Number expansions over imaginary quadratic Euclidean domains

PhD thesis

Péter Varga

Supervisor: Dr. Attila Pethő

University of Debrecen

Science and Informatics Doctoral Council

Doctoral School of Informatics

Debrecen, 2017.

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Debrecen, 23rd August 2017.

candidate's signature

I certify that *Péter Varga* PhD candidate worked under my supervision from September 2008 to June 2011 as a PhD student of the Doctoral Program Theoretical computer science, data security and cryptography at Doctoral School of Informatics, Science and Informatics Doctoral Council, University of Debrecen. The creative activity of the candidate decisively contributed to the preparation of the results in this thesis. I recommend acceptance of the thesis.
Debrecen, 11th September 2017.

supervisor's signature

Number expansions over imaginary quadratic Euclidean domains

Dissertation for obtaining doctoral (PhD) degree in Mathematics

Written by: Péter Varga mathematician and computer scientist

This thesis has been made during the Doctoral Program Theoretical computer science, data security and cryptography at Doctoral School of Informatics, Science and Informatics Doctoral Council, University of Debrecen in order to obtain PhD degree at University of Debrecen.

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Introduction

„Die Mathematik ist die Königin der Wissenschaften und die Zahlentheorie ist die Königin der Mathematik.“

Carl Friedrich Gauß

Number systems have an important role in our life. As ancient as the mankind itself this concept has changed and evolved. Choosing 10 as base number for radix representation of the integers is just one of the infinitely many possibilities. Choosing non-negative numbers as digits is also just a convention. B. Pascal in 1658 first stated in print that any integer greater than 1 could serve as radix. A.-L. Cauchy in 1840 pointed out that negative numbers as digits make it unnecessary for a person to memorize the multiplication table past $5x5$.

V. Grünwald [35] introduced the radix representation with respect to negative bases in 1885 on the following way. Let $g \leq -2$ be an integer. Then every $n \in \mathbb{Z}$ can be represented in the form

$$\sum_{i=0}^{l} n_i g^i, 0 \leq n_i < |g|. \tag{1}$$

He investigated how to perform the four basic operations in such number systems. In this concept there is no distinguish between positive and negative elements, thus it allows far reaching generalizations. It's started by D. E. Knuth [47] in 1960. His number system is known as quarter-imaginary numeral system which uses $2i$ as its base and $0, 1, 2, 3$ as its digits. All of the Gaussian integers $a + bi$ $(a, b \in \mathbb{Z})$ can be represented in this number system. Another similar number system was analyzed by W. Penney [69] in 1965. He used the number

$-1+i$ as basis and $0,1$ as digits. I. Kátai and J. Szabó [44] in 1975 generalized W. Penney's result. They proved that the only numbers which are suitable bases for all Gaussian integers, using $0,1,\ldots,N-1$ as digits, are $-n\pm i$, where n is a positive integer and $N=n^2+1$, the norm of $-n\pm i$. W. J. Gilbert [34] in 1981 generalized I. Kátai and J. Szabó's result to find all the bases for quadratic number fields using $0,1,\ldots,N-1$ as digits. I. Kátai and J. Szabó also stated that if $\{\alpha,\mathcal{N}\}$ is a canonical number system (CNS for short, α is the base number, $\mathcal{N}$ is the digit set) in the ring of Gaussian integers, then any complex number γ can be written in the form (canonical in a sense that digits are in ascending order): $\gamma=a_k\alpha^k+a_{k-1}\alpha^{k-1}+\cdots+a_0+a_{-1}\alpha^{-1}+\ldots$, $a_i\in\mathcal{N}$. This is called $\alpha\mathcal{N}$-expansion of γ which has been studied by I. Kátai and B. Kovács [43], B. Kovács [50], B. Kovács and Gy. Maksa [57], I. Kátai and I. Környei [41], B. Kovács and I. Környei [56] and by A. Pethő [73]. S. Ito [38] in 1989 investigated Kátai and Szabó's number systems and showed that the boundary curve is a fractal curve. Later in 2001, W. Müller, J. M. Thuswaldner and R. F. Tichy [62] generalized the investigation of the boundary fractal curve for number systems over n-dimensional real vector space. M.-A. Jacob and J.-P. Reveilles [39] in 1995 defined an integer division for Gaussian Integers, which linked two different objects: discrete affine applications and Gaussian numeration systems. A. Kovács [51] in 1999 analyzed the structure of the expansions in the ring of Gaussian integers with canonical digits. In 2001 he extended this result to the integers in imaginary quadratic fields [53]. Another generalization of CNS, namely for polynomials over imaginary quadratic Euclidean domains was studied by A. Pethő and P. Varga in [76], and these results are also presented in this dissertation in Chapter 1.

For a given positive integer base b, A. M. Odlyzko [68] in 1978 gave necessary and sufficient conditions for a set S of positive real numbers to have the property that every real number can be represented in the form

$$\pm\sum_{i=-N}^{\infty} s_i b^{-i}, s_i\in S.$$

The integers' unique representation was investigated by D. W. Matula [67] in 1982.

D. E. Knuth [48] in 1981 described numerous reference to alternative number systems, and he gave results about radix representation of integers with negative bases. He also analyzed the $-1+i$ based number system, which is related to the „twin dragon“ fractal. The connection between fractals and CNS has been investigated by S. Akiyama and J. M. Thuswaldner [15], [16], [17],

K. H. Indlekofer, I. Kátai and P. Racskó [37] and by K. Scheicher and J. M. Thuswaldner [79]. J. M. Thuswaldner [81] calculated fractal dimensions of sets generated by CNS over imaginary quadratic fields in 1998.

B. Kovács and A. Pethő [58] in 1983 proved that for a given rational integer basis there exists infinitely many finite digit sets. They proved in 1991 [59] that if $g(t)$ is irreducible then it is decidable whether the pair $\{g(t), \mathcal{N}\}$ is a number system in the ring $\mathbb{Z}[t]/g(t)\mathbb{Z}[t]$. Later, in 2006, H. Brunotte, A. Huszti and A. Pethő used this result in [23] to compute canonical number systems of some quartic fields. Also, A. Pethő [72] in 1991 generalized this result for arbitrary polynomials, and he defined CNS as follows. Let $P(x) = x^{n+1} + p_n x^n + p_{n-1}x^{n-1} + \cdots + p_0 \in \mathbb{Z}[x]$ and $D = \{0, 1, \ldots, |p_0| - 1\}$. The polynomial $P(x)$ is called CNS polynomial if for every $0 \neq A(x) \in \mathbb{Z}[x]$ there exist $h \geq 0$ and $a_0, \ldots, a_h \in D$ such that

$$A(x) \equiv a_h x^h + a_{h-1}x^{h-1} + \cdots + a_1 x + a_0 \pmod{P(x)}. \tag{2}$$

If $P(x)$ is irreducible one gets the concept of canonical number systems in algebraic number fields, which was introduced and studied by I. Kátai and J. Szabó (see [44]). This result was generalized to quadratic integers by I. Kátai and B. Kovács [42], [43], [49] and independently in W. J. Gilbert [34]. S. Körmendi [40] in 1986 established all CNS bases in a class of pure cubic number fields. B. Kovács and A. Pethő [59] presented a general algorithm for the computation of all CNS bases in an algebraic number field and used their method in some cases in 1991. S. Akiyama, H. Brunotte and A. Pethő [4], [5] disproved a conjecture of W. J. Gilbert about the structure of the set of cubic CNS polynomial. H. Brunotte [20] investigated the totally real cubic CNS polynomials. Families of quartic CNS polynomials were studied by H. Brunotte, A. Huszti and A. Pethő [23] and by A. Pethő [71].

By changing appropriately the bases $1, x, \ldots, x^{n-1}$ of the $\mathbb{Q}$-vector space of polynomials of degree at most $n-1$, H. Brunotte [18] found a very efficient algorithm for the decision of the CNS property. He used it in [22] for the characterization all CNS whose bases are roots of trinomials.

In 1993 B. Kovács and A. Pethő [60] gave an asymptotic estimate for the number $(L(\beta))$ of required digits for a given β to be represented in a number system. In 2001 A. Kovács analyzed binary number systems and number systems with small digit set over algebraic number fields. The characterization of CNS polynomials is complicated already for degree three, as indicated in [7]. It is still unsolved. A. Kovács [54] dealt with binary number systems in 2001, also G. Farkas and A. Kovács [31] analyzed the expansion $\mathbb{Q}(\sqrt{2})$ in 2003. A necessary condition for a polynomial to be a CNS-polynomial is to be expanding, which has been investigated by P. Burcsi and A. Kovács [26] in 2005. Additive functions for CNS polynomials has been studied by M. G. Madritsch and J. M. Thuswaldner [66], M. G. Madritsch [63] and by M. G. Madritsch and A. Pethő [64].

Rational based number systems has been studied by S. Akiyama, C. Frougny and J. Sakarovitch [10] in 2008.

P. Burcsi and A. Kovács [27] called $P(x)$ a *semi-CNS polynomial* if the finite expansions (2) form an additive semigroup. This is a generalization of the usual radix representations of natural numbers. They were able to prove some sufficient properties for $P(x)$ being a semi-CNS polynomial. Moreover they generalized Brunotte's algorithm for semi-CNS polynomials. I have conducted an enquiry into cubic semi-CNS polynomials (see [83]), I was able to fully characterize them. H. Brunotte generalized this result for semi-CNS polynomials with any degree in [22].

An interesting alternative concept of CNS polynomials are symmetric-CNS polynomials, where the digit set is balanced on the way that it contains negative and positive elements as well. This concept can also generalized to symmetric SRSes. This topic has been studied by H. Brunotte [19], [21], S. Akiyama and K. Scheicher [14], and by A. Huszti, K. Scheicher, P. Surer and J. M. Thuswaldner [36].

K. H. Indlekofer, I. Kátai and P. Racskó [37] initiated simultaneous number systems in 1993. A. Kovács [55] analyzed this concept of simultaneous number systems over Eisenstein integers in 2013.

Inspired by [76], A. Pethő and J. M. Thuswaldner [75] generalized the CNS concept to number systems over number field orders.

The shift radix systems, SRS, for real vectors were introduced by S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. M. Thuswaldner [2] in 2005. Generalizing SRS, H. Brunotte, P. Kirschenhofer and J. Thuswaldner [24] defined Gaussian shift radix systems (GSRS) for Hermitian vector spaces as follows. Let $r \in \mathbb{C}^n$ be given ($n \in \mathcal{N}$). Let the mapping $\gamma_r : \mathbb{Z}[i]^d \to \mathbb{Z}[i]^d$ be defined by

$$x = (x_1, x_2, \ldots, x_n) \mapsto (x_2, x_3, \ldots, x_n, -\lfloor rx \rfloor), \tag{3}$$

where rx is the inner product of r and x, and $\lfloor z \rfloor := \lfloor Re(z) \rfloor + i \lfloor Im(z) \rfloor, z \in \mathbb{C}$. For $\mathbf{r} \in \mathbb{R}^n$ the mapping $\tau_{\mathbf{r}} \; : \; \mathbb{Z}^n \mapsto \mathbb{Z}^n$, defined as

$$\tau_{\mathbf{r}}((a_1, \ldots, a_n)) = (a_2, \ldots, a_n, -\lfloor \mathbf{ra} \rfloor), \tag{4}$$

where $\mathbf{ra}$ denotes the inner product, is called *shift radix system*, shortly SRS. In [2] it is also proved that SRS is a common generalization of canonical number systems (CNS) and the β-expansions, defined by A. Rényi [78]. This concept has been studied in many articles by S. Akiyama and J. M. Thuswaldner [16], S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. Thuswaldner [1], [3], S. Akiyama and K. Scheicher [13], S. Akiyama and J. M. Thuswaldner [17], S. Akiyama, H. Brunotte, A. Pethő and J. M. Thuswaldner [7], [8], [9], P. Surer [80], P. Kirschenhofer, A. Pethő, P. Surer and J. M. Thuswaldner [46], H. Brunotte, P. Kirschenhofer and J. M. Thuswaldner [24], [25], M. G. Madritsch and A. Pethő [65], S. Akiyama, H. Brunotte, A. Pethő, W. Steiner and J. M. Thuswaldner [6], P. Kirschenhofer and J. M. Thuswaldner [82], M. Weitzer [86], [85] and by A. Pethő [74]. Another generalization of SRS for Hermitian vector spaces, namely for vectors over imaginary quadratic Euclidean domains was studied by A. Pethő, P. Varga and M. Weitzer in [77], and these results are also presented in this dissertation in Chapter 2. It is well known that there are exactly five imaginary quadratic Euclidean domains, which are the ring of integers of the imaginary quadratic fields $\mathbb{Q}(\sqrt{d}), d = 1, 2, 3, 7, 11$. The Euclidean norm function allows not only the division by remainder, but also to define a floor function for complex numbers. This generalization, which I call ESRS, is uniform for the five imaginary quadratic Euclidean domains. This has the consequence that in case of the Gaussian integers the floor function differs from that used in [24].

The SRS $\tau_{\mathbf{r}}$ is said to have the finiteness property if and only if for all $\mathbf{a} \in \mathbb{Z}^{\mathbf{n}}$ there exists a $k \geq 1$ such that $\tau_{\mathbf{r}}^k(\mathbf{a}) = \mathbf{0}$. Denote by $\mathcal{D}_n^{(0)}$ the set of $\mathbf{r} \in \mathbb{R}^n$ such that $\tau_{\mathbf{r}}$ has the finiteness property. From numeration point of view these real vectors are most important. It turned out that the structure of $\mathcal{D}_n^{(0)}$ is very complicated already for $n = 2$, see [7], [80] and [86].

The analogue of the two dimensional SRS is the one dimensional GSRS and ESRS. H. Brunotte, P. Kirschenhofer and J. M. Thuswaldner [24] studied first the set of one dimensional GSRS with finiteness property, which I denote by $\mathrm{GSRS}^{(0)}$. It turned out that its structure is quite complicated as well. Recently a more precise investigation of M. Weitzer [85] showed that the structure of $GSRS^{(0)}$ is much simpler as that of $\mathcal{D}_2^{(0)}$. Based on extensive computer investigations he conjectures a finite description of $\mathrm{GSRS}^{(0)}$.

Analogously to $\mathcal{D}_n^{(0)}$ one can define $\mathcal{D}_{n,d}^{(0)}$ for $d = 1, 2, 3, 7, 11$ in a straightforward way. I present how a good approximations of $\mathcal{D}_{n,d}^{(0)}$ can be computed. Performing the computation it turned out that the shape of these objects are quite different. The subjective impression can be misleading, but Theorem 2.2.12 shows that $\mathcal{D}_{n,d}^{(0)}$ has no critical points in the cases $d = 2, 11$. More specifically this theorem shows that the circle of radius 0.99 around the origin contains

$\mathcal{D}_{n,d}^{(0)}$. In the other cases this is probably not true. It is certainly not true for $\mathcal{D}_2^{(0)}$ and $\mathrm{GSRS}^{(0)}$.

In this dissertation I will define and analyze a number system over norm-Euclidean domains (ENS), and I will generalize the shift radix systems to finite dimensional Hermitian vector spaces (ESRS) using a similar structure. One of the main features of this construction is that the remainder set is the subset of the opened unit disc, which gives us the property that $|r| < 1$ for every reminder r.

The first section describes the basic concepts, while the second section defines a number system over norm-Euclidean domains (ENS), and examines some of its properties. One of the most important properties is that the ENS property is always algorithmically decidable, this is the result of Theorem 1.2.13. The third section presents the properties which are specific to the number systems over imaginary quadratic Euclidean domains. The fourth section is to characterize the linear ENS polynomials over imaginary quadratic Euclidean domains. The main result can be found in Theorem 1.4.6. The fifth section is about the quadratic case and its properties. The sixth section investigates a kind of infinite sequences of ENS polynomials, and shows an interesting result about the connection between the CNS and symmetric-CNS polynomials over rational integers and the ENS polynomials over imaginary quadratic Euclidean domains in Theorem 1.6.1.

The last three sections are about the ESRS concept. In this concept even the one dimensional case is a hard problem, its characterization is still an open question. The last section shows that Brunotte's algorithm [18] can be generalized to the ESRS concept, but with some restrictions (Theorem 2.3.5). The proof of this generalization borrows ideas from S. Akiyama's proof in [12].

Chapter 1

ENS

In this chapter a number system concept over Euclidean domains will be defined. I will investigate some properties on norm-Euclidean domains, then specifically for imaginary quadratic Euclidean domains.

1.1 Basic concepts

Definition 1.1.1. *Let $\mathbb{E}$ be an integral domain. The function $N : \mathbb{E} \mapsto \mathbb{N}$ with the following properties:*

1. *$N(a) = 0$ for an $a \in \mathbb{E}$, if and only if $a = 0$,*
2. *if $a \in \mathbb{E}$ and $b \in \mathbb{E} \setminus \{0\}$, then there are $q, r \in \mathbb{E}$ such that $a = bq + r$ and $N(r) < N(b)$*

is called ***Euclidean function****.*

Remark 1.1.2. *In 2. above, we say that q is the quotient and r is the remainder part of the Euclidean division of a by b.*

Definition 1.1.3. *The integral domain $\mathbb{E}$ is called* ***Euclidean domain*** *if it is endowed with a Euclidean function.*

Definition 1.1.4. *The Euclidean domain $\mathbb{E}$ is called* ***norm-Euclidean*** *if its Euclidean function is derived from the corresponding field's absolute value function.*

Remark 1.1.5. *In this dissertation the following notations will be used:*

$\mathbb{Q}$	*field of rational numbers,*
$\mathbb{Z}$	*ring of integers,*
$\mathbb{C}$	*field of complex numbers,*
i	*the imaginary unit,*
$\lvert z\rvert$	*complex absolute value:* $\lvert z\rvert := \sqrt{z_1^2 + z_2^2}$, *where* $z \in \mathbb{C}, z_1, z_2 \in \mathbb{R}, z = z_1 + z_2 i$,
$K[x]$	*the set of polynomials with coefficients belonging to* K.

Definition 1.1.6. *Let* $\mathbb{K}$ *denote the quotient field of* $\mathbb{E}$. *Then all elements* $\alpha \in \mathbb{K}$ *can be written in the form* $\alpha = \frac{a}{b}$ *with* $a, b \in \mathbb{E}$*;* $b \neq 0$. *Let* q *be the quotient and* r *be the remainder of the Euclidean division of* a *by* b. *Then* q *is called the* ***integer part of*** α *and is denoted by* $\lfloor \alpha \rfloor = \left\lfloor \frac{a}{b} \right\rfloor$ *and* r *is called the* ***remainder part of*** α *and is denoted by* $\{\alpha\} = \left\{\frac{a}{b}\right\}$. *The function* $\alpha \mapsto \lfloor \alpha \rfloor$ *is called the* ***floor function****, and the integer part* $\lfloor \alpha \rfloor$ *is also called the* ***floor of*** α.

1.2 ENS over Euclidean domains

This section is devoted to definitions and theorems about the ENS (Euclidean number system) concept which needs only a Euclidean domain and a digit set. The digit set is sufficient for an unambiguous definition of a number system over the Euclidean domain.

Definition 1.2.1. *Let* $\mathbb{E}$ *be an Euclidean domain with a Euclidean function* N. *Let* $P(x) = x^{n+1} + p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0 \in \mathbb{E}[x] (n \in \mathbb{N})$ *be a monic irreducible polynomial over* $\mathbb{E}$ *such that* $N(p_0) \geq 2$*, and let* $\mathbb{D}_{p_0} \subset \mathbb{E}$ *be a so called digit set with* $|\mathbb{D}_{p_0}| = N(p_0)$. *The elements of the factor ring* $\mathbb{E}[x]/P(x)\mathbb{E}[x]$ *can be represented by polynomials over* $\mathbb{E}$ *of degree at most* n. *This set is denoted by* $\mathbb{E}^n[x]$.
If for an $A(x) \in \mathbb{E}^n[x]$ *there exists* $a(x) \in \mathbb{D}_{p_0}[x]$ *such that*

$$A(x) \equiv a(x) \pmod{P(x)},$$

then $A(x)$ *has an* ***expansion****. If all* $A(x) \in \mathbb{E}^n[x]$ *have an expansion, then the pair* $(P, \mathbb{D}_{p_0})$ *is called* ***ENS*** *and the polynomial* $P(x) \in \mathbb{E}[x]$ *is called an* ***ENS polynomial****.*

Let $\mathbb{K}$ denote the quotient field of $\mathbb{E}$. Using irreducible ENS polynomials numeration systems can be defined in $\mathbb{E}$ and in some of its extensions. Indeed, let $P(x)$ be an ENS polynomial over $\mathbb{E}$ and let γ denote one of its roots. Then $\mathbb{K}[x]/P(x)\mathbb{K}[x]$ is isomorphic to the field $\mathbb{K}(\gamma)$. Moreover $\mathbb{E}[x]/P(x)\mathbb{E}[x]$ is isomorphic to the ring $\mathbb{E}[\gamma]$. Thus every element $0 \neq \beta \in \mathbb{E}[\gamma]$ can be written uniquely in the form

$$\beta = \sum_{j=0}^{h} b_j \gamma^j, \quad b_j \in \mathbb{D}_{p_0}, b_h \neq 0.$$

From this point $\mathbb{E}$ denotes a norm-Euclidean domain, absolute value of its elements is defined by the complex absolute value function, and $\forall e \in \mathbb{E} : N(e) = |e|^2$.

Definition 1.2.2. *Let $P(x) \in \mathbb{C}[x]$ be a monic complex polynomial. $P(x)$ is called expanding if all of its roots lie outside the closed unit circle, i.e.*

$$P(\gamma) = 0 \Rightarrow |\gamma| > 1.$$

Theorem 1.2.3 is a consequence of A. Vince's result [84].

Theorem 1.2.3. *If $P(x) \in \mathbb{E}[x]$ is an ENS polynomial then it is expanding.*

Proof. (This proof only covers the case, when $|\gamma| < 1$. For the case of $|\gamma| = 1$, see [84]. For polynomials over $\mathbb{Z}$ this has been proved by A. Pethő in [72] in the proof of Theorem 6.1.) $\mathbb{E}[x]/P(x)\mathbb{E}[x]$ is isomorphic to the ring $\mathbb{E}[\gamma]$, $P(\gamma) = 0$. This is true for all roots γ_i of the polynomial $P(x)$. If the absolute value of one of these is less then or equal to 1, then the representation of the elements $0 \neq \beta \in \mathbb{E}[\gamma]$ is bounded, so this cannot represent all elements β:

$$|\beta| = \left|\sum_{j=0}^{h} b_j \gamma^j\right| \leq \sum_{j=0}^{h} \left|b_j \gamma^j\right| = \sum_{j=0}^{h} |b_j| \left|\gamma^j\right| <$$

$$< \sum_{j=0}^{h} |p_0| \left|\gamma^j\right| = |p_0| \sum_{j=0}^{h} \left|\gamma^j\right| = |p_0| \sum_{j=0}^{h} |\gamma|^j \leq$$

$$\leq |p_0| \lim_{h\to\infty} \sum_{j=0}^{h} |\gamma|^j = |p_0| \frac{1}{1-|\gamma|} (\text{if } |\gamma| < 1).$$

$\square$

Lemma 1.2.4. *If p_0 is the constant term of the expanding monic polynomial $P(x) \in \mathbb{E}[x]$, then*

$$N(p_0) \geq 2,$$

Proof. $P(x)$ is an expanding monic polynomial, so for all its roots $|\gamma_i| > 1$. p_0 is the product of the roots and $N(p_0) \in \mathcal{N}$. $\square$

Definition 1.2.5. *Let $P(x) = x^{n+1}+p_n x^n+p_{n-1}x^{n-1}+\cdots+p_1 x+p_0 \in \mathbb{E}^{n+1}[x]$ be such that $N(p_0) \geq 2$. Let the mapping $T_P : \mathbb{E}^n[x] \mapsto \mathbb{E}^n[x]$ be defined as follows: for $A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{E}^n[x]$ let*

$$T_P(A) = \frac{A - qP - r}{x},$$

*where $q = \left\lfloor \frac{a_0}{p_0} \right\rfloor$ and $r = a_0 - qp_0 \in \mathbb{D}_{p_0}$. The mapping T_P is called **Backward division**.*

The mapping backward division can be iterated, which means

$$T_P^k(A) = \begin{cases} A, & \text{if} \quad k = 0; \\ T_P(T_P^{k-1}(A)), & \text{if} \quad k > 0. \end{cases}$$

Let $q_k \in \mathbb{E}$ and $r_k \in \mathbb{D}_{p_0}$ be defined by the equation

$$T_P^{k+1}(A) = \frac{T_P^k(A) - q_k P - r_k}{x},$$

where $a_0^{(k)} = T_P^k(A)|_{x=0}, q_k = \left\lfloor \frac{a_0^{(k)}}{p_0} \right\rfloor$ and $r_k = a_0^{(k)} - q_k p_0, k \in \mathbb{N}$. Let $A_k := T_P^k(A)$.

The orbit of T_P starting from A will be denoted as follows:

$$A \xRightarrow[P]{(q_1,r_1)} A_1 \xRightarrow[P]{(q_2,r_2)} A_2 \xRightarrow[P]{(q_3,r_3)} A_3 \ldots,$$

if it is not necessary to know the multipliers, it will simply be denoted by:

$$A \xRightarrow[P]{r_1} A_1 \xRightarrow[P]{r_2} A_2 \xRightarrow[P]{r_3} A_3 \ldots,$$

or if it is not necessary to know even the remainders, it will simply be denoted by:

$$A \underset{P}{\Rightarrow} A_1 \underset{P}{\Rightarrow} A_2 \underset{P}{\Rightarrow} A_3 \ldots.$$

If for $A, B \in \mathbb{E}^n[x]$ there exists $k \in \mathbb{N}$ such that $T_P^k(A) = B$ then I write:

$$A \underset{P}{\overset{*}{\Longrightarrow}} B.$$

Plainly the orbits of T_P are either ultimately periodic or consist of infinitely many pairwise different elements and both cases may occur. Moreover in the first case the orbit is ultimately 0 or not. One of the most important aim of the investigations on ENS polynomials is the distinction between these possibilities. Theorem 1.2.6 is my result (see [76]). It is a direct consequence of the previous definitions and it states that investigating the orbits of T_P can decide the ENS property of a polynomial $P(x) \in \mathbb{E}[x]$.

Theorem 1.2.6. *$P(x) \in \mathbb{E}^{n+1}[x]$ is an ENS polynomial if and only if for all $A(x) \in \mathbb{E}^n[x]$*

$$A \underset{P}{\overset{*}{\Longrightarrow}} 0.$$

Proof. This theorem is a direct consequence of Definitions 1.2.1 and 1.2.5. □

Theorem 1.2.7 is my result (see [76]). It can be used to find sets of polynomials which are not ENS polynomials.

Theorem 1.2.7. *Let $P(x) := x^{n+1} + p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0 \in \mathbb{E}^{n+1}[x]$ be such that $N(p_0) \geq 2$. Assume that the orbit of T_P starting from $A(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{E}^n[x]$ is periodic and let $l > n$ be a multiple of the period length, as follows:*

$$A = A_0 \underset{P}{\xrightarrow{(q_0,r_0)}} A_1 \underset{P}{\xrightarrow{(q_1,r_1)}} A_2 \underset{P}{\xrightarrow{(q_2,r_2)}} A_3 \ldots \underset{P}{\xrightarrow{(q_{l-2},r_{l-2})}} A_{l-1} \underset{P}{\xrightarrow{(q_{l-1},r_{l-1})}} A.$$

Then

$$-\sum_{m=0}^{n+1} q_{l+h-m} p_m \in \mathbb{D}_{p_0}$$

holds for $h = 0, 1, \ldots, l-1$.

Proof. Let $A_h(x) = \sum_{j=0}^{\infty} a_j^{(h)} x^j$, where $a_j^{(h)} = 0$ for all $h \geq 0$ and $j > n$. Similarly $P(x) = \sum_{j=0}^{\infty} p_j x^j$ with $p_{n+1} = 1$ and $p_j = 0$ for $j > n+1$. With these notations

$$a_j^{(h)} = a_{j+h}^{(0)} - \sum_{k=0}^{h-1} q_k p_{j+h-k}. \tag{1.1}$$

Indeed, the claim is true for $h=0$ because the empty sum is 0. Assume that it is true for a $h \geq 0$. Then

$$A_{h+1} = T_P(A_h) = \frac{A_h - q_h P - r_h}{x}.$$

Comparing the coefficients and using the induction hypothesis one can get

$$\begin{aligned}
a_j^{(h+1)} &= a_{j+1}^{(h)} - q_h p_{j+1} \\
&= a_{j+h+1}^{(0)} - \sum_{k=0}^{h-1} q_k p_{j+1+h-k} - q_h p_{j+1} \\
&= a_{j+h+1}^{(0)} - \sum_{k=0}^{h} q_k p_{j+1+h-k},
\end{aligned}$$

which proves the claim.

Consider equation (1.1) for $j=0$ and $h = l, \ldots, 2l-1$. By the assumption $A_h(x) = A_{h+l}(x), h = 0, \ldots, l-1$, especially $a_0^{(h+l)} = a_0^{(h)}, h = 0, \ldots, l-1$. Thus

$$q_{h+l} = \left\lfloor \frac{a_0^{(h+l)}}{p_0} \right\rfloor = \left\lfloor \frac{a_0^{(h)}}{p_0} \right\rfloor = q_h.$$

As $l > n$, $a_{l+h}^{(0)} = 0$ for $h \geq 0$. Summarizing (1.1) leads to

$$a_0^{(h)} = a_0^{(l+h)} = -\sum_{k=0}^{l+h-1} q_k p_{l+h-k}, \ h = 0, \ldots, l-1.$$

By the construction $a_0^{(l+h)} - q_{l+h} p_0 = r_{l+h} \in \mathbb{D}_{p_0}$, hence

$$-\sum_{k=0}^{l+h} q_k p_{l+h-k} \in \mathbb{D}_{p_0}, \ h = 0, \ldots, l-1.$$

Replacing the summation variable k by $m = l+h-k$ and taking into account that $p_m = 0$ for $m > n+1$ one can obtain

$$-\sum_{m=0}^{n+1} q_{l+h-m} p_m \in \mathbb{D}_{p_0}, \ h = 0, \ldots, l-1,$$

as it was stated. $\square$

Remark 1.2.8. *Applying Theorem 1.2.7 with length* 1*, one can get the following restriction for the coefficients of an ENS polynomial:*

$$-q \sum_{m=0}^{n+1} p_m \notin \mathbb{D}_{p_0},$$

where $q = \frac{a}{p_0}$*, for all* $a \in \mathbb{E} \setminus \{0\}$*.*

Remark 1.2.9. *The polynomial* $P(x) = x + p_0, 0 \neq p_0 \in \mathbb{E}$ *is obviously irreducible. This implies that* $(x + p_0, \mathbb{D}_{p_0})$ *is a numeration system (ENS) in* $\mathbb{E}$ *if and only if* $x + p_0$ *is an ENS polynomial.*

The next definition and two lemmata are necessary to prove the result of Theorem 1.2.13.

Definition 1.2.10. *Let* $Z = \{z_1, z_2, z_3, \ldots, z_n\} \subset \mathbb{C}$ *be a set of different complex numbers (if* $i \neq j$*, then* $z_i \neq z_j$*). Let*

$$s_k = \sum_{i_1, i_2, \cdots, i_k} z_{i_1} z_{i_2} \cdots z_{i_k} \quad \textit{(if } k \neq l\textit{, then } i_k \neq i_l\textit{)}$$

be the kth ***Viète sum*** *of the set* Z*. For the case* $k = 0$*, let* $s_0 = 1$ *by definition.*

The next lemma's proof is trivial.

Lemma 1.2.11. *Let* $Z = \{z_1, z_2, z_3, \ldots, z_n\} \subset \mathbb{C}$ *be a set of different complex numbers (if* $i \neq j$*, then* $z_i \neq z_j$*). Then the following equation is true for all* z_j*:*

$$\sum_{i=0}^{n} (-z_j)^i s_{n-i} = 0.$$

Lemma 1.2.12 describes the Lacunary Vandermonde determinant, which is a classical, well known result.

Lemma 1.2.12. *(generalization of Vandermonde determinant)*
Let $Z = \{z_1, z_2, z_3, \ldots, z_n\} \subset \mathbb{C}$ *be a set of different complex numbers (if* $i \neq j$*, then* $z_i \neq z_j$*), let* $0 \leq i \leq n$*, and let*

$$V_{n,i} := \begin{vmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{i-1} & z_1^{i+1} & \cdots & z_1^n \\ 1 & z_2 & z_2^2 & \cdots & z_2^{i-1} & z_2^{i+1} & \cdots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{i-1} & z_n^{i+1} & \cdots & z_n^n \end{vmatrix}$$

be called ***Lacunary Vandermonde determinant****. Then*

$$V_{n,i} = s_{n-i} \prod_{1 \le k < l \le n} (z_k - z_l).$$

Proof. If $i = n$, then it is the well known Vandermonde determinant ($s_0 = 1$). Let's assume that the equation is true for $V_{n,k}$. We will see that $V_{n,k-1} = \frac{s_{n-k+1}}{s_{n-k}} V_{n,k}$.

$$\frac{s_{n-k+1}}{s_{n-k}} V_{n,k} = \frac{s_{n-k+1}}{s_{n-k}} \begin{vmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{k-1} & z_1^{k+1} & \cdots & z_1^n \\ 1 & z_2 & z_2^2 & \cdots & z_2^{k-1} & z_2^{k+1} & \cdots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{k-1} & z_n^{k+1} & \cdots & z_n^n \end{vmatrix} =$$

$$= \frac{1}{s_{n-k}} \begin{vmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{k-2} & z_1^{k-1} s_{n-k+1} & z_1^{k+1} & \cdots & z_1^n \\ 1 & z_2 & z_2^2 & \cdots & z_2^{k-2} & z_2^{k-1} s_{n-k+1} & z_2^{k+1} & \cdots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{k-2} & z_n^{k-1} s_{n-k+1} & z_n^{k+1} & \cdots & z_n^n \end{vmatrix}$$

Let $c_0, c_1, c_2, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$ be the column vectors of this matrix respectively. Let's add to the column c_{k-1} the linear combination of the other columns

$$\sum_{\substack{i=0, \\ i \neq k-1, \\ i \neq k}}^{n} (-1)^{k-1+i} c_i s_{n-i},$$

this way c_{k-1}'s values are

$$\sum_{\substack{i=0, \\ i \neq k}}^{n} (-1)^{k-1+i} z_j^i s_{n-i}, \text{ (the value of the } j\text{th row).}$$

Due to the Lemma 1.2.11, this value is equal to $z_j^k s_{n-k}$, so

$$\frac{1}{s_{n-k}} \cdot \begin{vmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{k-2} & z_1^k s_{n-k} & z_1^{k+1} & \cdots & z_1^n \\ 1 & z_2 & z_2^2 & \cdots & z_2^{k-2} & z_2^k s_{n-k} & z_2^{k+1} & \cdots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{k-2} & z_n^k s_{n-k} & z_n^{k+1} & \cdots & z_n^n \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{k-2} & z_1^k & \cdots & z_1^n \\ 1 & z_2 & z_2^2 & \cdots & z_2^{k-2} & z_2^k & \cdots & z_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{k-2} & z_n^k & \cdots & z_n^n \end{vmatrix} = V_{n,k-1}.$$

□

The following result is a common result with A. Pethő (see [76]). It states that ENS property of a polynomial $P(x) \in \mathbb{E}[x]$ is algorithmically decidable. The set of $A(x)$ polynomials which is sufficient to be investigated is finite.

Theorem 1.2.13. *Let $P(x) \in \mathbb{E}^{n+1}[x]$ be an expanding polynomial, i.e. all of its roots lie outside the closed unit circle. There exists a constant c depending only on $P(X)$ such that this is an ENS polynomial if and only if for every $A(x) = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0 \in \mathbb{E}^n[x]$ with $|A_j| < c, j = 0, \ldots, n$ there exists $a(x) \in \mathbb{D}_{p_0}[x]$ such that*

$$A(x) \equiv a(x) \pmod{P(x)}.$$

Proof. (This proof is valid only for the case of simple roots. The general case is treated by A. Kovács in [52]!) Let γ be one of the roots of the polynomial P ($P(\gamma) = 0$). $\beta \in \mathbb{E}[\gamma]$, and $A(x) \in \mathbb{E}[x]$ is the representative polynomial of β. Then

$$\beta = A(\gamma) = \sum_{h=0}^{l-1} r_h \gamma^h + T_P^l(\beta)\gamma^l,\ |\gamma| > 1.$$

Let's express $T_P^l(\beta)$.

$$T_P^l(\beta) = \frac{\beta}{\gamma^l} - \sum_{h=1}^{l} \frac{r_h}{\gamma^h}.$$

This is true for all of the roots $\gamma_i, i \in \{0, 1, 2, \ldots, n\}, P(\gamma_i) = 0$. If all of them is simple root, then one can have $n + 1$ different equation. Let

$$T_P^l(\beta_i) = A'(\gamma_i) = \sum_{j=0}^{n} A'_j \gamma_i^j,\ i \in \{0, 1, 2, \ldots, n\}.$$

Then

$$\sum_{j=0}^{n} A'_j \gamma_i^j = \frac{A(\gamma_i)}{\gamma_i^l} - \sum_{h=1}^{l} \frac{r_h}{\gamma_i^h}.$$

Using Cramer's rule:

$$A'_j = \frac{\begin{vmatrix} 1 & \gamma_0 & \gamma_0^2 & \cdots & \gamma_0^{j-1} & \frac{A(\gamma_0)}{\gamma_0^l} - \sum_{h=1}^{l} \frac{r_h}{\gamma_0^h} & \gamma_0^{j+1} & \cdots & \gamma_0^n \\ 1 & \gamma_1 & \gamma_1^2 & \cdots & \gamma_1^{j-1} & \frac{A(\gamma_1)}{\gamma_1^l} - \sum_{h=1}^{l} \frac{r_h}{\gamma_1^h} & \gamma_1^{j+1} & \cdots & \gamma_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma_n & \gamma_n^2 & \cdots & \gamma_n^{j-1} & \frac{A(\gamma_n)}{\gamma_n^l} - \sum_{h=1}^{l} \frac{r_h}{\gamma_n^h} & \gamma_n^{j+1} & \cdots & \gamma_n^n \end{vmatrix}}{\begin{vmatrix} 1 & \gamma_0 & \gamma_0^2 & \cdots & \gamma_0^n \\ 1 & \gamma_1 & \gamma_1^2 & \cdots & \gamma_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma_n & \gamma_n^2 & \cdots & \gamma_n^n \end{vmatrix}},$$

where $j \in \{0, 1, 2, \ldots, n\}$.
The denominator is the Vandermonde determinant. Let's use Laplace expansion of the counter matrix along the $(j+1)$th column. Let's use the result of the Lemma 1.2.12 and Minkowski's inequality:

$$\left|A'_j\right| = \left| \frac{\sum_{i=0}^{n} \left((-1)^{j+i} V_{n,j} \left(\frac{A(\gamma_i)}{\gamma_i^l} - \sum_{h=1}^{l} \frac{r_h}{\gamma_i^h} \right) \right)}{\prod_{0 \le k < l \le n} (\gamma_k - \gamma_l)} \right| \le$$

(γ_i is missing from the set of $V_{n,j}$.)

$$\le \frac{\sum_{i=0}^{n} \left(|V_{n,j}| \left(\frac{|A(\gamma_i)|}{|\gamma_i|^l} + |p_0| \sum_{h=1}^{l} \frac{1}{|\gamma_i|^h} \right) \right)}{\prod_{0 \le k < l \le n} |\gamma_k - \gamma_l|} \le$$

$$\le \frac{\sum_{i=0}^{n} \left(|s_{n-i}| \prod_{0 \le k < l \le n : k, l \ne i} |\gamma_k - \gamma_l| \left(1 + |p_0| \frac{|\gamma_i|}{|\gamma_i| - 1} \right) \right)}{\prod_{0 \le k < l \le n} |\gamma_k - \gamma_l|} =$$

$$= \sum_{i=0}^{n} \left(\frac{|s_{n-i}|}{\prod\limits_{0 \le k \le n} |\gamma_k - \gamma_i|} \left(1 + |p_0| \frac{|\gamma_i|}{|\gamma_i| - 1} \right) \right).$$

(γ_i is missing from the Viète sum s_{n-i}.)

□

Remark 1.2.14. *Theorem 1.2.13 shows that the ENS property is algorithmically decidable, because only finitely many polynomials have to be tested. Lemma 1.2.16 gives a more practical bound for the coefficients.*

Definition 1.2.15. *The **length** of a polynomial $A(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{E}^n[x]$ is defined by $\lambda(A) := \sum_{i=0}^{n} |a_i|$.*

Lemma 1.2.16. *Let $P(x) \in \mathbb{E}^{n+1}[x]$. Let*

$$\lambda_{i,P}(A) = \sum_{j=i}^{n} |a_j| + \sum_{j=0}^{i-1} \sum_{k=i-j}^{n+1} \sum_{l=0}^{j} \frac{|a_l| + \sqrt{|p_0|^2 - 1}}{|p_0|} f_{j-l}(P)\, |p_k|$$

be the parameterized approximate length of polynomial $A \in \mathbb{E}^n[x]$, where $f_i(P) = \sum_{j=1}^{i} \frac{|p_j|}{|p_0|} f_{i-j}(P)$, $f_0(P) = 1$, $f_1(P) = \frac{|p_1|}{|p_0|}$.

$$\lambda(T_P^i(A)) \le \lambda_{i,P}(A)$$

for all $i \in \{0, 1, \ldots, n\}$.

Proof.

$$\lambda(T_P^i(A)) = |a_i - q_{i-1}p_1 - q_{i-2}p_2 - q_{i-3}p_3 - \cdots - q_0 p_i| +$$

$$+ |a_{i+1} - q_{i-1}p_2 - q_{i-2}p_3 - q_{i-3}p_4 - \cdots - q_0 p_{i+1}| + \cdots +$$

$$+ |a_n - q_{i-1}p_{n-i+1} - q_{i-2}p_{n-i+2} - \cdots - q_0 p_n| +$$

$$+ |-q_{i-1}p_{n-i+2} - q_{i-2}p_{n-i+3} - \cdots - q_0 p_{n+1}| +$$

$$+ |-q_{i-1}p_{n-i+3} - q_{i-2}p_{n-i+4} - \cdots - q_1 p_{n+1}| + \cdots +$$

$$+\left|-q_{i-1}p_n - q_{i-2}p_{n+1}\right| + \left|-q_{i-1}p_{n+1}\right| \le$$

$$\le \sum_{j=i}^{n} |a_j| + \sum_{j=0}^{i-1} \sum_{k=i-j}^{n+1} |q_j| \, |p_k| \text{, where}$$

$$|q_j| = \left| \frac{a_j - q_{j-1}p_1 - q_{j-2}p_2 - q_{j-3}p_3 - \cdots - q_0 p_j - r_j}{p_0} \right| \le$$

$$\le \frac{|a_j| + \sqrt{|p_0|^2 - 1}}{|p_0|} + \sum_{l=0}^{j-1} |q_l| \frac{|p_{j-l}|}{|p_0|} = \sum_{l=0}^{j} \frac{|a_l| + \sqrt{|p_0|^2 - 1}}{|p_0|} f_{j-l}(P).$$

□

Remark 1.2.17. *Lemma 1.2.16 and Theorem 1.2.6 can be used to give an upper bound for the terms of polynomial A. Those polynomials A where the length of the polynomial is strictly decreasing by applying the backward division mapping is not necessary to be investigated in order to determine the ENS property, so one can use the following inequality to get bounds of the coefficients of A:*

$$\mathop{\forall}_{m=1}^{n+1} \lambda(A) \le \lambda_{m,P}(A).$$

Lemma 1.2.18. *If* $\lambda(P) < 2|p_0|$*, then*

$$|A_i| \le \sqrt{|p_0|^2 - 1} \frac{\sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{k=i+1-j}^{n+1} f_{j-l}(P) \, |p_k|}{2p_0 - \lambda(P)}, \text{ for all } i = 0, 1, 2, \ldots, n$$

Proof. Let's investigate the $(i+1)$th inequality of Remark 1.2.17, which will be used to give an upper bound of the ith coefficient A_i:

$$\lambda(A) \le \lambda_{i+1,P}(A) = \sum_{j=i+1}^{n} |A_j| + \sum_{j=0}^{i} \sum_{k=i+1-j}^{n+1} \sum_{l=0}^{j} \frac{|A_l| + \sqrt{|p_0|^2 - 1}}{|p_0|} f_{j-l}(P) \, |p_k| \,,$$

$$|p_0| \sum_{j=0}^{i} |A_j| \le \sum_{j=0}^{i} \sum_{k=i+1-j}^{n+1} \sum_{l=0}^{j} \left(\left(|A_l| + \sqrt{|p_0|^2 - 1} \right) f_{j-l}(P) \, |p_k| \right),$$

$$|A_i|\left(|p_0|-\sum_{k=1}^{n+1}|p_k|\right)\leq\sum_{l=0}^{i-1}\left(|A_l|\left(-|p_0|+\sum_{j=l}^{i}f_{j-l}(P)\sum_{k=i+1-j}^{n+1}|p_k|\right)\right)+$$

$$+\sqrt{|p_0|^2-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{k=i+1-j}^{n+1}f_{j-l}(P)\,|p_k|\,.$$

Since $\lambda(P)<2\,|p_0|$, the expression $\left(-|p_0|+\sum_{j=l}^{i}f_{j-l}(P)\sum_{k=i+1-j}^{n+1}|p_k|\right)\leq 0$, so

$$|A_i|\leq\sqrt{|p_0|^2-1}\frac{\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{k=i+1-j}^{n+1}f_{j-l}(P)\,|p_k|}{2p_0-\lambda(P)}.$$

$\square$

Remark 1.2.19. *Algorithmically a better bound can be obtained via the method of A. Kovács in [52] and P. Burcsi, A. Kovács, Zs. Papp-Varga in [28].*

1.3 Imaginary quadratic Euclidean domains

This section describes the results of Section 2 and 3 in [76]. It was proved by L. E. Dickson [30] and O. Perron [70], see also H. Davenport [29] and H. L. Keng [45] (Theorem 15.3), that the ring of integers of an imaginary quadratic number field $\mathbb{Q}[\sqrt{-d}]$ is Euclidean if and only if $d\in\{1,2,3,7,11\}$. These will be called **imaginary quadratic Euclidean domains** and will be denoted by $\mathbb{E}_d$. Here the Euclidean function is the absolute value function:

$$N(z_1+z_2i):=|z_1+z_2i|^2=z_1^2+z_2^2,\ \text{where } z_1,z_2\in\mathbb{R}.$$

Definition 1.3.1. *Let $\mathbb{E}_d$ be an imaginary quadratic Euclidean domain. Its **canonical integer basis** is: $\{1,\omega_d\}$, where $\omega_d\in\mathbb{E}_d$ and*

$$\omega_d:=\begin{cases}\sqrt{-d} & \text{, if } d\in\{1,2\},\\ \frac{1+\sqrt{-d}}{2} & \text{, otherwise } (d\in\{3,7,11\}).\end{cases}$$

(In the case of $d=1$ $\omega_1=\sqrt{-1}$, so that the imaginary unit i is used.)
(For ω_d during these investigations simply ω is used.)

For fixed d*, the complex numbers* 1 *and* ω *form a basis of* $\mathbb{C}$*, as a two dimensional vector space over* $\mathbb{R}$*. Thus all* $z \in \mathbb{C}$ *can be uniquely written in the form* $z = e_1 + e_2\omega$ *with* $e_1, e_2 \in \mathbb{R}$*. This representation will be denoted by* $(e_1, e_2)_d$*. Plainly* $z \in \mathbb{E}_d$ *if and only if* $e_1, e_2 \in \mathbb{Z}$*. Let the functions* $Re_d : \mathbb{C} \mapsto \mathbb{R}$ *and* $Im_d : \mathbb{C} \mapsto \mathbb{R}$ *be defined as:*

$$Re_d(z) := e_1, \ Im_d(z) := e_2.$$

$Re_d(z)$ *and* $Im_d(z)$ *are called the* ***Euclidean real and Euclidean imaginary part of*** z*.*

Remark 1.3.2. *For all* $z \in \mathbb{C}$ *(and* $d \in \{1, 2, 3, 7, 11\}$*),*

$$\begin{aligned} Im_d(z) &= \frac{Im(z)}{Im(\omega)}, \\ Re_d(z) &= Re(z) - Im(z)\frac{Re(\omega)}{Im(\omega)}. \end{aligned}$$

For all $z_1, z_2 \in \mathbb{C}$*,*

$$\begin{aligned} Im_d(z_1 \pm z_2) &= Im_d(z_1) \pm Im_d(z_2), \\ Re_d(z_1 \pm z_2) &= Re_d(z_1) \pm Re_d(z_2). \end{aligned}$$

For all $z \in \mathbb{C}$ *and* $n \in \mathbb{Z}$*,*

$$\begin{aligned} Im_d(nz) &= nIm_d(z), \\ Re_d(nz) &= nRe_d(z). \end{aligned}$$

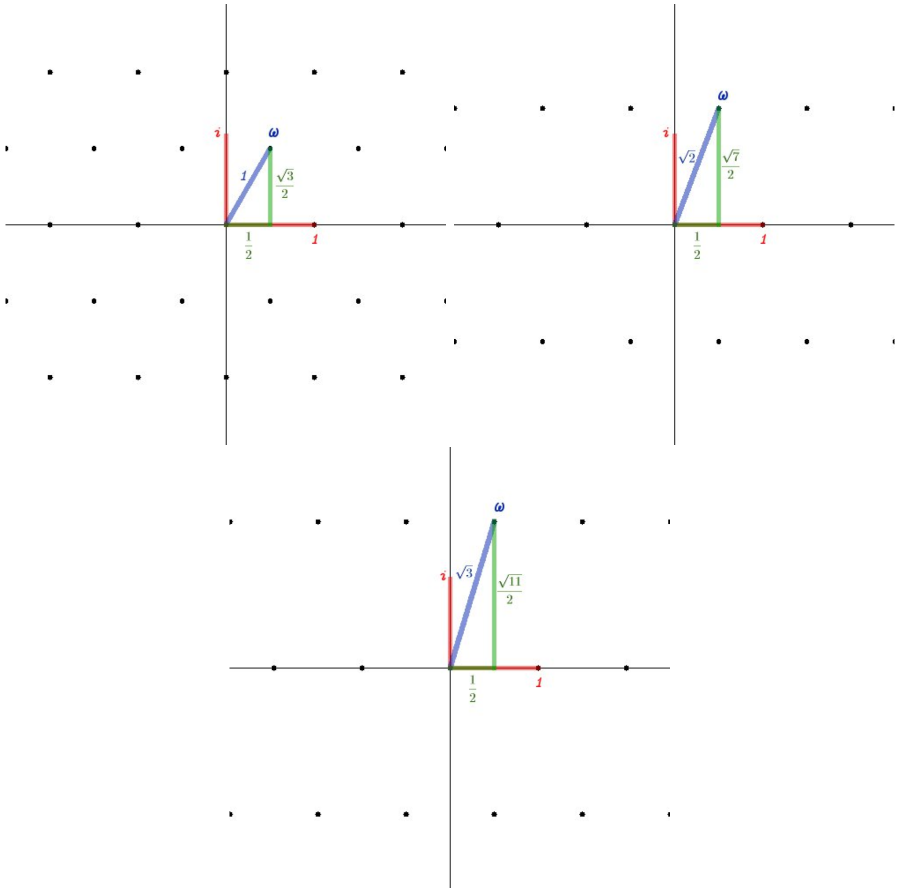
Remark 1.3.3. *Let* $\mathbb{E}_d$ *be a Euclidean domain. The norm of the elements* $z \in \mathbb{E}_d$ *is calculated as follows:*
If $d \in \{1, 2\}$*,* $N(z) = N(e_1 + e_2\sqrt{-d}) = e_1^2 + de_2^2$*, in the other cases* $N(z) = N(e_1 + e_2\frac{1+\sqrt{-d}}{2}) = e_1^2 + e_1e_2 + \frac{d+1}{4}e_2^2$*. Thus one can get*

$$N(z) = \begin{cases} e_1^2 + e_2^2 & , \text{if } d = 1, \\ e_1^2 + 2e_2^2 & , \text{if } d = 2, \\ e_1^2 + e_1e_2 + e_2^2 & , \text{if } d = 3, \\ e_1^2 + e_1e_2 + 2e_2^2 & , \text{if } d = 7, \\ e_1^2 + e_1e_2 + 3e_2^2 & , \text{if } d = 11. \end{cases}$$

Table 1.1. Elements of $\mathbb{E}_d$ with specific norm.

$N \backslash d$	1	2	3	7	11
0	0	0	0	0	0
1	± 1, $\pm i$	± 1	$\pm 1, \pm\omega$, $\pm(1-\omega)$	± 1	± 1
2	$1 \pm i$ $-1 \pm i$	$\pm\omega$	-	$\pm\omega$, $\pm(1-\omega)$	-
3	-	$1 \pm \omega$, $-1 \pm \omega$	$\pm(1+\omega)$, $\pm(2-\omega)$, $\pm(1-2\omega)$	-	$\pm\omega$, $\pm(1-\omega)$
4	± 2, $\pm 2i$	± 2	$\pm 2, \pm 2\omega$, $\pm 2(1-\omega)$	$\pm 2, \pm(2-\omega)$, $\pm(1+\omega)$	± 2
5	$2 \pm i$, $-2 \pm i$, $1 \pm 2i$, $-1 \pm 2i$	-	-	-	$\pm(1+\omega)$, $\pm(2-\omega)$
6	-	$2 \pm \omega$, $-2 \pm \omega$	-	-	-
7	-	-	$\pm(2+\omega)$, $\pm(1+2\omega)$, $\pm(1-3\omega)$, $\pm(2-3\omega)$, $\pm(3-2\omega)$, $\pm(3-\omega)$	$\pm(1-2\omega)$	-
8	$2 \pm 2i$, $-2 \pm 2i$	$\pm 2\omega$	-	$\pm 2\omega$, $\pm(2+\omega)$, $\pm(2-2\omega)$, $\pm(3-\omega)$	- -
9	± 3, $\pm 3i$	± 3, $1 \pm 2\omega$, $-1 \pm 2\omega$	± 3, $\pm 3\omega$, $\pm 3(1-\omega)$	± 3	± 3, $\pm(2+\omega)$, $\pm(3-\omega)$
10	$3 \pm i$, $-3 \pm i$, $1 \pm 3i$, $-1 \pm 3i$	-	-	-	-

Figure 1.1. Graphical representation of ω, when $d = 3, 7, 11$.



Remark 1.3.4. *Assume that* $a, b \in \mathbb{E}_d, b \neq 0$. *Let* $\mathbb{E}_d^*$ *be the set of units in* $\mathbb{E}_d$ *(*$\varepsilon \in \mathbb{E}_d^*$*, if and only if* $N(\varepsilon) = 1$*). Let* $q, r \in \mathbb{E}_d$ *be such that* $a = bq + r$ *and* $N(r) < N(b)$. *Then* $a = b(q+\varepsilon) + (r - b\varepsilon)$ *and* $a = b(q + \varepsilon\omega) + (r - b\varepsilon\omega)$ *hold for any* $\varepsilon \in \mathbb{E}_d^*$. *It means, in some cases the remainder* r *is not uniquely defined, i.e., not only* $N(r) < N(b)$*, but also* $N(r - b\varepsilon) < N(b)$ *or* $N(r - b\varepsilon\omega) < N(b)$ *holds for some* $\varepsilon \in \mathbb{E}_d^*$. *This problem has already arisen in the case of rational integers, where the uniqueness of the remainder is ensured by the assumption that the remainder is non-negative. In order to make the floor function uniquely*

defined, the solution is to define a special set of reminders which is a complete residue system modulo b.

In the next definition I propose specific digit sets for each d. The definition and its properties has been published in [76], in Section 3.

Definition 1.3.5. *Let $\mathbb{E}_d$ be an imaginary quadratic Euclidean domain and $0 \neq b \in \mathbb{E}_d$. The set*

$$\mathbb{D}_{d,b} := \left\{ z \in \mathbb{E}_d \;\middle|\; |z| < |b| \text{ and } |z+b| \geq |b| \text{ and } -\frac{1}{2} \leq Im_d\left(\frac{z}{b}\right) < \frac{1}{2} \right\}$$

*be called the **(Sail) digit set** for b and $b \in \mathbb{E}_d$ the **base number**.*

Remark 1.3.6. *In Definition 1.3.5 there are three conditions. The first is to make sure the norm of the digits are smaller than the norm of the base number. The second is to rule out the numbers which are "negative" in a sense (generalization of the assumption that the remainder should be non-negative). The last one is to reach a complete residue system (uniqueness).*

Remark 1.3.7. *The assumptions in Definition 1.3.5 ensure that if $b \in \mathbb{Z} \subseteq \mathbb{E}_d$ then $\{sgn(b)j \mid j = 0, \ldots, |b|-1\} \subseteq \mathbb{D}_{d,b}$.*

Remark 1.3.8. *The equation $Im_d\left(\frac{z}{b}\right) = s$ defines a line on the complex plane with the direction $arg(b)$ and offset $s \cdot Im(\omega) \cdot i$ $(s \in \mathbb{R}, b \in \mathbb{C})$. The equation $|z - a| = r$ defines a circle on the complex plane with the center $a \in \mathbb{C}$ and radius $r \in \mathbb{R}$.*

Definition 1.3.9. *For $0 \neq b \in \mathbb{E}_d$ the set*

$$\mathbb{V}_{d,b} := \left\{ z \in \mathbb{E}_d \;\middle|\; -\frac{1}{2} \leq Im_d\left(\frac{z}{b}\right) < \frac{1}{2} \right\}$$

*is called the **real band**.*

Theorem 1.3.10 is a common result with A. Pethő (see [76]). It states that the sail digit set $\mathbb{D}_{d,b}$ is a complete residue system modulo b.

Theorem 1.3.10. *Let $0 \neq b \in \mathbb{E}_d$. Then the set $\mathbb{D}_{d,b}$ is a complete residue system modulo b containing 0. Moreover for any $a \in \mathbb{E}_d$ there exist $q, r \in \mathbb{E}_d$ such that $a = bq + r$ and $r \in \mathbb{D}_{d,b}$, in particular $N(r) < N(b)$.*

Proof. As $a/b \in \mathbb{C}$ there exist $u_1, u_2 \in \mathbb{R}$ such that $\frac{a}{b} = u_1 + u_2\omega$. Write $u_i = q_i + r_i, i = 1,2$ such that $q_1, q_2 \in \mathbb{Z}$ and $-\frac{1}{2} \leq r_i < \frac{1}{2}$ and put $q' = q_1 + q_2\omega, r' = r_1 + r_2\omega$ and $r'' = br'$. Then $a = bq' + r''$ and $q' \in \mathbb{E}_d$, thus $r'' \in \mathbb{E}_d$. Further $Im_d\left(\frac{r''}{b}\right) = Im_d(r')$. Thus $-\frac{1}{2} \leq Im_d\left(\frac{r''}{b}\right) < \frac{1}{2}$.

Further $N(r'') = N(b)N(r')$, and by Remark 1.3.3 $N(r') \leq \frac{3}{4}$, if $d \leq 3$ and $N(r') \leq \frac{5}{4}$ in the remaining two cases. If $N(r') < 1$, then we have also the inequality $N(r'') < N(b)$. Assume that $N(r') \geq 1$, which can happen only if $d = 7, 11$ and $r_1 r_2 > 0$. Then redefine $r'' = b(r' + (-1)\frac{r_1}{|r_1|})$. Plainly we have $r'' \in \mathbb{E}_d$ such that $N(r'') < N(b)$ and $-\frac{1}{2} \leq Im_d\left(\frac{r''}{b}\right) < \frac{1}{2}$ hold.

Finally consider the sequence $r'' + mb, m = 0, 1, \ldots$. As the function $f(x) = N(r'' + xb)$ tends to infinity with $x \to \infty$ and $f(0) < N(b)$ there exists an $x_0 > 0$ such that $f(x_0) = N(b)$. Taking $m = \lfloor x_0 \rfloor$ we get $f(r'' + mb) < b$ and $f(r'' + (m+1)b) \geq b$. Putting $r = r'' + mb$ and $q = (a - r)/b$ we get $a = bq + r, q, r \in \mathbb{E}_d$ and $r \in \mathbb{D}_{d,b}$. As a was arbitrary $\mathbb{D}_{d,b}$ includes a complete residue system modulo b.

It remains to prove that the elements of $\mathbb{D}_{d,b}$ are incongruent modulo b. Assume that $a \in \mathbb{D}_{d,b}$ and $e := (e_1, e_2)_d \in \mathbb{E}_d \setminus \{0\}$ such that $a + eb \in \mathbb{D}_{d,b}$ holds too. Then both inequalities

$$-\frac{1}{2} \leq Im_d \frac{a}{b} < \frac{1}{2},\ -\frac{1}{2} \leq Im_d \frac{a+eb}{b} < \frac{1}{2}$$

hold. On the other hand $Im_d \frac{a+eb}{b} = Im_d \frac{a}{b} + e_2$, where e_2 is an integer. Thus both inequalities can hold only if $e_2 = 0$.

If $e_2 = 0$ then $eb = e_1 b$ with an integer e_1. Assume that $e_1 \neq 0$. If $|e_1| \geq 2$ then using $|a| < |b|$ we obtain $|a + e_1 b| \geq |e_1 b| - |a| > 2|b| - |b| \geq |b|$, which contradicts $a + e_1 b \in \mathbb{D}_{d,b}$. Hence $e_1 = \pm 1$.

If $e_1 = -1$ then as $a - b \in \mathbb{D}_{d,b}$ we get $|a| = |(a - b) + b| \geq |b|$, which contradicts $a \in \mathbb{D}_{d,b}$. Finally if $e_1 = 1$ then as $a + b \in \mathbb{D}_{d,b}$ we have $|a + b| < |b|$, which again contradicts $a \in \mathbb{D}_{d,b}$. The proof is completed.

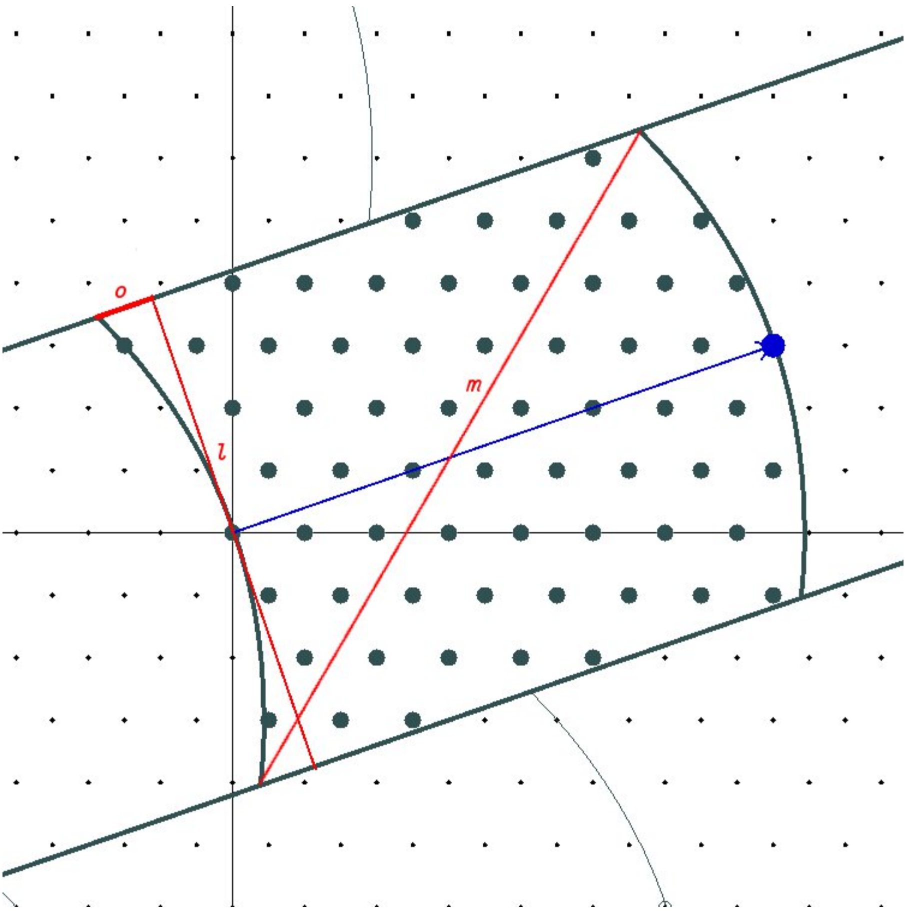
□

Let $a, b \in \mathbb{E}_d$ with $b \neq 0$. There exist by Theorem 1.3.10 uniquely defined $q \in \mathbb{E}_d$ and $r \in \mathbb{D}_{d,b}$ such that $a = bq + r$, so the sail digit set can be used as a complete residue system for the floor function defined in Definition 1.1.6.

Definition 1.3.11. *Let $\mathbb{D}_{d,b}$ be a sail digit set.*
Let's define the following notations for this digit set:
line distance: $l := Im(\omega)|b|$.

corner offset: $o := |b| - \sqrt{|b|^2 - \left(\frac{l}{2}\right)^2}$.
maximum distance between digits: $m := \sqrt{|b|^2 + l^2}$.

Figure 1.2. Digit set measures in $\mathbb{E}_3$, when the base of the digit set is $(6,3)_3$ (large dot). 'o': corner offset, 'l': line distance, 'm': maximum distance between digits



Remark 1.3.12. *Let $\mathbb{E}_d$ be a Euclidean domain, and let $b \in \mathbb{E}_d$ be the base of the digit set $\mathbb{D}_{d,b}$.*

d	*Corner offset (o)*	*Line distance (l)*	*Maximum distance between digits (m)*
1	$\lvert b\rvert\left(1-\frac{\sqrt{3}}{2}\right)$	$\lvert b\rvert$	$\lvert b\rvert\sqrt{2}$
2	$\lvert b\rvert\left(1-\frac{\sqrt{2}}{2}\right)$	$\lvert b\rvert\sqrt{2}$	$\lvert b\rvert\sqrt{3}$
3	$\lvert b\rvert\left(1-\frac{\sqrt{13}}{4}\right)$	$\lvert b\rvert\frac{\sqrt{3}}{2}$	$\lvert b\rvert\frac{\sqrt{7}}{2}$
7	$\lvert b\rvert\frac{1}{4}$	$\lvert b\rvert\frac{\sqrt{7}}{2}$	$\lvert b\rvert\frac{\sqrt{11}}{2}$
11	$\lvert b\rvert\left(1-\frac{\sqrt{5}}{4}\right)$	$\lvert b\rvert\frac{\sqrt{11}}{2}$	$\lvert b\rvert\frac{\sqrt{15}}{2}$

Lemma 1.3.13. *Let $a, b \in \mathbb{E}_d, b \neq 0$. If $|a| < \frac{Im(\omega)|b|}{2}$, then $a \in \mathbb{V}_{d,b}$.*

Proof. The assumption $|a| < \frac{Im(\omega)|b|}{2}$ implies $\left|\frac{a}{b}\right| < \frac{Im(\omega)}{2}$. As $|z| \geq |Im(z)|$ we get $\frac{|Im(a/b)|}{Im(\omega)} < \frac{1}{2}$, i.e. $a \in \mathbb{V}_{d,b}$. $\square$

Lemma 1.3.14. *Let $a, b \in \mathbb{E}_d$ with $N(b) \geq 2$. If $|a| < \frac{l}{2}$ and $q = \left\lfloor \frac{a}{b} \right\rfloor$ then $q \in \{0; -1\}$.*

Proof. Let $a = bq + r$ with $q \in \mathbb{E}_d$ and $r \in \mathbb{D}_{d,b}$. Then

$$Im_d(q) = Im_d\left(\frac{a}{b}\right) + Im_d\left(\frac{-r}{b}\right).$$

By the assumption on a and as $r \in \mathbb{D}_{d,b}$ we get

$$|Im_d(q)| < \frac{1}{2} + \frac{1}{2} = 1,$$

i.e. $q \in \mathbb{Z}$. Further we have

$$|bq| \leq |a| + |r| < \frac{Im\,(\omega)\,|b|}{2} + |b|.$$

Dividing by $|b|$ we obtain $|q| < \frac{\sqrt{11}}{4} + 1 < 2$, thus $q \in \{0, \pm 1\}$. If $q = 1$ then $a = b + r$. The assumption $r \in \mathbb{D}_{d,b}$ implies $|a| = |b + r| \geq |b|$, which contradicts the assumption on a. $\square$

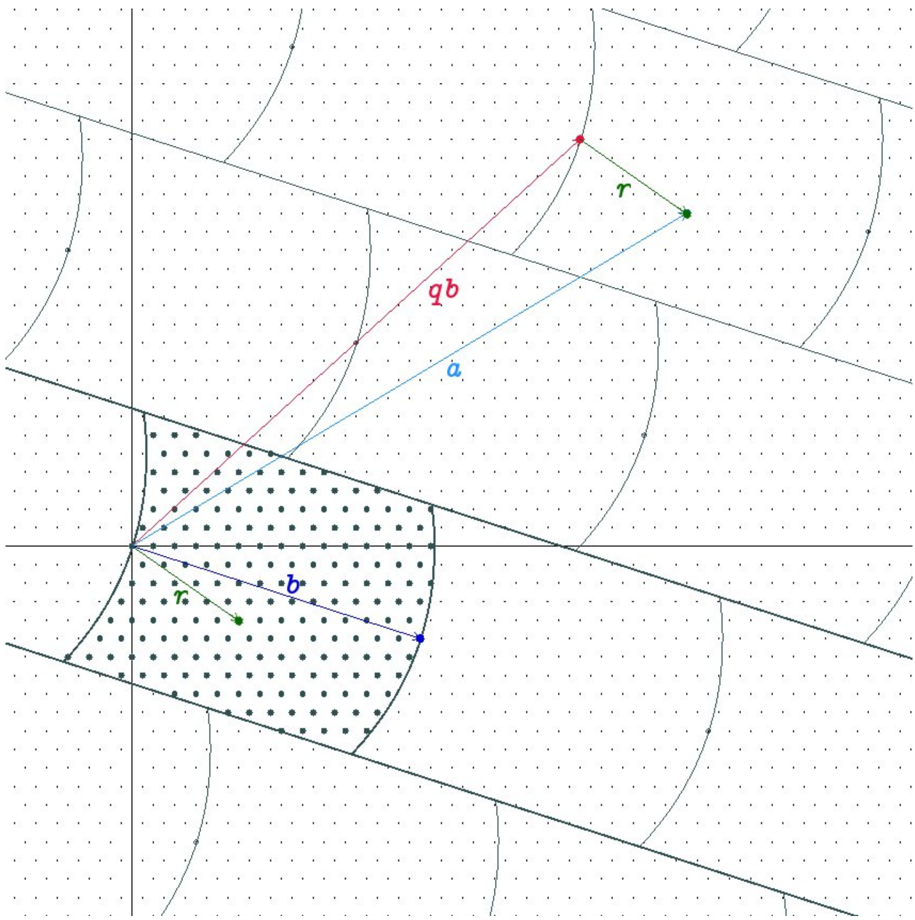
Lemma 1.3.15. *If $z \in \mathbb{V}_{d,b}$ and $a \in \mathbb{Z}$, then $z + a \cdot b \in \mathbb{V}_{d,b}$.*

Proof. We have

$$Im_d\left(\frac{z+ab}{b}\right) = Im_d\left(\frac{z}{b}\right) + Im_d(a) = Im_d\left(\frac{z}{b}\right),$$

which proves the assertion. □

Figure 1.3. Floor function results in $\mathbb{E}_3$ when $b = (16, -5)_3$ and $a = (17, 18)_3$. $\alpha \approx 1.41 + 1.61i$ which is approximately $\alpha \approx 0.48 + 1.86\omega$. The floor function will return $q = (0, 2)_3$ and $r = (7, -4)_3$.



1.4 Linear ENS over imaginary quadratic Euclidean domains

This section describes the results of Section 5 in [76]. Investigating the linear case, Theorem 1.4.6 below shows that the ENS property of linear polynomials is easily decidable over imaginary quadratic Euclidean domains. In this section I will often refer to the real band $\mathbb{V}_{d,p}$, which will be called, for simplicity, band.

Lemma 1.4.1. *Let $P(x) := x + p$ be a polynomial over $\mathbb{E}_d$ with $N(p) \geq 2$ and let $\mathbb{D}_{d,p}$ be the sail digit set. If the Line distance $l = Im(\omega)|p|$ is greater than 2, a necessary condition for the ENS property is $1 \in \mathbb{D}_{d,p}$.*

Proof. Assume that $1 \notin \mathbb{D}_{d,p}$.
The assumption $l > 2$ and Lemma 1.3.13 mean that $\mathbb{V}_{d,p}$ includes the closed unit disc, thus $1 \in \mathbb{V}_{d,p}$. Lemma 1.3.14, $|p| > 1$ and $1 \notin \mathbb{D}_{d,p}$ mean that $\left\lfloor \frac{1}{p} \right\rfloor = 1$, so

$$1 \underset{P}{\Rightarrow} 1,$$

which is a cycle, thus P cannot be an ENS polynomial with its sail digit set. □

Remark 1.4.2. *It is easy to check that $1 \in \mathbb{D}_{d,p}$ is equivalent to $Re(p) \geq -1/2$ except when*

$$p = \begin{cases} 1-i, -2i & : \quad d=1 \\ -\sqrt{-2} & : \quad d=2 \\ \pm\sqrt{-3}, 1-\sqrt{-3} & : \quad d=3 \\ \frac{\pm 1-\sqrt{-7}}{2} & : \quad d=7. \end{cases}$$

Lemma 1.4.3. *Let $P(x) := x + p$ be a polynomial over $\mathbb{E}_d$ with $N(p) \geq 2$ and let $\mathbb{D}_{d,p}$ be the sail digit set. To decide the ENS property those and only those polynomials have to be investigated, where*

$$\begin{aligned} A(x) \quad &:= \quad a \text{ with } a \in \mathbb{E}_d \text{ and} \\ |a| \quad &\leq \quad \sqrt{\frac{|p|+1}{|p|-1}}. \end{aligned}$$

Proof. Let $A \in \mathbb{E}_d[x]$. Consider the orbit of T_P, which starts from A. If $\lambda(T_P(A)) < \lambda(A)$ then iterate T_P. As $\lambda(A)$ is a non-negative number we have to reach an element B of the orbit such that $\lambda(T_P(B)) \geq \lambda(B)$. We may assume without loss of generality that this happens already at the beginning, i.e. with A. The length of $T_P(A)$ is

$$\lambda(T_P(A)) = |q| = \left|\frac{a-r}{p}\right| \leq \frac{|a|+|r|}{|p|}.$$

Thus $\lambda(A) \leq \lambda(T_P(A))$ implies

$$|a| \leq \frac{|a|+|r|}{|p|},$$

which leads to

$$|a| \leq \sqrt{\frac{|p|+1}{|p|-1}}$$

since $N(r) < N(p)$, that is $|r|^2 \leq |p|^2 - 1$. $\square$

Remark 1.4.4. *Lemma 1.4.3 is a special case of Lemma 1.2.18.*

The following theorem is my result (see [76]).

Theorem 1.4.5. *Let $P(x) := x+p$ be a linear polynomial over $\mathbb{E}_d$ with $N(p) \geq 2$ and let $\mathbb{D}_{d,p}$ be the sail digit set. If $Im(\omega)|p| = l > 2\sqrt{\frac{|p|+1}{|p|-1}}$, a sufficient and necessary condition for the ENS property is $1 \in \mathbb{D}_{d,p}$.*

Proof. From Lemma 1.4.3, those and only those constant polynomials a have to be investigated for the ENS property, where $|a| \leq \sqrt{\frac{|p|+1}{|p|-1}}$.
Since $\frac{l}{2} > \sqrt{\frac{|p|+1}{|p|-1}}$ we have $q = \left\lfloor \frac{a}{p} \right\rfloor \in \{0, -1\}$ by Lemma 1.3.14, thus all orbits of T_P reach either 0 or -1. If $1 \in \mathbb{D}_{d,p}$ then P is an ENS polynomial, and since $l > 2$ is also satisfied, from Lemma 1.4.1, this is not just sufficient but necessary condition as well. $\square$

Theorem 1.4.6 is my result (see [76]). It gives a sufficient and necessary condition for linear ENS polynomials with the sail digit set.

Theorem 1.4.6. *Let $P(x) := x+p$ be a linear polynomial over $\mathbb{E}_d$ and $N(p) \geq 2$ and let $\mathbb{D}_{d,p}$ be the sail digit set. $P(x)$ is an ENS polynomial with $\mathbb{D}_{d,p}$ if and only if $1 \in \mathbb{D}_{d,p}$ or*

$$p \in \left\{1-i, -2i, -\sqrt{-2}, \sqrt{-3}, -\sqrt{-3}, 1-\sqrt{-3}, \frac{1-\sqrt{-7}}{2}, \frac{-1-\sqrt{-7}}{2}\right\}.$$

Proof. By Lemma 1.4.3 it is enough to check the representability only those constant polynomials $A(x) = a$ with

$$|a| \leq \sqrt{\frac{|p|+1}{|p|-1}},$$

which implies

$$N(a) \leq \frac{|p|+1}{|p|-1} = 1 + \frac{2}{|p|-1} = 1 + \frac{2}{\sqrt{N(p)}-1}.$$

Table 1.2 presents the possible values of $N(a)$ for each $N(p)$.

$N(p)$	$1+\frac{2}{\sqrt{N(p)}-1}$	$N(a)$
2	≈ 5.8284	$\in \{0,1,2,3,4,5\}$
3	≈ 3.7321	$\in \{0,1,2,3\}$
4	$= 3.0000$	$\in \{0,1,2,3\}$
5	≈ 2.6180	$\in \{0,1,2\}$
6	≈ 2.3798	$\in \{0,1,2\}$
7	≈ 2.2153	$\in \{0,1,2\}$
8	≈ 2.0938	$\in \{0,1,2\}$
9	$= 2.0000$	$\in \{0,1,2\}$
≥ 10	< 2	$\in \{0,1\}$

Table 1.2

The necessary constant polynomials for the case $2 \leq N(p)$ will be investigated. By Theorem 1.4.5 if $\frac{l}{2} > \sqrt{\frac{|p|+1}{|p|-1}}$, a sufficient and necessary condition for the ENS property is $1 \in \mathbb{D}_{d,p}$. Thus it is enough to check the case $\frac{l}{2} \leq \sqrt{\frac{|p|+1}{|p|-1}}$.

Then

$$\begin{aligned}
\frac{l^2}{4} &\leq 1+\frac{2}{|p|-1}, \\
(l^2-4)(|p|-1) &\leq 8, \\
|p| &\leq \frac{l^2+4}{l^2-4}, \\
|p| &\leq \frac{(Im(\omega)|p|)^2+4}{(Im(\omega)|p|)^2-4}, \\
Im(\omega)^2|p|^3 - Im(\omega)^2|p|^2 - 4|p| - 4 &\leq 0, \\
Im(\omega)^2\sqrt{N(p)}^3 - Im(\omega)^2\sqrt{N(p)}^2 - 4\sqrt{N(p)} - 4 &\leq 0.
\end{aligned}$$

The cubic polynomial in $\sqrt{N(p)}$ staying on the left hand side of the last inequality has exactly one positive real root. The possible values of $N(p)$ lie between zero and this root. Let's present these in Table 1.3.

d	$N(p) <$	$N(p) \in$
1	8.2664	$\{2,3,4,5,6,7,8\}$
2	5.1508	$\{2,3,4,5\}$
3	10.1968	$\{2,3,4,5,6,7,8,9,10\}$
7	5.6206	$\{2,3,4,5\}$
11	4.2163	$\{2,3,4\}$

Table 1.3

For each triplets (d,p,a) with $d \in \{1,2,3,7,11\}$, $p,a \in \mathbb{E}_d$ such that $N(p)$ and $N(a)$ satisfy the conditions of Table 1.3 and Table 1.2 respectively I checked the representability of a. The investigation of all possible triplets (d,p,a) can be found in the Appendix.

To summarize, if $1 \in \mathbb{D}_{d,p}$ then $x+p$ is an ENS polynomial. If $1 \notin \mathbb{D}_{d,p}$ then $x+p$ is an ENS polynomial, if and only if

$$p \in \left\{1-i, -2i, -\sqrt{-2}, \sqrt{-3}, -\sqrt{-3}, 1-\sqrt{-3}, \frac{1-\sqrt{-7}}{2}, \frac{-1-\sqrt{-7}}{2}\right\}.$$

$\square$

1.5 Quadratic ENS over imaginary quadratic Euclidean domains

This section describes the results of Section 6 in [76]. The characterization of quadratic ENS polynomials with the sail digit set seems to be much more difficult than the characterization of the linear ones. In the present section this problem will be investigated.

The first theorem is my result (see [76]). It determines the set of possible quadratic ENS polynomials to a finite set for a fixed constant term p_0 using the sail digit set $\mathbb{D}_{d,p_0}$.

Theorem 1.5.1. *Let $P(x) := x^2 + p_1 x + p_0$ be a quadratic polynomial over $\mathbb{E}_d$, $N(p_0) \geq 2$. It is expanding, if*

$$\frac{|\bar{p_1} - \bar{p_0} p_1|}{|p_0|^2 - 1} < 1,$$

where $\bar{x}$ is the complex conjugate of x.

Proof. This result comes from the Lehmer-Schur [61] algorithm. Let

$$P^*(x) = \bar{p_0} x^2 + \bar{p_1} x + 1, \text{ and}$$

$$g(x) = \bar{p_0} P(x) - P^*(x) = (\bar{p_0} p_1 - \bar{p_1}) x + \bar{p_0} p_0 - 1.$$

The root of $g(x)$ is:

$$x_0 = \frac{1 - \bar{p_0} p_0}{\bar{p_0} p_1 - \bar{p_1}}.$$

Thus $P(x)$ is expanding if and only if $|x_0| > 1$, i.e.

$$1 < |x_0| = \left| \frac{1 - \bar{p_0} p_0}{\bar{p_0} p_1 - \bar{p_1}} \right| = \frac{||p_0|^2 - 1|}{|\bar{p_1} - \bar{p_0} p_1|} = \frac{|p_0|^2 - 1}{|\bar{p_1} - \bar{p_0} p_1|}.$$

□

Remark 1.5.2.
For a fixed p_0 the inequality of Theorem 1.5.1 determines a finite set of p_1. We have

$$\frac{|\bar{p_1} - \bar{p_0} p_1|}{|p_0|^2 - 1} \geq \frac{|p_0|\,|p_1| - |p_1|}{|p_0|^2 - 1} = \frac{|p_1|}{|p_0| + 1}.$$

Hence if $|p_1| \leq |p_0| + 1$, then the inequality of Theorem 1.5.1 follows.

Remark 1.5.3. *In order to investigate the ENS property for a given quadratic polynomial P using the sail digit set, Lemma 1.2.18 gives bounds for the coefficients of the polynomial A in case of $|p_0| \geq |p_1| + 1$, as follows:*

$$|A_0| \leq \sqrt{|p_0|^2 - 1}\frac{|p_1| + 1}{|p_0| - |p_1| - 1},$$

$$|A_1| \leq \sqrt{|p_0|^2 - 1}\frac{\frac{|p_1|}{|p_0|}(|p_1| + 1) + |p_1| + 2}{|p_0| - |p_1| - 1}.$$

The next theorem is my result (see [76]). It determines a set of quadratic ENS polynomials with the sail digit set $\mathbb{D}_{d,p_0}$.

Theorem 1.5.4. *Let $P(x) := x^2 + p_1 x + p_0$ be a quadratic polynomial over $\mathbb{E}_d$, $N(p_0) \geq 2$ and let $\mathbb{D}_{d,p_0}$ be the sail digit set. If*

$$|p_1| \leq \left(1 - \frac{1}{\sqrt{2}}\right)|p_0| - 1,$$

then the orbits of T_P are periodic for all $A \in \mathbb{E}_d[x]$. Moreover there are only four possible periods, the trivial $\{0\}$ cycle and the following ones:

$$x + (p_1 + 1) \xRightarrow[P]{(-1,r_0)} x + (p_1 + 1), \quad r_0 \in \mathbb{D}_{d,p_0},$$

$$1 \xRightarrow[P]{(-1,r_0)} x + p_1 \xRightarrow[P]{(0,r_1)} 1, \quad r_0, r_1 \in \mathbb{D}_{d,p_0},$$

$$1 \xRightarrow[P]{(-1,r_0)} x + p_1 \xRightarrow[P]{(-1,r_1)} x + (p_1 + 1) \xRightarrow[P]{(0,r_2)} 1, \quad r_0, r_1, r_2 \in \mathbb{D}_{d,p_0}.$$

Proof. Assume that $A(x) = a_1 x + a_0 \in \mathbb{E}_d[x]$ with T_P leads to a period of length $n \geq 2$. Then by Theorem 1.2.7 the inclusions

$$-q_{j-2} - p_1 q_{j-1} - p_0 q_j \in \mathbb{D}_{d,p_0}$$

hold for $j = 0, 1, \ldots, n-1$, where I used $q_{-2} = q_{n-2}$ and $q_{-1} = q_{n-1}$. For a fixed p_0 these conditions can be transformed to a restriction for the linear term p_1. In fact if $q_j \neq 0$ then

$$p_1 \in \frac{\mathbb{D}_{d,p_0} + p_0 q_k + q_i}{-q_j},$$

and this is a conformal mapping of the digit set. If $q_j = 0$, then

$$q_k = \left\lfloor \frac{-q_i}{p_0} \right\rfloor,$$

which is a restriction for the position of p_0.

Let's check p_1's minimal absolute value in the intersection of the sets $\frac{\mathbb{D}_{d,p_0}+p_0q_k+q_i}{-q_j}$. If the cycle does not contain 0 multiplier, let h be the index of the multiplier which has maximal absolute value: $|q_h| \geq |q_i|, i \in \{0, 1, \ldots, n-1\}$.

$$\begin{aligned}
\min |p_1| &= \min \left\{ |t| \ : \ t \in \bigcap \frac{\mathbb{D}_{d,p_0} + p_0q_k + q_i}{-q_j} \right\} \\
&\geq \min \left\{ |t| \ : \ t \in \frac{\mathbb{D}_{d,p_0} + p_0q_h + q_{h-2}}{-q_{h-1}} \right\} \\
&= \min \left\{ \frac{|t + p_0q_h + q_{h-2}|}{|q_{h-1}|} \ : t \in \mathbb{D}_{d,p_0} \right\} \\
&\geq \min \left\{ \frac{|p_0||q_h| - |q_{h-2}| - |t|}{|q_{h-1}|} \ : \ t \in \mathbb{D}_{d,p_0} \right\} \\
&> \frac{|p_0||q_h| - |q_{h-2}| - |p_0|}{|q_{h-1}|} \\
&\geq \frac{|p_0||q_h| - |q_h| - |p_0|}{|q_h|} \\
&= \left(1 - \frac{1}{|q_h|}\right)|p_0| - 1.
\end{aligned}$$

This value increases, if $|q_h|$ increases. If the period does not contain 0 and contains at least one element with absolute value greater than one then the smallest value of $|q_h|$ is $\sqrt{2}$, which implies

$$|p_1| > \left(1 - \frac{1}{\sqrt{2}}\right)|p_0| - 1.$$

If the period contains a 0 multiplier, the above inequality holds, except when $q_{h-1} = 0$. In such a case we have

$$q_h = \left\lfloor \frac{-q_{h-2}}{p_0} \right\rfloor,$$

thus $|q_h||p_0| = |-q_{h-2} - r| < |q_{h-2}| + |p_0|$. As $|q_{h-2}| \leq |q_h|$ we get

$$\frac{|q_h|}{|q_h| - 1} > |p_0|.$$

The expression $\frac{|q_h|}{|q_h|-1}$ decreases if $|q_h|$ increases. The lowest possible value of it is $|q_h| = \sqrt{2}$, whence

$$\sqrt{12} > \frac{\sqrt{2}}{\sqrt{2}-1} > |p_0|.$$

If $|p_0| < \sqrt{12}$, then the disc $|p_1| \leq \left(1 - \frac{1}{\sqrt{2}}\right)|p_0| - 1 \leq \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{11} - 1 \approx -0.02858$ has no element. With our assumption the expression $\frac{|q_h|}{|q_h|-1} > |p_0|$ has no solution, so there is no period with $q_{h-1} = 0$.

So the periods in this region can contain elements only with absolute value 0 or 1.

Let's check the conditions $p_1 \in \frac{\mathbb{D}_{d,p_0}+p_0q_k+q_i}{-q_j}$ and $q_k = \left\lfloor \frac{-q_i}{p_0} \right\rfloor$ $(q_j = 0)$ again. If $|q_j| = 1$, then $q_k \in \{0, -1\}$, because in every other cases the minimum absolute value of p_1 will be outside the examined region:

(If $|q_k| = 1$, but $q_k \neq -1$)

From Theorem 1.3.14 elements of the set $\mathbb{D}_{d,p_0}+p_0q_k$ have absolute value greater than $\frac{l}{2}$, and in every Euclidean domain $\frac{l}{2} \geq \left(1 - \frac{1}{\sqrt{2}}\right)|p_0|$.

$$\min|p_1| = \min\left\{|t| \;:\; t \in \bigcap \frac{\mathbb{D}_{d,p_0} + p_0q_n + q_l}{-q_m}\right\} \geq$$

$$\geq \min\left\{|t| \;:\; t \in \frac{\mathbb{D}_{d,p_0} + p_0q_k + q_i}{-q_j}\right\} = \min\left\{\left|\frac{t + p_0q_k + q_i}{-q_j}\right| \;:\; t \in \mathbb{D}_{d,p_0}\right\} \geq$$

$$\geq \min\left\{|t + p_0q_k| - |q_i| \;:\; t \in \mathbb{D}_{d,p_0}\right\} \geq \min\left\{|t + p_0q_k| - 1 \;:\; t \in \mathbb{D}_{d,p_0}\right\} \geq$$

$$\geq \left(1 - \frac{1}{\sqrt{2}}\right)|p_0| - 1.$$

If $|q_j| = 0$, then $q_k = -1$, because $q_k = \left\lfloor \frac{-q_i}{p_0} \right\rfloor$, q_i is a unit or zero, so in every Euclidean domain, for every sail digit set $\left\lfloor \frac{-q_i}{p_0} \right\rfloor \in \{-1, 0\}$ (Theorem 1.3.14), but zero is not possible, because then two 0-s are there next to each other, which means $p_0q_k \in \mathbb{D}_{d,p_0}$ and this is impossible.

If three equal values are next to each other in a cycle, then the whole period is constructed, because every multiplier is uniquely determined by the previous two values. So the periods with multipliers $(-1), (0, -1), (0, -1, -1)$ are the only possible periods in the examined region, these will be the witnesses for the ENS property. $\square$

1.6 Infinite sequences of ENS over imaginary quadratic Euclidean domains

This section describes the results of Section 7 in [76].

Polynomials with rational integer coefficients can be considered also elements of $\mathbb{E}_d[x]$. In this section I prove a necessary and sufficient condition under which such a polynomial is ENS with its sail digit set. The second aim is to prove a simple sufficient condition in terms of the coefficient. The later result implies that there exist for any degree infinitely many ENS polynomials.

To formulate the results I need some preparation. Let $P(x) \in \mathbb{Z}[x]$ with $P(0) = p_0$ and $I = \left[-\left\lfloor \frac{|p_0|-1}{2} \right\rfloor, |p_0| - 1 - \left\lfloor \frac{|p_0|-1}{2} \right\rfloor\right] \cap \mathbb{Z}$. S. Akiyama and K. Scheicher [14] called $P(x)$ symmetric-CNS if for any $A(x) \in \mathbb{Z}[x]$ there exists $a(x) \in I[x]$ such that $A(x) \equiv a(x) \pmod{P}$. Theorem 1.6.1 is a common result with A. Pethő (see [76]). It gives an interesting connection between CNS, symmetric-CNS and ENS polynomials with the sail digit set.

Theorem 1.6.1. *Let $P(x) \in \mathbb{Z}[x]$ with $p_0 > 0$. If $P(x)$ is a CNS and symmetric-CNS in $\mathbb{Z}[x]$ then it is ENS in $\mathbb{E}_d[x]$ with the sail digit set $\mathbb{D}_{d,p_0}$. The conversion is true if $d = 1, 2$.*

Proof. Assume first that $P(x)$ is a CNS and symmetric-CNS in $\mathbb{Z}[x]$. Let $A(x) \in \mathbb{E}_d[x]$. There exist $A_1(x), A_2(x) \in \mathbb{Z}[x]$ such that $A(x) = A_1(x) + \omega A_2(x)$. As $P(x)$ is a symmetric-CNS there exist $a_2(x) \in I[x], q_2(x) \in \mathbb{Z}[x]$ such that $A_2(x) = a_2(x) + q_2(x)P(x)$. Let

$$a_2(x) = \sum_{j=0}^{m_2} a_{2j} x^j.$$

Assume that the first $j \geq -1$ coefficients of $A_1(x) + \omega a_2(x)$ belong to $\mathbb{D}_{d,p_0}$. This is obviously true for $j = -1$ because the coefficient of our polynomial with index -1 is zero, which belongs to $\mathbb{D}_{d,p_0}$. Let its $j+1$-th coefficient be $\beta = A_{1,j+1} + \omega a_{2,j+1}$. There exists by Theorem 1.3.5 a $\beta_1 \in \mathbb{D}_{d,p_0}$ such that $\beta_1 \equiv \beta \pmod{p_0}$. We have $\beta_1 - \beta \in \mathbb{Z}$ because $a_{2,j+1} \in I$ and $p_0 \in \mathbb{Z}$. Thus $(\beta_1 - \beta)/p_0 \in \mathbb{Z}$. Denote it by q and set $A(x) \leftarrow A(x) + qP(x)x^{j+1}$. This transformation does not affect $a_2(x)$, but the first $j+1$ coefficients of $A(x)$ belong to $\mathbb{D}_{d,p_0}$.

Performing the transformation of the last paragraph $m_2 + 1$-times we obtain a polynomial $a_1^{(1)}(x) + a_1^{(2)}(x)x^{m_2+1} + \omega a_2(x) \equiv A(x) \pmod{P(x)}$ such that

$a_1^{(1)}(x)+\omega a_2(x) \in \mathbb{D}_{d,p_0}[x]$ and $a_1^{(2)}(x) \in \mathbb{Z}[x]$. As $P(x)$ is a CNS polynomial in $\mathbb{Z}[x]$ there exists $a_1^{(3)}(x)$ with coefficients in $\{0,1,\dots,p_0-1\}$, which is a subset of $\mathbb{D}_{d,p_0}$, such that $a_1^{(2)}(x) \equiv a_1^{(3)}(x) \pmod{P(x)}$. Setting $a_1(x) = a_1^{(1)}(x) + a_1^{(3)}(x)x^{m_2+1}$ and $a(x) = a_1(x)+\omega a_2(x)$ we have that $A(x) \equiv a(x) \pmod{P(x)}$ and the coefficients of $a(x)$ belong to $\mathbb{D}_{d,p_0}$. Thus the conditions are sufficient.

Assume that $P(x)$ is ENS in $\mathbb{E}_d[x]$. Then for any $A(x) \in \mathbb{E}_d[x]$ there exists $a(x) \in \mathbb{D}_{d,p_0}[x]$ such that $A(x) \equiv a(x) \pmod{P(x)}$. Write $a(x) = a_1(x) + \omega a_2(x)$. Then the coefficients of a_2 belong obviously to I. If $d = 1, 2$ then the coefficients of $a(x)$ have the form $e_1 + e_2\sqrt{-d}$, which absolute value is $\sqrt{e_1^2 + de_2^2} < p_0$. Thus $|e_1| < p_0$ and $e_1 \geq 0$ because $|(e_1+p_0)+e_2\sqrt{-d}| > p_0$. □

To characterize the CNS polynomials in $\mathbb{Z}[x]$ is a hard problem, see [2]. However there is a simple sufficient criterion proved by B. Kovács [49], which I cite now.

Theorem 1.6.2. *Let $P(x) = p_0 + p_1x + \cdots + p_{n-1}x^{n-1} + x^n \in \mathbb{Z}[x]$ be a polynomial. If $p_0 \geq 2$ and $p_i > p_{i+1}, i = 0,\dots,n-1$, then $P(x)$ is a CNS polynomial.*

L. Germán and A. Kovács in [33] investigated the case of symmetric-CNS. Let $P(x) = p_0 + p_1x + \cdots + p_{n-1}x^{n-1} + x^n \in \mathbb{Z}[x]$ be a polynomial. If $|p_0| > 2\sum_{i=1}^n |p_i|$, then $P(x)$ is a symmetric-CNS. Indeed, the polynomials $x^2 + ax + a, 3 \leq a \in \mathbb{Z}$ are CNS by Theorem 1.6.2, but they are not symmetric-CNS, however the polynomials $x^2 + ax + 3a, 3 \leq a \in \mathbb{Z}$ are CNS, symmetric-CNS and ENS polynomials as well with the sail digit set. In the next lemma I prove a condition, which depends only on the coefficients of $P(x)$. In its proof I borrowed ideas from [11]. In the sequel set $M = \left\lfloor \frac{p_0-1}{2} \right\rfloor$ and $I = \left[-\left\lfloor \frac{p_0-1}{2} \right\rfloor, p_0 - 1 - \left\lfloor \frac{p_0-1}{2} \right\rfloor\right] \cap \mathbb{Z}$.

Lemma 1.6.3. *Let $P(x) = p_0 + p_1x + \cdots + p_{n-1}x^{n-1} + p_nx^n \in \mathbb{Z}[x]$ be a polynomial such that $M \geq p_1 \geq p_2 \geq \cdots \geq p_n = 1$ and*

$$\sum_{j=2}^{n} p_j \leq M.$$

Then $P(x)$ is a symmetric-CNS.

Proof. By Theorem 1.6.2 we may assume that $a_j \in [0, p_0 - 1], j = 0,\dots,k$. Let $J = [-p_0, p_0 + M - 1] \cap \mathbb{Z}$. For polynomials $a(x) \in \mathbb{Z}[x]$ with constant term a_0

define the mapping $U(a) = U_P(a) : \mathbb{Z}[x] \mapsto \mathbb{Z}[x]$ as

$$U(a) = \frac{a - \varepsilon P - (a_0 - \varepsilon p_0)}{x},$$

where ε denotes the unique integer with

$$\varepsilon p_0 \leq a_0 + M < (\varepsilon + 1)p_0.$$

Notice that if the coefficients of a belong to J then

$$a = r + xU(a) + \varepsilon P, \tag{1.2}$$

where $r = a_0 - \varepsilon p_0$ and $\varepsilon \in \{0, \pm 1\}$ is the coefficient of P in the definition of $U(a)$. Further it is clear that if $a_0 \in J$ then $r \in I$. Thus the lemma will be proved when we are able to show that for all $a \in J[x]$ there exists $m > 0$ such that $U^m(a) \equiv 0 \pmod{P}$.

We claim that if the coefficients of $a(x) \in \mathbb{Z}[x]$ belong to $[0, p_0 - 1]$ then $U^\ell(a) \in J[x]$ hold for $\ell \geq 0$. To prove the claim we have to examine the coefficients of $U^\ell(a)$ carefully.

Let $U^\ell(a) = \sum_{j=0}^\infty a_j^{(\ell)} x^\ell$. (Of course the number of non-zero coefficients of $U^\ell(a)$ is finite, thus there exists $j_0 = j_0()\ell$ such that $a_j^{(\ell)} = 0$ for all $j > j_0$. We use the same convention for $U^0(a) = a$ and for P too, i.e. we set $p_j = 0$ for $j > n$. Then we have

$$a_j^{(\ell)} = a_{\ell+j} - \sum_{h=1}^{\ell} \varepsilon^{(h)} p_{\ell+j-h+1},\ j, \ell \geq 0, \tag{1.3}$$

where $\varepsilon^{(s)} = 0$, if $s < 0$ and for $s \geq 0$ it is defined by the equation

$$U^{(s-1)}(a) = r_s + xU^{(s)} + \varepsilon^{(s)} P,$$

with $r_s \in I$.

Equation (1.3) is obviously true for $\ell = 0$. Assume that it is true for all $s \leq \ell$. Set $\varepsilon^{(\ell+1)}$ according to the size of $a_0^{(\ell)}$. Then we have

$$\begin{aligned} U^{(\ell+1)}(a) &= \frac{U^{(\ell)}(a) - a_0^{(\ell)} - \varepsilon^{(\ell+1)}(P - p_0)}{x} \\ &= \sum_{j=1}^\infty a_j^{(\ell)} x^{j-1} - \varepsilon^{(\ell+1)} \sum_{j=1}^\infty p_j x^{j-1} \\ &= \sum_{j=0}^\infty (a_{j+1}^{(\ell)} - \varepsilon^{(\ell+1)} p_{j+1}) x^j. \end{aligned}$$

Comparing coefficients and using (1.3) we obtain

$$\begin{aligned} a_j^{(\ell+1)} &= a_{\ell+j+1} - \sum_{h=1}^{\ell} \varepsilon^{(h)} p_{\ell+j+2-h} - \varepsilon^{(\ell+1)} p_{j+1} \\ &= a_{j+\ell+1} - \sum_{h=1}^{\ell+1} \varepsilon^{(h)} p_{\ell+j+2-h}, \end{aligned}$$

which is (1.3) for $\ell+1$, i.e. (1.3) is true for all $\ell, j \geq 0$.

Now we are in the position to prove the claim. Assume that the coefficients of $a(x) \in \mathbb{Z}[x]$ belong to $[0, p_0 - 1]$, i.e. $0 \leq a_j = a_j^{(0)} < p_0$. Thus the claim is true for $\ell = 0$ and $\varepsilon^{(1)} \in \{0, \pm 1\}$. Let $\ell \geq 1$ and assume that the claim and $\varepsilon^{(j)} \in \{0, \pm 1\}$ hold for $1 \leq j < \ell$. Then

$$\varepsilon^{(\ell)} = \left\lfloor \frac{a_0^{(\ell-1)} - p_0/2}{p_0} \right\rfloor,$$

which belongs to he set $\{0, \pm 1\}$ because by the induction hypothesis $-p_0 \leq a_0^{(\ell-1)} \leq p_0 + M - 1$. By (1.3) we have

$$a_j^{(\ell)} = a_{\ell+j} - \sum_{h=1}^{\ell} \varepsilon^{(h)} p_{\ell+j-h+1}.$$

Plainly the sum of the right hand side is at least

$$0 - \sum_{h=1}^{n} p_h = -(p_1 + \sum_{h=2}^{n} p_h) \geq -2M > -p_0.$$

To finish the induction we have to prove the upper bound for $a_j^{(\ell)}$. Assume that $\varepsilon^{(m)} = -1$ for some $m \leq \ell$. Then $a_0^{(m-1)} < M$. We have

$$\begin{aligned} a_0^{(m-1)} &= a_{m-1} - \sum_{h=1}^{m-1} \varepsilon^{(h)} p_{m-h} \\ &\geq 0 - \varepsilon^{(m-1)} p_1 - \sum_{h=2}^{n} p_h \\ &\geq -\varepsilon^{(m-1)} p_1 - M. \end{aligned}$$

Thus $a_0^{(m-1)} < -M$ can hold only if $\varepsilon^{(m-1)} = 1$. Applying again (1.3) and using the induction hypothesis and this observation we get

$$a_j^{(\ell)} \le p_0 - 1 + \sum_{h=1}^{n} (-1)^{h+1} p_h = p_0 - 1 + p_1 - (p_2 - p_3) - \cdots \le p_0 + M - 1. \quad (1.4)$$

Here we used the monotonicity of the coefficients as well. The claim is proved completely.

If $U^{(k+1)}(a) = 0$ then the Lemma is proved. Assume in the sequel $U^{(k+1)}(a) \neq 0$. Then the inequality in (1.4) can be considerably improved. Indeed as $a_\ell = 0, \ell > k$ we get

$$a_j^{(\ell)} \le M$$

for all $j \ge 0$. The degree of the polynomial $U^{(k+1)}(a)$ is at most n and its coefficients belong to $[-2M, M]$. Thus $U^{n+k+2}(a) \in I[x]$ and the lemma is proved. □

The following theorem is a common result with A. Pethő (see [76]). It gives infinite sequences of ENS polynomials over $\mathbb{Z}$ with the sail digit set.

Theorem 1.6.4. *Let* $P(x) := \sum_{i=0}^{n} p_i x^i \in \mathbb{Z}[x]$ *be a monic polynomial of degree* n. *Put* $M = \left\lfloor \frac{p_0 - 1}{2} \right\rfloor$ *and assume* $p_0 \ge M \ge p_1 \ge p_2 \ge \cdots \ge p_n = 1$ *and*

$$\sum_{j=2}^{n} p_j \le M.$$

Then $P(x)$ *is an ENS polynomial with the sail digit set* $\mathbb{D}_{d,p_0}$.

Proof. By Lemma 1.6.3, starting from a general polynomial one can determine a polynomial which is equivalent to the original modulo $P(x)$, and the imaginary part of the coefficients of the new polynomial belong to the interval $]-\left\lfloor \frac{p_0-1}{2} \right\rfloor, p_0 - 1 - \left\lfloor \frac{p_0-1}{2} \right\rfloor]$ (coefficients on the real band property).

For the real part an iteration can be started using the following transformation. In every step the investigated polynomial $A(x)$ will be changed, such that

$$A := T_P(A) := \frac{A - q \cdot P - r}{x},$$

where $q := \left\lfloor \frac{a_0}{p_0} \right\rfloor$. It is easy to see that $q \in \mathbb{Z}$, because of the coefficients on the real band property, this means that if one wants to move a coefficient to the digit set an integer times p_0 has to be added. After some iteration of this transformation all of the original coefficients of the polynomial $A(x)$ will be moved into the digit set, in every step the newly created coefficients are rational integers. So after finitely many steps $A(x)$ becomes a polynomial with rational integer coefficients. Polynomial $P(x)$ satisfies the assumptions of Theorem 1.6.2, thus it is CNS. From this point on we can use Theorem 1.6.2 to get an $A(x) \in \mathbb{D}_{d,p_0}[x]$ because the integer canonical digit set of the integer CNS polynomial $P(x)$ is a subset of $\mathbb{D}_{d,p_0}$ (see Remark 1.3.7).

$\square$

Remark 1.6.5. *Theorem 1.6.4 is a consequence of Theorem 1.6.1, Theorem 1.6.2 and [33].*

Chapter 2

ESRS

The results of this chapter are essentially the same as those of [77]. The concept *shift radix system* (SRS) was introduced by S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. M. Thuswaldner [2] for real numbers as follows. For $\mathbf{r} \in \mathbb{R}^n$ the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^n \mapsto \mathbb{Z}^n$, defined as

$$\tau_{\mathbf{r}}((a_1, \ldots, a_n)) = (a_2, \ldots, a_n, -\lfloor \mathbf{ra} \rfloor),$$

where $\mathbf{ra}$ denotes the inner product, is called SRS. This chapter generalizes this concept for hermitian vector spaces.

2.1 Basic concepts

In order to establish a shift radix system over the complex numbers, an imaginary quadratic Euclidean domain will be used as the set of integers, and a floor function is needed which can be determined by making its Euclidean function unique, so choosing the set of fractional numbers from the possible values.

In order to define a floor function, a set of fractional numbers has to be defined. Regarding generalization purposes the absolute value of a fractional number should be less than 1, a fractional number should not be negative in a sense, it is a superset of the fractional numbers for the reals, and the floor function should be unambiguous. From these considerations the following definition will be used to specify the floor function with the set of fractional numbers which will be called fundamental sail tile.

Definition 2.1.1. *Let $d \in \{1, 2, 3, 7, 11\}$. Let the set*

$$\mathbb{D}_d := \left\{ c \in \mathbb{C} \;\middle|\; |c| < 1 \text{ and } |c+1| \geq 1 \text{ and } -\frac{1}{2} \leq Im_d(c) < \frac{1}{2} \right\}$$

*be defined as the **fundamental sail tile** (the set of fractional numbers). Let $p \in \mathbb{E}_d$. The set*

$$\mathbb{D}_d(p) := \left\{ p + c \;\middle|\; c \in \mathbb{C} \text{ and } |c| < 1 \text{ and } |c+1| \geq 1 \text{ and } -\frac{1}{2} \leq Im_d(c) < \frac{1}{2} \right\}$$

*is called p-**sail tile** and p is called its **representative integer**.*

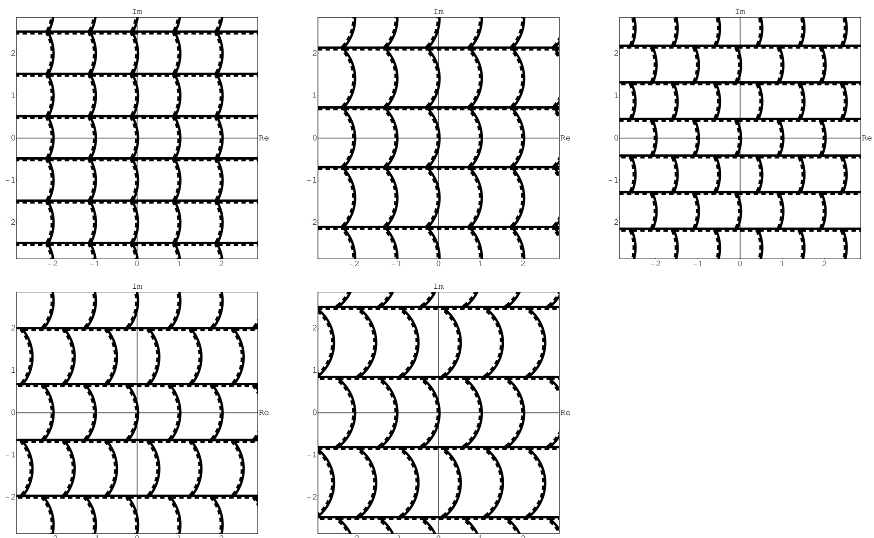


Figure 2.1. Tilings of $\mathbb{C}$ given by the sets $\mathbb{D}_d(p)$, $d \in \{1, 2, 3, 7, 11\}$.

By using Theorem 1.3.10 one can show that the sets $\mathbb{D}_d(p)$, where p runs through $\mathbb{E}_d$ do not overlap and cover the complex plain $\mathbb{C}$. This justifies the following definition:

Definition 2.1.2. *Let the function $\lfloor\ \rfloor_d : \mathbb{C} \to \mathbb{E}_d$ be defined as the **floor function**. The floor of e is the representative integer p of the unique p-sail tile that contains e.*

The next lemma shows that the above defined floor function can be described with the well-known floor function over the real numbers.

Lemma 2.1.3.

$$\lfloor e \rfloor_d = \begin{cases} \left\lfloor Re(e) - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Re(\omega) \right\rfloor + \omega \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor, \textit{ if} \\ \qquad \Big(Re(e) - \left\lfloor Re(e) - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Re(\omega) \right\rfloor - \\ \qquad - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Re(\omega) \Big)^2 + \\ \qquad + \left(Im(e) - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Im(\omega) \right)^2 < 1, \\ \left\lfloor Re(e) - \left\lfloor Im_d(e) + \frac{1}{2} \right\rfloor Re(\omega) \right\rfloor + \omega \left\lfloor Im_d(e) + \frac{1}{2} + 1 \right\rfloor, \textit{ otherwise.} \end{cases}$$

Proof. This lemma is a trivial consequence of the Definition 2.1.1 and the Definition 2.1.2. □

Equipped with the appropriate floor functions let's define shift radix systems for Hermitian vectors. The notion depends on the imaginary Euclidean domain.

Definition 2.1.4. *Let $C := (c_1, \ldots, c_n) \in \mathbb{C}^n$ be a complex vector. Let $d \in \{1, 2, 3, 7, 11\}$ and let $\lfloor x \rfloor_d$ denote the floor function defined above.*
For all vectors $A := (a_1, a_2, \ldots, a_n) \in \mathbb{E}_d^n$ let

$$\mathcal{T}_{d,C}(A) := (a_2, \ldots, a_n, -q),$$

*where $q = \lfloor c_1 a_1 + c_2 a_2 + \cdots + c_n a_n \rfloor_d$. The mapping $\mathcal{T}_{d,C} : \mathbb{E}_d^n \mapsto \mathbb{E}_d^n$ is called **Euclidean shift radix system with parameter** d or $ESRS_d$ respectively, ESRS for short. If $B := \mathcal{T}_{d,C}(A)$, this mapping will be denoted by*

$$A \underset{d,C}{\Rightarrow} B.$$

If for $A, B \in \mathbb{E}_d^n$ there exists $k \in \mathbb{N}$, such that $\mathcal{T}_{d,C}^k(A) = B$ then this will be indicated by:

$$A \overset{*}{\underset{d,C}{\Longrightarrow}} B.$$

*$\mathcal{T}_{d,C}$ is called **ESRS with finiteness property** if and only if for all vectors $A \in \mathbb{E}_d^n$*

$$A \overset{*}{\underset{d,C}{\Longrightarrow}} 0,$$

where 0 denotes the zero vector.

Definition 2.1.5. *The following sets form a generalization of the corresponding sets defined in [2]:*

$$\mathcal{D}_{n,d}^{(0)} := \left\{ C \in \mathbb{C}^n \,\middle|\, \forall A \in \mathbb{E}_d^n : A \underset{d,C}{\overset{*}{\Longrightarrow}} 0 \right\},$$

$$\mathcal{D}_{n,d} := \left\{ C \in \mathbb{C}^n \,\middle|\, \forall A \in \mathbb{E}_d^n \text{ the sequence } \left\{ \mathcal{T}_{d,C}^k(A) \right\}_{k \geq 0} \text{ is ultimately periodic} \right\}.$$

$\mathcal{T}_{d,C}$ is ESRS with finiteness property if and only if $C \in \mathcal{D}_{n,d}^{(0)}$.

Remark 2.1.6. *The construction defined in this section can be generalized by using a complex number for d.*

2.2 Basic properties of the one dimensional Euclidean shift radix systems

This section and the following ones will consider C as a one dimensional vector, i.e. a complex number, which will be denoted by c. In this section I will investigate some properties of the one dimensional case.

The following theorem is my result (see [77]). Theorem 2.2.1 can be considered as the generalization of the *cutout polyhedra* defined in [2]. These are areas defined by a closed curve (arcs and lines). Let this area be denoted by P. Let's consider this as **cutout area**.

Theorem 2.2.1. *Let $c \in \mathbb{C}$ and let's say that applying the mapping $\mathcal{T}_{d,c}$ by l times on the number $a_0 \in \mathbb{E}_d$, it admits a period as follows:*

$$a_0 \underset{d,c}{\Rightarrow} a_1 \underset{d,c}{\Rightarrow} a_2 \underset{d,c}{\Rightarrow} a_3 \ldots \underset{d,c}{\Rightarrow} a_{l-1} \underset{d,c}{\Rightarrow} a_0, \text{ if and only if}$$

$$c \in \left(\frac{\mathbb{D}_d - a_1}{a_0} \right) \cap \left(\frac{\mathbb{D}_d - a_2}{a_1} \right) \cap \cdots \cap \left(\frac{\mathbb{D}_d - a_{l-1}}{a_{l-2}} \right) \cap \left(\frac{\mathbb{D}_d - a_0}{a_{l-1}} \right).$$

The number l will be called the length of the period.

Proof. Let's investigate $a_i \underset{d,c}{\Rightarrow} a_j$ for an $i, j \in \{0; 1; \ldots; l-1\}$. Let's see what are the conditions for c in order to get a_j by applying the mapping $\mathcal{T}_{d,c}$ on a_i.

$$\mathcal{T}_{d,c}(a_i) := (-\lfloor ca_i \rfloor) = -(ca_i - d) = -ca_i + r,$$

for an $r \in \mathbb{D}_d$. This means that $a_j = -ca_i + r$, so

$$c = \frac{r - a_j}{a_i},$$

which proves the theorem. □

Theorem 2.2.2 is my result (see [77]). It shows that if the ESRS associated to c has the finiteness property then it must lie in the closed unit circle.

Theorem 2.2.2. *Let $|c| > 1$, $d \in \{1, 2, 3, 7, 11\}$ then $\mathcal{T}_{d,c}$ doesn't have the finiteness property.*

Proof. The basic idea is that we ignore those values of a where the length decreases after applying $\mathcal{T}_{d,c}$, since after finitely many steps it will end in 0 or another value a' the absolute value of which increases by applying the mapping. Investigating the length of a vector after applying the shift radix mapping:

$$a \underset{d,c}{\Rightarrow} ac - r.$$

For the length

$$|a| > |ac - r| \geq |a||c| - |r| > |a||c| - 1,$$

$$|a| < \frac{1}{|c| - 1}.$$

If this inequality holds the length decreases. This is a finite open disk around the origin. For any other a the length will increase, so starting from a applying the shift radix mapping leads to a divergent sequence. □

Plainly $\mathcal{T}_{d,1}$ doesn't have the finiteness property for any d. For finding ESRS with finiteness property, one has to use a well chosen complex number c. Based on Theorem 2.2.2, let's start from the closed unit disc around the origin, and let's ignore these cutout areas in order to reach those points which are good to define ESRS with finiteness property:

Remark 2.2.3. *The set $\mathcal{D}^0_{n,d}$ can be defined in the following way. Let $S := \{c \in \mathbb{C} | \ |c| \leq 1\}$ and let's consider the areas defined by Theorem 2.2.1 as P_i. Then*

$$\mathcal{D}^0_{n,d} = S \setminus \cup P_i.$$

Since there can be infinitely many cutout areas, they can be disjoint, overlapped by each other or superset and subset of each other, finding the union area of all is a hard problem. The following definition helps to estimate how many cutout areas are around some point in $\mathcal{D}_{n,d}$.

Definition 2.2.4. *Let $c \in \mathcal{D}_{n,d}$.*

- *If there exists an open neighborhood of c which contains only finitely many cutout areas then I call c a **regular point**.*
- *If each open neighborhood of c has nonempty intersection with infinitely many cutout areas then I call c a **weak critical point** for $\mathcal{D}_{n,d}$.*
- *If for each open neighborhood U of c the set $U \setminus \mathcal{D}^0_{n,d}$ cannot be covered by finitely many cutout areas then c is called a **critical point**.*

Let's check what are the conditions to reach a cutout area in the one dimensional case.

Remark 2.2.5. *Theorem 2.2.1's result for one dimensional case can be used to define cutout areas with periods of any length. $\mathcal{T}_{d,c}$ admits a period $a_0 \underset{d,c}{\Rightarrow} a_1 \underset{d,c}{\Rightarrow} a_2 \underset{d,c}{\Rightarrow} \dots \underset{d,c}{\Rightarrow} a_n \underset{d,c}{\Rightarrow} a_0$ if and only if*

$$c \in \left(\frac{\mathbb{D}_d - a_1}{a_0}\right) \cap \left(\frac{\mathbb{D}_d - a_2}{a_1}\right) \cap \dots \cap \left(\frac{\mathbb{D}_d - a_n}{a_{n-1}}\right) \cap \left(\frac{\mathbb{D}_d - a_0}{a_n}\right).$$

The one-step and the two-step cases are really important, since the one-step periods define large sets around -1, and the two-step cases appear most likely around 1. The following two lemmata speak about these special cases.

Lemma 2.2.6. *Let $c \in \mathbb{C}$ be with $|c| < 1$. $\mathcal{T}_{d,c}$ admits a one-step period, if and only if $c \in \left(\frac{\mathbb{D}_d}{a}\right) - 1$ for an $a \in \mathbb{E}_d \setminus \{0\}$.*

Proof. The shift radix mapping leads to the following:

$$a \underset{d,c}{\Rightarrow} -ac + r,$$

$a \in \mathbb{E}_d \setminus \{0\}$. This can be a one-step period, if and only if $c = \frac{r}{a} - 1$. r is a general element of the fundamental sail tile, so $c \in \left(\frac{\mathbb{D}_d}{a}\right) - 1$. $\square$

Lemma 2.2.7. *Let $c \in \mathbb{C}$ be with $|c| < 1$. $\mathcal{T}_{d,c}$ admits a two-step period, if and only if $c \in \left(\frac{\mathbb{D}_d - a'}{a}\right) \cap \left(\frac{\mathbb{D}_d - a}{a'}\right)$, where $a, a' \in \mathbb{E}_d \setminus \{0\}$.*

Proof. The shift radix mapping leads to the following:

$$a \underset{d,c}{\Rightarrow} -ac + r,$$

$a \in \mathbb{E}_d \setminus \{0\}$. Let $a' := -ac + r \in \mathbb{E}_d \setminus \{0\}$, $a' \underset{d,c}{\Rightarrow} -a'c + r$. This can be a two-step period, if and only if $a = -a'c + r$. This means that c has to be in the set

$$c \in \left(\frac{\mathbb{D}_d - a'}{a}\right) \cap \left(\frac{\mathbb{D}_d - a}{a'}\right).$$

□

Theorem 2.2.8 is my result (see [77]). It shows that only finitely many $a \in \mathbb{E}_d$ have to be investigated to decide the finiteness property of a specific value of c.

Theorem 2.2.8. *Let $c \in \mathbb{C}$ be with $|c| < 1$. $\mathcal{T}_{d,c}$ is an ESRS with finiteness property, if and only if for all $a \in \mathbb{E}_d$ where $|a| < \frac{1}{1-|c|}$*

$$a \overset{*}{\underset{d,c}{\Longrightarrow}} 0.$$

Proof.

$$a \underset{d,c}{\Rightarrow} -ac + r, \text{ where}$$

$r \in \mathbb{D}_d$. To decide the finiteness property one has to check only those numbers where the absolute value does not decrease.

$$|a| \leq |-ac + r| \leq |a||c| + |r| < |a||c| + 1, \text{ so}$$

$|a| < \frac{1}{1-|c|}$. □

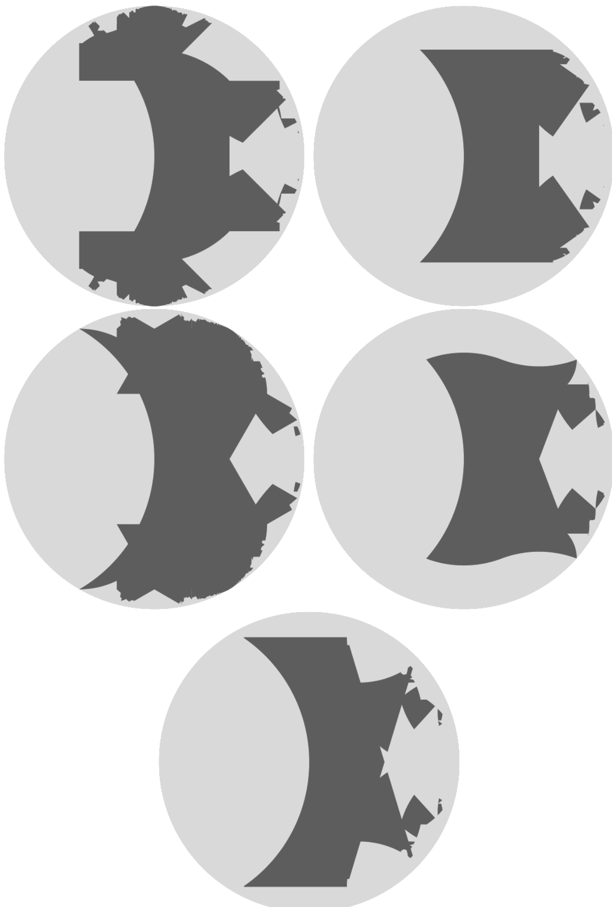
Now, let's see how the sets $\mathcal{D}^0_{1,d}$ ($d \in \{1, 2, 3, 7, 11\}$) look like.

Algorithm 1 is a common result with A. Pethő and M. Weitzer (see [77]). It defines a searching method, which will approximate the mentioned set using the results of Remark 2.2.3 and Theorem 2.2.8. The input parameters are $d \in \{1, 2, 3, 7, 11\}$ and rs, which sets how many points in the unit circle will be tested, the result is a superset of $\mathcal{D}^0_{1,d}$.

Algorithm 1 Approximation algorithm for the set $\mathcal{D}^0_{1,d}$

1: $d \in \{1, 2, 3, 7, 11\}$ (input parameter)
2: $rs := 1000000$ (input parameter)
3: $res := \frac{1}{\sqrt{rs}}$
4: $S := \{c \in \mathbb{C} | \ |c| \leq 1\}$
5: $S_{curr} := S$
6: **for** $rad \in \{0, res, 2res \dots, 1\}$ **do**
7:     **for** $ang \in \{0, res, 2res \dots, 2\pi\}$ **do**
8:         $c_{curr} := rad \cdot e^{i \cdot ang}$
9:         **if** $c_{curr} \in S_{curr}$ **then**
10:             $A_{curr} := \{a' | a' \in \mathbb{E}_d \ and \ |a'| < \frac{1}{1 - |c_{curr}|}\}$
11:             **for** $a_{curr} \in A_{curr}$ **do**
12:                 **if** $\mathcal{T}_{d, c_{curr}}$ admits a period P' starting from a_{curr} **then**
13:                     $S_{curr} = S_{curr} \setminus P'$
14:                     break operation *11*
15:                 **end if**
16:             **end for**
17:         **end if**
18:     **end for**
19: **end for**
20: **return** S_{curr}

Figure 2.2. Using Algorithm 1, these are the generated approximations of $\mathcal{D}^0_{1,1}, \mathcal{D}^0_{1,2}, \mathcal{D}^0_{1,3}, \mathcal{D}^0_{1,7}, \mathcal{D}^0_{1,11}$, respectively (black area).



The next theorem is my result (see [77]). The area close to the origin is the easiest part of the disc to decide the finiteness property, so let's consider the case $|c| < \frac{1}{2}$.

Theorem 2.2.9. *Let $c \in \mathbb{C}$ be with $|c| < 1 - \frac{1}{\sqrt{4}} = \frac{1}{2}$. The function $\mathcal{T}_{d,c}$ is an*

ESRS with finiteness property, if $c \in \mathbb{D}_d$. Additionally, if $d = 11$ then

$$c \notin \left\{ z \in \mathbb{C} \,\middle|\, |(-\omega)z + \omega - 1| \geq 1 \text{ and } -\frac{\sqrt{11}}{4} < Im((-\omega)z + \omega) \right\}, \text{ and}$$

$$c \notin \left\{ z \in \mathbb{C} \,\middle|\, |(-1+\omega)z - \omega| \geq 1 \text{ and } Im((-1+\omega)z + 1 - \omega) \leq \frac{\sqrt{11}}{4} \right\}.$$

Proof. The proof of this theorem only uses basic considerations and the results of this article. □

The following Lemma implies that $\mathcal{D}^0_{1,d}$ and $\mathcal{D}_{1,d}$ reflected at the real axis coincide almost everywhere. Parts where the two sets might not coincide are contained in the union of their respective boundaries.

Lemma 2.2.10. *Let $c \in \mathbb{C}$, $a, b \in \mathbb{E}_d$, and $\varphi = (a_1, a_2, \ldots, a_k) \in \mathbb{E}_d^k$. Then*
$2Im_d(ca)$ is not an odd integer $\Leftrightarrow (\mathcal{T}_c a = b \Leftrightarrow \mathcal{T}_{\overline{c}}\overline{a} = \overline{b})$,
$2Im_d(ca)$ is an odd integer $\Rightarrow (\mathcal{T}_c a = b \Rightarrow \mathcal{T}_{\overline{c}}\overline{a} - \overline{b} \in \{(0,-1)_d, (1,-1)_d\})$.
In particular, if c is contained in the interior of the cutout area corresponding to φ then
$(a_1, a_2, \ldots, a_k)$ period of $\mathcal{T}_c \Leftrightarrow (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_k})$ period of $\mathcal{T}_{\overline{c}}$.

Proof. The proof can be done the same way as the proof of Lemma 3.6 in [24]. □

Definition 2.2.11. *Let*
$\Big(((x_{2,1}, y_{2,1}), (a_{2,1}, b_{2,1})), \ldots, ((x_{2,45}, y_{2,45}), (a_{2,45}, b_{2,45})\Big) := \Big($
$\big((1,0), \ (-2,0)\big)$, $\big((-\frac{529}{4023}, \frac{22378908}{45415717}), \ (0,1)\big)$, $\big((-\frac{8413}{3862276}, \frac{6385}{8993}), \ (0,1)\big)$, $\big((-\frac{560}{3763}, \frac{166}{229}), \ (0,1)\big)$, $\big((\frac{11051}{36427}, \frac{12022}{16987}), \ (0,1)\big)$, $\big((-\frac{39833}{139318}, \frac{634841}{887952}), \ (0,1)\big)$, $\big((-\frac{587}{32542}, \frac{1260970}{1501501}), \ (0,1)\big)$, $\big((\frac{20911}{27059}, \frac{183}{517}), \ (0,1)\big)$, $\big((-\frac{3533}{7022}, \frac{1411}{1988}), \ (0,1)\big)$, $\big((\frac{645}{3757}, \frac{1432877}{1660169}), \ (0,1)\big)$, $\big((\frac{844688}{1266909}, \frac{2031}{3445}), \ (0,4)\big)$, $\big((\frac{44399}{51256}, \frac{4447}{14348}), \ (0,2)\big)$, $\big((\frac{781981}{1137704}, \frac{159}{260}), \ (0,4)\big)$,

$$\big(\big(\tfrac{3741}{6160}, \tfrac{2237}{3237}\big),\ (0,2)\big),\quad \big(\big(\tfrac{18563}{132052}, \tfrac{677269}{744909}\big),\ (0,1)\big),\quad \big(\big(-\tfrac{273}{461}, \tfrac{256}{357}\big),\ (0,1)\big),$$
$$\big(\big(-\tfrac{23531}{44649}, \tfrac{2367}{3041}\big),\ (0,1)\big),\ \big(\big(-\tfrac{2504}{4903}, \tfrac{53361}{66614}\big),\ (0,1)\big),\ \big(\big(\tfrac{2295978}{14352937}, \tfrac{128937}{134770}\big),\ (0,1)\big),$$
$$\big(\big(-\tfrac{22537}{155137}, \tfrac{19631}{20469}\big),\ (0,1)\big),\quad \big(\big(-\tfrac{1324}{2503}, \tfrac{85287}{104894}\big),\ (0,1)\big),\quad \big(\big(\tfrac{186647}{247677}, \tfrac{278}{433}\big),\ (0,2)\big),$$
$$\big(\big(\tfrac{81473}{111068}, \tfrac{86419}{129984}\big),\ (0,2)\big),\ \big(\big(-\tfrac{1087}{2004}, \tfrac{670}{809}\big),\ (0,1)\big),\ \big(\big(\tfrac{19}{25}, \tfrac{16}{25}\big),\ (0,2)\big),\ \big(\big(\tfrac{27}{37}, \tfrac{25}{37}\big),$$
$$(0,2)\big),\quad \big(\big(\tfrac{13}{17}, \tfrac{54}{85}\big),\ (0,2)\big),\quad \big(\big(\tfrac{7647}{10000}, \tfrac{16}{25}\big),\ (0,2)\big),\quad \big(\big(\tfrac{7339}{10000}, \tfrac{1347}{2000}\big),\ (0,2)\big),$$
$$\big(\big(\tfrac{1979}{20000}, \tfrac{4961}{5000}\big),\ (0,1)\big),\quad \big(\big(-\tfrac{1979}{20000}, \tfrac{397}{400}\big),\ (0,1)\big),\quad \big(\big(-\tfrac{2701}{5000}, \tfrac{8399}{10000}\big),\ (0,1)\big),$$
$$\big(\big(-\tfrac{1097}{2000}, \tfrac{4169}{5000}\big),\ (0,1)\big),\quad \big(\big(\tfrac{1527}{2000}, \tfrac{6429}{10000}\big),\ (0,2)\big),\quad \big(\big(\tfrac{3831}{5000}, \tfrac{6413}{10000}\big),\ (0,2)\big),$$
$$\big(\big(\tfrac{3699}{5000}, \tfrac{6711}{10000}\big),\ (0,2)\big),\quad \big(\big(\tfrac{7321}{10000}, \tfrac{6767}{10000}\big),\ (0,2)\big),\quad \big(\big(\tfrac{7419}{10000}, \tfrac{1339}{2000}\big),\ (0,2)\big),$$
$$\big(\big(\tfrac{3683}{5000}, \tfrac{3377}{5000}\big),\ (0,2)\big),\quad \big(\big(-\tfrac{1087}{2000}, \tfrac{4183}{5000}\big),\ (0,1)\big),\quad \big(\big(-\tfrac{1089}{2000}, \tfrac{8387}{10000}\big),\ (0,1)\big),$$
$$\big(\big(-\tfrac{1089}{2000}, \tfrac{1677}{2000}\big),\ (0,1)\big),\big(\big(\tfrac{1}{10}, \tfrac{7}{5\sqrt{2}}\big),\ (0,1)\big),\ \big(\big(\tfrac{1}{100}\big(50+\sqrt{1534}\big), -\tfrac{-100+\sqrt{1534}}{100\sqrt{2}}\big),$$
$$(0,1)\big),\ \big(\big(\tfrac{9}{10}, \tfrac{3}{5\sqrt{2}}\big),\ (0,1)\big)\Big),$$

$$\big(((x_{11,1}, y_{11,1}), (a_{11,1}, b_{11,1})), \ldots, ((x_{11,47}, y_{11,47}), (a_{11,47}, b_{11,47})\big) := \Big($$
$$((1,0),\ (-2,0))\ \big(\big(-\tfrac{529}{4023}, \tfrac{22378908}{45415717}\big),\ (0,1)\big),\qquad \big(\big(\tfrac{25699}{75158}, \tfrac{11951}{22586}\big),\ (2,0)\big),$$
$$\big(\big(\tfrac{122233}{192089}, \tfrac{5593}{12399}\big),\ (0,1)\big),\quad \big(\big(\tfrac{6229}{23994}, \tfrac{22353}{28738}\big),\ (0,9)\big),\quad \big(\big(\tfrac{2039}{57213}, \tfrac{17365}{20941}\big),\ (0,1)\big),$$
$$\big(\big(\tfrac{3099}{4183}, \tfrac{442047}{1060847}\big),\ (0,1)\big),\quad \big(\big(-\tfrac{39923}{156499}, \tfrac{22371}{26896}\big),\ (0,1)\big),\quad \big(\big(\tfrac{4038}{5203}, \tfrac{4722}{11383}\big),\ (0,1)\big),$$
$$\big(\big(\tfrac{285}{406}, \tfrac{752}{1417}\big),\ (0,1)\big),\quad \big(\big(\tfrac{15765}{22453}, \tfrac{431}{725}\big),\ (0,1)\big),\quad \big(\big(\tfrac{2023}{7895}, \tfrac{2634}{2981}\big),\ (0,1)\big),$$
$$\big(\big(-\tfrac{810241}{3496246}, \tfrac{662044}{743591}\big),\ (0,1)\big),\big(\big(\tfrac{127129}{185005}, \tfrac{42539}{67882}\big),\ (0,4)\big),\big(\big(-\tfrac{109151}{435226}, \tfrac{1106}{1235}\big),\ (0,1)\big),$$
$$\big(\big(\tfrac{1499}{5037}, \tfrac{10953}{12284}\big),\ (0,1)\big),\quad \big(\big(-\tfrac{8495}{29356}, \tfrac{259913}{290617}\big),\ (0,1)\big),\quad \big(\big(\tfrac{755}{851}, \tfrac{3083}{7406}\big),\ (0,1)\big),$$
$$\big(\big(-\tfrac{15483}{32584}, \tfrac{4513239}{5265740}\big),\ (0,1)\big),\ \big(\big(-\tfrac{39752315}{80135632}, \tfrac{1130}{1337}\big),\ (0,1)\big),\ \big(\big(-\tfrac{45318560}{90412991}, \tfrac{235960}{280199}\big),$$
$$(0,1)\big),\big(\big(-\tfrac{422566}{838723}, \tfrac{6443}{7665}\big),\ (0,1)\big),\big(\big(-\tfrac{7361}{14390}, \tfrac{105082}{125711}\big),\ (0,1)\big),\big(\big(-\tfrac{724614}{1438463}, \tfrac{2019}{2369}\big),$$
$$(0,1)\big),\ \big(\big(-\tfrac{4861}{9600}, \tfrac{1020}{1199}\big),\ (0,1)\big),\ \big(\big(-\tfrac{1064}{2059}, \tfrac{166081}{196678}\big),\ (0,1)\big),\ \big(\big(-\tfrac{545}{1034}, \tfrac{168253}{200773}\big),$$
$$(0,1)\big),\big(\big(\tfrac{13}{50}, \tfrac{24}{25}\big),\ (0,1)\big),\big(\big(\tfrac{13}{51}, \tfrac{49}{51}\big),\ (0,1)\big),\big(\big(-\tfrac{45}{82}, \tfrac{34}{41}\big),\ (0,1)\big),\big(\big(-\tfrac{1135}{2048}, \tfrac{1699}{2048}\big),$$
$$(0,1)\big),\big(\big(-\tfrac{1125}{2048}, \tfrac{851}{1024}\big),\ (0,1)\big),\big(\big(-\tfrac{1123}{2048}, \tfrac{1701}{2048}\big),\ (0,1)\big),\big(\big(-\tfrac{1083}{2048}, \tfrac{869}{1024}\big),\ (0,1)\big),$$
$$\big(\big(-\tfrac{1075}{2048}, \tfrac{433}{512}\big),\ (0,1)\big),\quad \big(\big(-\tfrac{1069}{2048}, \tfrac{873}{1024}\big),\ (0,1)\big),\quad \big(\big(-\tfrac{531}{1024}, \tfrac{1745}{2048}\big),\ (0,1)\big),$$
$$\big(\big(-\tfrac{529}{1024}, \tfrac{875}{1024}\big),\ (0,1)\big),\quad \big(\big(\tfrac{505}{2048}, \tfrac{991}{1024}\big),\ (0,1)\big),\quad \big(\big(\tfrac{511}{2048}, \tfrac{1983}{2048}\big),\ (0,1)\big),$$
$$\big(\big(\tfrac{513}{2048}, \tfrac{991}{1024}\big),\ (0,1)\big),\quad \big(\big(\tfrac{135}{512}, \tfrac{987}{1024}\big),\ (0,1)\big),\quad \big(\big(\tfrac{129106}{516339}, \tfrac{2147435}{2219844}\big),\ (0,1)\big),$$
$$\big(\big(\tfrac{1}{212}\big(-140+\sqrt{573}\big), \tfrac{\sqrt{11}}{4}\big),\ (0,3)\big),\big(\big(\tfrac{-550-\sqrt{42130}}{1500}, \tfrac{\sqrt{11}\left(-25+2\sqrt{42130}\right)}{1500}\big),\ (0,1)\big),$$
$$\big(\big(\tfrac{1}{48}\big(-33+\sqrt{93}\big), \tfrac{1}{48}\sqrt{11}\big(3+\sqrt{93}\big)\big),\ (0,1)\big),\qquad \big(\big(\tfrac{1639+\sqrt{10021}}{6600}, \tfrac{539+\sqrt{10021}}{200\sqrt{11}}\big),$$
$$(0,1)\big)\Big),$$

and let $C_0^{(2)}(k)$ *denote the ultimate period of the orbit of* $(a_{2,k}, b_{2,k})_2$ *under* $\mathcal{T}_{2,(x_{2,k},y_{2,k})}$ *for all* $k \in \{1, \ldots 45\}$ *and* $C_0^{(11)}(k)$ *the ultimate period of the orbit of* $(a_{11,k}, b_{11,k})_{11}$ *under* $\mathcal{T}_{11,(x_{11,k},y_{11,k})}$ *for all* $k \in \{1, \ldots 47\}$*. Furthermore let* *for all $k \in \mathbb{Z}$:*

$$\begin{aligned} C_1^{(d)}(k) &:= ((-k,1)_d,(k,-1)_d) \\ C_2^{(d)}(k) &:= ((-k,1)_d,(k+1,-1)_d). \end{aligned}$$

The next theorem is a common result with M. Weitzer (see [77]). It shows the critical points of the sets $\mathcal{D}_{1,2}^{(0)}$ and $\mathcal{D}_{1,11}^{(0)}$.

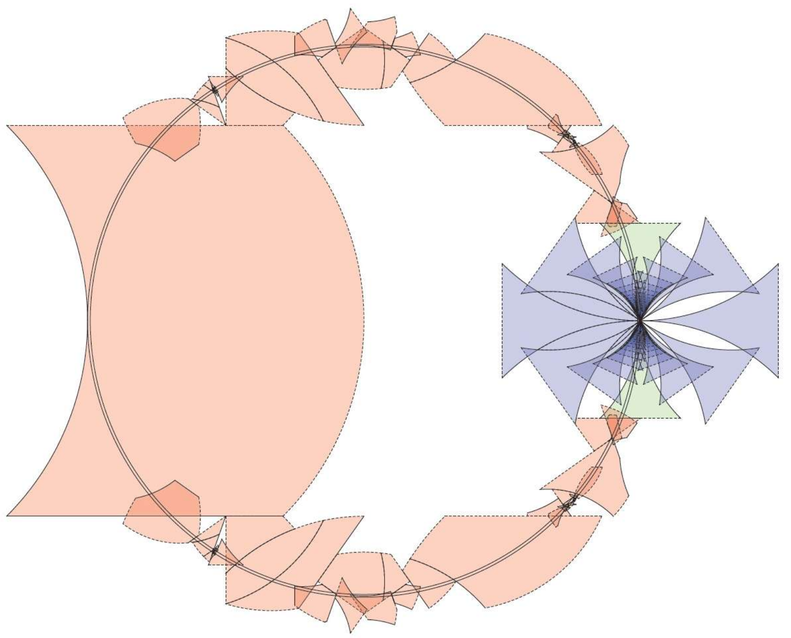


Figure 2.3. Cutout areas of $\mathcal{D}_{1,2}$ which covers the annulus with radii 99/100 and 1. The green area represents the first cutout area, the blue ones are the two infinite sequences.

Theorem 2.2.12. *The sets $\mathcal{D}_{1,2}^{(0)}$ and $\mathcal{D}_{1,11}^{(0)}$ do not contain any weakly critical points (and thus no critical points) r satisfying $r \in \overline{\mathcal{D}_{1,2}^{(0)}}$ and $r \in \overline{\mathcal{D}_{1,11}^{(0)}}$ respectively. More precisely the circle of radius 0.99 around the origin contains the sets $\mathcal{D}_{1,2}^{(0)}$ and $\mathcal{D}_{1,11}^{(0)}$.*

Proof. For any cycle π of complex numbers let $\overline{\pi}$ denote the cycle one gets if all elements of π are replaced by their complex conjugates. The cutout sets of the cycles $C_1^{(2)}(k), C_2^{(2)}(k),\ k \in \mathbb{Z},\ C_0^{(2)}(1), \ldots, C_0^{(2)}(45), \overline{C_0^{(2)}(1)}, \ldots, \overline{C_0^{(2)}(45)}$, and $C_1^{(11)}(k), C_2^{(11)}(k),\ k \in \mathbb{Z},\ C_0^{(11)}(1), \ldots, C_0^{(11)}(47),\ \overline{C_0^{(11)}(1)}, \ldots, \overline{C_0^{(11)}(47)}$ respectively, completely cover the ring centered at the origin in the complex plane with inner radius $\frac{99}{100}$ and outer radius 1. Figures 2.3 and 2.4 show the cutout sets for the cases $d = 2$ and $d = 11$ respectively. The list has been found by a combination of a variant of Algorithm 1 with manual search. $\square$

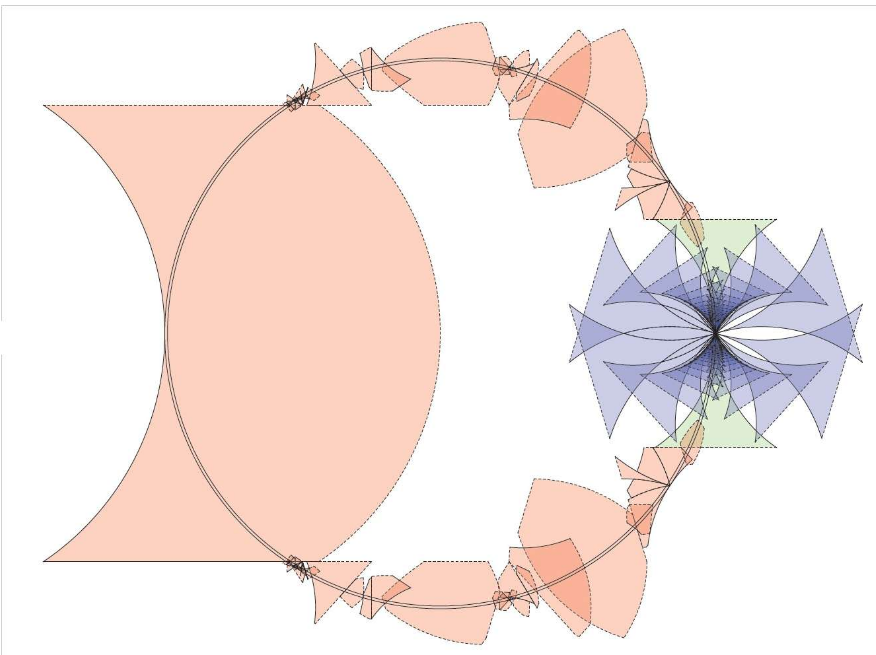


Figure 2.4. Cutout areas of $\mathcal{D}_{1,11}$ which covers the annulus with radii 99/100 and 1. The green area represents the first cutout area, the blue ones are the two infinite sequences.

2.3 Generalization of Brunotte's algorithm

This section describes a possible generalization of Brunotte's algorithm to one dimensional ESRS (see Theorem 2.3.5). Proving this theorem requires the following three lemmas. First, let's recall, how ω has been defined.

Definition 2.3.1. *Let $\mathbb{E}_d$ be an imaginary quadratic Euclidean domain. Its* ***canonical integer basis*** *is: $\{1, \omega_d\}$, where $\omega_d \in \mathbb{E}_d$ and*

$$\omega_d := \begin{cases} \sqrt{-d} & \text{, if } d \in \{1,2\}, \\ \frac{1+\sqrt{-d}}{2} & \text{, otherwise } (d \in \{3,7,11\}). \end{cases}$$

(In the case of $d = 1$ $\omega_1 = \sqrt{-1}$, so that the imaginary unit i is used.)
(For ω_d during these investigations simply ω is used.)

Lemma 2.3.2. *Let $c \in \mathbb{C}^n$ be a complex vector, $|c| < 1$, $a \in \mathbb{Z}$. If $-ca \in \mathbb{R}$ and the remainder part of $-ca$ is greater than 0, then $\mathcal{T}_c(a) = -\mathcal{T}_c(-a) + 1$.*

Proof. $\mathcal{T}_c(a) = -ca + r, r \in [0, 1)$. If $r \geq 0$, then $\mathcal{T}_c(-a) = ca + 1 - r$. $\square$

Lemma 2.3.3. *Let $c \in \mathbb{C}$ be a complex number, $|c| < 1$, $a \in \mathbb{Z}$. If $-ca \in \mathbb{R}$ then $\mathcal{T}_c(a\omega) = \pm\mathcal{T}_c(\pm a)\omega$. If the remainder part of $-ca$ is less than $\frac{1}{2}$, then $\mathcal{T}_c(a\omega) = \mathcal{T}_c(a)\omega$.*

Proof. $\mathcal{T}_c(a\omega) = -ca\omega + r\omega, r \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. If $r \geq 0$, then $\mathcal{T}_c(a) = -ca + r$, so $\mathcal{T}_c(a\omega) = \mathcal{T}_c(a)\omega$. If $r < 0$, then $\mathcal{T}_c(-a) = ca - r$, so $\mathcal{T}_c(a\omega) = -\mathcal{T}_c(-a)\omega$. $\square$

Lemma 2.3.4. *Let $c \in \mathbb{C}$ be a complex number, $|c| < 1$, $a \in \mathbb{Z}$. If $-ca \in \mathbb{R}$ then $\mathcal{T}_c(a\omega) = \pm\mathcal{T}_c(\pm a)\omega \mp \omega$. If the remainder part of $-ca$ is less than $\frac{1}{2}$, then $\mathcal{T}_c(a\omega) = -\mathcal{T}_c(-a)\omega + \omega$.*

Proof. $\mathcal{T}_c(a\omega) = -ca\omega + r\omega, r \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. If $r \geq 0$, then $\mathcal{T}_c(-a) = ca + 1 - r$, so $\mathcal{T}_c(a\omega) = (-\mathcal{T}_c(-a) + 1)\omega$. If $r < 0$, then $\mathcal{T}_c(a) = -ca + 1 + r$, so $\mathcal{T}_c(a\omega) = (\mathcal{T}_c(a) - 1)\omega$. $\square$

The next Theorem is my result and it is not published previously. It aims to generalize Brunotte's algorithm for ESRS. This theorem and its proof is inspired by S. Akiyama and H. Rao's result in [12].

Theorem 2.3.5. *Let $d \in \{1, 2, 3, 7, 11\}$, $c \in \mathbb{C}$ be a complex number, $|c| < 1$, let $c_E \in \mathbb{E}_d$ be a number from the Euclidean domain, $arg(c_E) = arg(c)$. $\mathcal{T}_{d,c}$ is an ESRS with finiteness property, if the set $\mathcal{E}$ exists for c and has finitely many elements:*

- $\overline{-c_E}, \overline{c_E} \in \mathcal{E}$,
- $a + b\omega \in \mathcal{E}$, *where* $a + b\omega \in \mathbb{D}_{d,\overline{c_E}}$, *(the sail digit set of the corresponding ENS),*
- *if* $z \in \mathcal{E}$ *then* $\mathcal{T}_{d,c}(z), -\mathcal{T}_{d,c}(-z) \in \mathcal{E}$,
- *for any* $z \in \mathcal{E}$ *there exists* $n \in \mathbb{Z}^+$ *such that* $\mathcal{T}^n_{d,c}(z) = 0$.

Proof. Let $z \in \mathbb{E}_d$ and $b \in \mathbb{Z}$. Let's assume that z has finiteness property and $b\overline{c_E} \in \mathcal{E}$, and

$$\mathcal{T}_{d,c}(z) = -cz + r_z,$$

$$\mathcal{T}_{d,c}(b\overline{c_E}\omega) = -cb\overline{c_E}\omega + r_b\omega,$$

$$r_z \in \mathbb{D}_d,\ r_b \in \left[-\frac{1}{2}, \frac{1}{2}\right).$$

It will be proven that $z + b\overline{c_E}\omega$ also has a finiteness property.

$$\mathcal{T}_{d,c}(z + b\overline{c_E}\omega) = -cz - cb\overline{c_E}\omega + r.$$

1. If $Im(r_z) + r_b \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, then $r_z + r_b\omega = r$,

$$\mathcal{T}_{d,c}(z + b\overline{c_E}\omega) = \mathcal{T}_{d,c}(z) + \mathcal{T}_{d,c}(b\overline{c_E}\omega).$$

2. If $Im(r_z) + r_b \in \left[-\frac{1}{2}, \frac{1}{2}\right) + 1$, then $r_z + r_b\omega - \omega = r$ and $r_b > 0$.

$$\mathcal{T}_{d,c}(z + b\overline{c_E}\omega) = \mathcal{T}_{d,c}(z) + \mathcal{T}_{d,c}(b\omega\overline{c_E}) - \omega = \mathcal{T}_{d,c}(z) - \mathcal{T}_{d,c}(-b\overline{c_E})\omega.$$

3. If $Im(r_z) + r_b \in \left[-\frac{1}{2}, \frac{1}{2}\right) - 1$, then $r_z + r_b\omega + \omega = r$ and $r_b < 0$.

$$\mathcal{T}_{d,c}(z + b\overline{c_E}\omega) = \mathcal{T}_{d,c}(z) + \mathcal{T}_{d,c}(b\overline{c_E}\omega) + \omega = \mathcal{T}_{d,c}(z) + \mathcal{T}_{d,c}(b\overline{c_E})\omega.$$

Since $b\overline{c_E} \in \mathcal{E}$, $-\mathcal{T}_{d,c}(-b\overline{c_E}) \in \mathcal{E}$ and $\mathcal{T}_{d,c}(b\overline{c_E}) \in \mathcal{E}$. It means that

$$\mathcal{T}^n_{d,c}(z + b\overline{c_E}\omega) = \mathcal{T}^n_{d,c}(z) + b^*\ (Re(b^*) = 0),$$

for all $n \in \mathbb{Z}^+$, where $b^* \in \mathcal{E}$. If this n is large enough, both $\mathcal{T}^n_{d,c}(z)$ and b^* will be zero, so $z + b\overline{c_E}\omega$ has a finiteness property.

In the second part of this proof let's turn to the other dimension. Let $z \in \mathbb{E}$ and $a \in \mathbb{Z}$. Let's assume that z has finiteness property and $a\overline{c_E} \in \mathcal{E}$, and

$$\mathcal{T}_{d,c}(z) = -cz + r_z,$$

$$\mathcal{T}_{d,c}(a\overline{c_E}) = -ca\overline{c_E} + r_a,$$

$$r_z \in \mathbb{D}_d,\ r_a \in [0,1).$$

It will be proven that $z + a\overline{c_E}$ also has a finiteness property.

$$\mathcal{T}_{d,c}(z + a\overline{c_E}) = -cz - ca\overline{c_E} + r.$$

It's easy to see that $Im(r) = Im(r_z)$.

- If $Re(r_z) + r_a \in [0,1)$, then $Re(r_z) + r_a = Re(r)$,

$$\mathcal{T}_{d,c}(z + a\overline{c_E}) = \mathcal{T}_{d,c}(z) + \mathcal{T}_{d,c}(a\overline{c_E}).$$

- If $Re(r_z) + r_a \in [0,1) + 1$, then $Re(r_z) + r_a - 1 = Re(r)$ and $r_a > 0$.

$$\mathcal{T}_{d,c}(z + a\overline{c_E}) = \mathcal{T}_{d,c}(z) + \mathcal{T}_{d,c}(a\overline{c_E}) - 1 = \mathcal{T}_{d,c}(z) - \mathcal{T}_{d,c}(-a\overline{c_E}).$$

  Since $a\overline{c_E} \in \mathcal{E}$, $-\mathcal{T}_{d,c}(-a\overline{c_E}) \in \mathcal{E}$ and $\mathcal{T}_{d,c}(a\overline{c_E}) \in \mathcal{E}$. It means that

$$\mathcal{T}^n_{d,c}(z + a\overline{c_E}) = \mathcal{T}^n_{d,c}(z) + a^*,$$

  for all $n \in \mathbb{Z}^+$, where $a^* \in \mathcal{E}$. If this n is large enough, both $\mathcal{T}^n_{d,c}(z)$ and a^* will be zero, so $z + a\overline{c_E}$ has a finiteness property.

□

Example 2.3.6. *Let*

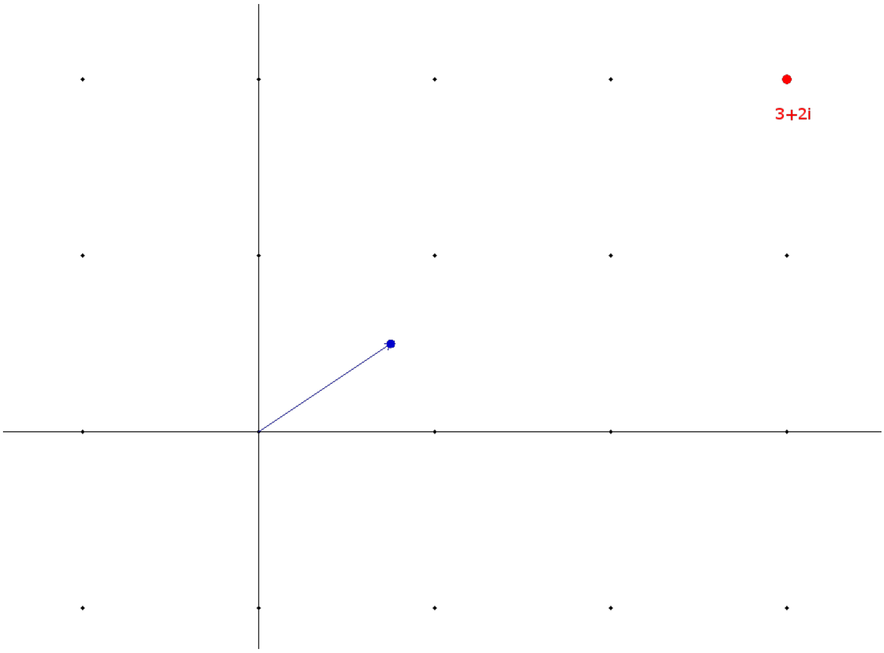
$$d = 1,$$

$$c = \frac{3+2i}{4},\ (|c| \approx 0.9014,\ arg(c) = arctan\tfrac{2}{3} \approx 0.588rad)$$

then

$$c_E = 3 + 2i.$$

Figure 2.5. c and c_E on the complex number field



$$\mathbb{D}_{d,\overline{c_E}} = \{0, 1, 2, 3, 1+i, -1-i, -i, 1-i, 2-i, 3-i, -2i, 1-2i, 2-2i\}.$$

It means the initial set of $\mathcal{E}$ is:

$$\mathcal{E}_0 = \{-3+2i, 3-2i, 0, 1, 2, 3, 1+i, -1-i, -i, 1-i, 2-i, 3-i, -2i, 1-2i, 2-2i\},$$

$$\mathcal{T}_c(\mathcal{E}_0) \setminus \mathcal{E}_0 = \{4, -3, -2-2i, i, -2, -2-i, -1+i\},$$
$$-\mathcal{T}_c(-\mathcal{E}_0) \setminus \mathcal{E}_0 = \{-4, -1, -2-i, -3-i, i, -2, -1+2i, -3+i\},$$

so the new elements are

$$\mathcal{E}_1 = \{4, -4, -3, -2-2i, i, -2, -2-i, -1+i, -1, -2-i, -3-i, -1+2i, -3+i\},$$

$$\mathcal{T}_c(\mathcal{E}_1) \setminus (\mathcal{E}_1 \cup \mathcal{E}_0) = \{-3-2i, 3+2i, 3+i, 1+2i, 2+i, 2+2i, 3+i\},$$
$$-\mathcal{T}_c(-\mathcal{E}_1) \setminus (\mathcal{E}_1 \cup \mathcal{E}_0) = \{3+2i, -3-2i, 2+2i, -1-2i, 1+2i, 2+i\},$$

thus

$$\mathcal{E}_2 = \{-3-2i, 3+2i, 3+i, 1+2i, 2+i, 2+2i, 3+i, -1-2i\},$$

$$\mathcal{T}_c(\mathcal{E}_2) \setminus (\mathcal{E}_2 \cup \mathcal{E}_1 \cup \mathcal{E}_0) = \{2+3i, -1-3i, -3i, 2i\},$$
$$-\mathcal{T}_c(-\mathcal{E}_2) \setminus (\mathcal{E}_2 \cup \mathcal{E}_1 \cup \mathcal{E}_0) = \{1+3i, -2-3i, -2-2i\},$$

consequently

$$\mathcal{E}_3 = \{2+3i, -1-3i, -3i, 2i, 1+3i, -2-3i, -2-2i\},$$

$$\mathcal{T}_c(\mathcal{E}_3) \setminus (\mathcal{E}_3 \cup \mathcal{E}_2 \cup \mathcal{E}_1 \cup \mathcal{E}_0) = \{3i, 1-3i\},$$
$$-\mathcal{T}_c(-\mathcal{E}_3) \setminus (\mathcal{E}_3 \cup \mathcal{E}_2 \cup \mathcal{E}_1 \cup \mathcal{E}_0) = \{-1+3i, -2+2i, 3i\}, \mathcal{E}_4 = \{3i, 1-3i, -1+3i, -2+2i\},$$
$$\mathcal{T}_c(\mathcal{E}_4) \setminus (\mathcal{E}_4 \cup \mathcal{E}_3 \cup \mathcal{E}_2 \cup \mathcal{E}_1 \cup \mathcal{E}_0) = \{\},$$
$$-\mathcal{T}_c(-\mathcal{E}_4) \setminus (\mathcal{E}_4 \cup \mathcal{E}_3 \cup \mathcal{E}_2 \cup \mathcal{E}_1 \cup \mathcal{E}_0) = \{\},$$

the iterative step finished. These are all the elements which needs to be investigated in order to decide the finiteness property. There is an orbit, namely

$$-1 \underset{1,\frac{3+2i}{4}}{\Rightarrow} 1 \underset{1,\frac{3+2i}{4}}{\Rightarrow} -i \underset{1,\frac{3+2i}{4}}{\Rightarrow} i \underset{1,\frac{3+2i}{4}}{\Rightarrow} 1-i \underset{1,\frac{3+2i}{4}}{\Rightarrow} -1,$$

thus $\mathcal{T}_c$ *is not an ESRS with finiteness property.*

Chapter 3

Summary, conclusion

To sum up, I have initiated a new approach how to define the digit set for number systems over imaginary quadratic Euclidean domains. This, so called „sail digit set“ is well defined on all of the five possible domains. I was able to prove several interesting properties of this set. The result is a number system with this digit set, which can be considered as generalization of canonical number systems over integers with some useful properties. I showed that there are infinitely many ENS polynomials. There is an interesting connection between CNS, symmetric CNS and ENS polynomials, which is described in Theorem 1.6.1. For a given polynomial the ENS property is always algorithmically decidable, this is the result of Theorem 1.2.13. I fully characterized the linear case (Theorem 1.4.6), however it turned out that the quadratic case is hard, and its characterization is still an open problem. I generalized the shift radix systems to finite dimensional Hermitian vector spaces using this structure. One of the main features of this construction is that the remainder set is the subset of the opened unit disc, thus for every remainder r we have the property $|r| < 1$. Theorem 2.2.1 can be considered as the generalization of the cutout polyhedra defined in [2]. I generalized and use Brunotte's algorithm [18] for the case of linear ESRS with some restrictions (Theorem 2.3.5). This can be continued to define Brunotte's algorithm for any number of dimensions, and it can be used to further investigate Euclidean shift radix systems.

Chapter 4

Appendix

In this chapter I've investigated all possible triplets (d, p, a), which is essential for the proof of the Theorem 1.4.6. d determines the imaginary quadratic Euclidean domain, p is the constant term of the linear polynomial $P(x) = x + p$, and $A(x) = a$ is a constant polynomial whose representability needs to be checked.

○ $d = 1$,

- $p = 1 - i$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; -i\}$,
  $a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i, \pm 2, \pm 2i, 2 \pm i, -2 \pm i, 1 \pm 2i, -1 \pm 2i\}$.

$$\begin{tikzcd}
 & & -2+i \arrow[d] & & -1+2i \arrow[d] & & & \\
 & & 2 \arrow[dr] & & 2-i \arrow[dl] & & & \\
 & & & -1-i \arrow[dr] & & -1-2i \arrow[dl] & & \\
2+i \arrow[d] & & 1+2i \arrow[d] & & i \arrow[d] & & 2i \arrow[dll] & \\
-2i \arrow[dr] & & 1-2i \arrow[dr] & & 1-i \arrow[dl] & & & \\
 & -1+i \arrow[dr] & & -1 \arrow[dl] & & -2 \arrow[dr] & & -2-i \arrow[dl] \\
 & & 1 \arrow[drr] & & & & 1+i \arrow[dll] & \\
 & & & & -i \arrow[d] & & & \\
 & & & & 0 & & &
\end{tikzcd}$$

$x + 1 - i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.

- $p = 1 + i$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; 1\}$,

$a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i, \pm 2, \pm 2i, 2 \pm i, -2 \pm i, 1 \pm 2i, -1 \pm 2i\}$.

$$\begin{tikzcd}
 & & -2+i \arrow[d] & & -1+2i \arrow[d] & & & \\
 & & 1-2i \arrow[dr] & & -2i \arrow[dl] & & & \\
 & & & 1+i \arrow[dr] & & 2+i \arrow[dl] & & \\
-1-2i \arrow[d] & & -2-i \arrow[d] & & -1 \arrow[d] & & -2 \arrow[dll] & \\
2 \arrow[dr] & & 2-i \arrow[dr] & & 1-i \arrow[dl] & & & \\
 & -1+i \arrow[dr] & & i \arrow[dl] & & 2i \arrow[dr] & & 1+2i \arrow[dl] \\
 & & -i \arrow[drr] & & & & -1-i \arrow[dll] & \\
 & & & & 1 \arrow[d] & & & \\
 & & & & 0 & & &
\end{tikzcd}$$

$x + 1 + i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.

- $p = -1 - i$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; -1\}$,
  $-i \underset{P}{\Rightarrow} -i$ and this is a cycle($1 \notin \mathbb{D}_{d,p}$).

- $p = -1 + i$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; i\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.

- There are no elements in $\mathbb{E}_1$ with norm 3.

- $p = 2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1; -i; 1 - i\}$,

$a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i\}$.

$$\begin{tikzcd}
-1 \arrow[dr] & -1-i \arrow[d] & -1+i \arrow[d] & i \arrow[d] & 1+i \arrow[dl] \\
& 1 \arrow[dr] & 1-i \arrow[d] & -i \arrow[dl] & \\
& & 0 & &
\end{tikzcd}$$

$x+2$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.

- $p = -2i$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1; -i; -1-i\}$,
  $a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i\}$.

$$\begin{tikzcd}
1+i \arrow[dr] & & -1+i \arrow[dr] & & i \arrow[dl] \\
& 1-i \arrow[dr] & & 1 \arrow[dl] & \\
-1 \arrow[drr] & & -i \arrow[d] & & -1-i \arrow[dll] \\
& & 0 & &
\end{tikzcd}$$

$x-2i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.

- $p = 2i$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1; i; 1+i\}$,
  $a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i\}$.

$$\begin{tikzcd}
-1-i \arrow[dr] & & -1+i \arrow[dr] & & -1 \arrow[dl] \\
& 1-i \arrow[dr] & & -i \arrow[dl] & \\
1+i \arrow[drr] & & 1 \arrow[d] & & i \arrow[dll] \\
& & 0 & &
\end{tikzcd}$$

$x+2i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.

- $p = -2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1; i; -1+i\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.

- $p = 2 - i$, $N(p) = 5$,
  $\mathbb{D}_{d,p} = \{0; 1; 2; -i; 1-i\}$,
  $a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i\}$.

$$\begin{tikzcd}
-1-i \arrow[d] & & & \\
1+i \arrow[d] & -1+i \arrow[dr] & -1 \arrow[d] & i \arrow[dl] \\
-i \arrow[dr] & 1-i \arrow[d] & 1 \arrow[dl] & \\
 & 0 & &
\end{tikzcd}$$

$x + 2 - i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.

- $p = 2 + i$, $N(p) = 5$,
  $\mathbb{D}_{d,p} = \{0; 1; 2; i; 1+i\}$,
  $a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i\}$.

$$\begin{tikzcd}
-1+i \arrow[d] & & & \\
1-i \arrow[d] & -1-i \arrow[dr] & -1 \arrow[d] & -i \arrow[dl] \\
i \arrow[dr] & 1+i \arrow[d] & 1 \arrow[dl] & \\
 & 0 & &
\end{tikzcd}$$

$x + 2 + i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.

- $p = 1 - 2i$, $N(p) = 5$,
  $\mathbb{D}_{d,p} = \{0; 1; -i; 1-i; -2i\}$,

$a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i\}$.

$$\begin{tikzcd}
 & -1-i \arrow[d] & & \\
1+i \arrow[d] & i \arrow[dr] & -1 \arrow[d] & -1+i \arrow[dl] \\
1-i \arrow[dr] & -i \arrow[d] & 1 \arrow[dl] & \\
 & 0 & &
\end{tikzcd}$$

$x + 1 - 2i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.

- $p = 1 + 2i$, $N(p) = 5$,
  $\mathbb{D}_{d,p} = \{0; 1; i; 1 + i; 2i\}$,
  $a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i\}$.

$$\begin{tikzcd}
 & -1+i \arrow[d] & & \\
1-i \arrow[d] & -i \arrow[dr] & -1 \arrow[d] & -1-i \arrow[dl] \\
1+i \arrow[dr] & i \arrow[d] & 1 \arrow[dl] & \\
 & 0 & &
\end{tikzcd}$$

$x + 1 + 2i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.

- $p = -2 - i$, $N(p) = 5$,
  $\mathbb{D}_{d,p} = \{0; -1; -2; -i; -1 - i\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -2 + i$, $N(p) = 5$,
  $\mathbb{D}_{d,p} = \{0; -1; -2; i; -1 + i\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -1 - 2i$, $N(p) = 5$,
  $\mathbb{D}_{d,p} = \{0; -1; -i; -1 - i; -2i\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -1 + 2i$, $N(p) = 5$,
  $\mathbb{D}_{d,p} = \{0; -1; i; -1 + i; 2i\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.

- There are no elements in $\mathbb{E}_1$ with norm 6 or 7.
- $p = 2-2i$, $N(p) = 8$,
  $\mathbb{D}_{d,p} = \{0; 1; -1-i; -i; 1-i; 2-i; -2i; 1-2i\}$,
  $a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i\}$.

$$\begin{tikzcd}
i \arrow[drr] & & -1 \arrow[d] & & -1+i \arrow[dll] & & 1+i \arrow[dll] \\
1-i \arrow[drrr] & & 1 \arrow[dr] & & -i \arrow[dl] & & -1-i \arrow[dlll] \\
& & & 0 & & &
\end{tikzcd}$$

  $x + 2 - 2i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.
- $p = 2+2i$, $N(p) = 8$,
  $\mathbb{D}_{d,p} = \{0; 1; 2; 1-i; i; 1+i; 2+i; 1+2i\}$,
  $a \in \{\pm 1, \pm i, 1 \pm i, -1 \pm i\}$.

$$\begin{tikzcd}
& & -1+i \arrow[d] & & & & \\
-1 \arrow[drr] & & -i \arrow[d] & & -1-i \arrow[dll] & & \\
1+i \arrow[drrr] & & 1 \arrow[dr] & & i \arrow[dl] & & 1-i \arrow[dlll] \\
& & & 0 & & &
\end{tikzcd}$$

  $x + 2 + 2i$ is an ENS polynomial with the sail digit set $\mathbb{D}_{1,p}$.
- $p = -2-2i$, $N(p) = 8$,
  $\mathbb{D}_{d,p} = \{0; -1; -2; -1+i; -i; -1-i; -2-i; -1-2i\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -2+2i$, $N(p) = 8$,
  $\mathbb{D}_{d,p} = \{0; -1; 1+i; i; -1+i; -2+i; 2i; -1+2i\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- If $N(p) \geq 9$, then $\frac{l}{2} > \sqrt{\frac{|p|+1}{|p|-1}}$.

◦ $d = 2$,

- $p = -\omega$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; -1\}$,
  $a \in \{\pm 1, \pm\omega, 1 \pm \omega, -1 \pm \omega, \pm 2\}$.

$$\begin{tikzcd}
-2 \arrow[d] & & & & & \\
\omega \arrow[dr] & & -1+\omega \arrow[dl] & & & 1+\omega \arrow[d] \\
& 1 \arrow[dr] & & 2 \arrow[dl] & & 1-\omega \arrow[dl] \\
& & -\omega \arrow[dr] & & -1-\omega \arrow[dl] & \\
& & & -1 \arrow[d] & & \\
& & & 0 & &
\end{tikzcd}$$

$x - \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{2,p}$.

- $p = \omega$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; 1\}$,
  $a \in \{\pm 1, \pm\omega, 1 \pm \omega, -1 \pm \omega, \pm 2\}$.

$$\begin{tikzcd}
2 \arrow[d] & & & & & \\
\omega \arrow[dr] & & 1+\omega \arrow[dl] & & & -1+\omega \arrow[d] \\
& -1 \arrow[dr] & & -2 \arrow[dl] & & -1-\omega \arrow[dl] \\
& & -\omega \arrow[dr] & & 1-\omega \arrow[dl] & \\
& & & 1 \arrow[d] & & \\
& & & 0 & &
\end{tikzcd}$$

$x + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{2,p}$.

- $p = 1 - \omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; 1; -\omega\}$,
  $a \in \{\pm 1, \pm\omega, 1 \pm \omega, -1 \pm \omega\}$.

$$\begin{tikzcd}
1+\omega \arrow[d] & & & \\
1-\omega \arrow[d] & & -1-\omega \arrow[d] & \\
-1 \arrow[drr] & & \omega \arrow[d] & -1+\omega \arrow[dl] \\
-\omega \arrow[dr] & & 1 \arrow[dl] & \\
& 0 & &
\end{tikzcd}$$

  $x + 1 - \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{2,p}$.
- $p = 1 + \omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; 1; \omega\}$,
  $a \in \{\pm 1, \pm\omega, 1 \pm \omega, -1 \pm \omega\}$.

$$\begin{tikzcd}
1-\omega \arrow[d] & & & \\
1+\omega \arrow[d] & & -1+\omega \arrow[d] & \\
-1 \arrow[drr] & & -\omega \arrow[d] & -1-\omega \arrow[dl] \\
\omega \arrow[dr] & & 1 \arrow[dl] & \\
& 0 & &
\end{tikzcd}$$

  $x + 1 + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{2,p}$.
- $p = -1 - \omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; -1; -\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -1 + \omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; -1; \omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.

- $p = 2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1; -\omega; 1-\omega\}$,
  $a \in \{\pm 1, \pm\omega, 1 \pm \omega, -1 \pm \omega\}$.

$$\begin{tikzcd}
-1-\omega \arrow[dr] & -1 \arrow[d] & -1+\omega \arrow[d] & \omega \arrow[d] & 1+\omega \arrow[dl] \\
 & 1 \arrow[dr] & 1-\omega \arrow[d] & -\omega \arrow[dl] & \\
 & & 0 & &
\end{tikzcd}$$

  $x + 1 + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{2,p}$.

- $p = -2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1; \omega; -1+\omega\}$, $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- There are no elements in $\mathbb{E}_2$ with norm 5.
- If $N(p) \geq 6$, then $\frac{l}{2} > \sqrt{\frac{|p|+1}{|p|-1}}$.

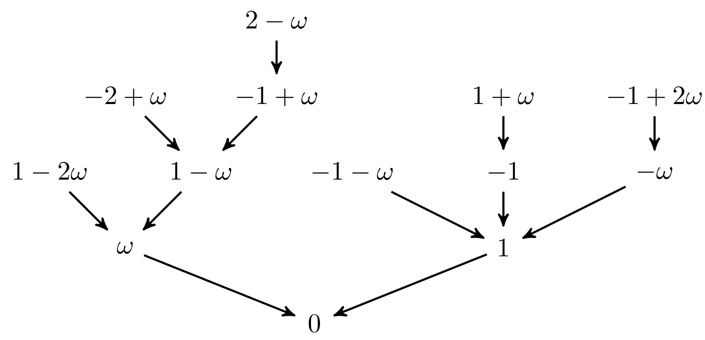
∘ $d = 3$,

- There are no elements in $\mathbb{E}_3$ with norm 2.
- $p = 2 - \omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; 1; 1-\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm(1+\omega), \pm(2-\omega), \pm(1-2\omega)\}$.

$$\begin{tikzcd}
 & & & 1+\omega \arrow[d] & & & \\
 & -1-\omega \arrow[dr] & & -\omega \arrow[dl] & & 2-\omega \arrow[d] & 1-2\omega \arrow[d] \\
-1+2\omega \arrow[dr] & & \omega \arrow[dl] & & -2+\omega \arrow[dr] & -1 \arrow[d] & -1+\omega \arrow[dl] \\
 & 1-\omega \arrow[drr] & & & & 1 \arrow[dll] & \\
 & & & 0 & & &
\end{tikzcd}$$

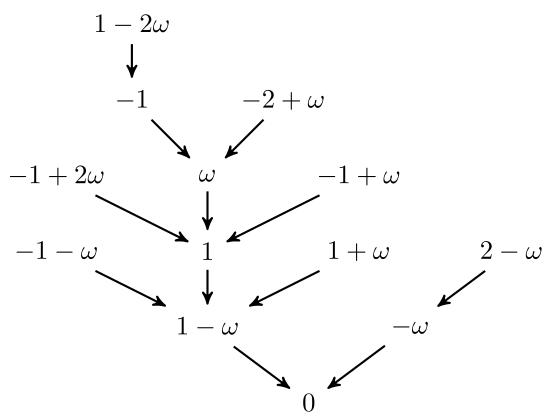
  $x + 2 - \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = 1 + \omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; 1; \omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm(1+\omega), \pm(2-\omega), \pm(1-2\omega)\}$.



  $x + 1 + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

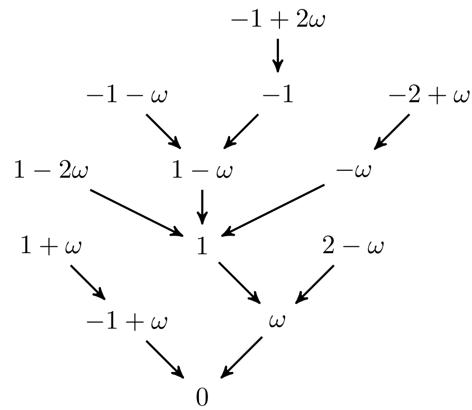
- $p = 1 - 2\omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; -\omega; 1-\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm(1+\omega), \pm(2-\omega), \pm(1-2\omega)\}$.



  $x + 1 - 2\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

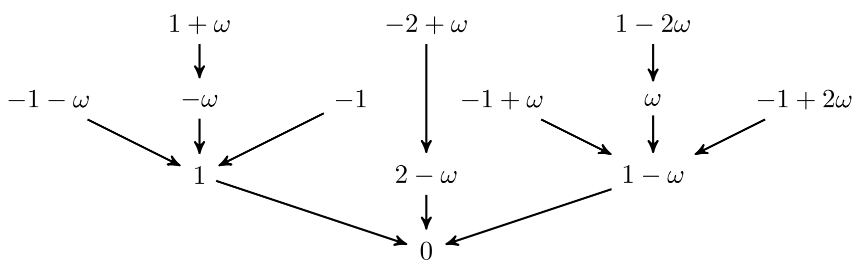
- $p = -1 + 2\omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; \omega; -1+\omega\}$,

$a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm(1+\omega), \pm(2-\omega), \pm(1-2\omega)\}$.



$x - 1 + 2\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = -1-\omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; -1; -\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -2+\omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{0; -1; -1+\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = 2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1; 1-\omega; 2-\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm(1+\omega), \pm(2-\omega), \pm(1-2\omega)\}$.



$x + 2$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = 2 - 2\omega$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1 - \omega; -\omega; 1 - 2\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm(1+\omega), \pm(2-\omega), \pm(1-2\omega)\}$.

$$\begin{tikzcd}
-1+2\omega \arrow[d] & -1+\omega \arrow[dr] & -1 \arrow[d] & -2+\omega \arrow[dl] & -1-\omega \arrow[dl] \\
2-\omega \arrow[dr] & 1+\omega \arrow[dr] & 1 \arrow[d] & \omega \arrow[dl] & \\
 & -\omega \arrow[dr] & 1-\omega \arrow[d] & 1-2\omega \arrow[dl] & \\
 & & 0 & &
\end{tikzcd}$$

  $x + 2 - 2\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = 2\omega$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1; \omega; 1 + \omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm(1+\omega), \pm(2-\omega), \pm(1-2\omega)\}$.

$$\begin{tikzcd}
-1-\omega \arrow[d] & -1+\omega \arrow[dr] & -2+\omega \arrow[d] & -1 \arrow[dl] & -1+2\omega \arrow[dl] \\
2-\omega \arrow[dr] & 1-2\omega \arrow[dr] & 1-\omega \arrow[d] & -\omega \arrow[dl] & \\
 & \omega \arrow[dr] & 1 \arrow[d] & 1+\omega \arrow[dl] & \\
 & & 0 & &
\end{tikzcd}$$

  $x + 2\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = -2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1; -1 + \omega; -2 + \omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -2 + 2\omega$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1 + \omega; \omega; -1 + 2\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -2\omega$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1; -\omega; -1 - \omega\}$,
  $1 - \omega \underset{P}{\Rightarrow} 1 - \omega$ and this is a cycle ($1 \notin \mathbb{D}_{3,-2\omega}$).

- There are no elements in $\mathbb{E}_3$ with norm 5 or 6.
- $p = 3 - \omega$, $N(p) = 7$,
  $\mathbb{D}_{d,p} = \{0; 1; 2; \omega; -\omega; 1-\omega; 2-\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
 & -1+\omega \arrow[dr] & & -1 \arrow[dl] & & \\
\omega \arrow[drrr] & & 1 \arrow[dr] & & 1-\omega \arrow[dl] & -\omega \arrow[dll] \\
 & & & 0 & &
\end{tikzcd}$$

  $x + 3 - \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.
- $p = 3 - 2\omega$, $N(p) = 7$,
  $\mathbb{D}_{d,p} = \{0; 1; \omega; -\omega; 1-\omega; 2-\omega; 2-2\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

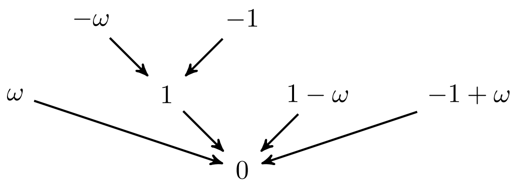
$$\begin{tikzcd}
 & -1+\omega \arrow[dr] & & -1 \arrow[dl] & & \\
\omega \arrow[drrr] & & 1 \arrow[dr] & & 1-\omega \arrow[dl] & -\omega \arrow[dll] \\
 & & & 0 & &
\end{tikzcd}$$

  $x + 3 - 2\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.
- $p = 2 + \omega$, $N(p) = 7$,
  $\mathbb{D}_{d,p} = \{0; 1; 2; -\omega; -1+\omega; \omega; 1+\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
 & -\omega \arrow[dr] & & -1 \arrow[dl] & & \\
\omega \arrow[drrr] & & 1 \arrow[dr] & & 1-\omega \arrow[dl] & -1+\omega \arrow[dll] \\
 & & & 0 & &
\end{tikzcd}$$

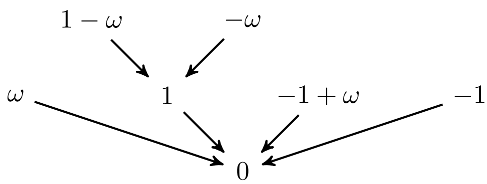
  $x + 2 + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.
- $p = 1 + 2\omega$, $N(p) = 7$,
  $\mathbb{D}_{d,p} = \{0; 1; 1-\omega; -1+\omega; \omega; 1+\omega; 2\omega\}$,

$a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.



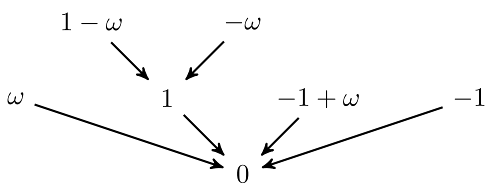
$x + 1 + 2\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = -1 + 3\omega$, $N(p) = 7$,
  $\mathbb{D}_{d,p} = \{-1; 0; 1; -1+\omega; \omega; -1+2\omega; 2\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.



$x - 1 + 3\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

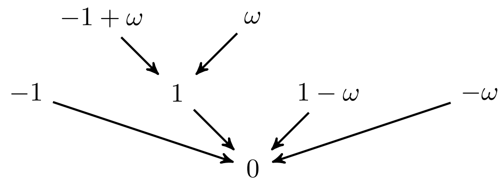
- $p = -2 + 3\omega$, $N(p) = 7$,
  $\mathbb{D}_{d,p} = \{-1; 0; 1; -1+\omega; \omega; -2+2\omega; -1+2\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.



$x - 2 + 3\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

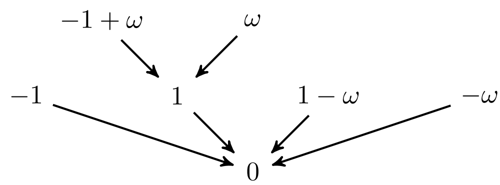
- $p = 2 - 3\omega$, $N(p) = 7$,
  $\mathbb{D}_{d,p} = \{-1; 0; 1; -\omega; 1-\omega; 1-2\omega; 2-2\omega\}$,

$a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.



$x + 2 - 3\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = 1 - 3\omega,\ N(p) = 7$,
  $\mathbb{D}_{d,p} = \{-1; 0; 1; -\omega; 1-\omega; -2\omega; 1-2\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.



  $x + 1 - 3\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = -1 - 2\omega,\ N(p) = 7$,
  $\mathbb{D}_{d,p} = \{0; -1; -1+\omega; -1-\omega; -\omega; 1-\omega; -2\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -2 - \omega,\ N(p) = 7$,
  $\mathbb{D}_{d,p} = \{0; -1; -2; -1+\omega; 1-\omega; -\omega; -1-\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -3 + \omega,\ N(p) = 7$,
  $\mathbb{D}_{d,p} = \{0; -1; -2; -\omega; \omega; -1+\omega; -2+\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -3 + 2\omega,\ N(p) = 7$,
  $\mathbb{D}_{d,p} = \{0; -1; -\omega; \omega; -1+\omega; -2+\omega; -2+2\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- There are no elements in $\mathbb{E}_3$ with norm 8.

- $p = 3$, $N(p) = 9$,
  $\mathbb{D}_{d,p} = \{0; 1; 2; \omega; 1+\omega; 2+\omega; 1-\omega; 2-\omega; 3-\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
-1+\omega \arrow[dr] & -1 \arrow[d] & -\omega \arrow[dl] \\
\omega \arrow[dr] & 1 \arrow[d] & 1-\omega \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

  $x + 3$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = 3 - 3\omega$, $N(p) = 9$,
  $\mathbb{D}_{d,p} = \{0; 1; -\omega; 1-\omega; 2-\omega; 1-2\omega; 2-2\omega; 3-2\omega; 2-3\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
-1+\omega \arrow[dr] & -1 \arrow[d] & \omega \arrow[dl] \\
-\omega \arrow[dr] & 1 \arrow[d] & 1-\omega \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

  $x + 3 - 3\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = 3\omega$, $N(p) = 9$,
  $\mathbb{D}_{d,p} = \{0; 1; -1+\omega; \omega; 1+\omega; -1+2\omega; 2\omega; 1+2\omega; -1+3\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
1-\omega \arrow[dr] & -1 \arrow[d] & -\omega \arrow[dl] \\
\omega \arrow[dr] & 1 \arrow[d] & -1+\omega \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

  $x + 3\omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{3,p}$.

- $p = -3\omega$, $N(p) = 9$,
  $\mathbb{D}_{d,p} = \{0; -1; -1-\omega; -\omega; 1-\omega; -1-2\omega; -2\omega; 1-2\omega; 1-3\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.

- $p = -3$, $N(p) = 9$,
  $\mathbb{D}_{d,p} = \{0; -1; -2; -\omega; -1-\omega; -2-\omega; -1+\omega; -2+\omega; -3+\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.

- $p = -3+3\omega$, $N(p) = 9$,
  $\mathbb{D}_{d,p} = \{0; -1; \omega; -1+\omega; -2+\omega; -1+2\omega; -2+2\omega; -3+2\omega; -2+3\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.

- There are no elements in $\mathbb{E}_3$ with norm 10.

- If $N(p) \geq 11$, then $\frac{l}{2} > \sqrt{\frac{|p|+1}{|p|-1}}$.

∘ $d = 7$,

- $p = 1-\omega$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; -1\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm 2, \pm(2-\omega), \pm(1+\omega)\}$.

$$
\begin{tikzcd}
 & & & 2-\omega \arrow[d] & & & \\
 & & & -1-\omega \arrow[d] & & & \\
 & -2+\omega \arrow[dr] & & -1+\omega \arrow[dl] & & -2 \arrow[dl] & \\
2 \arrow[dr] & & 1 \arrow[dl] & & \omega \arrow[dr] & & 1+\omega \arrow[dl] \\
 & -\omega \arrow[drr] & & & & 1-\omega \arrow[dll] & \\
 & & & -1 \arrow[d] & & & \\
 & & & 0 & & &
\end{tikzcd}
$$

$x + 1 - \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{7,p}$.

- $p = \omega$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; 1\}$,

$a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm 2, \pm(2-\omega), \pm(1+\omega)\}$.

$$
\begin{tikzcd}
 & & & -1-\omega \arrow[d] & & & \\
 & & & 2-\omega \arrow[d] & & & \\
 & 1+\omega \arrow[dr] & & \omega \arrow[dl] & & 2 \arrow[dl] & \\
-2 \arrow[dr] & & -1 \arrow[dl] & & -1+\omega \arrow[dr] & & -2+\omega \arrow[dl] \\
 & 1-\omega \arrow[drr] & & & & -\omega \arrow[dll] & \\
 & & & 1 \arrow[d] & & & \\
 & & & 0 & & &
\end{tikzcd}
$$

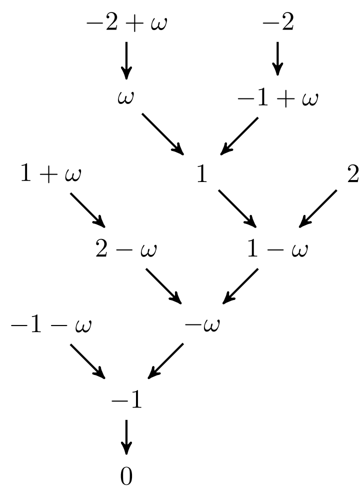
$x + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{7,p}$.

- $p = -1 + \omega$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; 1\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm 2, \pm(2-\omega), \pm(1+\omega)\}$.

$$
\begin{tikzcd}
 & 1+\omega \arrow[d] & & 2 \arrow[d] & \\
 & -1+\omega \arrow[dr] & & \omega \arrow[dl] & \\
-2+\omega \arrow[dr] & & -1 \arrow[dr] & & -2 \arrow[dl] \\
 & -1-\omega \arrow[dr] & & -\omega \arrow[dl] & \\
2-\omega \arrow[dr] & & 1-\omega \arrow[dl] & & \\
 & 1 \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}
$$

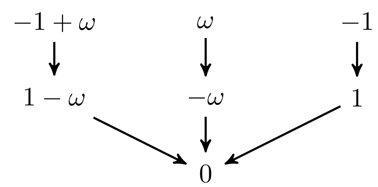
$x - 1 + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{7,p}$.

- $p = -\omega$, $N(p) = 2$,
  $\mathbb{D}_{d,p} = \{0; -1\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega), \pm 2, \pm(2-\omega), \pm(1+\omega)\}$.



$x - \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{7,p}$.

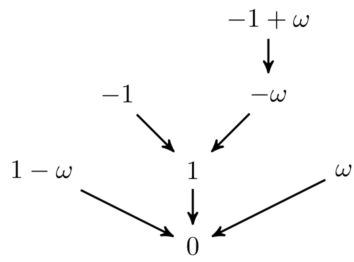
- There are no elements in $\mathbb{E}_7$ with norm 3.
- $p = 2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1; -\omega; 1-\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.



$x + 2$ is an ENS polynomial with the sail digit set $\mathbb{D}_{7,p}$.

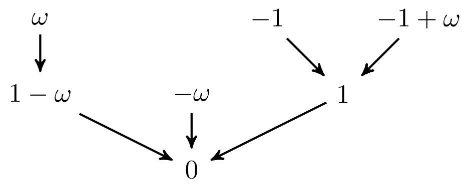
- $p = 1 + \omega$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1; \omega; 1-\omega\}$,

$a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.



$x + 1 + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{7,p}$.

- $p = 2 - \omega$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1; -\omega; 1-\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.



$x + 2 - \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{7,p}$.

- $p = -2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1; -1+\omega; \omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -2 + \omega$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1; -1+\omega; \omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- $p = -1 - \omega$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1; -\omega; -1+\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.
- There are no elements in $\mathbb{E}_7$ with norm 5.
- If $N(p) \geq 6$, then $\frac{l}{2} > \sqrt{\frac{|p|+1}{|p|-1}}$.

- $d = 11$,
  - There are no elements in $\mathbb{E}_{11}$ with norm 2.
  - $p = \omega$, $N(p) = 3$,
    $\mathbb{D}_{d,p} = \{-1; 0; 1\}$,
    $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
-\omega \arrow[dr] & & 1-\omega \arrow[dl] & & -1+\omega \arrow[dr] & & \omega \arrow[dl] \\
& 1 \arrow[drr] & & & & -1 \arrow[dll] & \\
& & & 0 & & &
\end{tikzcd}$$

    $x + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{11,p}$.
  - $p = -\omega$, $N(p) = 3$,
    $\mathbb{D}_{d,p} = \{-1; 0; 1\}$,
    $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
-\omega \arrow[dr] & & 1-\omega \arrow[dl] & & -1+\omega \arrow[dr] & & \omega \arrow[dl] \\
& -1 \arrow[drr] & & & & 1 \arrow[dll] & \\
& & & 0 & & &
\end{tikzcd}$$

    $x - \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{11,p}$.
  - $p = -1 + \omega$, $N(p) = 3$,
    $\mathbb{D}_{d,p} = \{-1; 0; 1\}$,
    $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
-\omega \arrow[dr] & & 1-\omega \arrow[dl] & & -1+\omega \arrow[dr] & & \omega \arrow[dl] \\
& 1 \arrow[drr] & & & & -1 \arrow[dll] & \\
& & & 0 & & &
\end{tikzcd}$$

    $x - 1 + \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{11,p}$.

- $p = 1 - \omega$, $N(p) = 3$,
  $\mathbb{D}_{d,p} = \{-1; 0; 1\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
-\omega \arrow[dr] & & 1-\omega \arrow[dl] & & -1+\omega \arrow[dr] & & \omega \arrow[dl] \\
& -1 \arrow[drr] & & & & 1 \arrow[dll] & \\
& & & 0 & & &
\end{tikzcd}$$

  $x + 1 - \omega$ is an ENS polynomial with the sail digit set $\mathbb{D}_{11,p}$.

- $p = 2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; 1; -\omega; 1-\omega\}$,
  $a \in \{\pm 1, \pm\omega, \pm(1-\omega)\}$.

$$\begin{tikzcd}
-1 \arrow[d] & \omega \arrow[d] & -1+\omega \arrow[d] \\
1 \arrow[dr] & -\omega \arrow[d] & 1-\omega \arrow[dl] \\
& 0 &
\end{tikzcd}$$

  $x + 2$ is an ENS polynomial with the sail digit set $\mathbb{D}_{11,p}$.

- $p = -2$, $N(p) = 4$,
  $\mathbb{D}_{d,p} = \{0; -1; \omega; -1+\omega\}$,
  $1 \underset{P}{\Rightarrow} 1$ and this is a cycle.

- If $N(p) \geq 5$, then $\frac{l}{2} > \sqrt{\frac{|p|+1}{|p|-1}}$.

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