ON THE EQUATION $1^k + 2^k + \cdots + x^k = y^n$ FOR FIXED x

A. BÉRCZES, L. HAJDU, T. MIYAZAKI, AND I. PINK

Dedicated to Kálmán Győry on the occassion of his 75th birthday.

Abstract. We provide all solutions of the title equation in positive integers x, k, y, n with $1 \leq x < 25$ and $n \geq 3$. For these values of the parameters, our result gives an affirmative answer to a related, classical conjecture of Schäffer. In our proofs we combine several tools: Baker's method (in particular, sharp bounds for the linear combinations of logarithms of two algebraic numbers), polynomial-exponential congruences and computational methods.

1. Introduction

Let x and k be positive integers. Write

$$S_k(x) = 1^k + 2^k + \cdots + x^k.$$

The equation

$$S_k(x) = y^n \tag{1}$$

in unknown positive integers k, n, x, y with $n \geq 2$ has a long history. The case $(k, n) = (2, 2)$ has already been considered by Lucas [9], [10], Watson [17] and others. Here we do not give details; the interested reader may consult to the book [15], the papers [4], [1], [3] and the references given therein.

It is long known that when (k, n) is one of the pairs

$$(1, 2), (3, 2), (3, 4), (5, 2), \tag{2}$$

then (1) has infinitely many solutions. These solutions can be described easily. As the first deep general result, in 1956 Schäffer [14] proved that

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if (k,n) is fixed and is not in the list (2), then equation (1) has only finitely many solutions. Schäffer's proof was ineffective. Still, for some (small) pairs (k,n) he was able to show that equation (1) has only the trivial solution $(x,y)=(1,1)$. Beside this, he conjectured that for (k,n) not in the list (2), equation (1) has the only nontrivial solution $(x,k,y,n)=(24,2,70,2)$.

Considerably later, Győry, Tijdeman and Voorhove [5] gave an effective proof for Schäffer's result, in the much more general case where the exponent n is also unknown. Moreover, Pintér [12] (under some mild assumptions) proved that for the nontrivial solutions $n < ck\log(2k)$ holds, where c is an effectively computable absolute constant. For further results about equation (1) and its generalizations we refer to the book [15] and the papers [4], [1], [3], and the references there.

The conjecture of Schäffer has been verified under certain assumptions for the parameters involved. Beside the "small" fixed pairs (k,n) considered by Schäffer [14], Jacobson, Pintér and Walsh [7] verified the conjecture for $n=2$ and even values of k with $2\le k\le 58$. Later, Bennett, Győry and Pintér [1] proved that the conjecture holds for any $n\ge 2$ with $1\le k\le 11$. Further, Pintér [13] verifed Schäffer's conjecture for the even values of n with $n>4$, provided that k is odd with $1\le k<170$.

Recently, Hajdu [6] proved that Schäffer's conjecture holds under certain assumptions made on x, letting all the other parameters free. Among other results, he has proved that the conjecture is true if $x\equiv 0,3 \pmod 4$ and $x<25$. The main tools in the proof of this result were the 2-adic valuation of $S_k(x)$ and local methods for polynomial-exponential congruences.

The purpose of the present paper is to extend the results in [6] for all values of x with $x<25$. It is important to mention that for this purpose we need different tools than those used in [6]. The reason is that for the remaining values of x with $x<25$ (i.e. those with $x\equiv 1,2 \pmod 4$) the methods used in [6] are not applicable. To prove our main theorem, we need to combine sharp upper bounds for linear forms in two logarithms and polynomial-exponential congruences, and we also make use of involved computational facilities. The reason why we stop at $x<25$ (though our method in principle is capable to cover larger intervals for x) is the following. The total running time of our computer calculations for $x=21$ (the value of x requiring heavy computations) was already around six days. For larger values of x, the bounds appearing in Table 1 would be significantly worse, resulting in much longer running times in the computational part. Since solving the equation for such values of x would rise questions more of technical and computational type, and also because of a nice property of $x = 24$ (being the only value with a non-trivial solution), we decided to stop at this point.

The structure of the paper is the following. In the next section we give our main result. In the third section we give an overview of our strategy to prove our main theorem, and we provide several lemmas. Finally, in the last section we give the proof of our main result.

2. The main result

Our main result is the following.

Theorem 2.1. *All solutions of equation* (1) *in positive integers* x, k, y, n *with* $x < 25$ *and* $n \geq 3$ *are given by*

$$(x, k, y, n) = (1, k, 1, n), \ (8, 3, 6, 4).$$

As a simple consequence we obtain the following immediate

Corollary 2.1. *For* $x < 25$ *and* $n \geq 3$*, Schäffer's conjecture is true.*

Remark. We mention that in case of $n = 2$, in view of the identity $S_3(x) = \left(\frac{x(x+1)}{2}\right)^2$, equation (1) has many more solutions with $x < 25$.

3. Lemmas

In this section we give some lemmas which are needed in the proof of Theorem 2.1. First we get rid of those values of x for which equation (1) is already solved.

Lemma 3.1. *Suppose that*

$$x \in \{1, 2, 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 23, 24\}.$$

Then equation (1) *with* $n \geq 3$ *has only the trivial solution with* $(x, y) = (1, 1)$.

Proof. The case $x = 1$ is trivial. When $x = 2$, the only solution to (1) is given by $(x, k, y, n) = (2, 3, 3, 2)$ (where we have $n = 2$). This fact is well-known; it follows e.g. from the nice result of Mihăilescu [11] concerning the Catalan equation. All the other cases are handled by Hajdu [6]. $\square$

In view of the above lemma, we may assume that we have

$$x \in \{5, 6, 9, 10, 13, 14, 17, 18, 21, 22\}.$$

In these cases, the strategy of our proof is the following. First, using Baker's method (for linear forms in two logarithms) we prove that one of the exponential variables k and n has to be "small". For this we use results of Laurent [8]. Then the remaining cases will be handled separately. It is important to mention that we need to provide rather sharp upper bounds for k and n (which makes the proofs of our corresponding lemmas rather technical). The reason is that the "small" values of k and n need to be handled separately, one by one, by a numerical method, and the running time of our algorithm is very sensitive for the initial upper bounds for these parameters.

When k is small, since x is fixed, the left hand side of equation (1) is fixed, and we only need to perform a simple check (which for "large" values of k can still be rather time consuming). When n is "small" then for each possible values of n, we solve (1) locally, as a polynomial-exponential congruence. At this stage we also make use of the program package Magma [2]. We note that this is the point where we need to require the assumption $n > 2$, since in case of $n = 2$ some of the occurring equations cannot be handled locally.

So we start with deriving upper bounds for the exponential variables k, n in equation (1). As we have mentioned, for this purpose we use Baker's method for linear forms in logarithms of two algebraic numbers. We need to introduce some notation.

For an algebraic number α of degree d over $\mathbb{Q}$, we define the *absolute logarithmic height* of α by the following formula:

$$\mathrm{h}(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\max\left\{1, |\alpha^{(i)}|\right\}\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and $\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(d)}$ are the conjugates of α in the field of complex numbers.

Let α_1 and α_2 be multiplicatively independent algebraic numbers with $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$. Consider the linear form in two logarithms:

$$\Lambda = b_2 \log\alpha_2 - b_1 \log\alpha_1,$$

where $\log\alpha_1, \log\alpha_2$ are any determinations of the logarithms of α_1, α_2 respectively, and b_1, b_2 are positive integers.

We shall use the following result due to Laurent [8].

Lemma 3.2 ([8], Theorem 2)**.** *Let ρ and μ be real numbers with $\rho > 1$ and $1/3 \leq \mu \leq 1$. Set*

$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda = \sigma\log\rho.$$

Let a_1, a_2 be real numbers such that

$$a_i \geq \max\{1,\ \rho|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i)\} \qquad (i = 1, 2),$$

where

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] \,/\, [\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}].$$

Let h be a real number such that

$$h \geq \max\left\{D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.75\right) + 0.06,\ \lambda,\ \frac{D\log 2}{2}\right\}.$$

We assume that

$$a_1 a_2 \geq \lambda^2.$$

Put

$$H = \frac{h}{\lambda} + \frac{1}{\sigma}, \quad \omega = 2 + 2\sqrt{1 + \frac{1}{4H^2}}, \quad \theta = \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H}.$$

Then we have

$$\log|\Lambda| \geq -Ch'^2 a_1 a_2 - \sqrt{\omega\theta}h' - \log\left(C'h'^2 a_1 a_2\right)$$

with

$$h' = h + \frac{\lambda}{\sigma}, \quad C = C_0\frac{\mu}{\lambda^3\sigma}, \quad C' = \sqrt{\frac{C\sigma\omega\theta}{\lambda^3\mu}},$$

where

$$C_0 = \left(\frac{\omega}{6} + \frac{1}{2}\sqrt{\frac{\omega^2}{9} + \frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1a_2}H^{1/2}} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\lambda\omega}{H}}\right)^2.$$

Using this lemma, we show the following.

Lemma 3.3. *Let $A = \{5, 6, 9, 10, 13, 14, 17, 18, 21, 22\}$ and consider equation (1) with $x \in A$ in integer unknowns (k, y, n) with $k \geq 83, y \geq 2$ and $n \geq 3$ a prime. Then for $y > x^2$ we have $n \leq n_0$, for $y > 10^6$ even $n \leq n_1$ holds, and for $y \leq x^2$ we have $k \leq k_1$, where $n_0 = n_0(x), n_1 = n_1(x)$ and $k_1 = k_1(x)$ are given in Table 1.*

Proof. In the course of the proof we will always assume that $x \in A$ and we distinguish three cases according to $y > x^2$, $y > 10^6$ or $y \leq x^2$.

Case I. $y > x^2$

We may suppose, without loss of generality, that n is large enough, that is

$$n > n_0. \tag{3}$$

Further, by $k \geq 83$ we easily deduce that for every $x \in A$ we have

$$1^k + 2^k + \cdots + x^k < 2x^k < (x+1)^k, \tag{4}$$

x	n_0 $(y > x^2)$	n_1 $(y > 10^6)$	k_1 $(y \leq x^2)$
5	$14,000$	$6,100$	$78,000$
6	$21,000$	$10,100$	$121,000$
9	$52,000$	$28,000$	$304,000$
10	$65,000$	$36,000$	$381,000$
13	$111,000$	$64,000$	$651,000$
14	$129,000$	$75,000$	$754,000$
17	$187,000$	$113,000$	$1,099,000$
18	$209,000$	$127,100$	$1,224,000$
21	$278,000$	$174,100$	$1,633,000$
22	$244,000$	$168,000$	$1,466,000$

TABLE 1. Bounding n and k under the indicated conditions

and

$$1^k + 2^k + \cdots + (x-1)^k < 2(x-1)^k. \tag{5}$$

Since $y > x^2$ by (1), (4) and $x \geq 5$ we get that

$$k \geq 2n. \tag{6}$$

Using (6) and the fact that n is odd we may write k in the form

$$k = Bn + r \text{ with } B \geq 1,\ 0 \leq |r| \leq \frac{n-1}{2}. \tag{7}$$

We show that in (7) we have $r \neq 0$. On the contrary, suppose $r = 0$. Then, using (1) and (5) we infer by (7) that

$$\begin{aligned} 2(x-1)^k &> 1 + 2^k + \cdots + (x-1)^k = y^n - x^k = y^n - x^{Bn} \\ &= (y - x^B)(y^{n-1} + \cdots + x^{B(n-1)}) \geq x^{B(n-1)}. \end{aligned}$$

Hence

$$n < \frac{\log x}{\log\left(\frac{x}{x-1}\right)} + \frac{\log 2}{B \log\left(\frac{x}{x-1}\right)}.$$

This together with $x \leq 22$ and $B \geq 1$ implies $n < 82$, which contradicts (3). Thus, $r \neq 0$.

On dividing equation (1) by y^n we obviously get

$$1 - \frac{x^k}{y^n} = \frac{s}{y^n}, \tag{8}$$

where $s = 1^k + 2^k + \ldots + (x-1)^k$. Using (7) and (8) we infer that

$$\left| x^r \cdot \left(\frac{x^B}{y}\right)^n - 1 \right| = \frac{s}{y^n}. \tag{9}$$

Put

$$\Lambda_r = \begin{cases} r\log x - n\log\frac{y}{x^B} & \text{if } r > 0, \\ |r|\log x - n\log\frac{x^B}{y} & \text{if } r < 0. \end{cases} \tag{10}$$

In what follows we find upper and lower bounds for $\log|\Lambda_r|$. We distinguish two subcases according to

$$1 - \frac{x^k}{y^n} \geq 0.795 \text{ or } 1 - \frac{x^k}{y^n} < 0.795,$$

respectively. If $1 - \frac{x^k}{y^n} \geq 0.795$ then by (1) and (4) we immediately obtain a contradiction, so we may assume that the latter case holds. It is well known (see Lemma B.2 of [16]) that for every $z \in \mathbb{R}$ with $|z - 1| < 0.795$ one has

$$|\log z| < 2|z - 1|. \tag{11}$$

On applying inequality (11) with $z = x^k/y^n$ we get by (8), (9), (10) and $x^k \neq y^n$ that

$$|\Lambda_r| < \frac{2s}{y^n}. \tag{12}$$

Observe that (1) implies

$$k < \frac{n\log y}{\log x}. \tag{13}$$

Thus by (12), (5) and (13) we infer that

$$\log|\Lambda_r| < -\frac{\log\left(\frac{x}{x-1}\right)}{\log x}(\log y)\, n + \log 4. \tag{14}$$

Next, for a lower bound for $\log|\Lambda_r|$, we shall use Lemma 3.2 with

$$(\alpha_1, \alpha_2, b_1, b_2) = \begin{cases} \left(\frac{y}{x^B}, x, n, r\right) & \text{if } r > 0, \\ \left(\frac{x^B}{y}, x, n, |r|\right) & \text{if } r < 0. \end{cases}$$

Using (1) and (4) one can easily check that $\alpha_1 > 1$ and $\alpha_2 > 1$. We show that α_1, α_2 are multiplicatively independent. Assume the contrary. Then the set of prime factors of y coincides with that of x. Since y is odd (as $x \not\equiv 0, 3 \pmod 4$), hence x is also odd, that is, $x \in \{5, 9, 13, 17, 21\}$. If x is a prime, i.e. $x \in \{5, 13, 17\}$, then y has to be a power of x, and equation (1) can be written as

$$1^k + 2^k + \cdots + x^k = x^m, \quad k \geq 2,\ m \geq 2.$$

One can verify that this equation has no solution (since $x^k < x^m < 2x^k < x^{k+1}$). If $x = 9$, then y has to be a power of 3, and equation (1) can be written as

$$1^k + 2^k + \cdots + 9^k = 3^m, \quad k \geq 2,\, m \geq 2.$$

Taking this equation modulo 4, we have

$$3 + 2 \cdot (-1)^k \equiv (-1)^m \pmod 4.$$

This implies that m is even, and we find

$$1^k + 2^k + \cdots + 9^k = 9^{m/2},$$

which, as we already know, has no solution. If $x = 21$, then the set of prime factors of the integer

$$1^k + 2^k + \cdots + 21^k$$

should be $\{3, 7\}$. However, we can observe that the above integer is not divisible by 3 if k is even, and that it is divisible by 11 provided that k is odd. This is a contradiction. To sum up, we may assume that α_1, α_2 are multiplicatively independent.

Now, we apply Lemma 3.2 with the following choice of parameters (ρ, μ): for every $x \in A$ we choose $\mu = 0.57$ uniformly, and set

$$\rho = \begin{cases} 7.7 & \text{if} \quad x \in A \setminus \{22\}, \\ 7 & \text{if} \quad x = 22. \end{cases} \tag{15}$$

In what follows we shall derive upper bounds for the quantities

$$\rho |\log \alpha_i| - \log |\alpha_i| + 2D\mathrm{h}(\alpha_i), \quad (i = 1, 2)$$

occurring in Lemma 3.2. Since $D = 1$ and $\alpha_2 > 1$, for $i = 2$ we get

$$\rho |\log \alpha_2| - \log |\alpha_2| + 2D\mathrm{h}(\alpha_2) = (\rho + 1) \log x. \tag{16}$$

For $i = 1$ we obtain

$$\rho |\log \alpha_1| - \log |\alpha_1| + 2D\mathrm{h}(\alpha_1) < \frac{\rho + 1}{2} \log x + 2 \log y - 2 \log g, \tag{17}$$

where $g = \gcd(x, y)$. To verify that (17) is valid we shall estimate $\log \alpha_1$ and $\mathrm{h}(\alpha_1)$ from above, by using equation (1), i.e. $s + x^{Bn+r} = y^n$. Observe

$$\mathrm{h}(\alpha_1) = \mathrm{h}\left(\frac{x^B}{y}\right) \leq \log \max\{x^B, y\} - \log g = \begin{cases} \log y - \log g & \text{if } r > 0, \\ \log x^B - \log g & \text{if } r < 0. \end{cases}$$

If $r > 0$, then

$$\alpha_1^n = \left(\frac{y}{x^B}\right)^n = x^r + \frac{s}{x^{Bn}} = x^r \left(1 + \frac{s}{x^k}\right) < 2x^r \text{ (as } s < x^k\text{)},$$

so

$$\log \alpha_1 < \frac{\log 2}{n} + \frac{r}{n} \log x \leq \frac{\log 2}{n} + \frac{n-1}{2n} \log x,$$

whence

$$\rho |\log \alpha_1| - \log |\alpha_1| + 2D\mathrm{h}(\alpha_1) <$$

$$< \left(\frac{\log 2}{n \log x} + \frac{n-1}{2n} \right) (\rho - 1) \log x + 2 \log y - 2 \log g$$

which by (15), (3) and $x \geq 5$ clearly implies (17).

If $r < 0$, then

$$\alpha_1^n = \left(\frac{x^B}{y} \right)^n = x^{-r} \left(1 - \frac{s}{y^n} \right) < x^{-r} = x^{|r|},$$

so

$$\log \alpha_1 < \frac{|r|}{n} \log x \leq \frac{n-1}{2n} \log x,$$

and

$$\log x^B = \log \alpha_1 + \log y < \frac{n-1}{2n} \log x + \log y,$$

and we get

$$\rho |\log \alpha_1| - \log |\alpha_1| + 2D\mathrm{h}(\alpha_1) <$$

$$< \left(\frac{n-1}{2n} (\rho - 1) + \frac{n-1}{n} \right) \log x + 2 \log y - 2 \log g,$$

which by (3) again implies (17).

In view of (16) we can obviously take for every $x \in A$

$$a_2 = (\rho + 1) \log x. \tag{18}$$

For the values a_1 we do the following. If we can calculate the exact value of $g = \gcd(x, y)$ then we use for a_1 the upper bound occurring in (17), while if we do not know the exact value of $g = \gcd(x, y)$ we use (17) with $g = 1$. Namely, we can take a_1 as

$$a_1 = \begin{cases} \frac{\rho+1}{2} \log x + 2 \log y & \text{if} \quad x \in A \setminus \{22\}, \\ \frac{\rho+1}{2} \log x + 2 \log y - 2 \log 11 & \text{if} \quad x = 22. \end{cases} \tag{19}$$

To see that the choice of a_1 for $x = 22$ is valid we observe that

$$\begin{cases} S_k(22) \equiv 0 \pmod 3 \text{ and } S_k(22) \not\equiv 0 \pmod 9 & \text{if } k \text{ is even}, \\ S_k(22) \equiv 0 \pmod{11} & \text{if } k \text{ is odd}, \end{cases} \tag{20}$$

Since by (3) n is large in equation (1), we may assume that k is odd, hence (20) together with (1), implies $y \equiv 0 \pmod{11}$. Thus, since y

must be odd we have $\gcd(x,y) = g = 11$ for $x = 22$, and from $y > x^2$, we additionally get that

$$y \geq 22^2 + 11 = 495. \tag{21}$$

Since $\mu = 0.57$ we get

$$\sigma = 0.90755 \text{ and } \lambda = 0.90755 \log \rho, \tag{22}$$

whence by (15), (18), (19), (22) and $y > x^2$ we easily check that for every $x \in A$

$$a_1 a_2 > \lambda^2$$

holds. Now, we are going to derive an upper bound h for the quantity

$$\max\left\{ D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.75\right) + 0.06,\ \lambda,\ \frac{D\log 2}{2} \right\}.$$

Using $D = 1$, (15), (18), (19), (22), and (21) for $x = 22$ and $y > x^2$ for $x \in A \setminus \{22\}$, for the values of h occurring in Lemma 3.2 we obtain $h = \log n + \varepsilon$, with $\varepsilon = \varepsilon(x)$ given in Table 2.

x	5	6	9	10	13
ε	0.2087	0.1014	−0.1026	−0.1494	−0.2573
x	14	17	18	21	22
ε	−0.2858	−0.3568	−0.3768	−0.4287	−0.3454

TABLE 2. Choosing the parameter $h = \log n + \varepsilon$ occurring in Lemma 3.2 if the case $y > x^2$

Further, by (3) we easily check that for the above values of h assumptions of Lemma 3.2 concerning the parameter h are satisfied. Using (3) again we obtain a lower bound for H and hence upper bounds for ω and θ. Moreover, using these values of ω and θ by (15), (18), (19), (22) and (21) for $x = 22$ and $y > x^2$ for $x \in A \setminus \{22\}$ we obtain Table 3. Now, on combining (19), (18), (21) for $x = 22$ and $y > x^2$ for $x \in A \setminus \{22\}$ with Table 3 we get

$$\frac{\log 4 + \log(C' a_1 a_2)}{\log y} < 4. \tag{23}$$

Further, by Lemma 3.2 we obtain

$$\log|\Lambda_r| > -Ch'^2 a_1 a_2 - \sqrt{\omega\theta}\, h' - \log(C' h'^2 a_1 a_2), \tag{24}$$

whence using (23) and comparing (14) with (24) we get

$$n < \left(\frac{Ch'^2 a_1 a_2}{\log y} + \frac{\sqrt{\omega\theta}}{\log y} h' + \frac{\log h'^2}{\log y} + 4 \right) \frac{\log x}{\log \frac{x}{x-1}}. \tag{25}$$

x	H	ω	θ	C_0	C	C'	h'
5	6.34	4.0063	1.0820	2.2802	0.2253	0.50	$\log n$+2.2500
6	6.51	4.0059	1.0800	2.2245	0.2198	0.50	$\log n$+2.1427
9	6.90	4.0053	1.0751	2.1331	0.2108	0.50	$\log n$+1.9387
10	6.99	4.0052	1.0741	2.1150	0.2090	0.50	$\log n$+1.8919
13	7.23	4.0048	1.0717	2.0757	0.2051	0.50	$\log n$+1.7840
14	7.29	4.0047	1.0710	2.0662	0.2042	0.50	$\log n$+1.7555
17	7.46	4.0045	1.0700	2.0435	0.2020	0.50	$\log n$+1.6845
18	7.50	4.0045	1.0689	2.0377	0.2014	0.50	$\log n$+1.6645
21	7.63	4.0043	1.0677	2.0224	0.2000	0.50	$\log n$+1.6126
22	7.92	4.0040	1.0652	2.0429	0.2330	0.55	$\log n$+1.6006

TABLE 3. Lower bounds for H and upper bounds for $\omega, \theta, C_0, C, C', h'$ occurring in Lemma 3.2 if $y > x^2$

Finally, using (18), (19) and (21) for $x = 22$ and $y > x^2$ for $x \in A\backslash\{22\}$, by Table 3 we obtain the desired bounds for n in this case.

Case II. $y > 10^6$

We work as in the previous case. Namely, we apply Lemma 3.2 again, the only difference is that in this case for y we may write $y > 10^6$. We may suppose, without loss of generality, that n is large enough, that is

$$n > n_1. \tag{26}$$

Further, we choose $\mu = 0.57$ uniformly, and set

$$\rho = \begin{cases} 9.6 & \text{if} \quad x = 5, 6, 9, 10, 13, 14, \\ 9.5 & \text{if} \quad x = 17, \\ 9.4 & \text{if} \quad x = 18, \\ 9.3 & \text{if} \quad x = 21, \\ 8.9 & \text{if} \quad x = 22. \end{cases} \tag{27}$$

As before, we may take a_1 and a_2 as in (19) and (18).
Thus by (27), (19), (18) and $y > 10^6$ for the values of h occurring in Lemma 3.2 we obtain $h = \log n - \varepsilon$, with $\varepsilon = \varepsilon(x)$ given in Table 4.
On combining (18), (19), (26), (27) with $y > 10^6$ and with Table 4 we obtain Table 5.

x	5	6	9	10	13
ε	0.0938	0.1851	0.3572	0.3964	0.4866
x	14	17	18	21	22
ε	0.5103	0.5662	0.5795	0.6195	0.5866

TABLE 4. Choosing the parameter $h = \log n - \varepsilon$ occurring in Lemma 3.2 if $y > 10^6$

x	H	ω	θ	C_0	C	C'	h'
5	5.25	4.0091	1.1000	2.1395	0.1554	0.4	$\log n + 2.1680$
6	5.47	4.0084	1.0956	2.1075	0.1531	0.4	$\log n + 2.0767$
9	5.90	4.0072	1.0884	2.0540	0.1492	0.4	$\log n + 1.9046$
10	6.01	4.0070	1.0867	2.0428	0.1484	0.4	$\log n + 1.8654$
13	6.25	4.0064	1.0832	2.0189	0.1467	0.4	$\log n + 1.7752$
14	6.31	4.0063	1.0824	2.0130	0.1462	0.4	$\log n + 1.7515$
17	6.51	4.0059	1.0800	1.9983	0.1472	0.4	$\log n + 1.6851$
18	6.59	4.0058	1.0788	1.9942	0.1490	0.4	$\log n + 1.6613$
21	6.75	4.0055	1.0769	1.9840	0.1504	0.4	$\log n + 1.6106$
22	6.86	4.0054	1.0756	1.9814	0.1594	0.4	$\log n + 1.5995$

TABLE 5. Lower bounds for H and upper bounds for $\omega, \theta, C_0, C, C', h'$ occurring in Lemma 3.2 if $y > 10^6$

Now, on combining (18), (19) and $y > 10^6$ with Table 5 we get

$$\frac{\log 4 + \log(C' a_1 a_2)}{\log y} < 4. \tag{28}$$

Further, by Lemma (3.2) we obtain

$$\log |\Lambda_r| > -Ch'^2 a_1 a_2 - \sqrt{\omega\theta} h' - \log(C' h'^2 a_1 a_2), \tag{29}$$

whence using (28) and comparing (14) with (29) we obtain

$$n < \left(\frac{Ch'^2 a_1 a_2}{\log y} + \frac{\sqrt{\omega\theta}}{\log y} h' + \frac{\log h'^2}{\log y} + 4 \right) \frac{\log x}{\log \frac{x}{x-1}}. \tag{30}$$

Finally, using also (18), (19) and $y > 10^6$, by Table 5 we obtain the desired bounds for n in this case.

Case III. $y \leq x^2$

In order to obtain the desired upper bounds for k we may clearly assume that k is large, namely

$$k > k_1. \tag{31}$$

Since $y \le x^2$ we have by (1) that

$$n > \lfloor k/2 \rfloor. \tag{32}$$

Hence by (32), we can write

$$n = Bk + r \quad \text{with} \quad B \ge 1,\ 0 \le |r| \le \left\lfloor \frac{k}{2} \right\rfloor. \tag{33}$$

Further, using the same argument as in Case I, by $x \in A$ and $k \ge 83$ we may suppose that in (33) we have $r \ne 0$.
We divide our equation (1) by x^k. Then, by (33) we infer

$$y^r \left(\frac{y^B}{x} \right)^k - 1 = \frac{s}{x^k}, \tag{34}$$

where $s = 1^k + 2^k + \cdots + (x-1)^k$. Thus, $y^r \left(\frac{y^B}{x} \right)^k > 1$. Put

$$\Lambda_r = b_2 \log \alpha_2 - b_1 \log \alpha_1, \tag{35}$$

where

$$(\alpha_1, \alpha_2, b_1, b_2) = \begin{cases} \left(\frac{x}{y^B}, y, k, r \right) & \text{if } r > 0, \\ \left(\frac{y^B}{x}, y, k, |r| \right) & \text{if } r < 0. \end{cases} \tag{36}$$

It is easy to see $\alpha_1 > 1$ and $\alpha_2 > 1$, moreover similarly to Case I we obtain that α_1 and α_2 are multiplicatively independent. We find upper and lower bounds for $\log |\Lambda_r|$. Since for every $z \in \mathbb{R}$ with $z > 1$ we have $|\log z| < |z - 1|$ it follows by (34), (35), (36) and (5) that

$$\log |\Lambda_r| < -k \log \left(\frac{x}{x-1} \right) + \log 2. \tag{37}$$

For a lower bound, we again use Lemma 3.2. We choose $\mu = 0.57$ uniformly (i.e. for every $x \in A$), and set

$$\rho = \begin{cases} 6.0 & \text{if} \quad x = 5, \\ 6.1 & \text{if} \quad x = 6, 9, 10, \\ 6.2 & \text{if} \quad x = 13, 14, 17, 18, 21, \\ 5.6 & \text{if} \quad x = 22. \end{cases} \tag{38}$$

Moreover, using the same argument as in Case I by $y \le x^2$ and $11 = \gcd(22, y)$ we may take

$$a_1 = \begin{cases} (\rho + 3) \log x & \text{if} \quad x \in A \setminus \{22\}, \\ (\rho + 1) \log x + 2 \log 2 & \text{if} \quad x = 22 \end{cases} \tag{39}$$

and

$$a_2 = 2 \cdot (\rho + 1) \log x. \tag{40}$$

Since $\mu = 0.57$ we get

$$\sigma = 0.90755 \text{ and } \lambda = 0.90755 \log \rho, \tag{41}$$

whence by (38), (39), (40), (41) we easily check that for every $x \in A$

$$a_1 a_2 > \lambda^2$$

holds. Now, we are going to derive an upper bound h for the quantity

$$\max\left\{D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.75\right) + 0.06,\ \lambda,\ \frac{D\log 2}{2}\right\}.$$

Using (38), (39), (40), (41) for h occurring in Lemma 3.2 we obtain $h = \log n + \varepsilon$, with $\varepsilon = \varepsilon(x)$ given in Table 6.

x	5	6	9	10	13
ε	0.1216	0.0112	−0.1928	−0.2397	−0.3507
x	14	17	18	21	22
ε	−0.3792	−0.4502	−0.4702	−0.5221	−0.3945

TABLE 6. Choosing the parameter $h = \log n + \varepsilon$ occurring in Lemma 3.2 if the case $y \le x^2$

On combining (31), (38), (39), (40), (41) with Table 6 we obtain Table 7.

x	H	ω	θ	C_0	C	C'	h'
5	8.09	4.0039	1.0638	2.0737	0.3030	0.70	$\log k$+1.9134
6	8.23	4.0037	1.0626	2.0412	0.2901	0.67	$\log k$+1.8195
9	8.67	4.0034	1.0594	1.9876	0.2825	0.66	$\log k$+1.6155
10	8.78	4.0033	1.0586	1.9768	0.2810	0.66	$\log k$+1.5686
13	8.97	4.0032	1.0573	1.9542	0.2704	0.64	$\log k$+1.4739
14	9.04	4.0031	1.0569	1.9486	0.2696	0.64	$\log k$+1.4454
17	9.23	4.0030	1.0557	1.9353	0.2678	0.64	$\log k$+1.3744
18	9.28	4.0030	1.0554	1.9318	0.2673	0.63	$\log k$+1.3544
21	9.42	4.0029	1.0545	1.9229	0.2661	0.63	$\log k$+1.3025
22	9.93	4.0026	1.0517	1.9338	0.3179	0.75	$\log k$+1.3283

TABLE 7. Lower bounds for H and upper bounds for $\omega, \theta, C_0, C, C', h'$ occurring in Lemma 3.2 if $y \le x^2$

Further, on using Table 7 and Lemma 3.2 we obtain

$$\log|\Lambda_r| > -Ch'^2 a_1 a_2 - \sqrt{\omega\theta}h' - \log(C'h'^2 a_1 a_2), \tag{42}$$

whence, on comparing (37) with (42) we get

$$k < \frac{Ch'^2 a_1 a_2 + \sqrt{\omega\theta}h' + \log(2C'h'^2 a_1 a_2)}{\log\left(\frac{x}{x-1}\right)}.$$

Finally, using (38), (39), (40), by Table 7 we obtain the desired bounds for k in this case. Thus our lemma is proved.

□

Lemma 3.3 gives us sharp estimates for either n or k, depending on some further assumptions. Now we give some lemmas which take care of the cases where one of n, k is "small".

Lemma 3.4. *Let $A = \{5, 6, 9, 10, 13, 14, 17, 18, 21, 22\}$. Then equation (1) with $x \in A$ has no solutions under the following assumption:*

$$3 \le n \le n_1, \ n \text{ prime or } n = 4,$$

where n_1 is the bound (depending on x) specified in Table 1.

Proof. We use the following strategy. Fix a value of $x \in A$ and also fix n such that either $n = 4$ or $3 \le n \le n_1$ is a prime. Take the smallest prime p_1 of the form $2in + 1$ with $i \in \mathbb{Z}$. Let $o_1 := p_1 - 1$. Consider all values of $k = 1, \dots, o_1$, and check whether $S_k(x) \pmod{p_1}$ is a perfect power or not. Let $K(o_1)$ be the set of all those values of $k \pmod{o_1}$ for which $S_k(x) \pmod{p}$ is a perfect power. Then take the next prime p_2 of the form $p_2 := 2in + 1$ with $i \in \mathbb{Z}$. Put $o_2 := \mathrm{lcm}(o_1, p_2 - 1)$, and construct the set $K_0(o_2)$ of all those numbers $1, \dots, o_2$ which are congruent to elements of $K(o_1)$ modulo o_1. Considering now equation (1) modulo p_2, exclude from the set $K_0(o_2)$ all those elements k for which $S_k(x) \pmod{p_2}$ is not a perfect power. Thus we get the set $K(o_2)$ of all possible values of $k \pmod{o_2}$ for which a solution is possible. Continue this procedure by taking new primes $p_3, p_4, \dots, p_l$ of the form $2in + 1$ with $i \in \mathbb{Z}$, until the set $K(o_l)$ becomes empty. Then we conclude that equation (1) has no solution for the given x and n. We have performed the above computation in the computer algebra package Magma [2] for every $x \in A$ and every prime n with $3 \le n \le n_1$ and $n = 4$ concluding the proof of our lemma. The Magma code and the list of primes $p_1, \dots, p_l$ can be downloaded from the link "http://math.unideb.hu/berczes-attila/linkek.html" under the name "Results for Skx".

□

Lemma 3.5. *Let* $A = \{5, 6, 9, 10, 13, 14, 17, 18, 21, 22\}$. *Assume that equation* (1) *has a solution* (x, k, y, n) *with* $x \in A$ *such that either:*

(i) $k \geq 83$ *and* $x^2 < y \leq 10^6$, *or*
(ii) $k \geq 83$ *and* $y \leq x^2$.

Then we have $n < 12$.

Proof. In the case (ii), by Lemma 3.3 we immediately get $k \leq k_1$. In the case of (i), by Lemma 3.3 we get $n \leq n_0$, which together with (1) and the assumption $y \leq 10^6$ gives the estimate

$$k < \frac{n \log y}{\log x} \leq \frac{6n_0 \log 10}{\log x}.$$

Put $k_0 := \max\left\{\frac{6n_0 \log 10}{\log x}, k_1\right\}$, which for given x is a fixed number.

For given $x \in A$ let us take any fix value of k with $2 \leq k \leq k_0$. For every prime $2 \leq p \leq 10^6$ we check whether $S_k(x)$ is divisible by p (in fact we compute $S_k(x) \pmod p$ and we check if it is 0 or not). If p divides $S_k(x)$, then we check if $S_k(x)$ is also divisible by p^{12} or not. During our computations for every possible pair (x, k) we either found that there is no prime $p \leq 10^6$ dividing $S_k(x)$ at all, which by $y \leq 10^6$ proves there is no solution, or we could find a prime divisor $p \leq 10^6$ of $S_k(x)$ with the property that p^{12} does not divide $S_k(x)$ leading to the conclusion that $n < 12$. The computations were performed again in Magma [2]. □

Lemma 3.6. *The only solution of equation* (1) *with* $5 \leq x < 25$ *and* $n \geq 3$ *under the assumption* $k \leq 100$ *is* $(x, k, y, n) = (8, 3, 6, 4)$.

Proof. A direct computation of $S_k(x)$ for all possible pairs (k, x) corresponding to the requirements of Lemma 3.6, and checking whether it is a perfect power can be done in Magma [2] in a few seconds. □

4. Proof of Theorem 2.1

In principle the proof is a simple combination of the above proved lemmas, namely of Lemma 3.3, Lemma 3.4, Lemma 3.5 and Lemma 3.6.

Proof of Theorem 2.1. Clearly, it is enough to prove the theorem for $n = 4$ and for odd prime values of n. Further, the cases

$$x \in \{1, 2, 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 23, 24\}$$

are handled by Lemma 3.1. So now we only need to prove Theorem 2.1 for $x \in A = \{5, 6, 9, 10, 13, 14, 17, 18, 21, 22\}$.

We split the proof into several subcases. The case $k \leq 100$ is completely covered by Lemma 3.6, so for the rest of the proof we may assume $k > 100$. If $y > 10^6$ then by Lemma 3.3 we have $n \leq n_1$, and by Lemma 3.4 we know that there is no solution for $3 \leq n \leq n_1$, n prime or $n = 4$. This concludes the proof of Theorem 2.1 whenever $y > 10^6$.

For $y \leq 10^6$ by Lemma 3.5 we have $n < 12$. Thus by $x \geq 5$ from (1) we get the estimate

$$k < \frac{n \log y}{\log x} \leq \frac{66 \log 10}{\log 5} < 100,$$

which has been treated already. □

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A. Bérczes and L. Hajdu,
University of Debrecen, Institute of Mathematics
H-4010 Debrecen, P.O. Box 12.
Hungary
E-mail address: berczesa@science.unideb.hu
E-mail address: hajdul@science.unideb.hu

T. Miyazaki
Faculty of Science and Engineering, Gunma University,
Gunma, 376-8515,
Japan
E-mail address: tmiyazaki@gunma-u.ac.jp

I. Pink
Institute of Mathematics, University of Debrecen
H-4010 Debrecen, P.O. Box 12, Hungary
and
University of Salzburg
Hellbrunnerstrasse 34/I
A-5020 Salzburg, Austria
E-mail address: pinki@science.unideb.hu; istvan.pink@sbg.ac.at