



## On the Jensen convex and Jensen concave envelopes of means

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**Abstract.** In recent papers, the convexity of quasiarithmetic means was characterized under twice differentiability assumptions. One of the main goals of this paper is to show that the convexity or concavity of a quasiarithmetic mean implies the twice continuous differentiability of its generator. As a consequence of this result, we can characterize those quasiarithmetic means which admit a lower convex and upper concave quasiarithmetic envelope.

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**1. Introduction.** For a given  $n \in \mathbb{N}$  and an interval  $I \subset \mathbb{R}$ , an  $n$ -variable mean on  $I$  is a function  $\mathcal{M}: I^n \rightarrow I$  such that

$$\min(x_1, \dots, x_n) \leq \mathcal{M}(x_1, \dots, x_n) \leq \max(x_1, \dots, x_n) \quad \text{for all } x_1, \dots, x_n \in I.$$

The family of all  $n$ -variable means on  $I$  will be denoted by  $\mathcal{M}_n(I)$ . We will also consider the family of means of any number of arguments

$$\mathcal{M}_\infty(I) := \left\{ \mathcal{M}: \bigcup_{n=1}^{\infty} I^n \rightarrow I \mid \mathcal{M}|_{I^n} \in \mathcal{M}_n(I) \text{ for all } n \in \mathbb{N} \right\}.$$

For example, the  $n$ -variable arithmetic mean (from now on denoted by  $\mathcal{A}_n$ ) belongs to  $\mathcal{M}_n(\mathbb{R})$  for all  $n \in \mathbb{N}$ , while the general arithmetic mean  $\mathcal{A}$  (which is defined for every finite vector of reals) is an element of  $\mathcal{M}_\infty(\mathbb{R})$ .

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Such a definition of  $\mathcal{M}_\infty$  causes an important drawback as there is no assertions binding the value of a given mean for different number of arguments. One of the plausible solutions has been recently proposed by the authors in [12]. In more details, a mean  $\mathcal{M} \in \mathcal{M}_\infty(I)$  is called *repetition invariant* provided

$$\mathcal{M}(\underbrace{x_1, \dots, x_1}_{m\text{-times}}, \dots, \underbrace{x_n, \dots, x_n}_{m\text{-times}}) = \mathcal{M}(x_1, \dots, x_n)$$

holds for all  $n, m \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in I^n$ . We also adopt the standard convention that properties of means like (*Jensen*) *convexity*, *symmetry*, *continuity*, etc. refer to the respective properties of the multivariable functions  $\mathcal{M}|_{I^n}$ . In this spirit, one can easily see that the arithmetic mean is continuous, affine, symmetric, etc.

Let us now proceed to the main issue of this paper. Jensen convex and Jensen concave means are two narrow families which play an important role in the investigation of inequalities involving means, especially the Ingham–Jessen property. Recall that two means  $\mathcal{M} \in \mathcal{M}_m(I)$  and  $\mathcal{N} \in \mathcal{M}_n(I)$  form an Ingham–Jessen pair if

$$\begin{aligned} &\mathcal{N}(\mathcal{M}(x_{11}, x_{12}, \dots, x_{1m}), \mathcal{M}(x_{21}, x_{22}, \dots, x_{2m}), \dots, \mathcal{M}(x_{n1}, x_{n2}, \dots, x_{nm})) \\ &\leq \mathcal{M}(\mathcal{N}(x_{11}, x_{21}, \dots, x_{n1}), \mathcal{N}(x_{12}, x_{22}, \dots, x_{n2}), \dots, \mathcal{N}(x_{1m}, x_{2m}, \dots, x_{nm})) \end{aligned}$$

for every matrix  $x \in I^{n \times m}$ . A pair  $\mathcal{M}, \mathcal{N} \in \mathcal{M}_\infty(I)$  is an Ingham–Jessen pair if their restrictions  $\mathcal{M}|_{I^n}, \mathcal{N}|_{I^m}$  admit this property for all  $m, n \in \mathbb{N}$ .

Whenever  $\mathcal{M}, \mathcal{N} \in \mathcal{M}_\infty(I)$  are symmetric, repetition-invariant, and  $(\mathcal{M}, \mathcal{N})$  is an Ingham–Jessen pair, then we can derive several interesting inequalities, among others the mixed-means inequality [2, 20] and the Kedlaya inequality [6] (see also [12])

$$\begin{aligned} &\mathcal{N}(x_1, \mathcal{M}(x_1, x_2), \dots, \mathcal{M}(x_1, x_2, \dots, x_n)) \\ &\leq \mathcal{M}(x_1, \mathcal{N}(x_1, x_2), \dots, \mathcal{N}(x_1, x_2, \dots, x_n)) \end{aligned}$$

which is valid for all  $n \in \mathbb{N}$  and  $x \in I^n$ ; see also the recent paper [4] for more examples.

In the simplest case when one of the means is the arithmetic mean, we easily obtain:

- (i)  $(\mathcal{M}, \mathcal{A})$  is an Ingham–Jessen pair if and only if  $\mathcal{M}$  is Jensen concave;
- (ii)  $(\mathcal{A}, \mathcal{N})$  is an Ingham–Jessen pair if and only if  $\mathcal{N}$  is Jensen convex.

Let us stress that, due to the Bernstein–Doetsch theorem [1], in the family of means, Jensen convexity and Jensen concavity coincide with convexity and concavity, respectively.

Following the ideas of convex embeddings (hulls, cones, and so on), there arises a natural problem: How can we associate a convex (or concave) mean to a given one? A quite similar and comprehensive study related to the homogeneity axiom has been presented recently by the authors [15].

The rest of this paper is split into two parts – we consider convex and concave envelopes in the abstract setting (Section 2) and in the quasiarithmetic setting (Section 3).

Let us now recall several elementary facts for the family of quasiarithmetic means. This family was axiomatized in the 1930s [5, 8, 10]. For a continuous and strictly monotone function  $f : I \rightarrow \mathbb{R}$ , we define the *quasiarithmetic mean*  $\mathcal{A}_f \in \mathcal{M}_\infty(I)$  by

$$\mathcal{A}_f(x) := f^{-1}\left(\frac{f(x_1) + \dots + f(x_n)}{n}\right), \quad \text{where } n \in \mathbb{N}, x = (x_1, \dots, x_n) \in I^n.$$

The family of all quasiarithmetic means on  $I$  will be denoted by  $\mathcal{Q}(I)$ . Obviously  $\mathcal{Q}(I) \subset \mathcal{M}_\infty(I)$ . Furthermore, it was Knopp [7] who noticed that for  $I = \mathbb{R}_+$  and  $\pi_p(x) := x^p$  ( $p \neq 0$ ) and  $\pi_0(x) := \ln x$ , the quasiarithmetic mean  $\mathcal{A}_{\pi_p}$  coincides with the  $p$ -th power mean  $\mathcal{P}_p$ .

An important subclass of this family (which contains power means) consists of the means which are generated by  $\mathcal{C}^2$  functions with a nowhere vanishing first derivative. This class of generating functions is denoted by  $\mathcal{C}^{2\#}$  or, more frequently,  $\mathcal{C}^{2\#}(I)$  if it is necessary to emphasize the domain. Indeed, in view of the Jensen inequality, one can easily show that for  $f, g \in \mathcal{C}^{2\#}$ , the comparison inequality  $\mathcal{A}_f \leq \mathcal{A}_g$  is equivalent to the inequality  $\frac{f''}{f'} \leq \frac{g''}{g'}$  which is the comparability of two single-variable functions. For a detailed discussion concerning this relationship, we refer the reader to the paper by Mikusiński [9]. The operator  $f \mapsto \frac{f''}{f'}$  was used in several contexts by Pasteczka [16–19].

There are few approaches to convexity (or concavity) within this family. First, in the late 1980s, Páles [11] characterized the convexity of so-called quasideviation means (this family contains quasiarithmetic means). Later, the convexity of quasiarithmetic means was characterized by the authors [14] under the assumption that the mean is generated by a function from  $\mathcal{C}^{2\#}$ . The purpose of this paper is to prove that whenever a quasiarithmetic mean is convex (or concave), then its generator must belong to the class  $\mathcal{C}^{2\#}$ —see Theorem 3.1 below. An extensive discussion concerning convexity and concavity in the weighted setting has been given recently by Chudziak et al. [3].

**2. Abstract approach to envelopes.** In this section, we prove a few preliminary results concerning convex and concave envelopes of means. Let us first introduce a formal definition of these operators.

**Definition 2.1.** Let  $\mathcal{S}$  be a set of real-valued functions which are defined on a convex set  $D$  of a linear space  $X$ . For a given function  $f : D \rightarrow I$ , we define its  $\mathcal{S}$ -convex (resp.  $\mathcal{S}$ -concave) envelopes  $\text{conv}_{\mathcal{S}}(f) : D \rightarrow [-\infty, +\infty)$  and  $\text{conc}_{\mathcal{S}}(f) : D \rightarrow (-\infty, +\infty]$  by

$$\begin{aligned} \text{conv}_{\mathcal{S}}(f)(x) &:= \sup\{g(x) : g \in \mathcal{S}, g \text{ is a Jensen convex function, and } g \leq f\}, \\ \text{conc}_{\mathcal{S}}(f)(x) &:= \inf\{g(x) : g \in \mathcal{S}, g \text{ is a Jensen concave function, and } g \geq f\}. \end{aligned}$$

Let us emphasize a few simple however important remarks. First,  $\text{conv}_{\mathcal{S}}(f)$  (resp.  $\text{conc}_{\mathcal{S}}(f)$ ) is either finite everywhere or  $\text{conv}_{\mathcal{S}}(f) \equiv -\infty$  (resp.  $\text{conc}_{\mathcal{S}}(f) \equiv +\infty$ ). Second,  $\text{conv}_{\mathcal{S}}(f)$  is a Jensen convex function (unless  $\text{conv}_{\mathcal{S}}(f) \equiv$

$-\infty$ ) and  $\text{conc}_{\mathcal{S}}(f)$  is a Jensen concave function (unless  $\text{conc}_{\mathcal{S}}(f) \equiv +\infty$ ). Third, these operators are monotone functions of both  $f$  (with the pointwise ordering) and  $\mathcal{S}$  (with the inclusion ordering). Fourth, for every  $f$  and  $\mathcal{S}$  like above, the inequality  $\text{conv}_{\mathcal{S}}(f) \leq f \leq \text{conc}_{\mathcal{S}}(f)$  holds. Fifth, a function  $f \in \mathcal{S}$  is Jensen convex (resp. Jensen concave) if and only if  $\text{conv}_{\mathcal{S}}(f) = f$  (resp.  $\text{conc}_{\mathcal{S}}(f) = f$ ).

**2.1. Envelopes in a family of means.** We say that a family  $\mathcal{S} \subseteq \mathcal{M}_n(I)$  is *permutation-closed* if, for every  $\mathcal{M} \in \mathcal{S}$  and every permutation  $\sigma$  of  $\{1, \dots, n\}$ , the mean  $\mathcal{M}_{\sigma} : I^n \rightarrow I$  given by

$$\mathcal{M}_{\sigma}(x_1, \dots, x_n) := \mathcal{M}(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad (x_1, \dots, x_n \in I)$$

also belongs to  $\mathcal{S}$ . We say that  $\mathcal{M} : I^n \rightarrow I$  is *cyclically symmetric* (resp. *symmetric*) if  $\mathcal{M}_{\sigma} = \mathcal{M}$  for all cyclic permutations (resp. for all permutations)  $\sigma$  of  $\{1, \dots, n\}$ .

**Theorem 2.2.** *Let  $\mathcal{S} \subseteq \mathcal{M}_n(I)$  with  $\mathcal{A}_n \in \mathcal{S}$ . A cyclically symmetric mean  $\mathcal{M} \in \mathcal{S}$  admits a Jensen convex (resp. Jensen concave) envelope in  $\mathcal{S}$  if and only if  $\mathcal{M} \geq \mathcal{A}_n$  (resp.  $\mathcal{M} \leq \mathcal{A}_n$ ).*

*Proof.* First assume that there exists a Jensen convex mean  $\mathcal{N} \in \mathcal{S}$  such that  $\mathcal{N} \leq \mathcal{M}$ . If we apply this inequality to all cyclic permutations of a fixed vector  $x = (x_1, \dots, x_n) \in I^n$ , we obtain

$$\frac{1}{n} \sum_{\sigma \in \text{Cyc}_n} \mathcal{M}(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \geq \frac{1}{n} \sum_{\sigma \in \text{Cyc}_n} \mathcal{N}(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

As  $\mathcal{M}$  is cyclically symmetric, the left hand side of the above inequality equals  $\mathcal{M}(x)$ . On the other hand, by the Jensen convexity of  $\mathcal{N}$ , we have

$$\begin{aligned} & \frac{1}{n} \sum_{\sigma \in \text{Cyc}_n} \mathcal{N}(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \\ & \geq \mathcal{N}\left(\frac{1}{n} \sum_{\sigma \in \text{Cyc}_n} x_{\sigma(1)}, \dots, \frac{1}{n} \sum_{\sigma \in \text{Cyc}_n} x_{\sigma(n)}\right) \\ & = \mathcal{N}\left(\frac{x_1 + \dots + x_n}{n}, \dots, \frac{x_1 + \dots + x_n}{n}\right) = \frac{x_1 + \dots + x_n}{n}. \end{aligned}$$

This implies  $\mathcal{M}(x) \geq \mathcal{A}_n(x)$ . As  $x$  was taken arbitrarily, we have  $\mathcal{M} \geq \mathcal{A}_n$ .

Conversely, if  $\mathcal{M} \geq \mathcal{A}_n$ , then, as  $\mathcal{A}_n$  is Jensen convex (in fact, Jensen affine), we have  $\mathcal{M} \geq \text{conv}_{\mathcal{S}}(\mathcal{M}) \geq \mathcal{A}_n$ .

The proof of the second (i.e. concave) part is analogous. □

**Lemma 2.3.** *Let  $\mathcal{S} \subset \mathcal{M}_n(I)$  be a permutation-closed family. If  $\mathcal{M} \in \mathcal{S}$  is symmetric with respect to some permutation  $\sigma$  (that is  $\mathcal{M}_{\sigma} = \mathcal{M}$ ), then so are  $\text{conv}_{\mathcal{S}}(\mathcal{M})$  and  $\text{conc}_{\mathcal{S}}(\mathcal{M})$ .*

*Proof.* One can easily show that  $(\text{conv}_{\mathcal{S}}(\mathcal{M}))_{\sigma}$  is Jensen convex and we have  $(\text{conv}_{\mathcal{S}}(\mathcal{M}))_{\sigma} \leq \mathcal{M}_{\sigma} = \mathcal{M}$ . This implies  $(\text{conv}_{\mathcal{S}}(\mathcal{M}))_{\sigma} \leq \text{conv}_{\mathcal{S}}(\mathcal{M})$ .

On the other hand,  $\mathcal{M}$  is symmetric with respect to the inverse permutation  $\sigma^{-1}$  as well. Thus  $(\text{conv}_{\mathcal{S}}(\mathcal{M}))_{\sigma^{-1}} \leq \text{conv}_{\mathcal{S}}(\mathcal{M})$ , which directly implies the

inequality  $\text{conv}_{\mathcal{S}}(\mathcal{M}) \leq (\text{conv}_{\mathcal{S}}(\mathcal{M}))_{\sigma}$ . Therefore  $\text{conv}_{\mathcal{S}}(\mathcal{M}) = (\text{conv}_{\mathcal{S}}(\mathcal{M}))_{\sigma}$ , and  $\text{conv}_{\mathcal{S}}(\mathcal{M})$  is symmetric with respect to  $\sigma$ . The proof for  $\text{conc}_{\mathcal{S}}(\mathcal{M})$  is analogous.  $\square$

**2.2. Reflected means.** Let us recall the notion of reflected means from the paper [13]. Let  $I$  be an interval,  $n \in \mathbb{N}$ , and  $\mathcal{M} \in \mathcal{M}_n(I)$  be a mean. We define the *reflected mean* of  $\mathcal{M}$  as the function  $\widehat{\mathcal{M}}: (-I)^n \rightarrow (-I)$  given by

$$\widehat{\mathcal{M}}(x_1, \dots, x_n) = -\mathcal{M}(-x_1, \dots, -x_n).$$

Then it is easy to check that  $\mathcal{M}$  is convex (or concave) if and only if  $\widehat{\mathcal{M}}$  is concave (or convex), respectively. Moreover  $\mathcal{M} \leq \mathcal{N}$  if and only if  $\widehat{\mathcal{M}} \geq \widehat{\mathcal{N}}$ . This notion can be extended to the means from  $\mathcal{M}_{\infty}(I)$  in a similar way.

For a family of means  $\mathcal{S}$ , we define its reflected family by  $\widehat{\mathcal{S}} := \{\widehat{\mathcal{M}}: \mathcal{M} \in \mathcal{S}\}$ . Let us emphasize that for most families of means such reflection preserves the family (but reflects the interval). For example, reflected quasiarithmetic means on  $I$  are exactly quasiarithmetic means on  $-I$ . The same is valid for deviation and quasideviation means. This duality swaps Jensen concave and Jensen convex envelopes.

**Lemma 2.4.** *Let  $\mathcal{S}$  be a family of means on  $I$  and  $\mathcal{M} \in \mathcal{S}$ . Then  $\widehat{\text{conc}_{\mathcal{S}}(\mathcal{M})} = \text{conv}_{\widehat{\mathcal{S}}}(\widehat{\mathcal{M}})$ .*

*Proof.* Indeed, for fixed  $n \in \mathbb{N}$  and  $x \in I^n$ , one has

$$\begin{aligned} \widehat{\text{conc}_{\mathcal{S}}(\mathcal{M})}(x) &= -\text{conc}_{\mathcal{S}}(\mathcal{M})(-x) \\ &= -\inf\{\mathcal{N}(-x): \mathcal{N} \in \mathcal{S}, \mathcal{N} \text{ is Jensen concave and } \mathcal{N} \geq \mathcal{M}\} \\ &= \sup\{\widehat{\mathcal{N}}(x): \mathcal{N} \in \mathcal{S}, \mathcal{N} \text{ is Jensen concave and } \mathcal{N} \geq \mathcal{M}\} \\ &= \sup\{\widehat{\mathcal{N}}(x): \widehat{\mathcal{N}} \in \widehat{\mathcal{S}}, \widehat{\mathcal{N}} \text{ is Jensen convex and } \widehat{\mathcal{N}} \leq \widehat{\mathcal{M}}\} \\ &= \text{conv}_{\widehat{\mathcal{S}}}(\widehat{\mathcal{M}})(x), \end{aligned}$$

which ends the proof.  $\square$

**3. Quasiarithmetic means.** In the next result, we establish a complete characterization of the convexity of quasiarithmetic means.

**Theorem 3.1.** *Let  $I$  be an open interval,  $f: I \rightarrow \mathbb{R}$  be a continuous, strictly monotone function. Then  $\mathcal{A}_f$  is convex if and only if the following two conditions are valid:*

- (1)  $f \in \mathcal{C}^{2\#}(I)$ ;
- (2) either  $f''$  is nonvanishing and  $\frac{f'}{f''}$  is positive and concave, or  $f'' \equiv 0$ .

*Proof.* The implication  $(\Leftarrow)$  was already proved by the authors in [14]. It was also proved that, under the assumption that  $\mathcal{A}_f$  is convex, (1) implies (2). Therefore the only remaining part is to show that every convex quasiarithmetic mean is generated by a  $\mathcal{C}^{2\#}$  function.

As  $\mathcal{A}_f$  is convex, by Theorem 2.2, we obtain  $\mathcal{A}_f \geq \mathcal{A}$ . Thus, by Jensen’s inequality and the well-known identity  $\mathcal{A}_f = \mathcal{A}_{-f}$ , one may assume without

loss of generality that  $f$  is strictly increasing and convex. Then  $f$  has strictly positive one-sided derivatives  $f'_+$  and  $f'_-$  at every point of  $I$ .

Applying some general results concerning quasideviation means (cf. [11, Theorem 11]), it can be shown that  $\mathcal{A}_f$  is convex if and only if there exist  $a, b: I^2 \rightarrow \mathbb{R}$  such that

$$f\left(\frac{x+y}{2}\right) - f\left(\frac{u+v}{2}\right) \leq a(u, v)(f(x) - f(u)) + b(u, v)(f(y) - f(v)) \quad (3.1)$$

is valid for all  $x, y, u, v \in I$ .

For  $x > u$  and  $y = v$ , we obtain

$$\frac{f\left(\frac{x+v}{2}\right) - f\left(\frac{u+v}{2}\right)}{x - u} \leq a(u, v) \cdot \frac{f(x) - f(u)}{x - u}.$$

Upon taking the limit  $x \searrow u$ , it follows that

$$\frac{1}{2} \cdot f'_+\left(\frac{u+v}{2}\right) \leq a(u, v)f'_+(u).$$

Therefore

$$\frac{f'_+\left(\frac{u+v}{2}\right)}{2f'_+(u)} \leq a(u, v).$$

Analogously, we obtain

$$a(u, v) \leq \frac{f'_-\left(\frac{u+v}{2}\right)}{2f'_-(u)},$$

which implies the double inequality

$$\frac{f'_+\left(\frac{u+v}{2}\right)}{2f'_+(u)} \leq a(u, v) \leq \frac{f'_-\left(\frac{u+v}{2}\right)}{2f'_-(u)}. \quad (3.2)$$

Take  $p \in I$  arbitrarily. As  $f$  is convex, we know that  $f'_-(p) \leq f'_+(p)$  and  $f$  is differentiable everywhere except at countably many points. In particular, one can take  $u_p \in I$  such that  $f$  is differentiable at  $u_p$  and  $v_p := 2p - u_p \in I$ . Then (3.2) with  $(u, v) := (u_p, v_p)$  simplifies to

$$\frac{f'_+(p)}{2f'_+(u_p)} \leq \frac{f'_-(p)}{2f'_-(u_p)}$$

which implies  $f'_+(p) \leq f'_-(p)$ . Consequently,  $f$  is differentiable at  $p$ . As  $f$  is convex, we get  $f \in \mathcal{C}^1(I)$  with  $f' \neq 0$  (in particular,  $f'$  is positive). Then, in view of (3.2) and the similar inequality for the function  $b$ , one gets

$$a(u, v) = \frac{f'\left(\frac{u+v}{2}\right)}{2f'(u)}, \quad b(u, v) = \frac{f'\left(\frac{u+v}{2}\right)}{2f'(v)} \quad (u, v \in I).$$

Now condition (3.1) can be equivalently rewritten as

$$\frac{f\left(\frac{x+y}{2}\right) - f\left(\frac{u+v}{2}\right)}{f'\left(\frac{u+v}{2}\right)} \leq \frac{1}{2} \left( \frac{f(x) - f(u)}{f'(u)} + \frac{f(y) - f(v)}{f'(v)} \right) \quad (x, y, u, v \in I). \quad (3.3)$$

This implies that the two-variable continuous function  $F: I^2 \rightarrow \mathbb{R}$  given by

$$F(x, u) := \frac{f(x) - f(u)}{f'(u)}$$

is convex on  $I^2$ . In particular, for all fixed  $x \in I$ , the mapping  $u \mapsto F_x(u) := F(x, u)$  is convex on  $I$ . Consequently,  $F_x$  is differentiable at every point of  $I$  from the left and from the right. However, as  $f \in \mathcal{C}^1(I)$ , the mapping  $u \mapsto f(x) - f(u)$  is differentiable. Therefore  $f'$  is differentiable at every point of  $I$  both from the left and from the right. We will denote its one-sided derivatives by  $f''_-$  and  $f''_+$ .

Let  $x \in I$  be fixed. By the convexity of  $F_x$ , for all  $v \in I$ , there exist a real number  $p(x, v)$  such that

$$\frac{f(x) - f(u)}{f'(u)} - \frac{f(x) - f(v)}{f'(v)} \geq p(x, v) \cdot (u - v) \quad \text{for all } u, v \in I.$$

Then, for all  $u, v \in I$ ,

$$\frac{(f(x) - f(v))(f'(v) - f'(u)) - f'(v)(f(u) - f(v))}{f'(u)f'(v)} \geq p(x, v) \cdot (u - v)$$

Now assume that  $u > v$  and divide by  $u - v$  side-by-side. Then we get

$$\frac{(f(x) - f(v))\frac{f'(v) - f'(u)}{u - v} - f'(v)\frac{f(u) - f(v)}{u - v}}{f'(u)f'(v)} \geq p(x, v).$$

By taking the limit as  $u \searrow v$ , we obtain

$$\frac{(f(v) - f(x))f''_+(v) - f'(v)^2}{f'(v)^2} \geq p(x, v).$$

Repeating the same argumentation for  $u < v$ , we similarly obtain

$$\frac{(f(v) - f(x))f''_-(v) - f'(v)^2}{f'(v)^2} \leq p(x, v).$$

The above inequalities then imply that

$$(f(v) - f(x))(f''_+(v) - f''_-(v)) \geq 0 \quad \text{for all } x, v \in I.$$

By taking  $x$  to be smaller and bigger than  $v$ , it follows that  $f''_+(v) = f''_-(v)$ , which proves the differentiability of  $f'$  at  $v$ .

The remaining part is to show that  $f''$  is continuous. However, as  $F_x$  is convex and differentiable, we know that  $F'_x$  is continuous and the continuity of  $f''$  is straightforward. □

**Theorem 3.2.** *Let  $I$  be an interval,  $f \in \mathcal{C}^{2\#}(I)$  be a strictly increasing and convex function. Then  $\text{conv}_{\mathcal{Q}(I)}(\mathcal{A}_f) = \mathcal{A}_g$  for some  $g \in \mathcal{C}^{2\#}(I)$ .*

*Moreover either  $g'' \equiv 0$  (and  $\mathcal{A}_g$  is the arithmetic mean) or  $g''$  is nowhere vanishing and  $\frac{g'}{g''} = \text{conc}_{\mathcal{C}(I)}\left(\frac{f'}{f''}\right)$ .*

*Proof.* Let  $P$  be the family of all strictly monotone, affine functions on  $I$  and denote

$$U := \{h \in \mathcal{C}(I) : \mathcal{A}_h \text{ is Jensen convex and } \mathcal{A}_h \leq \mathcal{A}_f\}.$$

Then, by the previous theorem,

$$U = \{h \in \mathcal{C}^{2\#}(I) : \mathcal{A}_h \text{ is Jensen convex and } \mathcal{A}_h \leq \mathcal{A}_f\}.$$

Obviously  $P \subseteq U$  as  $\mathcal{A} \leq \mathcal{A}_f$  and the arithmetic mean is convex. Moreover, by definition,

$$\text{conv}_{\mathcal{Q}(I)}(\mathcal{A}_f)(x) = \sup\{\mathcal{A}_h(x) : h \in U\} \quad \text{for all } x \in \bigcup_{n=1}^{\infty} I^n.$$

If  $U = P$ , then obviously  $\mathcal{A}_h = \mathcal{A}$  for all  $h \in U$  and  $\text{conv}_{\mathcal{Q}(I)}(\mathcal{A}_f) = \mathcal{A}$ . From now on assume that the set  $U_0 := U \setminus P$  is nonempty. Then, for every  $h \in U_0$ , we have  $\mathcal{A}_h \geq \mathcal{A}$  and  $\mathcal{A}_h \neq \mathcal{A}$ . In particular,

$$\text{conv}_{\mathcal{Q}(I)}(\mathcal{A}_f)(x) = \sup\{\mathcal{A}_h(x) : h \in U_0\} \quad \text{for all } x \in \bigcup_{n=1}^{\infty} I^n. \quad (3.4)$$

By virtue of Theorem 3.1, for all  $h \in U_0$ , we have  $h \in \mathcal{C}^{2\#}(I)$ ,  $h''$  is nowhere vanishing, and  $\frac{h''}{h'}$  is positive and concave. Moreover, applying a well-known comparability criterion, we have  $\frac{h''}{h'} \leq \frac{f''}{f'}$  for all  $h \in U_0$ . In particular,  $\frac{f''}{f'}$  is positive on its domain.

On the other hand, it is relatively easy to verify that each function  $h : I \rightarrow \mathbb{R}$  satisfying all the properties above belongs to  $U_0$ . Therefore

$$U_0 = \{h \in \mathcal{C}^{2\#}(I) : h'' \text{ is nonvanishing, } \frac{h'}{h''} \text{ is concave, and } \frac{h'}{h''} \geq \frac{f'}{f''}\}.$$

Now define  $m : I \rightarrow \mathbb{R}$  by  $m := \text{conc}_{\mathcal{C}(I)}\left(\frac{f'}{f''}\right)$ . By  $m \geq \frac{f'}{f''}$ , we know that  $m$  is positive. Thus the 2nd-order linear ordinary differential equation  $\frac{g'}{g''} = m$  has a solution  $g \in \mathcal{C}^{2\#}(I)$ .

Obviously  $g''$  is nowhere vanishing and  $\frac{g'}{g''}$  is positive and concave. Thus Theorem 3.1 implies that  $\mathcal{A}_g$  is convex.

On the other hand, by the definition of the concave envelope for every  $h \in U_0$ , we have  $\frac{h'}{h''} \geq \frac{g'}{g''}$  and therefore  $\mathcal{A}_h \leq \mathcal{A}_g$ . Applying this inequality to all  $h \in U_0$  in view of (3.4), one gets  $\text{conv}_{\mathcal{Q}(I)}(\mathcal{A}_f) \leq \mathcal{A}_g$ .

To verify the converse inequality, observe that  $\frac{g''}{g'} = \frac{1}{m} \leq \frac{f''}{f'}$  which implies  $\mathcal{A}_g \leq \mathcal{A}_f$ . Thus  $\mathcal{A}_g$  is a convex minorant of  $\mathcal{A}_f$ , equivalently  $g \in U_0$ . Applying the inequality (3.4), we obtain  $\text{conv}_{\mathcal{Q}(I)}(\mathcal{A}_f) \geq \mathcal{A}_g$ .  $\square$

Using Lemma 2.4, we can formulate the result concerning concave envelopes in a family of quasiarithmetic means.

**Corollary 3.3.** *Let  $I$  be an interval,  $f \in \mathcal{C}^{2\#}(I)$  be an increasing and concave function. Then  $\text{conc}_{\mathcal{Q}(I)}(\mathcal{A}_f) = \mathcal{A}_g$  for some  $g \in \mathcal{C}^{2\#}(I)$ .*

*Moreover either  $g'' \equiv 0$  (and  $\mathcal{A}_g$  is the arithmetic mean) or  $g''$  is nowhere vanishing and  $\frac{g'}{g''} = \text{conv}_{\mathcal{C}(I)}\left(\frac{f'}{f''}\right)$ .*

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### References

- [1] Bernstein, F., Doetsch, G.: Zur Theorie der konvexen Funktionen. *Math. Ann.* **76**(4), 514–526 (1915)
- [2] Carlson, B.C., Meany, R.K., Nelson, S.A.: Mixed arithmetic and geometric means. *Pac. J. Math.* **38**(2), 343–349 (1971)
- [3] Chudziak, J., Gazowska, D., Jarczyk, J., Jarczyk, W.: On weighted quasi-arithmetic means which are convex. *Math. Inequal. Appl.* **22**(4), 1123–1136 (2019)
- [4] Chudziak, J., Páles, Z., Pasteczka, P.: From the Ingham–Jessen property to mixed-mean inequalities. [arXiv:1909.13769](https://arxiv.org/abs/1909.13769) (2019)
- [5] de Finetti, B.: Sul concetto di media. *Giorn. Ist. Ital. Attuari* **2**, 369–396 (1931)
- [6] Kedlaya, K.S.: Proof of a mixed arithmetic-mean, geometric-mean inequality. *Am. Math. Mon.* **101**(4), 355–357 (1994)
- [7] Knopp, K.: Über Reihen mit positiven Gliedern. *J. Lond. Math. Soc.* **3**, 205–211 (1928)
- [8] Kolmogorov, A.N.: Sur la notion de la moyenne. *Atti Accad. Naz. Lincei. Rend.* **6**(12), 388–391 (1930)
- [9] Mikusinski, J.G.: Sur les moyennes de la forme  $\psi^{-1}[\sum q\psi(x)]$ . *Stud. Math.* **10**, 90–96 (1948)
- [10] Nagumo, M.: Über eine Klasse der Mittelwerte. *Japan J. Math.* **7**, 71–79 (1930)
- [11] Páles, Z.: General inequalities for quasideviation means. *Aequationes Math.* **36**(1), 32–56 (1988)
- [12] Páles, Z., Pasteczka, P.: Characterization of the Hardy property of means and the best Hardy constants. *Math. Inequal. Appl.* **19**(4), 1141–1158 (2016)
- [13] Páles, Z., Pasteczka, P.: On Kedlaya-type inequalities for weighted means. *J. Inequal. Appl.* **2018**(1), 1–22 (2018)
- [14] Páles, Z., Pasteczka, P.: On the best Hardy constant for quasi-arithmetic means and homogeneous deviation means. *Math. Inequal. Appl.* **21**(2), 585–599 (2018)
- [15] Páles, Z., Pasteczka, P.: On the homogenization of means. *Acta Math. Hung.* **159**(2), 537–562 (2019)

- [16] Pasteczka, P.: When is a family of generalized means a scale? *Real Anal. Exchange* **38**(1), 193–209 (2012/13)
- [17] Pasteczka, P.: A new estimate of the difference among quasi-arithmetic means. *Math. Inequal. Appl.* **18**(4), 1321–1327 (2015)
- [18] Pasteczka, P.: Lower estimation of the difference between quasi-arithmetic means. *Aequationes Math.* **92**(1), 7–24 (2018)
- [19] Pasteczka, P.: On a lattice-like property of quasi-arithmetic means. *J. Math. Anal. Appl.* **486**(1), 123892 (2020)
- [20] Sadikova, R.K.: Comparison of discrete mixed means that contain symmetric functions. *Mat. Zamet.* **80**(2), 262–269 (2006)

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