

α -Wiener bridges: singularity of induced measures and sample path properties

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Abstract

Let us consider the process $(X_t^{(\alpha)})_{t \in [0, T]}$ given by the SDE $dX_t^{(\alpha)} = -\frac{\alpha}{T-t} X_t^{(\alpha)} dt + dB_t$, $t \in [0, T)$, where $\alpha \in \mathbb{R}$, $T \in (0, \infty)$, and $(B_t)_{t \geq 0}$ is a standard Wiener process. In case of $\alpha > 0$ the process $X^{(\alpha)}$ is known as an α -Wiener bridge, in case of $\alpha = 1$ as the usual Wiener bridge. We prove that for all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, the probability measures induced by the processes $X^{(\alpha)}$ and $X^{(\beta)}$ are singular on $(C[0, T], \mathcal{B}(C[0, T]))$. Further, we investigate regularity properties of $X_t^{(\alpha)}$ as $t \uparrow T$.

1 Introduction

There has been a lot of work concerning questions of absolute continuity and singularity for various types of stochastic processes on finite and infinite time intervals, see for example Jacod and Shiryaev [8], Ben-Ari and Pinsky [4], and Prakasa Rao [15]. However, most of the literature deal with time homogeneous diffusion processes. In this paper we study absolute continuity and singularity of α -Wiener bridges which are time inhomogeneous diffusion processes. We also present some results on the sample path behavior of these processes.

Let $T \in (0, \infty)$ be fixed. For all $\alpha \in \mathbb{R}$, we consider the process $(X_t^{(\alpha)})_{t \in [0, T]}$ given by the stochastic differential equation (SDE)

$$(1.1) \quad \begin{cases} dX_t^{(\alpha)} = -\frac{\alpha}{T-t} X_t^{(\alpha)} dt + dB_t, & t \in [0, T), \\ X_0^{(\alpha)} = 0, \end{cases}$$

where $(B_t)_{t \geq 0}$ is a 1-dimensional standard Wiener process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. To our knowledge, these kind of processes have been first considered by Brennan and Schwartz [7], and see also Mansuy [14]. In Brennan and Schwartz [7] the SDE (1.1) is used to model the arbitrage profit associated with a given futures contract in the absence of transaction costs. In case of $\alpha > 0$ the process $X^{(\alpha)}$ is known as an α -Wiener bridge, in case of $\alpha = 1$ as the usual Wiener bridge. By formula (5.6.6) in Karatzas and Shreve [9],

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the SDE (1.1) has a unique strong solution, namely,

$$(1.2) \quad X_t^{(\alpha)} = \int_0^t \left(\frac{T-t}{T-s} \right)^\alpha dB_s, \quad t \in [0, T),$$

defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \in [0, T)}, \mathbb{P})$ constructed by the help of the standard Wiener process B , see, e.g., Karatzas and Shreve [9, Section 2.5.A]. This filtered probability space satisfies the so called usual conditions, i.e., $(\Omega, \mathcal{A}, \mathbb{P})$ is complete, the filtration $(\mathcal{A}_t)_{t \in [0, T)}$ is right-continuous, \mathcal{A}_0 contains all the \mathbb{P} -null sets in \mathcal{A} and $\mathcal{A} = \mathcal{A}_{T-}$, where $\mathcal{A}_{T-} := \sigma \left(\bigcup_{t \in [0, T)} \mathcal{A}_t \right)$.

In Section 2 we calculate the covariance of $X_s^{(\alpha)}$ and $X_t^{(\beta)}$ for all $s, t \in [0, T)$ and $\alpha, \beta \in \mathbb{R}$. Further, we recall a strong law of large numbers and a law of the iterated logarithm for continuous local martingales which will be used for proving regularity properties of $X^{(\alpha)}$.

In Section 3 we prove that $X_t^{(\alpha)} \rightarrow 0$ almost surely as $t \uparrow T$ in case of $\alpha > 0$, see, Lemma 3.1, and that's why we can use the expression ' α -Wiener bridge' for $X^{(\alpha)}$ in this case. Lemma 3.1 can be considered as a generalization of Lemma 5.6.9 in Karatzas and Shreve [9]. We will also examine what happens in case of $\alpha \leq 0$, see Remark 3.5. Further, we investigate regularity properties of $X_t^{(\alpha)}$ as $t \uparrow T$. In case of $\alpha \geq \frac{1}{2}$ we have theorems of type of the law of the iterated logarithm for $X^{(\alpha)}$, see Theorems 3.2 and 3.3.

In Section 4 we investigate the absolute continuity and singularity of the probability measures induced by the processes $X^{(\alpha)}$ with different values of α . Namely, we show that for all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, the probability measures induced by the processes $X^{(\alpha)}$ and $X^{(\beta)}$ on $(C[0, T), \mathcal{B}(C[0, T)))$ are singular, where $C[0, T)$ is the space of continuous functions from $[0, T)$ into \mathbb{R} , and $\mathcal{B}(C[0, T))$ denotes the Borel σ -algebra on $C[0, T)$, see Theorem 4.1. We note that Prakasa Rao [15, Theorem 5] proved a similar statement for fractional Wiener processes. Namely, he showed that if $(W_{H_i}(t))_{t \geq 0}$, $i = 1, 2$, are two fractional Wiener processes with Hurst indices $H_1, H_2 \in (0, 1)$, $H_1 \neq H_2$, then the probability measures induced by the processes W_{H_i} , $i = 1, 2$, on $(C[0, \infty), \mathcal{B}(C[0, \infty)))$ are singular, where $\mathcal{B}(C[0, \infty))$ denotes the Borel σ -algebra on $C[0, \infty)$. We also note that our technique for the proof of Theorem 4.1 differs from the technique of Prakasa Rao [15, Theorem 5]. Our proof is based on strong consistency of the maximum likelihood estimator of α , while the proof of Prakasa Rao [15, Theorem 5] is based on a Baxter type result of Kurchenko [10] for second order quadratic variations (second order increments) for a fractional Wiener process. By giving a second proof of Theorem 4.1, we also discuss the connections between strong consistency of the maximum likelihood estimator of α , Hellinger processes (see, e.g., Jacod and Shiryaev [8, Chapter IV]) and singularity of induced measures. Moreover, we study absolute continuity and singularity of probability measures induced by processes for which the diffusion coefficients in the SDE (1.1) are not identically one. Giving two different proofs, we prove that a so-called dichotomy holds, see Theorem 4.5.

2 Preliminaries

First we determine the covariance of $X_s^{(\alpha)}$ and $X_t^{(\beta)}$ for all $s, t \in [0, T)$ and $\alpha, \beta \in \mathbb{R}$.

2.1 Lemma. Let $T \in (0, \infty)$ be fixed. For $\alpha, \beta \in \mathbb{R}$, let us consider the processes $(X_t^{(\alpha)})_{t \in [0, T]}$ and $(X_t^{(\beta)})_{t \in [0, T]}$ given by the SDE (1.1). Then for all $s, t \in [0, T]$, the covariance of $X_s^{(\alpha)}$ and $X_t^{(\beta)}$ is

$$(2.1) \quad \text{Cov}(X_s^{(\alpha)}, X_t^{(\beta)}) = \begin{cases} \frac{(T-s)^\alpha (T-t)^\beta}{1-\alpha-\beta} (T^{1-\alpha-\beta} - (T - (s \wedge t))^{1-\alpha-\beta}) & \text{if } \alpha + \beta \neq 1, \\ (T-s)^\alpha (T-t)^\beta \ln \left(\frac{T}{T-(s \wedge t)} \right) & \text{if } \alpha + \beta = 1. \end{cases}$$

Epecially, for all $t \in [0, T]$, $X_t^{(\alpha)}$ is a normally distributed random variable with mean $\mathbb{E}X_t^{(\alpha)} = 0$ and with variance

$$\mathbb{E}(X_t^{(\alpha)})^2 = \begin{cases} \frac{T}{1-2\alpha} \left(\frac{T-t}{T} \right)^{2\alpha} - \frac{T-t}{1-2\alpha} & \text{if } \alpha \neq \frac{1}{2}, \\ (T-t)(\ln(T) - \ln(T-t)) & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

Proof. By Bauer [5, Lemma 48.2] and (1.2), $(X_t^{(\alpha)})_{t \in [0, T]}$ is a Gauss process with mean $\mathbb{E}X_t^{(\alpha)} = 0$, $t \in [0, T]$, and with variance $\mathbb{E}(X_t^{(\alpha)})^2$, $t \in [0, T]$. By Proposition 3.2.10 in Karatzas and Shreve [9], for all $s, t \in [0, T]$, we have

$$\begin{aligned} \text{Cov}(X_s^{(\alpha)}, X_t^{(\beta)}) &= \mathbb{E}(X_s^{(\alpha)} X_t^{(\beta)}) = \mathbb{E} \left(\int_0^s \left(\frac{T-u}{T-u} \right)^\alpha dB_u \int_0^t \left(\frac{T-v}{T-v} \right)^\beta dB_v \right) \\ &= (T-s)^\alpha (T-t)^\beta \int_0^{s \wedge t} \frac{1}{(T-u)^{\alpha+\beta}} du, \end{aligned}$$

and hence we obtain (2.1). \square

For proving regularity properties of $X^{(\alpha)}$, we recall a strong law of large numbers and a law of the iterated logarithm for continuous local martingales.

Let $T \in (0, \infty]$ be fixed. In all what follows, if $(M_t)_{t \in [0, T]}$ is a continuous local martingale satisfying $\mathbb{P}(M_0 = 0) = 1$, then $(\langle M \rangle_t)_{t \in [0, T]}$ denotes the quadratic variation of M .

The following theorem is a modification of Theorem 3.4.6 in Karatzas and Shreve [9] (due to Dambis, Dubins and Schwartz), see also Theorem 1.6 in Chapter V in Revuz and Yor [16]. In fact, our next Theorem 2.2 is Exercise 1.18 in Chapter V in Revuz and Yor [16].

2.2 Theorem. Let $T \in (0, \infty]$ be fixed and let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $(M_t)_{t \in [0, T]}$ be a continuous local martingale with respect to the filtration $(\mathcal{G}_t)_{t \in [0, T]}$ such that $\mathbb{P}(M_0 = 0) = 1$ and $\mathbb{P}(\lim_{t \uparrow T} \langle M \rangle_t = \infty) = 1$. For each $s \in [0, \infty)$, define the stopping time

$$\tau_s := \inf\{t \in [0, T] : \langle M \rangle_t > s\}.$$

Then the time-changed process

$$(B_s := M_{\tau_s}, \mathcal{G}_{\tau_s})_{s \geq 0}$$

is a standard Wiener process. In particular, the filtration $(\mathcal{G}_{\tau_s})_{s \geq 0}$ satisfies the usual conditions and

$$\mathbb{P}(M_t = B_{\langle M \rangle_t} \text{ for all } t \in [0, T]) = 1.$$

Now we formulate a strong law of large numbers for continuous local martingales. Compare with Lépingle [11, Theoreme 1] or with 3°) in Exercise 1.16 in Chapter V in Revuz and Yor [16]. We note that the above mentioned citations are about continuous local martingales with time interval $[0, \infty)$, but they are also valid for continuous local martingales with time interval $[0, T)$, $T \in (0, \infty)$, with appropriate modifications in the conditions, see as follows.

2.3 Theorem. *Let $T \in (0, \infty]$ be fixed and let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0, T]}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. Let $(M_t)_{t \in [0, T]}$ be a continuous local martingale with respect to the filtration $(\mathcal{G}_t)_{t \in [0, T]}$ such that $\mathbf{P}(M_0 = 0) = 1$ and $\mathbf{P}(\lim_{t \uparrow T} \langle M \rangle_t = \infty) = 1$. Let $f : [1, \infty) \rightarrow (0, \infty)$ be an increasing function such that*

$$\int_1^\infty \frac{1}{f(x)^2} dx < \infty.$$

Then

$$\mathbf{P} \left(\lim_{t \uparrow T} \frac{M_t}{f(\langle M \rangle_t)} = 0 \right) = 1.$$

Now we present a law of the iterated logarithm for continuous local martingales. Compare with Exercise 1.15 in Chapter V in Revuz and Yor [16]. We note that the above mentioned citation is about continuous local martingales with time interval $[0, \infty)$, but the result is also valid for continuous local martingales with time interval $[0, T)$, $T \in (0, \infty)$, with appropriate modifications in the conditions, see as follows.

2.4 Theorem. *Let $T \in (0, \infty]$ be fixed and let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0, T]}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. Let $(M_t)_{t \in [0, T]}$ be a continuous local martingale with respect to the filtration $(\mathcal{G}_t)_{t \in [0, T]}$ such that $\mathbf{P}(M_0 = 0) = 1$ and $\mathbf{P}(\lim_{t \uparrow T} \langle M \rangle_t = \infty) = 1$. Then*

$$\mathbf{P} \left(\limsup_{t \uparrow T} \frac{M_t}{\sqrt{2 \langle M \rangle_t \ln(\ln \langle M \rangle_t)}} = 1 \right) = \mathbf{P} \left(\liminf_{t \uparrow T} \frac{M_t}{\sqrt{2 \langle M \rangle_t \ln(\ln \langle M \rangle_t)}} = -1 \right) = 1.$$

Theorem 2.4 simply follows from Theorem 2.2 and the law of the iterated logarithm for a standard Wiener process (see, e.g., Karatzas and Shreve [9, Theorem 2.9.23]).

3 Sample paths properties

The following Lemma 3.1 can be considered as a generalization of Lemma 5.6.9 in Karatzas and Shreve [9]. Namely, our result with $\alpha = 1$ gives back Lemma 5.6.9 in Karatzas and Shreve [9]. Furthermore, Lemma 3.1 can be also considered as a generalization of Corollary 4.4 in Becker-Kern [6] in the 1-dimensional Brownian case. Namely, by (1.2), $(X_t^{(\alpha)})_{t \in [0, T]}$ coincides with the process $(U_t)_{t \in [0, T]}$ defined in (4.7) in Becker-Kern [6] in the 1-dimensional Brownian case for all $\alpha > 0$. In this case Becker-Kern proved that U_t converges in probability to 0 as $t \uparrow T$, while we prove convergence with probability one. For historical fidelity, we remark that something similar to the statement of our Lemma 3.1 is stated on page 1023 in Mansuy [14] but without any proof.

3.1 Lemma. Let $T \in (0, \infty)$ and $\alpha > 0$ be fixed, and let $(B_t)_{t \geq 0}$ be a 1-dimensional standard Wiener process. The process $(Y_t^{(\alpha)})_{t \in [0, T]}$ defined by

$$Y_t^{(\alpha)} := \begin{cases} \int_0^t \left(\frac{T-s}{T}\right)^\alpha dB_s & \text{if } t \in [0, T), \\ 0 & \text{if } t = T, \end{cases}$$

is a centered Gauss process with almost surely continuous paths.

Proof. By Bauer [5, Lemma 48.2], $(Y_t^{(\alpha)})_{t \in [0, T]}$ is a centered Gauss process. To prove almost surely continuity, we follow the method of the proof of Lemma 5.6.9 in Karatzas and Shreve [9]. For all $t \in [0, T)$ and $\alpha \in \mathbb{R}$, let

$$M_t^{(\alpha)} := \int_0^t \frac{1}{(T-s)^\alpha} dB_s.$$

Then $(M_t^{(\alpha)})_{t \in [0, T]}$ is a continuous, square-integrable martingale with respect to the filtration induced by B and with quadratic variation

$$(3.1) \quad \langle M^{(\alpha)} \rangle_t := \int_0^t \frac{1}{(T-s)^{2\alpha}} ds = \begin{cases} \frac{T^{1-2\alpha}}{1-2\alpha} \left(1 - \left(1 - \frac{t}{T}\right)^{1-2\alpha}\right) & \text{if } \alpha \neq \frac{1}{2}, \\ -\ln\left(1 - \frac{t}{T}\right) & \text{if } \alpha = \frac{1}{2}, \end{cases} \quad t \in [0, T).$$

Then

$$(3.2) \quad \lim_{t \uparrow T} \langle M^{(\alpha)} \rangle_t = \begin{cases} \infty & \text{if } \alpha \geq \frac{1}{2}, \\ \frac{T^{1-2\alpha}}{1-2\alpha} & \text{if } \alpha < \frac{1}{2}. \end{cases}$$

Hence in case of $\alpha \geq \frac{1}{2}$, Theorem 2.2 implies that there exists a standard 1-dimensional Wiener process $(W_t)_{t \geq 0}$ on $(\Omega, \mathcal{A}, \mathbf{P})$ such that

$$\mathbf{P}(M_t^{(\alpha)} = W_{\langle M^{(\alpha)} \rangle_t} \text{ for all } t \in [0, T)) = 1.$$

First we consider the case of $\alpha > \frac{1}{2}$. Let us define the function $f_\alpha : [1, \infty) \rightarrow (0, \infty)$ by $f_\alpha(x) := x^{\alpha/(2\alpha-1)}$, $x \geq 1$. Then f_α is strictly monotone increasing and

$$\int_1^\infty \frac{1}{f_\alpha(x)^2} dx = \int_1^\infty x^{-2\alpha/(2\alpha-1)} dx = 2\alpha - 1 < \infty,$$

hence we may apply Theorem 2.3 and then we obtain

$$\mathbf{P}\left(\lim_{t \uparrow T} \frac{M_t^{(\alpha)}}{f_\alpha(\langle M^{(\alpha)} \rangle_t)} = 0\right) = 1, \quad \alpha > \frac{1}{2}.$$

We have

$$Y_t^{(\alpha)} = (T-t)^\alpha M_t^{(\alpha)} = (T-t)^\alpha f_\alpha(\langle M^{(\alpha)} \rangle_t) \frac{M_t^{(\alpha)}}{f_\alpha(\langle M^{(\alpha)} \rangle_t)},$$

where $t \in [0, T)$ is such that $\langle M^{(\alpha)} \rangle_t \geq 1$. Here

$$(T-t)^\alpha f_\alpha(\langle M^{(\alpha)} \rangle_t) \leq (T-t)^\alpha f_\alpha\left(\frac{1}{2\alpha-1} \frac{1}{(T-t)^{2\alpha-1}}\right) = (2\alpha-1)^{-\alpha/(2\alpha-1)},$$

where $t \in [0, T)$ is such that $\langle M^{(\alpha)} \rangle_t \geq 1$. Hence we conclude $\mathbf{P} \left(\lim_{t \uparrow T} Y_t^{(\alpha)} = 0 \right) = 1$.

Now we consider the case of $\alpha = \frac{1}{2}$. Let us define the function $f_{1/2}(x) := e^{x/2}$, $x \in [1, \infty)$. Then $f_{1/2}$ is strictly monotone increasing and

$$\int_1^\infty \frac{1}{f_{1/2}(x)^2} dx = \int_1^\infty e^{-x} dx = e^{-1} < \infty,$$

hence we may apply Theorem 2.3 and then we obtain

$$\mathbf{P} \left(\lim_{t \uparrow T} \frac{M_t^{(1/2)}}{f_{1/2}(\langle M^{(1/2)} \rangle_t)} = 0 \right) = 1.$$

We have

$$Y_t^{(1/2)} = (T - t)^{1/2} M_t^{(1/2)} = (T - t)^{1/2} f_{1/2}(\langle M^{(1/2)} \rangle_t) \frac{M_t^{(1/2)}}{f_{1/2}(\langle M^{(1/2)} \rangle_t)},$$

where $t \in [0, T)$ is such that $\langle M^{(1/2)} \rangle_t \geq 1$. Here

$$(T - t)^{1/2} f_{1/2}(\langle M^{(1/2)} \rangle_t) = (T - t)^{1/2} \exp \left\{ \frac{1}{2} \ln \left(\frac{T}{T - t} \right) \right\} = T^{1/2},$$

where $t \in [0, T)$ is such that $\langle M^{(1/2)} \rangle_t \geq 1$. Hence we conclude $\mathbf{P} \left(\lim_{t \uparrow T} Y_t^{(1/2)} = 0 \right) = 1$.

Finally, we consider the case of $0 < \alpha < \frac{1}{2}$. Using (3.2) we have Proposition 1.26 in Chapter IV and Proposition 1.8 in Chapter V in Revuz and Yor [16] imply that the limit $M_T^{(\alpha)} := \lim_{t \uparrow T} M_t^{(\alpha)}$ exists almost surely. Since

$$Y_t^{(\alpha)} = (T - t)^\alpha M_t^{(\alpha)}, \quad t \in [0, T),$$

we get $\lim_{t \uparrow T} Y_t^{(\alpha)} = 0$ almost surely. \square

By Lemma 3.1, we can say that in case of $\alpha > 0$, the process $X^{(\alpha)}$ has an almost surely continuous extension. In the later Remark 3.5 we examine the possibility of such an almost surely continuous extension of $(X_t^{(\alpha)})_{t \in [0, T)}$ in case of $\alpha \leq 0$.

Now we prove some results about the asymptotic behavior of $X_t^{(\alpha)}$ as $t \uparrow T$. Theorem 2.4 has the following consequences on $X^{(\alpha)}$.

3.2 Theorem. *If $\alpha > \frac{1}{2}$, then*

$$(3.3) \quad \mathbf{P} \left(\limsup_{t \uparrow T} \frac{X_t^{(\alpha)}}{\sqrt{\frac{2(T-t)}{2\alpha-1}} \ln \left(\ln \frac{1}{T-t} \right)} = 1 \right) = \mathbf{P} \left(\liminf_{t \uparrow T} \frac{X_t^{(\alpha)}}{\sqrt{\frac{2(T-t)}{2\alpha-1}} \ln \left(\ln \frac{1}{T-t} \right)} = -1 \right) = 1.$$

Especially,

$$(3.4) \quad \mathbf{P} \left(\limsup_{t \uparrow T} \frac{X_t^{(\alpha)}}{(T - t)^\alpha} = \infty \right) = \mathbf{P} \left(\liminf_{t \uparrow T} \frac{X_t^{(\alpha)}}{(T - t)^\alpha} = -\infty \right) = 1.$$

Proof. With the notation introduced in the proof of Lemma 3.1, we have $\frac{X_t^{(\alpha)}}{(T-t)^\alpha} = M_t^{(\alpha)}$, $t \in [0, T)$, and the quadratic variation $\langle M^{(\alpha)} \rangle_t$, $t \in [0, T)$, of the continuous martingale $\frac{X_t^{(\alpha)}}{(T-t)^\alpha}$, $t \in [0, T)$, is given in (3.1). Using (3.2) we have $\lim_{t \uparrow T} \langle M^{(\alpha)} \rangle_t = \infty$. Then, by Theorem 2.4, in case of $\alpha > \frac{1}{2}$ we get

$$\mathbb{P} \left(\limsup_{t \uparrow T} \frac{X_t^{(\alpha)}}{\sqrt{D_{\alpha,T}(t)}} = 1 \right) = \mathbb{P} \left(\liminf_{t \uparrow T} \frac{X_t^{(\alpha)}}{\sqrt{D_{\alpha,T}(t)}} = -1 \right) = 1,$$

where for all $t \in [0, T)$,

$$D_{\alpha,T}(t) := 2(T-t)^{2\alpha} \frac{T^{1-2\alpha}}{1-2\alpha} \left(1 - \left(1 - \frac{t}{T} \right)^{1-2\alpha} \right) \ln \left(\ln \left(\frac{T^{1-2\alpha}}{1-2\alpha} \left(1 - \left(1 - \frac{t}{T} \right)^{1-2\alpha} \right) \right) \right).$$

Hence to prove (3.3) it is enough to check that

$$\lim_{t \uparrow T} \frac{2(T-t)^{2\alpha} \frac{T^{1-2\alpha}}{1-2\alpha} \left(1 - \left(1 - \frac{t}{T} \right)^{1-2\alpha} \right) \ln \left(\ln \left(\frac{T^{1-2\alpha}}{1-2\alpha} \left(1 - \left(1 - \frac{t}{T} \right)^{1-2\alpha} \right) \right) \right)}{\frac{2(T-t)}{2\alpha-1} \ln \left(\ln \frac{1}{T-t} \right)} = 1.$$

This is satisfied, since, by using L'Hospital's rule twice, we get

$$\lim_{t \uparrow T} \frac{\ln \left(\ln \left(\frac{T^{1-2\alpha}}{1-2\alpha} \left(1 - \left(1 - \frac{t}{T} \right)^{1-2\alpha} \right) \right) \right)}{\ln \left(\ln \frac{1}{T-t} \right)} = \lim_{t \uparrow T} \frac{(T-t)^{1-2\alpha} - T^{1-2\alpha}}{(T-t)^{1-2\alpha}} = 1.$$

Using (3.3) and the decomposition

$$\frac{X_t^{(\alpha)}}{(T-t)^\alpha} = \frac{X_t^{(\alpha)}}{\sqrt{\frac{2(T-t)}{2\alpha-1} \ln \left(\ln \frac{1}{T-t} \right)}} \sqrt{\frac{2(T-t)^{1-2\alpha}}{2\alpha-1} \ln \left(\ln \frac{1}{T-t} \right)}, \quad t \in [0, T),$$

we have (3.4). □

The next theorem is about the limit behavior of $X_t^{(1/2)}$ as $t \uparrow T$.

3.3 Theorem. *We have*

$$\begin{aligned} (3.5) \quad & \mathbb{P} \left(\limsup_{t \uparrow T} \frac{X_t^{(1/2)}}{\sqrt{2(T-t) \left(\ln \frac{1}{T-t} \right) \left(\ln \ln \ln \frac{1}{T-t} \right)}} = 1 \right) \\ &= \mathbb{P} \left(\liminf_{t \uparrow T} \frac{X_t^{(1/2)}}{\sqrt{2(T-t) \left(\ln \frac{1}{T-t} \right) \left(\ln \ln \ln \frac{1}{T-t} \right)}} = -1 \right) = 1. \end{aligned}$$

Especially,

$$\mathbb{P} \left(\limsup_{t \uparrow T} \frac{X_t^{(1/2)}}{\sqrt{T-t}} = \infty \right) = \mathbb{P} \left(\liminf_{t \uparrow T} \frac{X_t^{(1/2)}}{\sqrt{T-t}} = -\infty \right) = 1.$$

Proof. With the notation introduced in the proof of Lemma 3.1, we have $\frac{X_t^{(1/2)}}{\sqrt{T-t}} = M_t^{(1/2)}$, $t \in [0, T)$, and the quadratic variation $\langle M^{(1/2)} \rangle_t$, $t \in [0, T)$, of the continuous martingale $\frac{X_t^{(1/2)}}{\sqrt{T-t}}$, $t \in [0, T)$, is given in (3.1). Using (3.2) we have $\lim_{t \uparrow T} \langle M^{(1/2)} \rangle_t = \infty$. Then, by Theorem 2.4, we get

$$\begin{aligned} & \mathbb{P} \left(\limsup_{t \uparrow T} \frac{X_t^{(1/2)}}{\sqrt{2(T-t) \left(\ln \frac{T}{T-t} \right) \left(\ln \ln \ln \frac{T}{T-t} \right)}} = 1 \right) \\ &= \mathbb{P} \left(\liminf_{t \uparrow T} \frac{X_t^{(1/2)}}{\sqrt{2(T-t) \left(\ln \frac{T}{T-t} \right) \left(\ln \ln \ln \frac{T}{T-t} \right)}} = -1 \right) = 1, \end{aligned}$$

which yields (3.5). \square

The next theorem is about the limit behavior of $X_t^{(\alpha)}$ as $t \uparrow T$ in case of $\alpha < \frac{1}{2}$.

3.4 Theorem. *If $\alpha < \frac{1}{2}$, then*

$$(3.6) \quad \mathbb{P} \left(\lim_{t \uparrow T} \frac{X_t^{(\alpha)}}{(T-t)^\alpha} = M_T^{(\alpha)} \right) = 1,$$

where $M_T^{(\alpha)}$ is a normally distributed random variable with mean 0 and with variance $\frac{T^{1-2\alpha}}{1-2\alpha}$. Consequently,

$$(3.7) \quad \mathbb{P} \left(\lim_{t \uparrow T} \frac{X_t^{(\alpha)}}{(T-t)^\beta} = 0 \right) = 1 \quad \text{for all } \beta < \alpha,$$

$$(3.8) \quad \mathbb{P} \left(\lim_{t \uparrow T} \frac{X_t^{(\alpha)}}{(T-t)^\beta} = -\infty \right) = \mathbb{P} \left(\lim_{t \uparrow T} \frac{X_t^{(\alpha)}}{(T-t)^\beta} = \infty \right) = \frac{1}{2} \quad \text{for all } \beta > \alpha.$$

Proof. By (1.2), using the notations introduced in the proof of Lemma 3.1, we get

$$M_t^{(\alpha)} = \frac{X_t^{(\alpha)}}{(T-t)^\alpha} = \int_0^t \frac{1}{(T-s)^\alpha} dB_s, \quad t \in [0, T).$$

By (3.2), since $\alpha < \frac{1}{2}$,

$$\lim_{t \uparrow T} \langle M^{(\alpha)} \rangle_t = \frac{T^{1-2\alpha}}{1-2\alpha} < \infty,$$

and hence Proposition 1.26 in Chapter IV and Proposition 1.8 in Chapter V in Revuz and Yor [16] imply that the limit $M_T^{(\alpha)} := \lim_{t \uparrow T} M_t^{(\alpha)}$ exists almost surely. Using that $M_t^{(\alpha)}$ is normally distributed with mean 0 and with variance $\langle M^{(\alpha)} \rangle_t$ for all $t \in [0, T)$, we have the random variable $M_T^{(\alpha)}$ is also normally distributed with mean 0 and with variance $\frac{T^{1-2\alpha}}{1-2\alpha}$. Indeed, normally distributed random variables can converge in distribution only to a normally distributed random variable, by continuity theorem, see, e.g., page 304 in Shiryaev [17]. This implies (3.6). Hence for all $\alpha, \beta \in \mathbb{R}$, we get

$$(3.9) \quad \frac{X_t^{(\alpha)}}{(T-t)^\beta} = (T-t)^{\alpha-\beta} M_t^{(\alpha)}, \quad t \in [0, T).$$

If $\beta < \alpha$, then using (3.9) and that $\mathbf{P}(\lim_{t \uparrow T} M_t^{(\alpha)} = M_T^{(\alpha)}) = 1$, we get (3.7). If $\beta > \alpha$, using that $\mathbf{P}(M_T^{(\alpha)} = 0) = 0$, we have (3.9) implies that

$$\mathbf{P} \left(\lim_{t \uparrow T} \frac{X_t^{(\alpha)}}{(T-t)^\beta} \in \{-\infty, \infty\} \right) = 1.$$

Since $\mathbf{P}(M_T^{(\alpha)} > 0) = \mathbf{P}(M_T^{(\alpha)} < 0) = \frac{1}{2}$, we get (3.8). \square

3.5 Remark. In case of $\alpha = 0$, the process $(X_t^{(0)})_{t \in [0, T]}$ is a standard Wiener process and hence it can be extended to an almost surely continuous process $(Y_t^{(0)})_{t \in [0, T]}$ with the definition $Y_t^{(0)} := B_T$. In case of $\alpha < 0$, there does not exist an almost surely continuous process $(Y_t^{(\alpha)})_{t \in [0, T]}$ such that $\mathbf{P}(X_t^{(\alpha)} = Y_t^{(\alpha)}) = 1$ for all $t \in [0, T]$. Indeed, by (3.8), we get

$$\mathbf{P} \left(\lim_{t \uparrow T} X_t^{(\alpha)} = -\infty \right) = \mathbf{P} \left(\lim_{t \uparrow T} X_t^{(\alpha)} = \infty \right) = \frac{1}{2} \quad \text{for all } \alpha < 0.$$

\square

4 Singularity of induced measures

For probability measures \mathbf{P}_1 and \mathbf{P}_2 on a measurable space (Ω, \mathcal{G}) , equivalence and singularity of them will be denoted by $\mathbf{P}_1 \sim \mathbf{P}_2$ and $\mathbf{P}_1 \perp \mathbf{P}_2$, respectively.

Using that for all $\alpha \in \mathbb{R}$, the process $X^{(\alpha)}$ has continuous paths (by the definition of strong solution, see, e.g., Jacod and Shiryaev [8, Definition 2.24, Chapter III]), we have

$$(4.1) \quad \mathbf{P} \left(\int_0^t \frac{(X_u^{(\alpha)})^2}{(T-u)^2} du < \infty \right) = 1, \quad \forall \alpha \in \mathbb{R}, \quad t \in [0, T].$$

For all $\alpha \in \mathbb{R}$ and $t \in (0, T)$, let $\mathbf{P}_{X^{(\alpha)}, t}$ denote the law of the process $(X_s^{(\alpha)})_{s \in [0, t]}$ on $(C[0, t], \mathcal{B}(C[0, t]))$, where $\mathcal{B}(C[0, t])$ denotes the Borel σ -algebra on $C[0, t]$. Using Theorem 7.20 in Liptser and Shiryaev [12] and (4.1), we get $\mathbf{P}_{X^{(\alpha)}, t} \sim \mathbf{P}_{X^{(0)}, t}$ and

$$(4.2) \quad \frac{d\mathbf{P}_{X^{(\alpha)}, t}}{d\mathbf{P}_{X^{(0)}, t}}(X^{(\alpha)}|_{[0, t]}) = \exp \left\{ -\alpha \int_0^t \frac{X_u^{(\alpha)}}{T-u} dX_u^{(\alpha)} - \frac{\alpha^2}{2} \int_0^t \frac{(X_u^{(\alpha)})^2}{(T-u)^2} du \right\}.$$

Here $\mathbf{P}_{X^{(0)}, t}$ is nothing else but the Wiener measure on $(C[0, t], \mathcal{B}(C[0, t]))$.

We recall that for all $t \in (0, T)$, the maximum likelihood estimator (MLE) $\hat{\alpha}_t^{(X^{(\alpha)})}$ of the parameter α based on the observation $(X_s^{(\alpha)})_{s \in [0, t]}$ is defined by

$$\hat{\alpha}_t^{(X^{(\alpha)})} := \arg \max_{\alpha \in \mathbb{R}} \ln \left(\frac{d\mathbf{P}_{X^{(\alpha)}, t}}{d\mathbf{P}_{X^{(0)}, t}}(X^{(\alpha)}|_{[0, t]}) \right).$$

By (4.1) and (4.2), for all $t \in (0, T)$, there exists a unique MLE $\hat{\alpha}_t^{(X^{(\alpha)})}$ of the parameter α based on the observation $(X_s^{(\alpha)})_{s \in [0, t]}$ given by

$$\hat{\alpha}_t^{(X^{(\alpha)})} = - \frac{\int_0^t \frac{X_s^{(\alpha)}}{T-s} dX_s^{(\alpha)}}{\int_0^t \frac{(X_s^{(\alpha)})^2}{(T-s)^2} ds}, \quad t \in (0, T).$$

To be more precise, by (4.1), for all $t \in (0, T)$, the MLE $\hat{\alpha}_t^{(X^{(\alpha)})}$ exists \mathbf{P} -almost surely. As a special case of Theorem 3.12 in Barczy and Pap [2], the MLE of α is strongly consistent, i.e.,

$$(4.3) \quad \mathbf{P}\left(\lim_{t \uparrow T} \hat{\alpha}_t^{(X^{(\alpha)})} = \alpha\right) = 1, \quad \alpha \in \mathbb{R}.$$

For all $\alpha \in \mathbb{R}$, let $\mathbf{P}_{X^{(\alpha)}, T}^T$ be the law of the process $(X_t^{(\alpha)})_{t \in [0, T]}$ given by the SDE (1.1) on $(C[0, T], \mathcal{B}(C[0, T]))$.

4.1 Theorem. *For all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, we have $\mathbf{P}_{X^{(\alpha)}, T}^T \perp \mathbf{P}_{X^{(\beta)}, T}^T$. In other words, the laws of the processes $(X_t^{(\alpha)})_{t \in [0, T]}$ and $(X_t^{(\beta)})_{t \in [0, T]}$ on $(C[0, T], \mathcal{B}(C[0, T]))$ are singular for all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$.*

Proof. First we check that for all $\alpha \in \mathbb{R}$ and $t \in [0, T)$,

$$(4.4) \quad \int_0^t \frac{X_s^{(\alpha)}}{T-s} dX_s^{(\alpha)} = \frac{1}{2} \left(\frac{(X_t^{(\alpha)})^2}{T-t} - \int_0^t \frac{(X_s^{(\alpha)})^2}{(T-s)^2} ds - \ln \left(\frac{T}{T-t} \right) \right).$$

By Itô's rule (see, e.g., Liptser and Shiryaev [12, Theorem 4.4]), we get

$$(4.5) \quad \begin{aligned} d \left(\frac{X_t^{(\alpha)}}{T-t} \right) &= \frac{X_t^{(\alpha)}}{(T-t)^2} dt + \frac{1}{T-t} dX_t^{(\alpha)} \\ &= \frac{X_t^{(\alpha)}}{(T-t)^2} dt - \alpha \frac{X_t^{(\alpha)}}{(T-t)^2} dt + \frac{1}{T-t} dB_t, \quad t \in [0, T). \end{aligned}$$

Now we verify that $(X_t^{(\alpha)})_{t \in [0, T)}$ and $\left(\frac{X_t^{(\alpha)}}{T-t}\right)_{t \in [0, T)}$ are continuous semimartingales adapted to the filtration induced by B . Consider the decomposition

$$X_t^{(\alpha)} = (T-t)^\alpha \int_0^t \frac{1}{(T-s)^\alpha} dB_s, \quad t \in [0, T).$$

Here the deterministic function $(T-t)^\alpha$, $t \in [0, T)$, is monotone and hence has a finite variation over each finite interval of $[0, T)$, and then, by Jacod and Shiryaev [8, Proposition 4.28, Chapter I], it is a semimartingale. Since

$$\int_0^t \frac{1}{(T-s)^\alpha} dB_s, \quad t \in [0, T),$$

is a martingale with respect to the filtration induced by B , using Theorem 4.57 in Chapter I in Jacod and Shiryaev [8] with the function $f(x, y) := xy$, $x, y \in \mathbb{R}$, we have $(X_t^{(\alpha)})_{t \in [0, T)}$ is a continuous semimartingale adapted to the filtration induced by B . Similarly as above, using that $\frac{1}{T-t}$, $t \in [0, T)$, is continuously differentiable, and hence has a finite variation over each finite interval of $[0, T)$, one can get $\left(\frac{X_t^{(\alpha)}}{T-t}\right)_{t \in [0, T)}$ is a continuous semimartingale adapted to the filtration induced by B . Moreover, by (4.5), the cross-variation process of the continuous martingale parts of the processes $(X_t^{(\alpha)})_{t \in [0, T)}$ and $\left(\frac{X_t^{(\alpha)}}{T-t}\right)_{t \in [0, T)}$ equals

$$\int_0^t \frac{1}{T-s} ds = \ln \left(\frac{T}{T-t} \right), \quad t \in [0, T).$$

Hence, by integration by parts formula (see, e.g., Karatzas and Shreve [9, page 155]), we have for all $t \in [0, T)$,

$$\begin{aligned} \int_0^t \frac{X_s^{(\alpha)}}{T-s} dX_s^{(\alpha)} &= \frac{X_t^{(\alpha)}}{T-t} X_t^{(\alpha)} - \int_0^t X_s^{(\alpha)} d\left(\frac{X_s^{(\alpha)}}{T-s}\right) - \ln\left(\frac{T}{T-t}\right) \\ &= \frac{(X_t^{(\alpha)})^2}{T-t} - \int_0^t \frac{(X_s^{(\alpha)})^2}{(T-s)^2} ds - \int_0^t \frac{X_s^{(\alpha)}}{T-s} dX_s^{(\alpha)} - \ln\left(\frac{T}{T-t}\right), \end{aligned}$$

which yields (4.4). Hence $\hat{\alpha}_t^{(X^{(\alpha)})} = A_t(X^{(\alpha)})$ for all $t \in (0, T)$, where $A_t : C[0, T] \setminus \{0\} \rightarrow \mathbb{R}$, defined by

$$A_t(x) := \frac{-\frac{x(t)^2}{T-t} + \int_0^t \frac{x(s)^2}{(T-s)^2} ds + \ln\left(\frac{T}{T-t}\right)}{2 \int_0^t \frac{x(s)^2}{(T-s)^2} ds}, \quad x \in C[0, T] \setminus \{0\}, \quad t \in (0, T).$$

For all $\alpha \in \mathbb{R}$, let us introduce the following subset of $C[0, T]$,

$$S_\alpha := \left\{ x \in C[0, T] \setminus \{0\} : \lim_{t \uparrow T} A_t(x) = \alpha \right\}.$$

We check that $S_\alpha \in \mathcal{B}(C[0, T])$. By Problem 2.4.1 in Karatzas and Shreve [9], under the metric

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{u \in [0, n]} (|x(\Psi(u)) - y(\Psi(u))| \wedge 1), \quad x, y \in C[0, T],$$

the set $C[0, T]$ is a complete, separable metric space, where $\Psi : [0, \infty) \mapsto [0, T)$, $\Psi(u) := \frac{2T}{\pi} \arctan(u)$, $u \geq 0$. For all $t \in [0, T)$, let $L_t : C[0, T] \rightarrow \mathbb{R}$,

$$L_t(x) := \int_0^t \frac{x(s)^2}{(T-s)^2} ds, \quad x \in C[0, T].$$

Let $x \in C[0, T]$ be fixed. We show that for all $t \in [0, T)$, L_t is continuous at the point $x \in C[0, T]$. Indeed, for all $y \in C[0, T]$, we have

$$\begin{aligned} |L_t(x) - L_t(y)| &\leq \sup_{s \in [0, t]} (|x(s) + y(s)||x(s) - y(s)|) \int_0^t \frac{1}{(T-s)^2} ds \\ &\leq \sup_{s \in [0, t]} \left((2|x(s)| + |y(s) - x(s)|)|x(s) - y(s)| \right) \int_0^t \frac{1}{(T-s)^2} ds. \end{aligned}$$

If $y \in C[0, T]$ is such that $\delta := \sup_{s \in [0, t]} |y(s) - x(s)| < 1$ and $n_0 \in \mathbb{N}$ is such that $n_0 > \Psi^{-1}(t)$, then

$$\rho(x, y) \geq \frac{1}{2^{n_0}} \sup_{u \in [0, n_0]} (|y(\Psi(u)) - x(\Psi(u))| \wedge 1) \geq \frac{1}{2^{n_0}} \sup_{u \in [0, \Psi^{-1}(t)]} |y(\Psi(u)) - x(\Psi(u))| = \frac{\delta}{2^{n_0}},$$

and hence

$$|L_t(x) - L_t(y)| \leq \delta \left(1 + 2 \sup_{s \in [0, t]} |x(s)| \right) \int_0^t \frac{1}{(T-s)^2} ds \leq K(t) \rho(x, y),$$

where $K(t) := 2^{n_0}(1 + 2 \sup_{s \in [0, t]} |x(s)|) \int_0^t \frac{1}{(T-s)^2} ds$, which yields the continuity of L_t at x . Consequently, A_t is continuous for all $t \in (0, T)$. Consider the decomposition

$$\begin{aligned} S_\alpha &= \bigcap_{\varepsilon > 0} \bigcup_{t \in [0, T)} \bigcap_{s \in [t, T)} \left\{ x \in C[0, T) \setminus \{0\} : |A_s(x) - \alpha| \leq \varepsilon \right\} \\ &= \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{s \in [T - \frac{1}{m}, T) \cap \mathbb{Q}_+} \left\{ x \in C[0, T) \setminus \{0\} : |A_s(x) - \alpha| \leq \frac{1}{n} \right\}, \end{aligned}$$

where \mathbb{Q}_+ denotes the set of positive rational numbers. Since A_s is continuous for all $s \in (0, T)$, we have

$$\left\{ x \in C[0, T) \setminus \{0\} : |A_s(x) - \alpha| \leq \frac{1}{n} \right\} \in \mathcal{B}(C[0, T)), \quad s \in (0, T), \quad n \in \mathbb{N},$$

and hence $S_\alpha \in \mathcal{B}(C[0, T))$. For all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, we have $S_\alpha \cap S_\beta = \emptyset$ and, by (4.3),

$$\begin{aligned} \mathbf{P}_{X^{(\alpha)}, T}^T(S_\alpha) &= \mathbf{P}(\lim_{t \uparrow T} \hat{\alpha}_t^{(X^{(\alpha)})} = \alpha) = 1, & \mathbf{P}_{X^{(\beta)}, T}^T(S_\beta) &= \mathbf{P}(\lim_{t \uparrow T} \hat{\alpha}_t^{(X^{(\beta)})} = \beta) = 1, \\ \mathbf{P}_{X^{(\alpha)}, T}^T(S_\beta) &= \mathbf{P}_{X^{(\beta)}, T}^T(S_\alpha) = 0, \end{aligned}$$

which implies the assertion by definition of singularity. \square

In what follows we will study the connections between the technique of the proof of our Theorem 4.1 and the very general results on singularity and absolute continuity due to Jacod and Shiryaev [8, Chapter IV]. In fact, we also present a second proof of Theorem 4.1.

First we recall that the proof of Theorem 4.1 is based on the strong consistency of the MLE of α , see Barczy and Pap [2, Theorem 3.12]. A short outline of the proof of Theorem 3.12 in Barczy and Pap [2] (specifying for α -Wiener bridges) sounds as follows. Using the explicit form of the Laplace transform of $\int_0^t \frac{(X_u^{(\alpha)})^2}{(T-u)^2} du$, $t \in [0, T)$, due to Barczy and Pap [2, Theorem 4.1], one can check that

$$\lim_{t \uparrow T} \mathbf{E} \exp \left\{ - \int_0^t \frac{(X_u^{(\alpha)})^2}{(T-u)^2} du \right\} = 0, \quad \forall \alpha \in \mathbb{R}.$$

Hence

$$(4.6) \quad \mathbf{P} \left(\lim_{t \uparrow T} \int_0^t \frac{(X_u^{(\alpha)})^2}{(T-u)^2} du = \infty \right) = 1, \quad \forall \alpha \in \mathbb{R},$$

which easily implies strong consistency of the MLE of α . It will turn out that if we apply Theorem 4.23 in Jacod and Shiryaev [8, Chapter IV] for proving $\mathbf{P}_{X^{(\alpha)}, T}^T \perp \mathbf{P}_{X^{(\beta)}, T}^T$ with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, then we have to check condition (4.6). We also note that the fact that condition (4.6) has to be checked is in accordance with part (i) of Theorem 1 in Ben-Ari and Pinsky [4]. But we emphasize that Ben-Ari and Pinsky's result is valid for time-homogeneous diffusions and hence we can not use it for α -Wiener bridges. By giving a second proof of Theorem 4.1, we shed more light on the role of condition (4.6).

Second proof of Theorem 4.1. Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$ be fixed. Let us introduce the process $(\tilde{X}_t^{(\alpha)})_{t \geq 0}$ given by

$$\tilde{X}_t^{(\alpha)} := X_{\Psi(t)}^{(\alpha)}, \quad t \geq 0,$$

where $\Psi : [0, \infty) \mapsto [0, T)$, $\Psi(t) = \frac{2T}{\pi} \arctan(t)$, $t \geq 0$. Then, by the SDE (1.1) and a change of variable, we get for all $t \geq 0$,

$$\tilde{X}_t^{(\alpha)} = -\alpha \int_0^{\Psi(t)} \frac{X_s^{(\alpha)}}{T-s} ds + B_{\Psi(t)} = -\alpha \int_0^t \frac{X_{\Psi(u)}^{(\alpha)}}{T-\Psi(u)} \psi(u) du + B_{\Psi(t)},$$

where $\psi(t) := \frac{d}{dt} \Psi(t)$, $t \geq 0$, and $(B_{\Psi(t)})_{t \geq 0}$ is a Wiener process with variance function $\Psi(t)$, $t \geq 0$, see Definitions 4.9 in Chapter I in Jacod and Shiryaev [8].

Let us consider the filtered space $(C[0, \infty), \mathcal{B}, (\mathcal{B}_t)_{t \geq 0})$, where \mathcal{B} is the Borel σ -algebra $\mathcal{B}(C[0, \infty))$ on $C[0, \infty)$ and \mathcal{B}_t , $t \geq 0$, defined as follows. For all $t \geq 0$, let

$$\mathcal{B}_t := \bigcap_{\varepsilon > 0} \rho_{t+\varepsilon}^{-1}(\mathcal{B}),$$

where $\rho_t : C[0, \infty) \rightarrow C[0, \infty)$ defined by $(\rho_t x)(s) := x(t \wedge s)$ for $s \geq 0$, $x \in C[0, \infty)$. Then the filtration $(\mathcal{B}_t)_{t \geq 0}$ is right-continuous, since for all $t \geq 0$,

$$\mathcal{B}_t = \bigcap_{\varepsilon > 0} \rho_{t+\varepsilon}^{-1}(\mathcal{B}) = \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \rho_{t+\varepsilon+\delta}^{-1}(\mathcal{B}) = \bigcap_{\varepsilon > 0} \mathcal{B}_{t+\varepsilon}.$$

Moreover, since $\rho_t^{-1}(\mathcal{B}) \subset \mathcal{B}_t$ for all $t \geq 0$, by Problem 2.4.2 in Karatzas and Shreve [9], we get

$$\mathcal{B} = \sigma \left(\bigcup_{t \geq 0} \mathcal{B}_t \right).$$

Let $\mathbf{P}_{\tilde{X}^{(\alpha)}}$ and $\mathbf{P}_{\tilde{X}^{(\beta)}}$ denote the law of the processes $(\tilde{X}_t^{(\alpha)})_{t \geq 0}$ and $(\tilde{X}_t^{(\beta)})_{t \geq 0}$ on $(C[0, \infty), \mathcal{B})$, respectively. We check that

$$\mathbf{P}_{X^{(\alpha)}, T}^T \perp \mathbf{P}_{X^{(\beta)}, T}^T \iff \mathbf{P}_{\tilde{X}^{(\alpha)}} \perp \mathbf{P}_{\tilde{X}^{(\beta)}}.$$

Indeed, by definition, $\mathbf{P}_{X^{(\alpha)}, T}^T \perp \mathbf{P}_{X^{(\beta)}, T}^T$ means that there exist so called distinguishing sets S_α and S_β in $\mathcal{B}(C[0, T])$ such that $S_\alpha \cap S_\beta = \emptyset$ and

$$\mathbf{P}(\{\omega \in \Omega : (X_t^{(\alpha)}(\omega))_{t \in [0, T]} \in S_\alpha\}) = \mathbf{P}(\{\omega \in \Omega : (X_t^{(\beta)}(\omega))_{t \in [0, T]} \in S_\beta\}) = 1.$$

Similarly, $\mathbf{P}_{\tilde{X}^{(\alpha)}} \perp \mathbf{P}_{\tilde{X}^{(\beta)}}$ means that there exist \tilde{S}_α and \tilde{S}_β in \mathcal{B} such that $\tilde{S}_\alpha \cap \tilde{S}_\beta = \emptyset$ and

$$\mathbf{P}(\{\omega \in \Omega : (\tilde{X}_t^{(\alpha)}(\omega))_{t \geq 0} \in \tilde{S}_\alpha\}) = \mathbf{P}(\{\omega \in \Omega : (\tilde{X}_t^{(\beta)}(\omega))_{t \geq 0} \in \tilde{S}_\beta\}) = 1.$$

Using that

$$\begin{aligned} \{\omega \in \Omega : (\tilde{X}_t^{(\alpha)}(\omega))_{t \geq 0} \in \tilde{S}_\alpha\} &= \{\omega \in \Omega : (X_{\Psi(t)}^{(\alpha)}(\omega))_{t \geq 0} \in \tilde{S}_\alpha\} \\ &= \{\omega \in \Omega : (X_t^{(\alpha)}(\omega))_{t \in [0, T]} \in \Psi^{-1}(\tilde{S}_\alpha)\}, \end{aligned}$$

where $\Psi^{-1}(\tilde{S}_\alpha) := \{f \circ \Psi^{-1} : f \in \tilde{S}_\alpha\}$, singularity of $\mathbf{P}_{\tilde{X}^{(\alpha)}}$ and $\mathbf{P}_{\tilde{X}^{(\beta)}}$ with distinguishing sets $\tilde{S}_\alpha, \tilde{S}_\beta \in \mathcal{B}$ implies singularity of $\mathbf{P}_{X^{(\alpha)}, T}^T$ and $\mathbf{P}_{X^{(\beta)}, T}^T$ with distinguishing sets $\Psi^{-1}(\tilde{S}_\alpha), \Psi^{-1}(\tilde{S}_\beta) \in \mathcal{B}(C[0, T])$. The converse statement can be thought over similarly.

Hence by Corollary 2.8 in Chapter IV in Jacod and Shiryaev [8], to prove the assertion it is enough to check that the measures $\mathbf{P}_{\tilde{X}^{(\alpha)}}$ and $\mathbf{P}_{\tilde{X}^{(\beta)}}$ are locally equivalent with respect to each other (where the restrictions of the measures refers to the given filtration $(\mathcal{B}_t)_{t \geq 0}$) and that $\mathbf{P}_{\tilde{X}^{(\alpha)}}(\lim_{t \rightarrow \infty} h_t^{(1/2)} < \infty) = 0$, where $(h_t^{(1/2)})_{t \geq 0}$ is the Hellinger process of order 1/2 between $\mathbf{P}_{\tilde{X}^{(\alpha)}}$ and $\mathbf{P}_{\tilde{X}^{(\beta)}}$. Using that the continuity of the process $X^{(\alpha)}$ implies that the process

$$\int_0^t \frac{(X_{\Psi(u)}^{(\alpha)})^2}{(T - \Psi(u))^2} \psi(u)^2 du, \quad t \geq 0,$$

does not jump to infinity (for the definition of jumping to infinity, see, e.g., Definitions 5.8 (ii) in Chapter III in Jacod and Shiryaev [8]), by (4.1) and a generalization of part (b) and (c) of Theorem 4.23 in Chapter IV in Jacod and Shiryaev [8], we have the measures $\mathbf{P}_{\tilde{X}^{(\alpha)}}$ and $\mathbf{P}_{\tilde{X}^{(\beta)}}$ are locally equivalent with respect to each other and the process

$$(4.7) \quad \frac{(\alpha - \beta)^2}{8} \int_0^t \frac{x(\Psi(u))^2}{(T - \Psi(u))^2} \psi(u) du, \quad x \in C[0, \infty), \quad t > 0,$$

is a version of the Hellinger process $(h_t^{(1/2)})_{t \geq 0}$. Indeed, using the notations of Sections 3a and 4b in Chapter IV in Jacod and Shiryaev [8], we have $C(t) = \Psi(t) = \int_0^t \psi(s) ds$, $t \geq 0$, and

$$\begin{aligned} \beta_s(x) &= -\alpha \frac{x(\Psi(s))}{T - \Psi(s)} \psi(s), \quad x \in C[0, \infty), \quad s \geq 0, \\ \beta'_s(x) &= -\beta \frac{x(\Psi(s))}{T - \Psi(s)} \psi(s), \quad x \in C[0, \infty), \quad s \geq 0, \\ \tilde{\beta}_s(s) &= \frac{\beta_s(x) - \beta'_s(x)}{\psi(s)} = -(\alpha - \beta) \frac{x(\Psi(s))}{T - \Psi(s)}, \quad x \in C[0, \infty), \quad s \geq 0. \end{aligned}$$

Hence using the very same arguments given in the proof of Theorem 4.23 in Jacod and Shiryaev [8, Chapter IV], we get (4.7). By a change of variable, we have

$$\int_0^t \frac{x(\Psi(u))^2}{(T - \Psi(u))^2} \psi(u) du = \int_0^{\Psi(t)} \frac{x(s)^2}{(T - s)^2} ds, \quad x \in C[0, \infty), \quad t \geq 0,$$

and hence $\mathbf{P}_{\tilde{X}^{(\alpha)}}(\lim_{t \rightarrow \infty} h_t^{(1/2)} < \infty) = 0$, $\alpha \in \mathbb{R}$, is equivalent with (4.6), i.e., to prove the assertion it is enough to verify (4.6). As it was mentioned earlier, as a special case of the proof of Theorem 3.12 in Barczy and Pap [2] we get (4.6). \square

4.2 Remark. If $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, by Theorem 4.1, we have $\mathbf{P}_{X^{(\alpha)}, T}^T \perp \mathbf{P}_{X^{(\beta)}, T}^T$. Moreover, in the proof of the theorem, we also constructed disjoint sets S_α and S_β in $\mathcal{B}(C[0, T])$ that distinguish between the measures in the sense that $\mathbf{P}_{X^{(\alpha)}, T}^T(S_\alpha) = 1$ and $\mathbf{P}_{X^{(\beta)}, T}^T(S_\beta) = 1$. We note that for some special time-homogeneous (1-dimensional) diffusions Ben-Ari and Pinsky [4, Propositions 1, 2 and 3] also gave "illuminating" distinguishing sets. \square

4.3 Remark. In case of $\alpha \geq 0$, by Lemma 3.1, one can define a probability measure $\mathbf{P}_{Y^{(\alpha)},T}^T$ on $(C[0,T], \mathcal{B}(C[0,T]))$, as the law of the process $(Y_t^{(\alpha)})_{t \in [0,T]}$ given in Lemma 3.1. Then, by Theorem 4.1, for all $\alpha, \beta \geq 0$, $\alpha \neq \beta$, we get $\mathbf{P}_{Y^{(\alpha)},T}^T \perp \mathbf{P}_{Y^{(\beta)},T}^T$. Indeed, since $\mathbf{P}_{X^{(\alpha)},T}^T \perp \mathbf{P}_{X^{(\beta)},T}^T$, there exist sets S_α and S_β in $\mathcal{B}(C[0,T])$ such that $S_\alpha \cap S_\beta = \emptyset$ and $\mathbf{P}_{X^{(\alpha)},T}^T(S_\alpha) = \mathbf{P}_{X^{(\beta)},T}^T(S_\beta) = 1$. For all $B \in \mathcal{B}(C[0,T])$, let us introduce the notation

$$\tilde{B} := \left\{ x \in C[0,T] : x|_{[0,T)} \in B \text{ and } \exists \lim_{t \uparrow T} x(t) \in \mathbb{R} \right\}.$$

Then we have $\tilde{S}_\alpha, \tilde{S}_\beta \in \mathcal{B}(C[0,T])$, $\tilde{S}_\alpha \cap \tilde{S}_\beta = \emptyset$ and $\mathbf{P}_{Y^{(\alpha)},T}^T(\tilde{S}_\alpha) = \mathbf{P}_{Y^{(\beta)},T}^T(\tilde{S}_\beta) = 1$. As a special case, we also have for all $\alpha > 0$, the probability measure $\mathbf{P}_{Y^{(\alpha)},T}^T$ and the standard Wiener measure $\mathbf{P}_{Y^{(0)},T}$ on $(C[0,T], \mathcal{B}(C[0,T]))$ are singular. \square

4.4 Remark. We note that Theorem 4.1 is not an astonishing result. One can easily formulate conditions on a general time-inhomogeneous diffusion process under which the same kind of singularity holds. Namely, let us consider a process $(X_t^{(\theta)})_{t \geq 0}$ given by the SDE

$$(4.8) \quad \begin{cases} dX_t^{(\theta)} = \theta a(t, X_t^{(\theta)}) dt + dB_t, & t \geq 0, \\ X_0^{(\theta)} = 0, \end{cases}$$

where $a : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a known Borel-measurable function, $(B_t)_{t \geq 0}$ is a standard Wiener process, and $\theta \in \mathbb{R}$ is an unknown parameter. Let us suppose that the SDE (4.8) has a unique strong solution $(X_t^{(\theta)})_{t \geq 0}$ for all $\theta \in \mathbb{R}$. For all $\theta \in \mathbb{R}$, let us denote by \mathbf{P}_θ the law of $(X_t^{(\theta)})_{t \geq 0}$ on $(C[0, \infty), \mathcal{B}(C[0, \infty)))$. Let us suppose that for all $t > 0$ and all $\theta \in \mathbb{R}$,

$$\mathbf{P} \left(\int_0^t a(s, X_s^{(\theta)})^2 ds < \infty \right) = 1.$$

As it is explained in details in the second proof of Theorem 4.1, using Theorem 4.23 in Chapter IV in Jacod and Shiryaev [8], we get for all $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 \neq \theta_2$,

$$\mathbf{P}_{\theta_1} \perp \mathbf{P}_{\theta_2} \quad \Longleftrightarrow \quad \mathbf{P} \left(\lim_{t \rightarrow \infty} \int_0^t a(u, X_u^{(\theta_i)})^2 du = \infty \right) = 1 \quad \text{for } i = 1 \text{ or } i = 2.$$

Concerning singularity of \mathbf{P}_{θ_1} and \mathbf{P}_{θ_2} , the point is that whether the imposed conditions can be checked for a given diffusion process. And in this respect, time-inhomogeneous diffusions in general represent a hard task. \square

Concerning the SDE (1.1) one can ask why the diffusion coefficient in the SDE (1.1) is identically 1. The point is only that it is supposed to be a known and positive constant. Remember that in many cases the measures induced by processes with different diffusion coefficients are singular and continuous-time statistical inference for this type of model is often trivial. In what follows we consider this phenomenon in details. For all $T \in (0, \infty)$, $\alpha \in \mathbb{R}$ and $\sigma > 0$, let us introduce the time-inhomogeneous diffusion process $(X_t^{(\alpha, \sigma)})_{t \in [0, T]}$ given by the SDE

$$(4.9) \quad \begin{cases} dX_t^{(\alpha, \sigma)} = -\frac{\alpha}{T-t} X_t^{(\alpha, \sigma)} dt + \sigma dB_t, & t \in [0, T), \\ X_0^{(\alpha, \sigma)} = 0, \end{cases}$$

where $(B_t)_{t \geq 0}$ is a 1-dimensional standard Wiener process. By formula (5.6.6) in Karatzas and Shreve [9], the SDE (4.9) has a unique strong solution, namely,

$$X_t^{(\alpha, \sigma)} = \sigma \int_0^t \left(\frac{T-t}{T-s} \right)^\alpha dB_s, \quad t \in [0, T].$$

For all $t \in (0, T)$, let $\mathbb{P}_{X^{(\alpha, \sigma)}, t}$ be the law of the process $(X_s^{(\alpha, \sigma)})_{s \in [0, t]}$ given by the SDE (4.9) on $(C[0, t], \mathcal{B}(C[0, t]))$.

4.5 Theorem. *For all $\alpha_1, \alpha_2 \in \mathbb{R}$, $\sigma_1 > 0$, $\sigma_2 > 0$ and $t \in (0, T)$, the following dichotomy holds:*

$$\mathbb{P}_{X^{(\alpha_1, \sigma_1)}, t} \sim \mathbb{P}_{X^{(\alpha_2, \sigma_2)}, t} \quad \text{if } \sigma_1 = \sigma_2,$$

$$\mathbb{P}_{X^{(\alpha_1, \sigma_1)}, t} \perp \mathbb{P}_{X^{(\alpha_2, \sigma_2)}, t} \quad \text{if } \sigma_1 \neq \sigma_2.$$

First proof. In case of $\sigma_1 = \sigma_2$, the equivalence of $\mathbb{P}_{X^{(\alpha_1, \sigma_1)}, t}$ and $\mathbb{P}_{X^{(\alpha_2, \sigma_2)}, t}$ follows from Theorem 7.20 or Theorem 7.19 in Liptser and Shiryaev [13] and from (4.1).

Let us suppose now that $\sigma_1 \neq \sigma_2$. For all $\alpha \in \mathbb{R}$ and $\sigma > 0$, by giving a direct proof, we show the following Baxter type result

$$(4.10) \quad \mathbb{P} \left(\lim_{n \rightarrow \infty} \sum_{j=1}^n (X_{jt/n}^{(\alpha, \sigma)} - X_{(j-1)t/n}^{(\alpha, \sigma)})^2 = t\sigma^2 \right) = 1.$$

By the SDE (4.9), we have

$$(4.11) \quad \begin{aligned} & \sum_{j=1}^n (X_{jt/n}^{(\alpha, \sigma)} - X_{(j-1)t/n}^{(\alpha, \sigma)})^2 \\ &= \alpha^2 \sum_{j=1}^n \left(\int_0^{jt/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du - \int_0^{(j-1)t/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du \right)^2 \\ & \quad - 2\alpha\sigma \sum_{j=1}^n \left(\int_0^{jt/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du - \int_0^{(j-1)t/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du \right) (B_{jt/n} - B_{(j-1)t/n}) \\ & \quad + \sigma^2 \sum_{j=1}^n (B_{jt/n} - B_{(j-1)t/n})^2. \end{aligned}$$

It is known that

$$(4.12) \quad \mathbb{P} \left(\lim_{n \rightarrow \infty} \sum_{j=1}^n (B_{jt/n} - B_{(j-1)t/n})^2 = t \right) = 1,$$

see, e.g., Lemma 4.3 in Liptser and Shiryaev [12]. Moreover, by Lagrange's mean value theorem, one can think it over that for all $t \in [0, T)$ and for all continuous functions $f : [0, t] \rightarrow \mathbb{R}$, we have

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \left(\int_0^{jt/n} f(x) dx - \int_0^{(j-1)t/n} f(x) dx \right)^2 = 0.$$

Since for all $t \in (0, T)$, the process $\left(\frac{X_u^{(\alpha, \sigma)}}{T-u}\right)_{u \in [0, t]}$ is continuous, we have

$$(4.13) \quad \mathbb{P} \left(\lim_{n \rightarrow \infty} \sum_{j=1}^n \left(\int_0^{jt/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du - \int_0^{(j-1)t/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du \right)^2 = 0 \right) = 1.$$

Now we check that

$$(4.14) \quad \mathbb{P} \left(\lim_{n \rightarrow \infty} \sum_{j=1}^n \left(\int_0^{jt/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du - \int_0^{(j-1)t/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du \right) (B_{jt/n} - B_{(j-1)t/n}) = 0 \right) = 1.$$

By Cauchy-Schwartz's inequality, we have

$$\begin{aligned} & \sum_{j=1}^n \left(\int_0^{jt/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du - \int_0^{(j-1)t/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du \right) (B_{jt/n} - B_{(j-1)t/n}) \\ & \leq \sqrt{\sum_{j=1}^n \left(\int_0^{jt/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du - \int_0^{(j-1)t/n} \frac{X_u^{(\alpha, \sigma)}}{T-u} du \right)^2} \sqrt{\sum_{j=1}^n (B_{jt/n} - B_{(j-1)t/n})^2}, \end{aligned}$$

with probability one, and then (4.12) and (4.13) implies (4.14). By (4.12), (4.13) and (4.14), using (4.11) we have (4.10). Then, using the definition of singularity of measures, (4.10) implies that $\mathbb{P}_{X^{(\alpha, \sigma_1)}, t} \perp \mathbb{P}_{X^{(\alpha, \sigma_2)}, t}$ for all $\alpha \in \mathbb{R}$ and $\sigma_1 \neq \sigma_2$. In case of $\alpha_1 \neq \alpha_2$ and $\sigma_1 \neq \sigma_2$ we have $\mathbb{P}_{X^{(\alpha_1, \sigma_1)}, t} \sim \mathbb{P}_{X^{(\alpha_2, \sigma_1)}, t}$ and $\mathbb{P}_{X^{(\alpha_2, \sigma_1)}, t} \perp \mathbb{P}_{X^{(\alpha_2, \sigma_2)}, t}$, which imply that $\mathbb{P}_{X^{(\alpha_1, \sigma_1)}, t} \perp \mathbb{P}_{X^{(\alpha_2, \sigma_2)}, t}$.

Second proof. Using Baxter's theorem due to Baxter [3, Theorem 1], we show that for all $\alpha \in \mathbb{R}$ and $\sigma > 0$,

$$(4.15) \quad \mathbb{P} \left(\lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left(X_{kt/2^n}^{(\alpha, \sigma)} - X_{(k-1)t/2^n}^{(\alpha, \sigma)} \right)^2 = t\sigma^2 \right) = 1,$$

which is also enough (like (4.10)) to ensure that $\mathbb{P}_{X^{(\alpha, \sigma_1)}, t} \perp \mathbb{P}_{X^{(\alpha, \sigma_2)}, t}$ for all $\alpha \in \mathbb{R}$ and $\sigma_1 \neq \sigma_2$. For all $t \in (0, T)$, $(X_s^{(\alpha, \sigma)})_{s \in [0, t]}$ is a Gauss process with identically 0 mean function, and to have right to apply Baxter's theorem, we need to check that the covariance function of $(X_s^{(\alpha, \sigma)})_{s \in [0, t]}$ is continuous on $[0, t] \times [0, t]$ and has uniformly bounded second derivatives on $[0, t] \times [0, t] \setminus \{(s, s) : s \in [0, t]\}$.

In case of $\alpha \neq \frac{1}{2}$, by (2.1), we get for all $u, v \in [0, t]$,

$$r(v, u) := \text{Cov}(X_v^{(\alpha, \sigma)}, X_u^{(\alpha, \sigma)}) = \sigma^2 \frac{(T-v)^\alpha (T-u)^\alpha}{1-2\alpha} \left(T^{1-2\alpha} - (T-(v \wedge u))^{1-2\alpha} \right).$$

Then $r(v, u)$, $v, u \in [0, t]$, is continuous, and clearly, if $0 \leq v < u \leq t$, then

$$\begin{aligned} \frac{\partial r}{\partial v}(v, u) &= \sigma^2 \frac{T^{1-2\alpha}}{1-2\alpha} (T-u)^\alpha \alpha (T-v)^{\alpha-1} (-1) - \frac{\sigma^2}{1-2\alpha} (T-u)^\alpha (1-\alpha) (T-v)^{-\alpha} (-1), \\ \frac{\partial r}{\partial u}(v, u) &= \sigma^2 \frac{T^{1-2\alpha}}{1-2\alpha} (T-v)^\alpha \alpha (T-u)^{\alpha-1} (-1) - \frac{\sigma^2}{1-2\alpha} (T-v)^{1-\alpha} \alpha (T-u)^{\alpha-1} (-1). \end{aligned}$$

If $0 \leq u < v \leq t$, then

$$\begin{aligned}\frac{\partial r}{\partial v}(v, u) &= \sigma^2 \frac{T^{1-2\alpha}}{1-2\alpha} (T-u)^\alpha \alpha (T-v)^{\alpha-1} (-1) - \frac{\sigma^2}{1-2\alpha} (T-u)^{1-\alpha} \alpha (T-v)^{\alpha-1} (-1), \\ \frac{\partial r}{\partial u}(v, u) &= \sigma^2 \frac{T^{1-2\alpha}}{1-2\alpha} (T-v)^\alpha \alpha (T-u)^{\alpha-1} (-1) - \frac{\sigma^2}{1-2\alpha} (T-v)^\alpha (1-\alpha) (T-u)^{-\alpha} (-1).\end{aligned}$$

This implies that $(X_s^{(\alpha, \sigma)})_{s \in [0, t]}$ has uniformly bounded first derivatives on $[0, t] \times [0, t] \setminus \{(s, s) : s \in [0, t]\}$, and similarly one can check that the second derivatives also admit this property. Moreover, for all $u \in [0, t]$,

$$\begin{aligned}D^+(u) &:= \lim_{v \downarrow u} \frac{r(u, u) - r(v, u)}{u - v} = \lim_{v \downarrow u} \frac{r(v, u) - r(u, u)}{v - u} = \lim_{v \downarrow u} \frac{\partial r}{\partial v}(v, u) \\ &= -\frac{\sigma^2 \alpha}{1-2\alpha} T^{1-2\alpha} (T-u)^{2\alpha-1} + \frac{\sigma^2 \alpha}{1-2\alpha}.\end{aligned}$$

Similarly, for all $u \in (0, t]$,

$$\begin{aligned}D^-(u) &:= \lim_{v \uparrow u} \frac{r(u, u) - r(v, u)}{u - v} = \lim_{v \uparrow u} \frac{r(v, u) - r(u, u)}{v - u} = \lim_{v \uparrow u} \frac{\partial r}{\partial v}(v, u) \\ &= -\frac{\sigma^2 \alpha}{1-2\alpha} T^{1-2\alpha} (T-u)^{2\alpha-1} + \frac{\sigma^2 (1-\alpha)}{1-2\alpha}.\end{aligned}$$

Hence for all $t \in [0, T]$,

$$\int_0^t (D^-(u) - D^+(u)) du = \sigma^2 \int_0^t \left(\frac{1-\alpha}{1-2\alpha} - \frac{\alpha}{1-2\alpha} \right) du = \sigma^2 \int_0^t 1 du = \sigma^2 t,$$

and then Baxter's theorem due to Baxter [3, Theorem 1] yields (4.15).

In case of $\alpha = \frac{1}{2}$, similarly to the case $\alpha \neq \frac{1}{2}$, one can check that the covariance function of $(X_s^{(1/2, \sigma)})_{s \in [0, t]}$ is continuous on $[0, t] \times [0, t]$, and has uniformly bounded first and second derivatives on $[0, t] \times [0, t] \setminus \{(s, s) : s \in [0, t]\}$. Moreover, by (2.1), for all $u \in [0, t]$,

$$\begin{aligned}D^+(u) &= \lim_{v \downarrow u} \frac{\sigma^2 \sqrt{(T-v)(T-u)} \ln\left(\frac{T}{T-u}\right) - \sigma^2 (T-u) \ln\left(\frac{T}{T-u}\right)}{v - u} \\ &= \sigma^2 \ln\left(\frac{T}{T-u}\right) \sqrt{T-u} \frac{d}{du} \sqrt{T-u} = -\frac{\sigma^2}{2} \ln\left(\frac{T}{T-u}\right).\end{aligned}$$

Similarly, by (2.1), for all $u \in (0, t]$,

$$\begin{aligned}D^-(u) &= \lim_{v \uparrow u} \frac{\sigma^2 \sqrt{(T-v)(T-u)} \ln\left(\frac{T}{T-v}\right) - \sigma^2 (T-u) \ln\left(\frac{T}{T-u}\right)}{v - u} \\ &= \sigma^2 \sqrt{T-u} \frac{d}{du} \left(\sqrt{T-u} \ln\left(\frac{T}{T-u}\right) \right) = -\frac{\sigma^2}{2} \ln\left(\frac{T}{T-u}\right) + \sigma^2.\end{aligned}$$

Hence for all $t \in [0, T]$,

$$\int_0^t (D^-(u) - D^+(u)) du = \sigma^2 \int_0^t 1 du = \sigma^2 t,$$

and then Baxter's theorem due to Baxter [3, Theorem 1] yields (4.15). \square

We note that the same dichotomy that we have in Theorem 4.5 holds for Ornstein–Uhlenbeck processes, see, e.g., page 226 in Arató, Pap and van Zuijlen [1].

4.6 Remark. For all $\alpha \in \mathbb{R}$ and $\sigma > 0$, let $P_{X^{(\alpha, \sigma)}}$ denote the law of the process $(X_s^{(\alpha, \sigma)})_{s \in [0, T]}$ on $(C[0, T], \mathcal{B}(C[0, T]))$. By the proof of Theorem 4.1, we get $P_{X^{(\alpha_1, \sigma_1)}} \perp P_{X^{(\alpha_2, \sigma_2)}}$ for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\sigma_1 > 0, \sigma_2 > 0$. \square

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