INTEGERS REPRESENTED BY LUCAS SEQUENCES

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Abstract. In this paper, we study the sets of integers which are n-th terms of Lucas sequences. We establish lower- and upper bounds for the size of these sets. These bounds are sharp for n sufficiently large. We also develop bounds on the growth order of the terms of Lucas sequences that are independent of the parameters of the sequence, which is a new feature.

1. Introduction

In this paper, we study sets of integers which are n-th terms of Lucas sequences for some $n \geq 0$. For integers A, B, the sequence $U = (U_n)_{n=0}^{\infty}$ with $U_0 = 0$, $U_1 = 1$ satisfying the binary recursive relation

$$U_n = AU_{n-1} - BU_{n-2} \quad (n \geq 2),$$

is called a Lucas sequence. Note that the Fibonacci sequence is a Lucas sequence, corresponding to the choice $(A, B) = (1, -1)$. Furthermore, Mersenne numbers are generated by the choice $(3, 2)$, Pell numbers by $(2, -1)$ and Jacobsthal numbers by $(1, -2)$. Lucas sequences are well studied in the literature, see for example [4] and the references there.

Write $f(x) = x^2 - Ax + B$ for the characteristic polynomial of U, and let α, β be its roots. Throughout the paper, unless stated otherwise, we shall assume that the sequence is non-degenerate, that is, $AB \neq 0$ and α/β is not a root of unity. (It is easy to deal with the excluded cases; see Lemma 4.1.) Without loss of generality we assume that $|\alpha| \geq |\beta|$ throughout the paper. We have

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \geq 0). \tag{1}$$

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If $A^2 > 4B$, then α and β are real and we say that we are in the real case. If $A^2 < 4B$, then α and β are non-real complex numbers which are conjugates, and we say that we are in the non-real case.

There are numerous papers on common properties of non-degenerate Lucas sequences. For example, if p is an odd prime which divides $A^2 - 4B$, but not AB, then p divides U_n if and only if p divides n (see Theorem 1.9 of [18]). We also mention that Carmichael [9] showed that in the real case, if n is not 1, 2 or 6, then U_n has a primitive prime factor. In the general case, Bilu, Hanrot and Voutier [5] proved that, if $n > 30$, then U_n has a primitive prime factor. Stewart [28] showed that non-degenerate Lucas sequences grow exponentially (cf. Lemma 5 of [26]). He gave an explicit lower bound. These results depend on the sequence, and therefore on A and B. In this paper we study results on Lucas sequences which are independent of A and B. Such results are rare. There are a number of papers on n-th terms from Lucas sequences being squares (with n being fixed), see e.g. [6, 7, 8] and the references there. We prove as a variant of Stewart's result that $|U_n| \geq \frac{1}{2}(\frac{1+\sqrt{5}}{2})^{n-2}$ in the real case and $|U_n| \geq (\sqrt{2})^{n-c(\log n)^2}$ in the non-real case for $n > 1$, independently of the chosen Lucas sequence. Here c is an absolute constant, which we give explicitly. The proof in the real case is elementary, while in the non-real case it is based on an estimate of linear forms in two logarithms.

Next we give sharp upper and lower bounds for the number of Lucas sequences with $|\alpha| \leq t$.

As another way to measure the amount of numbers occurring in Lucas sequences, we study the sets

$$\mathcal{L}_n(N) = \{x \in \mathbb{Z} : 0 \leq x \leq N \text{ and } x = |U_n| \\ \text{for some non-degenerate Lucas sequence } U\}$$

and

$$\mathcal{L}_{\geq n}(N) = \bigcup_{m=n}^{\infty} \mathcal{L}_m(N).$$

We derive sharp, explicit upper- and lower bounds for the growth order of $|\mathcal{L}_n(N)|$ and $|\mathcal{L}_{\geq n}(N)|$. In the proofs we need to combine several tools, including our new bounds on the growth of Lucas sequences, extensions of theorems of Erdős and Mahler [11] and Lewis and Mahler [20] concerning representability of integers by binary forms $G(x, y)$ to representations of the type $G(x^2, y) = k$, and certain properties of the Fibonacci polynomials. For the introduction of these polynomials, observe that the first few terms of U are given by

$$0,\ 1,\ A,\ A^2-B,\ A^3-2AB,\ A^4-3A^2B+B^2,\ A^5-4A^3B+3AB^2.$$

We define $F_n(x,y)$ by $F_0(x,y)=0$, $F_1(x,y)=1$ and

$$F_n(x,y)=xF_{n-1}(x,y)-yF_{n-2}(x,y)\quad (n\geq 2).$$

Polynomials $F_n(A,B)$ correspond with Lucas sequences, polynomials $F_n(x,-1)$ are the Fibonacci polynomials, polynomials $F_n(2x,1)$ are the Chebyshev polynomials of the second kind.

The structure of the paper is the following. In Section 2 we formulate our principal results. In Section 3 we prove Theorem 2.1 which provides lower bounds on the growth of Lucas-sequences. In Section 4 we prove Theorem 2.2 which gives precise bounds for the number of Lucas sequences with bounded $|\alpha|$. In Section 5 we prove Theorem 2.3 on upper bounds for the sizes of the sets $\mathcal{L}_n$ and $\mathcal{L}_{\geq n}$. In Section 6 we prove Theorems 2.4 and 2.5 on lower bounds for them.

The obtained upper and lower bounds in Theorems 2.3 - 2.5 differ by a multiplicative constant depending only on n for $n\geq 7$ odd, and by a lower order factor for $n=5$ and for $n\geq 6$ even.

By $c_1,c_2,c_3,\ldots$ we denote effectively computable constants depending only on n.

2. Main results

Lemma 5 of [26] (taken from [28]) states that there exist positive constants N_0 and C_0 depending on A,B such that for $n>N_0$ we have $|U_n|>|\alpha|^{n-C_0\log n}$. Our first theorem (see its corollary and Remark 1) implies that N_0 and C_0 can be chosen independently of A and B.

Theorem 2.1. *For $n\geq 2$ we have in the real case,*

$$|U_n|\geq \frac{|\alpha|^{n-2}}{2}\quad \textit{if } B<0$$

$$|U_n|\geq |\alpha|^{n-1}\quad \textit{if } A^2>4B>0$$

and in the non-real case,

$$|U_n|\geq \begin{cases} \frac{1}{4}e^{-250(\log n)^2}|\alpha|^{n-2}, & \textit{if } n>5\cdot 10^8,\\ \frac{1}{4}e^{-100000}|\alpha|^{n-2}, & \textit{if } n\leq 5\cdot 10^8\end{cases}$$

for $B\leq 535$, and

$$|U_n|\geq \begin{cases} \frac{1}{4}|\alpha|^{n-2-88(\log n)^2}, & \textit{if } n>2.1\cdot 10^8,\\ \frac{1}{4}|\alpha|^{n-31710}, & \textit{if } n\leq 2.1\cdot 10^8\end{cases}$$

for $B\geq 536$.

We have in the real case,

$$|\alpha| = \frac{1+\sqrt{5}}{2} \text{ if } (A,B) = \pm(1,-1), \quad |\alpha| \geq 2 \quad \text{if } (A,B) \neq \pm(1,-1),$$

and in the non-real case, since $B = 1$ is excluded,

$$|\alpha| = \sqrt{B} \geq \sqrt{2}.$$

Thus Theorem 2.1 implies the following result.

Corollary 2.1. *For $n \geq 2$, we have in the real case*

$$|U_n| \geq \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \qquad \textit{if } (A,B) = (1,-1),$$

$$|U_n| \geq 2^{n-3} \quad \textit{if } B < 0 \textit{ with } (A,B) \neq (1,-1),$$

$$|U_n| \geq 2^{n-1} \quad \textit{if } A^2 > 4B > 0$$

and in the non-real case,

$$|U_n| \geq \begin{cases} 2^{\frac{n}{2}-361(\log n)^2-3}, & \textit{if } n > 5\cdot 10^8, \\ 2^{\frac{n}{2}-144273}, & \textit{if } n \leq 5\cdot 10^8 \end{cases}$$

for $B \leq 535$, and

$$|U_n| \geq \begin{cases} 2^{\frac{n}{2}-44(\log n)^2-3}, & \textit{if } n > 2.1\cdot 10^8, \\ 2^{\frac{n}{2}-15852}, & \textit{if } n \leq 2.1\cdot 10^8 \end{cases}$$

for $B \geq 536$.

Remark 1. In the proof of Theorem 2.1 in the non-real case, we use a result of Laurent [17] on linear forms in two logarithms. This causes the appearance of $(\log n)^2$ in the exponent, but with a rather small coefficient. If we use the Baker-type result of Matveev [21] instead, then the $(\log n)^2$ is replaced by $\log n$, however, with much larger constants as coefficients. The following argument, that we owe to an unknown colleague, shows that it is not true that there is an absolute constant $c > 0$ such that $|U_n| > c|\alpha|^n$ for all n in the non-real case. Write $\frac{\beta}{\alpha} = e^{2\pi i\theta}$. Let $\frac{m}{n}$ be a convergent of θ. Then $|n\theta - m| < \frac{1}{n}$. Hence $\left|\left(\frac{\beta}{\alpha}\right)^n - 1\right| \ll \frac{1}{n}$. Thus $|\alpha^n - \beta^n| \ll \frac{|\alpha|^n}{n}$ for infinitely many n.

The following result provides an upper and a lower bound for the number of Lucas sequences with bounded roots. It shows that this number is $4t^3 + O(t^2)$.

Theorem 2.2. *Let $t \in \mathbb{R}_{\geq 2}$. Then the number of non-degenerate Lucas sequences with $|\alpha| \leq t$ is at most $4t^3 - t^2 + 7t$ and at least $4t^3 - 10t^2 - 26t$.*

Remark 2. The lower bound in Theorem 2.2 can be improved to $4t^3 - 9t^2$, if t is an integer.

Observe the following facts. Here the degenerate cases are too few to influence the density (cf. Lemma 4.1).

- For every non-zero integer K, there exist infinitely many U with $U_2 = K$. For this, we may choose $A = K$ with B arbitrary.
- For every integer K, there exist infinitely many U with $U_3 = K$. For this we may choose $B = A^2 - K$ with A arbitrary.
- Let $U_4 = K$. Observe that A and $A^2 - 2B$ are both odd or both even, hence $K \not\equiv 2 \pmod 4$. If K is odd, we can take $A = 1$ and $B = \frac{1-K}{2}$. If 4 divides K, we can take $A = 2$ and $B = 2 - \frac{1}{4}K$.

We conclude that

$$\lim_{N\to\infty} \frac{\mathcal{L}_n(N)}{N} = 1, 1, \frac{3}{4}, \text{ for } n = 2, 3, 4, \text{respectively.}$$

In the sequel we restrict our attention to $n \geq 5$.

The following statement provides an upper bound for the number of n-th terms up to N counted over all non-degenerate Lucas sequences.

Theorem 2.3. i) $|\mathcal{L}_n(N)| \leq (612 + 2.5 \cdot 10^{11} \cdot 0.95^n)\, N^{\frac{3}{n-1}}$ *for* $n \geq 5$.
ii) $|\mathcal{L}_{\geq n}(N)| \leq (612 + 5 \cdot 10^{12} \cdot 0.95^n)\, N^{\frac{3}{n-1}} + 1836 \cdot N^{\frac{3}{n}} \log N$ *for* $n \geq 5$.

Theorem 2.3 implies that $\mathcal{L}_n(N) \ll N^{\frac{3}{n-1}}$ and that the density of $\mathcal{L}_{\geq 5}$ is zero. In fact, since $N^{\frac{3}{5}} \log N < 6N^{\frac{3}{4}}$ for all $N \geq 1$, we have

Corollary 2.2.

$|\mathcal{L}_n(N)| \leq (612 + o_n(1))N^{\frac{3}{n-1}}$ for $n \to \infty$, and $|\mathcal{L}_{\geq 5}(N)| \leq 5 \cdot 10^{12} \cdot N^{\frac{3}{4}}$.

The next statements show that the exponents $\frac{3}{n-1}$ in Theorem 2.3 are best possible.

Theorem 2.4. *Let* $n \geq 7$ *and odd. Then there exists a positive constant* c_1 *depending only on* n *such that*

$$|\mathcal{L}_n(N)| \geq c_1 N^{\frac{3}{n-1}}, \quad \text{hence} \quad |\mathcal{L}_{\geq n}(N)| \geq c_1 N^{\frac{3}{n-1}},$$

provided that $|\mathcal{L}_n(N)|$ *and* $|\mathcal{L}_{\geq n}(N)|$*, respectively, are positive.*

Thus, for odd values $n \geq 7$, the bounds are sharp apart from a multiplicative constant depending only on n.

Corollary 2.3. *For* $n \geq 7, n$ *odd, we have*

$$|\mathcal{L}_n(N)| \ll\gg_n N^{\frac{3}{n-1}}.$$

For other values of $n \geq 5$ our lower bound is slightly weaker, but of the same order of magnitude.

Theorem 2.5. *Let $n \geq 5$. Then there exists a positive constant c_2 depending only on n such that*

$$|\mathcal{L}_n(N)| \geq c_2 N^{\frac{3}{n-1}-\frac{4}{\log\log N}}, \quad \text{hence} \quad |\mathcal{L}_{\geq n}(N)| \geq c_2 N^{\frac{3}{n-1}-\frac{4}{\log\log N}},$$

provided that $|\mathcal{L}_n(N)|$ and $|\mathcal{L}_{\geq n}(N)|$, respectively, are positive.

Remark 3. We expect that, for all $n \geq 5$,

$$\mathcal{L}_n(N) \gg_n N^{\frac{3}{n-1}} \quad \text{and therefore} \quad \mathcal{L}_{\geq n}(N) \gg_n N^{\frac{3}{n-1}}.$$

Because of the symmetry of the polynomials $x^2 - Ax + B = (x-\alpha)(x-\beta)$ and $x^2 + Ax + B = (x+\alpha)(x+\beta)$, we assume that $A > 0$ in the sequel, unless stated otherwise. We further assume in the real case that $|\alpha| > |\beta|$. It follows that $\alpha > 0$.

3. Proof of Theorem 2.1

To prove Theorem 2.1 in the case where α, β are non-real, we shall use bounds for linear forms in two logarithms. For this, we introduce some notation. For an algebraic number γ of degree d over $\mathbb{Q}$, the absolute logarithmic height of γ is defined by

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{i=1}^{d} \log\max(1, |\gamma^{(i)}|)\right),$$

where a is the leading coefficient of the minimal polynomial of γ over $\mathbb{Z}$, and the $\gamma^{(i)}$'s are the algebraic conjugates of γ. Let α_1, α_2 be non-zero algebraic numbers. Consider the linear form

$$\Lambda = b_2 \log\alpha_2 - b_1 \log\alpha_1,$$

where b_1, b_2 are non-zero integers, and the logarithm of a non-zero complex number w (here and later on) is taken according to

$$\log w = \log|w| + \mathrm{i}\arg w,$$

with $-\pi < \arg w \leq \pi$. Set

$$D = \frac{[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]}{[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}]} \quad \text{and} \quad b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1},$$

where A_1, A_2 are real numbers greater than 1 such that

$$\log A_i \geq \max\left\{h(\alpha_i),\ \frac{|\log\alpha_i|}{D},\ \frac{1}{D}\right\} \quad (i = 1, 2).$$

The following result is due to Laurent, see Corollary 1 in [17].

Lemma 3.1. *Keeping the above notation, suppose that* α_1, α_2 *are multiplicatively independent. Then we have*

$$\log|\Lambda| \geq -25.2D^4\left(\max\left\{\log b' + 0.21,\ \frac{20}{D},\ 1\right\}\right)^2 \log A_1 \log A_2.$$

The following consequence of Lemma 3.1 will be useful.

Lemma 3.2. *Let* δ *be an algebraic number which is not a root of unity. Then, for any non-zero integer* ℓ, *we have*

$$\left|\delta^\ell - 1\right| \geq \frac{1}{2}\exp\left(-25.2\pi D_\delta^3\left(\max\left\{\log\frac{(\pi + D_\delta\log A_\delta)|\ell|}{\pi D_\delta \log A_\delta} + 0.21, \frac{20}{D_\delta}, 1\right\}\right)^2 \log A_\delta\right),$$

where

$$D_\delta = \frac{[\mathbb{Q}(\delta):\mathbb{Q}]}{[\mathbb{R}(\delta):\mathbb{R}]}$$

and A_δ *is a real number with*

$$\log A_\delta \geq \max\left\{h(\delta),\ \frac{|\log\delta|}{D_\delta},\ \frac{1}{D_\delta}\right\}.$$

Proof. Put $z = \delta^\ell - 1$. Since δ is not a root of unity, we have $z \neq 0$. If $|z| > \frac{1}{2}$, then our claim immediately follows. So we may assume that $|z| \leq \frac{1}{2}$. Then, as it is well-known, we have

$$|\log(1+z)| \leq 2|z|. \tag{2}$$

On the other hand, we also have

$$\log(1+z) = \ell\log(\delta) + 2k\log(-1), \tag{3}$$

with some integer k. As $|\log(1+z)| \leq 1$, we have $|\Im(\log(1+z))| \leq 1$. Hence, by the choice of the logarithm,

$$|\ell\Im(\log(\delta)) + 2k\pi| \leq 1,$$

which implies

$$|2k| \leq \frac{1+\pi|\ell|}{\pi} = |\ell| + \frac{1}{\pi}.$$

As $2k$ and ℓ are integers, this in fact gives

$$|2k| \leq |\ell|.$$

Hence, in view of (2) and (3), our claim follows from Lemma 3.1. □

Proof of Theorem 2.1. Let $n \geq 2$.

Real case. Suppose that $A^2 > 4B$. Then α and β are real. As stated at the end of Section 2, we assume $A \geq 1$ and $\alpha > |\beta|$. Put $D = \sqrt{A^2 - 4B}$.

Assume first that $B < 0$. If $(A, B) = (1, -1)$, then $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$ and $U_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$. Since $|\beta| < 1$, we have

$$U_n > \frac{\alpha^n - 1}{\sqrt{5}} > \frac{\alpha^{n-2}}{2}.$$

In all other cases we have $\alpha \geq 2$. Then we infer, by $\beta = \frac{B}{\alpha} < 0, |\beta| < \alpha$,

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \geq \frac{\alpha^n - |\beta|^n}{\alpha - |\beta|} \cdot \frac{\alpha - |\beta|}{\alpha + |\beta|} \geq \alpha^{n-1} \frac{A}{2\alpha} \geq \frac{\alpha^{n-2}}{2}. \tag{4}$$

Assume next that $B > 0$. Then $A \geq 3$, $\beta > 0$ and, on using that $D \geq 1$,

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \geq \alpha^{n-1}. \tag{5}$$

Non-real case. Suppose now $A^2 - 4B < 0$. Then $B > 0$ and α and β are complex conjugates. If $B = 1$, then $A = 1$ and $\frac{\alpha}{\beta}$ is a root of unity, which is excluded. So we may assume that $B > 1$. Then we have

$$|U_n| = \left| \frac{\alpha^n - \beta^n}{\alpha - \beta} \right| = \left| \frac{\alpha^n}{\alpha - \beta} \right| \left| \left(\frac{\beta}{\alpha} \right)^n - 1 \right|. \tag{6}$$

We shall use that $\frac{\beta}{\alpha}$ is a root of the polynomial

$$B \left(x - \frac{\alpha}{\beta} \right) \left(x - \frac{\beta}{\alpha} \right) = Bx^2 - (A^2 - 2B)x + B,$$

which is irreducible over $\mathbb{Q}$. Thus, on using that $|\alpha| = |\beta|$,

$$h \left(\frac{\beta}{\alpha} \right) = \frac{\log B}{2}. \tag{7}$$

Note that

$$\left| \log \frac{\beta}{\alpha} \right| \leq \left| \log \left| \frac{\beta}{\alpha} \right| \right| + \pi = \pi.$$

We distinguish two cases.

Assume first that $B < e^{2\pi} < 536$. Then we apply Lemma 3.2 for $\delta = \frac{\beta}{\alpha}$, with $D_\delta = 1$, $\ell = n$, $A_\delta = e^\pi$. We get

$$\left| \left(\frac{\beta}{\alpha} \right)^n - 1 \right| \geq \frac{1}{2} \exp \left(-25.2\pi^2 \left(\max \left\{ \log \frac{2n}{\pi} + 0.21, 20 \right\} \right)^2 \right).$$

This easily yields

$$\left| \left(\frac{\beta}{\alpha} \right)^n - 1 \right| \geq \frac{1}{2} \exp \left(-25.2\pi^2 \left(\max \left\{ \log n - 0.24, 20 \right\} \right)^2 \right).$$

A simple calculation shows that, for $n \leq 5 \cdot 10^8$, we have

$$\max\{\log n - 0.24, 20\} = 20.$$

On the other hand, for $n \geq 6.2 \cdot 10^8$,

$$\max\{\log n - 0.24, 20\} = \log n - 0.24 < \log n$$

holds. Finally, for $5 \cdot 10^8 < n < 6.2 \cdot 10^8$,

$$(\max\{\log n - 0.24, 20\})^2 < 20.01^2 < 1.001 \cdot (\log 5 \cdot 10^8)^2 < 1.001 \cdot (\log n)^2$$

is valid. Thus we get that

$$\left|\left(\frac{\beta}{\alpha}\right)^n - 1\right| \geq \begin{cases} \frac{1}{2}e^{-250(\log n)^2}, & \text{if } n > 5 \cdot 10^8, \\ \frac{1}{2}e^{-100000}, & \text{if } n \leq 5 \cdot 10^8. \end{cases} \tag{8}$$

Combining it with (6) we conclude that

$$|U_n| \geq \begin{cases} \frac{1}{4}e^{-250(\log n)^2} B^{\frac{n}{2}-1}, & \text{if } n > 5 \cdot 10^8, \\ \frac{1}{4}e^{-100000} B^{\frac{n}{2}-1}, & \text{if } n \leq 5 \cdot 10^8. \end{cases}$$

Suppose next that $B \geq e^{2\pi} > 535$. Now we apply Lemma 3.2 for $\delta = \frac{\beta}{\alpha}$ with $D_\delta = 1$, $\ell = n$, $A_\delta = \sqrt{B}$ to obtain

$$\left|\left(\frac{\beta}{\alpha}\right)^n - 1\right| \geq \frac{1}{2}\exp\left(-12.6\pi(\log B)\left(\max\left\{\log\frac{(2\pi + \log B)n}{\pi \log B} + 0.21, 20\right\}\right)^2\right).$$

This by $B \geq 536$ implies

$$\left|\left(\frac{\beta}{\alpha}\right)^n - 1\right| \geq \frac{1}{2}\exp\left(-12.6\pi(\log B)\left(\max\{\log n + 0.85, 20\}\right)^2\right).$$

Hence we get

$$\left|\left(\frac{\beta}{\alpha}\right)^n - 1\right| \geq \begin{cases} \frac{1}{2}B^{-44(\log n)^2}, & \text{if } n > 2.1 \cdot 10^8, \\ \frac{1}{2}B^{-15854}, & \text{if } n \leq 2.1 \cdot 10^8. \end{cases} \tag{9}$$

Thus from (6) we obtain

$$|U_n| \geq \begin{cases} \frac{1}{4}B^{\frac{n}{2}-1-44(\log n)^2}, & \text{if } n > 2.1 \cdot 10^8, \\ \frac{1}{4}B^{\frac{n}{2}-15855}, & \text{if } n \leq 2.1 \cdot 10^8, \end{cases}$$

and our claim follows. $\square$

4. Bounds for the number of Lucas sequences with bounded roots

In this section we prove Theorem 2.2. For this (and also later on), the following lemma describing degenerate Lucas sequences will be useful. In fact, this description is simple and well-known, however, for the convenience of the reader we provide its proof, as well.

Lemma 4.1. *Let* A, B *be integers with* $AB(A^2 - 4B) \neq 0$. *Then the Lucas sequence* U_n *belonging to* (A, B) *is degenerate precisely for*

$$(A,B) = (r, r^2),\ (2r, 2r^2),\ (3r, 3r^2) \quad (r \in \mathbb{Z} \setminus \{0\}). \tag{10}$$

Note that in each degenerate case $A^2 < 4B$, so degenerate Lucas sequences with $AB(A^2 - 4B) \neq 0$ are all non-real.

Proof. As before, let α, β be the roots of $x^2 - Ax + B$. As $B \neq 0$, $\alpha\beta \neq 0$. If $\varepsilon := \frac{\alpha}{\beta}$ is a root of unity, then as $A(A^2 - 4B) \neq 0$, we see that α, β are non-real. So ε is a quadratic non-real algebraic number which is a root of unity, implying

$$\varepsilon \in \left\{ \pm i, \pm\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \right\}.$$

In view of $\varepsilon\beta^2 = B$ and $(1 + \varepsilon)\beta = A$, we get that

$$\beta = \frac{B}{A} \cdot \frac{1+\varepsilon}{\varepsilon} = r(1 + \varepsilon^{-1}), \quad \alpha = \frac{B}{\beta} = \varepsilon\beta = r(\varepsilon + 1),$$

with $r = \frac{B}{A}$. Hence

$$A = \alpha + \beta = r(\varepsilon + 2 + \varepsilon^{-1}) \in \{2r, r, 3r\},\ B \in \{2r^2, r^2, 3r^2\}.$$

Since B is a non-zero integer, we have $r \in \mathbb{Z} \setminus \{0\}$. Thus (A, B) is given by (10).

On the other hand, one can easily check for (A, B) as in (10) that $\frac{\alpha}{\beta}$ is a root of unity, and the claim follows. □

Proof of Theorem 2.2. First we derive an upper bound for the number of Lucas sequences with $|\alpha| \leq t$, where $t \in \mathbb{R}_{\geq 2}$. Here, instead of assuming that U_n is non-degenerate, for simplicity we shall only use that $AB(A^2 - 4B) \neq 0$. Again we assume $A > 0$, and $\alpha > 0$ in the real case, and double the number of counted polynomials at the end.

Assume that α, β are non-real complex numbers. Then we have $A^2 - 4B < 0$, hence

$$0 < A < 2\sqrt{B}, \quad 0 < B = |\alpha|^2 \leq t^2.$$

Thus we obtain that the number of possible pairs (A,B) is at most

$$\sum_{1\le B\le t^2} 2\sqrt{B} \le 2\int_0^{t^2}\sqrt{x}\,dx + 2t = \frac{4}{3}t^3 + 2t.$$

Suppose next that α,β are real, so $A^2 > 4B$. Note that we have

$$\alpha = \frac{A+\sqrt{A^2-4B}}{2} \le t.$$

We distinguish two subcases. If $B>0$, then $\sqrt{A^2-4B} \le 2t-A$, hence

$$0<A<2t,\quad \max(0,At-t^2)\le B<\frac{1}{4}A^2.$$

In this subcase the number of pairs (A,B) is at most

$$\sum_{1\le A\le t}\left(\frac{1}{4}A^2\right) + \sum_{t<A<2t}\left(\frac{1}{4}A^2 - At + t^2 + 1\right) < \frac{1}{6}t^3 + \frac{4}{3}t.$$

(Note that here, and later on in similar situations in the proof, in view of that t is not necessarily integer, a careful calculation is needed.)

On the other hand, if $B<0$, then $A<t$ and $A+\sqrt{A^2-4B}<2t$, hence $A^2-4B<4t^2-4At+A^2$. From

$$0<A<t,\quad 0<-B\le t^2-At,$$

it follows that the number of such possible pairs (A,B) in this subcase is at most

$$\sum_{1\le A<t}(t^2-At) \le \frac{1}{2}t^3 - \frac{1}{2}t^2 + \frac{1}{8}t.$$

Adding up the three bounds for the possible pairs (A,B) and doubling this number, we obtain that the number of Lucas sequences with $AB(A^2-4B)\ne 0$ is at most $4t^3-t^2+7t$.

Now we give a lower bound for the number of non-degenerate Lucas sequences with $|\alpha|\le t$ and $A>0$, whence $\alpha>0$ in the real case. We apply a similar case-by-case analysis as for the upper bound, however, now we need to take care of the degenerate Lucas sequences. These sequences are completely characterized by Lemma 4.1. Recall that there are no degenerate Lucas sequences with $AB(A^2-4B)\ne 0$ in the real case. Observe that for fixed $A>0$, there are at most three B-s with $B(A^2-4B)\ne 0$ such that (A,B) generates a degenerate Lucas sequence, and for fixed $B\ne 0$, there are at most three A-s with $A(A^2-4B)\ne 0$ such that (A,B) generates a degenerate Lucas sequence. Note that different pairs (A,B) yield different Lucas sequences.

The non-real case. Choose $0 < B \leq t^2$, $0 < A < 2\sqrt{B}$. Then $AB \neq 0$, $A^2 - 4B < 0$ and $|\alpha| = \sqrt{B} \leq t$. The number of such non-degenerate sequences is at least

$$\sum_{1 \leq B \leq t^2-1} (2\sqrt{B} - 1) - 3t^2 \geq 2\int_0^{t^2} \sqrt{x}\, dx - 2t - 4t^2 = \frac{4}{3}t^3 - 4t^2 - 2t.$$

Next the real case with $B > 0$. Choose $0 < A \leq t$, $0 < B < \frac{1}{4}A^2$ and $t < A \leq 2t$, $At - t^2 \leq B < \frac{1}{4}A^2$. Then $AB \neq 0$, $A^2 - 4B > 0$ and

$$\alpha = \frac{A + \sqrt{A^2 - 4B}}{2} \leq t.$$

The number of such non-degenerate sequences is at least

$$\sum_{1 \leq A \leq t} \left(\frac{A^2}{4} - 1\right) + \sum_{t < A \leq 2t} \left(\frac{A^2}{4} - At + t^2 - 1\right) - 6t > \frac{1}{6}t^3 - \frac{1}{2}t^2 - \frac{17}{2}t.$$

Finally, the real case with $B < 0$. Choose $0 < A \leq t$, $0 < -B < t^2 - At$. Then $AB \neq 0$, $A^2 - 4B > 0$, and

$$\alpha = \frac{A + \sqrt{A^2 - 4B}}{2} < \frac{1}{2}(A + 2t - A) = t.$$

The number of such non-degenerate sequences is greater than

$$\sum_{1 \leq A \leq t} (t^2 - At - 1) \geq \frac{1}{2}t^3 - \frac{1}{2}t^2 - t.$$

So we have altogether more than $4t^3 - 10t^2 - 26t$ non-degenerate Lucas sequences with $|\alpha| \leq t$.

□

5. Proof of the upper bounds in Theorem 2.3

Proof. Suppose $|U_n| \leq N$ for the Lucas pair (A, B) and index n. Again we assume $A > 0$, hence $\alpha > 0$ in the real case, and double the number of polynomials at the end. We include the number of degenerate cases with $AB(A^2 - 4B) \neq 0$ in the upper bound. We distinguish four cases.

Case 1. $0 < A < 9N^{\frac{1}{n-1}}$ and $|B| < 17N^{\frac{2}{n-1}}$. Then the number of pairs (A, B) with $|F_n(A, B)| \leq N$ is at most $306N^{\frac{3}{n-1}}$.

Case 2. $A \geq 9N^{\frac{1}{n-1}}$, $|B| < 17N^{\frac{2}{n-1}}$. Since $A^2 > 4B$, we are in the real case. By [16] we know the zeros of the Fibonacci polynomial $F_n(x, -1)$ are given by

$$2\cos\frac{k\pi}{n} \quad (k = 1, \ldots, n-1). \tag{11}$$

Hence, if n is odd, then

$$F_n(A,B) = \prod_{k=1}^{\frac{n-1}{2}} \left(A^2 - 4B\cos^2\frac{k\pi}{n}\right),$$

and if n is even, then

$$F_n(A,B) = A\prod_{k=1}^{\frac{n-2}{2}} \left(A^2 - 4B\cos^2\frac{k\pi}{n}\right).$$

Since $A \geq 9N^{\frac{1}{n-1}}$ and $A^2 - 4B\cos^2\frac{k\pi}{n} \geq A^2 - 4|B| > N^{\frac{2}{n-1}}$, we obtain

$$N \geq |F_n(A,B)| > \min\left((N^{\frac{2}{n-1}})^{\frac{n-1}{2}}, 9N^{\frac{1}{n-1}}(N^{\frac{2}{n-1}})^{\frac{n-2}{2}}\right) = N.$$

Thus there are no pairs (A,B) with $|F_n(A,B)| \leq N$ in this case.

Case 3. $B \leq -17N^{\frac{2}{n-1}}$. We have, in view of (11),

$$F_n(A,B) = \prod_{k=1}^{n-1}\left(A - 2\sqrt{B}\cos\frac{k\pi}{n}\right).$$

As $\sqrt{B}$ is purely imaginary, $|A - 2\sqrt{B}\cos\frac{k\pi}{n}| \geq 2\sqrt{|B|}\,|\cos\frac{k\pi}{n}|$. Thus, by $|\cos\frac{k\pi}{n}| = |\cos\frac{(n-k)\pi}{n}|$,

$$N \geq |F_n(A,B)| \geq \left(\prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(2\sqrt{|B|}\cos\frac{k\pi}{n}\right)^2.$$

If n is odd, then we get, on using that $\cos x \geq 1 - \frac{2}{\pi}x$ for $0 \leq x \leq \frac{\pi}{2}$,

$$\prod_{k=1}^{\frac{n-1}{2}}\cos\frac{k\pi}{n} \geq \prod_{k=1}^{\frac{n-1}{2}}\left(1 - \frac{2k}{n}\right) = \frac{(n-1)!}{(\frac{n-1}{2})!(2n)^{\frac{n-1}{2}}} > \left(\frac{n+1}{4n}\right)^{\frac{n-1}{2}} > 2^{-n+1}.$$

Thus, by $B \leq -17N^{\frac{2}{n-1}}$,

$$N \geq (2\sqrt{|B|})^{n-1}4^{-n+1} \geq \left(\frac{1}{2}\sqrt{17}N^{\frac{1}{n-1}}\right)^{n-1} > N, \tag{12}$$

a contradiction. So there are no such pairs (A,B).

If n is even, then we get similarly, by $n \geq 6$,

$$\prod_{k=1}^{\frac{n-2}{2}}\cos\frac{k\pi}{n} \geq \prod_{k=1}^{\frac{n-2}{2}}\left(1 - \frac{2k}{n}\right) = \frac{2^{\frac{n-2}{2}}\left(\frac{n-2}{2}\right)!}{n^{\frac{n-2}{2}}} > \left(\frac{2}{n}\cdot\frac{n-2}{2e}\right)^{\frac{n-2}{2}} > 2^{-n+2}.$$

As in the case n odd, we find that there are no such pairs (A,B).

Case 4. $B \geq 17N^{\frac{2}{n-1}}$. We use the formula

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}.$$

Let k_0 be the value of k with $0 < k < n$ such that $\cos \frac{k\pi}{n}$ is nearest to $\frac{A}{2\sqrt{B}}$. Then

$$\left| \prod_{\substack{k=1 \\ k \neq k_0}}^{n-1} \left(A - 2\sqrt{B} \cos \frac{k\pi}{n} \right) \right| \geq$$

$$(2\sqrt{B})^{n-2} \left| \left(\cos \frac{(k_0 - 0.5)\pi}{n} - \cos \frac{(k_0 - 1)\pi}{n} \right) \left(\cos \frac{(k_0 + 0.5)\pi}{n} - \cos \frac{(k_0 + 1)\pi}{n} \right) \right|$$

$$\times \left| \prod_{k=1}^{k_0 - 2} \left(\cos \frac{(k_0 - 1)\pi}{n} - \cos \frac{k\pi}{n} \right) \prod_{k=k_0+2}^{n-1} \left(\cos \frac{(k_0 + 1)\pi}{n} - \cos \frac{k\pi}{n} \right) \right| \geq$$

$$(4\sqrt{B})^{n-2} \left(\sin \frac{\pi}{4n} \right)^4 \times \left| \prod_{k=1}^{k_0 - 2} \left(\sin \frac{(k + k_0 - 1)\pi}{2n} \cdot \sin \frac{(k_0 - k - 1)\pi}{2n} \right) \right|$$

$$\times \left| \prod_{k=k_0+2}^{n-1} \left(\sin \frac{(k + k_0 + 1)\pi}{2n} \cdot \sin \frac{(k - k_0 - 1)\pi}{2n} \right) \right|.$$

On using that $k_0 < n, n - k_0 < n$ and $\sin x \geq \frac{2}{\pi} x$ for $0 \leq x \leq \frac{\pi}{2}$, we obtain that

$$\prod_{k=1}^{k_0 - 2} \sin \frac{(k_0 - k - 1)\pi}{2n} \cdot \prod_{k=k_0+2}^{n-1} \sin \frac{(k - k_0 - 1)\pi}{2n} \geq \frac{(k_0 - 2)!}{n^{k_0 - 2}} \cdot \frac{(n - k_0 - 2)!}{n^{n - k_0 - 2}}.$$

Observe that $k + k_0 - 1$ for $k = 1, \ldots, k_0 - 2$ and $k + k_0 + 1$ for $k = k_0 + 2, \ldots, n - 1$ form a set of $n - 4$ integers in the interval $[k_0, k_0 + n] \subset [1, 2n - 1]$. Using that the total product for such a set is minimal if $k_0 = 1$ or $k_0 = n - 1$, we obtain

$$\prod_{k=1}^{k_0 - 2} \sin \frac{(k + k_0 - 1)\pi}{2n} \times \prod_{k=k_0+2}^{n-1} \sin \frac{(k + k_0 + 1)\pi}{2n} \geq \prod_{k=1}^{n-4} \sin \frac{k\pi}{2n} \geq \frac{(n-4)!}{n^{n-4}}.$$

On combining the inequalities, we obtain

$$\left| \prod_{\substack{k=1 \\ k \neq k_0}}^{n-1} \left(A - 2\sqrt{B} \cos \frac{k\pi}{n} \right) \right| \geq \frac{(4\sqrt{B})^{n-2}}{(2n)^4} \cdot \frac{(k_0 - 2)!(n - k_0 - 2)!}{n^{n-4}} \cdot \frac{(n-4)!}{n^{n-4}}.$$

By $m! \geq \left(\frac{m}{e}\right)^m$ for all $m \geq 0$ (which follows e.g. from the first inequality of 3.6.7 on p. 269 of [22]), we have $(n-4)! \geq \left(\frac{n-4}{e}\right)^{n-4}$. If n is even, then $(k_0-2)!(n-k_0-2)!$ is minimal for $k_0 = \frac{n}{2}$ and we get

$$(k_0-2)!(n-k_0-2)! \geq \left(\left(\frac{n}{2}-2\right)!\right)^2 > \left(\frac{n-4}{2e}\right)^{n-4}.$$

If n is odd, we use that $(k_0-2)!(n-k_0-2)!$ is minimal for $k_0 = \frac{n\pm1}{2}$. As $(x+y)\log(x+y)+(x-y)\log(x-y)$ is monotonically increasing on the interval $(0,x)$ in y, for any $x>0$, in this case we get

$$(k_0-2)!(n-k_0-2)! > \left(\frac{n-3}{2e}\right)^{\frac{n-3}{2}} \cdot \left(\frac{n-5}{2e}\right)^{\frac{n-5}{2}} \geq \left(\frac{n-4}{2e}\right)^{n-4}.$$

We conclude that

$$\left|\prod_{\substack{k=1\\k\neq k_0}}^{n-1}\left(A-2\sqrt{B}\cos\frac{k\pi}{n}\right)\right| \geq \frac{(4\sqrt{B})^{n-2}}{(2n)^4}\cdot\left(\frac{n-4}{2en}\right)^{n-4}\cdot\left(\frac{n-4}{en}\right)^{n-4}.$$

We apply that $(1+\frac{r}{s})^s < e^r$ for positive integers r,s (cf. e.g. (1) of 3.6.3 on p. 266 of [22]) to derive that $(\frac{n-4}{n})^{n-4} > e^{-4}$. Hence

$$\left|\prod_{\substack{k=1\\k\neq k_0}}^{n-1}\left(A-2\sqrt{B}\cos\frac{k\pi}{n}\right)\right| \geq \frac{(4\sqrt{B})^{n-2}}{(2n)^4}\cdot\frac{4e^{-4}}{(2e^2)^{n-2}} = \frac{1}{4e^4n^4}\cdot\left(\frac{2\sqrt{B}}{e^2}\right)^{n-2}.$$

If $|F_n(A,B)| \leq N$, then, via

$$\left|A-2\sqrt{B}\cos\frac{k_0\pi}{n}\right| < N\cdot 4e^4n^4\left(\frac{e^2}{2\sqrt{B}}\right)^{n-2},$$

we obtain, by putting $c_3 = c_3(n) = 2^{4-n}e^{2n}n^4$,

$$A \in \left(2\sqrt{B}\cos\frac{k_0\pi}{n} - c_3NB^{-\frac{1}{2}n+1}, 2\sqrt{B}\cos\frac{k_0\pi}{n} + c_3NB^{-\frac{1}{2}n+1}\right).$$

Therefore, for every B the number of possible integers A is at most

$$2c_3NB^{-\frac{1}{2}n+1}.$$

Thus in this case the number of possible pairs (A,B) with $|F_n(A,B)| \leq N$ is at most

$$2c_3N\sum_{B\geq 17N^{\frac{2}{n-1}}} B^{-\frac{1}{2}n+1} \leq 2c_3N\int_{17N^{\frac{2}{n-1}}-1}^{\infty} t^{-\frac{1}{2}n+1}dt =$$

$$\frac{4c_3}{n-4}N(17N^{\frac{2}{n-1}}-1)^{-\frac{1}{2}n+2} < 4c_3N(16N^{\frac{2}{n-1}})^{-\frac{1}{2}n+2} < 2^{14-3n}e^{2n}n^4N^{\frac{3}{n-1}}.$$

A calculation gives that $e^{2n}2^{14-3n}n^4(0.95)^{-n}$ has its maximum for $n = 142$, of value $< 1.23 \cdot 10^{11}$. Combining the Cases 1-4 and using that $e^{2n}2^{14-3n}n^4 < 1.23 \cdot 10^{11} \cdot 0.95^n$ for all $n \geq 5$, we conclude that there are at most

$$2 \cdot (306 + 1.23 \cdot 10^{11} \cdot 0.95^n)N^{\frac{3}{n-1}}$$

pairs (A, B) with $|F_n(A, B)| \leq N$. $\square$

Proof of the upper bound for $|\mathcal{L}_{\geq n}(N)|$. Let $m \geq n$ and suppose that $|U_m(A, B)| \leq N$. Observe that for an arbitrary Lucas sequence U, if $|U_m| \leq N$, then, by Corollary 2.1 and $\log(2) > \frac{2}{3}$,

$$m \leq 3 \log N$$

for N sufficiently large. Therefore

$$|\mathcal{L}_{\geq n}(N)| \leq |\mathcal{L}_n(N)| + \sum_{m=n+1}^{\lfloor 3\log N \rfloor} \left(612 + 2.46 \cdot 10^{11} \cdot 0.95^m\right) N^{\frac{3}{n}}.$$

Thus

$$|\mathcal{L}_{\geq n}(N)| \leq (612+2.46\cdot 10^{11}\cdot 0.95^n)N^{\frac{3}{n-1}}+1836N^{\frac{3}{n}}\log N+4.68\cdot 10^{12}\cdot 0.95^n N^{\frac{3}{n}}.$$

$\square$

6. Proof of the lower bounds in Theorems 2.4 and 2.5

We consider n to be fixed. By that, also the polynomials $F_n(x, y)$ and

$$G_m(x, y) := \begin{cases} F_n(\sqrt{x}, y) & \text{for } n \text{ even,} \\ F_n(\sqrt{x}, y)/\sqrt{x} & \text{for } n \text{ odd} \end{cases}$$

are fixed. Note that G_m is a binary form of degree $m = \frac{n-1}{2}$ if n is odd, and of degree $m = \frac{n-2}{2}$ if n is even. In our arguments we may clearly assume that N is large enough. If we write 'for sufficiently large N', it means that the lower bound for N may depend on n only.

For the proof of Theorem 2.4 we follow a paper by Erdős and Mahler [11], who proved a similar result for binary forms.

Write $F_n(x, y) = \sum_{h=0}^{\lfloor \frac{n-1}{2} \rfloor} a_h x^{n-2h-1} y^h$. Then $a_0 = 1$. The discriminant d of $F_n(x, 1)$ is non-zero [12]. Let $\gamma = \max(|d|, n)$, R a non-zero integer and θ a number satisfying $0 < \theta < 1$, to be fixed later. Let $h(R)$ be the arithmetical function defined by

$$h(R) = \prod_{\gamma<p\leq N^\theta,\ p^a || R,\ p^a \leq N^\theta} p^a.$$

Lemma 6.1.

$$H(N) := \prod_{|x|\le N,\ |y|\le N^2,\ F_n(x,y)\neq 0} h(F_n(x,y)) \le N^{24\theta n N^3}.$$

Proof. Since $d \neq 0$ and $a_0 = 1$, for given prime $p > \gamma$ and integers $a \ge 0$ and y, there are at most $n-1$ incongruent values of $x \pmod{p^a}$ for which $F_n(x,y) \equiv 0 \pmod{p^a}$. Therefore, for given p and a with $\gamma < p \le p^a \le N^\theta$, the conditions

$$|x| \le N,\ |y| \le N^2,\ F_n(x,y) \neq 0,\ F_n(x,y) \equiv 0 \pmod{p^a}$$

have less than

$$n(2N^2+1)\left\lceil \frac{2N+1}{p^a} \right\rceil \le \frac{6nN^3}{p^a}$$

solutions x, y. It follows that the exponent b with $p^b || H(N)$ satisfies the inequality

$$b \le \sum_{a=1}^{\infty} \frac{6nN^3}{p^a} = \frac{6nN^3}{p-1} \le \frac{12nN^3}{p}.$$

Hence, for sufficiently large N,

$$H(N) \le \exp\left(\sum_{\gamma<p\le N^\theta} \frac{12nN^3}{p} \log p \right) \le N^{24\theta n N^3},$$

since

$$\sum_{p\le u} \frac{\log p}{p} \le 2\log u$$

for sufficiently large u. □

Lemma 6.2. *If μ is the number of pairs x, y with*

$$|F_n(x,y)| \le \sqrt{N}, |x| \le N, |y| \le N^2$$

then $\mu \le N^3$ for sufficiently large N.

Proof. For a given m with $|m| \le \sqrt{N}$ and a given y with $|y| \le N^2$, the equation $F_n(x,y) = m$ has at most $n-1$ integer solutions x, and therefore

$$\mu \le (n-1)(2\sqrt{N}+1)(2N^2+1) \le N^3$$

for N sufficiently large. □

Lemma 6.3. *For sufficiently large N, there are at least $2N^3$ pairs of integers x, y with $|x| \le N, |y| \le N^2$ and $(x,y) = 1$.*

Proof. Obviously, the number of these pairs is at least $4Q$, where Q denotes the number of pairs with $1 \le x \le N, 1 \le y \le N^2, (x,y) = 1$. Using that $\sum_p p^{-2} = 0.4522472...$ (see e.g. [25] A085548), we have

$$Q \ge N^3 - \sum_p \frac{N^3}{p^2} > \frac{N^3}{2}.$$

$\square$

Lemma 6.4. *For sufficiently large N there are at least $\frac{1}{2}N^3$ pairs of integers (x,y) with*

$$|x| \le N,\ |y| \le N^2,\ F_n(x,y) \ne 0,\ (x,y) = 1,\ h(F_n(x,y)) \le |F_n(x,y)|^{96\theta n}.$$

Proof. By Lemmas 6.2 and 6.3, there are at least N^3 pairs (x,y) with

$$|x| \le N,\ |y|\ \le N^2,\ (x,y) = 1,\ |F_n(x,y)| \ge \sqrt{N}.$$

Hence, if Lemma 6.4 were false, there would be more than $\frac{1}{2}N^3$ such coprime pairs x, y with

$$h\left(F_n(x,y)\right) > |F_n(x,y)|^{96\theta n} \ge N^{48\theta n}$$

and therefore, for N sufficiently large,

$$H(N) > N^{48\theta n \cdot \frac{1}{2}N^3} = N^{24\theta n N^3},$$

in contradiction to Lemma 6.1. $\square$

Lemma 6.5. *For all x and y,*

$$|F_n(x,y)| \le F_n^* \cdot (\max(x^2,|y|))^{\frac{1}{2}(n-1)} \le (3\max(x^2,|y|))^{\frac{1}{2}(n-1)},$$

where F_n^* is the n-th Fibonacci number.

Proof. As the Fibonacci sequence corresponds to the choice $(A,B) = (1,-1)$, by the definition of $F_n(x,y)$ (observing that the coefficients a_h of $F_n(x,y)$ have alternating signs) we obtain

$$F_n^* = F_n(1,-1) = \sum_{h=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^h a_h = \sum_{h=0}^{\lfloor \frac{n-1}{2} \rfloor} |a_h|.$$

From this the claim immediately follows. $\square$

Lemma 6.6. *For sufficiently large N there are at least $\frac{1}{2}N^3$ pairs of integers x, y with*

$$|x| \le N,\ |y| \le N^2,\ (x,y) = 1,\ F_n(x,y) \ne 0 \tag{13}$$

such that $|F_n(x,y)| = k_1k_2$, where k_1 and k_2 are positive integers such that k_1 is divisible by at most $\gamma + \frac{n\log(2N)}{\theta \log N}$ different primes and $k_2 \le |F_n(x,y)|^{96\theta n}$.

Proof. We apply Lemma 6.4 with

$$k_2 = h(F_n(x,y)),\ \ k_1 = \frac{|F_n(x,y)|}{k_2}.$$

Then k_1 and k_2 are positive integers, since $h(F_n(x,y))$ is a positive integer which divides $F_n(x,y)$. By Lemma 6.4, for at least $\frac{1}{2}N^3$ pairs (x,y) satisfying (13),

$$k_2 \leq |F_n(x,y)|^{96\theta n}.$$

The factor k_1 is divisible by prime numbers of the form p with $p \leq \gamma$ or $p > N^{\theta}$. But since $|F_n(x,y)| < F_n^* N^n < (2N)^n$ by Lemma 6.5, there are at most

$$\gamma + \frac{n\log(2N)}{\theta \log N}$$

primes dividing k_1. □

It now suffices to prove that the number of relatively prime integer solutions (x,y) to equation $F_n(x,y) = k$ is bounded by is bounded by a constant depending only on n. Here we apply Lemma 6.7, which is Theorem 3(i) of Lewis and Mahler [20]. Recall that $c_4, c_5, c_6, \dots$ are constants depending only on n. For an integer x and a prime p we write $|x|_p = p^{-r}$ if $p^r \mid x$, $p^{r+1} \nmid x$.

Lemma 6.7. *Let $G(x,y)$ be a binary form of degree $m \geq 3$ with integer coefficients and non-zero discriminant satisfying*

$$G(1,0) \neq 0 \ \ \textit{and} \ \ G(0,1) \neq 0.$$

Let a be the canonical height of $G(x,y)$; let

$$\rho_m = \min_{h=1,2,\dots,m}\left(\frac{m}{h+1} + h\right), \quad \sigma_m = \rho_m + \frac{1}{m};$$

and let $p_1, p_2, \dots, p_t$ be any finite number of primes. There are not more than

$$2^{\frac{\sigma_m+2}{\sigma_m-2}}(2m^2a)^{\frac{4m}{\sigma_m-2}} + e(t+1)\left\lfloor \frac{\log(48a^2m^8)}{\log(a-1)} + 2 \right\rfloor (em(3\rho_m m+4))^{t+1} \tag{14}$$

pairs of integers x, y such that

$$x \neq 0,\ \ y > 0,\ \ (x,y) = 1,\ \ 0 < |G(x,y)| \prod_{\tau=1}^{t} |G(x,y)|_{p_\tau} \leq (\max(|x|,|y|))^{c_4},$$

where $c_4 = m - \rho_m - \frac{4}{3m}$ for all $m \geq 3$.

On using that $\rho_m \leq 2\sqrt{m}$, we obtain

$$c_4 = m - \rho_m - \frac{4}{3m} = \frac{1}{18}, \frac{2}{3}, \frac{37}{30}, \frac{16}{9}, \frac{52}{21}, \frac{19}{6}, \frac{104}{27}, \geq \frac{m}{3}$$

for $m = 3, 4, 5, 6, 7, 8, 9, \geq 10$, respectively. Recall that n is odd, $n \geq 7$, hence $F_n(\sqrt{x}, y) \equiv G_m(x, y)$ and $n - 1 = 2m$. Thus $m \geq 3$ and (14) can be written as $c_5 + c_6(t+1)c_7^{t+1}$, provided that a can be bounded by a constant depending only on n. By Lemma 6.5 we have $|a| < 2^n$ for our polynomial $G(x, y)$.

Lemma 6.8. *If k is an integer, $k > (F^*_{2m+1})^5$, and it can be written in the form $k = k_1k_2$ with positive integers k_1 and k_2 such that k_1 is divisible by at most t different primes, k_2 is not divisible by any of these primes and $k_2 \leq k^{\frac{2c_4}{5m}}$, then the equation $G_m(x, y) = k$ has not more than $c_5 + c_6(t+1)c_7^{t+1}$ different solutions in relatively prime integers x, y.*

Proof. Let k be an integer, for which $G_m(x, y) = k$ has at least one solution (x_0, y_0). Then, by Lemma 6.5,

$$k \leq F^*_{2m+1} \cdot (\max(|x|, |y|))^{2m}. \tag{15}$$

The integer $k = k_1k_2$ is the product of the integer k_1 with at most t distinct prime divisors and the integer k_2 which is coprime to the product of these primes. By (15),

$$k_2 \leq k^{\frac{2c_4}{5m}} \leq (F^*_{2m+1})^{\frac{2c_4}{5m}} (\max(|x|, |y|))^{\frac{4c_4}{5}}.$$

Thus, since $k > (F^*_{2m+1})^5$, we get, by (15),

$$\max(|x|, |y|) \geq \left(\frac{k}{F^*_{2m+1}}\right)^{\frac{1}{2m}} > (F^*_{2m+1})^{\frac{2}{m}},$$

hence

$$k_2 < (\max(|x|, |y|))^{\frac{c_4}{5} + \frac{4c_4}{5}} = (\max(|x|, |y|))^{c_4}.$$

The statement of Lemma 6.8 follows from Lemma 6.7. □

Proof of Theorem 2.4. Let $R = \left(\frac{N}{F^*_n}\right)^{\frac{1}{n-1}}$. Then $|F_n(x, y)| \leq N$ for $|x| \leq R$ and $|y| \leq R^2$ in view of Lemma 6.5. It follows from Lemma 6.6 that there are at least

$$\frac{1}{2}R^3 = \frac{1}{2}\left(\frac{N}{F^*_n}\right)^{\frac{3}{n-1}} \tag{16}$$

different pairs of relatively prime integers x, y with $|x| \leq R, |y| \leq R^2$ for which $|F_n(x, y)| = k \neq 0$ and k is a product of two positive integers

$k = k_1k_2$, such that k_1 is divisible by at most $\gamma + \frac{n\log(2R)}{\theta\log R}$ different primes, whereas $k_2 \le |F_n(x,y)|^{96\theta n}$. We choose

$$\theta = \frac{c_4}{240n^2} \ (< 1),$$

where c_4 is the constant in Lemma 6.7 for $m = \frac{n-1}{2}$. Since $n \ge 7$, $G_m(x,y)$ is a binary form of degree $m = \frac{1}{2}(n-1) \ge 3$. Thus there are at least (16) different pairs of relatively prime integers x, y with $|x| \le R, |y| \le R^2$ for which $|G_m(x^2,y)| = |F_n(x,y)| = k \ne 0$ and $k = k_1k_2$, such that k_1 is divisible by at most

$$\gamma + \frac{n\log(2R)}{\theta\log R} < c_8n^3$$

different primes for some positive integer c_8, while

$$k_2 \le |G_m(x^2,y)|^{96\theta n} = |G_m(x^2,y)|^{\frac{2c_4}{5n}}.$$

Here we have used that N, hence R, is sufficiently large. We apply Lemma 6.8 with $t = c_8n^3$ and $k_2 \le k^{\frac{2c_4}{5n}}$. We conclude that the number of different relatively prime solutions of $G_m(x,y) = k$ with $k > (F_n^*)^5$ is not larger than

$$c_9 := c_5 + c_6(c_8n^3)c_7^{c_8n^3}.$$

So the number of different coprime solutions of $F_n(x,y) = G_m(x^2,y) = k$ for a fixed k is not larger than c_9. The number of such solutions of the equation $F_n(x,y) = k$ with $k \le (F_n^*)^5$ is bounded by a constant c_{10}. Put $c_{11} = c_9 + c_{10}$. Thus, for sufficiently large N, by (16), there must be at least

$$\frac{1}{2c_{11}}\left(\frac{N}{F_n^*}\right)^{\frac{3}{n-1}} > c_{12}N^{\frac{3}{n-1}} + 9$$

different positive integers $k \le N$ for which $|F_n(x,y)| = k$ has at least one solution in integers x, y with $(x,y) = 1$. Observe that in view of Lemma 4.1, for these values of x, y, the Lucas sequence corresponding to $(A,B) = (x,y)$ is degenerate only for the nine values

$$(x,y) = (0,0),\ (0,\pm1),\ (\pm1,0),\ (\pm1,\pm1).$$

This has been proved for $N > N_1(n)$. By adjusting the constant c_{12} for the smaller values of N and calling the new constant c_1, the result follows. $\square$

Remark 4. Obviously, the same lower bound applies to $\mathcal{L}_{\ge n}$.

Proof of Theorem 2.5 - case n even. Let n be even. As we already mentioned, in this case $U_n = AG_m(A^2, B)$, where A divides U_n, and $G_m(x, y)$ is a binary form of degree $m = \frac{n-2}{2}$. By Lemma 6.5 for all the pairs (A, B) with

$$1 \leq A \leq \frac{1}{2}N^{\frac{1}{n-1}}, \quad 1 \leq B \leq \frac{1}{4}N^{\frac{2}{n-1}},$$

we have $|G_m(A, B)| \leq N$. Observe that the number of such pairs (A, B) for which the corresponding Lucas sequence is non-degenerate, by Lemma 4.1 is at least $\frac{1}{8}N^{\frac{3}{n-1}} - 2N^{\frac{1}{n-1}}$. Indeed, for A fixed, there exists at most one such B with $A^2 - 4B = 0$, and at most three such B-s with (10). We claim that for any k with $1 \leq k \leq N$ the equation

$$F_n(x, y) = AG_m(x^2, y) = k \tag{17}$$

can have at most $nN^{\frac{1.066}{\log\log N}}$ solutions among these pairs (A, B). From this the theorem immediately follows. In the first place, observe that if (A, B) is any solution of (17), then $A \mid k$. By Théorème 1 of [24], the number of divisors of k, and so the number of choices of A, is at most

$$\tau(k) \leq k^{\frac{1.5379 \log 2}{\log\log k}} \leq N^{\frac{1.066}{\log\log N}}, \tag{18}$$

where $\tau(k)$ denotes the number of positive divisors of k. Furthermore, for any $x = A$, (17) can have at most n solutions in y, and our claim follows.

Obviously, the same lower bound applies to $\mathcal{L}_{\geq n}$. Hence the statement is proved for even n. □

Theorem 2.5 is a consequence of Theorem 2.4 for $n \geq 7$ odd, while it has been proved above for $n \geq 6$ even. In order to deal with the remaining case $n = 5$, we apply the following lemma.

Lemma 6.9. *There exists an absolute constant T_0, such that if T is an integer with $T > T_0$ and t is a non-zero integer with $|t| \leq 4T$, then the equation*

$$x^2 - 5y^2 = t \tag{19}$$

has at most $T^{\frac{4}{\log\log T}}$ solutions in positive integers x, y with $|y| \leq \frac{1}{2}\sqrt{T}$.

Proof. It is standard (see e.g. Lemma 3.2 and its proof in [14]), that all positive solutions of (19) come from identities

$$x + \sqrt{5}y = (u_i + v_i\sqrt{5})(9 + 4\sqrt{5})^n \quad (n \geq 0,\ i = 1, \dots, I),$$

where $u_i, v_i \in \mathbb{Z}$ $(i = 1, \dots, I)$ for some I. Here $9 + 4\sqrt{5}$ is the fundamental solution of (19) with $t = 1$. By Theorem 8-9 of [19] (see pp.

147-148), we can take

$$0 < u_i < \sqrt{\frac{(22+9\sqrt{5})|t|}{8}} < 4.6\sqrt{T} \quad (i = 1, \dots, I), \tag{20}$$

which implies

$$|v_i| < 2.3\sqrt{T} \quad (i = 1, \dots, I). \tag{21}$$

Further, by Lemma 5 of Győry [13] we know that

$$I \le 2^{\omega(t)} \tau_2(t),$$

where $\omega(t)$ is the number of prime divisors of t, while $\tau_2(t)$ is the number of ways the principal ideal (t) can be factorized into the product of two ideals in $\mathbb{Q}(\sqrt{5})$. As every prime can split into the product of at most two prime ideals in $\mathbb{Q}(\sqrt{5})$, we have $\tau_2(t) \le (\tau(t))^2$. Thus, in view of (18) we have

$$I \le \tau(t)^3 \le |t|^{\frac{3.2}{\log\log|t|}} \le (4T)^{\frac{3.2}{\log\log(4T)}}. \tag{22}$$

Using Lemma 3.2 from [14] and its proof, we get that y belongs to a recurrence sequence $G^{(i)}$ ($i \in \{1, \dots, I\}$) with initial terms

$$(G_0^{(i)}, G_1^{(i)}) = (v_i, \mu_2 u_i + 2\mu_1 v_i) = (v_i, 4u_i + 18v_i)$$

and recurrence relation

$$G_{n+2}^{(i)} = 2\mu_1 G_{n+1}^{(i)} - G_n^{(i)} = 18G_{n+1}^{(i)} - G_n^{(i)} \quad (n \ge 0),$$

because $\mu_1 + \mu_2\sqrt{5} = 9 + 4\sqrt{5}$. Setting

$$\alpha = 9 + 4\sqrt{5} \quad \text{and} \quad \beta = 9 - 4\sqrt{5},$$

we have

$$y = a_i\alpha^n - b_i\beta^n \tag{23}$$

for some $n \ge 0$ and $1 \le i \le I$ with

$$a_i = \frac{4u_i + (9+4\sqrt{5})v_i}{8\sqrt{5}} \quad \text{and} \quad b_i = \frac{4u_i + (9-4\sqrt{5})v_i}{8\sqrt{5}}.$$

By (20) and (21) we have

$$|b_i| < 1.1\sqrt{T}. \tag{24}$$

On the other hand, letting

$$\gamma = \frac{1}{4u_i + (9+4\sqrt{5})v_i}$$

we have

$$(16u_i^2 + 72u_iv_i + v_i^2)\gamma^2 - (8u_i + 18v_i)\gamma + 1 = 0.$$

Clearly, $16u_i^2+72u_iv_i+v_i^2=(4u_i+9v_i)^2-80v_i^2\neq 0$. Thus letting

$$h:=\max(|8u_i+18v_i|,1),$$

we see that either $|\gamma|\leq 1$, or

$$|\gamma^2|\leq|16u_i^2+72u_iv_i+v_i^2|\gamma^2\leq h(|\gamma|+1)\leq\frac{h|\gamma^2|}{|\gamma|-1},$$

whence $|\gamma|\leq h+1$. From this and (20) and (21) we get that

$$|\gamma|<79\ T.$$

This implies, by $a_i\gamma=\frac{1}{8\sqrt{5}}$,

$$|a_i|>\frac{1}{1414\ T}.$$

Combining this with (23) and (24), we obtain

$$y\geq|a_i\alpha^n|-|b_i\beta^n|>\frac{1}{1414T}\ (17.9)^n-\sqrt{T},$$

and this exceeds $\frac{1}{2}\sqrt{T}$ if $n>\frac{1.5\log T+\log 1414}{\log 17.9}$. This implies that $y>\frac{1}{2}\sqrt{T}$ for $n>\log T+3$. So in view of (22), our claim follows by a simple calculation. □

Proof of Theorem 2.5 - case $n=5$. Recall $F_5(A,B)=A^4-3A^2B+B^2$, and for a given positive integer N consider the set

$$H_N:=\left\{(A,B)\in\mathbb{Z}^2\ :\ 1\leq A\leq\frac{1}{2}N^{\frac{1}{4}},\ 1\leq B\leq\frac{1}{2}N^{\frac{1}{2}}\right\}.$$

Then for $(A,B)\in H_N$ we have that the corresponding Lucas-sequence (by Lemma 4.1, similarly as in case of n odd) is degenerate at most for $2N^{\frac{1}{4}}$ pairs (A,B), $|F_5(A,B)|<N$ and also that $|H_N|=\frac{1}{4}N^{\frac{3}{4}}$.

We show that for any k with $|k|\leq N$, there are only 'few' $(A,B)\in H_N$ such that

$$F_5(A,B)=k. \tag{25}$$

As $4F_5(A,B)=(2B-3A^2)^2-5A^4$, (25) can be written as

$$x^2-5y^2=t$$

with $x=2B-3A^2$, $y=A^2$ and $t=4k$. So applying Lemma 6.9 with $T=N$, we see that (25) has at most $N^{\frac{4}{\log\log N}}$ solutions with $N>N_2(n)$. Hence for $N>N_2(n)$, we have

$$|\mathcal{L}_5(N)|\geq\frac{1}{5}N^{\frac{3}{4}-\frac{4}{\log\log N}},$$

and the statement follows. □

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References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer New York, 1976, XII+340 pp.

[2] A. Baker, *A sharpening of the bounds for linear forms in logarithms* I, Acta Arith. **21** (1972), 117-129.

[3] A. Baker, *The theory of linear forms in logarithms*, in: Transcendence Theory: Advances and Applications, ed. by A. Baker and D.W. Masser, Academic Press, 1977, pp. 1-27.

[4] Ch. J.-C. Ballot, H. C. Williams, *The Lucas Sequences*, Springer Cham, 2023, XVIII+301 pp.

[5] Yu. Bilu, G. Hanrot, P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers*, J. reine angew. Math. **539** (2001), 75–122.

[6] A. Bremner, N. Tzanakis, *Lucas sequences whose 12th or 9th term is a square*, J. Number Theory **107** (2004), 215–227.

[7] A. Bremner, N. Tzanakis, *Lucas sequences whose nth term is a square or an almost square*, Acta Arith. **126** (2007), 261–280.

[8] A. Bremner, N. Tzanakis, *On squares in Lucas sequences*, J. Number Theory **124** (2007), 511–520.

[9] R. D. Carmichael, *On the Numerical Factors of the Arithmetic Forms* $\alpha^n \pm \beta^n$, Annals of Math. **15** (1913 - 1914), 30–48.

[10] P. L. Cijsouw, *Transcendence measures*, Ph.D. thesis, University of Amsterdam, 1972.

[11] P. Erdős and K. Mahler, *On the number of integers which can be represented by a binary form*, J. London Math. Soc. 13 (1938), 134–139.

[12] R. Flórez, R. Higuita, A. Ramírez, *The resultant, the discriminant, and the derivative of generalized Fibonacci polynomials*, J. Int. Seq. **22** (2019), Article 19.4.4.

[13] K. Győry, *On the numbers of families of solutions of systems of decomposable form equations*, Publ. Math. Debrecen **42** (1993), 65–101.

[14] L. Hajdu, P. Sebestyén, *Sums of S-units in the solution sets of generalized Pell equations*, Arch. Math. **115** (2020), 279–287.

[15] L. Hajdu, M. Szikszai, V. Ziegler, *On arithmetic progressions in Lucas sequences*, J. Integer Seq. **20** (2017), Article 17.

[16] V. E. Hoggatt, M. Bicknell, *Roots of Fibonacci polynomials*, Fibonacci Quart. **11** (1973), 25–28.

[17] M. Laurent, *Linear forms in two logarithms and interpolation determinants II*, Acta Arith. 133 (2008), 325–348.

[18] D. H. Lehmer, *An extended theory of Lucas' functions*, Annals Math. **31** (1930), 419–448.

[19] W. J. LeVeque, *Topics in Number Theory, Vol. I*, Adison-Wesley Publ. Comp., Third printing, 1965.

[20] D. J. Lewis, K. Mahler, *On the representation of integers by binary forms*, Acta Arith. 6 (1961), 333-363.

[21] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II*, Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), 125–180. English transl. in Izv. Math. **64** (2000), 1217–1269.

[22] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag Berlin, Heidelberg, New York, XII+404 pp., 1970.

[23] A. Mosunov, *On the area of the fundamental region of a binary form associated with algebraic trigonometric quantities*, Mathematika **67** (2021), 532–551.

[24] J. L. Nicolas, G. Robin, *Majorations explicites pour le nombre de diviseurs de N*, Canad. Math. Bull. **26** (1983), 485–492.

[25] The Online Encyclopedia of Integer Sequences, `https://oeis.org/`.

[26] T. N. Shorey, C. L. Stewart, *On the Diophantine equation* $ax^{2t}+bx^ty+cy^2=d$ *and pure powers in recurrence sequences*, Math. Scand. **52** (1983), 24–36.

[27] T. N. Shorey, A. J. van der Poorten, R. Tijdeman, A. Schinzel, *Applications of the Gel'fond-Baker method to Diophantine equations*, in: Transcendence Theory: Advances and Applications, ed. by A. Baker and D.W. Masser, Academic Press, 1977, pp. 59-77.

[28] C. L. Stewart, *Divisor Properties of Arithmetical Progressions*, Ph.D. Thesis, Univ. Cambridge, 1976.

[29] C. L. Stewart, *On the number of solutions of polynomial congruences and Thue equations*, J. Amer. Math. Soc. **4** (1991), 793–835.

[30] C. L. Stewart, S. Y. Xiao, *On the representation of integers by binary forms*, Math. Ann. **375** (2019), 133–163.

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