Short Thesis for The Degree of Doctor of Philosophy (PhD)

On stretch Finsler metrics and six-dimensional filiform nilmanifolds

by

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Introduction

This thesis contains an extensive summary of the main results addressed in two chapters of the dissertation.

In [8], L. Berwald presented the concept of the stretch curvature tensor and showed that it vanishes if and only if a vector's length stays constant under the parallel displacement along an infinitesimal parallelogram. Then, the stretch curvature studied by C. Shibata in 1978 [28], and M. Matsumoto in 1997 and 2004 (see [21, 22]). Shibata proved that a Kropina metric is a Berwald metric if and only if the stretch curvature tensor vanishes under the condition that the Kropina metric is positive-definite. Matsumoto aimed to provide a fresh understanding of the stretch curvature within the framework of modern Finsler connection theory. S. Bácsó and M. Matsumoto proved that a Douglas metric with vanishing stretch curvature is R-quadratic if and only if its **E**-curvature vanishes (see [7]). It is interesting to find conditions under which Douglas metrics with vanishing stretch curvature reduce to Berwald metrics. A. Tayebi and T. Tabatabaeifar in 2015 [35] proved that every Douglas-Randers metric with vanishing stretch curvature is a Berwald metric. B. Najafi and A. Tayebi in 2017 [24] introduced a new non-Riemannian quantity named *mean stretch curvature*. A Finsler metric with vanishing mean stretch curvature is called *weakly stretch metric*. This class of Finsler metrics contains the class of stretch metrics. In 2018, they showed that every complete P-reducible weakly stretch metric with bounded Cartan torsion is a Landsberg metric. Furthermore, classified complete weakly stretch surfaces and show that every complete weakly stretch surface is Riemannian or Landsbergian [25].

In [32] A. Tayebi and B. Najafi showed that every homogeneous (α, β)-metric is a stretch metric if and only if it is a Berwald metric. A. Tayebi and H. Sadeghi characterized the stretch (α, β)-metrics with vanishing S-curvature, more precisely they proved a regular non-Randers type (α, β)-metric with vanishing S-curvature is stretchian if and only if it is Berwaldian (see [34]). In [31] A. Tayebi N. Izadian proved that every Douglas-square metric is a

Berwald metric if and only if it is a weakly stretch metric. In [30] with the authors M. Faghfuri and N. Jazer proved that every compact Finsler manifold with posi-tive (or negative) relatively isotropic mean stretch curvature is a weakly Landsbergmetric. In [33] with B. Najafi classified the almost regular weakly stretch non-Randers-type (α, β)-metrics with vanishing S-curvature. In [29] with M. Bahadori and H. Sadeghi proved that a spherically symmetric Finsler metric is a stretch metric if and only if it is R-quadratic. In [18] with F. Kamelaei and B. Najafi proved that every homogeneous Finsler metric has relatively isotropic stretch curvature if and only if it is a Landsberg metric. Recently, some properties of the weakly stretch Finsler metric of the spical Finsler metric were investigated (see[11, 36]).

The contribution of the first chapter of the dissertation is use the Berwald curvature instead of the Cartan torsion, and investigate the relationships among the classes obtained analogously to the Landsberg and the stretch curvatures. This will enhance the understanding of the role of the relevant tensors in characterizing the new classes of Finsler metrics.

Z. Shen [27, page 139] introduced a non-Riemannian quantity $\widetilde{\mathbf{B}}$ which is obtained from the Berwald curvature $\mathbf{B}$ by the covariant horizontal differentiation along Finslerian geodesics. For a vector $y \in \mathcal{T}_pM$, define $\widetilde{\mathbf{B}}_y : T_pM \times T_pM \times T_pM \to T_pM$ by $\widetilde{\mathbf{B}}_y(u, v, w) := \widetilde{B}^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i} |_x$, where

$$\widetilde{B}^i_{jkl} := B^i_{jkl|m} y^m.$$

The Finsler metric F is called $\widetilde{\boldsymbol{B}}$*-metric* if and only if $\widetilde{\mathbf{B}} = 0$.

The goal of section 1.1 is use the quantity $\widetilde{\mathbf{B}}$ to study a class of Finsler metrics containing the class of Berwald metric. The quantity resulted from $\widetilde{\mathbf{B}}$ has the following form:

$$\mathcal{K}^i_{jklm} := 2\left(\widetilde{B}^i_{jkl|m} - \widetilde{B}^i_{jkm|l}\right).$$

The family $\boldsymbol{K} := \{\boldsymbol{K}_y : y \in \mathcal{T}_pM\}$ is called the *stretch* $\widetilde{\boldsymbol{B}}$*-curvature.* A Finsler metric F is said to be $\widetilde{\boldsymbol{B}}$*-stretch metric* if and only if $\boldsymbol{K} = 0$, where $\boldsymbol{K}_y(u, v, w, z) := \mathcal{K}^i_{jklm}(y)u^j v^k w^l z^m \frac{\partial}{\partial x^i} |_x$. It is interesting to find some curvature properties conditions under which a $\widetilde{\mathbf{B}}$-stretch metric reduces to a $\widetilde{\mathbf{B}}$-metric (see Theorems 1.4, 1.5, 1.6, 1.7).

The aim of section 1.2 is to study a class of Finsler metrics that includes the class of Douglas metrics. Finsler metrics in this class are known as generalized Douglas metrics. We prove that every generalized Douglas metric with vanishing $\widetilde{\mathbf{B}}$-stretch tensor is a Douglas metric under the condition that the mean Berwald curvature is horizontally constant along geodesics of F. Then, we show that if (M, F) is a Douglas Finsler manifold then the Finsler metric F is an $\mathbf{H}$-stretch if and only if it is a $\widetilde{\mathbf{B}}$-metric. The results of this section are introduced in Theorems 1.12, 1.14, 1.16.

The second chapter of the dissertation is devoted to investigate homogeneous Riemannian nilmanifolds. Among nilpotent Lie groups with higher nilpotency class the filiform Lie groups play an essential role. An n-dimensional filiform Lie algebra has the maximal possible nilpotency class $n-1$. That is its lower central series has $n-2$ non-trivial subalgebra. The 3-dimensional non-abelian nilpotent Lie algebra, the Heisenberg Lie algebra, is two-step nilpotent. At the same time it is the unique filiform Lie algebra of dimension 3. One way to generalize the Heisenberg Lie algebra leads to the notion of the Heisenberg type Lie algebras (cf. [19]). These are two-step nilpotent Lie algebras but the dimension of the top step is enlarged. The n-dimensional filiform Lie algebras are another generalization of the Heisenberg Lie algebra. In this case the number of steps grows up as the dimension increases (cf. [20], p. 2). In [17] the isometry equivalence classes and the isometry groups of connected and simply connected filiform Riemannian nilmanifolds of arbitrary dimension were thoroughly studed. It turns out that every filiform metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ has a decomposition into orthogonal direct sum of 1-dimensional subspaces which is preserved by all orthogonal automorphisms of $(\mathfrak{n}, \langle ., . \rangle)$. Hence the isometry groups of the filiform nilmanifolds have the same dimension as the nilmanifolds (cf. [17, Corollary 5]). In particular the detailed description for the isometrically isomorphic equivalence classes of standard filiform metric Lie algebras and the isometry groups of the corresponding nilmanifolds are given in [17, Theorem 7, Corollary 8]. Using this approach in section 2.1 we make analogous consideration for the 6-dimensional connected simply connected filiform nilmanifolds. Applying the classification procedure given by [17, pp. 371-372], we find in Theorems 2.1, 2.2, 2.3, 2.4, and in the classes of isometrically isomorphic 6-dimensional filiform metric Lie algebras and the group of their orthogonal automorphisms. The group of all isometries of the corresponding connected and simply connected filiform nilmanifolds are given in Corollary 2.5.

The purpose of sections 2.2, 2.3 is to study the totally geodesic subgroups of connected simply connected 6-dimensional filiform Riemannian nilmanifolds. The Levi-Civita connection of the metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ is defined by

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X,Y], Z \rangle + \langle [Z,X], Y \rangle + \langle [Z,Y], X \rangle)$$

for left invariant vector fields $X, Y, Z \in \mathfrak{n}$. A subalgebra $\mathfrak{h}$ of $(\mathfrak{n}, \langle .,. \rangle)$ is totally geodesic if for all $X, Y \in \mathfrak{h}$ one has $\nabla_X Y \in \mathfrak{h}$. A subgroup H of $(N, \langle .,. \rangle_N)$ which passes through the identity $e \in N$ is totally geodesic precisely if the corresponding subalgebra $\mathfrak{h}$ is totally geodesic. The left cosets xH, $x \in N$, with respect to a totally geodesic subgroup H give a totally geodesic foliation on N. The investigation of the geometry of two-step nilmanifolds began with the works [14, 15] of P. Eberlein. He studies curvatures and totally geodesic subgroups in non-singular two-step nilmanifolds. His results were generalized to the case where N does not have the non-singularity condition in [13]. It is also discussed in [13, 15] criteria under which a subalgebra of a two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ is totally geodesic. The work [23] is devoted to investigate totally geodesic subalgebras in two-step nilpotent metric Lie algebras. P. T. Nagy and S. Homolya proved there that for isomorphic Lie algebras $\mathfrak{n}$ and $\mathfrak{n}^*$ there exists a bijective linear map $\mathfrak{n} \to \mathfrak{n}^*$ preserving the flat totally geodesic property of subalgebras. Although the totally geodesic property of subalgebras is very sensitive with respect to the change of the inner product of a metric Lie algebra their outstanding result shows that the linear structure of totally geodesic subalgebras of two-step nilpotent metric Lie algebras depends only on the isomorphism class of the Lie algebra. The generating vector of a one-dimensional totally geodesic subalgebra is called geodesic. They also determined geodesic vectors and flat totally geodesic subalgebras in two-step nilpotent metric Lie algebras of dimension less than or equal to 6. In [20] M. M. Kerr and T. L. Payne extended the study of the geometry of two-step nilmanifolds for two infinite families of nilmanifolds belonging to filiform nilpotent Lie algebras with special inner products. Their paper is an initial source to study curvatures and totally geodesic subgroups of filiform nilmanifolds.

The continuation of their studies was presented by G. Cairns, A. Hinić Galić and Y. Nikolayevsky in [9, 10]. There the authors gave several results on the possible dimensions of totally geodesic subalgebras of nilpotent metric Lie algebras. They also found examples, where the obtained bounds on the dimensions of totally geodesic subalgebras are attained. Furthermore, they gave an example of a 6-dimensional filiform nilpotent Lie algebra which does not allow any totally geodesic subalgebra of dimension greater than 2. Using the classification of the non two-step nilpotent metric Lie algebras of dimension at most 5 given in [23]. A. Al-Abayechi and Á. Figula determined the geodesic vectors and flat totally geodesic subalgebras in these metric Lie algebras. They found that the flat totally geodesic subalgebras and geodesic vectors in non-filiform metric Lie algebras with a one-dimensional center are independent of the choice of the inner product (see [5]).

The aim in section 2.2 is to determine the sets of the geodesic vectors and hence the one-dimensional totally geodesic subalgebras in the six-dimensional filiform metric Lie algebras. Applying the results (2.1), (2.2), (2.3), (2.4) describing the classes of isometrically isomorphic 6-dimensional filiform metric Lie algebras and the claim that a non-zero vector $Y \in (\mathfrak{n}, \langle .,. \rangle)$ is geodesic precisely if for all $X \in (\mathfrak{n}, \langle .,. \rangle)$ one has $\langle [X,Y], Y \rangle = 0$, in Theorem 2.6 the set of the geodesic vectors is decomposed as the disjoint union of subsets depending on the parameters α_i, β_j. In section 2.3 we deal with the question under which conditions on the parameters α_i, β_j exists a flat totally geodesic subalgebra of dimension greater than 1 in the class $\mathcal{C}$ of the six-dimensional filiform metric Lie algebras. It follows from [10, Proposition 1.13] that for filiform nilpotent metric Lie algebras there does not exist any totally geodesic subalgebra of codimension one. From [10, Theorems 2.17, 2.18] it turns out that only metric Lie algebras corresponding to the standard filiform Lie algebra can allow a flat totally geodesic subalgebra of codimension 2. Some filiform Lie algebras possess only small dimensional totally geodesic subalgebras independently of the choice of innerproduct.

Our investigation shows that in the class $\mathcal{C}$ with the exception of the metric Lie algebras corresponding to the standard filiform Lie algebra the flat totally geodesic subalgebras of all metric Lie algebras have dimension at most two (see Theorem 2.7). A metric Lie algebra corresponding to the 6-dimensional standard filiform Lie algebra has a 4-dimensional flat totally geodesic subalgebra (see Theorem 2.8). The possibilities for the existence of 3- and 2-dimensional flat totally geodesic subalgebras in a metric Lie algebra corresponding to the 6-dimensional standard filiform Lie algebra are described in Theorem 2.8.

Chapter 1

Stretch Finsler metrics

Let (M, F) be a Finsler manifold. The third-order derivative of $\left(\frac{1}{2}F_p^2\right)$ at $y \in \mathcal{T}_pM$ is called *Cartan torsion*. The Landsberg curvature $\mathbf{L}$ is a description of how Cartan torsion $\mathbf{C}$ along geodesics changes. A Landsberg metric is a Finsler metric that fulfills $\mathbf{L} = 0$. In 1926, L. Berwald [8] introduced a Finslerian quantity during the investigation on the generalization of Landsberg curvature. This quantity is called a *stretch curvature* and is denoted by Σ_y. Geometrically, a Finsler metric is stretch-type if and only if a vector's length stays constant under parallel displacement along an infinitesimal parallelogram. In this chapter, we study a class of Finsler metrics containing the class of Berwald (weakly Berwald) metric (respectively). A Finsler metric in this class is called $\widetilde{\mathbf{B}}$-stretch ($\mathbf{H}$-stretch) metric (respectively).

1.1 $\widetilde{\mathbf{B}}$-, and H-stretch metric

Z. Shen [27, page 139] introduced a non-Riemannian quantity $\widetilde{\mathbf{B}}$ which is obtained from the Berwald curvature $\mathbf{B}$ by the covariant horizontal differentiation along Finslerian geodesics. For a vector $y \in \mathcal{T}_pM$, define $\widetilde{\mathbf{B}}_y : T_pM \times T_pM \times T_pM \to T_pM$ by $\widetilde{\mathbf{B}}_y(u, v, w) := \widetilde{B}^i_{jkl}(y)u^jv^kw^l \frac{\partial}{\partial x^i} \mid_x$, where

$$\widetilde{B}^i_{jkl} := B^i_{jkl|m}y^m. \tag{1.1}$$

The Finsler metric F is called $\widetilde{\boldsymbol{B}}$*-metric* if and only if $\widetilde{\mathbf{B}} = 0$.

Definition 1.1. *For a vector $y \in \mathcal{T}_pM$, we define $\boldsymbol{K}_y : T_pM \times T_pM \times T_pM \times T_pM \to T_pM$ by*

$$\boldsymbol{K}_y(u, v, w, z) := \mathcal{K}^i_{jklm}(y)u^jv^kw^lz^m \frac{\partial}{\partial x^i} \mid_x,$$

where

$$\mathcal{K}^i_{jklm} := 2\left(\widetilde{B}^i_{jkl|m} - \widetilde{B}^i_{jkm|l}\right),$$

and " | " is the horizontal derivation with respect to the Berwald connection D *of* F*.*

The family $\boldsymbol{K} := \{\boldsymbol{K}_y : y \in \mathcal{T}_pM\}$ is called the *stretch* $\widetilde{\mathbf{B}}$*-curvature.* A Finsler metric F is said to be $\widetilde{\mathbf{B}}$*-stretch metric* if and only if $\boldsymbol{K} = 0$. Especially, every $\widetilde{\mathbf{B}}$-metric is a $\widetilde{\mathbf{B}}$-stretch metric. Therefore, on the contrary, it is interesting to find some topological condition on the manifold M such that every $\widetilde{\mathbf{B}}$-stretch metric on M reduces to a $\widetilde{\mathbf{B}}$-metric.

Let us introduce a non-trivial example, where $|.|$ and $\langle , \rangle$ denote the Euclidean norm and the inner product in $\mathbb{R}^n$, respectively.

Example 1.2. *The Finsler function* F

$$F(x,y) = \frac{\left(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \varepsilon\langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}},$$

on the unit ball $\mathbb{B}^n$ is a $\widetilde{\mathbf{B}}$-stretch metric when $n = 2$ and $n = 3$. This can be shown using the Finsler package and Maple program [37]. We guess it should work in the general dimension but the calculation is very tedious and a bit complicated.

We have the following inclusions:

$$\{\text{Berwald metric}\} \subset \left\{\widetilde{\mathbf{B}}\text{-metric}\right\} \subset \left\{\widetilde{\mathbf{B}}\text{-stretch metric}\right\}.$$

The Finslerian quantity $\mathbf{H}$ was introduced by H. Akbar-Zadeh in 1988 [4] to characterization of Finsler metrics of constant flag curvature which is obtained from the mean Berwald curvature $\mathbf{E}$ by the covariant horizontal differentiation along geodesics. For a vector $y \in \mathcal{T}_pM$, $\mathbf{H}_y : T_pM \times T_pM \to \mathbb{R}$ is given by $\mathbf{H}_y(u,v) := H_{jk}(y)u^jv^k$, where

$$H_{jk} := E_{jk|l}y^l.$$

The Finsler metric F is called $\boldsymbol{H}$*-metric* if and only if $\mathbf{H} = 0$.

Definition 1.3. *For a vector $y \in \mathcal{T}_pM$, we define $\boldsymbol{\kappa}_y : T_pM \times T_pM \times T_pM \to \mathbb{R}$, by*

$$\boldsymbol{\kappa}_y(u, v, w) := \kappa_{jkl}(y)u^j v^k w^l,$$

where

$$\kappa_{jkl} := 2\left(H_{jk|l} - H_{jl|k}\right).$$

The Finsler metric F is called $\boldsymbol{H}$*-stretch metric* if and only if $\boldsymbol{\kappa} = 0$. We have the following inclusion relations

$$\{\text{weakly Berwald metric}\} \subset \{\mathbf{H}\text{-metric}\} \subset \{\mathbf{H}\text{-stretch metric}\}.$$

In this chapter, we prove the following theorems.

Theorem 1.4. *Suppose that F is a positively complete $\widetilde{\boldsymbol{B}}$-stretch metric with bounded Berwald torsion. Then F must be a $\widetilde{\boldsymbol{B}}$-metric and the Berwald torsion is constant along any geodesic.*

Theorem 1.5. *Every complete $\boldsymbol{H}$-stretch metric with bounded mean Berwald torsion is $\boldsymbol{H}$-metric.*

Let (M, F) be a Finsler manifold. Then F is called a *relatively isotropic stretch $\widetilde{\boldsymbol{B}}$-curvature* if its stretch $\widetilde{\mathbf{B}}$-curvature is given by

$$\mathcal{K}^i_{jklm} := \lambda F\left(B^i_{jkl|m} - B^i_{jkm|l}\right),$$

where $\lambda := \lambda(x, y)$ is scalar function on TM. In this case, (M, F) is called a *relatively isotropic $\widetilde{\boldsymbol{B}}$-stretch manifold.* If $\lambda \geq 0$ ($\lambda \leq 0, \lambda =$ constant), then F is said to be *non-negative (non-positive or constant) relatively isotropic stretch $\widetilde{\boldsymbol{B}}$-curvature* (respectively).

If the stretch $\mathbf{H}$-curvature is given by

$$\boldsymbol{\kappa} := \lambda F\left(E_{jk|l} - E_{jl|k}\right),$$

then F is said to be *non-negative (non-positive, constant) relatively isotropic stretch $\boldsymbol{H}$-curvature* if we have $\lambda \geq 0$ ($\lambda \leq 0, \lambda =$ constant) (respectively).

By Theorem 1.4 every complete $\widetilde{\mathbf{B}}$-stretch Finsler manifold with bounded Berwald torsion is a $\widetilde{\mathbf{B}}$-manifold. Thus, a compact $\widetilde{\mathbf{B}}$-stretch Finsler manifold reduces to a $\widetilde{\mathbf{B}}$-manifold. We generalize this result as follows.

Theorem 1.6. *A compact Finsler manifold with non-negative (non-positive) relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature is $\widetilde{\mathbf{B}}$-Finsler manifold. More precisely, a complete Finsler manifold with constant relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature and bounded $\widetilde{\mathbf{B}}$-curvature is $\widetilde{\mathbf{B}}$-metric.*

By Theorem 1.5 every complete **H**-stretch Finsler manifold with bounded mean Berwald torsion is a **H**-manifold. Thus, a compact **H**-stretch Finsler manifold reduces to a **H**-manifold. We generalize this result as follows.

Theorem 1.7. *Every compact Finsler manifold with non-positive (non-negative) relatively isotropic stretch **H**-curvature is **H**-Finsler manifold. More precisely, a complete Finsler metric with constant relatively isotropic stretch **H**-curvature and bounded **H**-curvature is **H**-metric.*

In view of Theorem 1.5, a **H**-stretch metric reduces to a **H**-metric. Tayebi et al. in [30] proved that any **H**-metric is a $\widetilde{\mathbf{B}}$-metric for a Finsler surface (M, F). Then, we get the following corollary.

Corollary 1.8. *Let (M, F) be a Finsler surface. Then F is a **H**-stretch metric if and only if it is $\widetilde{\mathbf{B}}$-metric.*

Corollary 1.9. *Let (M, F) be Finsler manifold. If F is an non-negative (non-positive) relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature, then it is an **H**-metric.*

Corollary 1.10. *Let (M, F) be Finsler surface. If F is a non-negative (non-positive) relatively isotropic stretch **H**-curvature, then it is a $\widetilde{\mathbf{B}}$-metric.*

1.2 Generalized Douglas curvature

One of the important quantities in Finsler geometry is Douglas' curvature which is an invariant tensor by a projective change $\phi : F \to \bar{F}$. In 1997 S. Bácsó and M. Matsumoto [6] established the concept of Douglas space as an extension of Berwald spaces by studying the geodesics curves in a Finsler space. Additionally, they investigate the relationship between Douglas spaces and other particular Finsler spaces, including Wagner spaces and Landsberg spaces. Douglas metrics can be characterized by

$$\mathcal{G}^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i,$$

where $\Gamma^i_{jk}(x)$ are local functions on M and $P(x,y)$ is a local positively homogeneous function of degree one. The class of Douglas metrics is much larger than that of Berwald metrics. The aim of this section is to study a class of Finsler metrics that includes the classes of Douglas metrics.

Definition 1.11. *We call the Finsler metric F is a generalized Douglas metric if and only if the quantity $\widetilde{\boldsymbol{B}}$-curvature in (1.1) is given by*

$$\widetilde{B}^i_{jkl} := B^i_{jkl} + \omega_{jk}\delta^i_l + \omega_{jl}\delta^i_k + \omega_{kl}\delta^i_j + E_{jk;l}y^i, \tag{1.2}$$

where ω is a smooth map $M \to \wedge^2 \mathcal{T}_p M$ given by $\omega(p) := \omega_{ij}(p)dx^i \wedge dx^j$ at any point $p \in M$.

We have the following

$$\{\text{ Berwald metric}\} \subset \{\text{Douglas metric}\} \subset \{\text{generalized Douglas metric}\}.$$

In the next result, we show that every generalized Douglas metric with vanishing $\widetilde{\mathbf{B}}$-stretch tensor is a Douglas metric under the condition that the mean Berwald curvature is horizontally constant along geodesics of F.

Theorem 1.12. *Let (M,F) be a generalized Douglas Finsler manifold. Suppose that F is a $\widetilde{\boldsymbol{B}}$-stretch metric. Then F is a Douglas metric.*

The converse of Theorem 1.12 is not true. Let us introduce a famous example of Finsler metrics introduced by a physicist G. Randers in [26].

Example 1.13. *Let $|.|$ and $\langle .,. \rangle$ be the Euclidean norm and the inner product in $\mathbb{R}^n$ respectively. Let $F = \alpha + \beta$ be a Finsler metric with $\|\beta_x\|_\alpha < 1$ where*

$$\alpha = \frac{\sqrt{|y|^2 + \varepsilon\left(|x|^2|y|^2 - \langle x,y\rangle^2\right)}}{1+\varepsilon|x|^2}, \quad \beta = \frac{\sqrt{-\varepsilon}\langle x,y\rangle}{1+\varepsilon|x|^2}, \qquad \varepsilon < 0,$$

and $x \in \mathbb{B}^n(\delta) \subset \mathbb{R}^n$, $\delta = \frac{1}{\sqrt{-\varepsilon}}$, $y \in T_x\mathbb{B}^n(\delta)$. Putting $\varepsilon = -1$, we get a metric called *Funk metric* which satisfies $F_{x^s} - FF_{y^s} = 0$. A Funk metric F is a Douglas metric but it is not $\widetilde{\mathbf{B}}$-stretch.

Every Finsler metric with vanishing $\widetilde{\mathbf{B}}$-curvature has vanishing **H**-curvature. Thus, every $\widetilde{\mathbf{B}}$-metric is an **H**-metric then it is **H**-stretch metric. But the converse might not hold. A. Tayebi et al. in [30] showed that it is true on Finsler surfaces. We generalized this result as follows

Theorem 1.14. *Let (M, F) be a Douglas Finsler manifold with $n \geq 3$. Then every **H**-stretch metric is a $\widetilde{\mathbf{B}}$-metric.*

Bácsó and Matsumoto [6] showed that if the 1-form β is a closed, then any Randers metric $F = \alpha + \beta$ is a Douglas metric. Then by Theorem 1.14, we get the following.

Corollary 1.15. *Let $F = \alpha + \beta$ is a Randers metric on a manifold M with closed 1-form β. Then F is a $\widetilde{\mathbf{B}}$-metric if and only if it is a **H**-stretch metric.*

Exploiting the above statements, the next result gives a formula involving the relation between the quantities stretch $\widetilde{\mathbf{B}}$-curvature and stretch **H**-curvature. Namely, we have the following

Theorem 1.16. *Let (M, F) be a Douglas Finsler manifold. Then, the stretch $\widetilde{\mathbf{B}}$-curvature of F is given by*

$$\mathcal{K}^i_{jklm} := \frac{2}{n+1}\left\{\kappa_{jlm}h^i_k + \kappa_{klm}h^i_j\right\}.$$

Chapter 2

6-dimensional filiform nilmanifolds and the corresponding metric Lie algebras

In this chapter of the dissertation we determine the isometry equivalence classes of nilmanifolds on 6-dimensional filiform Lie groups. The representatives of these classes are 6-dimensional connected simply connected filiform Lie groups N equipped with a left invariant metric $\langle .,. \rangle_N$. We study the isometry groups of the obtained filiform Lie groups N having left invariant metric $\langle .,. \rangle_N$. Moreover, we investigate the geodesic vectors and the flat totally geodesic subalgebras of the metric Lie algebras corresponding to the received filiform Lie groups N with left invariant metric $\langle .,. \rangle_N$.

2.1 Isometry equivalence classes and isometry group of 6-dimensional filiform nilmanifolds

Our aim in this section is to classify the isometrically isomorphic equivalence classes of 6-dimensional filiform metric Lie algebras $(\mathfrak{n}, \langle .,. \rangle)$ and to determine the group $\mathcal{OA}(\mathfrak{n})$ of all orthogonal automorphisms of the representatives of the obtained classes. The semi-direct product $N \rtimes \mathcal{OA}(\mathfrak{n})$ of the connected simply connected Lie group N of $\mathfrak{n}$ and the group $\mathcal{OA}(\mathfrak{n})$ gives the isometry group of the corresponding connected simply connected Lie group N with left invariant metric $\langle .,. \rangle_N$ induced by the Euclidean inner product $\langle .,. \rangle$ of $\mathfrak{n}$. To proceed this classification we use the list of W. de Graaf in [12] to fix a basis $\mathfrak{B} = \{G_1, G_2, \cdots, G_6\}$ of the non-isomorphic 6-dimensional filiform Lie algebras $\mathfrak{l}_{6,k}, k = 14, \ldots, 18$ and apply the classification procedure given in [17], pp. 371-372. This procedure describes the representative of each isometrically isomorphic equivalence classes of the 6-dimensional filiform metric Lie algebras $(\mathfrak{l}, \langle .,. \rangle)$ as a filiform Lie algebra $\mathfrak{n}$ isomorphic to $\mathfrak{l}$ such that the non-trivial Lie brackets of $\mathfrak{n}$ are defined on the Euclidean vector space $\mathbb{E}^6$ with a distinguished orthonormal basis $\{E_1, E_2, \cdots, E_6\}$.

Our first aim is to study the 6-dimensional filiform Lie algebra $\mathfrak{l}_{6,14}$ defined by the following non-vanishing Lie brackets

$$\begin{aligned}\mathfrak{l}_{6,14}:&[G_1,G_2]=G_3,\ [G_1,G_3]=G_4,\ [G_1,G_4]=G_5,\ [G_2,G_3]=G_5,\\ &[G_2,G_5]=G_6,\ [G_4,G_3]=G_6.\end{aligned}$$

Theorem 2.1. *A metric Lie algebra $(\mathfrak{l}_{6,14}, \langle .,.\rangle)$ is isometrically isomorphic to a unique metric Lie algebra $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$, $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 7$, defined on $\mathbb{E}^6$ by the non-vanishing commutators*

$$\begin{aligned}
&[E_1,E_2]=\alpha_1E_3+\beta_1E_4+\beta_2E_5+\beta_3E_6, && [E_2,E_3]=\alpha_3E_5+\beta_6E_6,\\
&[E_1,E_3]=\alpha_2E_4-\Big(\frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4}\Big)E_5+\beta_4E_6, && [E_2,E_4]=\beta_7E_6,\\
& && [E_2,E_5]=\alpha_4E_6,\\
&[E_1,E_4]=\frac{\alpha_1\alpha_5}{\alpha_4}E_5+\beta_5E_6, && [E_4,E_3]=\alpha_5E_6
\end{aligned}\tag{2.1}$$

such that if the set $J = \{j \in \{1, 4, 7\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$.

Next we consider the 6-dimensional filiform metric Lie algebra $\mathfrak{l}_{6,15}$ defined by the following non-vanishing Lie brackets

$$\begin{aligned}\mathfrak{l}_{6,15}:&[G_1,G_2]=G_3,\ [G_1,G_3]=G_4,\ [G_1,G_4]=G_5,\ [G_2,G_3]=G_5,\\ &[G_1,G_5]=G_6,\ [G_2,G_4]=G_6.\end{aligned}$$

Theorem 2.2. *A metric Lie algebra $(\mathfrak{l}_{6,15}, \langle .,.\rangle)$ is isometrically isomorphic to a unique metric Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$, $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 7$, defined on $\mathbb{E}^6$ by the non-vanishing commutators*

$$\begin{aligned}
&[E_1,E_2]=\alpha_1E_3+\beta_1E_4+\beta_2E_5+\beta_3E_6, && [E_2,E_3]=\frac{\alpha_2\alpha_5}{\alpha_4}E_5+\beta_7E_6,\\
&[E_1,E_3]=\alpha_2E_4+\beta_4E_5+\beta_5E_6, && [E_2,E_4]=\alpha_5E_6.\\
&[E_1,E_4]=\alpha_3E_5+\beta_6E_6,\\
&[E_1,E_5]=\alpha_4E_6,
\end{aligned}\tag{2.2}$$

such that if the set $J = \{j \in \{1,3,4,6,7\} : \beta_j \neq 0\} \neq \emptyset$, *then* $\beta_{j_\circ} > 0$ *for the minimal element* $j_\circ \in J$.

Below, we deal with the 6-dimensional filiform Lie algebra $\mathfrak{l}_{6,16}$ defined by the following non-vanishing Lie brackets

$$\begin{aligned}\mathfrak{l}_{6,16} :&[G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_2, G_5] = G_6,\\ &[G_4, G_3] = G_6.\end{aligned}$$

Theorem 2.3. *A metric Lie algebra* $(\mathfrak{l}_{6,16}, \langle .,. \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$, $\alpha_i > 0$, $i = 1, \dots, 4$, $\beta_j \in \mathbb{R}$, $j = 1, \dots, 8$, *defined on* $\mathbb{E}^6$ *by the non-vanishing commutators*

$$\begin{aligned}
&[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_2, E_3] = \beta_7 E_6,\\
&[E_1, E_3] = \alpha_2 E_4 - \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) E_5 + \beta_4 E_6, && [E_2, E_4] = \beta_8 E_6,\\
& && [E_2, E_5] = \alpha_4 E_6,\\
&[E_1, E_4] = \alpha_3 E_5 + \beta_5 E_6, && [E_4, E_3] = \frac{\alpha_3\alpha_4}{\alpha_1} E_6.\\
&[E_1, E_5] = \beta_6 E_6, &&
\end{aligned} \tag{2.3}$$

such that one of the following cases is satisfied:

1. $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_8 = 0$,
2. $\beta_3 > 0$ *or* $\beta_5 > 0$, $\beta_1 = \beta_4 = \beta_6 = \beta_8 = 0$,
3. $\beta_6 > 0$ *or* $\beta_4 > 0$, $\beta_1 = \beta_3 = \beta_5 = \beta_8 = 0$,
4. $\beta_1 > 0$ *or* $\beta_8 > 0$, $\beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$,
5. *at least two elements of the set* $\{\beta_1, \beta_3, \beta_4, \beta_5, \beta_6, \beta_8\}$ *are positive with the exceptions* $(\beta_1 > 0, \beta_8 > 0)$, $(\beta_3 > 0, \beta_5 > 0)$, $(\beta_4 > 0, \beta_6 > 0)$,

In what follow, we consider the 6-dimensional filiform Lie algebras $\mathfrak{l}_{6,17}$ and $\mathfrak{l}_{6,18}$ with their canonical basis $\{G_1, G_2, \cdots, G_6\}$

$$\begin{aligned}\mathfrak{l}_{6,17} :&[G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_1, G_5] = G_6,\\ &[G_2, G_3] = G_6,\\ \mathfrak{l}_{6,18} :&[G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_1, G_5] = G_6.\end{aligned}$$

Theorem 2.4. *A metric Lie algebra* $(\mathfrak{l}_{6,17}, \langle .,. \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle .,. \rangle)$, $\alpha_i > 0, i = 1, \dots, 5$, $\beta_j \in \mathbb{R}$, $j = 1, \dots, 6$, *defined on* $\mathbb{E}^6$ *by the non-vanishing commutators*

$$\begin{aligned} &[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6, \\ &[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, && [E_1, E_5] = \alpha_4 E_6, \\ &[E_2, E_3] = \alpha_5 E_6, && \end{aligned} \tag{2.4}$$

such that if the set $J = \{j \in \{1,3,4,6\} : \beta_j \neq 0\} \neq \emptyset$, *then* $\beta_{j_\circ} > 0$ *for the minimal element* $j_\circ \in J$. *Moreover, a metric Lie algebra* $(\mathfrak{l}_{6,18}, \langle .,. \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle .,. \rangle)$ *defined on* $\mathbb{E}^6$ *by the non-vanishing commutators* (2.4) *such that the constant* α_5 *is missing.*

The groups of all isometries of the corresponding connected and simply connected filiform nilmanifolds are given in the following corollary:

Corollary 2.5. *Let* $(N_{6,k}(\alpha_i, \beta_j), \langle .,. \rangle)$ *be the connected and simply connected Riemannian nilmanifold corresponding to metric Lie algebra* $(\mathfrak{n}_{6,k}(\alpha_i, \beta_j), \langle .,. \rangle)$, *where* $k = 14, \dots, 18$. *The isometry group of* $(N_{6,k}(\alpha_i, \beta_j), \langle .,. \rangle)$ *is*
in case $k = 14, 15, 17$ *one has*

$$\mathcal{I}(N_{6,k}(\alpha_i, \beta_j)) = \begin{cases} \mathbb{Z}_2 \ltimes N_{6,k}(\alpha_i, \beta_j) & if\, J = \emptyset, \\ N_{6,k}(\alpha_i, \beta_j) & if\, J \neq \emptyset, \end{cases}$$

in case $k = 18$ *we obtain*

$$\mathcal{I}(N_{6,18}(\alpha_i, \beta_j) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes N_{6,18}(\alpha_i, \beta_j) & if\, J = \emptyset, \\ \mathbb{Z}_2 \ltimes N_{6,18}(\alpha_i, \beta_j) & if\, J \neq \emptyset, \end{cases}$$

in case $k = 16$ *we receive*

$$\mathcal{I}(\aleph_{6,16}(\alpha_i, \beta_j) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes N_{6,16}(\alpha_i, \beta_j) & \textit{if } \beta_j = 0, \\ & j \in \{1,3,4,5,6,8\}, \\ \mathbb{Z}_2 \ltimes N_{6,16}(\alpha_i, \beta_j) & \textit{if } \beta_5 > 0 \textit{ or } \beta_3 > 0, \\ & \beta_j = 0, j \in \{1,4,6,8\}, \\ & \textit{or } \ \beta_6 > 0 \textit{ or } \beta_4 > 0, \\ & \beta_j = 0, j \in \{1,3,5,8\}, \\ & \textit{or } \beta_8 > 0 \textit{ or } \beta_1 > 0, \\ & \beta_j = 0, j \in \{3,4,5,6\}, \\ N_{6,16}(\alpha_i, \beta_j) & \textit{if } \{\beta_i > 0, \beta_j > 0\} \\ & i, j \in \{1,3,4,5,6,8\} \\ & \textit{with exception} \\ & \{\beta_1, \beta_8\}, \{\beta_3, \beta_5\}, \{\beta_4, \beta_6\}. \end{cases}$$

2.2 Geodesic vectors of 6-dimensional filiform metric Lie algebras

This section is devoted to study the totally geodesic subgroups of connected simply connected 6-dimensional filiform Riemannian nilmanifolds $(N, \langle .,. \rangle_N)$. A subgroup H of $(N, \langle .,. \rangle_N)$ is totally geodesic if and only if the corresponding subalgebra $\mathfrak{h}$ is totally geodesic. The left cosets xH, $x \in N$, give a totally geodesic foliation on N. A subalgebra $\mathfrak{h}$ of $(\mathfrak{n}, \langle .,. \rangle)$ is totally geodesic if for all $X, Y \in \mathfrak{h}$ one has $\nabla_X Y \in \mathfrak{h}$, where $\nabla_X Y$ denotes the Levi-Civita connection of $(\mathfrak{n}, \langle .,. \rangle)$. Taking the representatives of the isometrically isomorphic equivalence classes of the 6-dimensional filiform metric Lie algebras given in Theorems 2.1, 2.2, 2.3 and 2.4 first we determine the sets of the geodesic vectors and hence the one-dimensional totally geodesic subalgebras in the six-dimensional filiform metric Lie algebras. To do this we systematically use the following claim: a non-zero vector $Y \in (\mathfrak{n}, \langle .,. \rangle)$ is geodesic if and only if for all $X \in (\mathfrak{n}, \langle .,. \rangle)$ we have $\langle [X, Y], Y \rangle = 0$. According [5, Corollary 2.4], any element of the disjoint union $C_0 = \langle E_1, E_2\rangle \cup \langle E_3\rangle \cup \langle E_4\rangle \cup \langle E_5\rangle \cup \langle E_6\rangle$ is geodesic.

Theorem 2.6. *Let the sets* C_i, $i = 1, \cdots, 9$, *be defined as follows:*

$$C_1 := \{bE_2 + cE_3 + dE_4 : b\,(\alpha_1 c + \beta_1 d) + c\alpha_2 d = 0 : b, c, d \in \mathbb{R}\},$$

such that at least two of the numbers b, c, d *are non-zero*
with exception of the cases:
1. $b = 0$,
2. $d = 0$,
3. $c = 0$ *with* $\beta_1 \neq 0$,

$$\begin{aligned}C_2 := \Big\{ & a(E_1 - \frac{\alpha_2}{\alpha_5}E_6) + cE_3 + dE_4 + eE_5 : a \neq 0, a, c, d, e \in \mathbb{R},\\ & \alpha_5\beta_1 + \alpha_2\beta_7 = 0 = \beta_4,\\ & e = (\beta_5 a - \alpha_5 c)\frac{\alpha_2\alpha_4}{\alpha_1\alpha_5^2},\\ & a(\alpha_1 c + \beta_1 d + \beta_2 e - \frac{\alpha_2\beta_3}{\alpha_5}) - c(\alpha_3 e - \frac{\alpha_2\beta_6}{\alpha_5}a) + \frac{a\alpha_2}{\alpha_5}(\beta_7 d + \alpha_4 e) = 0\Big\},\end{aligned}$$

$$\begin{aligned}C_3 := \Big\{ & a\Big(E_1 - \frac{\alpha_2}{\alpha_5}E_6 + (\frac{\beta_5}{\alpha_5} + \frac{\alpha_1\beta_4}{\alpha_5\beta_1 + \alpha_2\beta_7})E_3 - \frac{\alpha_2\alpha_4\beta_4}{\alpha_5(\alpha_5\beta_1 + \alpha_2\beta_7)}E_5\Big) +\\ & dE_4 : a \neq 0,\ a, d \in \mathbb{R},\\ & a(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c\,(\alpha_3 e + \beta_6 f) - f(\beta_7 d + \alpha_4 e) = 0\Big\},\end{aligned}$$

$$\begin{aligned}C_4 := \Big\{ & aE_1 + cE_3 + dE_4 + eE_5 + fE_6 : a \neq 0,\ f \neq -a\frac{\alpha_2}{\alpha_5},\ f \neq 0,\\ & a, c, d, e, f \in \mathbb{R},\quad e = (c\alpha_5 - a\beta_5)\frac{f\alpha_4}{a\alpha_1\alpha_5},\\ & d = \frac{a}{\alpha_5 f + \alpha_2 a}\Big(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}e - \beta_4 f\Big),\\ & a\,(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c\,(\alpha_3 e + \beta_6 f) - f(\beta_7 d + \alpha_4 e) = 0\Big\},\end{aligned}$$

$$C_5 := \Big\{ cE_3 + dE_4 + eE_5 + fE_6 : f \neq 0,\ c \neq 0,\ c, d, e, f \in \mathbb{R},$$
$$d = \frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big),$$
$$\frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big)(\alpha_3 e + \beta_6 f + c\alpha_2) + c(\beta_4 e + \beta_5 f) + e\alpha_4 f = 0\Big\},$$

$$C_6 := \Big\{ bE_2 + cE_3 + dE_4 + eE_5 : b(\alpha_1 c + \beta_1 d + \beta_2 e) + c(\alpha_2 d + \gamma e) +$$
$$d\alpha_3 e = 0,\ b, c, d, e \in \mathbb{R}\Big\},$$ *where* $\gamma = -\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}$ *in the case of the metric Lie algebra* $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$, *whereas* $\gamma = \beta_4$ *in the case of the metric Lie algebra* $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ *such that at least two of the numbers* b, c, d, e *are non-zero with exception of the cases:*

1. $b = c = 0$,
2. $b = e = 0$,
3. $d = e = 0$,
4. $c = d = 0,\ \beta_2 \neq 0$,
5. $c = e = 0,\ \beta_1 \neq 0$,
6. $b = d = 0$ *with* $\alpha_3\alpha_4\beta_1 + \alpha_1\alpha_2\beta_8 \neq 0$ *in the case of the metric Lie algebra* $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$,
7. $b = d = 0$ *with* $\beta_4 \neq 0$ *in the case of the metric Lie algebra* $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$,

$$C_7 := \Big\{ a\Big(E_1 - \frac{\beta_6}{\alpha_4}E_2\Big) + cE_3 + dE_4 + eE_5 + fE_6 : af \neq 0,\ a, c, d, e, f \in \mathbb{R},$$

$$ae = \frac{f}{\alpha_3}\Big(\frac{a\beta_6\beta_8}{\alpha_4} + c\frac{\alpha_3\alpha_4}{\alpha_1} - a\beta_5\Big),$$

$$ac = \frac{f}{\alpha_1}(c\beta_7 + d\beta_8 + e\alpha_4) - \frac{a}{\alpha_1}\left(\beta_1 d + \beta_2 e + \beta_3 f\right),$$

$$ad = \frac{a}{\alpha_2}\Big((\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e - \beta_4 f\Big) + \frac{f}{\alpha_2}\Big(\frac{a\beta_6\beta_7}{\alpha_4} - d\frac{\alpha_3\alpha_4}{\alpha_1}\Big),$$

$$d(\alpha_2 c + \alpha_3 e) = a\frac{\beta_6}{\alpha_4}\left(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f\right) + ce\Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) -$$

$$f\left(c\beta_4 + d\beta_5 + e\beta_6\right)\Big\},$$

$$C_8 := \{dE_4 + eE_5 + fE_6 : d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0,\ d, e, f \in \mathbb{R}\},$$

such that at least two of the numbers d, e, f *are non-zero*
with exception of the cases:
1. $f = 0$,
2. $d = 0$,
3. $e = 0$ *with* $\beta_6 \neq 0$,

$$C_9 := \{bE_2 + cE_3 + dE_4 + eE_5 + fE_6 : b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) +$$

$$c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0, b, c, d, e, f \in \mathbb{R}\},$$

such that at least two of the numbers b, c, d, e, f *are non-zero*
with exception of the cases:
1. $b = c = d = 0$,
2. $b = c = f = 0$,
3. $d = e = f = 0$,
4. $b = c = e = 0$ *with* $\beta_6 \neq 0$,
5. $c = d = f = 0$ *with* $\beta_2 \neq 0$,
6. $c = d = e = 0$ *with* $\beta_3 \neq 0$,

1. *the geodesic vectors of the metric Lie algebra* $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$, *with* $\alpha_i > 0$, $\beta_j \in \mathbb{R}$, $i \in \{1, \ldots, 5\}$, $j \in \{1, \ldots 7\}$ *not belonging to* C_0 *are*

the non-zero elements of the set $C_1 \cup C_2 \cup C_4$ *in the case* $\alpha_5\beta_1 + \alpha_2\beta_7 = 0 = \beta_4$, *for* $\alpha_5\beta_1 + \alpha_2\beta_7 \neq 0$ *these are the non-zero elements of the set* $C_1 \cup C_3 \cup C_4$,

2. *the geodesic vectors in the metric Lie algebra* $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ *with* $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 7$, *not belonging to* C_0 *are the non-zero elements of the set* $C_1 \cup C_5$,

3. *the geodesic vectors of the metric Lie algebra* $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ *with* $\alpha_i > 0$, $i = 1, \ldots, 4$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 8$, *not belonging to* C_0 *are the non-zero elements of the set* $C_6 \cup C_7$,

4. *the geodesic vectors in the metric Lie algebra* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *with* $\alpha_i > 0$, $\beta_j \in \mathbb{R}$, $i \in \{1, \ldots, 5\}$, $j \in \{1, \ldots 6\}$ *not belonging to* C_0 *are the non-zero elements of the set* $C_6 \cup C_8$,

5. *the geodesic vectors of the metric Lie algebra* $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ *with* $\alpha_i > 0$, $\beta_j \in \mathbb{R}$, $i \in \{1, \ldots, 4\}$, $j \in \{1, \ldots 6\}$ *not belonging to* C_0 *are the non-zero elements of the set* C_9.

2.3 Flat totally geodesic subalgebras of 6-dimensional filiform metric Lie algebras

In this section, we describe the flat totally geodesic subalgebras of the 6-dimensional filiform metric Lie algebras. Since any non-zero vector in $\cup_{i=1}^4 \mathfrak{a}_i \cup \zeta$ is geodesic we focus to the study of subalgebras not belonging to $\cup_{i=1}^4 \mathfrak{a}_i \cup \zeta$. As for any non-zero geodesic vector Y generates a 1-dimensional flat totally geodesic subalgebra $\{tY, t \in \mathbb{R}\}$ here we wish to find totally geodesic subalgebras of dimension > 1.

Theorem 2.7. *Let* $\mathcal{C}$ *be the class of the* 6*-dimensional filiform metric Lie algebras not corresponding to the standard filiform Lie algebra. The maximal dimension of the flat totally geodesic subalgebras of a metric Lie algebra in* $\mathcal{C}$ *is* 2. *A metric Lie algebra* $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ *has the subalgebra* $\mathfrak{h} = span(E_1, E_6)$ *as a* *flat totally geodesic subalgebra if and only if* $\beta_3 = \beta_4 = \beta_5 = 0$. *A metric Lie algebra* $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ *allows the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2, E_4)$ *precisely if* $\beta_1 = \beta_7 = 0$.

A metric Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ *possesses the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_3, E_6)$ *if and only if* $\beta_5 = \beta_7 = 0$.

A metric Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ *has the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_1 - \frac{\beta_6}{\alpha_4}E_2, E_6)$ *precisely if* $\beta_3 = 0$, $\beta_4 = \frac{\beta_6\beta_7}{\alpha_4}$ *and* $\beta_5 = \frac{\beta_6\beta_8}{\alpha_4}$. *A metric Lie algebra* $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ *allows the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_3, E_5)$ *if and only if* $\alpha_3\alpha_4\beta_1 + \alpha_2\alpha_1\beta_8 = 0$. *A metric Lie algebra* $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ *has the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2 + \frac{\beta_8\alpha_1}{\alpha_3\alpha_4}E_3 - \frac{1}{\alpha_3}(\beta_1 + \frac{\beta_8\alpha_1\alpha_2}{\alpha_3\alpha_4})E_5,\ E_4)$ *precisely if for* α_i, β_j, $i = 1, 2, 3, 4$, $j = 1, 2, 8$ *the equation*

$$\frac{(\alpha_1)^2\beta_8}{\alpha_3\alpha_4} + \Big(\beta_1 + \frac{\beta_8\alpha_1\alpha_2}{\alpha_3\alpha_4}\Big)\Big(- \frac{\beta_2}{\alpha_3} + \frac{\beta_8\alpha_1}{(\alpha_3)^2\alpha_4}(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})\Big) = 0$$

holds. A metric Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ *allows the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2 + k_1E_4 + k_2E_5,\ E_3 + l_1E_4 + l_2E_5)$ *if and only if the equations*

$$\begin{aligned}
&\beta_7 + l_1\beta_8 + l_2\alpha_4 + k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,\\
&\beta_1k_1 + \beta_2k_2 + k_1\alpha_3k_2 = 0,\\
&\alpha_2l_1 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})l_2 + l_1\alpha_3l_2 = 0,\\
&\alpha_1 + \beta_1l_1 + \beta_2l_2 + \alpha_2k_1 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})k_2 + \alpha_3(k_1l_2 + l_1k_2) = 0
\end{aligned}$$

are satisfied.

A metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *processes the flat totally geodesic subalgebras* $\mathfrak{h} = span(E_4 - \frac{\alpha_3}{\alpha_4}E_6,\ E_5)$ *and* $\mathfrak{h} = span\,(E_4, E_6)$ *if and only if* $\beta_6 = 0$. *A metric Lie algebra* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *has the flat totally geodesic subalgebras* $\mathfrak{h} = span\,(E_3, E_5)$ *and* $\mathfrak{h} = span(E_3 - \frac{\alpha_2}{\alpha_3}E_5,\ E_4)$ *precisely if* $\beta_4 = 0$. *A metric Lie algebra* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *allows the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2 + k_1E_3 - \frac{\beta_2+\beta_4k_1}{\alpha_3}E_4,\ E_5)$ *if and only if* k_1 *is a solution* *of the equation*

$$\alpha_2\beta_4k_1^2 + (\beta_1\beta_4 + \alpha_2\beta_2 - \alpha_1\alpha_3)k_1 + \beta_1\beta_2 = 0.$$

A metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *possesses the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2 - \frac{\beta_1+\alpha_3k_2}{\alpha_2}E_3 + k_2E_5,\ E_4)$ *precisely if* k_2 *is a solution of the equation*

$$\alpha_3\beta_4k_2^2 + (\alpha_1\alpha_3 + \beta_1\beta_4 - \alpha_2\beta_2)k_2 + \alpha_1\beta_1 = 0.$$

Now, we describe the flat totally geodesic subalgebras of dimension > 1 of the metric Lie algebras belonging to the 6-dimensional standard filiform Lie algebra.

Theorem 2.8. *The maximal dimension of the flat totally geodesic subalgebras of a metric Lie algebra* $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ *is* 4. *A metric Lie algebra* $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ *has the* 4*-dimensional subalgebras* $\mathfrak{h} = span(E_2 - \frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}E_6,\ E_3,\ E_4 - \frac{\alpha_3}{\alpha_4}E_6,\ E_5)$ *and* $\mathfrak{h} = span(E_2,\ E_3 - \frac{\alpha_2}{\alpha_3}E_5,\ E_4,\ E_6)$ *as flat totally geodesic subalgebras if and only if* $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $\beta_2 = \frac{\alpha_1\alpha_3}{\alpha_2}$.

The 3*-dimensional flat totally geodesic subalgebras of* $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ *are:*

$\mathfrak{h} = span(E_3 - \frac{\alpha_2}{\alpha_3}E_5,\ E_4,\ E_6)$ *and* $\mathfrak{h} = span(E_3,\ E_4 - \frac{\alpha_3}{\alpha_4}E_6,\ E_5)$ *precisely if* $\beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$,

$\mathfrak{h} = span(E_2 + k_1E_3 + k_2E_6,\ E_4 + s_1E_6,\ E_5)$ *such that one of the following cases is satisfied:*

1. $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_2 = -\frac{\beta_2+k_1\beta_4}{\alpha_4}$, $s_1 = -\frac{\alpha_3}{\alpha_4}$ *and* k_1 *is a solution of the equation*

$$\beta_4\beta_5k_1^2 + k_1(\beta_2\beta_5 + \beta_3\beta_4 - \alpha_1\alpha_4) + \beta_2\beta_3 = 0, \tag{2.5}$$

2. $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_1 = \frac{\alpha_3\beta_3-\alpha_4\beta_1}{\alpha_2\alpha_4-\alpha_3\beta_5}$, $k_2 = -\frac{\beta_2(\alpha_2\alpha_4-\alpha_3\beta_5)+\beta_4(\beta_3\alpha_3-\beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4-\alpha_3\beta_5)}$, $s_1 = -\frac{\alpha_3}{\alpha_4}$ *and the equation*

$$\begin{aligned}&(\alpha_2\alpha_4 - \alpha_3\beta_5)\Big((\alpha_1\alpha_4 - \beta_3\beta_4 - \beta_2\beta_5)(\beta_3\alpha_3 - \beta_1\alpha_4) - \\ &\beta_2\beta_3(\alpha_2\alpha_4 - \alpha_3\beta_5) - \beta_4\beta_5(\beta_3\alpha_3 - \beta_1\alpha_4)^2\Big) = 0\end{aligned} \tag{2.6}$$

holds,

3. $\mathfrak{h} = span(E_2 + k_1E_3 + k_2E_5,\ E_4,\ E_6)$ *such that one of the following cases is satisfied:*

   (a) $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_2 = -\frac{\beta_3+k_1\beta_5}{\alpha_4}$ *and* k_1 *is a solution of the equation* (2.5),

   (b) $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_1 = \frac{\alpha_3\beta_3-\alpha_4\beta_1}{\alpha_2\alpha_4-\alpha_3\beta_5}$, $k_2 = \frac{\beta_3(\alpha_2\alpha_4-\alpha_3\beta_5)+\beta_5(\beta_3\alpha_3-\beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4-\alpha_3\beta_5)}$ *and the equation* (2.6) *holds,*

4. $\mathfrak{h} = span\,(E_2 + l_1E_4 + l_2E_5,\ E_3 + k_1E_4 + k_2E_5,\ E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_1 l_1 + \beta_2 l_2 + l_1\alpha_3 l_2 = 0,\\
&\alpha_2 k_1 + \beta_4 k_2 + k_1 k_2\alpha_3 = 0,\\
&\alpha_1 + \beta_1 k_1 + \beta_2 k_2 + \alpha_2 l_1 + \beta_4 l_2 + l_1 k_2\alpha_3 + k_1 l_2\alpha_3 = 0,\\
&\beta_3 + \beta_6 l_1 + \alpha_4 l_2 = 0,\\
&\beta_5 + k_1\beta_6 + k_2\alpha_4 = 0
\end{aligned}$$

   are satisfied,

5. $\mathfrak{h} = span\,(E_2 + k_1E_4 + k_2E_6,\ E_3 + l_1E_4 + l_2E_6,\ E_5)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_1 k_1 + \beta_3 k_2 + k_1\beta_6 k_2 = 0,\\
&\alpha_2 l_1 + \beta_5 l_2 + l_1\beta_6 l_2 = 0,\\
&\alpha_1 + \beta_1 l_1 + \beta_3 l_2 + \alpha_2 k_1 + \beta_5 k_2 + k_1 l_2\beta_6 + l_1 k_2\beta_6 = 0,\\
&\beta_2 + \alpha_3 k_1 + \alpha_4 k_2 = 0,\\
&\beta_4 + \alpha_3 l_1 + \alpha_4 l_2 = 0
\end{aligned}$$

   hold,

6. $\mathfrak{h} = span(E_2 + k_1E_5 + k_2E_6,\ E_3 + l_1E_5 + l_2E_6,\ E_4 + s_1E_5 + s_2E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_2 k_1 + \beta_3 k_2 + k_1\alpha_4 k_2 = 0,\\
&\beta_4 l_1 + \beta_5 l_2 + l_1\alpha_4 l_2 = 0,
\end{aligned}$$

$$\alpha_3 s_1 + \beta_6 s_2 + s_1\alpha_4 s_2 = 0,$$
$$\alpha_1 + \beta_2 l_1 + \beta_3 l_2 + \beta_4 k_1 + \beta_5 k_2 + k_1 l_2 \alpha_4 + l_1 k_2 \alpha_4 = 0,$$
$$\beta_1 + \beta_2 s_1 + \beta_3 s_2 + \alpha_3 k_1 + \beta_6 k_2 + k_1 s_2 \alpha_4 + s_1 k_2 \alpha_4 = 0,$$
$$\alpha_2 + \beta_4 s_1 + \beta_5 s_2 + \alpha_3 l_1 + \beta_6 l_2 + l_1 s_2 \alpha_4 + s_1 l_2 \alpha_4 = 0,$$
$$\beta_2 k_1 + \beta_3 k_2 + \beta_4 l_1 + \beta_5 l_2 + \alpha_3 s_1 + \beta_6 s_2 + k_1 s_2 + k_1 k_2 \alpha_4 + l_1 l_2 \alpha_4 + s_1 s_2 \alpha_4 = 0$$

are satisfied.

The 2-dimensional flat totally geodesic subalgebras of $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., .\rangle)$ *are:*

(a) $\mathfrak{h} =$ *span*$(E_4 - \frac{\alpha_3}{\alpha_4} E_6,\ E_5)$ *and* $\mathfrak{h} =$ *span*$(E_4,\ E_6)$ *if and only if* $\beta_6 = 0$*,*

(b) $\mathfrak{h} =$ *span*$(E_2 + k_1 E_3 + k_2 E_4 - \frac{\beta_3 + k_1\beta_5 + k_2\beta_6}{\alpha_4} E_5,\ E_6)$ *and* $\mathfrak{h} =$ *span*$(E_2 + k_1 E_3 + k_2 E_4 - \frac{\beta_2 + k_1\beta_4 + k_2\alpha_3}{\alpha_4} E_6,\ E_5)$ *if and only if the equation*

$$(\alpha_3 k_2 + \beta_2 + \beta_4 k_1)(k_1\beta_5 + k_2\beta_6 + \beta_3) - \alpha_2\alpha_4 k_1 k_2 - \alpha_1\alpha_4 k_1 - \beta_1\alpha_4 k_2 = 0$$

is satisfied,

(c) $\mathfrak{h} =$ *span*$(E_3 + k_1 E_4 - \frac{\beta_4 + \alpha_3 k_1}{\alpha_4} E_6,\ E_5)$ *and* $\mathfrak{h} =$ *span*$(E_3 + k_1 E_4 - \frac{\beta_5 + \beta_6 k_1}{\alpha_4} E_5,\ E_6)$*, precisely if* k_1 *is a solution of the equation*

$$\alpha_3\beta_6 k_1^2 + k_1(\alpha_3\beta_5 + \beta_4\beta_6 - \alpha_2\alpha_4) + \beta_4\beta_5 = 0,$$

(d) $\mathfrak{h} =$ *span*$(E_2 + k_1 E_3 + k_2 E_5 + k_3 E_6,\ E_4 + l_1 E_5 + l_2 E_6)$ *if and only if the following equations*

$$\alpha_3 l_1 + \beta_6 l_2 + l_1\alpha_4 l_1 = 0,$$
$$\alpha_1 k_1 + \beta_2 k_2 + \beta_3 k_3 + k_1\beta_4 k_2 + k_1\beta_5 k_3 + k_2\alpha_4 k_3 = 0,$$
$$\beta_1 + \beta_2 l_1 + \beta_3 l_2 + \alpha_2 k_1 + k_1\beta_4 l_1 + k_1\beta_5 l_2 + \alpha_3 k_2 + \beta_6 k_3 + \alpha_4 k_2 l_2 + \alpha_4 l_1 k_3 = 0$$

hold,

(e) $\mathfrak{h} =$ *span*$(E_3 + k_1E_5 + k_2E_6,\ E_4 + l_1E_5 + l_2E_6)$ *if and only if the following equations*

$$\beta_4k_1 + \beta_5k_2 + k_1\alpha_4k_2 = 0,$$
$$\alpha_3l_1 + \beta_6l_2 + l_1\alpha_4l_2 = 0,$$
$$\alpha_2 + \beta_4l_1 + \beta_5l_2 + \alpha_3k_1 + \beta_6k_2 + k_1l_2\alpha_4 + l_1k_2\alpha_4 = 0$$

are satisfied,

(f) $\mathfrak{h} =$ *span*$(E_2 + k_1E_4 + k_2E_5 + k_3E_6, \quad E_3 + l_1E_4 + l_2E_5 + l_3E_6)$ *precisely if the following equations*

$$\beta_1k_1 + \beta_2k_2 + \beta_3k_3 + k_1\alpha_3k_2 + k_1\beta_6k_3 + k_2\alpha_4k_3 = 0,$$
$$\alpha_2l_1 + \beta_4l_2 + \beta_5l_3 + l_1\alpha_3l_2 + l_1\beta_6l_3 + l_2\alpha_4l_3 = 0,$$
$$\alpha_1 + \beta_1l_1 + \beta_2l_2 + \beta_3l_3 + \alpha_2k_1 + \beta_4k_2 + \beta_5k_3 + k_1l_2\alpha_3 + l_1k_2\alpha_3 + k_1l_3\beta_6 + l_1k_3\beta_6 + k_2l_3\alpha_4 + l_2l_3\alpha_4 = 0$$

hold.

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List of publications related to the dissertation

Foreign language scientific articles in international journals (4)

1. **Al-Janabi, S. A. A.**, Figula, Á.: Geodesic vectors and flat totally geodesic subalgebras of six-dimensional filiform metric Lie algebras.
   J. Geom. 115 (1), 1-39, 2023. ISSN: 0047-2468.
   DOI: http://dx.doi.org/10.1007/s00022-023-00703-4
   IF: 0.6 (2022)
2. Figula, Á., **Al-Janabi, S. A. A.**: Isometry groups of six-dimensional filiform nilmanifolds.
   Int. J. Group Theory. 12 (2), 67-80, 2023. ISSN: 2251-7650.
   DOI: http://dx.doi.org/10.22108/IJGT.2022.131891.1767
   IF: 0.2 (2022)
3. **Al-Janabi, S. A. A.**, Kozma, L.: On generalized Douglas curvature of Finsler metrics.
   Global Journal of Advanced Research on Classical and Modern Geometries. 12 (2), 229-236, 2023. ISSN: 2284-5569.
4. **Al-Janabi, S. A. A.**, Kozma, L.: On New Classes of Stretch Finsler Metrics.
   Journal of Finsler Geometry and its Applications 3 (1), 86-99, 2022. EISSN: 2783-0500.
   DOI: http://dx.doi.org/10.22098/jfga.2022.10115.1058

List of other publications

Foreign language scientific articles in international journals (4)

5. Al-khafaji, S. N., Hussain, A. H., **Al-Janabi, S. A. A.**: Third Hankel Determinant for Certain Class of Bazilević Functions Associated with Linear Differential Operato.
   Discontinuity, Nonlinearity, and Complexity. 10 (2), 323-331, 2021. ISSN: 2164-6376.
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6. Salman, A. M., Mouajeeb, E. K., Al-Janabi, A. N., Hussain, A. H., **Al-Janabi, S. A. A.**: Weakly w-compact spaces via w-open sets.
   J. Interdiscip. Math. 24 (3), 713-719, 2021. ISSN: 0972-0502.
   DOI: http://dx.doi.org/10.1080/09720502.2020.1861784

7. Hussain, A. H., **Al-Janabi, S. A. A.**, Salman, A. M., Hussein, N. A.: Semi soft local function which generated a new topology in soft ideal spaces.
   Journal of Interdisciplinary Mathematics. 22 (8), 1509-1517, 2019. ISSN: 0972-0502.
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8. **Al-Janabi, S. A. A.**, Hussein, N. A., Salman, A. M., Hussain, A. H.: Soft-local function via soft point.
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   DOI: http://dx.doi.org/10.1080/09720502.2019.1706856

Foreign language conference proceedings (2)

9. **Al-Janabi, S. A. A.**, Hussain, A. H., Salman, A. M., Hussein, N. A.: On Soft Closure Function in Soft Ideal Spaces.
   J. Phys. Conf. Ser. 1804 (1), 1-9, 2021. ISSN: 1742-6588.
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10. Al-khafaji, S. N., Al-Fayadh, A., Hussain, A. H., **Al-Janabi, S. A. A.**: Toeplitz Determinant whose Its Entries are the Coefficients for Class of Non-Bazilevi'c Functions.
    J. Phys. Conf. Ser. 1660 (1), 1-7, 2020. ISSN: 1742-6588.
    DOI: http://dx.doi.org/10.1088/1742-6596/1660/1/012091

Foreign language abstracts (1)

11. **Al-Janabi, S. A. A.**, Figula, Á.: Totally geodesic subalgebras of 6-dimensional nilmanifolds having nilpotency classes 3 and 4.
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List of Talks

1. **DGA, 17-23 July, 2022**: Conference Differential Geometry and its Applications- Hradec Kralove, Czech Republic. Title of the talk: *On New classes of stretch Finsler metrics.*

2. **CFGA, 12-16 June, 2023**: Colloquium on Finsler Geometry and its Applications-Debrecen, Hungary. Title of the talk: *Isometry groups and totally geodesic subalgebras of* 6*-dimensional filiform nilmanifolds.*

3. **RIGA, 22-24 September, 2023**: The International Conference Riemannian Geometry and Applications-Bucharest, Romania. Title of the talk: *Totally geodesic subalgebras of* 6*-dimensional nilmanifolds having nilpotency classes* 3 *and* 4.

4. **RIGA, 12-15 January, 2021**: The International Conference Riemannian Geometry and Applications-Bucharest, Romania. *Attendance.*

5. **DGDS, 1-4 September, 2022** : The XVI-th International Conference of Differential Geometry and Dynamical Systems-Bucharest, Romania. *Attendance.*