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Sub-Finsler Geometry and Non-positive Curvature in Hilbert Geometry

DISSERTATION FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (PHD)

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Doctoral Council of Natural Sciences and Informatics
Doctoral School of Mathematical and Computational Sciences
Debrecen, 2019

Ezen értekezést a Debreceni Egyetem Természettudományi Doktori Tanács Matematika- és Számítástudományok Doktori Iskola Differenciálgeometria programja keretében készítettem a Debreceni Egyetem természettudományi doktori (PhD) fokozatának elnyerése céljából.

Debrecen, 2019. március

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Layth M. Alabdulsada

doktorjelölt

Tanúsítom, hogy Layth M. Alabdulsada doktorjelölt 2015-2019 között a fent megnevezett Doktori Iskola Differenciálgeometria doktori program keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

Debrecen, 2019. március

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Dr. László Kozma

témavezető

Sub-Finsler Geometry and Non-positive Curvature in Hilbert Geometry

Értekezés a doktori (PhD) fokozat megszerzése érdekében
matematika tudományágban.

Írta: Layth Muhsin Habeeb Alabdulsada okleveles matematikus

Készült a Debreceni Egyetem Matematika- és Számítástudományok
Doktori Iskolája Matematikai analízis, függvényegyenletek és
egyenlőtlenségek programja keretében

Témavezető: Dr. László Kozma egyetemi docens

A doktori szigorlati bizottság:

Elnök:	Dr.	....
Tagok:	Dr.	....
	Dr.	....

A doktori szigorlat időpontja: 2018. szeptember .

Az értekezés bírálói:

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A bírálóbizottság:

Elnök:	Dr.	....
Tagok:	Dr.	....
	Dr.	....
	Dr.	....
	Dr.	....

Az értekezés védésének időpontja: ....

Acknowledgements

First and foremost, I would like to thank my supervisor Dr. László Kozma for his guidance, unlimited favors and consistent support since the first day I met him. I am very grateful for all the knowledge and advices he has delivered to me, I appreciate all the time he has spent with me and for being very patient in all the times. I owe him for his permanent willingness to listen to me and share his ideas with me.

I would also like to acknowledge all the members of the Institute of Mathematics for the huge support that I received since I started my PhD studies four years ago.

I want to express my deep gratitude to Dr. Szilasi József for all his efforts in teaching me for more than three continuous semesters as well as I am very thankful to Dr. Muzsnay Zoltán, Dr. Figula Ágota, Dr. Tran Quoc Binh and Dr. Vincze Csaba for all the knowledge that they have conveyed to me through the courses that they taught me, I wish to express my heartfelt appreciation to my colleagues Dr. Salah G. Elgendi, Dr. Kertész Dávid and Hubicska Balázs for their aid during my studies in Debrecen.
I would also like to thank the head of the Doctoral School of Mathematical and Computational Sciences in University of Debrecen, Dr. Zsolt Páles for his continuous support.
Last but not least, I cannot thank enough my family for their continuous motivation, encouragement and help throughout the years.

Contents

Introduction

This dissertation consists of two main parts. The first part has intended to clarify what is the relationships among the concepts of non-positive curvatures, especially, that in the case of Hilbert metric of a convex domain some of them are equivalent. Furthermore, we show a condition which implies the rigidity feature: if the Hilbert metric is Berwald, i.e., its Finslerian Chern connection reduces to a linear one, then the domain is an ellipsoid and the metric is Riemannian, where we consider different types of non-positive curvature properties of the Hilbert metric of a convex domain in $\mathbb{R}^n$.

During the more than one-hundred years elapsed, a plenty of geometrical properties of the Hilbert metric have been investigated from several aspects. The reader may find a comprehensive survey of its history in [36] by Papadopoulos.

The Hilbert geometry of a convex domain is just the generalization of the Cayley-Klein model of the hyperbolic plane, first introduced by D. Hilbert in 1908. The Hilbert metric is not only an example of Finsler metric with special metric but also a geodesic metric space, where several synthetic concepts of hyperbolicity, i.e., of non-positive curvature can be used. Our aim is to find what consequences are implied by the non-positive curvature properties of the Hilbert metric, and whether the mutual relationships of the different non-positivity concepts reduce to each other in this case.

In geodesic metric spaces, several types of non-positivity concepts were introduced by Alexandrov, see [25], Busemann [12], and others. In the general context, the Alexandrov's one is the strongest, implying Busemann's one. Then a weaker class of non-positively curved spaces is the spaces of peakless metrics, called Pedersen type metrics by some authors, and the weakest one is the class of geodesic metric spaces with convex capsules. (See the definitions in Chapter 2). Examples show that the inclusion among these classes is proper.

Turning to the Hilbert geometry, it was proved ([12]) that for any strictly convex domain, the Hilbert metric automatically satisfies the 2 last mentioned assumptions of non-positivity, namely, the Pedersen property and having convex capsules. Alexandrov's and Busemann's assumptions are more rigid [26]: if the Hilbert metric of a convex domain satisfies the Busemann (or Alexandrov) nonpositivity property, then the domain is an ellipsoid, and the metric is a Riemannian one, and equivalent to the Cayley-Klein model of hyperbolic plane.

It has been turned out that the Hilbert metric is also a special case of Finsler geometry, which was introduced later (by P. Finsler in 1918), and deeply analyzed by L. Berwald in the twenties-thirties of the last century. Nevertheless, about 60 years later T. Okada ([35]) proved in a transparent manner that the Hilbert metric is projectively flat, and its flag curvature is negative constant. The main results of this part have been published in ([4]) the Journal of Geometric Analysis (2018).

The second part and the most important, in terms of the effort, the time spent and the results obtained, is devoted to solve questions of sub-Finslerian geometry. A sub-Finslerian geometry is, roughly speaking, a manifold endowed with a Finsler type metric which is defined on a k-dimensional smooth distribution only, not on the whole tangent manifold. Our viewpoint sheds some new lights on how to construct a generalized non-linear connection for a sub-Finslerian manifold, called $\mathcal{L}$-connection by the Legendre transformation. We also investigate some of its properties like normal, adapted, partial and metrical.

Sub-Finsler manifolds are generalizations of sub-Riemannian manifolds. A sub-Riemannian geometry is a smooth manifold M of dimension n endowed with a subbundle $\mathcal{D}$ of the tangent bundle TM (i.e. a smooth distribution of constant rank) and a Riemannian metric h on the distribution $\mathcal{D}$. Sub-Riemannian structures and related concepts arise naturally in many areas of pure mathematics (algebra, geometry, analysis) and applied mathematics (mechanics, control theory, mathematical physics), as well as in applications (e.g., robotics), that was the reason this subject has been received much attention in recent years. Moreover, sub-Riemannian has been discussed, extensively in the literature, (see for instance [39], [33], [29]).

Lopez and Martínez introduced the notion of sub-Finsler geometry first in 2000 as a natural generalization of sub-Riemannian geometry ([30]), one motivation for the generalization comes from control theory. Here, instead of the sub-Riemannian metric h, a more general, called sub-Finslerian metric $F : \mathcal{D} \to [0, \infty)$ is considered, where $\mathcal{D}$ is a subbundle of the tangent bundle on M as above. As in Finsler geometry we suppose that F to be regular, positively homogenous, and strictly convex. Practically it gives a norm in all horizontal subspaces $\mathcal{D}_x, x \in M$. The main results of this part divided into two papers, the first one related to connection of sub-Finsler manifold submitted to the International Journal of Geometric Methods in Modern Physics for publication (2018) [5], the second paper about Hopf-Rinow theorem on the sub-Finsler manifold [6].

In this dissertation, we develop some steps to study the connections of sub-Finslerian geometry on a manifold, particularly, a generalized non-linear connection over a bundle mapping will be constructed on the cotangent bundle of $\mathcal{D}$ by the Legendre transformation. For the construction we apply the extension of F to the whole tangent manifold, being compatible with a chosen projection operator. This step is different from the sub-Riemannian case of Langerock ([29]) where an arbitrary Riemannian extension is taken, and then the orthogonal projection played the same role as ours. Using Barthel (canonical) non-linear connection on the manifold we can construct a generalized nonlinear $\mathcal{L}$-connection. Its properties such as normal, adapted, partial and metrical can be analyzed in an analogue manner.

Moreover, we prove a version of the Hopf-Rinow theorem in the case of sub-Finslerian manifolds, which relates the properties of completeness, geodesically completeness, and compactness. The sub-Finsler bundle, the exponential map and the orthonormal frame are deeply involved in this investigation.

This dissertation is divided into three Chapters, organized as follows:

The first Chapter of the dissertation presents the basic concepts of differentiable structures. Specifically, manifolds, tangent spaces, cotangent spaces, vector fields, 1-forms and distributions. We will present the distributions of the tangent and cotangent bundles with some examples. Furthermore, we give a brief overview of Finsler geometry, which is a fundamental and natural generalization of Riemannian geometry. The Finsler structure depends on both coordinates and velocities. It is defined as a function $\hat{F} : TM \longrightarrow [0, \infty)$ on tangent bundle of a manifold. Then we summarize the basic facts on Berwald spaces. A Finsler structure is said to be of Berwald type if the Chern connection coefficients Γ^i_{jk} in natural coordinates depend only on the base point (see [10, p. 258]), the Chern connection coefficients are given by (see also [9] for details)

$$\Gamma^i_{jk} = \frac{1}{2} g^{is} \Big(\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j} \Big).$$

We also give some well known examples of the above concepts.

In the second Chapter, first of all, we explain what the non-positive curvature means by showing their definitions, more precisely, we give the basic definitions of four types of non-positive curvature in geodesic metric spaces, some of their properties and their general relationships, furthermore, list some results important for the next steps. After defining the Hilbert metric of a convex domain, we show that in the case of Hilbert metric, the first two concepts of non-positivity are equivalent if and only if the domain is an ellipsoid. Stepping to the analytical considerations, we restrict ourselves to the special case of Berwald space in Finsler geometry. Then we present those relationships and known facts about these concepts which are used to prove our result. Finally, we show that if the Finsler metric induced by the Hilbert metric of a convex domain is Berwald, then it is Riemannian and the domain is an ellipsoid.

In third Chapter, we define the sub-Finsler metrics and introduce some of its notions. We provide some examples that satisfy the sub-Finsler property, further, we introduce basic properties of sub-Finsler manifolds. Also, we shall take a look at the horizontal path-connectedness of the manifold, especially for the Chow's theorem [18]. After that, we give some basics of generalized connections in sub-Riemannian geometry, with which the normal and abnormal extremal can be characterized in the sub-Riemannian case (see the local version in [39], and the coordinate free version in [29]). Afterwards, we give a brief description of the Legendre transformation, we show the relationship between the sub-Finsler geometry with Lagrange spaces as well as the Hamiltonian spaces. We introduce the symmetric bracket associated to a sub-Finsler metric, and the symmetric product of an $\mathcal{L}$-connection it is shown, they are coincident if and only if the $\mathcal{L}$-connection is normal. With the help of the generalization of the Bott connection for involutive distribution we can characterize $\mathcal{D}$-adapted and normal connections, resp.

Last we turn to the problem of Hopf-Rinow theorem. It is well known that in the Riemannian and Finslerian geometry, there are two concepts of completeness. The first is the completeness in the sense of metric spaces, using the Riemannian metric. Furthermore, a Riemannian or Finsler manifold M is called geodesically complete if any geodesic $\gamma(t)$ starting from $x \in M$ is defined for all values of $t \in \mathbb{R}$. On the other hand, the completeness in the Finsler geometry is divided according to forward and backward distance metric, and forward and backward geodesically completeness.

Hopf-Rinow theorem is a basic theorem of complete Riemannian manifolds, which connects the completeness properties with compactness, and the exponential map. Its consequence says that any two points of the complete manifold can be connected by a length minimizing geodesic. In 1931, H. Hopf and W. Rinow showed their theorem only for surfaces, but the proof in higher dimensions is not significantly different. Hopf-Rinow theorem has been studied in detail in both Riemannian and Finslerian geometries in the literature, the best general references here are ([10], [16], [34]). In the Finsler case forward geodesic completeness is involved, only.

After a development of the sub-Riemannian geometry as well as its generalization, namely sub-Finslerian geometry, the generalization of core theorems of Riemannian geometry has been started. Relating to our issue, Strichartz ([39]), Rifford ([37]) and Agrachev et al ([2]) gave an extension for a sub-Riemannian case, while Bao et al. ([10]) showed the Finslerian version of Hopf-Rinow theorem. It turned out that in sub-Riemannian geometry, for general complete sub-Riemannian structures, the exponential mapping is not surjective. This is due to the fact that we may have abnormal minimizing curves and this is the case in the sub-Finslerian context, too.

To prove the statements of Hopf-Rinow theorem in sub-Finsler manifolds: we define the sub-Finsler metric on the distribution of the cotangent space $\mathcal{D}^*$ with help of the Legendre transformation, where we look more closely at a sub-Hamiltonian H defined on the cotangent bundle, induced by the sub-Finslerian metric on $\mathcal{D}^*$. Afterwards, we construct a sub-Finsler bundle, which plays a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry. Moreover, the sub-Finsler bundle allows an orthonormal frame for the sub-Finsler structure. Also, we introduce the notion of an exponential map in sub-Finsler geometry. At the end our main theorem, namely Hopf-Rinow theorem is stated and proved.

Chapter 1

The Background

1.1 Differentiable Structures

1.1.1 Manifolds

In spite of there are numerous types of manifolds, for example, analytic manifolds, topological manifolds, C^k-manifolds, and complex manifolds, in this dissertation we are interested mainly in smooth manifolds i.e. C^∞-manifolds. Let us start with topological manifolds, which are Hausdorff, second countable, locally Euclidean spaces.

Definition 1.1.1. We call a topological space M a *locally Euclidean* of dimension n if for each point x in M there is a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. A pair $(U, \phi : U \longrightarrow \mathbb{R}^n)$ is said to be a chart, U a coordinate neighborhood or a coordinate open set, and ϕ a coordinate map or a coordinate system on U. Moreover, a chart (U, ϕ) is called centered at $x \in U$ if $\phi(x) = 0$.

Definition 1.1.2. We say that a space is a *topological manifold* if it is locally Euclidean, Hausdorff, second countable space. It is of dimension n if it is locally Euclidean of dimension n.

Definition 1.1.3. A *smooth* or C^∞ *manifold* is a topological manifold M together with its differentiable structure on M. A manifold is of the dimension n if all of its connected components have the same dimension, i.e. n. Furthermore, a manifold of dimension 1 is called a curve, a manifold of dimension 2 is called a surface and an n-dimensional manifold is an n-manifold. We will use the symbol $\mathcal{F}(M)$ to denote the set of (real-valued) smooth functions on M.

Example 1.1.4.

- (Matrices). Let the $m \times n$ space matrices be given by $M(m \times n, \mathbb{R})$ with real elements. It is an mn-dimensional vector space under matrix scalar multiplication and addition. Thus $M(m \times n, \mathbb{R})$ is a smooth mn-dimensional manifold. Likewise, the space $M(m \times n, \mathbb{C})$ of $m \times n$ complex matrices is a $4mn$-dimension vector space over $\mathbb{R}$, and therefore, a real manifold of dimension $4mn$. In the particular case $m = n$ (square matrices), we will shorten $M(n \times n, \mathbb{R})$ and $M(n \times n, \mathbb{C})$ by $M(n, \mathbb{R})$ and $M(n, \mathbb{C})$, respectively.

- (The General Linear Group). Let the set of invertible $n \times n$ matrices with real members be a general linear group and denoted by $GL(n, \mathbb{R})$. It is a manifold of dimension n^2 due to an open subset of the n^2-dimensional vector space $M(n, \mathbb{R})$, in other words, the set where the determinant function is nonzero.

- (Finite-dimensional vector spaces). Suppose that V is n-dimensional vector space. One can define the standard topology by any norm on V, that is the topology is not related to the choice of norm. With the aid of the topology obtained in the above way, V possess a natural smooth structure given as follows. Any ordered basis $(e_1, ..., e_n)$ for V determines a linear isomorphism $e : \mathbb{R}^n \longrightarrow V$ by

$$e(x) = \sum_{i=1}^{n} x^i e_i$$

In addition, it is a homeomorphism, so the atlas including the single chart (V, e^{-1}) gives a smooth structure. It is fairly easy to check that this smooth structure is not dependent on the choice of basis. Assume that $(\widetilde{e}_1, ..., \widetilde{e}_n)$ is any other basis and suppose

$$\widetilde{e}(x) = \sum_j x^j \widetilde{e}_j$$

is the corresponding isomorphism. There is some invertible matrix (A_i^j) such that

$$e_i = \sum_j A_i^j \widetilde{e}_j \quad \text{for all} \quad j.$$

The transition map between the two charts is then defined by $\widetilde{e}^{-1} \circ e(x) = \widetilde{x}$ where $\widetilde{x} = (\widetilde{x}_1, ..., \widetilde{x}_n)$ is determined by

$$\sum_{j=1}^{n} \widetilde{x}^j \widetilde{e}_j = \sum_{i=1}^{n} x^i e_i = \sum_{i,j=1}^{n} x^i A_i^j \widetilde{e}_j.$$

Consequently, $\widetilde{x}^j = \sum_i A_i^j x^i$. So, the map from x to $\widetilde{x}$ is an invertible linear map and for this reason it is a diffeomorphism, therefore, the two charts are smoothly compatible. From above, we have that the union of two charts given by any two bases is yet a smooth atlas, and it follows that all bases give the same smooth structure. In the end, the earlier structure is said to be *standard smooth structure* on V.

1.1.2 The Tangent and Cotangent Bundle of a Manifold

A *tangent vector* to the manifold M at the point $x \in M$ is $\mathbb{R}$-linear function

$$v : C^\infty(M) \longrightarrow \mathbb{R}$$

such that

$$v(fg) = v(f)g(x) + f(x)v(g) \quad \text{for all} \quad f, g \in C^\infty(M).$$

The tangent vectors of M at point $x \in M$ form a real vector space under the natural definitions of scalar multiplication by a real number and addition of functions. This vector space is denoted by T_xM and called the *tangent space* to M at x. As a set, a *tangent bundle* is given by the disjoint union of the tangent spaces of M. That is

$$\begin{aligned} TM &:= \bigsqcup_{x\in M} T_xM \\ &= \bigcup_{x\in M} \{x\} \times T_xM = \bigcup_{x\in M} \{(x,y) \mid \quad y \in T_xM\} \\ &= \{(x,y) \mid \quad x \in M, y \in T_xM\}, \end{aligned}$$

and let $\pi : TM \longrightarrow M$ be the foot map which associates to each tangent vector $v \in T_xM$ the point $x \in M$. There exists a unique topology and $2n$-dimensional smooth structure on TM such that

$$\pi : TM \longrightarrow M$$

is a vector bundle where this vector bundle is called the *tangent bundle* of M. Sometimes, more accurately, we write π_M instead of π. For instance, the tangent bundle of TM is denoted by

$$\pi_{TM} : TTM \longrightarrow TM.$$

The cotangent bundle of a manifold is similar to the tangent bundle, more precisely, the cotangent space T_x^*M is just the dual space of a tangent space, i.e. a cotangent vector $p \in T_x^*M$ means a linear form $p : T_xM \longrightarrow \mathbb{R}$ on the tangent space T_xM. The *cotangent bundle* T^*M is the disjoint union of cotangent spaces:

$$\begin{aligned} T^*M &:= \bigsqcup_{x\in M} T_x^*M \\ &= \bigcup_{x\in M} \{x\} \times T_x^*M = \bigcup_{x\in M} \{(x,p)| \quad p \in T_x^*M\} \\ &= \{(x,p)| \quad x \in M, p \in T_x^*M\}. \end{aligned}$$

Note that there is $2n$-dimensional smooth structure on T^*M which is a natural projection (the cotangent bundle projection)

$$\hat{\pi} : T^*M \longrightarrow M,$$

which sends a cotangent vector $p \in T^*M$ to the corresponding point x of M. The cotangent bundle projection is vector bundle of rank n which is the dimension of M.

The simplest example for the tangent bundle is that of $\mathbb{R}^n$. In this case the tangent bundle is trivial $T_x\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$: each $T_x\mathbb{R}^n$ is canonically isomorphic to $T_0\mathbb{R}^n$ via the map $\{x\} \times \mathbb{R}^n \longrightarrow \{0\} \times \mathbb{R}^n$ which subtracts x, giving a diffeomorphism

$$T\mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n.$$

Our next example is the unit circle, i.e. S^1. The tangent bundle of the circle is very clear and isomorphic to $S^1 \times \mathbb{R}$. In a geometric viewpoint, it is a cylinder of infinite height.

One can easily be imagine the tangent bundles the real line $\mathbb{R}$ and the unit circle S^1, both are trivial. For manifolds of a dimension 2 the tangent bundle is 4-dimensional and thus hard to imagine.

A straightforward example of a nontrivial tangent bundle is that of the unit sphere S^2. This tangent bundle is nontrivial as a result of the hairy ball theorem. Thus, the sphere is not parallelizable.

Regarding cotangent bundle, one can give an example on a smooth manifold M (where M is the vanishing locus of a smooth function f) embedded into $\mathbb{R}^n$. Now, the set

$$\{(x, dx) \in T^*\mathbb{R}^n : f(x) = 0; \ dx \cdot f = 0\}$$

is a cotangent bundle of M. For instance, assume the 3-sphere presented by the vanishing locus of $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$. Afterwards, we can calculate a presentation of its cotangent bundle as the set

$$\{(x, y, z, w, dx, dy, dz, dw) \in T^*\mathbb{R}^4 : \\ x^2 + y^2 + z^2 + w^2 = 1; 2xdx + 2ydy + 2zdz + 2wdw = 0\}.$$

Example 1.1.5. Here, we show an example for the tangent and the cotangent bundle. Assume that we have a product structure as a manifold such that

$$(S^1)^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}^2, \|x_i\| = 1, i = 1, \ldots, n\}$$

which is an n-dimensional manifold embedded in $\mathbb{R}^{2n}$. We denote the product of n copies of 1-dimensional unite sphere in $\mathbb{R}^2$ by $(S^1)^n = S^1 \times \ldots \times S^1$. The tangent space to $(S^1)^n$ of dimensional n for any $x \in (S^1)^n$ defined by

$$T_x(S^1)^n = \{(u_1, \ldots, u_n) \in \mathbb{R}^{2n} : (x_i, u_i) = 0, i = 1, \ldots, n\}.$$

Further, the tangent bundle is given by

$$T(S^1)^n = \{(x, u) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} : x \in (S^1)^n, u \in T_x(S^1)^n\}.$$

Thus, for $x \in (S^1)^n$ the cotangent space to $(S^1)^n$ is the n-dimensional dual of the above tangent space, namely,

$$T_x^*(S^1)^n = (T_x(S^1)^n)^*.$$

This gives the cotangent bundle as follows:

$$T^*(S^1)^n = \{(x, \alpha) \in \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^* : x \in (S^1)^n, \alpha \in T_x^*(S^1)^n\}.$$

Indeed, the dimension of the cotangent bundle is $2n$.

1.1.3 Vector fields and 1-forms

Let M be an n-dimensional manifold, a smooth section of π which associates a vector in TM to each point $x \in M$ is called a *vector field*. More precisely, a vector field on a manifold M is a smooth mapping

$$X : M \longrightarrow TM$$

where the image of x, denoted by X_x, lies in T_xM, the tangent space at x. In the setting of fiber bundles, such a map is called a section of the tangent bundle. A vector field on M is therefore a section of the tangent bundle $\pi : TM \longrightarrow M$ of M. We use the notation $\mathfrak{X}(M) := \Gamma(TM)$ for C^∞-module of vector fields on M. Vector fields can be added pointwise

$$(X+Y)_x = X_x + Y_x.$$

Moreover, it can be multiplied by smooth functions on M

$$(fX)_x = f(x)X_x.$$

The collections of all vector fields $\Gamma(TM)$, therefore, takes on the structure of a module through the commutative algebra of smooth functions on M, denoted $C^\infty(M)$.

We call the smooth section of the cotangent bundle $\hat{\pi} : T^*M \longrightarrow M$ a 1-*form* on manifold M. We use the notation $\mathcal{A}_1(M) := \Gamma(T^*M)$ for the C^∞-module of 1-forms on M.
If $\alpha \in \mathcal{A}_1(M)$ and $X \in \Gamma(TM)$, we can form the smooth function

$$\alpha(X) : M \longrightarrow \mathbb{R}, \quad x \mapsto \alpha(X)(x) := \alpha_x(X_x).$$

In particular, the differential $df : M \longrightarrow T^*M, \quad x \mapsto (df)_x$ of smooth function f on M is the unique 1-form such that

$$(df)(X) = Xf \quad \text{for all } X \in \mathfrak{X}(M).$$

Let $\mathfrak{X}^*(M)$ be the dual of the C^∞-module $\mathfrak{X}(M)$. Then the mapping

$$\begin{cases} \mathcal{A}_1(M) \longrightarrow \mathfrak{X}^*(M), & \alpha \mapsto \widetilde{\alpha} \\ \widetilde{\alpha}(X) := \alpha(X), & \text{for all} \quad X \in \mathfrak{X}(M) \end{cases}$$

is a canonical isomorphism of C^∞-modules. We identify these isomorphic modules, and sometimes simply write $\mathcal{A}_1(M) = \mathfrak{X}^*(M)$. In most cases in this subsection the conventions of the book [40] are followed.

1.1.4 Distribution of Tangent and Cotangent Bundles

Let M be a smooth n-dimensional manifold, and $k \leq n$. Suppose that for any $x \in M$, we specify a subspace of the tangent space of dimension k, i.e. $\mathcal{D}_x \subset T_xM$ in such a way that for a neighbourhood $U_x \subset M$ of x there exist k linearly independent smooth vector fields $X_1, \dots, X_k$. Moreover, for each point $y \in U_x$,

$$\mathcal{D}_y = \operatorname{span}\{X_1(y), \dots, X_k(y)\}.$$

Let $\mathcal{D}$ refer to the collection of all the linear k-dimensional subspaces $\mathcal{D}_x$ for all $x \in M$, then $\mathcal{D}$ is a *distribution* of dimension k on M, or some other mathematicians call it a *k-plane distribution.* The set of smooth vector fields $\{X_1, \ldots, X_k\}$ is said to be a local basis of $\mathcal{D}$.

Example 1.1.6. The simplest example for the distribution on M is that distribution $\mathcal{D}$ of the rank n defined by $\mathcal{D}_x = T_xM$ for all $x \in M$. Sometimes, such a distribution may not admit non-vanishing sections and this is due to the topological reasons (e.g., by the hairy ball theorem, there is no non-vanishing continuous vector fields on any even dimensional sphere).

Example 1.1.7. In $\mathbb{R}^3$ with coordinates (x, y, z), the distribution $\mathcal{D}$ defined by

$$\mathcal{D}(x,y,z) = \operatorname{span}\{X(x,y,z), Y(x,y,z)\} \quad \forall (x,y,z) \in \mathbb{R}^3$$

with

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},$$

is a rank 2 (or co-rank 1) distribution on $\mathbb{R}^3$. More generally, if

$$x = (x_1, ..., x_k, y_1, ..., y_k, z)$$

denotes the coordinates in $\mathbb{R}^{2k+1}$ and the $2k$ smooth vector fields

$$X_1, ..., X_k, Y_1, ..., Y_k$$

are defined by

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial z} \quad \text{and} \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial z}, \quad i = 1, ..., k$$

then the distribution $\mathcal{D}$ defined by

$$\mathcal{D}(x) = \operatorname{span}\{X_1(x), ..., X_k(x), Y_1(x), ..., Y_k(x)\} \quad \forall x \in \mathbb{R}^{2k+1},$$

is a co-rank 1 distribution on $\mathbb{R}^{2k+1}$.

On the other hand, the distribution of a cotangent bundle $\mathcal{D}^*$ is the dual space of $\mathcal{D} \subset TM$ given as follows: Let M be an n-dimensional smooth manifold, and suppose that at each $x \in M$ there exists a k-dimensional subspace $\mathcal{D}^*_x$ of the cotangent space T^*_xM. Assume there is a neighbourhood $U_x \subset M$ of each x subset from M then there exists a k linearly independent smooth covector fields $\alpha_1, \ldots, \alpha_k$ which form a basis of $\mathcal{D}^*_x$ for $y \in U$. In this way we obtain what is known as a distribution $\mathcal{D}^*$ which is smooth of dimension k on M and $\alpha_1, \ldots, \alpha_k$ is local basis of $\mathcal{D}^*$.

The restriction map $\alpha \mapsto \alpha_{|\mathcal{D}}$ is naturally a covector bundle map

$$T^*M \longrightarrow \mathcal{D}^*.$$

Its kernel is called the *annihilator bundle* $\mathcal{D}^0 \subseteq T^*M$. Then the factor bundle $T^*M/\mathcal{D}^0$ is isomorphic to $\mathcal{D}^*$. In other words, the annihilator of the distribution $\mathcal{D}$ is the set of all covectors which annihilate the vectors in $\mathcal{D}_x$:

$$\mathcal{D}^0_x = \{\alpha \in T^*_x M : \alpha(v) = 0 \ \forall \ v \in \mathcal{D}_x\}.$$

Similarly, we define the annihilator of a distribution $\mathcal{D}^*$:

$$(\mathcal{D}^*)^0_x = \{u \in T_x M : u(p) = 0 \ \forall \ p \in \mathcal{D}^*_x\}.$$

In this way, a distribution of the tangent bundle associates a special distribution, namely, its annihilator bundle in the cotangent bundle.

A distribution $\mathcal{D}$ on M is called *involutive* if for every point $x \in M$ there exists a local basis of vector fields $\{X_1, \ldots, X_k\}$ in a neighbourhood of each point $x \in M$ and C^m_{ij} is a smooth function on it such that for all

$$[X_i, X_j] = \sum_{m=1}^{k} C^m_{ij} X_m, \quad \forall \ 1 \leq i, j \leq k.$$

Further, the Lie bracket of two vector fields i.e. $[X_i, X_j]$, is in the span of $\mathcal{D}$. In other words, if the Lie bracket of two vector fields is a linear combination of $\{X_1, \ldots, X_k\}$. In general, this is written as $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$. Involutive distributions are the tangent spaces to foliations. Also, they are important because they satisfy the statements of the Frobenius theorem, and consequently lead to integrable systems. Nevertheless, most distributions are not involutive, for example, let $\mathcal{D}$ be a distribution on $\mathbb{R}^3$, and

$$X = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

is not involutive since $[X, Y] = -\frac{\partial}{\partial x}$ which is not a $C^\infty(\mathbb{R}^3)$-linear combination of X and Y.

On the other hand, the smooth distribution $\mathcal{D}$ on M is called *flat* if for each point $x_0 \in M$ there exists a local diffeomorphism $\varphi : U_{x_0} \longrightarrow \mathbb{R}^n$ such that $\varphi_{*,x}(\mathcal{D}_x) = \mathbb{R}^k \times \{0\}$ for all $x \in U_{x_0}$.

Proposition 1.1.8. *(Frobenius Theorem). A smooth distribution is involutive if and only if it is flat* [2].

1.2 Finsler geometry

1.2.1 Finsler spaces

In this section, we touch only a few aspects of the Finsler spaces. For a deeper discussion of Finsler spaces can be found in [17], [9] and [10].

Definition 1.2.1. Let M be an n-dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle. If the continuous function $\hat{F} : TM \longrightarrow [0,\infty)$ satisfies the following conditions:

(i) (**Regularity**) $\hat{F}$ is continuous (C^0) on TM and C^∞ on $TM \setminus \{0\}$.

(ii) (**Positive homogeneity**) $\hat{F}(tu) = t\hat{F}(u)$ for all $t > 0$ and $u \in TM$, i.e., $\hat{F}$ is positively homogeneous of degree one.

(iii) (**Strong convexity**) The matrix $\dfrac{\partial^2 \hat{F}^2}{\partial y^i \partial y^j}(u)$ is positive definite for all $u \in TM \setminus \{0\}$.

Then we say that $\hat{F}$ is a Finsler fundamental function and $(M, \hat{F})$ is a *Finsler manifold*.

Every Finsler fundamental function naturally determines a metric $d_{\hat{F}}$ as follows: Let $\gamma : [0, r] \longrightarrow M$ be a piecewise C^∞ curve. Its *integral length* is defined as

$$L(\gamma) = \int_0^r \hat{F}(\gamma(t), \dot{\gamma}(t))\, dt.$$

For $x_0, x_1 \in M$ denote by $\Gamma(x_0, x_1)$ the set of all piecewise C^∞ curves $\gamma : [0, r] \longrightarrow M$ such that $\gamma(0) = x_0$ and $\gamma(r) = x_1$. Define a map $d_{\hat{F}} : M \times M \longrightarrow [0,\infty)$ by

$$d_{\hat{F}}(x_0, x_1) = \inf_{\gamma\in\Gamma(x_0,x_1)} L(\gamma).$$

Of course, we have $d_{\hat{F}}(x_0, x_1) \geq 0$, where equality holds if and only if $x_0 = x_1$; and

$$d_{\hat{F}}(x_0, x_2) \leq d_{\hat{F}}(x_0, x_1) + d_{\hat{F}}(x_1, x_2).$$

In general, since $\hat{F}$ is only a positively homogeneous function,

$$d_{\hat{F}}(x_0, x_1) \neq d_{\hat{F}}(x_1, x_0),$$

therefore $(M, d_{\hat{F}})$ is a non-reversible metric space, in general.

Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM \setminus \{0\} \longrightarrow M$. Unlike the Levi-Civita connection in Riemann geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM, we choose the *Chern connection* whose coefficients are denoted by Γ^i_{jk} (see [10, p. 38]). This connection induces the *hh-curvature tensor*, denoted by R (see [10, Chapter 3]).

Let $(x, y) \in TM \setminus \{0\}$ and V a section of the pulled-back bundle π^*TM. Then

$$\kappa(y, V) = \frac{g(R(V, y)y, V)}{g(y, y)g(V, V) - [g(y, V)]^2},$$

is the *flag curvature* with flagpole y and transverse edge V. Here

$$g_{(x,y)} := g_{ij}dx^i \otimes dx^j := (\frac{1}{2}\hat{F}^2)_{y^i y^j} dx^i \otimes dx^j$$

is the Riemannian metric on the pulled-back bundle π^*TM (see [10, p. 68]). When $\hat{F}$ is Riemannian, then the flag curvature coincides with the sectional curvature. Let κ abbreviate the collection of flag curvatures

$$\{\kappa(V, W) : 0 \neq V, W \in T_xM, x \in M, V \text{ and } W \text{ are not collinear}\}.$$

We say that the flag curvature of $(M, \hat{F})$ is *non-positive* if $\kappa \leq 0$.

Example 1.2.2.

- One can find a Finsler structure on C^∞-submanifolds (including open subsets) of a normed vector space of n-dimension if the norm of the vector space is smooth outside the origin.

- Riemannian metrics are special cases of Finsler metrics except for the pseudo-Riemannian metrics.

Let us just recall a little bit of background information on the history of connection, the main connection is Levi-Civita connection which plays a leading part in the metric geometry of Riemannian manifolds by using the Christoffel symbols, which has two remarkable attributes: metric-compatibility and torsion freeness. Therefore, without a doubt, it is a matter of great significance to introduce connections in Finsler geometry with analogue properties. In 1918, P. Finsler studies Finsler metrics and his PhD's dissertation reprinted and published in 1951. In the dissertation Finsler mainly studied the variational problem of Finsler metrics. Many mathematicians have made great contributions to the connection theory of Finsler manifolds. The first interesting linear connection was introduced by L. Berwald [3] in 1926, the Berwald connection on the slit tangent bundle $TM\setminus\{0\}$

is torsion free, but is not necessarily metric-compatible. Later on, another linear connection was discovered by E. Cartan [15] in 1934 that is metric-compatible but has torsion. The Cartan connection remains, to this day, extremely favoured with the Matsumoto and Debrecen school of Finsler geometry. Since then, many kinds of connections have been discussed [31]. In the last century , S.S. Chern [17] has found a new torsion-free linear connection. In natural coordinate on the slit tangent bundle $TM \setminus \{0\}$, the Chern connection coefficients are given by (see also [9] for details)

$$\Gamma^i_{jk} = \frac{1}{2} g^{is} \Big(\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j} \Big).$$

After that, Z. Shen in 1994 investigated a special connection in a Finsler space [38]. This connection is torsion-free, but not metrical. The importance of this connection is that the vanishing of its *hv*-curvature tensor characterizes the Riemannian spaces among Finsler spaces.

1.2.2 Berwald spaces

In this section, we give a brief exposition of some known facts about Berwald spaces. For more details we refer the reader to [10].

Definition 1.2.3. A Finsler manifold is of *Berwald type* if the Chern connection coefficients Γ^i_{jk} in natural coordinates depend only on the base point (see [10, p. 258]).

Kristály et al in [28], showed the equivalence of the non-positivity of the flag curvature with the non-positive curvature properties of Busemann and Pedersen for Berwald manifolds. So an analytical property is characterized by synthetic concepts of non-positively curved metric spaces. Next are auxiliary results from Finsler geometry Kristály et al used to prove their main theorem. The Chern connection defines the covariant derivative $D_V U$ of a vector field $U \in \mathfrak{X}(M)$ in the direction $V \in T_x M$. Since, generally, the Chern connection coefficients Γ^i_{jk} in natural coordinates have a directional dependence, we must say explicitly that $D_V U$ is defined with a fixed reference vector. Specially, let $\sigma : [0, r] \longrightarrow M$ be a smooth curve with velocity field $T = T(t) = \dot{\sigma}(t)$. Suppose that U and W are vector fields defined along σ. We define $D_T U$ with reference vector W as

$$D_T U = \left[\frac{dU^i}{dt} + U^i T^k (\Gamma^i_{jk})_{(\sigma, W)} \right] \frac{\partial}{\partial x^i}_{|\sigma(t)}.$$

We call a curve $\sigma : [0,r] \longrightarrow M$, where $T = \dot{\sigma}$ its velocity, a (Finslerian) geodesic if

$$D_T\left[\frac{T}{\hat{F}(T)}\right] = 0$$

with reference vector T. Now, if we consider U, V and W are vector fields along a curve σ, which has velocity $T = \dot{\sigma}$, we have the differential rule

$$\frac{d}{dt} g_W = g_W(D_T U, V) + g_W(U, D_T V)$$

such that $D_T U$ and $D_T V$ are with reference vector W and one of the following cases holds:

- U or V is proportional to W, or
- $W = T$ and σ is a geodesic.

A vector field J along a geodesic $\sigma : [0,r] \longrightarrow M$ such that T its velocity field, is said to be a Jacobi field if the following equation holds:

$$D_T D_T J + R(J,T)T = 0,$$

where R is the curvature tensor. Here, the covariant derivative D_T is defined with reference vector T. Let $\Sigma : [0,r] \times [-1,1] \longrightarrow M$ be a piecewise C^∞ variation of a geodesic $\sigma : [0,r] \longrightarrow M$, with $\Sigma(.,0) = \sigma$. Suppose that

$$T = T(t,s) = \frac{\partial \Sigma}{\partial t}, \qquad U = U(t,s) = \frac{\partial \Sigma}{\partial s}$$

are the velocities of the t-curves (i.e., $\Sigma(t, s = \text{constant})$) and s-curves (i.e., $\Sigma(t = \text{constant}, s)$), respectively. The expression for the first variation of arc length gives us

$$L'_\Sigma(0) := \frac{d}{ds} L(\Sigma(.,s))_{|s=0} = g_T\Big(U, \frac{T}{\hat{F}(T)}\Big)\Big|_0^r.$$

The second variation of arc length can be expressed as

$$\begin{aligned} L''_\Sigma(0) &:= \frac{d^2}{ds^2} L(\Sigma(.,s))_{|s=0} \\ &= \int_0^r \frac{1}{\hat{F}(T)}[g_T(D_T U, D_T U) - g_T(R(U,T)T, U)]dt \\ &+ g_T\Big(U, \frac{T}{\hat{F}(T)}\Big)\Big|_0^r - \int_0^r \frac{1}{\hat{F}(T)}\Big(\frac{\partial \hat{F}(T)}{\partial s}\Big)^2 dt, \end{aligned}$$

where all covariant derivatives are with reference vector $T = T(t,0)$.

Example 1.2.4. Let us consider an example of y-global Berwald space of a dimension 3 that is neither Riemannian nor locally Minkowskian (notice that all Riemannian and locally Minkowskian spaces are examples of Berwald spaces). This space is defined as follows:

Suppose that we have the cartesian product as an underlying manifold such that

$$M := S^2 \times S^1,$$

it is compact and boundaryless. As local coordinates, we can use the standard spherical θ, ϕ on S^2, where t parameter of S^1. Specifically, we measure ϕ from the positive z axis down. Furthermore, t is where $(\cos t,\ \sin t,\ 0)$ parameterizes S^1.

In our example, the Riemannian metric G is defined by the product metric on $S^2 \times S^1$. In this case, S^2 and S^1 are defined the standard Riemannian metrics that they are inherited as submanifolds of Euclidean $\mathbb{R}^3$. Especially, one gets that

$$G := (\sin^2\phi \; d\theta \otimes d\theta + d\phi \otimes d\phi) + dt \otimes dt.$$

This metric is not flat because it has nonzero curvature tensor.

Here, we need a 1-form which is given by

$$\beta := \epsilon dt,$$

such that ϵ is any fixed positive constant less than 1. Such a β here is globally defined on M. However, the coordinate t is not. It is non-vanishing by examination, and has Riemannian norm $\|\beta\| = \epsilon < 1$.

An uncomplicated computation proves that it is parallel with respect to the Levi-Civita (Christoffel) connection of G. As a consequence of the above construction, we have a Randers metric. If x is any point on M, with coordinates (θ, ϕ, t), and the arbitrary tangent vector $y \in T_xM$ be extended as $y^\theta \partial_\theta + y^\phi \partial_\phi + y^t \partial_t$. Then we have the following form:

$$\hat{F}(x, y) := \sqrt{\sin^2\phi (y^\theta)^2 + (y^\phi)^2 + (y^t)^2} + \epsilon y^t.$$

After all, this $\hat{F}$ is is a Finsler metric of Randers type. One can compute easily the fundamental tensor (see [10, p. 19]).

Chapter 2

Non-Positive Curvature in Finsler Geometry

2.1 Non-positive curvature in metric spaces

Definition 2.1.1. Let M be an arbitrary set. A function $d : M \times M \longrightarrow \mathbb{R} \cup \{\infty\}$ is a metric on M such that for any $x, y, z \in M$ the following conditions are holds.

(1) **Positiveness**: $d(x, y) > 0$ if $x \neq y$, and $d(x, x) = 0$.

(2) **Symmetry**: $d(x, y) = d(y, x)$.

(3) **Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

A metric space is a set with a metric on it. In a formal manner, a metric space is a pair (M, d) such that d is a metric on the set M. The elements of M is said to be points of the metric space; $d(x, y)$ is indicated to as the distance between points x and y.

Example 2.1.2. There are various examples of metric spaces. Here we describe several important ones.

(i) The normed vector space always forms a metric space by $d(x, y) := \|x - y\|$; a pseudonormed vector space is a pseudometric space.

(ii) The hyperbolic plane is a metric space. More precisely: Let us consider any connected Riemannian manifold M, then we can turn M into a metric space by interpreting the distance of two points as the infimum of the lengths of the paths (continuously differentiable curves) connecting them.

(iii) Another example for metric space, we are going to take the unit circle, S^1, which is a set of points in the plane and has radius one centered at the origin (0, 0). The circle S^1 takes the restriction of the Euclidean metric on it because it is a subset of the plane. If we define another metric by fixing the distance between two points as the length of the short arc between them. For instance, the arc-length distance between two opposite points of the circle is equal to π. The distance between adjacent vertices of regular n-gon (inscribed into the circle) is equal to $2\pi/n$.

2.1.1 Geodesics

Let (M, d) a metric space, and $\gamma : [0, 1] \to M$ a curve in M. The length of γ is

$$\ell(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : 0 = t_0 < t_1 < ... < t_n = 1, n \in \mathbb{N} \right\}.$$

A curve $\gamma : [0, 1] \to M$ is called a *geodesic* if there exists $\epsilon > 0$ such that

$$\ell(\gamma_{\restriction [t_1, t_2]}) = d(\gamma(t_1), \gamma(t_2)) \text{ whenever } |t_1 - t_2| < \epsilon, t_1, t_2 \in [0, 1].$$

This property is independent of the choice of parametrization, although the value of ϵ may change. A geodesic $\gamma : [0, 1] \to M$ is called a *shortest geodesic* if

$$\ell(\gamma) = d(\gamma(0), \gamma(1)).$$

Example 2.1.3. The well-known examples are the straight lines in Euclidean geometry. On a sphere, the images of geodesics are the large circles. The infimum of the lengths of the paths between two points x and y on a sphere is given by the shorter arc of the large circle passing through x and y. If x and y are opposite points, then there are infinitely many shortest paths between them. Geodesics on an ellipsoid act properly in a more complicated way than on a sphere; especially, they are not closed in general (see Figure 2.1).

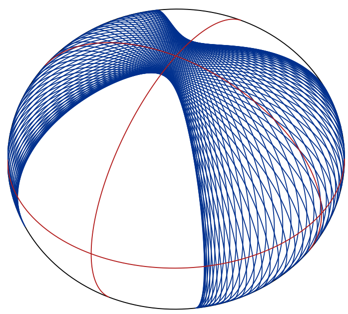

Figure 2.1. A geodesic on an ellipsoid

Definition 2.1.4. A metric space (M, d) is called a *geodesic length space*, or simply a *geodesic space*, if for any two points $x, y \in M$ there exists a shortest geodesic joining x and y. It is called *locally geodesic space* if this property holds in an appropriate neighborhood of any point. For $x, y \in M$, we call $m(x, y)$ a *midpoint* of x and y if

$$m(x, y) = \gamma(\frac{1}{2})$$

for a shortest geodesic $\gamma : [0, 1] \to M$ from x to y, where γ is supposed to be parametrized proportionally to arc-length.

Example 2.1.5. Let (H^2, g) be a Riemannian manifold with a Riemannian metric where H^2 is the 2-dimensional hyperbolic space. This manifold is locally compact, complete and there exist a rectifiable geodesic between every two points. Then there exist a shortest geodesic (not necessary unique) between every pair of points in H^2.

2.1.2 Alexandrov Non-positive Curvature

A locally geodesic space (M, d) is said to be an *Alexandrov non-positive curvature space* if for every $p \in M$ there exists $\delta_p > 0$ such that for every $x, y, z \in B(p, \delta_p)$ and any shortest geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = x, \gamma(1) = z$, we have for $0 \leq t \leq 1$

$$d^2(y, \gamma(t)) \leq (1 - t)\ d^2(y, x) + t\ d^2(y, z) - t(1 - t)\ d^2(x, z).$$

(Alexandrov non-positive curvature inequality).

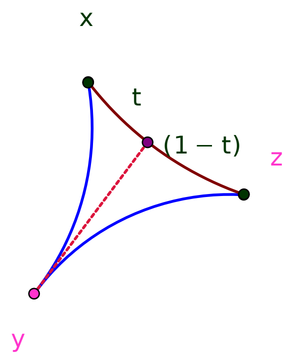

Figure 2.2. Alexandrov Non-positive Curvature

Remark 2.1.6. In some literature, the Alexandrov non-positive curvature spaces are called CAT(0)–spaces as well. Furthermore, the complete CAT(0)–spaces are the Hadamard manifolds, see [7].

2.1.3 Busemann Non-positive Curvature

A locally geodesic space (M, d) is said to be a *Busemann non-positive curvature space* if for every $p \in M$ there exists $\delta_p > 0$ such that for all $x, y, z \in B(p, \delta_p)$ we have

$$d(m(x,y), m(x,z)) \leq \frac{1}{2} d(y,z).$$

(Busemann non-positive curvature inequality).

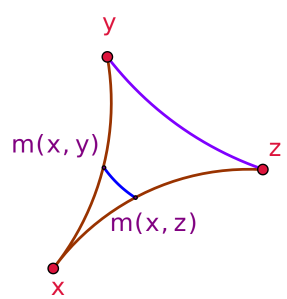

Figure 2.3. Busemann Non-positive Curvature

In other words, for any two shortest geodesic $\gamma_1, \gamma_2 : [0,1] \to M$, with $\gamma_1(0) = x = \gamma_2(0)) \in B(p, \delta_p)$ and with endpoints of $\gamma_1, \gamma_2 \in B(p, \delta_p)$, we have

$$d(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq \frac{1}{2} d(\gamma_1(1), \gamma_2(1)).$$

2.1.4 Pedersen Non-positive Curvature

Now, define the distance of a curve γ and a point $q \in M$ as

$$\operatorname{dist}(\gamma, q) = \inf\{d(\gamma(t), q) \ : \ 0 \leq t \leq 1\}.$$

A locally geodesic space is said to be a *Pedersen non-positive curvature space* if for every $p \in M$ there exists $\delta_p > 0$ such that for any two shortest geodesics $\gamma_1, \gamma_2 : [0,1] \to B(p, \delta_p)$ the function $f : [0,1] \to \mathbb{R}$, defined by

$$f(t) = \operatorname{dist}(\gamma_1, \gamma_2(t))$$

is quasiconvex, i.e., for every $t \in [0,1]$, $f(t) \leq \max\{f(0), f(1)\}$.

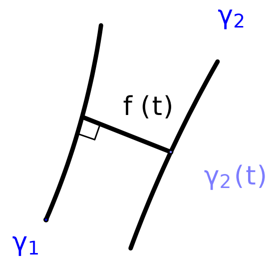

Figure 2.4. Pedersen Non-positive Curvature

2.1.5 Convex Capsules

Let $\gamma : [a,b] \to M$ be a shortest geodesic and $\alpha > 0$. Attached to γ and α, we define the *capsule* as

$$\mathcal{C}_\gamma(\alpha) = \{q \in M : \operatorname{dist}(\gamma, q) \leq \alpha\}.$$

Let M_0 be a non-empty subset of M. The pair (γ, α) is said to be M_0-admissible if $\mathcal{C}_\gamma(\alpha) \subset M_0$.

We say that a locally geodesic space (M, d) has *convex capsules* if for every $p \in M$ there exists $\delta_p > 0$ such that for every $B(p, \delta_p)$-admissible pair (γ, α), the capsule $\mathcal{C}_\gamma(\alpha)$ is convex.

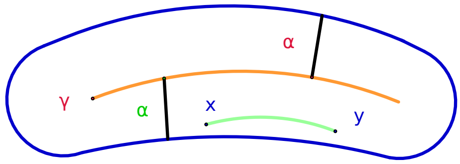

Figure 2.5. Convex Capsules

Remark 2.1.7. In general, an Alexandrov non-positive curvature space is a Busemann non-positive curvature space (see [25], Chapter 2). Moreover, Busemann non-positive curvature property meets the of Pedersen non-positive curvature space and accordingly has a convex capsule. Nevertheless, on Riemannian structures, all the above non-positive curvature notions coincide, and they characterize the non-positivity of the sectional curvature. For a deeper discussion of the non-positive curvature spaces, we indicate the reader to the classical work of Bridson and Haefliger [11] and Busemann [12].

Example 2.1.8. Let d_p be the metric on $\mathbb{R}^2$ defined by

$$d_p(x,y) := (|x^1 - y^1|^p + |x^2 - y^2|^p)^{\frac{1}{p}}$$

for $x = (x^1, x^2), y = (y^1, y^2)$. For $1 < p < \infty$, the unique geodesic in $(\mathbb{R}^2, d_p)$ between any two points is given by a straight line. The homogeneity of the metric d_p implies that the Busemann non-positive curvature property holds with equality above. Similarly, the Lebesgue spaces $L^P(\mathbb{R}), 1 < p < \infty$, constitute examples of Busemann non-positive curvature spaces that are not locally compact. Note, however, that for $p = 1$ or ∞, in the above examples, geodesics between two given points are not unique, and so the Busemann non-positive curvature inequality does not hold.

Example 2.1.9. The spaces $(\mathbb{R}^2, d_p)$ in the example above are not Alexandrov non-positive curvature spaces for $p \neq 2$. For $p < 2$, consider the points $x = (1,0), Y = (0,0), z = (0,1)$. Then the distance from $y = (0,0)$ to $(\frac{1}{2}, \frac{1}{2})$ (the midpoint of x and z) is $2^{\frac{1}{p}-1} > 2^{-\frac{1}{2}}$. For $p > 2$, consider the points $x = (-1,1), y = (0,0), z = (1,1)$. Then $d_p(x,y) = d_p(z,y) = 2^{\frac{1}{p}} < 2^{\frac{1}{2}}$ whereas the midpoint $(0,1)$ of x and z has distance 1 from y. We conclude that not all Busemann non-positive curvature spaces are Alexandrov non-positive curvature spaces.

2.1.6 Average angle

Definition 2.1.10. Given $\epsilon > 0$, let $\gamma_1 : [0, \epsilon) \longrightarrow X$ and $\gamma_2 : [0; \epsilon) \longrightarrow X$ be two geodesics in a length space X emanating from the same point $p = \gamma_1(0) = \gamma_2(0)$. We define the *angle* $\angle(\gamma_1, p, \gamma_2)$ between γ_1 and γ_2 as

$$\angle(\gamma_1, p, \gamma_2) = \lim_{s,t\to 0} \widetilde{\angle}(\gamma_1(s), p, \gamma_2(t)),$$

if the limit exists, where

$$\widetilde{\angle}(\gamma_1(s), p, \gamma_2(t)) := \arccos \frac{s^2 + t^2 - d(\gamma_1(s), \gamma_2(t))^2}{2st}.$$

Let $\gamma_1 : [0, a] \to M$ and $\gamma_2 : [0, b] \to M$ be two shortest geodesics with $p = \gamma_1(0) = \gamma_2(0)$. *The average angle* between γ_1 and γ_2 at p is defined by

$$\measuredangle(\gamma_1, p, \gamma_2) = \lim_{n\to\infty} A_{\gamma_1,\gamma_2}(\frac{a}{2^n}, \frac{b}{2^n}),$$

if the limit of the sequence exists, where the comparison angle is given by

$$A_{\gamma_1,\gamma_2}(a, b) := \arccos \frac{a^2 + b^2 - d(\gamma_1(a), \gamma_2(b))^2}{2ab}.$$

Let q be an inner point of a shortest geodesic pr, and qs be a shortest geodesic. It is clear that for an Alexandrov non-positive curvature space the sum of adjacent average angles is at least π, i.e., $\measuredangle(p, q, s) + \measuredangle(s, q, r) \geq \pi$.

Remark 2.1.11. In a locally geodesic space the Alexandrov and Busemann non-positive curvature properties are equivalent if and only if the sum of adjacent average angles $\geq \pi$ [19].

2.2 The Hilbert metric of a convex domain and its curvature

2.2.1 Hilbert metrics

Let K be a bounded convex open set in $\mathbb{R}^n$ $(n \geq 2)$. The Hilbert metric d_K on K is defined as follows. For any $x \in K$, let $d_K(x, x) = 0$. For distinct points $x, y \in K$, assume that the straight line passing through x, y intersects the boundary ∂K at two points a, b such that the order of these four points on the line is a, x, y, b as in Figure 2.6.

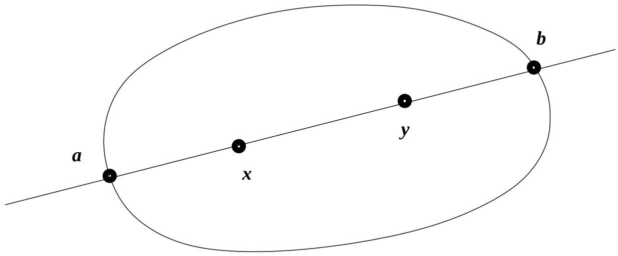

Figure 2.6. Hilbert geometry

Denote the cross-ratio of the points by

$$[a, x, y, b] = \frac{\|b - x\|}{\|b - y\|} \frac{\|a - y\|}{\|a - x\|},$$

where $\|.\|$ is the Euclidean norm of $\mathbb{R}^n$. Then the *Hilbert metric* is

$$d_K(x, y) = \frac{1}{2} \ln[a, x, y, b],$$

and the metric space (K, d_K) is called a *Hilbert geometry* of the domain K.

Example 2.2.1.

- Let us consider the unit ball in $\mathbb{R}^n$ as the domain K, the expression for d_K is coincident with the formula for the distance between points in the Cayley–Klein model of hyperbolic geometry, up to a multiplicative constant.

- Assume that C is a cone and it is positive orthant in $\mathbb{R}^n$, in that case, the induced metric on the projectivization of C is frequently said to be simply Hilbert's projective metric. This cone determines a domain K which is a regular simplex of dimension $n - 1$.

Concerning the curvature properties of the Hilbert metric, Busemann ([12], page 108) showed that for any strictly convex domain K, the Hilbert metric d_K satisfies the Pedersen non-positivity curvature property, and, consequently, has a convex capsules. Kelly and Straus published two papers on the curvature of Hilbert geometry in 1958 and 1968. They used the concept of Busemann non-positive curvature, which is a pleasant geometric approach and weaker than Alexandrov's one. Nevertheless in the case of Hilbert metric of a convex domain it has a strong consequence, namely it implies the reduction of the domain to an ellipsoid. In details, it was proved in [26]:

Proposition 2.2.2. *If Hilbert metric (K, d_K) has Busemann non-positive curvature, then the domain K is an ellipsoid and the Hilbert metric d_K is hyperbolic, i.e., Riemannian.*

Corollary 2.2.3. (see [20, Corollary 5.6]) *A Hilbert metric (K, d_K) satisfies the Alexandrov non-positive curvature condition if and only if K is an ellipsoid.*

Proposition 2.2.4. *Let (K, d_K) be the Hilbert metric of a convex domain K. Then the Busemann non-positive curvature is equivalent to the Alexandrov non-positive curvature.*

Proof. First, it is true in general that all Alexandrov non-positive curvature is Busemann non-positive curvature ([25]). Conversely, if the Hilbert metric is Busemann non-positive curvature, then by a paper of Kelly and Straus proved in 1958 ([26]) the domain is an ellipsoid, and the metric is a Riemannian. The latter property, however, implies by the above corollary that the Alexandrov non-positive curvature property is also satisfied. □

Corollary 2.2.5. *In Hilbert geometry, the sum of adjacent average angles is $\geq \pi$.*

Proof. Gu proved in [19] that the sum of adjacent average angles is $\geq \pi$ if and only if the Alexandrov and Busemann non-positive curvature properties are equivalent. Conversely, by our proposition, we get the corollary immediately. □

2.2.2 Finsler structure of the Hilbert metric

The first chapter of this dissertation constitutes sufficient preparation of Berwald space, as a continuation we have the following.

Kristály et al. proved in [28] that all mentioned non-positivity properties are equivalent to the analytical condition $\kappa \leq 0$ in the case of Berwald space:

Proposition 2.2.6. [28] *Let $(M, \hat{F})$ be a Berwald space where $\hat{F}$ is positively (but perhaps not absolutely) homogeneous of degree one.The following assertions are equivalent:*

a) The flag curvature κ of $(M, \hat{F})$ is non-positive.

b) $(M, d_{\hat{F}})$ is a Busemann non-positive curvature space.

c) $(M, d_{\hat{F}})$ is a forward Pedersen non-positive curvature space.

d) $(M, d_{\hat{F}})$ is a backward Pedersen non-positive curvature space.

e) $(M, d_{\hat{F}})$ has convex forward capsules.

f) $(M, d_{\hat{F}})$ has convex backward capsules.

The Hilbert metric d_K of the convex open domain K naturally determines its Hilbert Finsler fundamental function $\hat{F}_K$ as follows ([41]: First the asymmetric Finsler metric, called Funk metric $\widetilde{F}_K$ is defined by

$$p + \frac{1}{\widetilde{F}_K(u)} u \in \partial K \qquad \text{for any } u \in T_pK, \text{and } p \in K,$$

and then $\hat{F}_K$ is obtained by symmetrization:

$$\hat{F}_K(u) = \frac{1}{2}(\widetilde{F}_K(u) + \widetilde{F}_K(-u)).$$

Naturally, $\hat{F}_K$ is a reversible Finsler metric, therefore the forward and backward concepts coincide. It is easy to check that the induced distance of $\hat{F}_K$ is just the Hilbert distance d_K defined above in Definition 2.2.1.

The flag curvature of the Hilbert metrics was computed in 1929 by Funk in dimension 2 and by Berwald in all dimensions. Later T. Okada proposed a more direct computation:

Proposition 2.2.7. [35] *The Hilbert geometry (K, d_K) is projectively flat Finsler space of negative constant curvature -1.*

Theorem 2.2.8. *If the Hilbert metric d_K of a convex domain K is a Berwald metric, then it reduces to a Riemannian metric, and the domain is an ellipsoid.*

Proof. By a Proposition of Okada ([35]) the flag curvature is negative constant for the Hilbert metric of a convex domain. Kristály and Kozma showed in [28], among others, that for any Berwald space the non-positivity of the flag curvature is equivalent to the property of Busemann non-positive curvature. Moreover, Kelly and Straus proved in 1958 ([26]) that if the Hilbert metric satisfies Busemann's non-positive curvature property, then the domain is an ellipsoid, and the metric is a Riemannian one. □

From our proposition, we get immediately the next corollary, because all Riemannian metric is Berwald.

Corollary 2.2.9. (see [41, Theorem 11.6]) *The Hilbert metric d_K of a bounded convex domain $K \subset \mathbb{R}^n$ with smooth strongly convex boundary is Riemannian if and only if K is an ellipsoid.*

Remark 2.2.10. Conversely, if the domain is not an ellipsoid, then d_K is a non-Berwaldian projectively flat metric.

Chapter 3

Sub-Finslerian geometry

3.1 Setting

3.1.1 A setting of sub-Finslerian geometry

Definition 3.1.1. Let M be an n–dimensional connected manifold. A sub-Finslerian structure on M is a triple $(\mathcal{D}, \sigma, F)$ where:

(1) $(\mathcal{D}, \pi_{\mathcal{D}})$ is a vector bundle on M.

(2) $\sigma : \mathcal{D} \longrightarrow TM$ is a morphism of vector bundles. In particular, the following diagram is commutative

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\sigma"] \arrow[dr, "\pi_{\mathcal{D}}"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

such that $\pi_{\mathcal{D}} : \mathcal{D} \longrightarrow M$ and $\pi : TM \longrightarrow M$ are the canonical projection.

(3) A function $F : \widetilde{\mathcal{D}} \to \mathbb{R}$, where $\widetilde{\mathcal{D}} = \mathcal{D} \setminus \{0\}$, called a *sub-Finsler metric*, which satisfies the following properties:
- $F_x(v) > 0$ for all $v \in \widetilde{\mathcal{D}}$, $x \in M$.
- F is C^∞ on $\widetilde{\mathcal{D}}$.
- $F_x(\lambda v) = \lambda F_x(v)$ for all $v \in \widetilde{\mathcal{D}}_x$ and $\lambda \in \mathbb{R}_+$.
- The Hessian of F^2 with respect to the vector variables is positive definite.

A *sub-Finsler manifold* $(M, \mathcal{D}, F)$ is a smooth manifold M endowed with a sub-Finslerian structure i.e. the triple $(\mathcal{D}, \sigma, F)$.

Let $\mathcal{D}_x$ be the fiber over $x \in M$. The last condition means that the matrix $\frac{\partial^2 F^2}{\partial v^i \partial v^j}(x, v)$ is positive definite for all $v = (v^1, \dots, v^k) \in \mathcal{D}_x$. Equivalently, the corresponding indicatrix

$$I_x = \{v \,|\, v \in \mathcal{D}_x,\ F_x(v) = 1\}$$

is strictly convex.

The following technique describes the association between the sub-Finsler manifold $(\mathcal{D}, \sigma, F)$ and a Finsler metric $\hat{F}$ on $\mathrm{Im}(\sigma) \subset TM$:

For each $u \in \mathrm{Im}(\sigma)_x \subset T_xM$ and $x \in M$, we have

$$\hat{F}_x(u) = \inf_v \{F_x(v)|\ v \in D_x,\ \sigma(v) = u\}.$$

From now on we suppose that $\mathcal{D} \subset TM$, $\sigma : \mathcal{D} \longrightarrow TM$ is the inclusion $i : \mathcal{D} \longrightarrow TM$ and F is a sub-Finsler metric on $\mathcal{D}$.

As in the sub-Riemannian case, we call $\mathcal{D}$ the *horizontal distribution*. A curve $\gamma : [0, T] \to M$ is called *horizontal*, or *admissible* if $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in [0, T]$, that is $\gamma(t)$ is tangent to $\mathcal{D}$. The length of a smooth horizontal curve γ is defined as usual by

$$\ell(\gamma) = \int_0^T F(\dot{\gamma}(t))dt.$$

Equivalently and as in Finslerian case, we observe that it suffices to minimize the energy

$$E(\gamma) = \frac{1}{2}\int_0^T F^2(\dot{\gamma}(t))dt.$$

instead of length $\ell(\gamma)$.

The length induces a sub-Finslerian distance $d(x_0, x_1)$ between two points x_0 and x_1 as in Finsler geometry:

$$d(x_0, x_1) = \inf \ell(\gamma),$$

where we consider the infimum over all smooth horizontal curves joining x_0 and x_1. The distance is infinite if there is no such a horizontal curve between x_0 and x_1.

The length functional will be studied on the absolutely continuous curves, in other words, on the curves $\gamma : [0, T] \to M$ which have derivative for almost all $t \in [0, T]$, and the components of the derivative $\dot{\gamma}$ are measurable curves. This property is independent of the coordinate maps. An absolutely continuous curve γ is horizontal if its tangent vector field $\dot{\gamma}(t)$ lies in $\mathcal{D}_{\gamma(t)}$ whenever it exists. The length of an absolutely continuous curve exists, but it can be infinite. The distance $d(x_0, x_1)$ between two points is defined in the same way as for the continuous curves, the infimum being taken on the class of absolutely continuous curves joining x_0 and x_1.

Definition 3.1.2. An absolutely continuous horizontal curve between x_0 and x_1 which realizes the distance $d(x_0, x_1)$ is called a *minimizing* geodesic between x_0 and x_1.

A question naturally arises: given two points x_0 and x_1 in a sub-Finsler manifold, is there a horizontal curve that joins x_0 and x_1?

The answer is not always positive. In the case of an involutive distribution $\mathcal{D}$ the Frobenius theorem asserts that the set of horizontal path through S form a smooth immersed submanifold, the leaf through x_0, of dimension equal to the rank of distribution k. In this case, if $\mathcal{D}$ is involutive and x_1 is not contained in the leaf through x_0, there is no any horizontal curve joining x_0 and x_1.

A positive answer is given by Chow's theorem in the case of bracket generating distributions, which are the "contrary" of the involutive distributions.

Definition 3.1.3. [33] A distribution $\mathcal{D}$ is said to be *bracket generating* if any local frame X_i of $\mathcal{D}$, together with all of its iterated Lie brackets spans the whole tangent bundle TM.

Theorem 3.1.4. [33]*(Chow's theorem). If $\mathcal{D}$ is a bracket generating distribution on a connected manifold M then any two points of $\mathcal{D}$ can be joined by a horizontal path.*

The problem of minimizing the length of joining two given points x_0 and x_1 is equivalent to a time optimal problem: where the control bundle is $(\mathcal{D}, \pi, M)$ and we are searching for such a curve $\gamma(t)$ and a control curve $v(t) \in \mathcal{D}_{\gamma(t)}$ minimizing the time T needed to connect x_0 and x_1.

Remark 3.1.5. Of course, sub-Riemannian manifolds are the simplest example for sub-Finslerian manifold. In the following example, a Riemannian metric h is given in the distribution $\mathcal{D}$. If the distribution $\mathcal{D}$ admits a frame $X^1, \dots, X^m$ on an open set $U \subseteq M$, then the family $\mathcal{F} = \{X^1, \dots, X^m\}$ is called an orthonormal family of vector fields or an orthonormal frame for $(\mathcal{D}, h)$ in U if there holds

$$h_x(X^i(x), X^j(x)) = \delta_{ij} \quad \forall i, j = 1, \dots, \quad m \ \forall x \in U$$

where δ_{ij} denotes the Kronecker symbol, that is,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Sub-Riemannian structures admit local orthonormal frames in a neighborhood of each point of M.

Example 3.1.6. The space $\mathbb{R}^3$, with coordinates (x, y, z), equipped with the rank two distribution $\mathcal{D}$ given as follows

$$\mathcal{D} = \mathbb{R}^3 \times \{(v_1, v_2, 0)\} \subseteq \mathbb{R}^3 \times \mathbb{R}^3.$$

However, the metric

$$h = dx^2 + dy^2$$

is the simplest sub-Riemannian structure we can imagine. The norm of the vector $v = (v_1, v_2, 0) \in \mathcal{D}(x, y, z)$ is given by

$$\ell(v) = \sqrt{(v_1)^2 + (v_2)^2}.$$

Since v is an horizontal vector, the latter quantity does not vanishes unless $v = 0$.

Example 3.1.7. As a good example of the sub-Finsler manifold is a Kinematic penny on a plane. Assume that a wheel of radius 1 is rotating without sliding on the Euclidean plane $\mathbb{E}^2$. For the wheel's arrangement one can give a description by the vector ${}^t(x_1, x_2, \phi, \psi)$ such that

(*) The wheel's point (x_1, x_2) is the touching point to the plane,

(*) The angle of rotation given by ϕ of a designated point on the wheel from the vertical,

(*) Here, the wheel's heading angle denoted by ψ, which mean, the angle of the tangent line to the curve traced by the wheel on the plane with the x-axis.

Consequently, the concerned space which has dimension 4 surely is isomorphic to $\mathbb{R}^2 \times S^1 \times S^1$. Our assumption that the wheel moves without sliding implies the assertion that for its path ${}^t(x_1(t), x_2(t), \phi(t), \psi(t))$, in the state space the following differential equation holds:

$$y = \begin{bmatrix} x_1' \\ x_2' \\ \phi' \\ \psi' \end{bmatrix} = \phi'(t) \begin{bmatrix} \cos\psi \\ \sin\psi \\ 1 \\ 0 \end{bmatrix} + \psi'(t) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.1}$$

such that the functions $\phi'(t), \psi'(t)$ are control functions given arbitrarily [23]. Hence the velocity vector ${}^t(x_1', x_2', \phi', \psi')$ of any solution curve should lie in the distribution $\mathcal{D}$ spanned by the following vector fields:

$$\begin{aligned} X_1 &= \cos\psi \frac{\partial}{\partial x_1} + \sin\psi \frac{\partial}{\partial x_2} + \frac{\partial}{\partial \phi} \\ X_2 &= \frac{\partial}{\partial \psi}. \end{aligned}$$

In a natural way, a sub-Riemannian metric on $\mathcal{D}$ is gained by asserting the vector fields X_1, X_2 to be orthonormal,

$$\langle \phi' X_1 + \psi' X_2, \phi' X_1 + \psi' X_2 \rangle = \phi'^2 + \psi'^2.$$

The integral of this quadratic form measures the work completed in rolling the heading angle ψ at the rate ψ' and propelling the wheel ahead at the rate ϕ'. The sub-Riemannian structure will be adjusted as specified by the notion that curvature is costly: namely, it needs more attempt to steer the wheel in a tight circle with little forward or backward movement than to steer it in a wide arc. Therefore, the curvature of the projection γ given by $\kappa = \frac{\psi'}{\phi'}$ this brings us to assume a sub-Finsler metrics of the body

$$F = f\Big(\frac{\psi'}{\phi'}\Big)\sqrt{d\phi^2 + d\psi^2},$$

such that f finds larger but stays bounded like $\mid \frac{\psi'}{\phi'} \mid$ increases, we point out the interested readers to [24].

Example 3.1.8. The most interesting well known example in $\mathbb{R}^3$ is the sub-Finslerian structure of the Heisenberg group, where it is given as follows:

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x_2 & x_3 + \frac{1}{2}x_1 x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} : (x_1, x_2, x_3) \in \mathbb{R}^3 \right\}. \tag{3.2}$$

Then

$$\mathcal{D} = \{dx_3 + \frac{1}{2}(x_1 dx_2 - x_2 dx_1)\}^{\perp}$$

is a rank two distribution, a contact structure on $\mathbb{H}$.

By an appropriate sub-Riemannian metric on $(\mathbb{H}, \mathcal{D})$, one can assume that

$$X_1 = \frac{\partial}{\partial x_1} + \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2} - \frac{x_1}{2}\frac{\partial}{\partial x_3}$$

are orthonormal vectors. It is clear that the frame $\{X_1, X_2\}$ is left invariant and gives a global basis of the horizontal bundle.

Let Σ_1 be the unit circle bundle for this sub-Riemannian structure on $(\mathbb{H}, \mathcal{D})$, and define a coordinate θ on Σ_1 by the condition that

$$u = (\cos\theta)X_1 + (\sin\theta)X_2 \quad \text{for } u \in \Sigma_1.$$

Then any scaling function $r(\theta)$ which depends on θ alone defines a homogeneous sub-Finslerian metric on $\mathbb{H}$:

$$F(u) = r(\theta)g^{1/2}(u,u) \quad \text{for any } u \in \mathcal{D},$$

where g denotes the sub-Riemannian metric above, and θ is the angle of

$$\frac{u}{g^{1/2}(u,u)} \in \Sigma_1.$$

The indicatrix bundle of this metric is given as follows: $\Sigma = \{r(\theta)^{-1}u \mid u \in \Sigma_1\}$. For more details we refer the reader to [23].

3.1.2 Sub-Finslerian structure in control theory

In [8], D. Barilari et al studied the sub-Finsler geometry as a time-optimal control problem. Especially, they looked at non-smooth and non-strictly convex sub-Finsler geometries linked with groups distributions, namely, Heisenberg, Grushin, and Martinet. The purpose of this subsection is to show the settings of a sub-Finsler geometry as a time-optimal control problem and the reader will see the difference compared to our settings of the sub-Finsler geometry.

A control system is given by the differential equation: $\dfrac{dx^i}{dt} = f^i(x,u)$, where x^i are the state variables, t is the parameter of evolution (time); u^a's are the controls. It geometrically means that a fibred mapping X :

$(U, \tau, M) \longrightarrow (TM, \pi, M)$ is given, and the allowed curves $\varrho : I \subset \mathbb{R} \longrightarrow U$ of the control system satisfy $(H \circ \varrho)' = X \circ \varrho$.

In optimal control theory a cost functional $C(\varrho) = \int f^0(x(t), u(t))dt$ is also given, and the aim is to minimize the cost functional under boundary conditions.

In the notation of [8], the function on $\mathbb{R}^k$ is called a norm if it is subadditive, absolutely homogeneous and vanishes only at 0. A sub-Finsler structure of rank less than or equal to k (trivialized and of constant-type norm) on a C^∞ manifold M is a pair $(f, \|\cdot\|)$ such that $\|\cdot\|$ is a norm on $\mathbb{R}^k$ and

$$f : M \times \mathbb{R}^k \longrightarrow TM$$

is a smooth morphism of bundles where

$$f(\{x\} \times \mathbb{R}^k) \subseteq T_x M,$$

for all $x \in M$. We link distribution

$$\mathcal{D} = f(M \times \mathbb{R}^k)$$

and a norm on $\mathcal{D}$ with every sub-Finsler structure given by

$$\|v\| = \inf\{\|w\| \mid f(x, w) = v\}, \quad \text{for all } v \in \mathcal{D}_x. \tag{3.3}$$

Notice that the norm on $\mathbb{R}^k$ as stated above, does not satisfies all the conditions of Finsler geometry because it is not smooth away from the origin. An outcome to the above, even when $\mathcal{D} = TM$, the function $v \mapsto \|v\|$ does not necessarily supply M with a Finsler structure in the traditional sense.

We introduced the concept of sub-Finsler structure as above when the norm did not allow the smoothness. In other words, in a particular case of the following more general class: A partially smooth sub-Finsler structure on M is a triple $(V, \|\cdot\|_V, f)$ such that V is a vector bundle over M, $\|\cdot\|_V$ is a partially smooth Finsler structure on V ([8]), and

$$f : V \longrightarrow TM$$

is a smooth morphism of bundles where $f(V_x) \subseteq T_x M$, for all $x \in M$. The norm on the obtained distribution can be determined in analogy with 3.3, replacing $\|w\|$ by $\|(x, w)\|_V$.

As we defined the horizontal curve for the sub-Finsler geometry, one can define an absolutely continuous curve $\gamma : [0, T] \longrightarrow M$ is horizontal for any sub-Finsler structure with distribution $\mathcal{D}$ and norm $\|\cdot\|$ if $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ and in this case its length is given by

$$\ell(\gamma) = \int_0^T \|\dot\gamma(t))\| dt.$$

If the distribution $\mathcal{D}$ is Lie bracket generating, then Chow's theorem implies that the length induces a sub-Finslerian structure distance

$$d(x_0, x_1) = \inf\{\ell(\gamma) \mid \quad \gamma : [0, T] \longrightarrow M \text{ horizontal and } \gamma(0) = x_0, \gamma(T) = x_1\},$$

which is well defined and finite for any two points x_0 and x_1.

If M is a smooth n-dimensional manifold then a sub-Finsler structure on M is a pair $(f, \|\cdot\|)$, where $\|\cdot\|$ is a norm of $\mathbb{R}^k$ and $f : M \times \mathbb{R}^k \to TM$ is a smooth morphism of bundle where the latter can determine k vector fields $X_1, ..., X_k$ defined by $X_i(x) = f(x, e_i)$, where $e_1, ..., e_k$ is an orthonormal basis for $\mathbb{R}^k$. (Conversely, given k linearly independent smooth vector fields $X_1, ..., X_k$ on M, then there exists a unique bundle morphism $f : M \times \mathbb{R}^k \to TM$ for which $X_i(x) = f(x, e_i)$.)

The norm $\|\cdot\|$ identifies the set

$$B := \{w \in \mathbb{R}^k : \|w\| \leq 1\},$$

which is closed, convex, centrally symmetric, and with the origin in its interior. Conversely, any such a set is the closed until ball of a norm on $\mathbb{R}^k$.

The difficulty here is to find a sub-Finsler geodesics, i.e., curves that minimize the length between two points x_0 and x_1, one can explicated as a time-optimal control problem, that is the problem of minimizing the time $T \geq 0$ for which $\gamma : [0, T] \to M$ absolutely continuous and $u : [0, T] \to \mathbb{R}^k$ measurable such that

$$\begin{cases} \dot\gamma(t) = u_1(t)X_1(\gamma(t)) + ... + u_k(t)X_k(\gamma(t)), & \text{for almost every } t \in [0, T], \\ u(t) \in B, & \text{for almost every } t \in [0, T], \\ \gamma(0) = x_0, \ \gamma(T) = x_1. \end{cases} \tag{3.4}$$

Observe that the measurable (control) function u perhaps not uniquely determined the trajectory γ because of the vector field perhaps not linearly independent at each point. Nevertheless, taking u into account, γ is a unique trajectory satisfying

$$\dot\gamma(t) = \sum_{j=1}^{k} u_j(t) X_j(\gamma(t)), \tag{3.5}$$

in addition to $\gamma(0)) = p$. A sub-Finsler structure on M is locally given by a control system in (3.5) of constant rank k with controls $u(t)$. The controlled trajectories are obtained by integrating the above system (3.4).

To any optimal control system one can associate a sub-Finslerian structure $(M, \mathcal{D}, F)$ in such a way that the cost value of a curve solution of the system is the length of the curve [30]. Hence, the solutions of the optimal control problem, i.e., the curves that make the cost minimal, are geodesics of the sub-Finslerian manifold.

Example 3.1.9. Consider a time optimal problem $(f^0(x,u) = 1)$ in $\mathbb{R}^2$ with the control equations

$$\frac{dx}{dt} = 2 + u^1\cos(u^2), \qquad \frac{dy}{dt} = u^1\sin(u^2) \qquad 0 \le u^1 \le 1,\ u^2 \in S^1.$$

Then $S_{(x_0,y_0)}$ is the unit disk centered at $(2,0)$, and the horizontal set

$$\mathcal{D}_{(x_0,y_0)} = \{(v_x, v_y) \,|\, |\frac{v_y}{v_x}| \le \frac{1}{\sqrt{3}},\ v_x > 0\}$$

forms now only a cone, not a subspace on the tangent space. Then the sub-Finsler metric is given as follows:

$$F_{(x_0,y_0)}(v_x, v_y) = \frac{v_x^2 + v_y^2}{2v_x + \sqrt{v_x^2 - 3v_y^2}}.$$

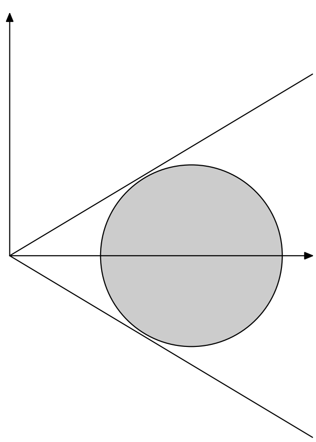

Figure 3.1. Example 3.1.9

Example 3.1.10. Again, $\mathbb{R}^2$ there are given the control equation as follows:

$$\frac{dx}{dt} = u^1|\cos(u^2)|\cos(u^2), \qquad \frac{dy}{dt} = u^1|\cos(u^2)|\sin(u^2)$$

$$0 \leq u^1 \leq 2, \quad 0 \leq u^2 \leq 2\pi$$

Then the horizontal sets are $\mathcal{D}_{(x_0,y_0)} = \{(v_x, v_y) \,|\, v_x \neq 0\}$, and one can see that the sub-Finsler metric is as follows: $F_{(x,y)}(v_x, v_y) = \dfrac{v_x^2 + v_y^2}{2|v_x|}$. This is a Kropina space (see Matsumoto, 1991, [32]).

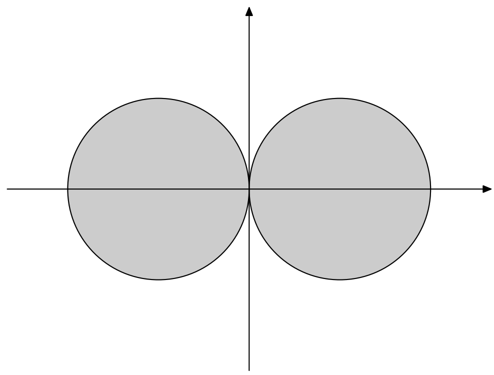

Figure 3.2. Example 3.1.10

3.2 Connection

3.2.1 Generalized connection for the sub-Riemannian manifold

Langerock ([29]) gave a coordinate free approach to sub-Riemannian geometry, based on a notion of generalized connection over a bundle map: The main idea was to consider the natural notion of connection on a Lie algebroid and extend it to bundles with an anchor map. It was shown that a generalized connection is associated with a sub-Riemannian structure. In this approach one can study length-minimizing curves and, in particular, give necessary and sufficient conditions for the existence of abnormal extremals.

Let (V, π, M) be a vector bundle, and (A, ν, M) the anchor bundle over the same base manifold M, while there is given a vector bundle map $\varrho : A \longrightarrow TM$, called the anchor map such that the diagram is commutative:

$$\begin{tikzcd} A \arrow[d, "\varrho"'] \arrow[dr, "\nu"] & V \arrow[d, "\pi"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

Definition 3.2.1. The map $D : Sec\,\nu \times Sec\,\pi \longrightarrow Sec\,\pi, \quad (s, \sigma) \mapsto D_s\sigma$ is called *a generalized linear connection over the anchor map* ϱ if

- $\mathbb{R}$-linear in s and σ;
- $\mathcal{F}(M)$-linear in s;
- for any $f \in \mathcal{F}(M)$, $D_s(f\sigma) = fD_s\sigma + \varrho(s)(f)\sigma$.

It follows from the Pontryagin maximum principle that sub-Riemannian minimizers fall into two non-disjoint classes, the "normal" and the "abnormal" extremals. In 1986 Strichartz [39] characterized first the normal and abnormal extremals of the sub-Riemannian structure. "Normal" extremals satisfy differential equations similar to the geodesic equations in Riemannian geometry, and they are smooth. On the other hand, abnormal extremals which happen to be minimizers were not known to actually exist until the first such example was given by R. Montgomery in 1994 (see [33]).

Proposition 3.2.2. *Given a sub-Riemannian structure* $(M, \mathcal{D}, h)$*, if a horizontal* $c : [0, T] \longrightarrow M$ *is length minimizing, then*
either there exists $\psi : [0, T] \longrightarrow T^*M$ *along* c *(normal extremal)*

$$\begin{aligned} \dot{\psi}_i(t) &= -1/2\partial_i g^{jk}(c(t))\psi_j(t)\psi_k(t) \\ g(\psi(t)) &= \dot{c}(t) \end{aligned}$$

or there exists $\psi : [0, T] \longrightarrow T^*M$ *along* c *(abnormal extremal)*

$$\begin{aligned} \dot{\psi}_i(t) &= -1/2\partial_i g^{jk}(c(t))\psi_j(t)\alpha_k(t) \\ g(\psi(t)) &= 0 \end{aligned}$$

where $\alpha(t)$ *is any* g*-admissible curve in* T^*M *along* c*, and* $g := i \circ \#_h \circ i^* : T^*M \longrightarrow M$ *[29].*

The notion of generalized connections can be applied to sub-Riemannian geometry with the choices: $V = A = T^*M$, $\varrho = g : T^*M \longrightarrow TM$. Then it was proved that there exists a unique normal $\mathcal{D}$-adapted (metrical) linear g-connection D of T^*M normal $\mathcal{D}$-adapted.

The main theorems of ([29]) imply that normal extremals are the base curve of autoparallel curves of D, and the abnormal extremals are the base curve of parallel sections in the annihilator bundle $\mathcal{D}^0$.

3.2.2 Legendre transformation of a sub-Finslerian manifold

The *sub-Lagrange function*

$$L : \mathcal{D} \longrightarrow \mathbb{R},$$

determined by F is given in the following way:

$$L = \frac{1}{2}F^2.$$

The fiber derivative of L defines the map

$$\mathcal{L}_L : \mathcal{D} \longrightarrow \mathcal{D}^*,$$

$$\mathcal{L}_L(v)(w) = \frac{d}{dt}L_x(v + tw), \text{ such that } v, w \in \mathcal{D}_x,$$

which is known in the literature as the *Legendre transformation* (see [1], [10]). We use the Legendre transformation to carry over the geometrical objects of a sub-Lagrange space from $\mathcal{D}$ onto $\mathcal{D}^*$. Let us denote by (x^i) the coordinate in a neighborhood $U \subset M$ with (x^i, v^a) in $\mathcal{D}|_U \subset TM$, and (x^i, p_a) in $\mathcal{D}^*|_U \subset T^*M$, respectively, where $i = 1, \dots, n,\ a = 1, \dots, k$. Then the relation of the distribution $\mathcal{D}$ of tangent bundle and the distribution $\mathcal{D}^*$ of cotangent bundle is given by Legendre transformation in local coordinates as follows

$$\mathcal{L}_L(x^i, v^a) = (x^i, \frac{\partial L}{\partial v^a}).$$

Then sub-Hamiltonian is given by

$$H : \mathcal{D}^* \longrightarrow \mathbb{R},$$

$$H = \iota_{\mathcal{L}_L^{-1}} - L \circ \mathcal{L}_L^{-1},$$

or locally, $H(x^i, p_a) = v^a p_a - L(x^i, v^a)$, where $p_a = \frac{\partial L}{\partial v^a}$.

Secondly, for fiber derivative of H, we define the Legendre transformation of the sub-Hamiltonian H in the following way

$$\mathcal{L}_H : \mathcal{D}^* \longrightarrow \mathcal{D},$$

for any $\alpha, \beta \in \mathcal{D}^*_x$, it holds

$$\beta(\mathcal{L}_H(\alpha)) = \frac{d}{dt} H_x(\alpha + t\beta).$$

This relates the distribution $\mathcal{D}^*$ of cotangent bundle and the distribution $\mathcal{D}$ of tangent bundle locally according to the next expression

$$\mathcal{L}_H(x^i, p_a) = (x^i, \frac{\partial H}{\partial p_a}),$$

we say H is a Hamiltonian if and only if $\mathcal{L}_H$ is local diffeormorphism ([1]).

It is clear that $\mathcal{L}_L$ and $\mathcal{L}_H$ are inverses of each other, so we have the following relations:

$$\begin{aligned} \mathcal{L}_H \circ \mathcal{L}_L &= 1_{\mathcal{D}}, & \mathcal{L}_L \circ \mathcal{L}_H &= 1_{\mathcal{D}^*}, \\ \frac{\partial H}{\partial x^i} &= -\frac{\partial L}{\partial x^i}, & g^{ab} g_{bc} &= \delta^a_c, \\ \frac{\partial^2 L}{\partial x^i \partial y^b} &= -\frac{\partial^2 H}{\partial x^i \partial p_a} g_{ab}, & \text{where} \quad g_{ab} &= \frac{\partial^2 L}{\partial v^a \partial v^b}. \end{aligned}$$

In other hand, for every $p \in \mathcal{D}^*$, one can define the sub-Finsler metric on the cotangent space $\mathcal{D}^*$ with help of the indicatrix I as follows:

$$F^*(p) := \sup_{w \in I} p(w) = \sup_{0 \neq v \in \mathcal{D}} p[\frac{v}{F(v)}].$$

Moreover, a function F^* defined in the following way

$$F^* : \widetilde{\mathcal{D}}^* \longrightarrow \mathbb{R},$$

where $\widetilde{\mathcal{D}}^*$ is the subbundle of the cotangent bundle obtained by removing the zero cotangent vector from each fibre. In fact, F^* turns out to meet the same properties that mentioned as in Definition 3.1.1, but on $\mathcal{D}^*$ instead of $\mathcal{D}$ such that the positive homogeneity is more obvious.

$$F^*(p) = F(v), \text{ where } p = \mathcal{L}_L(v),$$

and
$$H := \frac{1}{2}(F^*)^2,$$
see details in [10].

We recall here the basic relations of non-linear connections of Lagrange and Hamiltonian spaces. For a Lagrange space, there exists a canonical non-linear connection given by (see [21])

$$N^i_j = \frac{1}{2}\frac{\partial G^i}{\partial v^j}; \quad G^i = g^{ij}\left(\frac{\partial^2 L}{\partial v^j \partial x^k}v^k - \frac{\partial L}{\partial x^i}\right). \tag{3.6}$$

In the homogeneous case, i.e. for Finsler manifolds this connection is also called as the Barthel non-linear connection. For Hamiltonian spaces, the non-linear connection is the image of the non-linear connection of the sub-Finsler spaces (Lagrangian spaces) by Legendre transformation.

A Hamilton space (M, H) for which the Hamiltonian is 2-homogeneous with respect to p_i is called a *Cartan space*.

Let us consider γ^i_{jk} the Christoffel symbols (for more details see [10, p. 34]) of g_{ij}:

$$\begin{aligned} \gamma^i_{jk} &= \frac{1}{2}g^{ih}\Big(\frac{\partial g_{hj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h} + \frac{\partial g_{kh}}{\partial x^j}\Big), \\ \gamma^0_{jk} &= p_i\gamma^i_{jk}, \quad \gamma^0_{j0} = \frac{1}{2}\gamma^0_{jk}g^{kl}p_l. \end{aligned}$$

Then the coefficients for the non-linear connection of a Cartan space are given by [21]
$$N_{ij} = \gamma^0_{ij} - \gamma^0_{k0}\dot{\partial}^k g_{ij}.$$
This non-linear connection can be obtained as the Legendre transformation of the canonical non-linear connection of a Finsler space.

For the subsequent theorem, let us clarify some facts which will play an important role later on.

Definition 3.2.3. A linear transformation $P : TM \longrightarrow \mathcal{D}$ is called a *projection operator* in $\mathcal{D}$, if $P^2 = P$ and $\mathcal{D} = \mathrm{Im}(P)$. If P is a projection operator in $\mathcal{D}$ then

$$TM = \mathrm{Ker}(P) \oplus \mathrm{Im}(P), \quad \text{and} \quad P = 0_{\mathrm{Ker}(P)} \oplus \mathrm{I}_{\mathrm{Im}(P)},$$

such that $\mathcal{D}^\perp := \mathrm{Ker}(P)$. Its complement projection is $P^c = id - P$.

For the sake of notation, we shall use the symbols P^* to denote the projection on T^*M and $(P^*)^c$ to denote its complement, respectively, namely

$$P^* : T^*M \longrightarrow (\mathcal{D}^{\perp})^0.$$

where $P^*(\alpha)(u) = \alpha(P(u))$ for all $\alpha \in T^*M$, $P(u) \in \mathcal{D}$ and u is a vector in TM.
Next the projection complement which is given by

$$(P^*)^c : T^*M \longrightarrow \mathcal{D}^0,$$

satisfies the condition $(P^*)^c = id - P^*$, such that for all $(P^*)^c(\alpha) \in \mathcal{D}^0$ we have

$$(P^*)^c(\alpha)(u) = \alpha(u - P(u)) = 0, \text{ if } u \in \mathcal{D}.$$

Furthermore,

$$P^*(\alpha) + (P^*)^c(\alpha) = \alpha, \; P^* + (P^*)^c = id_{T^*M}.$$

Now T^*M can be written as the direct sum of $(\mathcal{D}^{\perp})^0$ and $\mathcal{D}^0$.

After all, one can imagine a picture from above which through it one has a complete conception of generating a Finsler metric from sub-Finsler one by using the upcoming technique. Starting with a sub-Finsler metric F in the subbundle $\mathcal{D}$, we choose an arbitrary $\widetilde{F}$ which is defined in the complement $\mathcal{D}^{\perp}$ associated to P. Now if we take the sum of both metrics we will obtain a *Finsler metric* $\hat{F}$ in TM, specifically

$$\hat{F}^2(u) = F^2(P(u)) + \widetilde{F}^2(P^c(u)) \text{ for all } u \in TM.$$

Comparing the Legendre transformations of the sub-Finsler metric F and the extended Finsler metric $\hat{F}$ one can easily see that the following relations hold:

$$\begin{aligned} \mathcal{L}_L &= \mathcal{L}_{\hat{L}}|_{\mathcal{D}}, \quad P^* \circ \mathcal{L}_L = \mathcal{L}_{\hat{L}} \circ i, \\ \mathcal{L}_H &= \mathcal{L}_{\hat{H}}|_{\mathcal{D}^*}, \quad \mathcal{L}_H \circ i = P^* \circ \mathcal{L}_{\hat{H}}. \end{aligned}$$

3.2.3 Generalized non-linear connection for a sub-Finslerian manifold

Suppose that $(M, \mathcal{D}, F)$ is a sub-Finsler geometry such that M is a smooth manifold of dimension n equipped with distribution $\mathcal{D} \subset TM$ and F is a sub-Finsler metric on $\mathcal{D}$. The natural inclusion $i : \mathcal{D} \longrightarrow TM$ is then a linear bundle mapping fibered over the identity of M. A distribution is also completely characterized by its annihilator, i.e., giving $\mathcal{D}$ is equivalent to specifying the subbundle $\mathcal{D}^0$ of the cotangent bundle T^*M whose fibre over $x \in M$ consists of all covector at x which annihilates all vectors in the subspace $\mathcal{D}_x$ of T_xM. With a sub-Finsler structure one can associate a smooth mapping, defined by

$$E : T^*M \longrightarrow TM, \quad E(\alpha_x) = i(\mathcal{L}_H(i^*(\alpha_x))) \in TM, \tag{3.7}$$

where $i^* : T^*M \longrightarrow \mathcal{D}^*$ is the adjoint mapping of i, i.e., for any $\alpha_x \in T_x^*M$, $i^*(\alpha_x)$ is determined by

$$\langle X_x, i^*(\alpha_x)\rangle = \langle i(X_x), \alpha_x\rangle \text{ for all } X_x \in \mathcal{D}_x,$$

such that

$$\langle v, \alpha\rangle := \alpha(v)$$

for all $v \in \mathcal{D}, \alpha \in \mathcal{D}^*$.

Clearly, E is a bundle mapping whose image set is precisely the subbundle $\mathcal{D}$ of TM and whose kernel is the annihilator $\mathcal{D}^0$ of $\mathcal{D}$. To simplify notations we shall often identify an arbitrary vector in $\mathcal{D}$ with its image in TM under i and smooth section of $\mathcal{D}$ (i.e., element of $\Gamma(\mathcal{D})$) will often be regarded as a vector field on M. To E we can further associate a section $\bar{E}$ of $TM \otimes TM \longrightarrow M$ according to

$$\begin{aligned}\bar{E}(x)(\alpha_x, \beta_x) &= \langle E(\alpha_x), \beta_x\rangle \\ &= \langle \mathcal{L}_H(i^*(\alpha_x)), i^*(\beta_x)\rangle,\end{aligned}$$

for all $x \in M$ and $\alpha_x, \beta_x \in T_x^*M$.

Recall that a curve $\sigma : [0,1] \to M$ is called *horizontal*, or *admissible* if $\dot{\sigma}(t) \in \mathcal{D}_{\sigma(t)}$ for all $t \in [0,1]$. Let us consider a curve $\psi : [0,1] \longrightarrow T^*M$ in the cotangent bundle and put $\sigma = \pi_M \circ \psi$, with $\pi_M : T^*M \longrightarrow M$ the natural cotangent bundle projection. Then, we say that ψ is a E-admissible if $E(\psi(t)) = \dot{\sigma}(t)$, for all $t \in [0,1]$.

One can check the invariance of $\bar{E}$ under the projection for all $\alpha \in \mathfrak{X}^*(M)$, more explicitly

$$\begin{aligned}(P^*\ \bar{E})(\alpha, \alpha) =&\langle E(\alpha(P)), \alpha(P)\rangle \\ =&\langle E(P^*(\alpha)), P^*(\alpha)\rangle \\ =&\langle E(\alpha - (P^*)^c(\alpha)), \alpha - (P^*)^c(\alpha)\rangle \\ =&\langle E(\alpha), \alpha\rangle - \langle E((P^*)^c(\alpha)), \alpha\rangle - \langle E(\alpha), (P^*)^c(\alpha)\rangle \\ &+ \langle E((P^*)^c(\alpha)), (P^*)^c(\alpha)\rangle \\ =&\langle E(\alpha), \alpha\rangle = \bar{E}(\alpha, \alpha), \qquad \text{for all } (P^*)^c(\alpha) \in \Gamma(\mathcal{H}^0).\end{aligned} \tag{3.8}$$

In order to construct a connection for sub-Finsler geometry, we need to introduce the concept of generalized *non-linear* connection over an anchor map in the same vector bundle setting as in Section 3.2.1:

Definition 3.2.4. The map $\nabla : Sec\,\nu \times Sec\,\pi \longrightarrow Sec\,\pi, \quad (s,\sigma) \mapsto \nabla_s \sigma$ is called *a generalized non-linear connection over the anchor map* ϱ if

- $\mathbb{R}$-linear in s and σ;
- additive in s;
- for any $f \in \mathcal{F}(M), \nabla_s(f\sigma) = f\nabla_s\sigma + \varrho(s)(f)\sigma$.

This will applied to sub-Finslerian geometry with the following choices: $V = A = T^*M, \varrho = E : T^*M \longrightarrow TM$.

Definition 3.2.5. An *$\mathcal{L}$-connection* ∇ on a sub-Finsler manifold is a generalized non-linear connection over the induced mapping $E : T^*M \longrightarrow TM$ constructed by Legendre transformation $\mathcal{L}_H : \mathcal{D}^* \longrightarrow \mathcal{D}$ by (3.7).

Definition 3.2.6. The *symmetric bracket* associated to sub-Finsler geometry is mapping

$$\{.,.\} : \mathfrak{X}^*(M) \times \mathfrak{X}^*(M) \longrightarrow \mathfrak{X}^*(M),$$

$$\{\alpha, \beta\} = \bar{\mathcal{L}}_{E(\alpha)}\beta + \bar{\mathcal{L}}_{E(\beta)}\alpha - d(\bar{E}(\alpha,\beta)) - d(\bar{E}(\beta,\alpha)),$$

where $\bar{\mathcal{L}}_X$ is the *Lie derivative* with respect to $X \in \mathfrak{X}^*(M)$.

In the following proposition we list some properties of the above bracket, the first of which justifies the denomination "symmetric bracket". The proofs are straightforward.

Proposition 3.2.7. [29] *For any symmetric bracket the following properties are satisfied for any $\alpha, \beta \in \mathfrak{X}^*(M)$:*

(1) $\{\alpha, \beta\} = \{\beta, \alpha\}$*;*

(2) the bracket is $\mathbb{R}$*-linear;*

(3) $\{f\alpha, \beta\} = E(\beta)(f)\alpha + f\{\alpha, \beta\}$*;*

(4) $\{\alpha, \gamma\} = \bar{\mathcal{L}}_{E(\alpha)}\gamma$*, for any* $\gamma \in \Gamma(\mathcal{D}^0)$ *and* $\{\alpha, \gamma\} = 0$ *if both* α *and* γ *belong to* $\Gamma(\mathcal{D}^0)$*.*

The first three properties justify the next definition.

Definition 3.2.8. An $\mathcal{L}$-connection ∇ on sub-Finsler manifold $(M, \mathcal{D}, F)$ is said to be a *normal* if the associated symmetric product equals to the symmetric bracket, i.e. if

$$\langle \alpha, \beta \rangle_\nabla = \{\alpha, \beta\}$$

holds for all $\alpha, \beta \in \mathfrak{X}^*(M)$, where the symmetric product of ∇ is given by

$$\langle \alpha, \beta \rangle_\nabla = \nabla_\alpha \beta + \nabla_\beta \alpha, \text{ for all } \alpha, \beta \in \mathfrak{X}^*(M).$$

One can associate a mapping δ to any sub-Finsler manifold $(M, \mathcal{D}, F)$ according to

$$\delta : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}^0) \longrightarrow \mathfrak{X}^*(M), \quad (X, \gamma) \mapsto \delta_X \gamma = i_X d\gamma.$$

It is clear that δ generalizes the Bott connection of involutive distribution to our general case of non-involutive distribution ([13]).

Definition 3.2.9. An $\mathcal{L}$-connection ∇ on sub-Finsler space $(M, \mathcal{D}, F)$ is said to be an *adapted* to the bundle $\mathcal{D}$ (shortly $\mathcal{D}$-adapted) if $\nabla_\alpha \gamma = \delta_\alpha \gamma$ for all $\alpha \in \mathfrak{X}^*(M)$ and $\gamma \in \Gamma(\mathcal{D}^0)$.

We define the *Barthel non-linear connection* $\overline{\nabla}^B$ of the cotangent bundle as follows

$$\overline{\nabla}^B_X \alpha(Y) = X(\alpha(Y)) - \alpha(\nabla^B_X Y),$$

where the Barthel non-linear connection ∇^B on the tangent bundle was locally given in (3.6). Recall that in some literature, the Barthel non-linear connection plays the same role instead of the Levi-Civita connection in case of positively homogeneous.

Theorem 3.2.10. *Let ∇ be an $\mathcal{L}$-connection, then the following assertions are equivalent:*

(i) ∇ is a normal $\mathcal{L}$-connection;

(ii) For any $\alpha \in \mathfrak{X}^*(M)$, ∇ satisfies:

$$\nabla_\alpha \alpha = \overline{\nabla}^B_{E(\alpha)} P^*(\alpha) + \delta_{E(\alpha)} (P^*)^c(\alpha).$$

Proof. (i) $\Longleftrightarrow$ (ii) If we apply the equality between the symmetric product and the symmetric bracket in the definition of normal $\mathcal{L}$-connections, then for all $\alpha \in \mathfrak{X}^*(M)$ we get

$$\langle \alpha, \alpha \rangle_\nabla = \nabla_\alpha \alpha + \nabla_\alpha \alpha = 2\nabla_\alpha \alpha = \{\alpha, \alpha\}$$

which is obviously equivalent to the mentioned definition above. Using the property of the symmetric bracket for all $\alpha \in \mathfrak{X}^*(M)$, we conclude

$$(\overline{\nabla}^B_{E(\alpha)}\ \alpha)(P) = \frac{1}{2}\{\alpha, \alpha\}(P) = \bar{\mathcal{L}}_{E(\alpha)}\alpha(P) - d(\langle E(\alpha(P)), \alpha(P)\rangle).$$

Taking into account equation (3.8), we deduce the following

$$(\overline{\nabla}^B_{E(\alpha)}\ \alpha)(P) = \bar{\mathcal{L}}_{E(\alpha)}\alpha(P) - d(\bar{E}(\alpha, \alpha)).$$

Keeping in mind that $P^*(\alpha) = \alpha(P)$ and the Barthel non-linear connection preserves the metric, i.e. $\nabla^B \circ \mathcal{L}_H = \mathcal{L}_H \circ \overline{\nabla}^B$, we obtain

$$\begin{aligned}
\overline{\nabla}^B_{E(\alpha)} P^*(\alpha) &= \bar{\mathcal{L}}_{E(\alpha)} P^*(\alpha) - d(\bar{E}(\alpha, \alpha)) \\
\overline{\nabla}^B_{E(\alpha)} P^*(\alpha) &= \frac{1}{2}\{\alpha, \alpha\} - \bar{\mathcal{L}}_{E(\alpha)}(P^*)^c(\alpha) \\
\frac{1}{2}\{\alpha, \alpha\} = \nabla_\alpha \alpha &= \overline{\nabla}^B_{E(\alpha)} P^*(\alpha) + \bar{\mathcal{L}}_{E(\alpha)}(P^*)^c(\alpha).
\end{aligned}$$

Since $(P^*)^c(\alpha) \in \Gamma(\mathcal{D}^0)$ and $E(\alpha) \in \Gamma(\mathcal{D})$. The last term on the right hand side reduces to $\delta_{E(\alpha)}(P^*)^c(\alpha)$, as wanted to be shown. □

The above theorem clarifies, in particular that

$$\nabla_\alpha \beta = \overline{\nabla}^B_{E(\alpha)} P^*(\beta) + \delta_{E(\alpha)}(P^*)^c(\beta),$$

is a non-linear $\mathcal{L}$-connection and it is normal. Moreover, for all $\beta \in \Gamma(\mathcal{D}^0)$ one can verify $\nabla_\alpha \beta = \delta_{E(\alpha)}(\beta)$, i.e., the connection under consideration is also $\mathcal{D}$-adapted. Consequently, we have

Theorem 3.2.11. *Given a sub-Finsler structure $(M, \mathcal{D}, F)$, one can always construct a normal and a $\mathcal{D}$-adapted $\mathcal{L}$-connection.*

3.2.4 Some related properties

Definition 3.2.12. A non-linear $\mathcal{L}$-connection is called *partial* if for any $\alpha \in \mathfrak{X}^*(M)$ and $\beta \in \Gamma(\mathcal{D}^0)$ we have $\nabla_\beta \alpha = 0$.

Proposition 3.2.13. *Let ∇ be a normal $\mathcal{L}$-connection. Then ∇ is partial if and only if ∇ is $\mathcal{D}$-adapted.*

Proof. Suppose that ∇ is a normal $\mathcal{L}$-connection, namely

$$\begin{aligned}\{\alpha, \beta\} &= \langle \alpha, \beta \rangle_\nabla \\ &= \nabla_\alpha \beta + \nabla_\beta \alpha.\end{aligned}$$

Let us consider that ∇ is partial. Then for all $\beta \in \Gamma(\mathcal{D}^0)$ the above equalities change as follows

$$\{\alpha, \beta\} = \nabla_\alpha \beta = \bar{\mathcal{L}}_{E(\alpha)} \beta = \delta_{E(\alpha)} \beta,$$

from which it is clear that ∇ satisfies the condition of $\mathcal{D}$-adapted. Conversely, assume that ∇ is normal and $\mathcal{D}$-adapted, then for all $\alpha, \beta \in \mathfrak{X}^*(M)$ ∇ can be written in such a way

$$\nabla_\alpha \beta = \{\alpha, \beta\} - \nabla_\beta \alpha.$$

Moreover, it may be shown that the right hand side of the previous equation is zero, more precisely, for any $\alpha \in \Gamma(\mathcal{D}^0)$ and $\beta \in \mathfrak{X}^*(M)$ one has that $\nabla_\alpha \beta = 0$, which completes the proof. □

Proposition 3.2.14. *An $\mathcal{D}$-adapted $\mathcal{L}$-connection is not metrical.*

Proof. Assume that ∇ is an $\mathcal{D}$-adapted $\mathcal{L}$-connection. Let us consider for all $\alpha, \beta \in \mathfrak{X}^*(M)$

$$E(\nabla_\alpha \beta) = \nabla_\alpha E(\beta)$$

which means, roughly speaking, that ∇ leaves $\bar{E}$ invariant. Then it follows that ∇ is $\mathcal{D}$-adapted and for all $\alpha \in \mathfrak{X}^*(M)$ and $\gamma \in \Gamma(\mathcal{D}^0)$: $E(\delta_{E(\alpha)}) = 0$. Nonetheless, this is equivalent to saying that $\mathcal{D}$ is involutive and taking into account the latter relation, we have

$$\begin{aligned}0 = \langle E(\delta_{E(\alpha)}), \beta \rangle &= \langle E(\beta), \delta_{E(\alpha)} \rangle \\ &= -\langle [E(\alpha), E(\beta)], \gamma \rangle,\end{aligned}$$

for every $\alpha, \beta \in \mathfrak{X}^*(M)$ and $\gamma \in \Gamma(\mathcal{D}^0)$, hence $[E(\alpha), E(\beta)] \in \Gamma(\mathcal{D})$.

□

3.3 Sub-Finsler bundle

We define in this section a sub-Finsler vector bundle which will play a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry. Let us consider first the covector subbundle $(\mathcal{D}^*, \tau, M)$ with the projection

$\tau : \mathcal{D}^* \longrightarrow M$, which is a covector subbundle of rank k $(= \dim \mathcal{D}^*)$ in the cotangent bundle T^*M. The illustrious role in our consideration will play by the pullback bundle $\tau^*(\tau)$ of τ by τ as follows:

$$\mathrm{pr}_1 : \mathcal{D}^* \times_M \mathcal{D}^* \longrightarrow \mathcal{D}^*, \ (p, q) \mapsto p.$$

Throughout, we shall call the above pullback bundle as the *sub-Finsler bundle* over $\mathcal{D}^*$. The following is summarized for the pullback construction in our case:

$$\mathcal{D}^* \times_M \mathcal{D}^* = \{(p, q) \in \mathcal{D}^* \times \mathcal{D}^* | \ \ \tau(p) = \tau(q)\}.$$

Now, if p is fixed, then

$$\begin{aligned}(\mathrm{pr}_1)^{-1}(p) &= \{(p, q) \in \mathcal{D}^* \times \mathcal{D}^* | \ \ q \in \mathcal{D}^*_{\tau(q)}\} \\ &= \{p\} \times \mathcal{D}^*_{\tau(p)},\end{aligned}$$

is a fiber of the sub-Finsler bundle over $p \in \mathcal{D}^*$.

We can introduce a Riemannian metric g^* in the sub-Finsler vector bundle induced by the Hamiltonian H as follows:

$$g^*_p(q, r) = \frac{\partial^2 H(p + tq + sr)}{\partial t \partial s}|_{t,s=0} \qquad \text{for all } q, r \in \mathcal{D}^*_{\tau(p)},$$

which locally means

$$g^{*ij} = \frac{\partial^2 H}{\partial p_i \partial p_j}.$$

Now the sub-Finsler bundle $\tau^*(\tau) = (\mathcal{D}^* \times \mathcal{D}^*, \mathrm{pr}_1), \mathcal{D}^*$ allows k covector fields $\alpha_1, \alpha_2, \ldots, \alpha_k$ which form an orthonormal frame with respect to the induced Riemannian metric g^*.

Notice that $\alpha_i(p)$ is a covector field that depends on the position $x \in M$ and the direction $p \in \mathcal{D}^*$. Moreover, one can choose in a way that $\alpha_i(p)$ is a homogeneous of degree zero in p i.e. $\alpha_i(tp) = t^0 \alpha_i(p) = \alpha_i(p)$. According to the above metric g^{*ij} on M which is a homogeneous of degree zero, we could generate a sub-Hamiltonian formalism in the components p_i (induces naturally by the inner product) as (see [14])

$$H_x(p) = \frac{1}{2} \sum_{i,j=1}^{n} g^{*ij} p_i p_j, \tag{3.9}$$

such that this metric defined in the extended Finsler metric which was shown in [5].

Since the sub-Hamiltonian $H_x(p)$ is homogeneous of degree 2 in the variable p. Consequently, we shall write the sub-Hamiltonian formalism (3.9) in a more useful way using the orthonormality of α_i as follows

$$H_x(p) = \frac{1}{2}\sum_{i=1}^{k}\langle p, \alpha_i(p)\rangle^2, \qquad p \in \mathcal{D}^*. \tag{3.10}$$

One can check the homogeneity of the sub-Hamiltonian formalism $H_x(p)$ easily

$$\begin{aligned} H_x(tp) &= \frac{1}{2}\sum_{i=1}^{k}\langle tp, \alpha_i(tp)\rangle^2 \\ &= \frac{1}{2}\sum_{i=1}^{k}\langle tp, t^0\alpha_i(p)\rangle^2 \\ &= \frac{t^2}{2}\sum_{i=1}^{k}\langle p, \alpha_i(p)\rangle^2 \\ &= t^2 H_x(p). \end{aligned}$$

The importance of $H_x(p)$ lies on to define sub-Finslerian geodesics.

Definition 3.3.1. A *normal geodesic* between the points A and B is a solution $x(t)$ of the sub-Hamiltonian system

$$\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial p_i}(x, p), \\ \dot{p}_i &= -\frac{\partial H}{\partial x^i}(x, p), \quad i = 1, \dots, n \end{aligned}$$

with the boundary conditions $x(0) = A$ and $x(T) = B$.

Remark 3.3.2. Let $\xi(t) = (x(t), p(t))$ be a solution of the sub-Hamiltonian system and let $x(t)$ be its projection to M. Then every sufficient short subarc of the normal geodesic $x(t)$ is a minimizer sub-Finslerian geodesic. This subarc is the unique minimizer joining its end points. (see [22, Theorem 1]).

Remark 3.3.3. In the sub-Finslerian geometry, not all the sub-Finslerian geodesics are normal (contrary to the Finsler geometry). This is due to the fact that the sub-Finslerian geodesics which admits a minimizing geodesic might not be solved the sub-Hamiltonian system. Those minimizer that are not normal geodesics called *singular* or *abnormal* geodesics (see [22], [33]).

Moreover, we call the extremal pair $\xi(t) := (x(t), p(t))$ a *normal extremal* if it is a solution for the sub-Hamiltonian system, otherwise it is called an *abnormal extremal*.

Turning to the relationship between the normal geodesic and the locally length-minimizing horizontal curves, Calin et al. proved in [14] that any normal geodesic is a horizontal curve and a locally length-minimizing horizontal curve.

After all, by using (3.10) one can generate the following form of the system of differential equations in term of canonical coordinates (x, p):

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = \sum_{j=1}^{k} \langle p, \alpha_j(p)\rangle \ (\delta_i(\alpha_j(p)) + \langle p, D_{p_i}\alpha_j(p)\rangle), \tag{3.11}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = -\sum_{j=1}^{k} \langle p, \alpha_j(p)\rangle \langle p, D_{x^i}\alpha_j(p)\rangle, \tag{3.12}$$

where δ_i is the i-th coordinate function.

3.4 Exponential map in sub-Finsler geometry

Let (M, d) be a non-reversible metric space. Since the function d is not necessarily symmetric, we define the forward metric balls and forward metric spheres, with center $x \in M$ and radius $r > 0$ as follows:

$$B_x(r) = \{\ y \in M :\ d(x, y) < r\},$$

$$S_x(r) = \{\ y \in M :\ d(x, y) = r\}.$$

The cotangent balls and the cotangent spheres in $\mathcal{D}^*$ are defined as follows:

$$\mathcal{B}_x^*(r) = \{\ p \in \mathcal{D}^* :\ F_x^*(p) < r\},$$

$$\mathcal{S}_x^*(r) = \{\ p \in \mathcal{D}^* :\ F_x^*(p) = r\},$$

for any fix $x \in M$ and radius r.

We say that a sequence $\{x_i\}$ converges to $x \in M$, if for any $\epsilon > 0$ there exist a positive number $\mathbb{N}$ such that

$$|x_n - x| < \epsilon,\ n \geq \mathbb{N}.$$

In other hand, we call a sequence $\{x_i\}$ forward Cauchy sequence if for any $\epsilon > 0$, there exist a positive number $\mathbb{N}$ such that

$$d(x_i, x_j) < \epsilon,\ j > i \geq \mathbb{N}.$$

Definition 3.4.1. (Lipschitz continuous) A sequence $\gamma_k : [0,T] \longrightarrow M$ is *Lipschitz continuous* if there exists a real $C \geq 0$ that for every $t_1, t_2 \in [0,T]$ we have

$$d(\gamma_k(t_1) - \gamma_k(t_2)) \leq C|t_1 - t_2|.$$

Moreover, C is called a *Lipschitz constant* of γ_k if it is satisfying the above definition.

Proposition 3.4.2. *Let M be a sub-Finsler manifold and $x \in M$. Assume that $\bar{B}_x(r)$ is compact, for some $r > 0$. Then for every $y \in B_x(r)$ there is a minimizing geodesic between x and y, i.e.,*

$$d(x,y) = \min\{\ell(\gamma) \;|\gamma : [0,T] \longrightarrow M \text{ horizontal}, \; \gamma(0) = x, \gamma(T) = y\}.$$

Proof. Fix $y \in B_x(r)$ and suppose that $\gamma_k : [0,T] \longrightarrow M$ is a minimizing sequence of horizontal paths with constant speed joining x and y and such that

$$\lim_{k\to\infty} \gamma_k(0) = x, \quad \lim_{k\to\infty} \gamma_k(T) = y, \quad \lim_{k\to\infty} \ell(\gamma_k) = d(x,y).$$

For the reason that $d(x,y) < r$, we get $\ell(\gamma_k) \leq r$ for all $k \geq k_0$ large enough. Thus, there is no loss of generality in assuming the form of γ_k is contained in the compact set $K = \bar{B}_x(r)$ for all k. In particular, by Definition 3.4.1

$$d(\gamma_k(t_1) - \gamma_k(t_2)) \leq \int_{t_1}^{t_2} F(\dot{\gamma}_k(t))dt \leq C|t_1 - t_2| \quad \text{for all } t_1, t_2 \in [0,T],$$

where C depends only on K. It can be seen that all horizontal paths in the sequence γ_k are Lipschitz which is identical Lipschitz constant. Therefore, the sequence γ_k is uniformly bounded and equicontinuous. Any sequence γ_k of curves which have uniformly bounded lengths has an uniformly convergent subsequence, we denote this subsequence by the same symbol. Let $\gamma : [0,T] \longrightarrow M$ be a Lipschitz curve such that $\gamma_k \longrightarrow \gamma$ uniformly for every $t \in [0,T]$. Further, from continuity of the distance and semicontinuity of the length

$$\ell(\gamma) \leq \lim_{k\to\infty} \inf \, d(\gamma_k(0), \gamma_k(T)) = d(\gamma(0), \gamma(T)).$$

This implies that $d(x,y) = \ell(\gamma)$, i.e., γ is length minimizer. □

Next, we explain in depth the exponential map. For the general case, if M is a smooth Finsler manifold, x a point in M and $u \in T_xM$. Then the exponential map given by

$$\exp_x : T_xM \longrightarrow M,$$

such that $\exp_x(u) = \gamma_u(1)$ for the unique geodesic γ that starts at x and has initial speed vector u. Furthermore, in the dual space the exponential map for every $x \in M$ and $p \in T_x^* M$ defined by

$$\exp_x^* : T_x^* M \longrightarrow M,$$

such that $\exp_x^*(p) = \gamma_p(1)$ for the unique geodesic γ that starts at x and has initial speed vector $u = \mathcal{L}_L^{-1}(p)$.

The exponential map is an essential object in sub-Finslerian geometry, that is the map that parametrizes normal extremals through their initial covectors. We are going to define the exponential map in both of the distribution $\mathcal{D}, \mathcal{D}^*$ of tangent and cotangent bundle, respectively.

Definition 3.4.3. Let $\Omega_x \subset \mathcal{D}_x$ be the domain of the exponential map over $x \in M$ such that Ω_x given by

$$\Omega_x = \{v \in \Omega_x |\ \xi \text{ is defined on the interval } [0,1]\},$$

where $v = \mathcal{L}_H(p)$ by the Legendre transformation of sub-Hamiltonian H. Then the *sub-Finsler exponential map* is defined as follows

$$\exp_x : \Omega_x \subset \mathcal{D}_x \longrightarrow M,\ v \mapsto \pi_{\mathcal{D}}(\mathcal{L}_H(\xi(1))),$$

here $\xi(t)$ is the normal extremal for every $t \in [0,1]$.

We can do the same in the dual distribution $\mathcal{D}_x^*$, consider $\Omega_x^* \subset \mathcal{D}_x^*$ be the domain of the exponential map over $x \in M$ such that Ω_x^* given by

$$\Omega_x^* = \{p \in \Omega_x^* |\ \xi \text{ is defined on the interval } [0,1]\}.$$

Consequently, the *sub-Hamiltonian exponential map* is given by

$$\exp_x^* : \Omega_x^* \subset \mathcal{D}_x^* \longrightarrow M,\ p \mapsto \tau(\xi(1)),$$

where $\xi(t)$ is the same normal extremal as above. The set Ω_x^* containing the origin and star-shaped with respect to 0. Moreover, with the help of Legendre transformation it is fairly easy to see that

$$\exp_x(v) = \exp_x^*(p).$$

Remark 3.4.4. It is clear that in the case of sub-Finsler exponential map the following expressions hold:

$$\exp_x^*[\mathcal{B}_x^*(r)] = B_x(r),$$

$$\exp_x^*[\mathcal{S}_x^*(r)] = S_x(r),$$

which are analogous to the Finslerian context, see Bao et al. [10] for more details. There is also shown that for general Finsler manifolds, the exponential map is C^∞ away from the origin of TM and only C^1 at the origin such that for any $x \in M$,

$$d(\exp_x)_{|0} : T_xM \longrightarrow T_xM$$

is the identity map at the origin $0 \in T_xM$. Moreover, Akbar-Zadeh [3] proved that the exponential map is C^2 map near the origin if and only if the Finsler metric is Berwald type (manifolds are called Berwald spaces if the Chern connection coefficients

$$\Gamma^i_{jk} = \frac{1}{2} g^{is}\Big(\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\Big).$$

in natural coordinates depend only on the base point). Furthermore, $\exp_x$ map is indeed C^∞ over TM.

Remark 3.4.5. Turning to sub-Riemannian case, Strichartz in [39] stated that for the bracket generating distributions in the case of a sub-Riemannian manifold the exponential map is a local diffeomorphism (that is, it takes a small neighborhood of the origin at $\mathcal{D}_x$ only, to a neighborhood of x in the manifold M). This is due to the fact that the solutions of the Hamiltonian equations depend differentially on the initial data. But this is a difference from the Riemannian context, the exponential map is not a diffeomorphism at the origin just like the Finslerian case.

As for our case, the sub-Finsler exponential map is a C^∞ away from the zero section of $\mathcal{D}$ and only C^1 at the zero section such that for each $x \in M$

$$d(\exp_x)_{|0} : \Omega_x \subset \mathcal{D}_x \longrightarrow \Omega_x \subset \mathcal{D}_x,$$

is the identity map at the origin $0 \in \mathcal{D}_x$. This is due to the same reason as in the case of the Finsler geometry (see [10], p. 125).

3.5 Hopf-Rinow Theorem in sub-Finslerian geometry

In the following, one can see the explanation of the terms that will be used in Hopf-Rinow theorem. A sub-Finsler manifold is said to be *forward complete* if every forward Cauchy sequence converges, and it is a *forward geodesically* *complete* if every geodesic $\gamma(t), t \in [0,T)$ parametrized to have constant speed, can be extended to a geodesic for all $t \in [0,\infty)$. A subset is said to be *forward bounded* if it is contained in some forward metric ball $B_x(r)$.

Theorem 3.5.1. *Let $(M,\mathcal{D},F)$ be any connected sub-Finsler manifold, where $\mathcal{D}$ is bracket generating distribution. The following conditions are equivalent:*

(i) The metric space (M,d) is forward complete.

(ii) The sub-Finsler manifold $(M,\mathcal{D},F)$ is forward geodesically complete.

(iii) $\Omega_x^* = \mathcal{D}_x^*$, additionally, the exponential map is onto if there are no strictly abnormal minimizer.

(iv) Every closed and forward bounded subset of (M,d) is compact.

Furthermore, for any $x,y \in M$ there exists a minimizing geodesic γ joining x to y, i.e. the length of this geodesic is equal to the distance between these points.

Proof. (i) $\Longrightarrow$ (ii) Let $\gamma(t) : [0,T) \longrightarrow M$ be a unit speed and maximally forward extended geodesic, $t \in [0,T)$. If we assume that $T \neq \infty$, and choose a $\{t_i\} \longrightarrow T$ in $[0,T)$ then $\gamma(t_i)$ is forward Cauchy, since

$$d(\gamma(t_i),\gamma(t_j)) \leq |t_j - t_i|, \text{ for all } i \leq j.$$

Now, (i) makes it obvious that $\gamma(t_i)$ converges to $y \in M$. On the one hand, let us define $\gamma(T)$ to be y. On the other hand, Lemma 4.1 in [39] told us that $\gamma(t)$ can be extended beyond $t = T$. This contradicts our assumption the fact that $T \neq \infty$. Thus, $T = \infty$ for sure, so we have the forward geodesically completeness.

(ii) $\Longrightarrow$ (iii) It is sufficient (for first part $\Omega_x^* = \mathcal{D}_x^*$) to prove that any normal extremal pair, starting from the initial conditions, is defined for all $t \in \mathbb{R}$. Suppose that the normal extremal is not extendable to the some interval $[0,T+\delta)$ for all $\delta > 0$ and suppose that it is defined on $[0,T)$. Let $\{t_i\}_i$ be any increasing sequence such that the limit of this sequence is T. Hence, $\gamma(t_i)$ is on $[0,T)$ is an horizontal curve with unit speed, thus the sequence is a forward Cauchy sequence on M, since

$$d(\gamma(t_i),\gamma(t_j)) \leq |t_i - t_j|.$$

By completeness, it follows that the sequence $(\gamma(t_i))_i$ converges to some point $y \in M$. We suppose there are coordinates around the point y and an orthonormal frame $\alpha_1, \alpha_2, ..., \alpha_k$ in small ball $\mathcal{B}^*_y(r)$ in the sub-Finsler bundle. We will show that the horizontal curve is uniformly bounded. This grants a contradiction that the normal extremal is not extendable. In fact, for every $p \in \mathcal{D}^*$, we consider the following non-negative form (3.10) of the sub-Hamiltonian function H:

$$H_x(p) = \frac{1}{2}\sum_{i=1}^{k}\langle p, \alpha_i(p)\rangle^2.$$

Then, the sub-Hamiltonian system have the form:

$$\dot{x}^i(t) = \frac{\partial H}{\partial p_i}(x(t), p(t)) = \sum_{j=1}^{k}\langle p(t), \alpha_j(p(t))\rangle(\delta_i(\alpha_j(p)) + \langle p, D_{p_i}\alpha_j(p)\rangle),$$
$$\dot{p}_i(t) = -\frac{\partial H}{\partial x^i}(x(t), p(t)) = -\sum_{j=1}^{k}\langle p(t), \alpha_j(p(t))\rangle\langle p(t), D_{x^i}\alpha_j(p(t))\rangle,$$

for $t \in [T-\delta, T)$ with $\delta > 0$ small enough. Since $D_{x^i}\alpha_j$ and $D_{p_i}\alpha_j$ are given in a compact small ball $\bar{\mathcal{B}}^*_y(r)$, they are bounded, so there is a constant $\mathcal{C} > 0$ such that

$$|\dot{p}(t)| \leq \mathcal{C}|p(t)| \quad \forall t \in [T-\delta, T).$$

If we apply Gronwall's Lemma (see [37], p.122), it leads us to that $|p(t)|$ is uniformly bounded on a bounded interval. This contradicts our assumption that the normal extremal can not be extended beyond T.

(iii) $\Longrightarrow$ (iv) Assume that $\bar{A}$ is a closed and forward bounded subset of (M, d). Applying the bracket generating assumption, for every $y \in \bar{A}$, Proposition 3.4.2 asserts that there is a minimizing geodesic $\exp^*_x(tp_y)$, $0 \leq t \leq T$, from x to y. The set of all p_y is subset A of $\mathcal{D}^*_x$. Due to $F^*_x(p_y) = d(x, y)$, this p_y is bounded which by the forward boundedness of $\bar{A}$, say by r independent of y, i.e. A is contained in the compact set $\mathcal{B}^*_x(r) \cup \mathcal{S}^*_x(r)$. By Remark 3.4.4, $\exp^*_x[\mathcal{B}^*_x(r) \cup \mathcal{S}^*_x(r)]$ is compact and contained in the closed set $\bar{A}$, then $\bar{A}$ it must be compact.

(iv) $\Longrightarrow$ (i) Let $\{x_i\}$ be a forward Cauchy sequence in M, and by the triangle inequality it must be forward bounded. In particular, its closure in the manifold topology aspects is still forward bounded. Then the closure $\bar{A} := \{x_i | i \in \mathbb{N}\}$ must be compact, in consequence of the Heine-Borel statements. Thus the sequence $\{x_i\}$ contains a convergent subsequence.

Let $\{x_k\}$ be a convergent subsequence, consider it converges to some $y \in \bar{A} \subset M$. In other hand, we need to check that $\{x_i\}$ converges to $y \in \bar{A} \subset M$. To do this, fix $\epsilon > 0$, since the $\{x_i\}$ is forward Cauchy, there exist a positive number n_0 such that $j > i \geq n_0$, then

$$d(x_i, x_j) < \frac{\epsilon}{2}.$$

At the same time $\{x_k\}$ converge to y. So there is a positive number n_1 such that if $k \geq n_1$, then

$$d(x_k, y) < \frac{\epsilon}{2}.$$

One can assume that n is greater than n_0 and n_1. If needed, by expanding n further, there is no loss of generality in assuming that n indeed equals some index of the convergent subsequence. Then $d(x_n, y) \leq \frac{\epsilon}{2}$, so, for $i > n$, we get

$$d(x_i, y) \leq d(x_i, x_n) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So, we have been shown that every forward Cauchy sequence is convergent. Hence (M, d) is forward complete.

At the end, we can use the same proof of Proposition 3.4.2 to verify that for every $x, y \in M$ there exists a length minimizer joining x and y, and these have to be geodesic by Remark 3.3.2. Finally, the property of compactness and completeness with help of Proposition 3.4.2, proves the second part of (iii).

$\square$

Chapter 4

Summary in English

This dissertation consists of two main parts. The first part has intended to clarify what is the relationships among the non-positive curvatures, especially, that in the case of Hilbert metric of a convex domain. The second part and the most important, in terms of the effort, the time spent and the results obtained, is devoted to solve questions of sub-Finslerian geometry. This dissertation is divided into three Chapters, organized as follows:

The first Chapter of the dissertation presents the basic concepts of Differentiable structure (manifolds, tangent spaces, cotangent spaces, vector fields and 1-forms, distributions). We present the distributions of the tangent and cotangent bundles with some examples. Furthermore, we give a brief overview of Finsler geometry, which is a fundamental and natural generalization of Riemannian geometry. The Finsler structure depends on both coordinates and velocities. It is defined as a function $\hat{F} : TM \longrightarrow [0, \infty)$ on tangent bundle of a manifold. Then we summarize the basic facts on Berwald spaces. We also give some well known examples of the above concepts.

In the second Chapter, first of all, we explain what the non-positive curvature means by showing their definitions, more precisely, we give the basic definitions of four types of non-positive curvature in geodesic metric spaces, some of their properties and their general relationships, furthermore, list some results important for the next steps. After defining the Hilbert metric of a convex domain, we show that in the case of Hilbert metric, the first two concepts of non-positivity are equivalent if and only if the domain is an ellipsoid. Stepping to the analytical considerations, we restrict ourselves to the to the special case of Berwald space in Finsler geometry. Then we present those relationships and known facts about these concepts which are used to prove our result. Finally, we show that if the Finsler metric induced by the Hilbert metric of a convex domain is Berwald, then the domain should be an ellipsoid.

In third Chapter, we define the sub-Finsler metrics and introduce some of its notions. We provide some examples that satisfy the sub-Finsler property, further, we introduce basic properties of sub-Finsler manifolds. Also, we take a look at the horizontal path-connectedness of the manifold, especially for the Chow's theorem [18]. In the next, we give some basics of generalized connections in sub-Riemannian geometry, with which the normal and abnormal extremal can be characterized in the sub-Riemannian case (see the local version in [39], and the coordinate free version in [29]. Afterwards, we give a brief description of the Legendre transformation, we show the relationship between the sub-Finsler geometry with Lagrange spaces as well as the Hamiltonian spaces. We introduce the symmetric bracket associated to a sub-Finsler metric, and the symmetric product of an $\mathcal{L}$-connection it is shown, they are coincident if and only if the $\mathcal{L}$-connection is normal. With the help of the generalization of the Bott connection for involutive distribution we can characterize $\mathcal{D}$-adapted and normal connections, resp.

Last we turn to the problem of Hopf-Rinow theorem. It is well known that in the Riemannian and Finslerian geometry, there are two concepts of completeness. The first is the completeness in the sense of metric spaces, using the Riemannian metric. Furthermore, a Riemannian or Finsler manifold M is called geodesically complete if any geodesic $\gamma(t)$ starting from $x \in M$ is defined for all values of $t \in \mathbb{R}$. On the other hand, the completeness in the Finsler geometry is divided according to forward and backward distance metric, and forward and backward geodesically completeness.

Hopf-Rinow theorem is a basic theorem of complete Riemannian manifolds, which connects the completeness properties with compactness, and the exponential map. Its consequence says that any two points of the complete manifold can be connected by a length minimizing geodesic. In 1931, H. Hopf and W. Rinow showed their theorem only for surfaces, but the proof in higher dimensions is not significantly different. Hopf-Rinow theorem has been studied in detail in both Riemannian and Finslerian geometries in the literature, the best general references here are ([10], [16], [34]). In the Finsler case forward geodesic completeness is involved, only.

To prove the statements of Hopf-Rinow theorem in sub-Finsler manifolds: we define the sub-Finsler metric on the cotangent space $\mathcal{D}^*$ with help of the Legendre transformation, where we look more closely at a sub-Hamiltonian H defined on the cotangent bundle, induced by the sub-Finslerian metric on $\mathcal{D}^*$. Afterwards, we construct a sub-Finsler bundle, which plays a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry.

Moreover, the sub-Finsler bundle allows an orthonormal frame for the sub-Finsler structure. Also, we introduce the notion of an exponential map in sub-Finsler geometry. At the end our main theorem, namely Hopf-Rinow theorem is stated and proved.

5. fejezet

Magyar nyelvű összefoglaló (Summary in Hungarian)

A disszertáció két fő részből áll. Az első rész célja, hogy tisztázzuk, mi a kapcsolat a nem-pozitív görbületi fogalmak között, különösen a Hilbert metrika esetében. A második rész – a ráfordított kutatási idő, és az elért eredmények szempontjából talán jelentősebb rész, - a szub-Finsler geometria kérdéseinek vizsgálatára irányul. A disszertáció három fejezete a következőképpen tagolódik: Az első fejezet bemutatja a differenciálgeometria alapfogalmait struktúra (sokaságok, érintőterek, kotangens terek, vektormezők és 1-formák, disztribúciók), továbbá példákat mutat be a tangens és kotangens nyalábokra. Ezt követően rövid áttekintést adunk a Finsler geometriáról, amely alapvető és természetes általánosítása a Riemann geometriának. A Finsler struktúra mindkét koordinátától függ, a ponttól és az érintővektortól (sebességtől) is, tehát a sokaság érintőterén megadott $\hat{F} : TM \longrightarrow [0, \infty)$ függvénnyel van meghatározva. Ezt követően összefoglaljuk a Berwald-terek alapvető tényeit. A fenti fogalmakra néhány jól ismert példát is bemutatunk. A második fejezetben először tisztázzuk, hogy mit jelent a nem-pozitív a görbület: bemutatjuk a definícióikat, pontosabban, a nem-pozitív görbület négy típusának definícióját a geodetikus metrikus terekben, azok egyes tulajdonságait és kapcsolatukat, továbbá a következő lépések szempontjából fontos, ismert eredményeket. Egy konvex tartomány Hilbert metrikájának definiálása után, azt mutatjuk meg, hogy a Hilbert metrika esetében az első két nem-pozitivitási fogalom pontosan akkor egyenértékű, ha a tartomány egy ellipszoid. Az analitikai megfontolásokra térve, a Finsler geometriában a Berwald tér speciális esetére szorítkozunk. Majd azokat az összefüggéseket és ismert tényeket mutatjuk be, melyek szükségesek a fő eredmény igazolásához. Végül megmutatjuk, hogy amennyiben a konvex tartomány Hilbert metrikája a Berwald típusú Finsler metrika, akkor a tartomány szükségképpen egy ellipszoid. A harmadik fejezetben definiáljuk a szub-Finsler metrikákat és néhány alapfogalmát. Példákat mutatunk be, amelyek teljesítik a szub-Finsler tulajdonságot, továbbá a szub-Finsler sokaságok alapvető tulajdonságait tárgyaljuk. Áttekintjük a pontpárok horizontális görbékkel való összeköthetőségének kérdését (Chow-tétele [18]). A következőkben bemutatjuk az általánosított konnexiók szerepét a szub-Riemann-geometriában, amellyel a normál és az abnormális extrémum-görbéket jellemezhetjük a szub-Riemann esetben (lásd a lokális verziót [39]-ben, és a koordináta-mentes változatát a [29]-ban.) Röviden ismertetjük a Legendre transzformációt a szub-Finsler geometriai esetben, mind a Lagrange, mind a Hamilton metrikából levezetve. Definiáljuk a szub-Finsler metrikához tartozó szimmetrikus zárójelet és egy L-konnexió szimmetrikus szorzatát, Megmutatjuk, hogy azok akkor és csak akkor esnek egybe, ha az L-konnexió normális. Az involutív disztribúciók Bott konnexiójának általánosításával jellemezhetjük a D-adaptált és a normál konnexiókat. Végül a Hopf-Rinow tételre térünk. Jól ismert, hogy a Riemann és a Finsler geometriában két teljességi fogalom van. Az első a metrikus terek teljessége. Másrészt egy Riemann vagy Finsler sokaságot geodetikusan teljesnek nevezzük, ha bármely $x \in M$-től kezdődő $\gamma(t)$ geodetikus a $t \in \mathbb{R}$ minden értékére van meghatározva. A Finsler geometriában ezen túlmenően beszélhetünk az előre, illetve visszafelé mért távolságról, és előre, illetve visszafelé tekintett geodetikus teljességről. A Hopf-Rinow tétel a teljes Riemann-sokaságok alaptétele, amely a teljesség tulajdonságot a kompaktsággal és az exponenciális leképezéssel kapcsolja össze. Ennek következménye szerint egy teljes sokaság bármely két pontja összeköthető minimális geodetikus görbével. 1931-ben H. Hopf és W. Rinow csak felületekre mutatta meg a tételt, de a bizonyítás magasabb dimenzióban sem különbözik jelentősen. A Hopf-Rinow-tételt részletesen tanulmányozták mind a Riemann, mind a Finsler geometriákban (ld. [10], [16], [34]). A Finsler esetben elegendő csak előre-teljességet feltételezni. A Hopf-Rinow-tétel állításainak bizonyítása érdekében a szub-Finsler-sokaságok esetén először egy szub-Finsler metrikát definiálunk a $\mathcal{D}^*$ kotangens térben a Legendre transzformáció segítségével, mely egy szub-Hamilton függvényt határoz meg. Ezután egy szub-Finsler nyalábot konstruálunk, amely jelentős szerepet játszik kap a szub-Hamilton függvény előállításában. Ugyanis, a szub-Finsler nyaláb lehetővé teszi egy ortonormált bázismező alkalmazását. Ezután áttekintjük a szub-Finsler geometriai exponenciális leképezés fogalmát és alapvető tulajdonságait. Végül a fő tételt, Hopf-Rinow tételét mondjuk ki és bizonyítjuk a szub-Finsler geometria esetében.

List of Symbols

C^∞	infinitely differentiable
$\mathrm{Ker}(P)$	kernel of P
$\mathrm{Im}(P)$	image of P
$\mathbb{R}$	real numbers
TM	tangent bundle
T_xM	tangent space
T^*M	cotangent bundle
T_x^*M	cotangent space
$\mathcal{D}$	distribution of TM
$\mathcal{D}^*$	distribution of T^*M
$\mathcal{D}^\perp$	orthogonal of distribution $\mathcal{D}$
$\mathcal{D}^0$	annihilator of $\mathcal{D}$.
(x^i, v^a)	coordinates in $\mathcal{D}\|_U \subset TM$ where U a neighborhood in M
(x^i, p_a)	coordinates of $\mathcal{D}^*\|_U \subset T^*M$
$(M, \hat{F})$	Finsler manifold
$\kappa(y, V)$	flag curvature
(M, d)	metric space
$m(x, y)$	midpoint
$\angle(\gamma_1, p, \gamma_2)$	angle between γ_1 and γ_2
$\measuredangle(\gamma_1, p, \gamma_2)$	average angle between γ_1 and γ_2
$d_K(x, y)$	Hilbert metric
(K, d_K)	Hilbert geometry
NPC	non-positive curvature space
$\widetilde{F}_K$	Funk metric
$\hat{F}_K$	Hilbert Finsler fundamental function

K	bounded convex open set in $\mathbb{R}^n$
∂K	boundary of K
h	smooth Riemannian metric on $\mathcal{D}$
δ_{ij}	Kronecker symbol
i	canonical injection
$(M, \mathcal{D}, F)$	sub-Finsler geometry
$\mathbb{H}$	Heisenberg group
L	sub-Lagrange space
$\mathcal{L}_L$	Legendre transformation of sub-Lagrange space
H	sub-Hamiltonian space
$\mathcal{L}_H$	Legendre transformation of sub-Lagrange space
α, β	1-form
X, Y	vector field
Σ	indicatrix bundle
N^i_j	canonical non-linear connection
γ^i_{jk}	Christoffel symbol
$\mathfrak{X}(M)$	module of vector fields on M
$\mathfrak{X}^*(M)$	module of 1-form on M
E	bundle mapping
∇	$\mathcal{L}$- connection
$\{.,.\}$	symmetric bracket
$\bar{\mathcal{L}}_X$	Lie derivative
δ	generalized Bott connection
∇^B	Barthel non-linear connection on TM
$\overline{\nabla}^B$	Barthel non-linear connection on T^*M
P	projection operator on TM
P^*	projection operator on T^*M
P^c	complement projection on P
$(P^*)^c$	complement projection on P^*

Irodalomjegyzék

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