g,$$

as it turns out from Section 13.3 of [7] as well. Finally, by the construction, P is the left parallelism of G. □

The proof above is based on the argument of the above cited [54, Proposition 2.5]. For other proofs we refer to [9], [19], [27], [28].

2.7 Conjugate parallelisms

Definition 2.43. Two parallelisms P_1 and P_2 on M are called *conjugate* if $[X, Y] = 0$ whenever X is P_1-parallel and Y is P_2-parallel.

Obviously, this is a symmetric relation on the set of parallelisms on M.

The motivation for introducing conjugate parallelisms obviously comes from the theory of Lie groups: the parallelisms P_L and P_R defined in Lemma and definition 2.39 are clearly conjugate by (1.10).

Lemma 2.44. *On a connected manifold every parallelism has at most one conjugate parallelism.*

Proof. Let P_1 be a parallelism on M. Arguing by contradiction, suppose that the parallelisms P_2 and $\bar{\mathsf{P}}_2$ are both conjugate with P_1. Consider a frame field $(E_i)_{i=1}^n$ associated to P_2, a P_1-parallel vector field X on M and a $\bar{\mathsf{P}}_2$-parallel vector field Y on M. Let Y^i, $i \in \{1, \dots, n\}$, be the component functions of Y with respect to $(E_i)_{i=1}^n$. Since

$$0 = [X, Y] = [X, Y^i E_i] = Y^i [X, E_i] + (XY^i)E_i = (XY^i)E_i,$$

the vector field $X \in \mathfrak{X}_{\mathsf{P}_1}(M)$ is arbitrary, and M is connected, the functions Y^i must be constant. Thus the $\bar{\mathsf{P}}_2$-parallel vector fields are P_2-parallel as well, from which the assertion follows. □

So we can state that a conjugate parallelism, if exists, is unique. Lemma 2.46 below will show us in which cases the existence is guaranteed. Beforehand, we compare the induced covariant derivatives of conjugate parallelisms.

Lemma 2.45. *Let P_1 and P_2 be two conjugate parallelisms on M with induced covariant derivatives ∇_1 and ∇_2, respectively. Then*

$$(\nabla_2)_X Y = (\nabla_1)_Y X + [X, Y], \text{ for any } X, Y \in \mathfrak{X}(M). \tag{2.23}$$

Proof. Define a mapping $\widehat{\nabla} \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$\widehat{\nabla} \colon (X, Y) \mapsto \widehat{\nabla}_X Y := (\nabla_1)_Y X + [X, Y].$$

It is easy to check that $\widehat{\nabla}$ is a covariant derivative on M. Choose a P_1-parallel frame field $(E_i)_{i=1}^n$ on M. Then we can write $X = X^i E_i$ with some functions $X^i \in C^\infty(M)$. If $Y \in \mathfrak{X}_{\mathsf{P}_2}(M)$, we obtain

$$\begin{aligned} \widehat{\nabla}_X Y &= (\nabla_1)_Y X + [X, Y] = (\nabla_1)_Y (X^i E_i) + [X^i E_i, Y] \\ &= (Y X^i) E_i + X^i (\nabla_1)_Y E_i + X^i [E_i, Y] - (Y X^i) E_i \\ &= X^i [E_i, Y] = 0, \end{aligned}$$

taking into account that P_1 and P_2 are conjugate parallelisms. Hence for every P_2-parallel vector field Y we have $\widehat{\nabla}Y = 0$, which proves that $\widehat{\nabla}$ is the covariant derivative induced by P_2. □

Lemma 2.46 ([20], Problem 14 in Chapter IV)**.** *If (M, P) is a parallelized manifold and P admits a conjugate parallelism, then the torsion of P is parallel. Conversely, if the torsion of a parallelism P on M is parallel, then every point has a neighbourhood in which a conjugate parallelism exists.*

Proof. To prove the first part of the lemma, let $\bar{\mathsf{P}}$ denote the parallelism conjugate to P, and consider the covariant derivatives ∇ and $\bar{\nabla}$ induced by P and $\bar{\mathsf{P}}$, respectively. If $X, Y \in \mathfrak{X}_{\mathsf{P}}(M)$ and $Z \in \mathfrak{X}_{\bar{\mathsf{P}}}(M)$, then for the torsion T of P we have

$$\begin{aligned}(\nabla_Z T)(X, Y) &= \nabla_Z(T(X, Y)) - T(\nabla_Z X, Y) - T(X, \nabla_Z Y)\\ &\overset{(2.8)}{=} \nabla_Z[Y, X] \overset{(2.23)}{=} \bar{\nabla}_{[Y,X]}Z + [Z, [Y, X]]\\ &= -[X, [Z, Y]] - [Y, [X, Z]] = 0,\end{aligned}$$

where we used also the Jacobi identity and the definition of conjugacy of parallelisms. Since both $\mathfrak{X}_{\mathsf{P}}(M)$ and $\mathfrak{X}_{\bar{\mathsf{P}}}(M)$ generate $\mathfrak{X}(M)$, it follows that $\nabla T = 0$, as desired.

Conversely, suppose that the torsion T of a parallelized manifold (M, P) is parallel, and let $(E_i)_{i=1}^n$ be a frame field on M associated to P. According to Lemma 2.19, the vector field $[E_i, E_j]$ is also P-parallel for every $i, j \in \{1, \dots, n\}$. If ∇ denotes the covariant derivative induced by P, then define

$$\bar{\nabla}_X Y := \nabla_Y X + [X, Y], \quad X, Y \in \mathfrak{X}(M). \tag{2.24}$$

It can easily be checked that $\bar{\nabla}$ is a covariant derivative on M. Furthermore, it has vanishing curvature $\bar{R}$, since

$$\begin{aligned}\bar{R}(E_i, E_j)E_k &= \bar{\nabla}_{E_i}\bar{\nabla}_{E_j}E_k - \bar{\nabla}_{E_j}\bar{\nabla}_{E_i}E_k - \bar{\nabla}_{[E_i,E_j]}E_k\\ &\overset{(2.24)}{=} \bar{\nabla}_{E_i}[E_j, E_k] - \bar{\nabla}_{E_j}[E_i, E_k] - (\nabla_{E_k}[E_i, E_j] + [[E_i, E_j], E_k])\\ &\overset{(2.24),\text{ cond.}}{=} [E_i, [E_j, E_k]] + [E_j, [E_k, E_i]] + [E_k, [E_i, E_j]] = 0.\end{aligned}$$

Applying Theorem 5.10.3 of [6], it follows that every point of M has a neighbourhood such that $\bar{\nabla}$ is the induced covariant derivative of a parallelism on this neighbourhood. For an arbitrary $p \in M$, let $\mathcal{U}$ denote such a neighbourhood, and let $\bar{\mathsf{P}}$ be the parallelism on $\mathcal{U}$ guaranteed by the cited result. Choose a $\bar{\mathsf{P}}$-parallel vector field Y on $\mathcal{U}$. If X is P-parallel, then

$$[X, Y] \overset{(2.24)}{=} \bar{\nabla}_X Y - \nabla_Y X = 0,$$

thus P and $\bar{\mathsf{P}}$ are conjugate. This concludes the proof. □

Remark 2.47. If, in particular, M is simply connected, and a parallelism is given on M with parallel torsion, then there exists a *global* conjugate parallelism; it can be seen if we compare [6, Theorem 5.10.3] and the proof above.

It is clear from (2.23), that the induced covariant derivative of the conjugate parallelism P_2 of P_1 is given on two P_1-parallel vector fields by their Lie bracket. Taking into account the previous lemma we obtain:

Corollary 2.48. *If P_1 and P_2 are conjugate parallelisms on M, then the covariant derivative ∇_1^+ associated to P_1 according to Proposition 2.22 exists, and it is the induced covariant derivative of P_2.*

Consequently, conjugate parallelisms have the same geodesics, cf. Corollary 2.24. In fact, we can state a little more.

Lemma 2.49. *Two conjugate parallelisms generate the same spray.*

Proof. Let P_1 and P_2 be conjugate parallelisms on M, and, as above, let ∇_1 and ∇_2 denote their induced covariant derivatives. Since for any vector fields X, Y on M we have

$$(\nabla_1)_X Y - (\nabla_2)_X Y \overset{(2.23)}{=} (\nabla_1)_X Y - (\nabla_1)_Y X - [X, Y] = T_1(X, Y),$$

where T_1 is the torsion of ∇_1, the difference tensor $\nabla_1 - \nabla_2$ is skew-symmetric, thus their geodesic sprays coincide by [14, (18.6.2)(i)]. □

Lemma 2.50. *Let P_1 and P_2 be conjugate parallelisms on M, with torsions T_1 and T_2, respectively. Then $T_2 = -T_1$.*

Proof. This is an immediate consequence of Corollary 2.48 and Proposition 2.22. □

Lemma 2.51. *On a connected manifold conjugate parallelisms have the same automorphism group.*

Proof. Let P_1 and P_2 be two conjugate parallelisms on M, and assume that $\varphi \in \mathrm{Diff}(M)$ is an automorphism of P_1. If ∇_i denotes the induced covariant derivative of P_i, $i \in \{1,2\}$, as above, then Lemma 2.34 implies $\varphi \in \mathrm{Aut}(\nabla_1)$, thus, for all $X, Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned}\varphi_\#\big((\nabla_2)_X Y\big) &\overset{(2.23)}{=} \varphi_\#\big((\nabla_1)_X Y\big) + \varphi_\#[X,Y] \\ &\overset{(1.3)}{=} (\nabla_1)_{\varphi_\# X}(\varphi_\# Y) + [\varphi_\# X, \varphi_\# Y] \overset{(2.23)}{=} (\nabla_2)_{\varphi_\# X}(\varphi_\# Y).\end{aligned}$$

Hence $\varphi \in \mathrm{Aut}(\nabla_2)$, thus, by the connectedness of M, Lemma 2.34 implies $\varphi \in \mathrm{Aut}(\mathsf{P}_2)$. This concludes the proof since the roles of P_1 and P_2 are symmetric. □

Having the results of this section in our hand, Lemmas 2.40 and 2.41 are easy consequences of the fact that the left and right parallelisms are conjugate. The following lemma about the flows of conjugate parallelisms is already known for the case of the natural parallelisms on Lie groups, as it can be seen from (1.11).

Lemma 2.52. *If* P *and* $\bar{\mathsf{P}}$ *are two conjugate parallelisms on* M, X *is a* P*-parallel vector field and* $\varphi^X : \mathcal{D}_X \subset \mathbb{R} \times M \to M$ *is the flow of* X*, then for any* $(t,p) \in \mathcal{D}_X$ *we have* $((\varphi_t^X)_*)_p = \bar{\mathsf{P}}(p, \varphi_t^X(p))$.

Proof. Choose a tangent vector v at p and consider the unique $\bar{\mathsf{P}}$-parallel vector field Y satisfying $Y_p = v$. Since $[X,Y] = 0$, by Proposition 3.2.32 in [49] we have $(\varphi_t^X)_* \circ Y = Y \circ \varphi_t^X$. Therefore

$$\begin{aligned}((\varphi_t^X)_*)_p(v) &= ((\varphi_t^X)_*)_p(Y_p) = Y(\varphi_t^X(p)) \\ &= \bar{\mathsf{P}}(p, \varphi_t^X(p))(Y_p) = \bar{\mathsf{P}}(p, \varphi_t^X(p))(v),\end{aligned}$$

as desired. □

2.8 Conformal change of parallelisms

Throughout this section, let σ be a positive smooth function on M.

Lemma and definition 2.53. *Let* (M, P) *be a parallelized manifold and define the mapping*

$$\widetilde{\mathsf{P}} \colon (p,q) \in M \times M \mapsto \widetilde{\mathsf{P}}(p,q) := \frac{\sigma(p)}{\sigma(q)} \mathsf{P}(p,q) \in L(T_pM, T_qM).$$

Then $\widetilde{\mathsf{P}}$ *is a parallelism on* M *obtained by a* conformal change of P with conformal factor σ.

Proof. A simple calculation shows that the conditions for $\widetilde{\mathsf{P}}$ to be a parallelism hold. □

Lemma 2.54. *Let* (M, P) *be a manifold with parallelism and consider a* P*-parallel vector field* X*. If* $\widetilde{\mathsf{P}}$ *is a parallelism on* M *obtained by a conformal change of* P *with conformal factor* σ*, then* $\widetilde{X} := \frac{1}{\sigma} X$ *is a* $\widetilde{\mathsf{P}}$*-parallel vector field on* M*, and each* $\widetilde{\mathsf{P}}$*-parallel vector field on* M *can be obtained in this way.*

Proof. Let $(p, q) \in M \times M$. Since, in particular, the mapping $\mathsf{P}(p, q)$ is $\mathbb{R}$-homogeneous, we find that

$$\begin{aligned}\widetilde{\mathsf{P}}(p,q)(\widetilde{X}_p) &= \frac{\sigma(p)}{\sigma(q)} \mathsf{P}(p,q) \left(\frac{1}{\sigma(p)} X_p \right) \\ &= \frac{1}{\sigma(q)} \mathsf{P}(p,q)(X_p) = \frac{1}{\sigma(q)} X_q = \widetilde{X}_q\end{aligned}$$

so $\widetilde{X}$ is indeed $\widetilde{\mathsf{P}}$-parallel.

Conversely, if $\widetilde{Y}$ is a $\widetilde{\mathsf{P}}$-parallel vector field on M, then a similar calculation shows that $\sigma \widetilde{Y}$ is a P-parallel vector field. □

Remark 2.55. If $(E_i)_{i=1}^n$ is a frame field on M associated to P, then $(\widetilde{E}_i)_{i=1}^n$, where $\widetilde{E}_i := \frac{1}{\sigma} E_i$, is a frame field on M associated to the conformal change $\widetilde{\mathsf{P}}$ of P with conformal factor σ.

Proposition 2.56. *Let* P *and* $\widetilde{\mathsf{P}}$ *be two parallelisms on* M *such that* $\widetilde{\mathsf{P}}$ *is obtained by a conformal change of* P *with conformal factor* σ*, as above. If* ∇ *and* $\widetilde{\nabla}$ *are the covariant derivatives induced by* P *and* $\widetilde{\mathsf{P}}$*, respectively, then we have*

$$\widetilde{\nabla} = \nabla + d\sigma \otimes 1_{\mathfrak{X}(M)}. \tag{2.25}$$

The torsion T *of* ∇ *and* $\widetilde{T}$ *of* $\widetilde{\nabla}$ *are related by*

$$\widetilde{T} = T + d\sigma \wedge 1_{\mathfrak{X}(M)}. \tag{2.26}$$

Proof. Define a mapping $\widehat{\nabla}\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ by

$$(X,Y)\mapsto \widehat{\nabla}_X Y := \nabla_X Y + (d\sigma\otimes 1_{\mathfrak{X}(M)})(X,Y) = \nabla_X Y + (X\sigma)Y.$$

A straightforward calculation shows that $\widehat{\nabla}$ is a covariant derivative on M.

Now consider a P-parallel vector field Y and an arbitrary vector field X on M. If $\widetilde{Y} := \frac{1}{\sigma}Y$, then we have

$$\begin{aligned}\widehat{\nabla}_X\widetilde{Y} &= \nabla_X\Big(\frac{1}{\sigma}Y\Big) + \Big(\frac{1}{\sigma}(X\sigma)\Big)Y\\ &= X\Big(\frac{1}{\sigma}\Big)Y + \frac{1}{\sigma}\nabla_X Y + \Big(\frac{1}{\sigma}(X\sigma)\Big)Y\\ &= -\frac{1}{\sigma^2}\sigma(X\sigma)Y + \frac{1}{\sigma}\nabla_X Y + \Big(\frac{1}{\sigma}(X\sigma)\Big)Y\\ &= \frac{1}{\sigma}\nabla_X Y = 0,\end{aligned}$$

since ∇ is the covariant derivative induced by P. However, by Lemma 2.54, each $\widetilde{\mathsf{P}}$-parallel vector field $\widetilde{Z}$ is of the form $\widetilde{Z} = \frac{1}{\sigma}Z$, where Z is a P-parallel vector field. Thus $\widehat{\nabla}$ is the covariant derivative induced by $\widetilde{\mathsf{P}}$, that is, $\widehat{\nabla} = \widetilde{\nabla}$ and so relation (2.25) holds.

Since for any vector fields $X, Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned}\widetilde{T}(X,Y) &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]\\ &= \nabla_X Y + (X\sigma)Y - \nabla_Y X - (Y\sigma)X - [X,Y]\\ &= T(X,Y) + (d\sigma\wedge 1_{\mathfrak{X}(M)})(X,Y),\end{aligned}$$

the torsion $\widetilde{T}$ of $\widetilde{\nabla}$ is indeed of the form (2.26). □

Remark 2.57. Relation (2.25) holds also if the covariant derivatives ∇ and $\widetilde{\nabla}$ are constructed from *covering* parallelisms $(\mathcal{U}_\alpha, \mathsf{P}^\alpha)_{\alpha\in\mathcal{A}}$ and $(\mathcal{U}_\alpha, \widetilde{\mathsf{P}}^\alpha)_{\alpha\in\mathcal{A}}$, respectively (for the construction of ∇ and $\widetilde{\nabla}$, we refer to the first paragraph of the proof of Theorem 3.17). We can prove this by using a partition of unity subordinate to the open covering $(\mathcal{U}_\alpha)_{\alpha\in\mathcal{A}}$. Consequently, we have an analogous result for the torsions as well.

Definition 2.58. We say that two parallelisms P_1 and P_2 are *conformally conjugate with conformal factor* σ if $\widetilde{\mathsf{P}}_1$ and $\widetilde{\mathsf{P}}_2$ are conjugate parallelisms, where $\widetilde{\mathsf{P}}_i$ is obtained by a conformal change of P_i with conformal factor σ, $i\in\{1,2\}$.

Lemma 2.59. *The parallelisms* P_1 *and* P_2 *are conformally conjugate with conformal factor* σ *if, and only if, for every* P_1*-parallel vector field* X *and* P_2*-parallel vector field* Y *we have*

$$[X, Y] = (d\sigma \wedge 1_{\mathfrak{X}(M)})(X, Y).$$

Proof. Since $\widetilde{X} := \frac{1}{\sigma}X$ is $\widetilde{\mathsf{P}}_1$-parallel and $\widetilde{Y} := \frac{1}{\sigma}Y$ is $\widetilde{\mathsf{P}}_2$-parallel,

$$\begin{aligned}
[\widetilde{X}, \widetilde{Y}] = \Big[\frac{1}{\sigma}X, \frac{1}{\sigma}Y\Big] &= \frac{1}{\sigma}\Big(X\frac{1}{\sigma}\Big)Y - \frac{1}{\sigma}\Big(Y\frac{1}{\sigma}\Big)X + \frac{1}{\sigma^2}[X, Y] \\
&= -\frac{1}{\sigma^2}(X\sigma)Y + \frac{1}{\sigma^2}(Y\sigma)X + \frac{1}{\sigma^2}[X, Y] \\
&= \frac{1}{\sigma^2}\big((Y\sigma)X - (X\sigma)Y + [X, Y]\big) \\
&= \frac{1}{\sigma^2}\big([X, Y] - (d\sigma \wedge 1_{\mathfrak{X}(M)})(X, Y)\big),
\end{aligned}$$

whence our claim. $\square$

Chapter 3

Finsler functions and parallelisms

Throughout this chapter (M, F) is a Finsler manifold as in Section 1.4. If in our considerations the base manifold is a Lie group G, then F is a Finsler function for G.

3.1 Finsler functions compatible with a parallelism

Definition 3.1. Let P be a parallelism on M. We say that a Finsler function F on TM is *compatible with* P or P*-invariant* if

$$F_q \circ \mathsf{P}(p, q) = F_p \text{ for all } p, q \in M. \tag{3.1}$$

More generally, F is *compatible with a covering parallelism* $(\mathcal{U}_\alpha, \mathsf{P}^\alpha)_{\alpha \in \mathcal{A}}$ of M if the restricted function $F \upharpoonright \tau^{-1}(\mathcal{U}_\alpha)$ is compatible with P^α, for all $\alpha \in \mathcal{A}$.

Example 3.2. Consider a Lie group G and the natural parallelisms of G investigated in Section 2.6. A Finsler function F on G is compatible with P_L if, and only if, F is left invariant, that is, $F \circ (\lambda_g)_* = F$ for all $g \in G$. Indeed, the sufficiency is clear from (2.20), while the necessity follows from

$$F_{gp} \circ ((\lambda_g)_*)_p \overset{(2.20)}{=} F_{gp} \circ \mathsf{P}_L(p, gp) = F_p \quad (p, g \in G).$$

Analogously, F is compatible with P_R if, and only if, F is right invariant.

Remark 3.3. If there exists a parallelism on M which is compatible with a Finsler function for M, then we obtain a special type of Finsler manifolds appearing in the literature under a variety of names, for example, 'Finsler space with 1-form metric' [39], 'metrically homogeneous metric' [35], 'Finsler manifold modeled on a Minkowski space' [30].

We can generalize the notion 'P-parallel tensor field' (Definition 2.2) for Finsler tensor fields in a natural way, thus, for example, for the metric tensor g of (M, F). Then we use the term 'compatibility', as in the case of the Finsler function.

Definition 3.4. Suppose that there exists a parallelism P on M. We say that the metric tensor g of (M, F) is *compatible with* P or P*-invariant*, if for any points $p, q \in M$ and tangent vectors $u \in \overset{\circ}{T}_pM$, $v, w \in T_pM$ we have

$$g_{\mathsf{P}(p,q)u}((\mathsf{P}(p,q)u, \mathsf{P}(p,q)v), (\mathsf{P}(p,q)u, \mathsf{P}(p,q)w)) = g_u((u,v),(u,w)).$$

Remark 3.5. The compatibility of a Finsler function F and a parallelism P can be equivalently defined in the following way: for every P-parallel vector field X on M the function $F \circ X \colon M \to \mathbb{R}$ is constant. Indeed, if F is P-invariant, then

$$F_q(X_q) = F_q(\mathsf{P}(p,q)X_p) \overset{(3.1)}{=} F_p(X_p) \quad \text{for all } p, q \in M.$$

On the other hand, assume that $F \circ X$ is constant, and let $v \in T_pM$. With the notation of Remark 2.3(1),

$$F_q(\mathsf{P}(p,q)(v)) = F_q(\mathsf{P}(p,q)(v_{\mathsf{P}})_p) = F_q((v_{\mathsf{P}})_q) \overset{\text{cond.}}{=} F_p((v_{\mathsf{P}})_p) = F_p(v).$$

Similarly, the metric tensor g of (M, F) is P-invariant if, and only if, for any P-parallel vector fields $X, Y, Z \in \mathfrak{X}(M)$, where X is nowhere zero, the function

$$g(\widehat{Y}, \widehat{Z}) \circ X \colon M \to \mathbb{R},\ p \mapsto g_{X_p}((X_p, Y_p), (X_p, Z_p))$$

is constant.

Lemma 3.6. *Let (M, F) be a Finsler manifold with metric tensor g. If there exists a parallelism P on M, then the P-invariance of F and g are equivalent.*

Proof. First assume that F is P-invariant. Then the energy $E = \frac{1}{2}F^2$ is obviously P-invariant as well, that is,

$$E_q \circ \mathsf{P}(p,q) = E_p, \text{ for } p,q \in M. \tag{3.2}$$

Let $p, q \in M$ and $u, v, w \in T_pM$ such that $u \neq 0$. We can abbreviate $g_u((u,v),(u,w))$ as $g_u(v,w)$, according to **1.33**. By (1.17),

$$\begin{aligned}
&g_{\mathsf{P}(p,q)u}(\mathsf{P}(p,q)v, \mathsf{P}(p,q)w) = E_q''(\mathsf{P}(p,q)u)(\mathsf{P}(p,q)v, \mathsf{P}(p,q)w)\\
&= \lim_{s,t\to 0} \frac{1}{st}\big(E_q(\mathsf{P}(p,q)u + s\mathsf{P}(p,q)v + t\mathsf{P}(p,q)w)\\
&\quad - E_q(\mathsf{P}(p,q)u + s\mathsf{P}(p,q)v) - E_q(\mathsf{P}(p,q)u + t\mathsf{P}(p,q)w)\\
&\quad + E_q(\mathsf{P}(p,q)u)\big)\\
&= \lim_{s,t\to 0} \frac{1}{st}\big(E_q(\mathsf{P}(p,q)(u + sv + tw))\\
&\quad - E_q(\mathsf{P}(p,q)(u+sv)) - E_q(\mathsf{P}(p,q)(u+tw)) + E_q(\mathsf{P}(p,q)u)\big)\\
&\overset{(3.2)}{=} \lim_{s,t\to 0} \frac{1}{st}\big(E_p(u+sv+tw) - E_p(u+sv) - E_p(u+tw) + E_p(u)\big)\\
&= E_p''(u)(v,w) = g_u(v,w),
\end{aligned}$$

so g is indeed P-invariant.

Conversely, suppose that g is compatible with P. Choose two points $p, q \in M$ and a non-zero tangent vector $v \in T_pM$. Then

$$F_p^2(v) \overset{(1.18)}{=} g_v(v,v) = g_{\mathsf{P}(p,q)v}(\mathsf{P}(p,q)v, \mathsf{P}(p,q)v) \overset{(1.18)}{=} F_q^2(\mathsf{P}(p,q)v).$$

Of course, if v is the zero vector in T_pM, then both sides of the equality above are zero and (3.1) holds. This completes the proof. □

We shall see in a while that the following, seemingly innocent, but far not trivial result has important consequences when we apply it to the particular case of Lie groups.

Theorem 3.7. *If a Finsler function F on a parallelizable manifold M is compatible with two conjugate parallelisms, then their (common) generated spray is just the canonical spray of the Finsler manifold.*

Proof. Let S be the spray associated to the conjugate parallelisms P and $\bar{\mathsf{P}}$ (see Corollary and definition 2.12 and Lemma 2.49). We show that

(i) $SF = 0$,

(ii) $S(X^{\mathsf{v}}F) - X^{\mathsf{c}}F = 0$ for all $X \in \mathfrak{X}(M)$;

then **1.37** implies that S is the canonical spray of the Finsler manifold.

(i) Given a tangent vector v at $p \in M$, consider the P-parallel vector field $X := v_{\mathsf{P}}$. Then $S_v = (X_*)_p(v)$, and, consequently,

$$(SF)(v) = S_v(F) = (X_*)_p(v)(F) = v(F \circ X) = 0. \tag{3.3}$$

In the last step we used the fact that $F \circ X$ is constant by Remark 3.5.

(ii) It is sufficient to prove the equality for P-parallel vector fields, because these generate the module $\mathfrak{X}(M)$. So let $X = v_{\mathsf{P}} \in \mathfrak{X}_{\mathsf{P}}(M)$, as above. Since F is compatible with $\bar{\mathsf{P}}$ as well, and the derivatives of the stages of the flow φ^X of X can be expressed in terms of $\bar{\mathsf{P}}$ as in Lemma 2.52, we obtain for any (t,p) in the domain of φ^X that

$$F \circ ((\varphi_t^X)_*)_p = F \circ \bar{\mathsf{P}}(p, \varphi_t^X(p)) \overset{(3.1)}{=} F.$$

Thus the stages of φ^X are isometries of (M,F), and hence X is a Killing vector field by **1.38**. It also follows that for any P-parallel vector field X we have $X^{\mathsf{c}}F = 0$, therefore

$$(S(X^{\mathsf{v}}F) - X^{\mathsf{c}}F)(w) = S_w(X^{\mathsf{v}}F) \overset{(2.5)}{=} ((w_{\mathsf{P}})_*)(w)(X^{\mathsf{v}}F) = w(X^{\mathsf{v}}F \circ w_{\mathsf{P}})$$

for all $w \in T_pM$. We show that the function $X^{\mathsf{v}}F \circ w_{\mathsf{P}}$ is constant. To see this, for brevity, use the notation $Y := w_{\mathsf{P}} \in \mathfrak{X}_{\mathsf{P}}(M)$, and recall that the (global) flow of X^{v} is given by

$$\varphi^{X^{\mathsf{v}}} : \mathbb{R} \times TM \to TM,\ (t,u) \mapsto \varphi^{X^{\mathsf{v}}}(t,u) = u + tX(\tau(u))$$

[49, Lemma 4.1.35]. With the help of the dynamical interpretation of the Lie derivative of a function (see [49, (3.2.8)]) we obtain for any $q \in M$, that

$$\begin{aligned}(X^{\mathsf{v}}F \circ Y)(q) = X^{\mathsf{v}}_{Y(q)}(F) &= \lim_{t\to 0} \frac{1}{t}\big(F(Y_q + tX_q) - F(Y_q)\big) \\ &= \left(\lim_{t\to 0} \frac{1}{t}(F \circ (Y + tX) - F \circ Y)\right)(q).\end{aligned}$$

If we take into account Remark 3.5 and that the vector field $Y + tX$ is P-parallel for all $t \in \mathbb{R}$, we obtain that $X^{\mathsf{v}}F \circ w_{\mathsf{P}}$ is indeed constant, consequently $S(X^{\mathsf{v}}F) - X^{\mathsf{c}}F = 0$. □

If G is a Lie group, then a bi-invariant Finsler function on G is just a Finsler function which is compatible with the conjugate parallelisms P_L and P_R. Thus we can apply Theorem 3.7 in the context of Lie groups to conclude the following two results.

Corollary 3.8. *If F is a bi-invariant Finsler function on a Lie group G, then the one-parameter subgroups of G are the geodesics of F starting at e. All other geodesics are left translations of them.*

Proof. If F is a bi-invariant Finsler function, then its canonical spray is the (common) induced spray of P_L and P_R by Theorem 3.7. However, the geodesics of a parallelism and its generated spray coincide (Lemma 2.16), so the geodesics of F are just the geodesics of P_L and P_R. Taking into account Lemma 2.40, the assertion follows. □

Remark. We found this lemma (with a different proof) in a manuscript of Libing Huang and Xiaohuan Mo on the website of the School of Mathematical Sciences, Peking University, however, it is now unavailable. A simpler argument can be found in [1], but neither proof uses conjugate parallelisms in general. The Riemannian version of the result is well-known, see, e.g., [41], Proposition 9 in Chapter 11. A further important consequence of Theorem 3.7 is the next result, discovered originally by D. Latifi and A. Razavi [34].

Corollary 3.9. *If F is a bi-invariant Finsler function on a Lie group G, then (G, F) is a Berwald manifold.*

Proof. Theorem 3.7 shows that the canonical spray of a bi-invariant Finsler function is generated by a parallelism (either P_L or P_R), hence it is an affine spray by Lemma and definition 2.12. Thus **1.42** implies the statement. □

Notice that the latter result can also be obtained as a corollary of Theorem 3.17 in the next section, as it is indicated in our joint paper [2]. Nevertheless, the condition prescribed on the Finsler function cannot be weakened to left (or right) invariance, as will be shown in Example 3.16.

In the case of Abelian Lie groups things are simpler.

Corollary 3.10 ([2]). *Let G be an Abelian Lie group. If F is a left (or right) invariant Finsler function on G, then (G, F) is a locally Minkowski manifold.*

Proof. Since G is Abelian, the left invariance of F implies bi-invariance, thus, by the previous corollary, (G, F) is a Berwald manifold. It can be seen from the proof of Theorem 3.17, that the unique torsion-free covariant derivative of this Berwald manifold is $\bar{\nabla} := \frac{1}{2}(\nabla_1 + \nabla_2)$, where ∇_1 and ∇_2 are the covariant derivatives attached to the compatible conjugate parallelisms on G. However, in our case, these parallelisms are P_L and P_R, and their induced covariant derivatives are ∇^+ and ∇^-, respectively (see Lemma 2.41 and the comments afterwards). Thus

$$\bar{\nabla}_X Y := \frac{1}{2}(\nabla^- + \nabla^+)(X, Y) = \frac{1}{2}[X, Y] =: \nabla^0_X Y$$

for any left invariant vector fields X, Y on G (cf. Proposition 2.22). The curvature of ∇^0 is given by (2.12). Since on an Abelian Lie group the Lie brackets vanish identically [21, 1.11], the proof is completed. □

3.2 Covering parallelisms and generalized Berwald manifolds

In this section we show that the holonomy invariance of a Finsler function and its compatibility with a covering parallelism are equivalent. Consequently, by **1.40**, we obtain a characterization of generalized Berwald manifolds (Theorem 3.13). To do this, we begin with two preparatory results. The first one is simple, it states that the compatibility of a Finsler function (actually, any function on TM) and a parallelism P can be expressed also in terms of a global trivialization associated to P.

Lemma 3.11. *Assume that (M, F) is a Finsler manifold and P is a parallelism on M compatible with F. Consider a fixed trivializing map φ associated to P, as in Remark 2.9, and for $p \in M$ let $\varphi_p := \varphi(p, \cdot)$. Then there exists a function f on $\mathbb{R}^n$ such that $f = F_p \circ \varphi_p$ for all $p \in M$.*

Proof. Let $p, q \in M$. Notice first, that Remark 2.9 implies

$$\mathsf{P}(p, q) \circ \varphi_p = \varphi_q. \tag{3.4}$$

Now consider the diagram

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{\varphi_p} & T_pM & \xrightarrow{F_p} & \mathbb{R} \\ {\scriptstyle 1_{\mathbb{R}^n}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{P}(p,q)} & & \big\downarrow{\scriptstyle 1_{\mathbb{R}}} \\ \mathbb{R}^n & \xrightarrow{\varphi_q} & T_qM & \xrightarrow{F_q} & \mathbb{R} \end{array} .$$

The left part of the diagram commutes by (3.4), while the right part commutes by the compatibility of F and P. Therefore the entire diagram is commutative and we have $F_p \circ \varphi_p = F_q \circ \varphi_q$. Thus the function $F_p \circ \varphi_p$ is independent of the chosen point p of M, so we can set $f := F_p \circ \varphi_p$. □

The following lemma will be crucial in the proof of Theorem 3.13. To state this result, we need some notation.

Let (M, P) be a parallelized manifold and ∇ an *arbitrary* covariant derivative on M. For each $p \in M$ and $v \in T_pM$ we define an endomorphism $(\nabla\mathsf{P})_v$ of T_pM by

$$(\nabla\mathsf{P})_v \colon w \in T_pM \mapsto (\nabla\mathsf{P})_v(w) := \nabla_v w_{\mathsf{P}} \in T_pM.$$

Clearly, if $\Gamma^i_{jk} \in C^\infty(M)$ are the Christoffel symbols of ∇ with respect to a P-parallel frame field $(E_i)_{i=1}^n$ and $v = v^j E_j(p)$, $w = w^k E_k(p)$, then

$$(\nabla\mathsf{P})_v(w) = v^j w^k \Gamma^i_{jk}(p) E_i(p). \tag{3.5}$$

Lemma 3.12. *Let (M, F) be a Finsler manifold, and suppose that there exists a parallelism P on M such that F and P are compatible. Then F is holonomy invariant with respect to a covariant derivative ∇ on M if, and only if, the endomorphism $(\nabla\mathsf{P})_v$ is in the Lie algebra $\mathsf{Lie}(\mathrm{iso}(F_p))$ of $\mathrm{iso}(F_p)$ for any $p \in M$ and $v \in T_pM$.*

Proof. We note first that $\mathrm{iso}(F_p)$ is a Lie group by **1.23**, thus we can speak of its Lie algebra $\mathsf{Lie}(\mathrm{iso}(F_p))$. Furthermore, since $\mathrm{iso}(F_p)$ is a closed submanifold of the vector space $\mathrm{End}(T_pM)$, the Lie algebra $\mathsf{Lie}(\mathrm{iso}(F_p))$ can be regarded as a linear subspace of $\mathrm{End}(T_pM)$, so the assertion that $(\nabla\mathsf{P})_v$ is in $\mathsf{Lie}(\mathrm{iso}(F_p))$ also makes sense.

Let $\gamma \colon I \to M$ be a curve, φ a global trivialization of TM associated to P (see Remark 2.9), and define the function $f := F_{\gamma(0)} \circ \varphi_{\gamma(0)}$ on $\mathbb{R}^n$. Our first claim is that F is invariant under $(P_\gamma)_0^t$ for any parameter $t \in I$ (which is equivalent to the holonomy invariance of F, cf. **1.39**) if, and only if, the curve $\Phi\colon I \to \mathrm{GL}_n(\mathbb{R})$ given by

$$\Phi(t) := \varphi_{\gamma(t)}^{-1} \circ (P_\gamma)_0^t \circ \varphi_{\gamma(0)} \tag{3.6}$$

runs in $\mathrm{iso}(f)$. Indeed, since we also have $f = F_{\gamma(t)} \circ \varphi_{\gamma(t)}$ by Lemma 3.11, equality (3.6) implies $f \circ \Phi(t) = F_{\gamma(t)} \circ (P_\gamma)_0^t \circ \varphi_{\gamma(0)}$ for each $t \in I$. If we compare this to the definition of f, we find that the relations $f \circ \Phi(t) = f$ and $F_{\gamma(t)} \circ (P_\gamma)_0^t = F_{\gamma(0)}$ are equivalent.

Next we prove that Φ takes values only in $\mathrm{iso}(f)$ if, and only if, $(\nabla\mathsf{P})_{\dot\gamma(t)}$ is in $\mathsf{Lie}(\mathrm{iso}(F_{\gamma(t)}))$ for any $t \in I$. This will conclude the proof, since any vector in TM is the velocity of a curve in M.

Consider a vector $w \in T_{\gamma(0)}M$. We have $(P_\gamma)_0^t(w) = X(t)$, where X is the unique vector field along γ such that $\nabla_\gamma X = 0$ and $X(0) = w$. Let $(E_i)_{i=1}^n$ be the P-parallel frame field on M given by $E_i(p) := \varphi(p, e_i)$. Then we can write $X = X^i(E_i \circ \gamma)$ and $\dot\gamma = (\dot\gamma)^i(E_i \circ \gamma)$ with some smooth functions X^i, $(\dot\gamma)^i$ on I, and for all $t \in I$ we have

$$\begin{aligned} 0 = \nabla_\gamma X(t) &= \nabla_\gamma(X^i(E_i \circ \gamma))(t) \\ &= (X^i)'(t)(E_i \circ \gamma)(t) + X^i(t)(\nabla\mathsf{P})_{\dot\gamma(t)} E_i(\gamma(t)) \\ &\overset{(3.5)}{=} \big((X^i)'(t) + (\dot\gamma)^j(t) X^k(t) \Gamma^i_{jk}(\gamma(t))\big)(E_i \circ \gamma)(t). \end{aligned}$$

Let $\Phi(t) = (\Phi^i_j(t))$. By (3.6) and by $(P_\gamma)_0^t(w) = X(t)$ we obtain $w^l \Phi^i_l(t) = X^i(t)$, which, together with the calculation above, lead to

$$0 = w^l (\Phi^i_l)' + w^l \Phi^k_l (\dot\gamma)^j (\Gamma^i_{jk} \circ \gamma), \quad i \in \{1, \dots, n\}.$$

Since the vector w is arbitrary, we see that Φ satisfies an ODE of the form (1.13) with $A(t) = \big(-(\dot\gamma)^j(t)\Gamma^i_{jk}(\gamma(t))\big)$. **1.24** implies that Φ runs in $\mathrm{iso}(f)$ if, and only if, the matrices $\big(-(\dot\gamma)^j(t)\Gamma^i_{jk}(\gamma(t))\big)$ are in the Lie algebra $\mathsf{Lie}(\mathrm{iso}(f))$ of $\mathrm{iso}(f)$ for each $t \in I$.

It remains to show that the assertions $((\dot\gamma)^j(t)\Gamma^i_{jk}(\gamma(t))) \in \mathsf{Lie}(\mathrm{iso}(f))$ and $(\nabla\mathsf{P})_{\dot\gamma(t)} \in \mathsf{Lie}(\mathrm{iso}(F_{\gamma(t)}))$ are equivalent for all $t \in I$. We regard $\mathsf{Lie}(\mathrm{iso}(f))$ and $\mathsf{Lie}(\mathrm{iso}(F_{\gamma(t)}))$ as linear subspaces of $\mathrm{M}_n(\mathbb{R})$ and $\mathrm{End}(T_{\gamma(t)}M)$, respectively. The mapping

$$\psi(t)\colon B \in \mathrm{M}_n(\mathbb{R}) \mapsto \varphi_{\gamma(t)} \circ B \circ \varphi_{\gamma(t)}^{-1} \in \mathrm{End}(T_{\gamma(t)}M) \quad (t \in I)$$

is a linear isomorphism because it acts as

$$B = (B^i_j) \in \mathrm{M}_n(\mathbb{R}) \mapsto B^i_k(E^k \otimes E_i)(\gamma(t)) \in \mathrm{End}(T_{\gamma(t)}M),$$

where $(E^i)_{i=1}^n$ is the dual frame of $(E_i)_{i=1}^n$. Thus we obtain

$$\begin{aligned}\psi(t)((\dot\gamma)^j(t)\Gamma^i_{jk}(\gamma(t))) &= (\dot\gamma)^j(t)\Gamma^i_{jk}(\gamma(t))E^k(\gamma(t))E_i(\gamma(t)) \\ &\overset{(3.5)}{=} (\nabla\mathsf{P})_{\dot\gamma(t)}.\end{aligned} \tag{3.7}$$

It can be checked easily that $\psi(t) \restriction \mathrm{iso}(f)$ is a group isomorphism from $\mathrm{iso}(f)$ to $\mathrm{iso}(F_{\gamma(t)})$ because $F_{\gamma(t)} \circ \varphi_{\gamma(t)} = f$. Thus its derivative at the unit element is a linear isomorphism from $\mathsf{Lie}(\mathrm{iso}(f))$ onto $\mathsf{Lie}(\mathrm{iso}(F_{\gamma(t)}))$. However, $\psi(t)$ is linear, so its derivative is itself. From this we conclude that $\psi(t)$ is a bijection from $\mathsf{Lie}(\mathrm{iso}(f))$ onto $\mathsf{Lie}(\mathrm{iso}(F_{\gamma(t)}))$, hence (3.7) implies our claim. □

Now we are ready to state the already announced characterization of generalized Berwald manifolds.

Theorem 3.13 ([4])**.** *A Finsler manifold is a generalized Berwald manifold if, and only if, the Finsler function is compatible with a covering parallelism.*

Proof. Let (M, F) be a Finsler manifold. In the following, for simplicity, we assume that M is connected, however, the constructions can be accomplished on the connected components of M.

(1) First, let us assume that (M, F) is a generalized Berwald manifold, that is, F is holonomy invariant with respect to a covariant derivative ∇ on M (see **1.40**). Fix a point $p \in M$ and let $\mathcal{U}$ be a normal neighbourhood of p. Then for any point $q \in \mathcal{U}$ there is a unique geodesic of ∇, denoted by γ_q, such that $\gamma_q(0) = p$ and $\gamma_q(1) = q$. If $q, r \in \mathcal{U}$, let

$$\mathsf{P}(q, r) := (P_{\gamma_r})^1_0 \circ (P_{\gamma_q})^0_1,$$

where $(P_\gamma)^{t_2}_{t_1}$ is the parallel translation along γ from $T_{\gamma(t_1)}M$ to $T_{\gamma(t_2)}M$ induced by ∇. Then P is a parallelism on $\mathcal{U}$. Indeed, $\mathsf{P}(q, r)$ is a linear mapping $T_qM \to T_rM$, condition (2.1) is obviously satisfied, and the smoothness of P follows from the theorem on the smooth dependence on the initial condition (see, e.g., [49, Theorem 3.2.7]). Finally, the holonomy invariance of F implies that for any $q, r \in \mathcal{U}$ we have

$$F_r \circ \mathsf{P}(q, r) = F_r \circ (P_{\gamma_r})^1_0 \circ (P_{\gamma_q})^0_1 = F_p \circ (P_{\gamma_q})^0_1 = F_q,$$

which means that F is indeed compatible with P.

To obtain a covering parallelism of M compatible with F, we can apply the same method for sufficiently many points in M.

(2) In this part we assume that F is compatible with a covering parallelism $(\mathcal{U}_\alpha, \mathsf{P}^\alpha)_{\alpha\in\mathcal{A}}$ of M, and we construct a covariant derivative ∇ on M, such that the parallel translations with respect to ∇ preserve the Finsler norms of tangent vectors.

Let ∇^α be the covariant derivative induced by P^α on $\mathcal{U}_\alpha$ ($\alpha \in \mathcal{A}$). Then for each $v \in \tau^{-1}(\mathcal{U}_\alpha)$ the endomorphism $(\nabla^\alpha \mathsf{P}^\alpha)_v$ is zero. These covariant derivatives are compatible with (the restrictions of) F by Lemma 3.12.

If $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ intersect, $p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $v \in T_pM$, then the endomorphisms $(\nabla^\alpha \mathsf{P}^\beta)_v$ and $(\nabla^\beta \mathsf{P}^\alpha)_v$ are no longer zero in general, but they are still in the Lie algebra $\mathsf{Lie}(\mathrm{iso}(F_p))$ of $\mathrm{iso}(F_p)$, since F is holonomy invariant with respect to ∇^α over $\mathcal{U}_\alpha$ and ∇^β over $\mathcal{U}_\beta$. Thus, if we choose a partition of unity $(f_\alpha)_{\alpha\in\mathcal{A}}$ subordinate to the open covering $(\mathcal{U}_\alpha)_{\alpha\in\mathcal{A}}$, the covariant derivative $\nabla := f_\alpha \nabla^\alpha$ on M still has the property that the endomorphisms $(\nabla \mathsf{P}^\alpha)_v$ are in $\mathsf{Lie}(\mathrm{iso}(F_p))$. Hence, by Lemma 3.12 again, ∇ is compatible with F over each $\mathcal{U}_\alpha$. However, if F is invariant under the parallel translation along pieces of a curve, it is invariant along the entire curve, thus F is holonomy invariant with respect to ∇. This concludes the proof. □

An immediate consequence is that if a Finsler function F is compatible with a parallelism on M, then (M, F) is a generalized Berwald manifold. By applying the theorem for Lie groups and taking into account Example 3.2, we obtain the following important result.

Theorem 3.14 ([1]). *A Lie group equipped with a left invariant Finsler function is a generalized Berwald manifold.*

With the help of the characterization theorem above, we give an example of a proper generalized Berwald manifold. The idea is to define a Finsler function on a manifold which is compatible with a unique covariant derivative, and to show that this particular covariant derivative has non-vanishing torsion.

Example 3.15. [4] Our example will be a two-dimensional Randers manifold [49, Lemma and Definition 9.6.2]. We are going to define the covariant derivative with the help of a (global) parallelism, and use the natural correspondence between parallelisms and 2-frames on the manifold described in Lemma and definition 2.5.

Consider the two-dimensional manifold $\mathbb{R}^2$ and its standard global chart $(\mathbb{R}^2, (e^1, e^2))$ (see **1.2**). We define a 2-frame on $\mathbb{R}^2$ by

$$E_1 := e^1 \frac{\partial}{\partial e^1} + \frac{\partial}{\partial e^2} \quad \text{and} \quad E_2 := -\frac{\partial}{\partial e^1};$$

its dual frame (E^1, E^2) is $E^1 = de^2$, $E^2 = -de^1 + e^1 de^2$. Consider the Finsler norm $f := \sqrt{4(e^1)^2 + 12(e^2)^2} - e^1$ on $\mathbb{R}^2$. Then

$$F := f \circ (E^1, E^2) = \sqrt{4(de^2)^2 + 12(-de^1 + e^1 de^2)^2} - de^2$$

is a Finsler function for $\mathbb{R}^2$ of Randers type. The frame field (E_1, E_2) determines a parallelism P, which is compatible with the Finsler function F by the construction.

Let ∇ be the covariant derivative induced by P on $\mathbb{R}^2$ (see Remark 2.26). Then Remark 2.28 and the compatibility of F and P imply that F is holonomy invariant with respect to ∇, therefore $(\mathbb{R}^2, F, \nabla)$ is a generalized Berwald manifold. Furthermore, ∇ has non-vanishing torsion, since $[E_1, E_2] = \frac{\partial}{\partial e^1}$. Thus $(\mathbb{R}^2, F, \nabla)$ is not a Berwald manifold.

It remains to show that ∇ is the only covariant derivative on $\mathbb{R}^2$ such that F is holonomy invariant with respect to ∇; then $(\mathbb{R}^2, F)$ cannot be a Berwald manifold. To see this, notice first that the isometry group of F_p has only two elements for any $p \in \mathbb{R}^2$. More precisely, in the basis $(E_1(p), E_2(p))$ the elements of $\mathrm{iso}(F_p)$ are represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Indeed, if we assume that a linear mapping $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry of the Finsler norm $f := \sqrt{4(e^1)^2 + 12(e^2)^2} - e^1$, then the four conditions that A preserves the norms of the vectors $(1,0)$, $(-1,0)$, $(0,1)$ and $(0,-1)$ imply that A is either the identity or the reflection about the axis $\mathrm{span}(e_1)$.

Now suppose that F is holonomy invariant with respect to another covariant derivative $\bar{\nabla}$, and let $\gamma\colon I \to \mathbb{R}^2$ be a curve. Then for the parallel translation $(\bar{P}_\gamma)_0^t$ with respect to $\bar{\nabla}$ we have

$$((\bar{P}_\gamma)_0^t)^{-1} \circ (P_\gamma)_0^t \in \mathrm{iso}(F_{\gamma(0)}), \quad t \in I.$$

The parallel translations are smooth, hence the linear automorphism $((\bar{P}_\gamma)_0^t)^{-1} \circ (P_\gamma)_0^t$ of $T_{\gamma(0)}\mathbb{R}^2$ depends continuously on t. Since with the choice $t := 0$ this automorphism is the identity transformation of $T_{\gamma(0)}\mathbb{R}^2$, it follows that $(P_\gamma)_0^t = (\bar{P}_\gamma)_0^t$ for all $t \in I$. Then $\nabla = \bar{\nabla}$, because a covariant derivative is determined by its induced parallel translations (see, e.g., [49, Proposition 6.1.59]).

We present another example for a proper generalized Berwald manifold constructed by Huang and Mo in the manuscript mentioned already (see the remark after Corollary 3.8). Again, we construct a Randers function, but this time the base manifold is the upper half plane of $\mathbb{R}^2$ endowed with a Lie group structure. In the cited manuscript it served as an example for a left invariant Finsler function with different geodesics than the one-parameter subgroups and their left translations (cf. Corollary 3.8). We showed in [1], as an application of Theorem 3.14, that the construction yields a proper generalized Berwald manifold.

Example 3.16 ([1]). Let $G = \{(g^1, g^2) \in \mathbb{R}^2 \mid g^2 > 0\}$ and define a multiplication on it by

$$(g^1, g^2) \times (h^1, h^2) := (h^1 g^2 + g^1, g^2 h^2), \quad (g^1, g^2), (h^1, h^2) \in G.$$

Then G is a Lie group with unit element $e = (0, 1)$; the inverse of an element (g^1, g^2) is $(g^1, g^2)^{-1} = \left(-\frac{g^1}{g^2}, \frac{1}{g^2}\right)$. Actually, G can be regarded as a subgroup of the group of positive affine transformations $\mathrm{Aff}^+(\mathbb{R}^2)$ of $\mathbb{R}^2$. Indeed, the mapping

$$(g^1, g^2) \in G \mapsto \begin{pmatrix} g^2 & g^1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}^+(\mathbb{R}^2),$$

is an injective group homomorphism.

First we compute the derivative of a left translation $\lambda_{(a,b)}$ of G. Given a tangent vector $v \in T_pG$, choose a curve $\gamma = (\gamma^1, \gamma^2)\colon I \to G$ such that $\dot{\gamma}(0) = v$. Then

$$\lambda_{(a,b)}(\gamma(t)) = (a, b) \times (\gamma^1(t), \gamma^2(t)) = (b\gamma^1(t) + a, b\gamma^2(t)), \quad t \in I.$$

Thus

$$\begin{aligned} (\lambda_{(a,b)})_*(v) &= (\lambda_{(a,b)})_*(\dot{\gamma}(0)) = (\lambda_{(a,b)} \circ \gamma)'(0) \\ &= (b(\gamma^1)'(0), b(\gamma^2)'(0)) = bv. \end{aligned} \tag{3.8}$$

Now we define an appropriate Finsler function F on TG. Let

$$\alpha_p(v,w) := \frac{1}{(p^2)^2}(v^1\ v^2)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}, \quad \beta_p(v) := \frac{1}{p^2}(v^1+v^2),$$

and

$$\begin{aligned} F(v) &:= \sqrt{\alpha_p(v,v)} + \beta_p(v) \\ &= \frac{1}{p^2}\left(\sqrt{2(v^1)^2 + 2v^1v^2 + 2(v^2)^2} + v^1 + v^2\right), \end{aligned}$$

where $v = (v^1, v^2), w = (w^1, w^2) \in T_pG$ and $p = (p^1, p^2) \in G$. Since

$$\begin{aligned} \|\beta\|_\alpha(p) &= \sqrt{\alpha^{ij}(p)\beta_i(p)\beta_j(p)} \\ &= \sqrt{\frac{(p^2)^2}{3}\left(2\cdot\frac{1}{(p^2)^2} - \frac{1}{(p^2)^2} - \frac{1}{(p^2)^2} + 2\cdot\frac{1}{(p^2)^2}\right)} \\ &= \sqrt{\frac{2}{3}} < 1, \end{aligned}$$

F is in fact a Randers function.

This Randers function F is left invariant. To see this let $g := (a,b) \in G$ and $v \in T_pG$. Then

$$\begin{aligned} F((\lambda_g)_*(v)) &= F_{g\times p}((\lambda_{(a,b)})_*(v)) \overset{(3.8)}{=} F_{(p^1b+a,bp^2)}(bv) \\ &= \frac{1}{bp^2}\left(\sqrt{2b^2(v^1)^2 + 2b^2v^1v^2 + 2b^2(v^2)^2} + bv^1 + bv^2\right) \\ &= \frac{1}{p^2}\left(\sqrt{2(v^1)^2 + 2v^1v^2 + 2(v^2)^2} + v^1 + v^2\right) \\ &= F(v), \end{aligned}$$

as we claimed. Thus (G,F) is a generalized Berwald manifold by Theorem 3.14.

Finally, we show that $\nabla\beta \neq 0$, where ∇ is the Levi-Civita derivative of α, and hence, by a theorem of S. Kikuchi [32], (G,F) is surely not a Berwald manifold. Indeed, the Christoffel symbols of the Riemannian metric α are

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{22} = -\frac{2}{3}\frac{1}{e^2},\ \Gamma^1_{12} = -\frac{4}{3}\frac{1}{e^2},\ \Gamma^2_{11} = \frac{4}{3}\frac{1}{e^2},\ \Gamma^2_{12} = \frac{2}{3}\frac{1}{e^2},$$

where (e^1, e^2) is the dual of the canonical basis of $\mathbb{R}^2$. Thus, for example,

$$\nabla\beta\left(\frac{\partial}{\partial e^1}, \frac{\partial}{\partial e^1}\right) = \frac{\partial}{\partial e^1}\left(\beta\left(\frac{\partial}{\partial e^1}\right)\right) - \beta\left(\Gamma^1_{11}\frac{\partial}{\partial e^1}\right) - \beta\left(\Gamma^2_{11}\frac{\partial}{\partial e^2}\right)$$
$$= \frac{\partial}{\partial e^1}\left(\frac{1}{e^2}\right) - \Gamma^1_{11}\frac{1}{e^2} - \Gamma^2_{11}\frac{1}{e^2} = -\frac{2}{3}\frac{1}{(e^2)^2} \neq 0.$$

3.3 Conjugate and conformally conjugate parallelisms on Finsler manifolds

Theorem 3.17 ([2]). *Let (M, F) be a Finsler manifold and suppose that there exist two covering parallelisms $(\mathcal{U}_\alpha, \mathsf{P}^\alpha_1)_{\alpha\in\mathcal{A}}$ and $(\mathcal{U}_\alpha, \mathsf{P}^\alpha_2)_{\alpha\in\mathcal{A}}$ (with the same open covering) of M such that*

(i) F is compatible with both covering parallelisms;

(ii) for all $\alpha \in \mathcal{A}$ the parallelisms P^α_1 and P^α_2 are conjugate.

Then (M, F) is a Berwald manifold.

Proof. First we choose a partition of unity $(f_\alpha)_{\alpha\in\mathcal{A}}$ subordinate to the open covering $(\mathcal{U}_\alpha)_{\alpha\in\mathcal{A}}$. For any $\alpha \in \mathcal{A}$ we have the induced covariant derivative ∇^α_1 of $(\mathcal{U}_\alpha, \mathsf{P}^\alpha_1)$. Put them into a covariant derivative $\nabla_1 := f_\alpha\nabla^\alpha_1$ on M. Since F is compatible with $(\mathcal{U}_\alpha, \mathsf{P}^\alpha_1)_{\alpha\in\mathcal{A}}$, it is holonomy invariant with respect to ∇_1, as it turns out from the second part of the proof of Theorem 3.13. In other words, the parallel translations induced by ∇_1 preserve the Finsler norms of tangent vectors.

By setting $\nabla_2 := f_\alpha\nabla^\alpha_2$ and using the same argument as above, we obtain another covariant derivative ∇_2 on M inducing parallel translations under which the Finsler norms of tangent vectors are invariant. These imply by Theorem 3.13 that both (M, F, ∇_1) and (M, F, ∇_2) are generalized Berwald manifolds.

Consider the linear Ehresmann connections $\mathcal{H}_1$ and $\mathcal{H}_2$ induced by ∇_1 and ∇_2, respectively (see **1.16**), and their associated horizontal projections $\mathbf{h}_1$ and $\mathbf{h}_2$. According to **1.41**, we have

$$\mathbf{h}_i = \mathbf{h}_0 + \frac{1}{2}\,\mathbf{t}^\circ_i + \frac{1}{2}[\mathbf{J}, (dE \circ \mathbf{t}^\circ_i)^\#] \quad \text{for } i \in \{1, 2\}. \tag{3.9}$$

First we show that if T_1 and T_2 are the torsions of ∇_1 and ∇_2, respectively, then $T_1 = -T_2$. We have already seen in Lemma 2.50 that over every open set $\mathcal{U}_\alpha$ relation $T_1^\alpha + T_2^\alpha = 0$ is satisfied. So, if $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}(T_1 + T_2)(X,Y) &= (\nabla_1)_X Y - (\nabla_1)_Y X - [X,Y] \\ &\quad + (\nabla_2)_X Y - (\nabla_2)_Y X - [X,Y] \\ &= (f_\alpha \nabla_1^\alpha)_X Y - (f_\alpha \nabla_1^\alpha)_Y X - \big(\sum_\alpha f_\alpha\big)[X,Y] \\ &\quad + (f_\alpha \nabla_2^\alpha)_X Y - (f_\alpha \nabla_2^\alpha)_Y X - \big(\sum_\alpha f_\alpha\big)[X,Y] \\ &= \big(f_\alpha((\nabla_1^\alpha)_X Y - (\nabla_1^\alpha)_Y X - [X,Y] \\ &\quad + (\nabla_2^\alpha)_X Y - (\nabla_2^\alpha)_Y X - [X,Y])\big) \\ &= \big(f_\alpha(T_1^\alpha(X,Y) + T_2^\alpha(X,Y))\big) = 0.\end{aligned}$$

Next we check that the covariant derivative

$$\bar{\nabla} := \frac{1}{2}(\nabla_1 + \nabla_2) \tag{3.10}$$

on M induces a linear Ehresmann connection with associated horizontal projection $\bar{\mathbf{h}} = \frac{1}{2}(\mathbf{h}_1 + \mathbf{h}_2)$. Let $\bar{\mathbf{h}}$ be the horizontal projection of the Ehresmann connection induced by $\bar{\nabla}$. Then we have

$$((\nabla_1)_X Y)^\mathsf{v} = [\mathbf{h}_1(X^\mathsf{c}), Y^\mathsf{v}], \quad ((\nabla_2)_X Y)^\mathsf{v} = [\mathbf{h}_2(X^\mathsf{c}), Y^\mathsf{v}]$$

and

$$(\bar{\nabla}_X Y)^\mathsf{v} = \left[\bar{\mathbf{h}}(X^\mathsf{c}), Y^\mathsf{v}\right],$$

for all $X, Y \in \mathfrak{X}(M)$. By adding the first two equalities and taking into account (3.10), we find that

$$[(\mathbf{h}_1 + \mathbf{h}_2)(X^\mathsf{c}), Y^\mathsf{v}] = [2\bar{\mathbf{h}}(X^\mathsf{c}), Y^\mathsf{v}].$$

Thus, using [46, Lemma 1.5], it follows that the horizontal projection associated to $\bar{\nabla}$ is indeed

$$\bar{\mathbf{h}} = \frac{1}{2}(\mathbf{h}_1 + \mathbf{h}_2). \tag{$*$}$$

Finally, we show that (M, F) is a Berwald manifold. Indeed, our first observation $T_1 = -T_2$ and (1.5) imply that $\mathbf{t}_1^\circ + \mathbf{t}_2^\circ = 0$. Adding the equalities in (3.9), we find that $2\,\mathbf{h}_0 = \mathbf{h}_1 + \mathbf{h}_2$, so $\bar{\mathbf{h}} = \mathbf{h}_0$ by $(*)$. Thus the horizontal projection of the canonical connection of (M, F) is $\bar{\mathbf{h}}$. Since $\bar{\mathbf{h}}$ is the associated horizontal projection of a *linear* Ehresmann connection, it follows from **1.42** that (M, F) is a Berwald manifold. □

Now we turn to some Finslerian applications of conformal changes of parallelisms. For the rest of this section, let σ be a positive smooth function on M, and $\sigma^{\mathsf{v}} := \sigma \circ \tau$ its vertical lift. Then, as it is well-known, $\widetilde{F} = \sigma^{\mathsf{v}} F$ is also a Finsler function, obtained from F by a *conformal change with conformal factor* σ. We note that if F and $\widetilde{F}$ are two Finsler functions for M such that $\widetilde{F} = \varphi F$ with some function φ on TM, then the positive-homogeneity of the Finsler functions immediately implies that φ is the vertical lift of a smooth function on M ('Knebelman's observation'; see, e.g., [50, Lemma 6]).

The following result is due to Hashiguchi and Ichijyō [24]. Now, using conformal change of parallelisms and our characterization of generalized Berwald manifolds (Theorem 3.13) we deduce it as an easy corollary.

Proposition 3.18. *By a conformal change of a generalized Berwald manifold we obtain again a generalized Berwald manifold; that is, the class of generalized Berwald manifolds is closed under conformal changes.*

Proof. Let (M, F) be a generalized Berwald manifold and choose a conformal factor σ as above. By Theorem 3.13, there is a covering parallelism $(\mathcal{U}_\alpha, \mathsf{P}^\alpha)_{\alpha\in\mathcal{A}}$ compatible with F. Let us construct $\widetilde{\mathsf{P}}^\alpha$ from P^α according to Lemma and definition 2.53 for every $\alpha \in \mathcal{A}$, and consider the conformal change of F with the same conformal factor σ yielding $\widetilde{F}$. Since for any $p, q \in M$ we have

$$\begin{aligned}\widetilde{F}_q \circ \widetilde{\mathsf{P}}^\alpha(p, q) &= \sigma(q) F_q \circ \left(\frac{\sigma(p)}{\sigma(q)} \mathsf{P}^\alpha(p, q)\right) \\ &= \sigma(p) F_q \circ \mathsf{P}^\alpha(p, q) = \sigma(p) F_p = \widetilde{F}_p,\end{aligned}$$

it follows that $\widetilde{F}$ is compatible with the covering parallelism $(\mathcal{U}_\alpha, \widetilde{\mathsf{P}}^\alpha)_{\alpha\in\mathcal{A}}$, thus, again by Theorem 3.13, $(M, \widetilde{F})$ is also a generalized Berwald manifold. □

The next result is a counterpart of Theorem 3.17 for Wagner manifolds.

Theorem 3.19. *Let (M, F) be a Finsler manifold and suppose that there exist two covering parallelisms $(\mathcal{U}_\alpha, \mathsf{P}^\alpha_1)_{\alpha\in\mathcal{A}}$ and $(\mathcal{U}_\alpha, \mathsf{P}^\alpha_2)_{\alpha\in\mathcal{A}}$ (with the same open covering) of M such that*

(i) F is compatible with both covering parallelisms;

(ii) *for all* $\alpha \in \mathcal{A}$ *the parallelisms* P_1^α *and* P_2^α *are conformally conjugate with the same conformal factor* σ.

Then (M, F) *is a Wagner manifold.*

Proof. Let $\widetilde{\mathsf{P}}_1^\alpha$ and $\widetilde{\mathsf{P}}_2^\alpha$ denote the parallelisms obtained by a conformal change of P_1^α and P_2^α, respectively, with conformal factor σ for each $\alpha \in \mathcal{A}$. Then $\widetilde{\mathsf{P}}_1^\alpha$ and $\widetilde{\mathsf{P}}_2^\alpha$ are conjugate parallelisms on $\mathcal{U}_\alpha$. Let $\widetilde{F} := \sigma^{\mathsf{v}} F$. Since our Finsler function F is compatible with $(\mathcal{U}_\alpha, \mathsf{P}_1^\alpha)_{\alpha\in\mathcal{A}}$ and $(\mathcal{U}_\alpha, \mathsf{P}_2^\alpha)_{\alpha\in\mathcal{A}}$, the Finsler function $\widetilde{F}$ is compatible with $(\mathcal{U}_\alpha, \widetilde{\mathsf{P}}_1^\alpha)_{\alpha\in\mathcal{A}}$ and $(\mathcal{U}_\alpha, \widetilde{\mathsf{P}}_2^\alpha)_{\alpha\in\mathcal{A}}$ as it can be seen from the proof of Proposition 3.18. Then $(M, \widetilde{F})$ is a Berwald manifold by Theorem 3.17. Then, however, F can be obtained by a conformal change of the Berwaldian Finsler function $\widetilde{F}$ (with the conformal factor $\frac{1}{\sigma}$), so (M, F) is a Wagner manifold by a result of Hashiguchi and Ichijyō [24, Theorem B]. □

3.4 Strong compatibility of Finsler functions and parallelisms

On the analogy of the consistency of a Riemannian metric and a parallelism defined in [54, (3.1)], we introduce the following notion.

Definition 3.20. Let (M, F) be a Finsler manifold and P a parallelism on M. We say that P and F (or P and (M, F)) are *strongly compatible*, if they are compatible and their pregeodesics coincide.

Theorem 3.21. *Let* (M, F) *be a Finsler manifold, and let* P *be a parallelism on* M *such that* P *and* F *are compatible. Then the following conditions are equivalent:*

(i) F *is strongly compatible with* P;

(ii) *the geodesics of* P *and* F *coincide;*

(iii) *the* P*-parallel vector fields are Killing vector fields of* (M, F).

Proof. *(i)⇒(ii):* Let S_0 denote the canonical spray of (M, F), and let S be the spray generated by P. Our assumption is that the pregeodesics of F and P, and hence those of S_0 and S, coincide (cf. Proposition 2.16). However, as seen in (3.3), F is a first integral of S, that is, $SF = 0$. These imply by **1.37**, that $S = S_0$, thus the geodesics of P and F coincide.

(ii)⇒(iii): Consider a P-parallel vector field X, and let $\gamma\colon I \to M$ be a non-constant geodesic of (M, F). Since, by our assumption, γ is also a geodesic of P, there exists a P-parallel vector field Y such that γ is its integral curve (see Lemma 2.14). Taking into account Remark 3.5 and Lemma 3.6, the compatibility of F and P implies that the function

$$g_{Y\circ\gamma}(Y \circ \gamma, X \circ \gamma) = g_{\dot\gamma}(\dot\gamma, X \circ \gamma)\colon I \to \mathbb{R}$$

is constant, and thus that X is a Killing vector field of (M, F) by **1.38**.

(iii)⇒(i): We show that (iii) yields that the geodesics of F and P coincide, this will automatically imply the strong compatibility. Let $p \in M$ and $v \in T_pM$. There exist

(a) a unique maximal geodesic $\gamma\colon I \to M$ of (M, F) satisfying $\dot\gamma(0) = v$;
(b) a unique maximal integral curve $\alpha\colon J \to M$ of the P-parallel vector field v_{P} such that $\dot\alpha(0) = v$.

We show that $I = J$ and $\alpha = \gamma$, thus the geodesics of P and F coincide, cf. Lemma 2.14. By assumption, the vector field v_{P} is a Killing vector field of (M, F), hence the function

$$g_{\dot\gamma}(\dot\gamma, v_{\mathsf{P}} \circ \gamma)\colon t \in I \mapsto g_{\dot\gamma(t)}\big(\dot\gamma(t), v_{\mathsf{P}}(\gamma(t))\big) \in \mathbb{R}$$

is constant. Since γ is a geodesic of (M, F), for any parameter $t \in I$ we have $F(\dot\gamma(t)) = F(\dot\gamma(0)) = F(v)$, and, due to the compatibility of F and P, the function $F \circ v_{\mathsf{P}}$ is constant as well. So we have

$$F\big(v_{\mathsf{P}}(\gamma(t))\big) = F\big(v_{\mathsf{P}}(\gamma(0))\big) = F(v_{\mathsf{P}}(p)) = F(v). \tag{3.11}$$

These imply that for every $t \in I$,

$$\begin{aligned} g_{\dot\gamma(t)}\big(\dot\gamma(t), v_{\mathsf{P}}(\gamma(t))\big) &= g_{\dot\gamma(0)}\big(\dot\gamma(0), v_{\mathsf{P}}(\gamma(0))\big) \\ &= g_v(v, v) \overset{(1.18)}{=} F^2(v) = F(\dot\gamma(t)) \cdot F\big(v_{\mathsf{P}}(\gamma(t))\big). \end{aligned}$$

It follows by **1.34** that there exists a non-negative function $c\colon I \to \mathbb{R}$ such that $v_{\mathsf{P}}(\gamma(t)) = c(t)\dot\gamma(t)$ holds for all $t \in I$. But then, on the one hand,

$$F\big(v_{\mathsf{P}}(\gamma(t))\big) \overset{(3.11)}{=} F(v) = F(\dot\gamma(t)).$$

On the other hand, $F\big(v_{\mathsf{P}}(\gamma(t))\big) = F(c(t)\dot\gamma(t)) = c(t)F(\dot\gamma(t))$, by the positive-homogeneity of F. Hence $c(t) = 1$, so $v_{\mathsf{P}}(\gamma(t)) = \dot\gamma(t)$. By the uniqueness of integral curves and the maximality of α, we must have $I \subset J$ and $\gamma = \alpha \restriction I$.

Now suppose that ε is a positive number such that $]-\varepsilon, \varepsilon[\subset I$ and 2ε is in $J \setminus I$. (It can happen that we only have a positive ε such that $-2\varepsilon \in J \setminus I$ is satisfied; in this case the reasoning is similar which we do not repeat here.) Then $(t + \varepsilon, p)$ is in the domain of the flow φ of v_{P} for all $t \in]-\varepsilon, \varepsilon[$, and

$$\varphi_\varepsilon(\gamma(t)) = \varphi_\varepsilon(\alpha(t)) = \varphi(\varepsilon, \varphi(t, p)) = \varphi(\varepsilon + t, p) = \alpha(t + \varepsilon).$$

Since, by assumption, v_{P} is a Killing vector field of (M, F), the stages of its flow are isometries. Because isometries send geodesics to geodesics, the curve

$$t \in]-\varepsilon, \varepsilon[\mapsto \varphi_\varepsilon(\gamma(t)) = \alpha(t + \varepsilon),$$

is a geodesic of (M, F). After a reparametrization by the translation $\lambda_{-\varepsilon} \colon t \in \mathbb{R} \mapsto t - \varepsilon \in \mathbb{R}$, we get the geodesic

$$t \in]0, 2\varepsilon[\mapsto \alpha(t)$$

of (M, F). However, we have already seen that α is an F-geodesic on $]-\varepsilon, \varepsilon[\subset I = I \cap J$, thus the curve $\alpha \restriction]-\varepsilon, 2\varepsilon[$ is also a geodesic of the Finsler manifold. Its initial velocity is $\dot{\alpha}(0) = v = \dot{\gamma}(0)$, so it is an extension of the maximal geodesic γ of (M, F), which is a contradiction. Hence, necessarily, $I = J$ and $\gamma = \alpha$. □

Theorem 3.22. *If (M, F) is a Finsler manifold and P is a parallelism on M such that F and P are strongly compatible, then (M, F) is a Berwald manifold.*

Proof. Let S denote the spray generated by P (see Corollary and definition 2.12). The compatibility of P and F implies $SF = 0$, as in (3.3). By the strong compatibility, the pregeodesics of S and F coincide. According to **1.37**, S is the canonical spray of (M, F), and, since this spray is affine, (M, F) is a Berwald manifold by **1.42**. □

Corollary 3.23. *Let G be a Lie group endowed with a left invariant Finsler function F. If the pregeodesics of F are the one-parameter subgroups of G and their translations, then (G, F) is a Berwald manifold.*

Proof. By Lemma 2.40, the pregeodesics of F and P_L coincide. However, F and P_L are compatible, thus also strongly compatible, therefore the statement follows from the previous theorem. □

The following theorem is the Finslerian version of Theorem 3.8 in [54].

Theorem 3.24 (Structure theorem of parallelized Finsler manifolds)**.** *Let (M, F) be a connected Finsler manifold. Suppose that there exists a complete parallelism* P *on M such that its torsion is parallel and it is strongly compatible with F. Then*

(i) M is diffeomorphic to $H\backslash G$, where H is a discrete subgroup of a simply connected Lie group G;

(ii) the parallelism P *is induced by the left parallelism* P_L *on G;*

(iii) the Finsler function F is induced by a bi-invariant Finsler function on the Lie group G (and hence, (M, F) is a Berwald manifold).

Proof. The first two assertions are repetitions of the statements of Proposition 2.42. The left invariance of F follows immediately from the compatibility of F and $\mathsf{P} \cong \mathsf{P}_L$, see Example 3.2. The strong compatibility of P and F implies that the P-parallel vector fields are Killing vector fields by Theorem 3.21, consequently, the stages of their flows are isometries of the Finsler manifold by **1.38**. So if X is a P-parallel vector field, then it can be considered as a left invariant vector field which has right translations as the stages of its flow by **1.21**, so the right translations are isometries of (M, F) and hence F is right invariant as well. $\square$

Chapter 4

Summary

Here we summarize the contents of the Thesis chapter by chapter. The numbering of theorems, propositions, etc. is the same as that of in the Thesis. However, for the sake of conciseness, in some cases they are formulated slightly differently (but equivalently).

Chapter 1. Preliminaries. In this chapter we fix our basic notation and terminology, and collect the necessary tools and results (or at least most of them) from the theories of smooth manifolds, Lie groups, Lie group and Lie algebra actions and Finsler manifolds.

Below (as well as in the Thesis), M stands for an n-dimensional smooth manifold, G denotes a Lie group with unit element e, and $\mathsf{Lie}(G)$ is the Lie algebra of G. Then $\mathsf{Lie}(G)$ is the tangent space to G at e together with the Lie bracket induced from the Lie algebra $\mathfrak{X}_L(G)$ of left invariant vector fields on G by the isomorphism $X \in \mathfrak{X}_L(G) \mapsto X(e) \in T_eG$.

Chapter 2. Parallelisms. Our aim in this chapter is to provide a systematic account on the general theory of parallelisms, whose elements can be found only in scattered form in some papers and books (mainly in refs. [7, 19, 20, 21, 54]). Thus most of the results presented here are not new, but our approach to their proofs is somewhat novel. The only truly new concepts introduced here are those of ‘covering parallelism’, ‘conformally conjugate parallelisms’ and ‘associated Ehresmann connection’.

Coming to the details, let $\mathcal{P} \to M \times M$ be the vector bundle whose fibre at a point (p, q) is the real vector space $L(T_pM, T_qM)$ of linear mappings between the tangent spaces at p and q to M. Following ref. [20], by a *parallelism* on M we mean a smooth section P of $\mathcal{P} \to M \times M$ such that

$$\mathsf{P}(r,q) \circ \mathsf{P}(p,r) = \mathsf{P}(p,q) \text{ and } \mathsf{P}(p,p) = 1_{T_pM}; \quad p,q,r \in M.$$

A *covering parallelism* of M is a family $(\mathcal{U}_\alpha, \mathsf{P}^\alpha)_{\alpha\in\mathcal{A}}$, where $(\mathcal{U}_\alpha)_{\alpha\in\mathcal{A}}$ is an open covering of M and P^α is a parallelism on $\mathcal{U}_\alpha$ for each $\alpha \in \mathcal{A}$. A *parallelized manifold* is a manifold together with a parallelism. There are some other possibilities to introduce a parallelism, which we also discuss in the Thesis. An overview of them can be found in **2.10**.

Throughout the summary we let (M, P) *be a parallelized manifold.*

P-parallel vector fields and frames. A vector field X on M is called P-*parallel* if $\mathsf{P}(p,q)(X_p) = X_q$ for all $p,q \in M$. Given a tangent vector $v \in T_pM$, the vector field

$$v_\mathsf{P}\colon q \in M \mapsto v_\mathsf{P}(q) := \mathsf{P}(p,q)(v) \in T_qM$$

is the unique P-parallel vector field such that $v_\mathsf{P}(p) = v$. If $(b_i)_{i=1}^n$ is a basis of T_pM, then the P-parallel vector fields $E_i := (b_i)_\mathsf{P}$ form a (global) frame field $(E_i)_{i=1}^n$ of M; we say that this frame field is *associated to* P. Conversely, if M admits a global frame field $(E_i)_{i=1}^n$, then the mapping

$$\mathsf{P}\colon (p,q) \in M \times M \mapsto \mathsf{P}(p,q) \in L(T_pM, T_qM)$$
$$\mathsf{P}(p,q)(v) = v^i E_i(q) \text{ if } v = v^i E_i(p) \in T_pM$$

is a parallelism on M, and the vector fields E_i are then P-parallel. Thus a parallelism and a global frame field on M are two sides of the same coin.

It follows that P-parallel vector fields generate the $C^\infty(M)$-module $\mathfrak{X}(M)$, and a vector field on M is P-parallel if, and only if, it is an $\mathbb{R}$-linear combination of the members of a P-parallel frame field. The real vector space of P-parallel vector fields is denoted by $\mathfrak{X}_\mathsf{P}(M)$.

The Ehresmann connection generated by P is the linear Ehresmann connection

$$\mathcal{H}\colon TM \times_M TM \to TTM,\ (v,w) \mapsto \mathcal{H}(v,w) := (v_\mathsf{P})_*(w).$$

Then the vector field

$$S\colon TM \to TTM,\ v \mapsto S(v) := \mathcal{H}(v,v) = (v_\mathsf{P})_*(v)$$

is an affine spray, called also *generated by* (or *associated to*) P.

A (smooth) curve $\gamma\colon I \to M$ is a *geodesic* of (M, P) if

$$\mathsf{P}(\gamma(t_1), \gamma(t_2))\dot{\gamma}(t_1) = \dot{\gamma}(t_2) \quad \text{for all } t_1, t_2 \in I.$$

A curve in M is a *pregeodesic* of (M, P) if it has a reparametrization as a geodesic. A parallelized manifold (or a parallelism) is called *complete* if all of its geodesics are defined on the entire real line.

We have the following results:

(i) The geodesics of (M, P) are precisely the integral curves of P-parallel vector fields (Lemma 2.14).

(ii) The parallelism P is complete if, and only if, the P-parallel vector fields are complete (Corollary 2.15).

(iii) The geodesics of P and the geodesics of the spray associated to P coincide (Proposition 2.16 and Proposition 10.3.1 in [7]).

Torsion of P (Lemma and definition 2.17)**.** Let $p \in M$ be an arbitrarily fixed point. The mapping

$$\theta\colon q \in M \mapsto \theta_q \in L(T_qM, T_pM),\ \theta_q(w) := \mathsf{P}(q,p)(w) \in T_pM$$

is a T_pM-valued 1-form on M. The *torsion* of P is the type $(1, 2)$ tensor field

$$T^{\mathsf{P}}\colon (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto T^{\mathsf{P}}(X, Y) \in \mathfrak{X}(M)$$

given by

$$(T^{\mathsf{P}}(X, Y))_q := \mathsf{P}(p, q)(d\theta)_q(X_q, Y_q), \quad q \in M.$$

If $X, Y \in \mathfrak{X}_{\mathsf{P}}(M)$, then $T^{\mathsf{P}}(X, Y) = -[X, Y] = [Y, X]$.

Lemma 2.19. *The torsion of P is parallel, i.e., we have for all $p, q \in M$ the equality*

$$\mathsf{P}(p, q)(T^{\mathsf{P}}(X, Y))_p = T^{\mathsf{P}}_q(\mathsf{P}(p, q)(X_p), \mathsf{P}(p, q)(Y_p))$$

if, and only if, $\mathfrak{X}_{\mathsf{P}}(M)$ is a subalgebra of the Lie algebra $\mathfrak{X}(M)$.

Parallelisms and Lie groups. Every Lie group admits two natural parallelisms. The left parallelism on G is the mapping

$$\mathsf{P}_L\colon (p,q) \in G \times G \mapsto \mathsf{P}_L(p,q) := ((\lambda_{qp^{-1}})_*)_p \in L(T_pG, T_qG),$$

where λ_a ($a \in G$) is the left translation by a. The right parallelism P_R on G is defined analogously. Then

$$\mathfrak{X}_{\mathsf{P}_L}(G) = \mathfrak{X}_L(G),\ \mathfrak{X}_{\mathsf{P}_R}(G) = \mathfrak{X}_R(G)(:= \text{the module of right invariant vector fields on } G),$$

therefore (by Lemma 2.19) P_L and P_R have parallel torsions. The geodesics of P_L and P_R coincide: they are the one-parameter subgroups of G and their left and right translations. This implies that the natural parallelisms of a Lie group are complete.

The following result (see Proposition 2.42 and its proof) is a reformulation of a classical theorem for which we refer to [54, Proposition 2.5], and also [8, 19, 21, 27, 28].

Suppose that (M, P) is a connected, complete parallelized manifold with parallel torsion. Choose and fix a point $p \in M$, and define a Lie bracket on T_pM by

$$[u,v] := [u_{\mathsf{P}}, v_{\mathsf{P}}]_p; \quad u, v \in T_pM.$$

Let G be the connected and simply connected Lie group such that $\mathsf{Lie}(G) = (T_pM, [\cdot,\cdot])$. *Consider the right action*

$$A\colon M \times G \to M,\ (m,g) \mapsto m \cdot g$$

induced by the Lie algebra action

$$T_pM = T_eG \to \mathfrak{X}(M),\ v \mapsto v_{\mathsf{P}}.$$

Then the isotropy subgroup G_p of A at p is a discrete subgroup of G, and the quotient manifold $G_p\backslash G$ is canonically diffeomorphic to M by the mapping

$$G_p\backslash G \to M,\ G_p \cdot g \mapsto p \cdot g.$$

Then $\mathfrak{X}_L(G)$ and $\mathfrak{X}_{\mathsf{P}}(M)$ can be identified canonically.

P-invariant covariant derivatives. A covariant derivative ∇ on (M, P) is P-*invariant* if $\nabla_X Y \in \mathfrak{X}_{\mathsf{P}}(M)$ for all $X, Y \in \mathfrak{X}_{\mathsf{P}}(M)$.

The next observation is a transcription of a well-known result (see, e.g., [26], Proposition 1.4 in Chapter II) in the context of parallelized manifolds.

Proposition 2.21. *There is a one-to-one correspondence between the set of* P-*invariant covariant derivatives* ∇ *on* (M, P) *and the vector space of* $\mathbb{R}$-*bilinear mappings* $\alpha\colon \mathfrak{X}_{\mathsf{P}}(M) \times \mathfrak{X}_{\mathsf{P}}(M) \to \mathfrak{X}_{\mathsf{P}}(M)$ *such that*

$$\alpha(X, Y) = \nabla_X Y \quad \textit{for all } X, Y \in \mathfrak{X}_{\mathsf{P}}(M).$$

Then (see Proposition 2.22) the following natural choices are possible:

(A) $\alpha = 0$. *In this case* ∇ *is flat and*

$$T := \textit{torsion of } \nabla = \textit{torsion of } \mathsf{P}.$$

The covariant derivative specified in this way is called the covariant derivative induced by P.

(B) Under the condition that $\mathfrak{X}_{\mathsf{P}}(M)$ *is a Lie subalgebra of the Lie algebra* $\mathfrak{X}(M)$*:*

(i) $\alpha(X, Y) := [X, Y]$ *for* $X, Y \in \mathfrak{X}_{\mathsf{P}}(M)$. *The corresponding covariant derivative* ∇^+ *is flat and its torsion is* $-T$.

(ii) $\alpha(X, Y) := \frac{1}{2}[X, Y]$ *for* $X, Y \in \mathfrak{X}_{\mathsf{P}}(M)$. *The corresponding covariant derivative* ∇^0 *is torsion-free, and its curvature* R^0 *is given by*

$$R^0(X, Y)Z = -\frac{1}{4}[[X, Y], Z]; \quad X, Y, Z \in \mathfrak{X}_{\mathsf{P}}(M).$$

Proposition 2.23. *The geodesics of a* P-*invariant covariant derivative on* (M, P) *coincide with the geodesics of* (M, P) *if, and only if, the corresponding* $\mathbb{R}$-*bilinear mapping is skew-symmetric.*

Corollary 2.24. *The covariant derivatives* ∇, ∇^+ *and* ∇^0 *defined above have the same geodesics as* (M, P).

Corollary 2.25. *Given any tangent vector* $v \in T_pM$, *there is a unique maximal geodesic* γ *of* (M, P) *such that* $\dot\gamma(0) = v$.

Isomorphisms and automorphisms. An *isomorphism* of (M, P) onto a parallelized manifold $(\bar{M}, \bar{\mathsf{P}})$ is a diffeomorphism $\varphi\colon M \to \bar{M}$ such that

$$(\varphi_*)_q \circ \mathsf{P}(p,q) = \bar{\mathsf{P}}(\varphi(p), \varphi(q)) \circ (\varphi_*)_p \quad \text{for } p, q \in M.$$

Equivalently,

$$\varphi_* \circ v_{\mathsf{P}} = (\varphi_*(v))_{\bar{\mathsf{P}}} \circ \varphi \quad \text{for any } v \in TM.$$

An isomorphism of (M, P) onto itself is an *automorphism of* (M, P) (or of P). We write $\mathrm{Aut}(\mathsf{P})$ for the automorphism group of (M, P). A subgroup of $\mathrm{Aut}(\mathsf{P})$ is the group

$$\mathrm{Sym}(\mathsf{P}) := \{\varphi \in \mathrm{Diff}(M) \mid \varphi_\# X = X \text{ for all } X \in \mathfrak{X}_{\mathsf{P}}(M)\}$$

of *symmetries* (or *translations*) of (M, P).

If G is a connected Lie group, then

$$\mathrm{Sym}(\mathsf{P}_L) = \{\lambda_g \in \mathrm{Diff}(G) \mid g \in G\}.$$

The group $\mathrm{Aut}(\mathsf{P}_L)$ is larger than $\mathrm{Sym}(\mathsf{P}_L)$: it contains also the right translations and conjugations by the elements of G.

Propositions 2.34 and 2.35. *The group* $\mathrm{Aut}(\mathsf{P})$ *is a subgroup of the automorphism group of the covariant derivative induced by* P. *If* M *is connected, then these groups are equal. The group* $\mathrm{Aut}(\mathsf{P})$ *is a subgroup also of the automorphism group of the spray generated by* P.

Conjugate parallelisms. Two parallelisms P_1 and P_2 on M are *conjugate* if

$$[X, Y] = 0 \quad \text{for all } X \in \mathfrak{X}_{\mathsf{P}_1}(M) \text{ and } Y \in \mathfrak{X}_{\mathsf{P}_2}(M).$$

On a connected manifold every parallelism has at most one conjugate parallelism. If a parallelism P on M admits a conjugate parallelism, then the torsion of P is parallel. Conversely, if (M, P) is a parallelized manifold with parallel torsion, then every point of M has a neighbourhood in which a conjugate parallelism exists (Lemmas 2.44, 2.46, cf. [20], Chapter IV, Problems).

Here we can make the following observations:

(i) If P_1 *and* P_2 *are conjugate parallelisms on* M*, then their induced covariant derivatives* ∇_1 *and* ∇_2 *are related by*

$$(\nabla_2)_X Y = (\nabla_1)_Y X + [X, Y]; \quad X, Y \in \mathfrak{X}(M).$$

The torsions of these parallelisms differ in sign (Lemmas 2.45 and 2.50). The covariant derivative ∇_1^+ *associated to* P_1 *(see (B)/(i) above) exists, and* $\nabla_2 = \nabla_1^+$ *(Corollary 2.48).*

(ii) Conjugate parallelisms generate the same spray, and, if M *is connected, have the same automorphism group (Lemmas 2.49 and 2.51).*

(iii) If P *and* $\bar{\mathsf{P}}$ *are conjugate parallelisms on* M*,* X *is a* P*-parallel vector field, and* $\varphi^X : \mathcal{D}_X \subset \mathbb{R} \times M \to M$ *is the flow of* X*, then*

$$((\varphi_t^X)_*)_p = \bar{\mathsf{P}}(p, (\varphi_t^X)_p) \quad \textit{for any } (t, p) \in \mathcal{D}_X$$

(Lemma 2.52).

Conformal conjugacy. If P is a parallelism on M and $\sigma \in C^\infty(M)$ is a positive smooth function, then the mapping

$$\widetilde{\mathsf{P}} \colon (p, q) \in M \times M \mapsto \widetilde{\mathsf{P}}(p, q) := \frac{\sigma(p)}{\sigma(q)} \mathsf{P}(p, q) \in L(T_p M, T_q M)$$

is also a parallelism on M, obtained by a *conformal change of* P *with conformal factor* σ. Then the covariant derivatives ∇ induced by P and $\widetilde{\nabla}$ induced by $\widetilde{\mathsf{P}}$ furthermore their torsions T and $\widetilde{T}$ are related by

$$\widetilde{\nabla} = \nabla + d\sigma \otimes 1_{\mathfrak{X}(M)} \quad \text{and} \quad \widetilde{T} = T + d\sigma \wedge 1_{\mathfrak{X}(M)},$$

respectively (Proposition 2.56).

Two parallelisms P_1 and P_2 on M are called *conformally conjugate* with conformal factor $\sigma \in C^\infty(M)$ if the parallelisms $\widetilde{\mathsf{P}}_1$ and $\widetilde{\mathsf{P}}_2$, obtained from P_1 and P_2 by conformal change with conformal factor σ, are conjugate. This holds if, and only if,

$$[X, Y] = (d\sigma \wedge 1_{\mathfrak{X}(M)})(X, Y) \quad \text{for all } X \in \mathfrak{X}_{\mathsf{P}_1}(M), Y \in \mathfrak{X}_{\mathsf{P}_2}(M)$$

(Lemma 2.59).

Chapter 3. Finsler functions and parallelisms. Let (M, P) be a parallelized manifold. We say that a Finsler function F on TM is *compatible with* P if

$$F_q \circ \mathsf{P}(p,q) = F_p, \text{ for all } (p,q) \in M \times M.$$

More generally, F is *compatible with a covering parallelism* $(\mathcal{U}_\alpha, \mathsf{P}^\alpha)_{\alpha \in \mathcal{A}}$ of M if $F \restriction \tau^{-1}(\mathcal{U}_\alpha)$ is compatible with P^α for all $\alpha \in \mathcal{A}$.

Theorem 3.7. *If M is a parallelizable manifold and a Finsler function on TM is compatible with two conjugate parallelisms, then their common generated spray is the canonical spray of the Finsler function.*

Theorem 3.13. *A Finsler manifold is a generalized Berwald manifold if, and only if, the Finsler function is compatible with a covering parallelism.*

Theorem 3.14. *A Lie group equipped with a left invariant Finsler function is a generalized Berwald manifold.*

As an application of Theorems 3.13 and 3.14, two examples of proper generalized Berwald manifolds are given, thus fulfilling a suggestion of Hashiguchi in [23]: '. . . find much more interesting examples'.

Theorem 3.17. *Suppose that there exist two covering parallelisms $(\mathcal{U}_\alpha, \mathsf{P}_1^\alpha)_{\alpha \in \mathcal{A}}$ and $(\mathcal{U}_\alpha, \mathsf{P}_2^\alpha)_{\alpha \in \mathcal{A}}$ (with the same open covering) of a Finsler manifold (M, F) such that*

(i) the Finsler function F is compatible with both covering parallelisms;
(ii) for all $\alpha \in \mathcal{A}$, the parallelisms P_1^α and P_2^α are conjugate.

Then (M, F) is a Berwald manifold.

Theorem 3.19. *Suppose that there exist two covering parallelisms $(\mathcal{U}_\alpha, \mathsf{P}_1^\alpha)_{\alpha \in \mathcal{A}}$ and $(\mathcal{U}_\alpha, \mathsf{P}_2^\alpha)_{\alpha \in \mathcal{A}}$ (with the same open covering) of a Finsler manifold (M, F) such that*

(i) the Finsler function F is compatible with both covering parallelisms;
(ii) for all $\alpha \in \mathcal{A}$, the parallelisms P_1^α and P_2^α are conformally *conjugate with the same conformal factor σ.*

Then (M, F) is a Wagner manifold.

A Finsler function F is called *strongly compatible* with P if they are compatible and their pregeodesics coincide.

Theorem 3.21. *If P is a parallelism on M compatible with a Finsler function F for M, then the following conditions are equivalent:*

(i) *F is strongly compatible with P;*
(ii) *the geodesics of P and F coincide;*
(iii) *the P-parallel vector fields are Killing vector fields of (M, F).*

Theorem 3.22. *If P is a parallelism on M such that it is strongly compatible with a Finsler function F on TM, then (M, F) is a Berwald manifold.*

Corollary 3.23. *Let G be a Lie group endowed with a left invariant Finsler function F. If the pregeodesics of F are the one-parameter subgroups of G and their translations, then (G, F) is a Berwald manifold.*

Theorem 3.24 (Structure theorem of parallelized Finsler manifolds)**.** *Assume that (M, F) is a connected Finsler manifold such that there exists a complete parallelism P on M such that its torsion is parallel, and it is strongly compatible with F. Then:*

(i) *M is diffeomorphic to $H\backslash G$, where H is a discrete subgroup of a simply connected Lie group G;*
(ii) *the parallelism P is induced by the left parallelism P_L on G;*
(iii) *the Finsler function F is induced by a bi-invariant Finsler function on the Lie group G (and hence, (M, F) is a Berwald manifold).*

Chapter 5

Magyar nyelvű összefoglaló
(Summary in Hungarian)

Ebben a részben fejezetenként áttekintjük a disszertáció tartalmát. A tételek, lemmák, stb. számozása megegyezik az értekezésbeli számozással, megfogalmazásuk azonban célszerűségi okokból és a tömörség érdekében időnként enyhén eltér a főszövegben találhatótól (de azzal mindig ekvivalens).

1. fejezet. Előzmények. A disszertáció a szükséges előzmények tárgyalásával indul: rögzítjük az alapvető jelöléseket és terminológiát, valamint összegyűjtjük a későbbiekben felhasználásra kerülő eszközöket és eredményeket a sima sokaságok, Lie-csoportok, Lie-csoport és Lie-algebra hatások, valamint a Finsler-sokaságok elméletéből.

A továbbiakban – miként az egész értekezésben – M végig egy n-dimenziós sima sokaságot, G pedig egy Lie-csoportot jelöl, amelynek egységeleme e, Lie-algebrája $\mathsf{Lie}(G)$. Utóbbin az egységelembeli érintőteret értjük, ellátva azzal a Lie-zárójellel, amelyet az

$$X \in \mathfrak{X}_L(G) \mapsto X(e) \in T_eG$$

izomorfizmus által a bal-invariáns vektormezők $\mathfrak{X}_L(G)$ Lie-algebrája származtat.

2. fejezet. Párhuzamosítások. E fejezet célja a párhuzamosítások elméletének szisztematikus tárgyalása. A témakör irodalma igen gazdag; a számunkra legfontosabb források [7, 19, 20, 21, 54]. Nem találtunk azonban olyan átfogó munkát (monográfiát vagy tankönyvet), amelyet referenciaként használhattunk volna. Pótlandó ezt a hiányosságot, megkíséreltük a párhuzamosított sokaságok *geometriájának* egységes felépítését adni, és a fontosabb – bár többnyire egyszerű – eredményeket teljes bizonyításukkal együtt bemutatni. E fejezet remélt értékei tehát az újszerű megközelítésben, a párhuzamosításhoz csatolható geometriai struktúrák összekapcsolásában és számos esetben a korábbiaktól különböző bizonyítási módszerekben rejlenek. Kifejezetten új a "lefedő párhuzamosítás" és a "konform-konjugált párhuzamosítások" fogalma. Ugyancsak új ebben a kontextusban a csatolt Ehresmann-konnexió használata, ami nagyban elősegítette az egységes tárgyalást.

Rátérve a részletekre, legyen $\mathcal{P} \to M \times M$ az a vektornyaláb, amelynek egy (p,q) pontpár feletti fibruma a p és q pontok érintőterei között ható lineáris leképezések $L(T_pM, T_qM)$ valós vektortere. Werner Greub és Joseph A. Wolf megközelítését követve [19, 20, 54], M egy *párhuzamosításán* a $\mathcal{P} \to M \times M$ vektornyaláb olyan P sima szelését értjük, amelyre a

$$\mathsf{P}(r,q) \circ \mathsf{P}(p,r) = \mathsf{P}(p,q) \text{ és } \mathsf{P}(p,p) = 1_{T_pM} \quad (p,q,r \in M)$$

feltételek teljesülnek. M egy *lefedő párhuzamosítása* olyan $(\mathcal{U}_\alpha, \mathsf{P}^\alpha)_{\alpha \in \mathcal{A}}$ család, ahol $(\mathcal{U}_\alpha)_{\alpha \in \mathcal{A}}$ nyílt lefedése M-nek, és tetszőleges $\alpha \in \mathcal{A}$ esetén P^α párhuzamosítás $\mathcal{U}_\alpha$-n. Egy párhuzamosítással ellátott sokaságot *párhuzamosított sokaságnak* nevezünk, a továbbiakban (M, P) mindig egy ilyen sokaságot jelöl.

P-párhuzamos vektormezők és n-élmezők. Egy M-en adott X vektormezőt P-*párhuzamos*nak mondunk, ha minden $p,q \in M$ pont esetén $\mathsf{P}(p,q)(X_p) = X_q$. Rögzítve egy $v \in T_pM$ érintővektort, a

$$v_{\mathsf{P}} \colon q \in M \mapsto v_{\mathsf{P}}(q) := \mathsf{P}(p,q)(v) \in T_qM$$

vektormező az az egyetlen P-párhuzamos vektormező M-en, amelyre teljesül, hogy $v_{\mathsf{P}}(p) = v$. Ha rögzítjük T_pM-nek egy $(b_i)_{i=1}^n$ bázisát, akkor az $E_i := (b_i)_{\mathsf{P}}$ (P-párhuzamos) vektormezők egy globális n-élmezőt szolgáltatnak az alapsokaságon; az így kapott n-élmezőt P-*hez csatolt*nak mondjuk. Megfordítva, ha $(E_i)_{i=1}^n$ egy globális n-élmező M-en, akkor a

$$\mathsf{P}\colon (p,q) \in M \times M \mapsto \mathsf{P}(p,q) \in L(T_pM, T_qM)$$
$$\mathsf{P}(p,q)(v) = v^i E_i(q) \text{ ahol } v = v^i E_i(p) \in T_pM$$

leképezés párhuzamosítása M-nek, és az E_i vektormezők P-párhuzamosak. "Egy párhuzamosítás és egy globális n-élmező egy sokaságon: ugyanazon érem két oldala."

Következik ily módon, hogy a P-párhuzamos vektormezők generálják M vektormezőinek $C^\infty(M)$-modulusát, és egy M-en adott vektormező pontosan akkor P-párhuzamos, ha egy P-párhuzamos n-élmező tagjaiból $\mathbb{R}$-lineárisan kombinálható. A P-párhuzamos vektormezők valós vektorterét $\mathfrak{X}_{\mathsf{P}}(M)$-mel jelöljük.

A P által generált Ehresmann-konnexió a

$$\mathcal{H}\colon TM \times_M TM \to TTM,\ (v,w) \mapsto \mathcal{H}(v,w) := (v_{\mathsf{P}})_*(w).$$

lineáris Ehresmann-konnexió. Ekkor az

$$S\colon TM \to TTM,\ v \mapsto S(v) := \mathcal{H}(v,v) = (v_{\mathsf{P}})_*(v)$$

vektormező affin spray M számára, amelyet a P *által generált* (vagy P-*hez csatolt*) spray-nek nevezünk.

Egy $\gamma\colon I \to M$ (sima) görbét (M, P) *geodetikusá*nak mondunk, ha

$$\mathsf{P}(\gamma(t_1), \gamma(t_2))\dot\gamma(t_1) = \dot\gamma(t_2) \quad \text{minden } t_1, t_2 \in I \text{ esetén.}$$

Egy M-beli görbe *pregeodetikusa* (M, P)-nek, ha átparaméterezhető geodetikussá. Egy párhuzamosított sokaságot (vagy egyszerűen egy párhuzamosítást) akkor mondunk *teljes*nek, ha az összes geodetikusa értelmezve van a teljes valós számegyenesen.

Párhuzamosítások geodetikusaival kapcsolatban a következő egyszerűbb eredményeket nyertük:

(i) (M, P) *geodetikusai éppen a* P*-párhuzamos vektormezők integrálgörbéi (2.14. lemma).*

(ii) Egy P *párhuzamosítás akkor, és csak akkor, teljes, ha a* P*-párhuzamos vektormezők teljesek (2.15. következmény).*

(iii) Egy párhuzamosítás geodetikusai egybeesnek az általa generált spray geodetikusaival (2.16. állítás és [7, Proposition 10.3.1]).

P torziója (2.17. lemma és definíció)**.** Rögzítsünk tetszőlegesen egy $p \in M$ pontot. Ekkor a

$$\theta \colon q \in M \mapsto \theta_q \in L(T_qM, T_pM),\ \theta_q(w) := \mathsf{P}(q,p)(w) \in T_pM$$

leképezés T_pM-értékű 1-forma M-en. Azt a

$$T^{\mathsf{P}} \colon (X,Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto T^{\mathsf{P}}(X,Y) \in \mathfrak{X}(M)$$

$(1,2)$-típusú tenzormezőt, amelyet a

$$(T^{\mathsf{P}}(X,Y))_q := \mathsf{P}(p,q)(d\theta)_q(X_q,Y_q), \quad q \in M,$$

előírás értelmez, P *torziójá*nak nevezzük. Ha X és Y P-párhuzamos vektormezők, akkor $T^{\mathsf{P}}(X,Y) = -[X,Y] = [Y,X]$.

2.19. lemma. *Annak szükséges és elégséges feltétele, hogy* P *torziója párhuzamos legyen, azaz teljesítse a*

$$\mathsf{P}(p,q)(T^{\mathsf{P}}(X,Y))_p = T^{\mathsf{P}}_q(\mathsf{P}(p,q)(X_p), \mathsf{P}(p,q)(Y_p)) \quad (p,q \in M)$$

feltételt, az, hogy $\mathfrak{X}_{\mathsf{P}}(M)$ *részalgebrája legyen az* $\mathfrak{X}(M)$ *Lie-algebrának.*

Lie-csoportok és párhuzamosításaik. Minden Lie-csoportnak természetes módon létezik két párhuzamosítása. G bal-párhuzamosítása a

$$\mathsf{P}_L \colon (p,q) \in G \times G \mapsto \mathsf{P}_L(p,q) := ((\lambda_{qp^{-1}})_*)_p \in L(T_pG, T_qG),$$

leképezés, ahol λ_a ($a \in G$) az a elemmel való baleltolást jelenti. A Lie-csoport P_R jobb-párhuzamosítása analóg módon értelmezhető. Ekkor

$$\mathfrak{X}_{\mathsf{P}_L}(G) = \mathfrak{X}_L(G),\ \mathfrak{X}_{\mathsf{P}_R}(G) = \mathfrak{X}_R(G)(:= G \text{ jobb-invariáns vektormezőinek modulusa}),$$

így (a 2.19. lemma miatt) P_L és P_R torziója is párhuzamos. A két párhuzamosítás geodetikusai megegyeznek: ezek a Lie-csoport egyparaméteres részcsoportjai, valamint ezek jobb, illetve baleltoltjai. Ebből az is következik, hogy egy Lie-csoport természetes párhuzamosításai teljesek.

Az alábbi eredmény (lásd a 2.42. állítást és annak bizonyítását) egy klasszikus tétel részben új és részletes átfogalmazása. A fontosabb források [54, Proposition 2.5], továbbá [8, 19, 21, 27, 28].

Tegyük fel, hogy (M, P) *összefüggő, teljes és párhuzamos torzióval rendelkező párhuzamosított sokaság. Rögzítsünk tetszőlegesen egy* $p \in M$ *pontot, és adjunk meg Lie-zárójelet az*

$$[u, v] := [u_{\mathsf{P}}, v_{\mathsf{P}}]_p; \quad u, v \in T_pM$$

előírással T_pM*-en. Legyen* G *az az összefüggő és egyszeresen összefüggő Lie-csoport, amelynek Lie-algebrája* $\mathsf{Lie}(G) = (T_pM, [\cdot, \cdot])$*, és tekintsük* G*-nek a*

$$T_pM = T_eG \to \mathfrak{X}(M),\ v \mapsto v_{\mathsf{P}}$$

Lie-algebra hatás által indukált

$$A\colon M \times G \to M,\ (m, g) \mapsto m \cdot g$$

jobboldali hatását M*-en. Ekkor a* G_p *p-beli izotrópia-részcsoport diszkrét részcsoportja* G*-nek, és a* $G_p\backslash G$ *hányadossokaság kanonikusan diffeomorf* M*-mel a*

$$G_p\backslash G \to M,\ G_p \cdot g \mapsto p \cdot g$$

leképezés által. Az $\mathfrak{X}_L(G)$ *és* $\mathfrak{X}_{\mathsf{P}}(M)$ *Lie-algebrák természetes módon beazonosíthatók egymással.*

P-invariáns kovariáns deriváltak. Egy, az (M, P) párhuzamosított sokaságon adott ∇ kovariáns deriváltat P-*invariáns*nak mondunk, ha

$$\nabla_X Y \in \mathfrak{X}_{\mathsf{P}}(M) \quad \text{minden } X, Y \in \mathfrak{X}_{\mathsf{P}}(M) \text{ esetén.}$$

A következő eredmény egy jól ismert tény (lásd, pl. [26, Chapter II, Proposition 1.4]) kézenfekvő általánosítása párhuzamosított sokaságok esetére.

2.21. és 2.22. állítás. *A* $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ P-*invariáns kovariáns deriváltak halmaza és az* $\alpha\colon \mathfrak{X}_{\mathsf{P}}(M) \times \mathfrak{X}_{\mathsf{P}}(M) \to \mathfrak{X}_{\mathsf{P}}(M)$ $\mathbb{R}$-*bilineáris leképezések vektortere között egy-egyértelmű megfeleltetés létesíthető, amelyet az*

$$\alpha(X, Y) = \nabla_X Y, \quad \textit{minden } X, Y \in \mathfrak{X}_{\mathsf{P}}(M) \textit{ esetén},$$

reláció karakterizál. Ekkor a következő természetes választások lehetségesek:

(A) $\underline{\alpha = 0}$ *Ekkor a* ∇ *kovariáns derivált lapos és*

$$T := \nabla \textit{ torziója} = \mathsf{P} \textit{ torziója}.$$

Az így nyert kovariáns deriváltat P indukált kovariáns derivált*jának nevezzük.*

(B) Feltesszük, hogy $\mathfrak{X}_{\mathsf{P}}(M)$ Lie-részalgebrája $\mathfrak{X}(M)$-nek. *Ekkor:*

(i) Az

$$\alpha(X, Y) := [X, Y], \quad \textit{ha } X, Y \in \mathfrak{X}_{\mathsf{P}}(M)$$

feltétellel definiált α $\mathbb{R}$*-bilineáris leképezésnek megfelelő* ∇^+ *kovariáns derivált szintén lapos, a torziója pedig* $-T$.

(ii) Ha az α $\mathbb{R}$*-bilineáris leképezést az*

$$\alpha(X, Y) := \frac{1}{2}[X, Y], \quad \textit{ha } X, Y \in \mathfrak{X}_{\mathsf{P}}(M)$$

előírással adjuk meg, akkor az ennek megfelelő ∇^0 *kovariáns derivált torziómentes, de nem lapos;* R^0 *görbületi tenzorára az*

$$R^0(X, Y)Z = -\frac{1}{4}[[X, Y], Z]; \quad X, Y, Z \in \mathfrak{X}_{\mathsf{P}}(M)$$

kiszámítási formula érvényes.

2.23. állítás. *Egy* (M, P) *párhuzamosított sokaságon adott* P*-invaráns kovariáns deriválás geodetikusai pontosan akkor esnek egybe* (M, P) *geodetikusaival, ha a neki megfelelő* $\mathbb{R}$*-bilineáris leképezés ferdeszimmetrikus.*

2.24. következmény. *A* ∇, ∇^+ *és* ∇^0 *kovariáns deriváltak geodetikusai éppen* (M, P) *geodetikusai.*

2.25. következmény. *Tetszőlegesen kiválasztva egy* $v \in T_pM$ *érintővektort, egyértelműen létezik* (M, P)*-nek olyan* γ *maximális geodetikusa, amelyre* $\dot{\gamma}(0) = v$ *teljesül.*

Izomorfizmusok és automorfizmusok. Egy $\varphi\colon M \to \bar{M}$ diffeomorfizmust az (M, P) és $(\bar{M}, \bar{\mathsf{P}})$ párhuzamosított sokaság közötti *izomorfizmus*nak nevezünk, ha M minden p, q pontja esetén érvényes a

$$(\varphi_*)_q \circ \mathsf{P}(p,q) = \bar{\mathsf{P}}(\varphi(p), \varphi(q)) \circ (\varphi_*)_p$$

felcserélhetőség. Ekvivalens módon: minden $v \in TM$ érintővektor esetén

$$\varphi_* \circ v_{\mathsf{P}} = (\varphi_*(v))_{\bar{\mathsf{P}}} \circ \varphi.$$

Az (M, P) párhuzamosított sokaságnak egy önmagára való izomorfizmusát (M, P) (vagy P) *automorfizmusának* mondjuk, az automorfizmusok által alkotott csoportot $\mathrm{Aut}(\mathsf{P})$-vel jelöljük. Ennek egy részcsoportja

$$\mathrm{Sym}(\mathsf{P}) := \{\varphi \in \mathrm{Diff}(M) \mid \varphi_{\#} X = X \text{ ha } X \in \mathfrak{X}_{\mathsf{P}}(M)\},$$

amelynek elemeit (M, P) *szimmetriái*nak (vagy *transzlációi*nak) hívjuk.

Ha G összefüggő Lie-csoport, akkor

$$\mathrm{Sym}(\mathsf{P}_L) = \{\lambda_g \in \mathrm{Diff}(G) \mid g \in G\}.$$

Az $\mathrm{Aut}(\mathsf{P}_L)$ automorfizmus-csoport a bővebb csoport: ez tartalmazza a G elemeivel való jobbeltolásokat és konjugálásokat is.

2.34. és 2.35. állítás. *Az* $\mathrm{Aut}(\mathsf{P})$ *csoport a* P *által indukált kovariáns deriválás automorfizmus-csoportjának részcsoportja. Ha* M *összefüggő, akkor ez a két csoport egybeesik. Teljesül továbbá, hogy* $\mathrm{Aut}(\mathsf{P})$ *a* P *által generált spray automorfizmus-csoportjának is részcsoportja.*

Konjugált párhuzamosítások. Az M-en adott P_1 és P_2 párhuzamosítások *konjugáltak*, ha $X \in \mathfrak{X}_{\mathsf{P}_1}(M)$ és $Y \in \mathfrak{X}_{\mathsf{P}_2}(M)$ esetén $[X, Y] = 0$.

Összefüggő sokaságon minden párhuzamosításhoz legfeljebb egy konjugált párhuzamosítás létezik. Ha a P párhuzamosításhoz van konjugált párhuzamosítás, akkor P torzója párhuzamos. Megfordítva, ha egy (M, P) párhuzamosított sokaság torziója párhuzamos, akkor M minden pontjának van olyan környezete, amelyen létezik P-hez konjugált párhuzamosítás (2.44. és 2.46. lemma, ld. [20, Chapter IV, Problems]).

Konjugált párhuzamosításokkal kapcsolatban a következő egyszerűbb eredményeket nyertük:

(i) *Ha* P_1 *és* P_2 *konjugált párhuzamosítások* M*-en, akkor* ∇_1 *és* ∇_2 *indukált kovariáns deriváltjaik között a*

$$(\nabla_2)_X Y = (\nabla_1)_Y X + [X,Y]; \quad X,Y \in \mathfrak{X}(M)$$

összefüggés áll fenn, következésképpen P_1 *és* P_2 *torziója egymás ellentettje (2.45. és 2.50. lemma). A* P_1*-hez tartozó* ∇_1^+ *kovariáns deriválás (lásd fentebb (B)/(i)) létezik, és* $\nabla_2 = \nabla_1^+$ *(2.48. következmény).*

(ii) *A* P_1 *és* P_2 *konjugált párhuzamosítások ugyanazt a spray-t származtatják, és* M *összefüggősége esetén az automorfizmus-csoportjaik is megegyeznek (2.49. és 2.51 lemma).*

(iii) *Ha* P *és* $\bar{\mathsf{P}}$ *konjugált párhuzamosítások* M*-en,* X *pedig* P_1*-párhuzamos vektormező amelynek folyama a* $\varphi^X : \mathcal{D}_X \subset \mathbb{R} \times M \to M$ *leképezés, akkor*

$$((\varphi_t^X)_*)_p = \bar{\mathsf{P}}(p, (\varphi_t^X)_p) \quad \textit{minden } (t,p) \in \mathcal{D}_X \textit{ esetén}$$

(2.52. lemma).

Párhuzamosítások konform változtatása. Legyen σ pozitív, sima függvény M-en, és tekintsük M-nek egy P párhuzamosítását. Ekkor a

$$\widetilde{\mathsf{P}} \colon (p,q) \in M \times M \mapsto \widetilde{\mathsf{P}}(p,q) := \frac{\sigma(p)}{\sigma(q)} \mathsf{P}(p,q) \in L(T_pM, T_qM)$$

leképezés is párhuzamosítás M-en, amelyet a P párhuzamosítás σ konform faktorral való *konform változtatásának* nevezünk. A P által indukált ∇ és a $\widetilde{\mathsf{P}}$ által indukált $\widetilde{\nabla}$ kovariáns derivált, valamint ezek torziói között a

$$\widetilde{\nabla} = \nabla + d\sigma \otimes 1_{\mathfrak{X}(M)}, \quad \text{ill. a} \quad \widetilde{T} = T + d\sigma \wedge 1_{\mathfrak{X}(M)}$$

összefüggés áll fenn (2.56. állítás).

Legyen $\widetilde{\mathsf{P}}_1$ és $\widetilde{\mathsf{P}}_2$ a P_1, ill. P_2 párhuzamosítás σ faktorú konform változtatottja. Ha $\widetilde{\mathsf{P}}_1$ és $\widetilde{\mathsf{P}}_2$ konjugáltak, akkor azt mondjuk, hogy P_1 és P_2 (σ faktorú) *konform-konjugált párhuzamosítások*. A P_1 és a P_2 párhuzamosítás pontosan akkor áll ilyen kapcsolatban, ha minden $X \in \mathfrak{X}_{\mathsf{P}_1}(M)$ és $Y \in \mathfrak{X}_{\mathsf{P}_2}(M)$ esetén $[X,Y] = (d\sigma \wedge 1_{\mathfrak{X}(M)})(X,Y)$ (2.59. lemma).

3. fejezet. Finsler-függvények és párhuzamosítások.

Megállapodunk abban, hogy a továbbiakban (M,F) *Finsler-sokaság.*

Tekintsünk egy P párhuzamosítást M-en. Az F Finsler-függvény *kompatibilis* P-vel, ha

$$F_q \circ \mathsf{P}(p,q) = F_p, \text{ minden } p, q \in M \text{ esetén.}$$

Azt mondjuk, hogy F egy $(\mathcal{U}_\alpha, \mathsf{P}^\alpha)_{\alpha \in \mathcal{A}}$ *lefedő párhuzamosítással kompatibilis*, ha minden $\alpha \in \mathcal{A}$ esetén az F Finsler-függvény $\tau^{-1}(\mathcal{U}_\alpha)$ nyílt halmazra való leszűkítése kompatibilis P^α-val.

3.7. tétel. *Ha az F Finsler-függvény kompatibilis két konjugált párhuzamosítással, akkor a párhuzamosítások által generált (szükségképpen közös) spray éppen a Finsler-sokaság kanonikus spray-je.*

3.13. tétel. *Egy Finsler-sokaság pontosan akkor általánosított Berwald-sokaság, ha a Finsler-függvény kompatibilis egy lefedő párhuzamosítással.*

3.14. tétel. *Minden bal-invariáns Finsler-függvénnyel ellátott Lie-csoport általánosított Berwald-sokaság.*

3.17. tétel. *Tegyük fel, hogy M-en adva van két, ugyanazon $(\mathcal{U}_\alpha)_{\alpha \in \mathcal{A}}$ nyílt lefedéshez tartozó lefedő párhuzamosítás, $(\mathcal{U}_\alpha, \mathsf{P}_1^\alpha)_{\alpha \in \mathcal{A}}$ és $(\mathcal{U}_\alpha, \mathsf{P}_2^\alpha)_{\alpha \in \mathcal{A}}$ oly módon, hogy*

(i) az F Finsler-függvény kompatibilis mindkét lefedő párhuzamosítással;
(ii) minden $\alpha \in \mathcal{A}$ esetén a P_1^α és P_2^α párhuzamosítások konjugáltak.

Ekkor (M, F) Berwald-sokaság.

3.19. tétel. *Tegyük fel, hogy M-en adva van két, ugyanazon $(\mathcal{U}_\alpha)_{\alpha \in \mathcal{A}}$ nyílt lefedéshez tartozó lefedő párhuzamosítás, $(\mathcal{U}_\alpha, \mathsf{P}_1^\alpha)_{\alpha \in \mathcal{A}}$ és $(\mathcal{U}_\alpha, \mathsf{P}_2^\alpha)_{\alpha \in \mathcal{A}}$, eleget téve a következőknek:*

(i) az F Finsler-függvény kompatibilis mindkét lefedő párhuzamosítással;
(ii) minden $\alpha \in \mathcal{A}$ esetén a P_1^α és P_2^α párhuzamosítások konform-konjugáltak ugyanazzal a konform faktorral.

Ekkor (M, F) Wagner-sokaság.

Azt mondjuk, hogy egy, az M sokaságon adott P párhuzamosítás *erősen kompatibilis* az F Finsler-függvénnyel, ha kompatibilis vele, továbbá

F és P pregeodetikusai megegyeznek.

3.21. tétel. *Ha egy M-en adott P párhuzamosítás kompatibilis az F Finsler-függvénnyel, akkor a következő állítások ekvivalensek:*

(i) *F és P erősen kompatibilis;*
(ii) *F és P geodetikusai megegyeznek;*
(iii) *a P-párhuzamos vektormezők Killing-vektormezői a Finsler-sokaságnak.*

3.22. tétel. *Ha F erősen kompatibilis egy M-en adott P párhuzamosítással, akkor (M, F) Berwald-sokaság.*

3.23. következmény. *Ha F bal-invariáns Finsler-függvény egy G Lie-csoporton, és F pregeodetikusai az egyparaméteres részcsoportok valamint ezek eltoltjai, akkor (G, F) Berwald-sokaság.*

3.24. tétel (a párhuzamosított Finsler-sokaságok struktúratétele)**.** *Tegyük fel, hogy (M, P) teljes, összefüggő párhuzamosított sokaság, amelynek torziója párhuzamos. Ekkor*

(i) *M diffeomorf egy olyan $H \backslash G$ hányadossokasággal, ahol G összefüggő és egyszeresen összefüggő Lie-csoport, H diszkrét részcsoportja G-nek;*
(ii) *M párhuzamosítását a G Lie-csoport bal-párhuzamosítása származtatja.*

Amennyiben F P-vel erősen kompatibilis Finsler-függvény M-en, úgy (M, F) Berwald-sokaság és F-et egy G-n adott biinvariáns Finsler-függvény indukálja.

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