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Lajos Tamássy

ANGLE IN MINKOWSKI AND FINSLER SPACES

Summary

Area in Minkowski spaces was given by Busemann [Bu] and studied and often used by others. Infinitesimally a Finsler space is a Minkowski space. So if we can measure area in a Minkowski space, then by integration we obtain the area (of a domain) of a Finsler space. The same holds also for submanifolds. We consider the angle of two vectors in a tangent space of the base manifold of a Finsler space. This angle in Minkowski (or Finsler) spaces attracted less interest. Since the Finsler space makes its tangent space into a Minkowski space, measuring of angles in a Finsler space reduces to that in a Minkowski space. We show that they are applicable in measuring the deviation of a Finsler space from being Riemanian. Also it can be proved that a diffeomorphism between two Finsler spaces is an isometry iff it keeps angle (in the above sense) and area, similary to the well known result of Riemannian geometry.

1. Angle

Given a Finsler space $F^n = (M, \mathcal{F})$ we consider an angle $\alpha = \measuredangle(a, b)$ between two rays $a, b \in T_{p_0}M$ out of the origin $0 = p_0$ of $T_{p_0}M$. $T_{p_0}M$ is an n-dimensional vector space $\mathcal{V}^n$. a and b span a two-dimensional linear subspace Σ of $T_{p_0}M$, provided a is not parallel to $b : a \nparallel b$. If $a \parallel b$, then we assign to the pair a, b a 2-dimensional linear subspace Σ of T_pM trough the straight line $g \supset a, b$. The convex domain of Σ bounded by a and b will be denoted by A. This is unambigous if $a \nparallel b$. If $a = b$, then $A = \emptyset$. If $a, b \subset g$, $a \neq b$, then g cuts Σ into Σ^+ and Σ^-. Then $A = \Sigma^+$ or $A = \Sigma^-$.

$B^n_{x_0}(1) := \{y \mid \mathcal{F}(x_0, y) \leq 1\} \subset T_{x_0}M$ is the indicatrix body of F^n at $x_0 \in M$. F^n makes each $T_{x_0}M$ into a Minkowski space $\mathcal{M}^n_{x_0}$ with indicatrix body $B^n_{x_0}(1)$ and with the Minkowski functional $\mathcal{F}(y) = \mathcal{F}(x_0, y) : T_{x_0}M \to R^+$. $B^n_x(1)$ is then a

Minkowski ball of radius 1, and $\partial B^n_x(1) = \mathcal{I}$ is the indicatrix (hyper) surface. By $B^2_x(1) = B^n_x(1) \cap \Sigma$, $\mathcal{M}^n_x$ (or F^n) induces on $\Sigma \subset T_xM$ a two-dimensional Minkowski metric and thus an $\mathcal{M}^2_x$. $B^2_x(1) \cap A = D$ is a segment of the indicatrix body of $\mathcal{M}^2_x$ belonging to $\measuredangle\alpha(a,b)$.

Let $\{e_1, e_2\}$ be an arbitrary basis in the real vector space $\mathcal{V}^2 \approx \Sigma \subset T_xM$. Then $y = \sum_{i=1}^2 y^i e_i$. Let $\Psi : \Sigma \to R^2$ be a mapping given by $\Psi(y) = (y^1, y^2) \in R^2$. Considering $\Psi(e_i)$ as an orthonormal system, R^2 becomes a Euclidean space E^2. We denote the Minkowski area in $\mathcal{M}^2_x$ by $\|\cdot\|_M$, and the Euclidean area in E^2 by $\|\cdot\|_E$. Then the 2-dimensional Minkowski area of D in $\mathcal{M}^n_x$ is the Minkowski area of D in $\mathcal{M}^2 \equiv \mathcal{M}$:

$$\|D\|_{\mathcal{M}} = \int_{\mathbb{D}} \sigma dy^1\, dy^2, \quad \sigma = \frac{\pi}{\|\mathbb{B}^2\|_E}, \quad \mathbb{B}^2 = \Psi(B^2_x), \quad \mathbb{D} = \Psi(D). \tag{1}$$

(Z. Shen [S1] §1.3, or H. Busemann [Bu], H. Rund [Ru] Chap. I, §8, D. Bao – S. S. Chern – Z. Shen [BCS], §1.4, and many other places.) Since $\int_{\mathbb{D}} dy^1 dy^2$ is the Euclidean area of $\mathbb{D}$, (1) is equivalent to

$$\|D\|_{\mathcal{M}} = \frac{\|\mathbb{D}\|_E}{\|\mathbb{B}^2\|_E}. \tag{1'}$$

(1) and (1') are true for any domain $\mathcal{G} \subset \Sigma$.

The Minkowski measure of the angle $\measuredangle\alpha(a,b)$ can be defined as follows:

Definition 1.

$$\measuredangle_{\mathcal{M}}\alpha(a,b) := \epsilon 2\|D\|_{\mathcal{M}}. \quad \epsilon = 1 \text{ or } -1. \tag{2}$$

The sign ϵ depends on the orientation of the angle.

$\measuredangle_{\mathcal{M}}\alpha$ can be expressed by the Minkowski functional $\mathcal{F}$ and the data of the two legs a and b. Let (r,φ) be a polar coordinate system in E^2, $e(\varphi)$ a unit vector in E^2 with polar coordinates $(1,\varphi)$, and $\mathcal{I}^2 := \partial B^2_x(1)$ the indicatrix curve of $\mathcal{M}^2_x$. Let $y \in \mathcal{I}^2$, $\Psi(\mathcal{I}^2) = r(\varphi)$, and $\Psi^{-1}(e(\varphi)) = \overline{e}(\varphi)$. Then $1 = \mathcal{F}(y(\varphi)) = \mathcal{F}(r(\varphi)\overline{e}(\varphi)) = r(\varphi)\mathcal{F}(\overline{e}(\varphi))$. Thus $r(\varphi) = \mathcal{F}^{-1}(\overline{e}(\varphi))$. The Euclidean area of $\mathbb{B}^2$ is

$$\|\mathbb{B}^2\|_E = \int_{\varphi=0}^{2\pi} \frac{1}{2} r^2(\varphi) d\varphi = \frac{1}{2}\int_0^{2\pi} \frac{1}{\mathcal{F}^2(\overline{e}(\varphi))} d\varphi$$

or

$$= \frac{1}{2}\int_0^{2\pi} \left(y(\varphi) \wedge \frac{dy}{d\varphi} \right) d\varphi = \frac{1}{2}\int_0^{2\pi} (y^1 dy^2 - y^2 dy^1).$$

Hence

$$\epsilon\|D\|_{\mathcal{M}} = 2\pi \left[\int_0^{2\pi} \mathcal{F}^{-2}(\overline{e}(\varphi)) d\varphi \right]^{-1} \cdot \int_{\mathbb{D}} r\, dr\, d\varphi. \tag{3.a}$$

In another from

$$\epsilon\|D\|_{\mathcal{M}} = 2\pi \left[\int_0^{2\pi} \mathcal{F}^{-2}(\overline{e}(\varphi)) d\varphi \right]^{-1} \cdot \int_{\varphi=\varphi_1}^{\varphi_2} \left[\int_{r=0}^{r(\varphi)} r\, dr \right] d\varphi, \tag{3.b}$$

$$\epsilon = \begin{array}{ll} +1 & if \quad \varphi_1 < \varphi_2 \\ -1 & if \quad \varphi_2 < \varphi_1, \end{array}$$

where $0 \leq \varphi_1, \varphi_2 \leq 2\pi$ denote the directions of the two legs a, b of α.

If $\mathcal{M}_x^n$ is a Euclidean space E^n, then (2) reduces to the Euclidean measure $\measuredangle_E \alpha$ of the angle α. Indeed, if $\mathcal{M}_x^n = E^n$, then $\mathbb{B}^2$ is the Euclidean unit ball. Now $\Psi^{-1}(e(\varphi)) = y(\varphi) \in \mathcal{I}^2 \Rightarrow \mathcal{F}^{-1}(y(\varphi)) = 1 \Rightarrow \|D\|_{\mathcal{M}} \overset{(3.b)}{=} 2\pi(2\pi)^{-1}(\varphi_2 - \varphi_1)$ and by (2) $\measuredangle_{\mathcal{M}}\alpha = \varphi_2 - \varphi_1$, which is the Euclidean measure of α. Thus $\measuredangle_{\mathcal{M}}\alpha = 2\epsilon\|D\|_{\mathcal{M}}$ is a generalization of the Euclidean measure of α.

If $\partial\mathbb{B}^2$ is an ellipse $\mathcal{E}$, then there is a linear (Minkowski) isomorphism i of $\Sigma = \mathcal{V}^2$, which takes $\mathcal{E}$ into the unit circle of E^2, and $\mathcal{M}_x^2$ into a $\overline{\mathcal{M}}_x^2$. Since $\|\cdot\|_{\mathcal{M}}$ is a Haar measure which is preserved by linear isomorphisms, we obtain that $\measuredangle_{\mathcal{M}}\alpha = \measuredangle_{\overline{\mathcal{M}}}\overline{\alpha} = \measuredangle_E i\alpha$.

$\|D\|_{\mathcal{M}}$ of (3.b) is positive if $\varphi_2 > \varphi_1$, and negative if $\varphi_2 < \varphi_1$. Therefore $\measuredangle_{\mathcal{M}}\alpha$ has a sign, and because of the additivity of the second integral in (3.b), $\measuredangle_{\mathcal{M}}\alpha$ is additive: $\measuredangle_{\mathcal{M}}\alpha_1 + \measuredangle_{\mathcal{M}}\alpha_2 = \measuredangle_{\mathcal{M}}(\alpha_1 + \alpha_2)$. Also $\measuredangle_{\mathcal{M}}\alpha$ is symmetric in the sense that $|\measuredangle_{\mathcal{M}}(a,b)| = |\measuredangle_{\mathcal{M}}(b,a)|$.

The case of the straight angle: In this case $a \cup b = g$ is a line through $0 \in T_xM$. Let $\measuredangle(a,b) = \alpha^+$ be the straight angle with the domain $\Sigma_g^+ = A^+$, and $\measuredangle(b,a) = \alpha^-$ the straight angle with the domain $\Sigma_g^- = A^-$. Because of the additivity

$$\measuredangle_{\mathcal{M}}\alpha^+ + \measuredangle_{\mathcal{M}}\alpha^- = 2\pi \qquad \forall g.$$

Therefore the equality $\measuredangle_{\mathcal{M}}\alpha^+ = \measuredangle_{\mathcal{M}}\alpha^-$ of the Minkowski measure of the two straight angles α^+ and α^- implies $\|B^2 \cap A^+\|_E = \|{}^+\mathbb{D}^2\|_E = \|{}^-\mathbb{D}^2\|_E = \|B^2 \cap A^-\|_E$, and conversely. In other words: $\measuredangle_{\mathcal{M}}\alpha^+ = \measuredangle_{\mathcal{M}}\alpha^-$ iff g bisects B^2.

If $\mathbb{B}^2$ is symmetric, then every line g through O bisects B^2. We show that also conversely, if every g through O bisects B^2, then $\mathbb{B}^2$ is symmetric. Suppose that $\mathbb{B}^2$ is non-symmetric. Then there exists a φ_0, such that in the applied polar coordinate system $r(\varphi_0) > r(\varphi_0 + \pi)$, where $(r(\varphi), \varphi) \in \partial\mathbb{B}^2$, $\forall \varphi$. A g is fixed by its direction φ. Then for every $g(\varphi)$

$$\frac{1}{2}\int_{\varphi}^{\varphi+\pi} r^2(\varphi)d\varphi = \frac{1}{2}\|\mathbb{B}^2\|_E, \quad \forall 0 \leq \varphi < \pi.$$

Especially

$$\int_{\varphi_0-\epsilon}^{\varphi_0-\epsilon+\pi} r^2(\varphi)d\varphi = \int_{\varphi_0+\epsilon}^{\varphi_0+\epsilon+\pi} r^2(\varphi)d\varphi.$$

Hence

$$\int_{\varphi_0-\epsilon}^{\varphi_0+\epsilon} r^2(\varphi)d\varphi = \int_{\varphi_0-\epsilon+\pi}^{\varphi_0+\epsilon+\pi} r^2(\varphi)d\varphi.$$

By the integral mean value theorem

$$2\epsilon r^2(\varphi_1) = 2\epsilon r^2(\varphi_2), \qquad \begin{array}{l} \varphi_0 - \epsilon \leq \varphi_1 \leq \varphi_0 + \epsilon \\ \\ \varphi_0 - \epsilon + \pi \leq \varphi_2 \leq \varphi_0 + \epsilon + \pi, \end{array}$$

and because of the continuity of $r(\varphi)$, $\epsilon \to 0$ yields $r(\varphi_0) = r(\varphi_0+\pi)$ in contradiction to our assumption. Therefore $\mathbb{B}^2$, and thus also B^2 is symmetric. This is equivalent to the absolute homogeneity of $\mathcal{F}$.

These statements are summed up in

Theorem 1. *$\measuredangle_{\mathcal{M}}\alpha = \epsilon 2\|D\|_{\mathcal{M}}$ is an additive, symmetric measure of the angles in Finsler spaces. In a Euclidean space this reduces to the Euclidean measure of α. $\measuredangle_{\mathcal{M}}\widetilde{\alpha} = \pm\pi$ for every straight angle $\widetilde{\alpha}$ if and only if the Finsler metric is absolute homogeneous.*

2. Isometry between F^n and $\overline{F}^n$

Let $F^n = (M, \mathcal{F})$ and $\overline{F}^n = (\overline{M}, \overline{\mathcal{F}})$ be two Finsler spaces, $\varphi : M \to \overline{M}$ a diffeomorphism, $\mathcal{I}(p_0) := \{y \in T_{p_0}M \mid \mathcal{F}(p_0, y) = 1\}$ and $\overline{\mathcal{I}}(\overline{p}_0) := \{\overline{y} \in T_{\overline{p}_0}\overline{M} \mid \overline{\mathcal{F}}(\overline{p}_0, \overline{y}) = 1\}$, $\overline{p}_0 = \varphi(p_0)$ are indicatrix hypersurfaces (indicatrices) of F^n and $\overline{F}^n$, respectively. $\mathcal{I}(p) \cap \Sigma = \mathcal{I}^2(p)$ is the indicatrix of $\mathcal{M}^2_p$, and $\overline{\mathcal{I}}(\overline{p}) \cap \overline{\Sigma} = \overline{\mathcal{I}}^2(\overline{p})$, $\overline{p} = \varphi(p)$, $\overline{\Sigma} = \varphi(\Sigma)$ is the indicatrix of $\overline{\mathcal{M}}^2_{\overline{p}}$. φ is an isometry iff

$$(d\varphi)\mathcal{I}(p) = \overline{\mathcal{I}}(\overline{p}), \quad \forall\, p \in M. \tag{4}$$

Theorem 2. *The diffeomorphism $\varphi : M \to \overline{M}$ is an isometry between the Finsler spaces F^n and $\overline{F}^n$ iff φ keeps angle and (2-dimensional) area.*

A) Suppose that φ is an isometry. By (4)

$$(d\varphi)\mathcal{I}^2(p) = (d\varphi)\mathcal{I}(p) \cap (d\varphi)\Sigma = \overline{\mathcal{I}}(\overline{p}) \cap \overline{\Sigma} = \overline{\mathcal{I}}^2(\overline{p}). \tag{5}$$

Σ and $\overline{\Sigma}$ equipped with Euclidean metrics are $E^2$and $\overline{E}^2$, respectively. Then by (1')

$$\|D\|_{\mathcal{M}} = \|D\|_{\mathcal{M}^2} = \pi\frac{\|\mathbb{D}\|_E}{\|\mathbb{B}^2\|_E},$$

and, since $d\varphi$ is a linear mapping which keeps the ratio of areas

$$\|D\|_{\mathcal{M}} = \pi\frac{\|(d\varphi)\mathbb{D}\|_{\overline{E}}}{\|(d\varphi)\mathbb{B}^2\|_{\overline{E}}}.$$

(Strictly speaking, $d\varphi$ should be replaced here by $(d\varphi)^* := \overline{\Psi} \circ d\varphi \circ \Psi^{-1}$; $\overline{\Psi} : \overline{\Sigma} \to \overline{E}^2$, $\overline{y} \mapsto (\overline{y}^1, \overline{y}^2)$.)

Finally, in consequence of (5), we obtain

$$\|D\|_{\mathcal{M}} = \pi\frac{\|\overline{\mathbb{D}}\|_{\overline{E}}}{\|\overline{\mathbb{B}}^2\|_E} = \|\overline{D}\|_{\overline{\mathcal{M}}}, \quad \overline{D} = (d\varphi)D.$$

This means that φ keeps (2-dimensional) area. (It is easy to see that an isometry keeps also the k-dimensional ($1 \leq k \leq n$) area.)

According to (2) $\measuredangle_{\mathcal{M}}\alpha$ is defined by area. Thus, if φ keeps area, then φ keeps angle too. Indeed, we know that $\measuredangle_{\mathcal{M}}\alpha \overset{(2)}{=} 2\epsilon\|D\|_{\mathcal{M}}$ and $\measuredangle_{\overline{M}}\overline{\alpha} = 2\epsilon\|\overline{D}\|_{\overline{\mathcal{M}}}$, where

$\overline{\alpha} = (d\varphi)\alpha$. Then, from $\|D\|_{\mathcal{M}} = \|(d\varphi)D\|_{\overline{\mathcal{M}}}$ (φ keeps area) we obtain $\measuredangle_{\mathcal{M}}\alpha = \measuredangle_{\overline{\mathcal{M}}}\overline{\alpha}$, that is φ keeps angle too.

B) Suppose that φ keeps area (a) and angle (n). Let us denote $(d\varphi)\mathcal{I}^2(p) =: \widetilde{\mathcal{I}}(p)$. $\overline{F}^n$ determines the indicatrix $\overline{\mathcal{I}}^2(\overline{p}) = \overline{\mathcal{I}}(\overline{p}) \cap \overline{\Sigma}$. We denote $(d\varphi)^{-1}\overline{\mathcal{I}}^2(\overline{p}) =: \widehat{\mathcal{I}}(p) \subset \Sigma$ and $\widehat{\mathcal{I}}(p) \cap A =: \widehat{D}$ (A is the domain of the angle $\measuredangle\alpha(a,b)$).

$d\varphi$ maps a, b into $\widetilde{a}$, $\widetilde{b}$, and the domain A into $\widetilde{A}$. Furthermore $\widetilde{D} := \widetilde{A} \cap \widetilde{\mathcal{I}}(\overline{p})$, and $\overline{D} := \widetilde{A} \cap \overline{\mathcal{I}}^2(\overline{p})$. Moreover $(d\varphi)^{-1}$ takes $\widetilde{a}$, $\widetilde{b}$ into a, b; $\widetilde{\mathcal{I}}(\overline{p})$ into $\mathcal{I}^2(p)$.

By our assumption (a) and (n) we obtain

$$\|D\|_{\mathcal{M}} \overset{(a)}{=} \|\widetilde{D}\|_{\overline{\mathcal{M}}}$$
$$\|D\|_{\mathcal{M}} \overset{(n)}{=} \|\overline{D}\|_{\overline{\mathcal{M}}} \overset{(a)}{=} \|\widehat{D}\|_{\mathcal{M}}.$$

Thus

$$\frac{\|\mathbb{D}\|_E}{\|\mathbb{B}^2(p)\|_E} = \|D\|_{\mathcal{M}} = \|\widehat{D}\|_{\mathcal{M}} = \frac{\|\mathbb{D}\|_E}{\|\mathbb{B}^2(p)\|_E} \Longrightarrow \|\mathbb{D}_E\| = \|\widehat{\mathbb{D}}\|_E. \tag{6}$$

Let c be a ray in Σ, $c \cap \mathcal{I}^2(p) = C$, and $c \cap \widehat{\mathcal{I}} = \widehat{C}$. Suppose that there exists p and c such that $C \neq \widehat{C}$, and let us say that $\widehat{C}$ is outside B_p^2. Then, because of the continuity, there exists a ray $h (\neq c)$, such that the whole arc $\widehat{C}\widehat{H}$ ($\widehat{H} := c \cap \widehat{\mathcal{I}}$) is outside B_p^2. Then the segment $D(c,h)$ of B_p^2 is a proper part of the segment $\widehat{D}(c,h)$ bounded by c, h and $\widehat{\mathcal{I}}$. Then $\|D(c,h)\|_E < \|\widehat{D}(c,h)\|_E$, what contradicts (6). Therefore $C = \widehat{C}$, $\forall\, c, p$. Then $\partial B_p^2 = \mathcal{I}^2(p) = \widehat{\mathcal{I}}$. Consequently $(d\varphi)\mathcal{I}^2(p) = \widetilde{\mathcal{I}} = (d\varphi)\widehat{\mathcal{I}} = \overline{\mathcal{I}}^2(\overline{p})$, $\forall\, p \in M$. This yields (4), and thus φ is an isometry.

3. Deviation of Finsler spaces from Riemannian spaces

There are known several conditions which imply the reduction of an F^n to a Riemannian space V^n. Such a condition is the vanishing of the Cartan tensor C_{ijk} or the constantness of the distortion $\tau(x,y)$ [S3]. Many other quantities, such as the S-curvature [S2], Landsberg curvature, Cartan torsion, etc. can be coupled with this problem. Also, a Finsler space is a Riemann space iff the indicatrices are ellipsoids. We want to present conditions expressed by the Minkowskian angle which imply the reduction of the indicatrices to ellipsoids.

We consider a Finsler space $F^n = (M, \mathcal{F})$ and its tangent space, as a Minkowski space $\mathcal{M}^n = (T_pM, \mathcal{F}(p,y))$, and a 2-dimensional linear subspace Σ of $T_{p_0}M$. $T_{p_0}M$ can be indentified with a vector space $\mathcal{V}^n$ or the coordinate space $R^n(x)$ which can be equipped with a Euclidean metric, yielding $E^n(x)$. B^n is the indicatrix body of $\mathcal{M}^n$, $\partial B^n = \mathcal{I}$ the indicatrix surface, and $\mathcal{I} \cap \Sigma = \mathcal{I}^2$ is the indicatrix of the $\mathcal{M}^2$ induced by $\mathcal{M}^n$ on Σ. If F^n is a Riemannian space V^n, then $\mathcal{M}^2$ is a Euclidean space, and $\mathcal{I}$ reduces to an ellipse. In this case Minkowskian and Euclidean angle are the same, $\measuredangle_{\mathcal{M}}\alpha(a,b) = \measuredangle_E\alpha(a,b)$, and it equals π iff α is a straight angle: its two legs a, b are two half lines of a straight line $g : a \cup b = g$. As we have seen

$$\measuredangle_{\mathcal{M}}\alpha(a,b)=\pi \quad \text{if} \quad a\cup b=g, \qquad \forall\, g\subset\Sigma\subset T_{p_0}M, \quad \forall\, p\in M$$

is necessary for an F^n to be a V^n.

Given an arbitrary ray $a\subset\Sigma$, let $\overline{a}$ be the other ray, such that $a\cup\overline{a}$ is a line g, and let $b\subset\Sigma$ be such that $\forall_{\mathcal{M}}\alpha(a,b)=\pi$. b depends on a, and $|\measuredangle_{\mathcal{M}}(b,\overline{a})|=:f(a)\geq 0$ is a function of $a\subset\Sigma$. $f(a)=0$, $\forall\, a\subset\Sigma$ is necessary for $F^n=V^n$. Let (r,ν) be a Minkowskian polar coordinate system in Σ, that is $r=\mathcal{F}(p_0,y)$ for a $y\in\Sigma$, and $\nu=\measuredangle_{\mathcal{M}}(0y,d_0)$ the Minkowskian angle between the ray $0y$ and an initial direction (initial ray) d_0. Then

$$(7) \quad \mathcal{G}(p,\Sigma):=\int_{\nu=0}^{2\pi}\overline{f}(\nu)d\nu=0, \quad \overline{f}(\nu)\equiv f(a(\nu)), \quad \forall\Sigma\subset T_{p_0}M, \quad \forall\, p\in M$$

is necessary for $F^n=V^n$. This and sec. 1 of this Chaper yield

Proposition 1. (7) *is equivalent to:* 1) $b=\overline{a}, \forall\, a$, 2) $\measuredangle_{\mathcal{M}}(a,\overline{a})=\pi, \forall\, a$, 3) $\forall\, g$ *bysects* $\mathcal{I}^2$, 4) $\mathcal{I}(p)$ *is symmetric,* 5) F^n *is absolutely homogeneous.*

All these are necessary for a Finsler space to be Riemannian. $\mathcal{G}(p,\Sigma)\geq 0$ measures the deviation of an F^n from being absolutely homogeneous in Σ at $p\in\Sigma$.

We want to obtain sufficient conditions for $F^n=V^n$. Our tool for this will be the difference between Minkowski orthogonality and transversality. Since the properties listed in the Proposition are necessary, we suppose that the indicatrices are symmetric. Let $g=a\cup\overline{a}$, $h=b\cup\overline{b}$ be lines and rays in $\Sigma\subset T_pM$, $\mathcal{M}_p^2=(T_pM,\mathcal{F}(p,y))$, and F^n as above. Our considerations will be restricted to Σ. Because of the symmetry of $\mathcal{I}^2(p)$ the Minkowskian perpendicularities $a\perp_{\mathcal{M}}b$, i.e. $\measuredangle_{\mathcal{M}}\alpha(a,b)=\frac{\pi}{2}$, $a\perp_{\mathcal{M}}\overline{b}$, $\overline{a}\perp_{\mathcal{M}}b$, $\overline{a}\perp_{\mathcal{M}}\overline{b}$ are equivalent. They mean $g\perp_{\mathcal{M}}h$. So, in the case of the symmetry of $\mathcal{I}^2(p)$ we can speak of the perpendicularity of lines in place of rays. Denoting by $g^{\perp}$ a line perpendicular to g, we obtain $(g^{\perp})^{\perp}\parallel g$.

Another notion is transversality. Let $g\cap\mathcal{I}^2(p)=\mathcal{G},\mathcal{G}'$. Then the tangent $T_{\mathcal{G}}\mathcal{I}^2(p)=:g^*$ is called transversal to g. Because of the symmetry of $\mathcal{I}^2(p)$, $T_{\mathcal{G}'}\mathcal{I}^2(p)=:(g')^*$ is parallel to g^*. Also, any line parallel to g^* is said to be transversal to g. So we can speak of transversality of a direction to another direction. Nevertheless, this relation is not symmetrical, that is the direction transversal to g^* is in general not $g:(g^*)^*\nparallel g$. The relation

$$(8) \qquad (g^*)^*\parallel g, \quad \forall\, g\subset\Sigma$$

means that in $\mathcal{M}_p^2$ the transversality operation * is involutive.

A strictly convex, closed, differentiable curve with O in its interior, and with the property (8) is called a Radon curve. Every ellipse is a Radon curve, but not conversely. This shows that if the indicatrices of an $F^2=(M,\mathcal{F})$ satisfy (8) at every point $p\in M$, then these indicatrices need not be ellipses, and thus F^2 needs not be a Riemannian space $V^2=(M,g)$.

We claim that if $n > 2$ and (8) is satisfied in every Σ with respect to $\mathcal{I}^2(p)$, then F^n is a V^n. Indeed, under these conditions every $\mathcal{I}^2(p) = \mathcal{I}(p) \cap \Sigma$, $\forall p, \Sigma$ is a Radon curve. Then in T_pM every cylinder osculating to B_p^n osculates along a planar curve [T]. In this case, according to W. Blaschke ([Bl] pp. 157–159), every $\mathcal{I}(p)$ is an ellipsoid, and thus $F^n = V^n$.

If $F^n = V^n$, then $\forall \mathcal{I}^2(p)$ is an ellipse, and (8) is satisfied. But $(g^\perp)^\perp \parallel g$ is always true if $\mathcal{I}^2(p)$ is symmetric. Hence in this case $g^\perp \parallel g^* . \forall g$.

If $\measuredangle_{\mathcal{M}}\alpha(g^*, g^\perp) = 0$, i.e. if $g^* = g^\perp$, then (8) is realized, for $(g^\perp)^\perp \parallel g$ is true. Hence

$$\begin{gathered} K(p,\Sigma) := \int_0^\pi |\measuredangle_{\mathcal{M}}\alpha(g^*(\nu), g^\perp(\nu))| d\nu = 0, \\ g \subset \Sigma, \quad \forall \Sigma \subset T_{p_0}M, \ \forall p \in M \end{gathered}$$

is sufficient for $F^n = V^n$. Conversely, (9) is always satisfied in a V^n. Thus we obtain

Theorem 3. *An absolutely homogeneous Finsler space F^n, $n > 2$ reduces to a Riemann space if and only if $K(p, \Sigma) = 0$, $\forall \Sigma \subset T_{p_0}M$, $\forall p \in M$.*

The deviation of an absolutely homogeneous F^n from being a Riemannian space on $\Sigma \subset T_{p_0}M$ can be measured by $K(\Sigma)$. Thus $K(\Sigma)$ can be considered as a kind of sectional curvature. The deviation at a point $p_0 \in M$ can be measured by the integral

$$\mathcal{G}(p_0) = \frac{1}{\int_{\mathcal{G}_{n,2}} d\sigma} \int_{\mathcal{G}_{n,2}} K(\Sigma) d\sigma \geq 0,$$

where $\mathcal{G}_{n,2}$ is the Grassmann manifold of the 2-dimensional linear subspaces of $T_{p_0}M$, and $d\sigma$ is a positive measure on $\mathcal{G}_{n,2}$, such that $\int_{\mathcal{G}_{n,2}} d\sigma$ is finite and invariant with respect to linear transformations in $T_{p_0}M$. The deviation of F^n from being Riemannian on M (the global case) can be measured by the integral

$$H(M) = \frac{1}{\int_M d\mu} \int_M \mathcal{G}(x) d\mu \geq 0,$$

where $d\mu$ is the Finsler volume element, and $\int_M d\mu$ is supposed to be finite.

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Institute of Mathematics
University of Debrecen
H-4010 Debrecen Pf. 12
Hungary

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KĄT W PRZESTRZENIACH MINKOWSKIEGO I FINSLERA

Streszczenie

Pole w przestrzeniach Minkowskiego zostało określone przez Busemanna [Bu] oraz było badane i często stosowane przez kolejnych autorów. Infinitezymalnie przestrzeń Finslera jest przestrzenią Minkowskiego. Jeśli zatem potrafimy zmierzyć pole w przestrzeni Minkowskiego, przez całkowanie otrzymamy pole (obszaru) w przestrzeni Finslera. To samo zachodzi dla podrozmaitości. W pracy rozpatrujemy kąt utworzony przez dwa wektory w przstrzeni stycznej do rozmaitości bazowej przestrzeni Finslera. Pojęcie kąta w przestrzeniach Minkowskiego (względnie Finslera) nie budziło dotąd specjalnego zainteresowania. Ponieważ przestrzeń Finslera ma swą przestrzeń styczną w przestrzeni Minkowskiego, mierzenie kątów w przestrzeni Finslera redukuje się do mierzenia kątów w przestrzeni Minkowskiego. Wykazujemy, że można tego dokonać przez pomiar odchylenia przestrzeni Finslera od odpowiedniej przestrzeni Riemanna. Również można wykazać, że dyfeomorfizm między dwiema przestrzeniami Finslera jest izometrią – wtedy i tylko wtedy, gdy pozostawia niezmienione kąty (w powyższym sensie) i pola, podobnie do dobrze znanego wyniku w geometrii riemanowskiej.