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TOPOLOGICAL LOOP WITH SOLVABLE MULTIPLICATION GROUP

Dissertation for the degree of Doctor of Philosophy (PhD)

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Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematics and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at Debrecen University.

The results published in the thesis are not reported in any other PhD theses.

Debrecen, June 14, 2021.

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Ameer Al-Abayechi
signature of the candidate

Hereby I confirm that Ameer Al-Abayechi candidate conducted his studies with my supervision within the Differential Geometry and its Applications Doctoral Program of the Doctoral School of Mathematics and Computational Sciences between 2017 and 2021. The independent studies and research work of the candidate significantly contributed to the results published in the thesis.

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TOPOLOGICAL LOOP WITH SOLVABLE MULTIPLICATION GROUP

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,,The length of the experiments is an increase in the mind."

Imam Hussain Ibn Ali

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Contents

List of Symbols

G	Lie group
$\mathbf{g}$	Lie algebra
$L_{\mathcal{F}}$	elementary filiform loop
$\mathcal{F}_n$	the n-dimensional elementary filiform Lie group
$\mathbf{f}_n$	the n-dimensional elementary filiform Lie algebra
$Mult(L)$	multiplication group of loop L
$\mathbf{mult}(L)$	Lie algebra of the group $Mult(L)$
$\mathcal{L}_2$	2-dimensional non-abelian Lie group
$\mathbf{l}_2$	2-dimensional non-abelian Lie algebra
$Z(L)$	centre of loop L
Z	centre of multiplication group $Mult(L)$
$\mathbf{z}$	centre of Lie algebra $\mathbf{g}$
$\mathbf{g}'$	commutator subalgebra of Lie algebra $\mathbf{g}$
$Inn(L)$	inner mapping group of L
$\mathbf{inn}(L)$	Lie algebra of inner mapping group $Inn(L)$
$\widetilde{PSL_2}(\mathbb{R})$	universal covering group of $PSL_2(\mathbb{R})$
$\mathbb{R}^n$	n-dimensional abelian Lie algebra
$\mathbf{n}_{\text{rad}}$	nilradical of Lie algebra $\mathbf{g}$
e	identity element of loop L
$\Lambda(L)$	set of all left translations of loop L
$P(L)$	set of all right translations of loop L
G_ℓ	the group generated by all left translations of loop L
G_r	the group generated by all right translations of loop L

Introduction

The dissertation is devoted to investigate the relations between non-associative binary systems loops L and the transformation groups $Mult(L)$ generated by all left and right translations of L. This group is called the multiplication group of L. The action of the group $Mult(L)$ on L is transitive and effective. The stabilizer of the identity element of L in the group $Mult(L)$ is the inner mapping group $Inn(L)$ of L. The initial steps to treat loops came from the study of coordinate systems of non-desarguesian planes and from the investigation of topological questions in differential geometry (cf. [3]). Firstly R. Baer considered loops in connection with the group G_ℓ or G_r generated by their left or right translations (cf. [2]). The studies of A. A. Albert ([1]) and R. H. Bruck ([5]) strengthened the algebraic features of loops. They proved that every normal subloop of L corresponds to a normal subgroup of the group $Mult(L)$ and the orbit of a normal subgroup of $Mult(L)$ with respect to the identity element $e \in L$ results a normal subloop of L (cf. Theorems 3, 4 and 5 in [1] and Lemma 1.3, IV.1, in [5]). Hence the group $Mult(L)$ and the subgroup $Inn(L)$ play an essential role for the investigation of the structure of the loop L (cf. [1], [5], [6], [22], [23], [32], [33], [37], [38]). In [4] it is proved that the nilpotency of the group $Mult(L)$ forces that the loop L is centrally nilpotent. In this case the group $Inn(L)$ is commutative. For finite loops A. Vesanen ([42]) proved that from the solvability of the group $Mult(L)$ follows the classical solvability of the loop L. Analogously as in the group case a loop L is classically solvable if there is a subnormal series of L such that every factor loop is commutative. Using congruences defining the decomposition of a loop L into its left cosets $xN, x \in L$, with respect to the normal subloop N of L, D. Stanovský and P. Vojtěchovský developed commutator theory for loops (cf. [37]). If there exists a normal series $\{e\} = L_0 \leq L_1 \leq \ldots \leq L_n = L$ of L with the property that for all $i = 1, \cdots, n$, the factor loop L_i/L_{i-1} is abelian in L/L_{i-1}, then the loop L is congruence solvable. In contrast to the group case the class of congruence solvable loops is a proper subclass of the class of classical solvable loops (cf. Exercise 10 in [18] and Construction 9.1 and Example 9.3 in [37]). Moreover, the iterated abelian, respectively central extensions, yield congruence solvable, respectively centrally nilpotent loops (cf. Corollaries 5.1 and 5.2 in [38]).

In this dissertation we deal with connected topological loops L. We follow the approach of P. T. Nagy and K. Strambach who consistently studied topological and differentiable loops using the tools of Lie theory. In [29] topological and differentiable loops L are realized as sharply transitive sections in Lie groups G_ℓ generated by the left translations of L. The subject of our investigation is connected topological loops L having a solvable Lie group G as the group $Mult(L)$ generated by all left and right translations of L. The action of the group $Mult(L)$ on the topological space L is transitive and effective. Each 1-dimensional connected topological loop having a locally compact group as its multiplication group is associative (cf. Theorem 18.18 in [29]). In the class of Lie groups the elementary filiform groups $\mathcal{F}_n$ with dimension $n \geq 4$ are the multiplication groups of 2-dimensional connected topological proper loops. Moreover, these loops are central extensions of a 1-dimensional Lie group by the group $\mathbb{R}$ (cf. [9]). Chapter 2 deals with the investigation of the classical and congruence solvable properties for topological loops. Using the results of Lie on transitive actions of Lie groups on the plane $\mathbb{R}^2$ (cf. [21]) and those on the groups $Mult(L)$ of L, if $\dim(L) \leq 2$, we obtain that all 3-dimensional connected topological loops L having solvable Lie groups as their multiplication groups are classically solvable (cf. Theorem 12). Applying the relation between iterated abelian extensions and congruence solvability we formulated necessary and sufficient conditions for 3-dimensional topological loops L to be congruence solvable (cf. Theorem 13). A particular interesting example (Example 1) illustrates that also for the topological case the class of congruence solvable loops forms a proper subclass of the class of classical solvable loops.

In Chapters $3, 4, 5, 6$ we discuss the question what solvable Lie groups can be represented as the multiplication groups of connected topological loops having dimension 3. Many authors investigated the general problem, what group can be realized as the group $Mult(L)$ of a loop L, in particular if L is a finite loop ([7], [8], [22], [27], [34]). Firstly, T. Kepka and M. Niemenmaa considered the latter question and answered it using group theoretical tools (cf. [33]). The conditions for a group G to be the multiplication group $Mult(L)$ of a loop L request the existence of two special left transversals S, T with respect to a subgroup K of G. The group K results in being the inner mapping group of L and the transversals S and

T can be taken as the set of the left and right of the translations of L, respectively. The transversals S, T are K-connected and generate the group G (see Lemma 7). These criterions can be fruitfully applied for the topological case too (cf. [9]-[17]). In [11] it is found the at most 5-dimensional solvable connected simply connected Lie groups which are not nilpotent and can be realized as the group $Mult(L)$ for a 3-dimensional topological loop L.

The isomorphism classes of solvable Lie algebras g are classified in [24], [26], [25], [36], [41], if dim(g) ≤ 6. Hence we restrict our consideration for these classes of Lie algebras. The main result of Chapter 3 says that each at most 3-dimensional connected topological loop L, such that the group $Mult(L)$ of L is a solvable Lie group of dimension ≤ 6, has nilpotency class 2 (cf. Theorem 15). To prove this result in Chapter 3 we describe the structure of the 3-dimensional connected simply connected topological loops L and their multiplication groups if $Mult(L)$ are solvable Lie groups. Theorem 16 deals with the case that $Mult(L)$ has discrete centre. Theorem 17, respectively Theorem 18 treat the case that $Mult(L)$ has 1-dimensional, respectively 2-dimensional, centre. In Chapter 3 we give the steps of the procedure for the classification of the 6-dimensional solvable Lie groups which are multiplication groups 3-dimensional connected simply connected topological loops L having a solvable Lie group G of dimension 6 as their multiplication group. Based on the results of Theorems 16, 17, 18 we formulated Proposition 19, which is applied in Chapter 4 to exclude some classes of 6-dimensional Lie algebras which are not the Lie algebras of the groups $Mult(L)$ of L. These Lie algebras are characterized by one of the following properties:

- they have discrete centre (cf. Propositions 21, 22, 23),
- they are indecomposable and have 2-dimensional centre (cf. Theorem 20),
- they have 4-dimensional non-abelian nilradical (cf. Proposition 21),
- their nilradical is either $\mathbb{R}^5$ or a 5-dimensional indecomposable nilpotent Lie algebra with exception of the Lie algebra $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$ (cf. Proposition 22).

In Chapters 5 and 6 we give the 6-dimensional solvable Lie algebras which are the Lie algebras of the multiplication groups of 3-dimensional topological loops L. Chapters 5 and 6 consist of Lie algebras having 1-dimensional and 2-dimensional centre, respectively.

In Chapter 5 we find that there are seven classes of 6-dimensional solvable indecomposable Lie algebras $\mathbf{g}$ with 5-dimensional nilradical which are the Lie algebras of $Mult(L)$ (cf. Theorem 24). The nilradical of the Lie algebras $\mathbf{g}$ is isomorphic either to $\mathbf{f}_3 \oplus \mathbb{R}^2$ or to $\mathbf{f}_4 \oplus \mathbb{R}$ or to the 5-dimensional indecomposable nilpotent Lie algebra such that its 2-dimensional centre coincides with its commutator ideal. Among the 6-dimensional solvable indecomposable Lie algebras having 4-dimensional nilradical there are three classes which are Lie algebras of the multiplication groups of L. The nilradical of these Lie algebras is $\mathbb{R}^4$. The corresponding simply connected Lie groups G and their subgroups K, which are the inner mapping groups of L, are listed in Theorem 25. In Theorem 26 we give the 18 families of decomposable solvable Lie algebras with 1-dimensional centre which are the Lie algebras of $Mult(L)$. In Theorems 24, 26 we determine also the abelian subalgebras $\mathbf{k}$ of the Lie algebras $\mathbf{g}$ which are the Lie algebras of the inner mapping group $Inn(L)$. In Chapter 5 the centre $Z(L)$ of all 3-dimensional connected simply connected topological loops L is the group $\mathbb{R}$. Moreover, the factor loop $L/Z(L)$ is the group $\mathbb{R}^2$. Hence these loops have nilpotency class 2.

In Chapter 6 all Lie algebras are decomposable solvable Lie algebras (cf. Theorem 20). Among the 6-dimensional Lie algebras there are 9 families which can be realized as the Lie algebra of the group $Mult(L)$ of a 3-dimensional connected simply connected topological proper loop L (cf. Theorems 30, 31). In this case the centre $Z(L)$ of the loop L is the group $\mathbb{R}^2$ and the factor loop $L/Z(L)$ is the group $\mathbb{R}$. Therefore L is centrally nilpotent of class 2.

Hence our main results in the dissertation are the following:

Theorem 1. *Let L be a proper connected simply connected topological loop of dimension 3 having a solvable Lie group as its multiplication group $Mult(L)$.*

(a) Then L is classically solvable. There is a normal subgroup $N \cong \mathbb{R}$ of L. Every normal subgroup $N \cong \mathbb{R}$ of L lies in a 2-dimensional normal subloop M of L. The factor loop L/M is isomorphic to $\mathbb{R}$, whereas the

loops M and L/N are isomorphic either to a 2-dimensional simply connected Lie group or to an elementary filiform loop.
(b) The loop L is congruence solvable if and only if either L has non-discrete centre or L is an abelian extension of a 1-dimensional normal subgroup $N \cong \mathbb{R}$ by the factor loop L/N isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$.

If the multiplication group $Mult(L)$ of an at most 3-dimensional connected topological proper loop L is a solvable Lie group of dimension ≤ 6, then in Chapter 3 we show the following:

Theorem 2. *If L is a connected topological proper loop L of dimension ≤ 3 such that its multiplication group $Mult(L)$ is an at most 6-dimensional solvable Lie group, then L has nilpotency class 2.*

Chapters $4, 5$ and 6 are devoted to classify the solvable Lie groups of dimension ≤ 6 which can be represented as the groups $Mult(L)$ of 3-dimensional topological loops L. Our main results are summarized in the following Theorems. To formulate these results we use the notation in [24], [26], [36], [41].

Theorem 3. *Let L be a connected simply connected topological proper loop of dimension 3 such that the Lie algebra of its multiplication group $Mult(L)$ is a 6-dimensional solvable Lie algebra $\mathbf{g}$ having 1-dimensional centre. Then L is centrally nilpotent of class 2 and for the Lie algebra $\mathbf{g}$ we obtain:*

- *If $\mathbf{g}$ is an indecomposable Lie algebra having 5-dimensional nilradical, then the Lie algebra $\mathbf{g}$ is one of the following:* $\mathbf{g}_1 = \mathbf{g}_{6,14}^{a=0=b}$, $\mathbf{g}_2 = \mathbf{g}_{6,22}^{a=0}$, $\mathbf{g}_3 = \mathbf{g}_{6,17}^{\delta=1,a=0=\varepsilon}$, $\mathbf{g}_4 = \mathbf{g}_{6,51}^{\varepsilon=\pm 1}$, $\mathbf{g}_5 = \mathbf{g}_{6,54}^{a=0=b}$, $\mathbf{g}_6 = \mathbf{g}_{6,63}^{a=0}$, $\mathbf{g}_7 = \mathbf{g}_{6,25}^{a=0=b}$.

- *If $\mathbf{g}$ is an indecomposable Lie algebra with 4-dimensional nilradical, then for the Lie algebra $\mathbf{g}$ we get one of the following:* $\mathbf{g}_1 = N_{6,23}^{a}$, $a \in \mathbb{R}$, $\mathbf{g}_2 = N_{6,22}^{a}$, $a \in \mathbb{R}\backslash\{0\}$, $\mathbf{g}_3 = N_{6,27}$.

- *If $\mathbf{g}$ is a decomposable Lie algebra, then for the Lie algebra $\mathbf{g}$ we have one of the following:* $\mathbf{g}_1 = \mathbb{R} \oplus \mathbf{g}_{5,19}^{\alpha=0,\beta\neq 0}$, $\mathbf{g}_2 = \mathbb{R} \oplus \mathbf{g}_{5,20}^{\alpha=0}$, $\mathbf{g}_3 = \mathbb{R} \oplus \mathbf{g}_{5,27}$, $\mathbf{g}_4 = \mathbb{R} \oplus \mathbf{g}_{5,28}^{\alpha=0}$, $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,32}$, $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,33}$,

$\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,34}$, $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,35}$, $\mathbf{g}_9 = \mathbf{l}_2 \oplus \mathbf{g}_{4,1}$, $\mathbf{g}_{10} = \mathbf{l}_2 \oplus \mathbf{g}_{4,3}$, $\mathbf{g}_{11} = \mathbf{f}_3 \oplus \mathbf{g}_{3,2}$, $\mathbf{g}_{12} = \mathbf{f}_3 \oplus \mathbf{g}_{3,3}$, $\mathbf{g}_{13} = \mathbf{f}_3 \oplus \mathbf{g}_{3,4}$, $\mathbf{g}_{14} = \mathbf{f}_3 \oplus \mathbf{g}_{3,5}^{p>0}$, $\mathbf{g}_{15} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,2}$, $\mathbf{g}_{16} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,3}$, $\mathbf{g}_{17} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,4}$, $\mathbf{g}_{18} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,5}^{p>0}$.

Theorem 4. *Let L be a 3-dimensional connected simply connected topological proper loop having a solvable Lie algebra* $\mathbf{g}$ *of dimension ≤ 6 with 2-dimensional centre as the Lie algebra of the multiplication group $Mult(L)$. Then L is centrally nilpotent of class 2 and the Lie algebra* $\mathbf{g}$ *is one of the following:*

1 *The nilpotent Lie algebras* $\mathbb{R} \oplus \mathbf{f}_4$, $\mathbb{R} \oplus \mathbf{f}_5$.

2 *The solvable and non-nilpotent Lie algebras:* $\mathbf{g}_1 = \mathbb{R}^2 \oplus \mathbf{g}_{4,2}^{\alpha\neq 0}$, $\mathbf{g}_2 = \mathbb{R}^2 \oplus \mathbf{g}_{4,4}$, $\mathbf{g}_3 = \mathbb{R}^2 \oplus \mathbf{g}_{4,5}^{-1\leq\gamma\leq\beta\leq 1,\gamma\beta\neq 0}$, $\mathbf{g}_4 = \mathbb{R}^2 \oplus \mathbf{g}_{4,6}^{p\geq 0,\alpha\neq 0}$, $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,8}^{0<|\gamma|\leq 1}$, $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,10}$, $\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,14}^{p\neq 0}$, $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,15}^{\gamma=0}$.

1 Preliminaries

In this Chapter we collect notions, tools and results, which we use in the later investigation.

A set L equipped with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if for all $x \in L$ the left translation map $\lambda_x : L \to L, \lambda_x(y) = x \cdot y$ as well as the right translation map $\rho_x : L \to L$, $\rho_x(y) = y \cdot x$ are bijections and there is an element $e \in L$ with the property $x = e \cdot x = x \cdot e$. A loop L is proper if it is not associative.

The relation between loops and sharply transitive sections in groups is described in Section 1.2. of [29] in the following way: Denote by G_ℓ the group generated by the left translations of a loop L and by H the stabilizer of $e \in L$ in G_ℓ. The set $\Lambda(L)$ of the left translations of L is a subset of G_ℓ and operates sharply transitively on the left cosets $xH; x \in G_\ell$. The latter property says that for any given left cosets aH, bH there is precisely one left translation λ_z with $\lambda_z aH = bH$.
The core $Co_{G_\ell}(H)$ of the subgroup H in the group G_ℓ is the largest normal subgroup of G_ℓ contained in H. If G_ℓ is a group, H is one of its subgroups with $Co_{G_\ell}(H) = \{1\}$ and $\sigma : G_\ell/H \to G_\ell$ is a section such that
1. the image $\sigma(G_\ell/H)$ is a subset of G_ℓ with $\sigma(H) = 1 \in G_\ell$,

2. the action of $\sigma(G_\ell/H)$ on the factor space G_ℓ/H is sharply transitive,
3. $\sigma(G_\ell/H)$ generates G_ℓ,
then the multiplication on G_ℓ/H given by $xH * yH = \sigma(xH)yH$ defines a loop $L(\sigma)$ having G_ℓ as the group generated by its left translations.
The left, respectively the right division map is defined by $L \times L \to L : (x,y) \mapsto x\backslash y = \lambda_x^{-1}(y)$, respectively $(x,y) \mapsto y/x = \rho_x^{-1}(y)$. Moreover, denote by $\mu_x : L \to L$ the map $\mu_x(y) = y\backslash x$. One has $\mu_x^{-1}(y) = x/y$. The groups $Mult(L) = \langle \lambda_x, \rho_x;\ x \in L\rangle$ and $TMult(L) = \langle \lambda_x, \rho_x, \mu_x;\ x \in L\rangle$ are called the multiplication group and the total multiplication group of L. We denote by $Inn(L)$ and $TInn(L)$ the stabilizer of the identity element $e \in L$ in $Mult(L)$ and in $TMult(L)$, respectively. These subgroups of $Mult(L)$ and $TMult(L)$ are called the inner mapping group and the total inner mapping group of L.

A normal subloop N of L is the kernel of a loop homomorphism $\alpha : (L,\cdot) \to (L',*)$. A word W is a formal product of letters $\lambda_{t(\bar{x})}$, $\rho_{t(\bar{x})}$ and their inverses, where $t(\bar{x}) = t(x_1, \cdots, x_n)$ is a loop term. If we substitute elements u_i of a particular loop L for x_i into a word W and interpret $\lambda_{t(\bar{x})}$, $\rho_{t(\bar{x})}$ as translations of L, then we get an element $W_{\bar{u}}$ of $Mult(L)$. The word W is inner if $W_{\bar{u}}(e) = e$ for each loop L with identity element e and each assignment of elements $u_i \in L$. The notion of tot-inner word is defined analogously allowing $\mu_{t(\bar{x})}$ as generating letters. Let $\mathcal{W}$ be a set of tot-inner words such that each loop L satisfies the property $TInn(L) = \langle W_{\bar{u}} : W \in \mathcal{W}, u_i \in L\rangle$. Let L be a loop and N_1, N_2 be normal subloops of L. The commutator $[N_1, N_2]_L$ is the smallest normal subloop of L containing the set $\{W_{\bar{u}}(a)/W_{\bar{v}}(a) : W \in \mathcal{W}, a \in N_1, u_i, v_i \in L, u_i/v_i \in N_2\}$. For the set $\mathcal{W}$ one can choose the set $\{T_x, U_x, L_{x,y}, R_{x,y}, M_{x,y}\}$ of the tot-inner words $T_x = \rho_x^{-1}\lambda_x$, $U_x = \rho_x^{-1}\mu_x$, $L_{x,y} = \lambda_{xy}^{-1}\lambda_x\lambda_y$, $R_{x,y} = \rho_{yx}^{-1}\rho_x\rho_y$, $M_{x,y} = \mu_{y\backslash x}^{-1}\mu_x\mu_y$ (cf. Theorem 2.1. in [38]).

A normal subloop N of L is said to be central in L, respectively abelian in L, if $[N, L]_L = \{e\}$, respectively $[N, N]_L = \{e\}$. The centre $Z(L)$ of a loop L is the normal subloop of L consisting of all elements $z \in L$ that satisfy the identities $zx = xz$, $zx \cdot y = z \cdot xy$, $x \cdot yz = xy \cdot z$, $xz \cdot y = x \cdot zy$ for all $x, y \in L$. A normal subloop N is central in L precisely if one has $N \leq Z(L)$. The centre $Z(L)$ of L is a commutative normal subgroup of L. A loop L is classically solvable if there is a series $\{e\} = L_0 \leq L_1 \leq \ldots \leq L_n = L$ of subloops of L such that L_{i-1} is normal in L_i and the factor loop L_i/L_{i-1} is an abelian group for all $i = 1, 2, \cdots, n$. A loop L is called congruence solvable, respectively nilpotent, if there exists a chain $\{e\} = L_0 \leq L_1 \leq \ldots \leq L_n = L$ of normal subloops of L such that every factor loop L_i/L_{i-1} is abelian in L/L_{i-1}, respectively central in L/L_{i-1}. Based on the above remark this definition of nilpotence is equivalent to the classical concept of central nilpotence in loop theory. If we put $Z_0 = \{e\}$, $Z_1 = Z(L)$ and $Z_i/Z_{i-1} = Z(L/Z_{i-1})$, then we obtain a series of normal subloops of L. If Z_{n-1} is a proper subloop of L but $Z_n = L$, then we say that L is centrally nilpotent of class n. The centrally nilpotent loops are congruence solvable. If $(A, +, 0)$ is a commutative group, $(F, \cdot, e)$ is a loop and $\varphi, \phi : F \times F \to \text{Aut}(A)$, $\theta : F \times F \to A$ are functions with $\varphi(y, e) = Id = \phi(e, y)$, $\theta(e, y) = 0 = \theta(y, e)$ for every $y \in F$, then on $F \times A$ a loop is defined by

$$(x, a) \oplus (y, b) = (x \cdot y, \varphi(x, y)(a) + \phi(x, y)(b) + \theta(x, y)).$$

This loop has identity element $(e, 0)$ and it is called the abelian extension of A by F determined by the factor system $\Gamma = (\varphi, \phi, \theta)$. We denote it by $L = F \oplus_\Gamma A$. An abelian extension is central if $\varphi(x, y) = \phi(x, y) = Id$ for all $x, y \in F$. A loop L is said to be an iterated abelian, respectively central extension, if it has the form

$$((((A_0 \oplus_{\Gamma_1} A_1) \oplus_{\Gamma_2} A_2) \oplus_{\Gamma_3} \ldots \oplus_{\Gamma_{k-2}} A_{k-2}) \oplus_{\Gamma_{k-1}} A_{k-1}) \oplus_{\Gamma_k} A_k,$$

where A_i, $i = 0, \cdots, k$, are abelian groups and all extensions are abelian, respectively central (cf. Section 5 in [38] and Definition in [23], p. 380).

Corollaries 5.1 and 5.2 in [38], p. 380, prove:

Lemma 5. *A loop L is congruence solvable, respectively centrally nilpotent, precisely if it is an iterated abelian, respectively an iterated central extension.*

We will use very often the following relations between normal subloops N, factor loops L/N of a loop L and their multiplication groups $Mult(N)$, $Mult(L/N)$ in connection with the multiplication group $Mult(L)$ of L (see in [1], Theorems 3, 4 and 5, in [5], IV.1, Lemma 1.3 and in [17], Lemma 2.3).

Lemma 6. *Let L be a loop having $Mult(L)$ as its multiplication group and e as its identity element.*
(i) A homomorphism α of L onto the loop $\alpha(L)$ with kernel N induces a homomorphism of the group $Mult(L)$ onto the group $Mult(\alpha(L))$. The set $M(N) = \{m \in Mult(L);\ xN = m(x)N$ for all $x \in L\}$ forms a normal subgroup of $Mult(L)$ containing the group $Mult(N)$ for the normal subloop N. The factor group $Mult(L)/M(N)$ is isomorphic to the multiplication group $Mult(L/N)$ of the factor loop L/N.
(ii) For each normal subgroup $\mathcal{N}$ of $Mult(L)$ the orbit $\mathcal{N}(e)$ is a normal subloop of L. We have $\mathcal{N} \leq M(\mathcal{N}(e))$.

If G is a group, and K is a subgroup of G, then a system S of representatives for the left cosets $xK, x \in G$, is called a left transversal to K in G. If S, T are two left transversals to K in G, then we say that these are K-connected, if for all $s \in S$ and $t \in T$ the product $s^{-1}t^{-1}st$ lies in K. For a loop L the sets $\Lambda(L) = \{\lambda_a;\ a \in L\}$, $P(L) = \{\rho_a;\ a \in L\}$ are $Inn(L)$-connected left transversals in the group $Mult(L)$. In Theorem 4.1 of [33] the following necessary and sufficient conditions are given for a group G to be the group $Mult(L)$ of a loop L.

Lemma 7. *A group G is isomorphic to the multiplication group of a loop precisely if there is a subgroup K with $Co_G(K) = \{1\}$ and there exist K-connected left transversals S and T such that $G = \langle S, T \rangle$.*

In the later investigation we will often use the following assertion (cf. Proposition 2.7. in [33]).

Lemma 8. *If L is a loop having $Mult(L)$ as its multiplication group and $Inn(L)$ as its inner mapping group, then one has $Co_{Mult(L)}(Inn(L)) = \{1\}$ and the normalizer $N_{Mult(L)}(Inn(L))$ is the direct product $Inn(L) \times Z$, where Z denotes the centre of $Mult(L)$.*

A topological loop is a topological space L such that the three binary operations $(x, y) \mapsto x \cdot y,\ (x, y) \mapsto x \backslash y, (x, y) \mapsto y/x : L \times L \to L$ are continuous. In this case the multiplication group of L is a topological transformation group such that in general it has no natural (finite dimensional) differentiable structure. The condition that the group $Mult(L)$ is a Lie group restricts strongly the isomorphic classes of $Mult(L)$ as well as those of L. In the dissertation we suppose that the group $Mult(L)$ is a Lie group.

In the further considerations the following lemma is often applied.

Lemma 9. *Each connected topological loop has a universal covering loop, which is simply connected. If L is a 3-dimensional connected simply connected topological loop such that the group $Mult(L)$ is a solvable Lie group, then L is homeomorphic to $\mathbb{R}^3$.*

The first assertion is shown in [20], IX.1, whereas the second one is showed in Lemma 3.3 of [10], p. 390.

An elementary filiform Lie group $\mathcal{F}_n$ is a connected simply connected Lie group of dimension $n \geq 3$ such that its Lie algebra $\mathbf{f}_n$ has a basis $\{e_1, \cdots, e_n\}$ with $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$. A 2-dimensional connected simply connected loop $L_{\mathcal{F}}$ is said to be elementary filiform, if its multiplication group is an elementary filiform group $\mathcal{F}_n$ with $n \geq 4$.

A transitive action of a Lie group G on a manifold M is primitive, if on M there does not exist any G-invariant foliation with connected fibres of positive dimension smaller than dim M. A Lie algebra is called indecomposable, if it is not the direct sum of two proper ideals. Otherwise, the Lie algebra is decomposable.

In the next Lemma we summarize the preliminary results if a connected topological loop of dimension 3 has a solvable Lie group as its multiplication group (see Theorem 11 in [1], Lemmata 3.4, 3.5, 3.6 and Propositions 3.7, 3.8 in [10], Theorem 6, Sections 4 and 5 in [11], Propositions 2.6, 2.7 in [17]).

Lemma 10. *Let L be a proper connected simply connected topological loop of dimension 3. Assume that the group $Mult(L)$ of L is a solvable Lie group.*
a) The centre Z of the group $Mult(L)$ is isomorphic to the centre $Z(L) = Z(e)$ of the loop L. Moreover, the centre Z is either discrete or has dimension 1 or 2.
b) If dim$(Z(L)) = 2$ or if dim$(Z(L)) = 1$ and the factor loop $L/Z(L)$ is the group $\mathbb{R}^2$, then L has nilpotency class 2 and the inner mapping group $Inn(L)$ of L is commutative.
c) If dim$(Z(L)) = 2$, then the group $Mult(L)$ is a semidirect product of the group $V \cong \mathbb{R}^m$, $m \geq 3$, by a group $Q \cong \mathbb{R}$. The centre Z of $Mult(L)$

is the group $\mathbb{R}^2$ *and* V *is the direct product* $Z \times Inn(L)$.
d) Each 1*-dimensional connected normal subloop* N *of* L *is the group* $\mathbb{R}$ *and one of the following holds:*
(i) If the factor loop L/N *is isomorphic to* $\mathbb{R}^2$*, then* N *is contained in the centre of* L *and the group* $Mult(L)$ *is a semidirect product of the group* $P \cong \mathbb{R}^m$, $m \geq 2$ *by a group* $Q \cong \mathbb{R}^2$ *such that* $P = C \times Inn(L)$*, where* $\mathbb{R} = C \cong N$ *is a central subgroup of* $Mult(L)$.
(ii) If the factor loop L/N *is isomorphic either to the group* $\mathcal{L}_2$ *or to a loop* $L_{\mathcal{F}}$*, then the group* $Mult(L)$ *has a normal subgroup* $\mathcal{S}$ *containing* $Mult(N) \cong \mathbb{R}$ *so that the factor group* $Mult(L)/\mathcal{S}$ *is isomorphic to the direct product* $\mathcal{L}_2 \times \mathcal{L}_2$*, if* $L/N \cong \mathcal{L}_2$*, or to a Lie group* $\mathcal{F}_n$*,* $n \geq 4$*, if* $L/N \cong L_{\mathcal{F}}$.

Lemma 11. *Each elementary filiform loop* $L_{\mathcal{F}}$ *has nilpotency class* 2.

The proof of this Lemma can be found in [9], p. 420.

2 Classical solvable, congruence solvable topological loops

In this Chapter we prove the following theorems:

Theorem 12. *If* L *is a* 3*-dimensional connected simply connected topological loop such that its multiplication group is a solvable Lie group, then* L *is classically solvable. The loop* L *has a* 1*-dimensional normal subgroup* N *isomorphic to* $\mathbb{R}$*. For each* 1*-dimensional normal subgroup* N *there exists a normal series* $\{e\} = L_0 \leq N = L_1 \leq M = L_2 \leq L = L_3$ *of* L *such that the loops* M *and* L/N *are isomorphic either to a* 2*-dimensional simply connected Lie group or to a loop* $L_{\mathcal{F}}$ *and the factor loop* L/M *is the group* $\mathbb{R}$.

Theorem 13. *Let* L *be a* 3*-dimensional connected simply connected topological loop with a solvable Lie multiplication group. The loop* L *is congruence solvable if and only if* L *has one of the following properties:*

- *the centre of* L *has dimension* 1 *or* 2*,*

- *L has discrete centre and is an abelian extension of a normal subgroup $N \cong \mathbb{R}$ by the factor loop L/N isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$.*

Proof of Theorem 12. According to Lemma 9 the loop L is homeomorphic to $\mathbb{R}^3$. Since the group $Mult(L)$ is solvable it has a connected normal subgroup N of dimension 1 or 2. The orbit $N(e)$ is a connected normal subloop of L such that $N(e) \neq \{e\}$ (see Lemmata 6 and 8). Therefore one has $\dim(N(e)) = 1$ or 2. The action of the group $Mult(L)$ on the topological space L is transitive, effective and imprimitive. According to [21], p. 141, the Lie groups G acting imprimitively on $\mathbb{R}^3$ form three classes with respect to their actions:
I. In $\mathbb{R}^3$ there exists a G-invariant foliation with 2-dimensional connected fibres D, but there is no G-invariant foliation of D with 1-dimensional connected fibres.
II. In $\mathbb{R}^3$ there is a G-invariant foliation with 1-dimensional connected fibres F, but there does not exist any G-invariant foliation in $\mathbb{R}^3$ with 2-dimensional fibres D which are unions of fibres F.
III. In $\mathbb{R}^3$ there exists a G-invariant foliation with 1-dimensional connected fibres F and there is a G-invariant foliation with 2-dimensional fibres D which are unions of fibres F.

Suppose that the group $Mult(L)$ belongs to the I. class. Then the loop L has a 2-dimensional connected normal subloop M such that M does not have any one dimensional connected normal subloop. Since the multiplication group of M is a Lie group too, M is either a 2-dimensional Lie group or an elementary filiform loop (see [9], p. 420). All these loops have a 1-dimensional normal subloop, which is a contradiction. Hence $Mult(L)$ is not in the I. class.

Assume that the group $Mult(L)$ belongs to the II. class. In this case L has a 1-dimensional connected normal subloop N but there is no 2-dimensional connected normal subloop M of L which contains N. Hence the Lie groups in the II. class act primitively on $\mathbb{R}^2$. Among the Lie algebras acting locally primitively on $\mathbb{R}^2$ the Lie algebras $\mathbf{g}_1 = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \alpha(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\rangle, \alpha \geq 0$, and $\mathbf{g}_2 = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\rangle$ are solvable (see [19], p. 341, also [21], Theorem 34, p. 378). Hence the Lie algebra $\mathbf{mult}(\mathbf{L})$ of $Mult(L)$ is either isomorphic to one of the Lie algebras $\mathbf{g}_i$, $i = 1, 2$, or it has a proper subalgebra isomorphic to $\mathbf{g}_i$, $i = 1, 2$.

The first case is impossible because the Lie algebras $\mathbf{g}_i$, $i = 1, 2$, are not the Lie algebras of the multiplication groups of 3-dimensional topological loops (see Section 4 in [11]). In the second case we get $\mathbf{mult}(\mathbf{L}) = \langle X_1 + \phi_1(x, y, z)\frac{\partial}{\partial z}, \dots, X_k + \phi_k(x, y, z)\frac{\partial}{\partial z}, F_1(x, y, z)\frac{\partial}{\partial z}, \dots, F_{n-k}(x, y, z)\frac{\partial}{\partial z}\rangle$, where $X_1, \dots, X_k$ are the basis elements of $\mathbf{g}_i$, $i = 1, 2$, according to whether $\mathbf{g}_i$ is the subalgebra of $\mathbf{mult}(\mathbf{L})$. Moreover, the $n-k$-dimensional Lie subgroup A of $Mult(L)$ belonging to the subalgebra $\mathbf{a} = \langle F_i(x, y, z)\frac{\partial}{\partial z}\rangle$, $i = 1, \dots, n-k$, leaves every 1-dimensional connected left coset xN, $x \in L$, invariant (see [21], p. 155). Hence the subgroup A is the normal subgroup $M(N)$ of $Mult(L)$ in Lemma 6 (i) and the multiplication group $Mult(L/N)$ of the factor loop L/N is isomorphic to the Lie group $Mult(L)/A$. Therefore the 2-dimensional connected topological loop L/N is isomorphic either to a 2-dimensional Lie group or to a loop $L_{\mathcal{F}}$ (cf. Lemma 10 d). The factor Lie algebra $\mathbf{mult}(\mathbf{L})/\mathbf{a}$ is isomorphic to $\mathbf{g}_i$, $i = 1$ or 2. But none of the Lie algebras $\mathbf{g}_i$, $i = 1, 2$, is the Lie algebra of the group $Mult(L)$ of a 2-dimensional topological loop (see Theorem 1 in [9]).

This contradiction gives that the group $Mult(L)$ belongs to the III. class. In this case the loop L has a 2-dimensional connected normal subloop M containing a 1-dimensional connected normal subloop N of L. Moreover, every 1-dimensional normal subloop of L lies in a 2-dimensional normal subloop of L because $Mult(L)$ is not in the II. class. According to Lemma 10 d) the loop N is isomorphic to the group $\mathbb{R}$ and every orbit of N is homeomorphic to $\mathbb{R}$. By Theorem 18.18 in [29] the factor loop L/M is isomorphic either to the Lie group $\mathbb{R}$ or to $SO_2(\mathbb{R})$. The loops M and L/N have dimension 2 and their multiplication groups are Lie groups too (see Lemma 6). Hence the topological spaces M and L/N are homeomorphic either to $\mathbb{R}^2$ or to $S^1 \times \mathbb{R}$ or to $S^1 \times S^1$ (cf. Theorem 19.1 in [29]). The manifold L is a fibering of $\mathbb{R}^3$ over L/N with fibers homeomorphic to N and it is also a fibering of $\mathbb{R}^3$ over L/M with fibers homeomorphic to M. Therefore the first fundamental group $\pi_1(\mathbb{R}^3)$ is isomorphic to the sum $\pi_1(L/N) + \pi_1(N)$ and also to the sum $\pi_1(L/M) + \pi_1(M)$. Since one has $\pi_1(\mathbb{R}^n) = 0$, $\pi_1(S^1) = \mathbb{Z}$ and N is homeomorphic to $\mathbb{R}$ the loops M and L/N are homeomorphic to $\mathbb{R}^2$, and the loop L/M is homeomorphic to $\mathbb{R}$. Each 2-dimensional topological loop which is homeomorphic to $\mathbb{R}^2$ and having a Lie group as its multiplication group is isomorphic either to a loop $L_{\mathcal{F}}$ or to one of the Lie groups $\{\mathbb{R}^2, \mathcal{L}_2\}$ (see Theorem 1 in [9]). This proves the assertion. $\square$

Proof of Theorem 13. According to Lemma 5 the loop L is congruence solvable precisely if it is obtained by iterated abelian extensions. By Lemma 10 a) the centre $Z(L)$ of L is either discrete or has dimension 1 or 2. If $\dim(Z(L)) = 2$, then L has nilpotency class 2 (see Lemma 10 b) and therefore it is congruence solvable. If $\dim(Z(L)) = 1$, then L is a central extension of the group $Z(L) \cong \mathbb{R}$ by a loop isomorphic to the factor loop $L/Z(L)$. Since every central extension is an abelian extension, the loop L is an abelian extension of $Z(L)$ by $L/Z(L)$. According to Theorem 12 the factor loop $L/Z(L)$ is isomorphic either to $\mathbb{R}^2$ or to $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$. Taking into account Lemma 11 and the fact that $\mathcal{L}_2$ is solvable, the factor loop $L/Z(L)$ is an abelian extension of the group $\mathbb{R}$ by $\mathbb{R}$ (see Lemmata 10, 11 in [23], p. 380-381). Hence L is an iterated abelian extension. Finally we consider the case that the group L has discrete centre. According to Theorem 12 and Lemma 10 d) (ii) the loop L has a normal subgroup $N \cong \mathbb{R}$ such that the factor loop L/N is isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$. As the factor loop L/N is an abelian extension, the loop L is an iterated abelian extension if and only if L is an abelian extension of N by L/N. This proves the assertions. $\square$

Schreier's extensions defined in [30], p. 761, of the group $\mathbb{R}$ by the group $\mathcal{L}_2$ or by a loop $L_{\mathcal{F}}$ are abelian extensions. Hence these constructions result into congruence solvable loops. The following construction for topological loops yields non-abelian extensions.

Example 1. *Let $(Q, \cdot, 1)$ be a topological loop of dimension n having a normal subloop Q_1 such that the factor loop Q/Q_1 is isomorphic to the group $\mathbb{R}$. Let $\phi : (Q, \cdot) \to (\mathbb{R}, +)$ be a homomorphism. We consider a one-parameter family of loops $\Gamma_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(a, b) \mapsto \Gamma_t(a, b) = a *_t b$, $t \in \mathbb{R}$, such that $\Gamma_0(a, b) = a + b$ and Γ_t is not commutative for some $t \in \mathbb{R}$. Suppose that for all $t \in \mathbb{R}$ the loops Γ_t have the same identity element 0. We denote by $\Delta_t(a, b) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the right division map $(a, b) \mapsto \Delta_t(a, b) = a/_t b$, $t \in \mathbb{R}$, of the loop Γ_t. For the loops Γ_t, $t \neq 0$, we can take loops defined by the sharply transitive section $\sigma_t : PSL_2(\mathbb{R})/\mathcal{L}_2 \to PSL_2(\mathbb{R})$ determined by the functions $f(u) = \exp[\frac{1}{6} \sin^2 t \cos u (\cos u - 1)]$, $g(u) =$*

$(f(u)^{-1} - f(u)) \cot u$ *(see Proposition 18.15 and its proof in [29], pp. 244-245). All loops* Γ_t, $t \neq 0$, *are proper and hence they are not commutative (cf. Corollary 18.19. in [29], p. 248). The multiplication*

$$(x, a) \circ (y, b) = (x \cdot y, \Gamma_{\phi(x \cdot y)}(a, b))$$

on $Q \times \mathbb{R}$ *defines a loop* L_ϕ *which is an extension of the group* $\mathbb{R}$ *by the loop* Q. *The loop* L_ϕ *has the identity element* $(1, 0)$ *since one has* $(1, 0) \circ (y, b) = (y, \Gamma_{\phi(y)}(0, b)) = (y, b) = (y, b) \circ (1, 0)$. *Hence the loop* L_ϕ *is an Albert extension of the group* $\mathbb{R}$ *by the loop* $(Q, \cdot)$ *given by the one-parameter family* Γ_t *of the loop multiplications on* $\mathbb{R}$ *(see [28], p. 4). Let* $x \in Q$ *with* $\phi(x) \neq 0$. *We obtain* $T(x, a)(1, c) = ((x, a) \circ (1, c))/(x, a) = (x, \Gamma_{\phi(x)}(a, c))/(x, a) = (1, \Delta_{\phi(x)}(\Gamma_{\phi(x)}(a, c), a))$, *which is not independent of* $a \in \mathbb{R}$ *because the loop* $\Gamma_{\phi(x)}$ *is not commutative. Hence the normal subgroup* $\mathbb{R}$ *is not abelian in the loop* L_ϕ *(see Proof of Theorem 4.1 in [38], p. 377). In particular if the loop* $(Q, \cdot)$ *is the group* $\mathcal{L}_2$ *or a loop* $L_{\mathcal{F}}$, *then this construction yields a* 3*-dimensional connected topological loop, which is a non-abelian extension of the group* $\mathbb{R}$ *by the loop* $(Q, \cdot)$.

Note 14. *We are very thankful to Péter T. Nagy for the construction in Example 1.*

3 Topological loops with solvable Lie multiplication groups of dimension at most 6 are centrally nilpotent

From now on we restrict us for those solvable Lie groups which have dimension at most 6. The reason for this restriction is that the classification of the corresponding Lie algebras is complete (cf. [25], [36], [41]). Using this restriction we show:

Theorem 15. *If* L *is a connected topological proper loop of dimension* ≤ 3 *such that its multiplication group* $Mult(L)$ *is an at most* 6*-dimensional solvable Lie group, then* L *has nilpotency class* 2.

For the following cases this theorem is true: The multiplication group of every proper 1-dimensional loop has infinite dimension (cf. [29], Theorem

18.18, p. 248). By Theorem 1 in [9], p. 420, each 2-dimensional connected topological proper loop having a Lie group as its multiplication group is centrally nilpotent of class 2. Every 3-dimensional connected topological proper loop L such that the group $Mult(L)$ is an at most 5-dimensional solvable non-nilpotent Lie group has nilpotency class 2 (see Proposition 17, Theorem 18 in [11]). In Theorem of [17] we proved that all 3-dimensional connected topological proper loops which have indecomposable nilpotent Lie groups of dimension ≤ 6 as their multiplication groups are centrally nilpotent of class 2. To achieve the assertion of Theorem 15 it remains to investigate the classes of solvable non-nilpotent Lie groups of dimension 6 and the decomposable nilpotent Lie groups of dimension at most 6. Hence from now on we deal with these classes of Lie groups. According to Lemma 10 a) the centre Z of $Mult(L)$ has either dimension 1 or 2 or it is discrete. In Chapter 4 we prove that the centre Z of $Mult(L)$ cannot be discrete. Theorems 24, 25, 26 deal with the case that $\dim(Z) = 1$. In this case the loop L has an upper central series $\{e\} < Z(L) \cong \mathbb{R} < L$ with $L/Z(L) \cong \mathbb{R}^2$. Hence L has nilpotency class 2. In Theorems 30, 31 we consider the case that $\dim(Z) = 2$. The corresponding loops L have an upper central series $\{e\} < Z(L) \cong \mathbb{R}^2 < L$ with $L/Z(L) \cong \mathbb{R}$. Therefore the loops L have nilpotency class 2. This proves Theorem 15.

Firstly, Theorem 16 describes the structure of the 3-dimensional connected simply connected topological loops and their multiplication groups $Mult(L)$, if $Mult(L)$ has discrete centre.

Theorem 16. *Let L be a proper connected simply connected topological loop of dimension 3 having a solvable Lie group with discrete centre as its multiplication group $Mult(L)$. The loop L is classically solvable. It has a connected normal subgroup N isomorphic to $\mathbb{R}$ and the factor loop L/N is isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$. The dimension of the group $Mult(L)$ is ≥ 6 and the group $Mult(L)$ has a normal subgroup $\mathcal{S}$ containing $Mult(N) \cong \mathbb{R}$ such that the factor group $Mult(L)/\mathcal{S}$ is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$, if $L/N \cong \mathcal{L}_2$, or to a group $\mathcal{F}_n$, $n \geq 4$, if $L/N \cong L_{\mathcal{F}}$. For each normal subgroup N of L the loop L has a normal subloop M isomorphic either to $\mathbb{R}^2$ or to $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$ such that $N < M$ and L/M is isomorphic to $\mathbb{R}$. The group $Mult(L)$ contains a normal subgroup V such that $Mult(L)/V \cong \mathbb{R}$ and the orbit $V(e)$ is the loop M. The inner mapping group $Inn(L)$ of L, the multiplication group*

$Mult(M)$ of M and the commutator subgroup of $Mult(L)$ are subgroups of V. The normalizer $N_{Mult(L)}(Inn(L))$ is $Inn(L)$.

Proof. By Theorem 12 there exists a normal subgroup N of L isomorphic to $\mathbb{R}$ and there is a 2-dimensional normal subloop M of L containing N. As the group $Mult(L)$ has discrete centre, the factor loop L/N is not isomorphic to $\mathbb{R}^2$ (cf. Lemma 10 d (i)). Hence it is isomorphic either to the Lie group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$ and the group $Mult(L)$ has a normal subgroup S as in the assertion (cf. Lemma 10 d (ii)). Since none of the at most 5-dimensional solvable Lie groups with discrete centre is isomorphic to the multiplication group of a topological loop homeomorphic to $\mathbb{R}^3$ (cf. [11]) we have $\dim(Mult(L)) \geq 6$. Since L/M is isomorphic to $\mathbb{R}$ there is a normal subgroup $V = \{v \in Mult(L); xM = v(x)M \text{ for all } x \in L\} < Mult(L)$ such that $V(e) = M$. By Lemma 6 the subgroup V contains the multiplication group $Mult(M)$ of M and $Mult(L/M) \cong Mult(L)/V \cong \mathbb{R}$. The latter property yields that V contains the commutator subgroup of $Mult(L)$. Since the group $Mult(L)/V$ acts sharply transitively on the orbits of M in L the inner mapping group $Inn(L)$ is a subgroup of V. By Lemma 8 we obtain that $N_{Mult(L)}(Inn(L)) = Inn(L)$. □

The 3-dimensional connected simply connected topological loops and their multiplication groups $Mult(L)$, if $Mult(L)$ has a 1-dimensional centre, are characterized by Theorem 17.

Theorem 17. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group $Mult(L)$ is a solvable Lie group with 1-dimensional centre Z. Then L is congruence solvable. The orbit $K(e)$, where K is a 1-dimensional connected normal subgroup of $Mult(L)$, is a normal subgroup of L isomorphic to $\mathbb{R}$. Moreover, one of the following possibilities holds:*
(a) If the factor loop $L/K(e)$ is isomorphic to $\mathbb{R}^2$, then L has nilpotency class 2. The orbit $K(e)$ coincides with the centre $Z(L)$ of L. The connected simply connected group $Mult(L)$ is a semidirect product of the abelian normal subgroup $P = Z \times Inn(L)$ by a group $Q \cong \mathbb{R}^2$ and the orbit $P(e)$ is $Z(L)$.
(b) If the factor loop $L/K(e)$ is isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$, then $Mult(L)$ has a normal subgroup $\mathcal{S}$ containing K such that the orbits $\mathcal{S}(e)$, $K(e)$ coincide. The factor group $Mult(L)/\mathcal{S}$ is isomorphic

to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$, if $L/K(e) \cong \mathcal{L}_2$, or to a Lie group $\mathcal{F}_n$, $n \geq 4$, if $L/K(e) \cong L_{\mathcal{F}}$.
The loop L contains a 2-dimensional normal subloop M with $K(e) < M$ and the group $Mult(L)$ has a normal subgroup V as in Theorem 16.
In particular, if $K(e) = Z(L)$ and $L/Z(L)$ is isomorphic to a loop $L_{\mathcal{F}}$, then L is centrally nilpotent.

Proof. The first assertion follows from Theorem 13. Denote by K a 1-dimensional connected normal subgroup of the group $Mult(L)$. By Lemma 6 (ii) the orbit $K(e)$ is a normal subloop of L isomorphic to $\mathbb{R}$ because $Mult(K(e))$ is a Lie subgroup of $Mult(L)$. By Lemmata 6, 8 the normal subloop $K(e)$ is different from $\{e\}$. The factor loop $L/K(e)$ is a 2-dimensional connected topological loop and the group $Mult(L/K(e))$ is a factor group of $Mult(L)$ (see Lemma 6 (i)). Applying Lemma 10 d) for the case $N = K(e)$ assertions (a) and (b) are showed. According to Theorem 12 there exists a normal subloop M of L containing $K(e)$ and the assertions about the group V follow from the proof of Theorem 16. If $K(e) = Z(L)$ and $L/Z(L)$ is isomorphic to the group $\mathbb{R}^2$, then the loop L is centrally nilpotent of class 2 (see case (a)). Using Lemma 11, if $K(e) = Z(L)$ and $L/Z(L)$ is isomorphic to a loop $L_{\mathcal{F}}$, then L has nilpotency class 3. This proves the last assertion. □

The next theorem deals with the case that the centre of the multiplication group $Mult(L)$ of a 3-dimensional connected simply connected topological loop has dimension 2.

Theorem 18. *If L is a proper connected simply connected topological loop of dimension 3 having a solvable Lie group with 2-dimensional centre Z as its multiplication group $Mult(L)$, then L has nilpotency class 2. The group $Mult(L)$ is a semidirect product of the normal subgroup $V = Z \times Inn(L) \cong \mathbb{R}^{m-1}$ by a group $Q \cong \mathbb{R}$, where $\mathbb{R}^2 = Z \cong Z(L)$ and $m = dim(Mult(L))$. For every 1-dimensional connected subgroup N of Z the orbit $N(e)$ is a connected central subgroup of L and the factor loop $L/N(e)$ is isomorphic either to $\mathbb{R}^2$ or to a loop $L_{\mathcal{F}}$. In particular, if the group $Mult(L)$ is indecomposable, then one has $L/N(e) \cong L_{\mathcal{F}}$. If $L/N(e) \cong \mathbb{R}^2$, then Theorem 17 (a) holds. If $L/N(e) \cong L_{\mathcal{F}}$, then the group $Mult(L)$ contains a normal subgroup $\mathcal{S}$ with $N < \mathcal{S}$. The factor group $Mult(L)/\mathcal{S}$ is isomorphic to a Lie group $\mathcal{F}_n$ with $n \geq 4$.*

Proof. According to Lemma 10 a), b), c) the loop L has nilpotency class 2. Hence the group $Mult(L)$ is a semidirect product as in the assertion. As $N < Z$ the orbit $N(e)$ is a 1-dimensional central subgroup of L. The multiplication group of the 2-dimensional connected simply connected factor loop $L/N(e)$ is a factor group of $Mult(L)$. If the loop $L/N(e)$ is isomorphic to $\mathbb{R}^2$, then by Lemma 10 d) (i) the group $Mult(L)$ satisfies Theorem 17 (a). If $L/N(e)$ would be isomorphic to $\mathcal{L}_2$, then by Lemma 10 d) (ii) the group $Mult(L)$ would have a proper factor group isomorphic to $\mathcal{L}_2 \times \mathcal{L}_2$. A semidirect product $V \rtimes Q$, where V is an abelian normal subgroup of codimension 1 does not have such factor group. Hence this case is excluded. If $L/N(e)$ is isomorphic to a loop $L_{\mathcal{F}}$, then the remaining part of the assertion follows from Lemma 10 d) (ii). This case happens if the group $Mult(L)$ is indecomposable (cf. Proposition 2.6 in [17]). □

Our next aim is to determine the 6-dimensional solvable Lie groups which are multiplication groups of 3-dimensional connected simply connected topological loops.

Procedure of the classification:

1. step: For each 6-dimensional solvable Lie algebra g we have to find a suitable linear representation of the corresponding connected simply connected Lie group G.

2. step: As $dim(L) = 3$ we determine those 3-dimensional Lie subgroups K of G which have no non-trivial normal subgroup of G and satisfy the condition that the normalizer $N_G(K)$ is the direct product $K \times Z$, where Z is the centre of G (cf. Lemma 8).

3. step: We have to find left transversals S and T to K in G such that for all $s \in S$ and $t \in T$ one has $s^{-1}t^{-1}st \in K$ and G is generated by $S \cup T$ (cf. Lemma 7).

3.1. Since the transversals S and T are continuous, they are determined by 3 continuous real functions of 3 variables. The condition that the products $s^{-1}t^{-1}st$, $s \in S$ and $t \in T$, are in K is formulated by functional equations. Solving these functional equations we obtain the possible forms of the left transversals S and T. The left transversals S and T are the set $\Lambda(L)$ of all left translations and the set $P(L)$ of all right translations of L, respectively. These sets play an important role for the construction of the loop multiplication using the group G_ℓ, respectively G_r (cf. [29], p. 17-18).

3.2. We check whether the set $S \cup T$ generates the group G. If this is the case, then G is the multiplication group $Mult(L)$ of a loop L and K is the inner mapping group of L.

We use the following proposition to exclude those 6-dimensional solvable Lie algebras which are not the Lie algebras of the groups $Mult(L)$ of 3-dimensional topological loops L.

Proposition 19. *Suppose L is a proper connected simply connected topological loop of dimension 3 such that the Lie algebra of its multiplication group is a 6-dimensional solvable Lie algebra $\mathbf{g}$.*
a) For all 1-dimensional ideals $\mathbf{i}$ of $\mathbf{g}$ the orbits $I(e)$, where I is the simply connected Lie group of $\mathbf{i}$, are normal subgroups of L isomorphic to $\mathbb{R}$. We have one of the following possibilities:
(i) The factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$. Then $\mathbf{g}$ contains the ideal $\mathbf{p} = \mathbf{c} \oplus \mathbf{inn(L)} \cong \mathbb{R}^4$ such that the commutator ideal $\mathbf{g}'$ of $\mathbf{g}$ lies in $\mathbf{p}$ and $\mathbf{c}$ is a 1-dimensional subalgebra of the centre $\mathbf{z}$ of $\mathbf{g}$.
(ii) The factor loop $L/I(e)$ is isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$. Then $\mathbf{g}$ has an ideal $\mathbf{s}$ such that $\mathbf{i} \le \mathbf{s}$ and the factor Lie algebra $\mathbf{g}/\mathbf{s}$ is isomorphic either to $\mathbf{l}_2 \oplus \mathbf{l}_2$ or to a Lie algebra $\mathbf{f}_n$, $n = 4, 5$.
b) If $\mathbf{a}$ is an ideal of $\mathbf{g}$ such that dim$(\mathbf{a}) = 2$, $\mathbf{a} \le \mathbf{g}'$ and the factor Lie algebra $\mathbf{g}/\mathbf{a}$ is isomorphic neither to $\mathbf{l}_2 \oplus \mathbf{l}_2$ nor to $\mathbf{f}_4$, then the orbit $A(e)$, where A is the simply connected Lie group of $\mathbf{a}$, is either a 2-dimensional connected normal subloop M of L or the factor loop $L/A(e)$ is isomorphic to $\mathbb{R}^2$.
(iii) Assume $A(e) = M$. Then there exists a 5-dimensional ideal $\mathbf{v}$ of $\mathbf{g}$ such that the Lie algebra $\mathbf{inn(L)}$, the Lie algebra $\mathbf{mult}(M)$ and the ideal $\mathbf{g}'$ are subalgebras of $\mathbf{v}$. Moreover, for all ideals $\mathbf{b}$ of $\mathbf{g}$ with dim$(\mathbf{b}) \ge 3$ and $\mathbf{a} < \mathbf{b} \le \mathbf{g}'$ the orbit $B(e)$, where B is the simply connected Lie group of $\mathbf{b}$, coincides with M. One has $\mathbf{a} \cap \mathbf{inn(L)} = \{0\}$ and the intersection $\mathbf{b} \cap \mathbf{inn(L)}$ has dimension dim$(\mathbf{b}) - 2$.
(iv) If the factor loop $L/A(e)$ is isomorphic to $\mathbb{R}^2$, then we have case (i).
c) If the Lie algebra $\mathbf{g}$ is indecomposable, then its centre $\mathbf{z}$ has dimension ≤ 1, the subalgebra $\mathbf{c}$ in case a) (i) coincides with $\mathbf{z}$ and the ideal $\mathbf{p}$ lies in the nilradical $\mathbf{n}_{rad}$.
d) If dim$(\mathbf{n}_{rad}) = 4$, then the ideal $\mathbf{p}$ equals to $\mathbf{n}_{rad}$. Moreover, if $\mathbf{n}_{rad}$ is not commutative or the centre $\mathbf{z}$ of $\mathbf{g}$ is trivial, then for each 2-dimensional abelian ideal $\mathbf{a}$ of $\mathbf{g}$ such that the factor Lie algebra $\mathbf{g}/\mathbf{a}$ is isomorphic neither to $\mathbf{l}_2 \oplus \mathbf{l}_2$ nor to $\mathbf{f}_4$ and for each nilpotent ideal $\mathbf{s}$ of $\mathbf{g}$ having dimension

> 2 *the orbits* $A(e)$, $\mathcal{S}(e)$, *where* A, $\mathcal{S}$ *are the simply connected Lie groups of* $\mathbf{a}$, $\mathbf{s}$, *respectively, are the same* 2*-dimensional normal subloop* M *of* L. *There is a* 5*-dimensional ideal* $\mathbf{v}$ *of* $\mathbf{g}$ *with the same properties as in case b) (iii). If* $\mathbf{g}$ *differs from the Lie algebra* $N_{6,28}$ *in Table III in [41], p. 1349, then the loop* M *is isomorphic to* $\mathbb{R}^2$.
e) If dim$(\mathbf{n}_{rad}) = 5$, *then the factor loop* $L/I(e)$ *in case a) is not isomorphic to the group* $\mathcal{L}_2$.

Proof. The simply connected Lie group $I = \exp(\mathbf{i})$ of the ideal $\mathbf{i}$ of $\mathbf{g}$ is a 1-dimensional connected normal subgroup of the multiplication group G. By Lemmata 6 and 10 d) the orbit $I(e)$ is a 1-dimensional normal subgroup N of L isomorphic to $\mathbb{R}$ and the assertions (i) and (ii) in case a) follow from Lemma 10 d).

Since $\dim(\mathbf{a}) = 2$, the orbit $A(e)$ is a normal subloop of L having dimension 1 or 2 (cf. Lemma 6).

If $\dim(A(e)) = 2$, then one has $A(e) = M$. If $\dim(A(e)) = 1$, then we obtain that the factor loop $L/A(e) \cong \mathbb{R}^2$ because the Lie algebra $\mathbf{g}/\mathbf{a}$ is not isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ or to a Lie algebra $\mathbf{f}_n$, $n = 4, 5$.

Since L/M is isomorphic to $\mathbb{R}$, the Lie algebra of the normal subgroup V of the multiplication group of L in Theorem 16 is the ideal $\mathbf{v}$ in case (iii).

Let $\mathbf{b}$ be an ideal of $\mathbf{g}$ such that $\dim(\mathbf{b}) \geq 3$, $\mathbf{a} < \mathbf{b} \leq \mathbf{g}'$. The orbit $B(e)$ contains the loop $M = A(e)$. Hence $\dim(B(e)) = 2$ or 3. Since $\mathbf{a} < \mathbf{b} \leq \mathbf{g}' \leq \mathbf{v}$ and the orbit $V(e)$ has dimension 2 we obtain that the orbits $A(e)$, $V(e)$ and hence $G'(e)$ and $B(e)$ coincide with M.

As $\dim(\mathbf{a}) = 2$, the simply connected Lie group A of $\mathbf{a}$ has dimension 2 and acts sharply transitively on the 2-dimensional orbit $A(e)$ homeomorphic to $\mathbb{R}^2$. Hence one has $A \cap Inn(L) = \{1\}$. Since $\dim(\mathbf{b}) \geq 3$ and $\dim(B(e)) = 2$, there is a subgroup of B of dimension $\dim(\mathbf{b}) - 2$, which fixes the identity element e of L. This proves assertion (iii).

If $A(e)$ is isomorphic to $\mathbb{R}$ and the factor loop $L/A(e)$ is isomorphic to $\mathbb{R}^2$, then we have case (i). This proves assertion (iv).

If $\mathbf{g}$ is indecomposable, then by Theorem 20 its centre $\mathbf{z}$ has dimension ≤ 1. Therefore we have in case a) (i) that $\mathbf{c} = \mathbf{z}$. Since the abelian ideal $\mathbf{p}$ is nilpotent, it lies in the nilradical of $\mathbf{g}$. This proves c).

If the indecomposable Lie algebra $\mathbf{g}$ has a 4-dimensional nilradical $\mathbf{n}_{\text{rad}}$, then in one has $\mathbf{p} = \mathbf{n}_{\text{rad}}$ in case a) (i). If the nilradical $\mathbf{n}_{\text{rad}}$ is not abelian or $\mathbf{g}$ has trivial centre, then the commutator Lie algebra $\mathbf{g}'$ coincides with the nilradical $\mathbf{n}_{\text{rad}}$ (see Tables I, III, IV, V of [41], pp. 1347-1350). Let $\mathbf{a}$ be a 2-dimensional abelian ideal of $\mathbf{g}$. According to Lemma 6 the orbit $A(e)$ is a normal subloop of L of dimension 2. Therefore we have $A(e) = M$. An analogous argument as in the proof of case b) (iii) shows that the orbit $S(e)$, where S is the simply connected Lie group of a nilpotent ideal $\mathbf{s}$ of $\mathbf{g}$ having dimension > 2, equals to $A(e) = M$. Hence case b) (iii) is satisfied. Since $\mathbf{n}_{\text{rad}} < \mathbf{v}$ and $\dim(\mathbf{v}) = 5$, the intersection of $\mathbf{v}$ with the complement of $\mathbf{n}_{\text{rad}}$ in $\mathbf{g}$ has dimension 1. Therefore $\mathbf{v}$ does not contain a subalgebra isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. The nilradical of the Lie algebras $\mathbf{g}$ which are different from $N_{6,28}$ is not the Lie algebra $\mathbf{f}_4$. Hence one has $M = V(e) = \mathbb{R}^2$.

If the indecomposable Lie algebra $\mathbf{g}$ has a 5-dimensional nilradical $\mathbf{n}_{\text{rad}}$, then we obtain $\mathbf{p} < \mathbf{n}_{\text{rad}}$. Moreover, these Lie algebras have only one non-nilpotent basis element (cf. [25]). Hence they have no subalgebra and no factor Lie algebra isomorphic to the direct sum $\mathbf{l_2} \oplus \mathbf{l_2}$. This fact and Theorems 16 and 17 yield the assertion e). Hence the proposition is proved. $\square$

4 6-dimensional solvable Lie groups which are not multiplication groups of 3-dimensional topological loops

In this Chapter, we focus our attention to the classes of the following 6-dimensional solvable Lie groups:

- Indecomposable solvable Lie groups with 2-dimensional centre.
- Indecomposable solvable Lie groups such that their Lie algebras have one of the following nilradicals: a 4-dimensional non-abelian nilpotent Lie algebra, $\mathbb{R}^5$, a 5-dimensional indecomposable nilpotent Lie algebra with exception of the Lie algebra $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$.
- Solvable Lie groups with discrete centre.

We prove that the Lie algebras of the above listed Lie groups are not the Lie algebras of the multiplication groups of 3-dimensional topological loops. Firstly in Theorem 20, we state that the at most 6-dimensional indecomposable solvable Lie algebras with 2-dimensional centre are not the Lie algebras of the groups $Mult(L)$ of 3-dimensional topological loops L.

Theorem 20. *There does not exist any 3-dimensional proper connected topological loop L having an at most 6-dimensional indecomposable solvable Lie group with 2-dimensional centre as the group $Mult(L)$ of L.*

Proof. We may assume that L is simply connected and hence homeomorphic to $\mathbb{R}^3$ (cf. Lemma 9). The indecomposable solvable non-nilpotent Lie groups of dimension ≤ 5 are not the multiplication groups of 3-dimensional topological loops (see [11]). The centre of the at most 6-dimensional indecomposable nilpotent Lie groups which are the groups $Mult(L)$ of L has dimension 1 (see [17]). Hence it remains to deal with the 6-dimensional indecomposable non-nilpotent Lie groups. By Theorem 18 the group $Mult(L)$ has the form $Q \ltimes V$ with the 5-dimensional abelian normal subgroup V. Hence the Lie algebra $\mathbf{mult(L)}$ of $Mult(L)$ has a 5-dimensional abelian nilradical. The unique Lie algebra with 2-dimensional centre in the list given in [36], p. 37, is the Lie algebra $\mathbf{g}_{6,6}$ with $a = 0 = b$ defined by the Lie brackets: $[e_1, e_6] = e_1$, $[e_3, e_6] = e_2$, $[e_5, e_6] = e_4$. A 1-dimensional subalgebra $\mathbf{n}$ of the centre $\mathbf{z} = \langle e_2, e_4 \rangle$ of $\mathbf{g}_{6,6}$ has either the form $\mathbf{n}_\alpha = \langle e_2 + \alpha e_4 \rangle$, $\alpha \in \mathbb{R}$, or $\mathbf{n} = \langle e_4 \rangle$. There does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,6}$ containing $\mathbf{n}_\alpha$ or $\mathbf{n}$ such that the factor algebra $\mathbf{g}_{6,6}/\mathbf{s}$ is isomorphic to a Lie algebra $\mathbf{f}_n$, $n = \{4, 5\}$. This contradiction to Theorem 18 proves the assertion. □

Now we show that the 6-dimensional solvable indecomposable Lie algebras with 4-dimensional nilradical having trivial centre or non-abelian nilradical are not the Lie algebras of the groups $Mult(L)$ of 3-dimensional topological loops L.

Proposition 21. *Let $\mathbf{g}$ be a 6-dimensional solvable indecomposable Lie algebra with 4-dimensional nilradical $\mathbf{n}_{rad}$ such that $\mathbf{n}_{rad}$ is not commutative or the centre of $\mathbf{g}$ is trivial. There does not exist any 3-dimensional connected topological loop L having $\mathbf{g}$ as the Lie algebra of the multiplication group of L.*

Proof. We may assume that L is simply connected and hence it is homeomorphic to $\mathbb{R}^3$ (cf. Lemma 9). The 6-dimensional solvable indecomposable Lie algebras with 4-dimensional nilradical having trivial centre or non-abelian nilradical are listed in Tables I, III, IV, V of [41], pp. 1347-1350. The Lie algebras $N_{6,i}$, $i = 4, 7, 30, 39, 40$, have the ideal $\mathbf{i} = \langle n_4 \rangle$. The Lie algebras $N_{6,i}$, $i = 5, 16, 17$, have the ideal $\mathbf{i} = \langle n_2 \rangle$. The Lie algebras $N_{6,i}$, $i = 8, 9, 10, 13, 14, 28, 35, 36, 37$, have the ideal $\mathbf{i} = \langle n_1 \rangle$. There does not exist any ideal $\mathbf{s}$ of the above Lie algebras $N_{6,i}$ which contains $\mathbf{i}$ and the factor Lie algebras $N_{6,i}/\mathbf{s}$ are isomorphic either to $\mathbf{f}_4$ or to $\mathbf{l}_2 \oplus \mathbf{l}_2$. For $i = 39, 40$, the nilradical of $N_{6,i}$ is not abelian. Hence the factor loop $L/I(e)$ is not isomorphic to $\mathbb{R}^2$. By Proposition 19 a) d) these Lie algebras are not the Lie algebras of the multiplication groups of 3-dimensional topological loops. The Lie algebras $N_{6,j}$, $j = 12, 15, 18, 19$, have no 1-dimensional ideal. The unique 2-dimensional abelian ideal of $N_{6,12}$, respectively $N_{6,19}$, is $\mathbf{s}_1 = \langle n_2, n_4 \rangle$, respectively $\mathbf{s}_2 = \langle n_3, n_4 \rangle$. The Lie algebras $N_{6,15}$, $N_{6,18}$ have two 2-dimensional abelian ideals $\mathbf{s}_2$ and $\mathbf{s}_3 = \langle n_1, n_2 \rangle$. None of the factor Lie algebras $N_{6,12}/\mathbf{s}_1$, $N_{6,19}/\mathbf{s}_2$, $N_{6,j}/\mathbf{s}_k$, $j = 15, 18$, $k = 2, 3$, is isomorphic to $\mathbf{f}_4$ or to $\mathbf{l}_2 \oplus \mathbf{l}_2$. Hence the orbits $\mathcal{S}_i(e)$, where $\mathcal{S}_i = \exp(\mathbf{s_i})$, $i = 1, 2, 3$, are 2-dimensional normal subloops of L (cf. Proposition 19 d). If $N_{6,j}$, $j = 12, 15, 18, 19$, would be the Lie algebra of the group $Mult(L)$ of L, then L would have no 1-dimensional normal subgroup. This contradiction to Theorem 12 excludes these Lie algebras.

The Lie algebras $N_{6,i}$, $i \in \{1, 2, 3, 6, 11\}$, have trivial centre. Neither a subalgebra nor a factor Lie algebra of $N_{6,i}$ is isomorphic to an elementary filiform Lie algebra. The Lie algebra $N_{6,1}$ depends on four real parameters α, β, γ, δ with $\alpha\beta \neq 0$, $\gamma^2 + \delta^2 \neq 0$. It has the ideals $\mathbf{i}_1 = \langle n_3 \rangle$, $\mathbf{i}_2 = \langle n_4 \rangle$. If $N_{6,1}$ is the Lie algebra of the group $Mult(L)$ of L, then there are 2-dimensional ideals $\mathbf{s}_j$ of $N_{6,1}$ containing $\mathbf{i}_j$, $j = 1, 2$, such that the factor Lie algebras $N_{6,1}/\mathbf{s}_j$, $j = 1, 2$, are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ (cf. Theorem 16 and Proposition 19 a) (ii). This is the case if and only if $\gamma = \delta = 0$. This contradiction excludes the Lie algebra $N_{6,1}$.

The Lie algebra $N_{6,2}$ depends on real parameters α, β, γ and the Lie algebra $N_{6,6}$ depends on α, β. In both cases one has $\alpha^2 + \beta^2 \neq 0$. The Lie algebras $N_{6,3}$, $N_{6,11}$ depend on the real parameter α. The Lie algebra $N_{6,2}$ has the ideals $\mathbf{i}_1 = \langle n_1 \rangle$, $\mathbf{i}_2 = \langle n_2 \rangle$, $\mathbf{i}_3 = \langle n_4 \rangle$ and the Lie algebras $N_{6,j}$, $j = 3, 6, 11$, have the ideals $\mathbf{i}_k$, $k = 2, 3$. If $N_{6,j}$, $j = 2, 3, 6, 11$, would be the Lie algebra of the group $Mult(L)$, then applying Theorem 16 and Proposition 19 a) (ii) there are 2-dimensional ideals $\mathbf{s}$ of $N_{6,j}$, $j = 2, 3, 6, 11$, containing $\mathbf{i}_k$, $k = 1, 2, 3$, such that the factor Lie algebras $N_{6,j}/\mathbf{s}$, $j = 2, 3, 6, 11$, are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. For the ideals $\mathbf{s}_1 = \langle n_1, n_4 \rangle$, $\mathbf{s}_2 = \langle n_2, n_4 \rangle$ of $N_{6,2}$, respectively for the ideal $\mathbf{s}_2$ of the Lie algebras $N_{6,j}$, $j = 3, 6, 11$, the factor Lie algebras $N_{6,2}/\mathbf{s}_i$, $i = 1, 2$, respectively $N_{6,j}/\mathbf{s}_2$, $j = 3, 6, 11$, are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ precisely if $\beta = \gamma = 0$, respectively $\alpha = 0$. Hence we have to consider the Lie algebras $N_{6,2}$ with $\beta = \gamma = 0$, $\alpha \neq 0$, $N_{6,j}$, $j = 3, 11$, with $\alpha = 0$ and $N_{6,6}$ with $\alpha = 0$, $\beta \neq 0$. These Lie algebras have the abelian ideals $\mathbf{s}_3 = \langle n_1, n_2 \rangle$, $\mathbf{s}_4 = \langle n_3, n_4 \rangle$ such that the factor Lie algebras $N_{6,j}/\mathbf{s}_l$, $j = 2, 3, 6, 11$, $l = 3, 4$, are not isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. The 3-dimensional abelian ideals $\mathbf{s}_5 = \langle n_1, n_2, n_4 \rangle$, $\mathbf{s}_6 = \langle n_2, n_3, n_4 \rangle$, $\mathbf{s}_7 = \langle n_1, n_3, n_4 \rangle$ of $N_{6,2}$ and the ideals $\mathbf{s}_m$, $m = 5, 6$, of $N_{6,j}$, $j = 3, 6, 11$, are in $\mathbf{n}_{\text{rad}}$. According to Proposition 19 d) the orbits $\mathcal{S}_l(e)$, where $\mathcal{S}_l = \exp(\mathbf{s}_l)$, $l \in \{3, 4, 5, 6, 7\}$, and the orbit $N(e)$, where N is the simply connected Lie group of $\mathbf{n}_{\text{rad}}$, are the same normal subgroup $M \cong \mathbb{R}^2$ of L. Since $\mathbf{i}_k \subset \mathbf{n}_{\text{rad}}$, $k = 1, 2, 3$, the group M contains the 1-dimensional normal subgroups $I_k(e)$ of L, where I_k are the simply connected Lie groups of $\mathbf{i}_k$, $k \in \{1, 2, 3\}$. The ideal $\mathbf{v}$ in Proposition 19 d) has one of the following forms: $\mathbf{v}_{1,k} = \langle n_1, n_2, n_3, n_4, x_1 + kx_2 \rangle$, $k \in \mathbb{R}$, $\mathbf{v}_2 = \langle n_1, n_2, n_3, n_4, x_2 \rangle$. For $l = 3, 4$ one has $\mathbf{s}_l \cap \mathbf{inn}(\mathbf{L}) = \{0\}$, for $m = 5, 6, 7$ the intersections $\mathbf{s}_m \cap \mathbf{inn}(\mathbf{L})$ have dimension 1 and $\dim(\mathbf{n}_{\text{rad}} \cap \mathbf{inn}(\mathbf{L})) = 2$. Hence the Lie subalgebra $\mathbf{inn}(\mathbf{L})$ of $N_{6,j}$, $j = 2, 3, 6, 11$, has either the basis elements $b_1 = n_2 + a_1 n_4$, $b_2 = n_1 + a_2 n_3 + a_4 n_4$, where $a_1 a_2 \neq 0$ or the basis elements $b_1' = n_1 + a_1 n_2 + a_2 n_4$, $b_2' = n_2 + a_3 n_3 + a_4 n_4$, where $a_2 a_3 \neq 0$. In the second case for the Lie algebra $N_{6,2}$ we have $a_1 = 0$. The third basis element of $\mathbf{inn}(\mathbf{L})$ is either $b_3 = x_2 + c_1 n_3 + c_2 n_4$ or $b_{3,k} = x_1 + kx_2 + c_1 n_3 + c_2 n_4$, $k, c_1, c_2 \in \mathbb{R}$. Among the subspaces generated by the above basis elements b_i, $i = 1, 2, 3$, only the subspace $\langle b_1, b_2, b_{3,k} \rangle$ of the Lie algebras $N_{6,j}$, $j = 3, 6, 11$, forms a 3-dimensional Lie algebra. Then the subalgebra $\mathbf{inn}(\mathbf{L})$ has the shape: $\mathbf{inn}(\mathbf{L})_{a,a_4} = \langle n_2 + a(1+\beta)n_4, n_1 + an_3 + a_4 n_4, x_1 + x_2 \rangle$, where $a \neq 0$, $a_4 \in \mathbb{R}$, $\beta \neq -1$ for $N_{6,6}$ and $\beta = 0$ for $N_{6,j}$, $j = 3, 11$. Using the automorphism $\alpha(n_i) = an_i$, $\alpha(x_i) = x_i$, $i = 1, 2$, $\alpha(n_4) = n_4$, $\alpha(n_3) = n_3 - \frac{a_4}{a} n_4$ of the Lie algebras $N_{6,j}$, $j = 3, 6, 11$, we can change the Lie algebra $\mathbf{inn}(\mathbf{L})_{a,a_4}$ onto $\mathbf{inn}(\mathbf{L})_{\beta} = \langle n_2 + (1+\beta)n_4, n_1 + n_3, x_1 + x_2 \rangle$ such that for the Lie algebra $N_{6,6}$ one has $\beta \neq -1$ and for the Lie algebra $N_{6,j}$, $j = 3, 11$ we have $\beta = 0$. Linear representations of the simply connected Lie groups G_j of $N_{6,j}$, $j = 3, 6, 11$, are given by: for $N_{6,3}^{\alpha=0}$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1 e^{x_6}, x_2 + y_2 e^{x_6} + x_5 y_1 e^{x_6}, x_3 + y_3 e^{x_5},$$
$$x_4 + y_4 e^{x_5} + y_3 x_6 e^{x_5}, x_5 + y_5, x_6 + y_6)$$

for $N_{6,11}^{\alpha=0}$

$$g(x_1 + y_1 e^{x_6}, x_2 + y_2 e^{x_6} + x_5 y_1 e^{x_6}, x_3 + y_3 e^{x_5},$$
$$x_4 + y_4 e^{x_5} + y_3 x_5 e^{x_5}, x_5 + y_5, x_6 + y_6)$$

for $N_{6,6}^{\alpha=0,\beta\neq-1}$

$$g(x_1 + y_1 e^{x_6}, x_2 + y_2 e^{x_6} + x_6 y_1 e^{x_6}, x_3 + y_3 e^{x_5},$$
$$x_4 + y_4 e^{x_5} + y_3(x_5 + \beta x_6) e^{x_5}, x_5 + y_5, x_6 + y_6).$$

One has $Inn(L) = \{g(u_1, u_2, u_1, (1+\beta)u_2, s, s); u_i, s \in \mathbb{R}\}$, $i = 1, 2$, where $\beta \neq -1$ for the Lie algebra $N_{6,6}$ and $\beta = 0$ in the cases $N_{6,j}$, $j = 3, 11$. Two arbitrary left transversals to the group $Inn(L)$ in G_j, $j = 3, 6, 11$, are

$$S = \{g(f_1(k,l,m), f_2(k,l,m), k, l, m, f_3(k,l,m)), k, l, m \in \mathbb{R}\},$$

$$T = \{g(h_1(u,v,w), h_2(u,v,w), u, v, w, h_3(u,v,w)), u, v, w \in \mathbb{R}\},$$

where $f_i(k,l,m) : \mathbb{R}^3 \to \mathbb{R}$ and $h_i(u,v,w) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, are continuous functions with $f_i(0,0,0) = h_i(0,0,0) = 0$. For all $s \in S$, $t \in T$ the condition $s^{-1}t^{-1}st \in Inn(L)$ holds if and only if in all three cases the equation

$$e^{-h_3(u,v,w)} h_1(u,v,w)(1 - e^{-f_3(k,l,m)}) -$$
$$e^{-f_3(k,l,m)} f_1(k,l,m)(1 - e^{-h_3(u,v,w)}) = u e^{-w}(1 - e^{-m}) - k e^{-m}(1 - e^{-w}), \tag{1}$$

and for $N_{6,3}^{\alpha=0}$ the equation

$$e^{-h_3(u,v,w)}(1 - e^{-f_3(k,l,m)})(h_2(u,v,w) - w h_1(u,v,w)) -$$
$$e^{-f_3(k,l,m)}(1 - e^{-h_3(u,v,w)})(f_2(k,l,m) - m f_1(k,l,m)) +$$
$$e^{-f_3(k,l,m) - h_3(u,v,w)}(m h_1(u,v,w) - w f_1(k,l,m)) =$$
$$e^{-w}(1 - e^{-m})(v - h_3(u,v,w)u) - e^{-m}(1 - e^{-w})(l - f_3(k,l,m)k) +$$

$$e^{-m-w}(f_3(k,l,m)u - h_3(u,v,w)k), \quad (2)$$

for $N_{6,11}^{\alpha=0}$ the equation

$$e^{-h_3(u,v,w)}(1-e^{-f_3(k,l,m)})(h_2(u,v,w) - wh_1(u,v,w)) -$$
$$e^{-f_3(k,l,m)}(1-e^{-h_3(u,v,w)})(f_2(k,l,m) - mf_1(k,l,m)) +$$
$$e^{-f_3(k,l,m)-h_3(u,v,w)}(mh_1(u,v,w) - wf_1(k,l,m)) =$$
$$e^{-w}(1-e^{-m})(v-wu) - e^{-m}(1-e^{-w})(l-mk) + e^{-m-w}(mu-wk), \quad (3)$$

for $N_{6,6}^{\alpha=0,\beta\neq-1}$ the equation

$$(1+\beta)[e^{-h_3(u,v,w)}(1-e^{-f_3(k,l,m)})(h_2(u,v,w) - h_3(u,v,w)h_1(u,v,w)) -$$
$$e^{-f_3(k,l,m)}(1-e^{-h_3(u,v,w)})(f_2(k,l,m) - f_1(k,l,m)f_3(k,l,m)) +$$
$$e^{-f_3(k,l,m)-h_3(u,v,w)}(f_3(k,l,m)h_1(u,v,w) - h_3(u,v,w)f_1(k,l,m))] =$$
$$e^{-w}(1-e^{-m})[v - u(w+\beta h_3(u,v,w))] -$$
$$e^{-m}(1-e^{-w})[l - k(m+\beta f_3(k,l,m))] +$$
$$e^{-m-w}[mu - wk + \beta(f_3(k,l,m)u - h_3(u,v,w)k)] \quad (4)$$

are satisfied for all $u, v, w, k, l, m \in \mathbb{R}$. Equation (1) holds precisely if we have $h_3(u,v,w) = w$, $h_1(u,v,w) = u$, $f_1(k,l,m) = k$, $f_3(k,l,m) = m$. Putting these into equations (2), (3), (4) we obtain in case (4)

$$e^{-w}(1-e^{-m})(v-(1+\beta)h_2(u,v,w)) = e^{-m}(1-e^{-w})(l-(1+\beta)f_2(k,l,m)) \quad (5)$$

and in cases (2), (3) we get equation (5) with $\beta = 0$. Equation (5) is satisfied if and only if one has $h_2(u,v,w) = \frac{v}{1+\beta}$, $f_2(k,l,m) = \frac{l}{1+\beta}$, where $\beta = 0$ in the cases $N_{6,j}^{\alpha=0}$, $j = 3, 11$, and $\beta \in \mathbb{R} \setminus \{-1\}$ in the case $N_{6,6}^{\alpha=0,\beta\neq-1}$. In all these cases $S \cup T$ does not generate the group G_j, $j = 3, 6, 11$. By Lemma 7 the Lie algebras $N_{6,j}$, $j = 3, 6, 11$, are not the Lie algebras of groups $Mult(L)$ of L.

The Lie algebras $N_{6,j}$, $j \in \{29, 31, 32, 33, 34, 38\}$, have non-abelian nilradical and neither a subalgebra nor a factor Lie algebra of $N_{6,j}$ are isomorphic to a Lie algebra $\mathbf{f}_n$, $n \geq 4$. The Lie algebras $N_{6,31}$ and $N_{6,32}^{\alpha}$ have the ideal $\mathbf{i} = \langle n_1 \rangle$. Both Lie algebras contain the nilpotent ideals: $\mathbf{s}_1 = \langle n_1, n_3 \rangle$, $\mathbf{s}_2 = \langle n_1, n_4 \rangle$, $\mathbf{s}_3 = \langle n_1, n_2 \rangle$, $\mathbf{s}_4 = \langle n_1, n_2, n_3 \rangle$,

$\mathbf{s}_5 = \langle n_1, n_2, n_4 \rangle$, $\mathbf{s}_6 = \langle n_1, n_3, n_4 \rangle$, $\mathbf{n}_{\text{rad}}$. If $N_{6,j}$, $j = 31, 32$, would be the Lie algebra of the group $Mult(L)$ of L, then by Theorem 16 and Proposition 19 a) (ii) there exist 2-dimensional ideals s of $N_{6,j}$, $j = 31, 32$, containing $\mathbf{i}$ such that the factor Lie algebras $N_{6,j}/\mathbf{s}$, $j = 31, 32$, are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. The factor Lie algebra $N_{6,31}/\mathbf{s}_1$ is isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$, whereas the factor Lie algebras $N_{6,31}/\mathbf{s_i}$, $i = 2, 3$, are not so. The factor Lie algebra $N^{\alpha}_{6,32}/\mathbf{s}_1$ is isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ if and only if $\alpha = 0$, but the factor Lie algebras $N^{\alpha=0}_{6,32}/\mathbf{s_i}$, $i = 2, 3$, are not so. The factor Lie algebra $N^{\alpha}_{6,32}/\mathbf{s}_3$ is isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ precisely if $\alpha = 1$, whereas the factor Lie algebras $N^{\alpha=1}_{6,32}/\mathbf{s_i}$, $i = 1, 2$, are not so. Let $\mathcal{S}_k$, respectively N be the simply connected Lie groups of $\mathbf{s_k}$, $k = 1, 2, \dots, 6$, respectively $\mathbf{n}_{\text{rad}}$. For $N_{6,31}$, $N^{\alpha=0}_{6,32}$ the orbits $\mathcal{S}_i(e)$, $i = 2, 3, \dots, 6$, and $N(e)$ are the same normal subgroup $M \cong \mathbb{R}^2$ of L and for $N^{\alpha=1}_{6,32}$ we have $\mathcal{S}_j(e) = N(e) := M$, $j = 1, 2, 4, 5, 6$ (cf. Proposition 19 d). The subgroup M contains the normal subgroup $I(e) \cong \mathbb{R}$, where I is the simply connected Lie group of $\mathbf{i}$, of L. For $m = 4, 5, 6$ the intersections $\mathbf{s}_m \cap \mathbf{inn}(\mathbf{L})$ have dimension 1 and $\mathbf{n}_{\text{rad}} \cap \mathbf{inn}(\mathbf{L})$ has dimension 2 (see Proposition 19 d). Since for $N_{6,31}$ and $N^{\alpha=0}_{6,32}$ one has $\mathbf{s}_i \cap \mathbf{inn}(\mathbf{L}) = \{0\}$, $i = 2, 3$ and for $N^{\alpha=1}_{6,32}$ we have $\mathbf{s}_j \cap \mathbf{inn}(\mathbf{L}) = \{0\}$, $j = 1, 2$, the Lie algebra $\mathbf{inn}(\mathbf{L})$ contains the elements $b_1 = n_1 + a_1 n_3$, $b_2 = n_2 + a_2 n_1 + a_3 n_4$, $a_1 a_3 \neq 0$, in cases $N_{6,31}$ and $N^{\alpha=0}_{6,32}$ and the elements $b_1 = n_1 + a_1 n_2$, $b_2 = n_3 + a_2 n_1 + a_3 n_4$, $a_1 a_3 \neq 0$ in case $N^{\alpha=1}_{6,32}$. As in both cases $[b_1, b_2] = a_1 n_1$, $a_1 \neq 0$, the Lie algebra $\mathbf{inn}(\mathbf{L})$ would contain the ideal $\langle n_1 \rangle$ of $N_{6,j}$, $j = 31, 32$. This contradicts Lemma 8.

The Lie algebras $N_{6,33}$, $N_{6,38}$, $N^{\alpha}_{6,34}$ and $N^{\alpha,\beta}_{6,29}$ have the ideals $\mathbf{i}_1 = \langle n_1 \rangle$, $\mathbf{i}_2 = \langle n_4 \rangle$. The Lie algebras $N_{6,i}$, $i = 29, 38$, have the nilpotent ideals $\mathbf{s}_1 = \langle n_1, n_2 \rangle$, $\mathbf{s}_2 = \langle n_1, n_4 \rangle$, $\mathbf{s}_3 = \langle n_1, n_3 \rangle$, $\mathbf{s}_4 = \langle n_1, n_2, n_4 \rangle$, $\mathbf{s}_5 = \langle n_1, n_3, n_4 \rangle$, $\mathbf{s}_6 = \langle n_1, n_2, n_3 \rangle$, $\mathbf{n}_{\text{rad}}$, whereas the nilpotent ideals of $N_{6,j}$, $j = 33, 34$, are $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_4$, $\mathbf{s}_5$, $\mathbf{n}_{\text{rad}}$. Denote by I_k, $\mathcal{S}_i$ and N the simply connected Lie groups of the ideals $\mathbf{i}_k$, $k = 1, 2$, $\mathbf{s}_i$, $i = 1, 2, \dots, 6$ and $\mathbf{n}_{\text{rad}}$. The factor Lie algebras $N_{6,k}/\mathbf{s}_2$, $k \in \{29, 33, 38\}$, are isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ and $N_{6,34}/\mathbf{s}_2$ is isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ precisely if $\alpha = 0$. If $N_{6,j}$, $j = 29, 33, 34, 38$, would be the Lie algebra of the multiplication group of a 3-dimensional topological loop, then the orbits $I_k(e)$, $k = 1, 2$, are normal subgroups of L isomorphic to $\mathbb{R}$ and the factor loops $L/I_k(e)$, $k = 1, 2$, are isomorphic to $\mathcal{L}_2$ since the nilradical of $N_{6,j}$ are not abelian (cf. Proposition 19 a) d).

For $j = 33, 34$, the factor Lie algebras $N_{6,j}/\mathbf{s}_1$ are not isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$. By Proposition 19 d) the orbits $\mathcal{S}_l(e)$, $l = 1, 4, 5$, and $N(e)$ are the same normal subgroup $M \cong \mathbb{R}^2$ of L such that $\mathcal{S}_1 \cap Inn(L) = \{1\}$, the intersections $\mathcal{S}_l \cap Inn(L)$ have dimension 1, $l = 4, 5$, and $\dim(N \cap Inn(L)) = 2$. For $N_{6,29}$ the factor Lie algebra $N_{29}/\mathbf{s}_1$ is isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ precisely if $\beta = 0$ and $N_{29}/\mathbf{s}_3$ is isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ if and only if $\alpha = 0$. If $\alpha \neq 0$, respectively $\beta \neq 0$, the orbits $\mathcal{S}_l(e)$, $l = 3, 4, 5, 6$, and $N(e)$, respectively the orbits $\mathcal{S}_k(e)$, $k = 1, 4, 5, 6$, and $N(e)$, are the normal subgroup $M \cong \mathbb{R}^2$ of L. For $\alpha \neq 0$ one has $\mathcal{S}_3 \cap Inn(L) = \{1\}$, whereas for $\beta \neq 0$ we have $\mathcal{S}_1 \cap Inn(L) = \{1\}$, for $l = 4, 5, 6$ the intersections $\mathcal{S}_l \cap Inn(L)$ have dimension 1 and $N \cap Inn(L)$ has dimension 2 (cf. Proposition 19 d). Since the factor Lie algebras $N_{6,38}/\mathbf{s}_k$, $k = 1, 3$, are not isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$, the orbits $\mathcal{S}_l(e)$, $l = 1, 3, 4, 5, 6$, and $N(e)$ are the same normal subgroup $M \cong \mathbb{R}^2$ of L and for $l = 1, 3$, one has $\mathcal{S}_l \cap Inn(L) = \{1\}$, for $l = 4, 5, 6$, the intersections $\mathcal{S}_l \cap Inn(L)$ have dimension 1, and $\dim(N \cap Inn(L)) = 2$ (cf. Proposition 19 d). In all cases the normal subgroup $I_k(e)$, $k = 1, 2$, are in M. For $j = 29, 33, 34, 38$, the Lie algebra $\mathbf{inn(L)}$ lies in one of the following ideals: $\mathbf{v}_1 = \langle n_1, n_2, n_3, n_4, x_1 \rangle$, $\mathbf{v}_{2,k} = \langle n_1, n_2, n_3, n_4, x_2 + kx_1 \rangle$, $k \in \mathbb{R}$. If for $N_{6,j}$, $j = 33, 34$, the Lie algebra $\mathbf{inn(L)}$ would contain the basis elements $b_1 = n_2 + a_1 n_4 + a_2 n_1$, $b_2 = n_3 + a_3 n_4 + a_4 n_1$, and for $N_{6,29}$ either the basis elements $b_1 = n_1 + a_1 n_2$, $b_2 = n_3 + a_2 n_4 + a_3 n_1$, or the basis elements $b_1 = n_1 + a_1 n_3$, $b_2 = n_2 + a_2 n_4 + a_3 n_1$ with $a_1 \neq 0$ would be in $\mathbf{inn(L)}$, then $\mathbf{inn(L)}$ would contain the ideal $\langle n_1 \rangle$ of $N_{6,j}$, $j = 29, 33, 34$, since one has $[b_1, b_2] = cn_1$, $c \neq 0$. This is a contradiction to Lemma 8. Otherwise for $N_{6,j}$, $j = 29, 33, 34, 38$, the Lie algebra $\mathbf{inn(L)}$ would contain the basis elements either $b'_1 = n_1 + a_1 n_4$, $b'_2 = n_2 + a_2 n_3 + a_3 n_4$, $b'_3 = x_1 + c_1 n_3 + c_2 n_4$ or b'_1, b'_2, $b'_{3,k} = x_2 + kx_1 + c_1 n_3 + c_2 n_4$, where $a_1 a_2 \neq 0$, $k, c_1, c_2, a_3 \in \mathbb{R}$. The subspaces $\langle b'_1, b'_2, b'_3 \rangle$, $\langle b'_1, b'_2, b'_{3,k} \rangle$ are not 3-dimensional Lie algebras. This proves that none of the Lie algebras $N_{6,j}$, $j = 29, 31, 32, 33, 34, 38$, are the Lie algebras of the group $Mult(L)$ of L. □

Now we show that the 6-dimensional solvable indecomposable Lie algebras having as nilradical either a 5-dimensional indecomposable nilpotent Lie algebra with exception of the Lie algebra $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$, or $\mathbb{R}^5$, are not the Lie algebras of the groups $Mult(L)$ of 3-dimensional topological loops L.

Proposition 22. *There does not exist any* 3*-dimensional connected topological loop* L *such that the Lie algebra of the group* $Mult(L)$ *is a* 6*-dimensional indecomposable solvable Lie algebra having one of the following nilradicals: (a)* $[e_2, e_4] = e_3$, $[e_2, e_5] = e_1$, $[e_4, e_5] = e_2$*; (b)* $[e_2, e_4] = e_1$, $[e_3, e_5] = e_1$*; (c)* $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$*; (d)* $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$*; (e) the Lie algebra* $\mathbf{f}_5$*; (f) the Lie algebra* $\mathbb{R}^5$.

Proof. We may assume that L is simply connected and hence it is homeomorphic to $\mathbb{R}^3$ (cf. Lemma 9). The 6-dimensional solvable indecomposable Lie algebras having nilradical as in cases (a) to (e) of the assertion are the Lie algebras $\mathbf{g}_{6,i}$, $i = 71, \dots, 99$, in [36], pp. 40-41. The Lie algebras $\mathbf{g}_{6,i}$, $i \in \{71, \dots, 99\} \setminus \{80, 81\}$, have the 1-dimensional ideal $\mathbf{i} = \langle e_1 \rangle$. There does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,i}$ such that $\mathbf{i} \leq \mathbf{s}$ and the factor Lie algebras $\mathbf{g}_{6,i}/\mathbf{s}$ are isomorphic to a Lie algebra $\mathbf{f}_n$, $n \in \{4, 5\}$. If $\mathbf{g}_{6,i}$, $i \in \{71, \dots, 99\} \setminus \{80, 81\}$, would be the Lie algebra of the group $Mult(L)$ of L, then by Proposition 19 e) the orbit $I(e)$ is isomorphic to $\mathbb{R}$ and the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$. In this case the nilradical would contain a 4-dimensional abelian ideal of $\mathbf{g}_{6,i}$. None of the Lie algebras $\mathbf{g}_{6,i}$, $i = 71, \dots, 99$, have a 4-dimensional abelian ideal in their nilradical. Hence these Lie algebras are excluded.

The Lie algebras $\mathbf{g}'^{***}_{6,i}$, $i \in \{80, 81\}$, have trivial centre and the unique minimal ideal $\mathbf{s} = \langle e_1, e_3 \rangle$. Let $\mathcal{S}$ be the simply connected Lie group of $\mathbf{s}$. By Proposition 19 e) and Theorem 16 the orbit $\mathcal{S}(e)$ is a normal subgroup of L isomorphic to $\mathbb{R}$ such that the factor loop $L/\mathcal{S}(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. Since the factor Lie algebras $\mathbf{g}'^{***}_{6,i}/\mathbf{s}$, $i = 80, 81$, are not isomorphic to the Lie algebra $\mathbf{f}_4$ we obtain a contradiction. Hence these Lie algebras cannot be the Lie algebra of the group $Mult(L)$ of L. This proves the assertion in cases (a) to (e).

The 6-dimensional solvable indecomposable Lie algebras having $\mathbb{R}^5$ as their nilradical are given in [36], p. 37. All these Lie algebras $\mathbf{g}_{6,i}$, $i = 1, \dots, 12$, have the 1-dimensional ideal $\mathbf{i} = \langle e_1 \rangle$. With the exception of the Lie algebra $\mathbf{g}^{a=0}_{6,4}$ there does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,i}$ containing $\mathbf{i}$ such that the factor Lie algebras $\mathbf{g}_{6,i}/\mathbf{s}$ are isomorphic to a Lie algebra $\mathbf{f}_n$, $n = 4, 5$. Let I be the simply connected Lie group of the ideal $\mathbf{i}$. If $\mathbf{g}_{6,i}$, $i = 1, \dots, 12$, would be the Lie algebra of the group $Mult(L)$ of L, then the orbit $I(e)$ is isomorphic to $\mathbb{R}$ and the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$ (cf. Proposition 19 e). By Theorem 17 a) the orbit $I(e)$ coincides with $Z(L)$. By Proposition 19 a), c) the Lie algebra $\mathbf{inn(L)}$ of the group $Inn(L)$ lies in the 5-dimensional abelian nilradical of $\mathbf{g}_{6,i}$ which contains $\mathbf{p} = \mathbf{z} \oplus \mathbf{inn(L)} \cong \mathbb{R}^4$ as a proper ideal. Then the normalizer $N_{\mathbf{g}_{6,i}}(\mathbf{inn(L)})$, $i = 1, \dots, 12$, is the nilradical of $\mathbf{g}_{6,i}$ which contradicts Lemma 8.

If the Lie algebra $\mathbf{g}_{6,4}^{a=0}$ would be the Lie algebra of the group $Mult(L)$ of L, then from the above discussion it follows that the factor loop $L/Z(L)$ is isomorphic to a loop $L_{\mathcal{F}}$. In fact, for the ideal $\mathbf{s} = \langle e_1, e_5 \rangle$ the factor Lie algebra $\mathbf{g}_{6,4}^{a=0}/\mathbf{s}$ is isomorphic to the Lie algebra $\mathbf{f}_4$. Since the orbit $\mathcal{S}(e)$, where $\mathcal{S} = \exp(\mathbf{s})$, has dimension 1 we obtain that $\dim(\mathbf{s} \cap \mathbf{inn(L)}) = 1$. For the simply connected Lie group $I_2 = \{\exp(te_5); t \in \mathbb{R}\}$ of the ideal $\mathbf{i}_2 = \langle e_5 \rangle$ we obtain that the orbit $I_2(e)$ is a normal subgroup of L isomorphic to $\mathbb{R}$. Hence one has $\mathbf{i}_2 \cap \mathbf{inn(L)} = 0$. The abelian ideals $\mathbf{a} = \langle e_1, e_2 \rangle$, $\mathbf{b} = \langle e_1, e_2, e_3 \rangle$, $\mathbf{g}_{6,4}^{\prime a=0} = \langle e_1, e_2, e_3, e_5 \rangle$ of $\mathbf{g}_{6,4}^{a=0}$ satisfy the conditions of Proposition 19 b). Let A, B and N be the simply connected Lie group of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{g}'_{6,4}$. Since $\langle e_1 \rangle = \mathbf{z} < \mathbf{a}$ the orbit $A(e)$ contains $Z(L)$. If $\dim(A(e)) = 1$, then one has $A(e) = Z(L)$. As the factor Lie algebra $\mathbf{g}_{6,4}^{a=0}/\mathbf{a}$ is not isomorphic to the Lie algebra $\mathbf{f}_4$, the factor loop $L/Z(L)$ is not isomorphic to a loop $L_{\mathcal{F}}$.

According to Proposition 19 b) the orbit $A(e)$ is a 2-dimensional connected normal subloop M of L containing $Z(L)$ and the orbits $B(e)$ and $N(e)$ coincide with M. Therefore the Lie algebra $\mathbf{inn(L)}$ contains the subalgebra $\langle e_3 + a_1 e_1 + a_2 e_2, e_5 + b_1 e_1 \rangle$, $a_i, b_1 \in \mathbb{R}$, $i = 1, 2$, $b_1 \neq 0$. The ideal $\mathbf{v}$ in Proposition 19 b) has one of the following forms: $\mathbf{v}_{1,k} = \langle e_1, e_2, e_3, e_5, e_4 + k e_6 \rangle$, $k \in \mathbb{R}$, $\mathbf{v}_2 = \langle e_1, e_2, e_3, e_5, e_6 \rangle$. Therefore the Lie algebra $\mathbf{inn(L)}$ has as generator either $e_4 + k e_6 + c_1 e_1 + c_2 e_2$ or $e_6 + c_1 e_1 + c_2 e_2$, $k, c_1, c_2 \in \mathbb{R}$. Only the subspace $\langle e_3 + a_1 e_1 + a_2 e_2, e_4 + c_1 e_1 + c_2 e_2, e_5 + b_1 e_1 \rangle \subset \mathbf{n}_{\text{rad}}$ is a 3-dimensional Lie algebra. Hence it would be the Lie algebra $\mathbf{inn(L)}$. The normalizer $N_{\mathbf{g}_{6,4}^{a=0}}(\mathbf{inn(L)})$ equals to $\mathbf{n}_{\text{rad}}$ which contains $\mathbf{z} \oplus \mathbf{inn(L)}$ as a proper ideal. This is a contradiction to Lemma 8. This prove the assertion in case (f). □

In the next Proposition we wish to prove that the 6-dimensional solvable decomposable Lie algebras with trivial centre are not the Lie algebras of the groups $Mult(L)$ of 3-dimensional topological loops L.

Proposition 23. *The* 6*-dimensional decomposable solvable Lie algebras with trivial centre are not the Lie algebras of the multiplication groups of* 3*-dimensional topological loops* L*.*

Proof. We may assume that the loop L is simply connected and hence it is homeomorphic to $\mathbb{R}^3$ (cf. Lemma 9). As the multiplication group $Mult(L)$ of L is a 6-dimensional decomposable solvable Lie group with discrete centre for the Lie algebra $\mathbf{mult(L)}$ we have the following possibilities: $\mathbf{l}_2 \oplus \mathbf{l}_2 \oplus \mathbf{l}_2$, $\mathbf{g}_{3,i} \oplus \mathbf{g}_{3,j}$, $\mathbf{l}_2 \oplus \mathbf{g}_{4,k}$, where $\mathbf{g}_{3,i}$, $\mathbf{g}_{3,j}$, $i, j \in \{2, 3, 4, 5\}$, are the 3-dimensional solvable Lie algebras with trivial centre (cf. §4 in [24], p. 119), $\mathbf{g}_{4,k}$, $k = 2, 4, 5, 6, 7, 10$, $\mathbf{g}_{4,8}^{h \neq -1}$, $\mathbf{g}_{4,9}^{p \neq 0}$ are the 4-dimensional solvable Lie algebras with trivial centre (see §5 in [24], pp. 120-121). These Lie algebras have trivial centre and neither a subalgebra nor a factor Lie algebra is isomorphic to a Lie algebra $\mathbf{f}_n$, $n = 4, 5$.

The Lie algebras $\mathbf{mult(L)} = \mathbf{l}_2 \oplus \mathbf{g}_{4,k}$, $k = 2, 4, 5, 6, 7, 10$, $\mathbf{l}_2 \oplus \mathbf{g}_{4,8}^{h \neq -1}$, $\mathbf{l}_2 \oplus \mathbf{g}_{4,9}^{p \neq 0}$, where $\mathbf{l}_2 = \langle f_1, f_2 \rangle$, have the 1-dimensional ideal $\mathbf{i} = \langle f_1 \rangle$. There does not exist any ideal $\mathbf{s}$ of $\mathbf{mult(L)}$ such that $\mathbf{i} \leq \mathbf{s}$ and $\mathbf{mult(L)}/\mathbf{s}$ is isomorphic to the Lie algebra $\mathbf{l}_2 \oplus \mathbf{l}_2$. By Theorem 16 these Lie algebras are not the Lie algebra of the group $Mult(L)$.

Now we consider the Lie algebras $\mathbf{g}_{i,j} = \mathbf{g}_{3,i} \oplus \mathbf{g}_{3,j} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle$, $i, j \in \{2, 3, 4, 5\}$. Let be $j = 5$. The Lie algebra $\mathbf{g}_{3,5}$, respectively $\mathbf{g}_{4,5}$, is defined by $[e_1, e_3] = e_1$, $[e_2, e_3] = he_2$, $[e_4, e_6] = pe_4 - e_5$, $[e_5, e_6] = e_4 + pe_5$, $p \geq 0$, where $h = 1$, respectively $-1 \leq h < 1$, whereas the Lie algebra $\mathbf{g}_{2,5}$ is given by $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$, $[e_4, e_6] = pe_4 - e_5$, $[e_5, e_6] = e_4 + pe_5$, $p \geq 0$. They have the 1-dimensional ideal $\mathbf{i} = \langle e_1 \rangle$. There does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{i,5}$, $i = 2, 3, 4$, such that $\mathbf{i} \leq \mathbf{s}$ and $\mathbf{g}_{i,5}/\mathbf{s}$ is isomorphic to the Lie algebra $\mathbf{l}_2 \oplus \mathbf{l}_2$. This excludes the Lie algebra $\mathbf{g}_{i,5}$, $i = 2, 3, 4$. The Lie algebra $\mathbf{g}_{5,5}$ defined by $[e_1, e_3] = p_1 e_1 - e_2$, $[e_2, e_3] = e_1 + p_1 e_2$, $[e_4, e_6] = p_2 e_4 - e_5$, $[e_5, e_6] = e_4 + p_2 e_5$ with $p_1, p_2 \geq 0$ has the minimal ideals $\mathbf{s}_1 = \langle e_1, e_2 \rangle$, $\mathbf{s}_2 = \langle e_4, e_5 \rangle$. Let $\mathcal{S}_i$, $i = 1, 2$, be the simply connected Lie groups of $\mathbf{s}_i$. If $\mathbf{g}_{5,5}$ would be the Lie algebra of the group $Mult(L)$ of L, then by Theorems 12 and 16 at least one of the orbits $\mathcal{S}_i(e)$, $i = 1, 2$, is a normal subloop of L isomorphic to $\mathbb{R}$. For this orbit the factor loop $L/\mathcal{S}_i(e)$ is isomorphic to the group $\mathcal{L}_2$. Since the factor Lie algebras $\mathbf{g}_{5,5}/\mathbf{s}_i$, $i = 1, 2$, are not isomorphic to the Lie algebra $\mathbf{l}_2 \oplus \mathbf{l}_2$, the Lie algebra $\mathbf{g}_{5,5}$ is excluded (cf. Proposition 19 (ii)).

The Lie algebras $\mathbf{g}_{3,3}$, $\mathbf{g}_{3,4}$, $\mathbf{g}_{4,4}$ are defined by $[e_1, e_3] = e_1$, $[e_2, e_3] =$

$h_1 e_2$, $[e_4, e_6] = e_4$, $[e_5, e_6] = h_2 e_5$ such that for $\mathbf{g}_{3,3}$ one has $h_1 = h_2 = 1$, for $\mathbf{g}_{3,4}$ we have $h_1 = 1$, $-1 \le h_2 < 1$ and for $\mathbf{g}_{4,4}$ one has $-1 \le h_1, h_2 < 1$. The Lie algebra $\mathbf{g}_{2,3}$, respectively $\mathbf{g}_{2,4}$, is given by $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$, $[e_4, e_6] = e_4$, $[e_5, e_6] = h_2 e_5$, where $h_2 = 1$, respectively $-1 \le h_2 < 1$. The Lie algebra $\mathbf{g}_{2,2}$ is defined by $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$, $[e_4, e_6] = e_4$, $[e_5, e_6] = e_4 + e_5$. All these Lie algebras have the ideals $\mathbf{i}_1 = \langle e_1 \rangle$, $\mathbf{i}_2 = \langle e_4 \rangle$. Additionally, the Lie algebra $\mathbf{g}_{3,3}$ has the ideals $\mathbf{i}_3 = \langle e_2 + l_1 e_1 \rangle$, $\mathbf{i}_4 = \langle e_5 + l_2 e_4 \rangle$, $l_1, l_2 \in \mathbb{R}$, the Lie algebra $\mathbf{g}_{4,4}$ has the ideals $\mathbf{i}_5 = \langle e_2 \rangle$, $\mathbf{i}_6 = \langle e_5 \rangle$, the Lie algebra $\mathbf{g}_{2,3}$ has the ideal $\mathbf{i}_4$, the Lie algebra $\mathbf{g}_{2,4}$ has the ideal $\mathbf{i}_6$ and the Lie algebra $\mathbf{g}_{3,4}$ has the ideals $\mathbf{i}_3$, $\mathbf{i}_6$. All Lie algebras have the ideal $\mathbf{s}_1 = \langle e_1, e_4 \rangle$ containing $\mathbf{i}_1$, $\mathbf{i}_2$, such that the factor Lie algebras $\mathbf{g}_{i,j}/\mathbf{s}_1$, $i, j \in \{2, 3, 4\}$ are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. Furthermore, the Lie algebra $\mathbf{g}_{3,3}$ has the ideal $\mathbf{s}_2 = \langle e_2 + l_1 e_1, e_5 + l_2 e_4 \rangle$, the Lie algebra $\mathbf{g}_{4,4}$ has the ideal $\mathbf{s}_3 = \langle e_2, e_5 \rangle$, the Lie algebra $\mathbf{g}_{2,3}$ has the ideal $\mathbf{s}_4 = \langle e_1, e_5 + l_2 e_4 \rangle$, the Lie algebra $\mathbf{g}_{2,4}$ has the ideal $\mathbf{s}_5 = \langle e_1, e_5 \rangle$ and the Lie algebra $\mathbf{g}_{3,4}$ has the ideal $\mathbf{s}_6 = \langle e_2 + l_1 e_1, e_5 \rangle$ such that the factor Lie algebras $\mathbf{g}_{3,3}/\mathbf{s}_2$, $\mathbf{g}_{4,4}/\mathbf{s}_3$, $\mathbf{g}_{2,3}/\mathbf{s}_4$, $\mathbf{g}_{2,4}/\mathbf{s}_5$, $\mathbf{g}_{3,4}/\mathbf{s}_6$ are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. If $\mathbf{g}_{i,j}$, $i, j \in \{2, 3, 4\}$ is the Lie algebra of the group $Mult(L)$ of a L, then the orbits $I_k(e)$, $k = 1, \cdots, 6$, where $I_k = \exp(\mathbf{i}_k)$, are 1-dimensional normal subgroups of L isomorphic to $\mathbb{R}$ and the factor loops $L/I_k(e)$ are isomorphic to $\mathcal{L}_2$ (cf. Proposition 19 (ii)). All Lie algebras $\mathbf{g}_{i,j}$, $i, j \in \{2, 3, 4\}$, have the ideals $\mathbf{s}_7 = \langle e_1, e_2 \rangle$, $\mathbf{s}_8 = \langle e_4, e_5 \rangle$ such that the factor Lie algebras $\mathbf{g}_{i,j}/\mathbf{s}_l$, $l = 7, 8$, are not isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. Hence the orbits $\mathcal{S}_l(e)$, where $\mathcal{S}_l = \exp(\mathbf{s}_l)$, $l = 7, 8$, are 2-dimensional normal subloops of L and therefore one has $\mathbf{s}_l \cap \mathbf{inn}(L) = \{0\}$, $l = 7, 8$ (cf. Proposition 19). All Lie algebras $\mathbf{g}_{i,j}$, $i, j \in \{2, 3, 4\}$, have the commutator subalgebra $\mathbf{g}'_{i,j} = \langle e_1, e_2, e_4, e_5 \rangle$. Their 5-dimensional ideals are $\mathbf{v}_1 = \langle e_1, e_2, e_4, e_5, e_3 \rangle$, $\mathbf{v}_{2,k} = \langle e_1, e_2, e_4, e_5, e_6 + k e_3 \rangle$, $k \in \mathbb{R}$. Denote by N_1 the simply connected Lie group of $\mathbf{g}'_{i,j}$. By Proposition 19 d) we have $N_1(e) = \mathcal{S}_l(e)$, $l = 7, 8$. Therefore the intersection $\mathbf{g}'_{i,j} \cap \mathbf{inn}(L)$ has dimension 2. Hence the Lie algebra $\mathbf{inn}(L)$ has the basis elements $r_1 = e_4 + a_1 e_1 + a_2 e_2$, $r_2 = e_5 + b_1 e_1 + b_2 e_2$ such that at least one of a_1, a_2 as well as b_1, b_2 are different from 0 and $a_1 b_2 - a_2 b_1 \neq 0$.

All Lie algebras $\mathbf{g}_{i,j}$, $i, j \in \{2, 3, 4\}$, have the ideals $\mathbf{n}_2 = \langle e_1, e_2, e_3 \rangle$, $\mathbf{n}_3 = \langle e_4, e_5, e_6 \rangle$. As $\mathbf{s}_7 < \mathbf{n}_2$ and $\mathbf{s}_8 < \mathbf{n}_3$, the orbits $N_j(e)$, where $N_j = \exp(\mathbf{n}_j)$, $j = 2, 3$, has dimension 2 or 3. If $\mathcal{S}_7(e) = N_2(e)$ or $\mathcal{S}_8(e) =$

$N_3(e)$, then one has $\dim(\mathbf{n}_2 \cap \mathbf{inn}(L)) = 1$ or $\dim(\mathbf{n}_3 \cap \mathbf{inn}(L)) = 1$. Hence the Lie algebra $\mathbf{inn}(L)$ has the basis element either $r_3 = e_3 + c_1e_1 + c_2e_2$ or $r'_3 = e_6 + d_1e_4 + d_2e_5$, $c_i, d_i \in \mathbb{R}$, $i = 1, 2$. Since $[r_1, r_3]$, respectively $[r_2, r'_3]$, is a non-zero element of the ideal $\mathbf{s}_7$, respectively $\mathbf{s}_8$, the subspaces $\langle r_1, r_2, r_3\rangle$, $\langle r_1, r_2, r'_3\rangle$ are not 3-dimensional subalgebras of $\mathbf{g}_{i,j}$, $i, j \in \{2, 3, 4\}$. This contradiction gives that $N_2(e) = L$ and $N_3(e) = L$. As $\mathbf{n}_2 < \mathbf{v}_1$ and $\mathbf{n}_3 < \mathbf{v}_{2,0}$, we obtain that $N_2(e) = V_1(e) = V_{2,0}(e) = N_3(e) = L$. By Theorem 16 there exists a parameter $k \in \mathbb{R} \setminus \{0\}$ such that $V_{2,k}(e)$ is the 2-dimensional normal subloop $\mathcal{S}_7(e) = \mathcal{S}_8(e)$. Hence one has $\dim(\mathbf{v}_{2,k} \cap \mathbf{inn}(L)) = 3$. Therefore the Lie algebra $\mathbf{inn}(L)$ has the basis element $r_4 = e_6 + ke_3 + l_1e_1 + l_2e_2$ for some $k \in \mathbb{R} \setminus \{0\}$, $l_i \in \mathbb{R}$, $i = 1, 2$.

The subspace $\langle r_1, r_2, r_4\rangle$ is not a 3-dimensional subalgebra of the Lie algebras $\mathbf{g}_{2,3}$, $\mathbf{g}_{2,4}$, $\mathbf{g}_{3,4}$. Hence these Lie algebras cannot be the Lie algebra of the group $Mult(L)$ of L.

The subspace $\langle r_1, r_2, r_4\rangle$ forms a 3-dimensional subalgebra of $\mathbf{g}_{2,2}$ if and only if $k = 1$, $a_2 = 0$ and $b_2 = a_1 \neq 0$. Hence the subalgebra $\mathbf{inn}(L) < \mathbf{g}_{2,2}$ has the form $\mathbf{inn}(L) = \langle e_4+a_1e_1, e_5+b_1e_1+a_1e_2, e_6+e_3+l_1e_1+l_2e_2\rangle$, $a_1 \neq 0$, $b_1, l_i \in \mathbb{R}$.

The subspace $\langle r_1, r_2, r_4\rangle$ forms a 3-dimensional subalgebra of $\mathbf{g}_{3,3}$ if and only if $k = 1$. Hence the subalgebra $\mathbf{inn}(L) < \mathbf{g}_{3,3}$ has the form $\mathbf{inn}(L) = \langle e_4+a_1e_1+a_2e_2, e_5+b_1e_1+b_2e_2, e_6+e_3+l_1e_1+l_2e_2\rangle$ such that at least one of a_1, a_2 as well as b_1, b_2 are different from 0 and $a_1b_2 - a_2b_1 \neq 0$.

The subspace $\langle r_1, r_2, r_4\rangle$ forms a 3-dimensional subalgebra of $\mathbf{g}_{4,4}$ if and only if one has either $a_1 = 0 = b_2$, $k = h_2 = \frac{1}{h_1}$, or $a_2 = 0 = b_1$, $k = 1$, $h_2 = h_1$. Therefore the subalgebra $\mathbf{inn}(L) < \mathbf{g}_{4,4}$ has either the form $\mathbf{inn}(L) = \langle e_4 + a_2e_2, e_5 + b_1e_1, e_6 + ke_3 + l_1e_1 + l_2e_2\rangle$ such that $a_2b_1 \neq 0$, $k = h_2 = \frac{1}{h_1}$, or $\mathbf{inn}(L) = \langle e_4 + a_1e_1, e_5 + b_2e_2, e_6 + e_3 + l_1e_1 + l_2e_2\rangle$ such that $a_1b_2 \neq 0$, $h_1 = h_2$. Using the automorphism $\phi(e_1) = \frac{b_2e_1 - a_2e_2}{a_1b_2 - a_2b_1}$, $\phi(e_2) = \frac{b_1e_1 - a_1e_2}{a_2b_1 - a_1b_2}$, $\phi(e_3) = e_3 - l_1\phi(e_1) - l_2\phi(e_2)$, $\phi(e_i) = e_i$, $i = 4, 5, 6$, of the Lie algebras $\mathbf{g}_{i,i}$, $i = 2, 3, 4$, such that for $\mathbf{g}_{2,2}$ one has $a_2 = 0, b_2 = a_1 \neq 0$ and for $\mathbf{g}_{4,4}$ we have $a_2 = b_1 = 0$, $h_2 = h_1$, we can reduce $\mathbf{inn}(L)$ to $\mathbf{inn}(L)_1 = \langle e_4 + e_1, e_5 + e_2, e_6 + e_3\rangle$. Moreover, the automorphism $\phi(e_1) = \frac{1}{b_1}e_1$, $\phi(e_2) = \frac{1}{a_2}e_2$, $\phi(e_3) = e_3 - \frac{h_1l_1}{b_1}e_1 - \frac{h_1l_2}{a_2}e_2$, $\phi(e_i) = e_i$, $i = 4, 5, 6$, of the Lie algebra $\mathbf{g}_{4,4}$ with $h_2 = \frac{1}{h_1}$ reduces $\mathbf{inn}(L)$ to $\mathbf{inn}(L)_2 = \langle e_4 + e_2, e_5 + e_1, e_6 + \frac{1}{h_1}e_3\rangle$. Linear representations of the simply connected Lie groups $G_{i,i}$, $i = 2, 3, 4$, are given as follows: for $G_{2,2}$

one has

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6) = g(x_1+(y_1+x_3y_2)e^{x_3},$$

$$x_2+y_2e^{x_3}, x_3+y_3, x_4+(y_4+x_6y_5)e^{x_6}, x_5+y_5e^{x_6}, x_6+y_6),$$

for $G_{3,3}$, where $h_1 = 1$, and for $G_{4,4}$ with $h_2 = h_1$ we have

$$g(x_1+y_1e^{x_3}, x_2+y_2e^{h_1x_3}, x_3+y_3, x_4+y_4e^{x_6}, x_5+y_5e^{h_1x_6}, x_6+y_6),$$

for $G_{4,4}$, where $h_2 = \frac{1}{h_1}$, one has

$$g(x_1+y_1e^{x_3}, x_2+y_2e^{h_1x_3}, x_3+y_3, x_4+y_4e^{x_6}, x_5+y_5e^{\frac{x_6}{h_1}}, x_6+y_6).$$

We get that the subgroup $Inn(L)_1$ of $G_{2,2}$, $G_{3,3}$ and $G_{4,4}$ with $h_2 = h_1$ has the form $Inn(L)_1 = \{g(u_1,u_2,u_3,u_1,u_2,u_3); u_i \in \mathbb{R}\}, i = 1,2,3$, and we have $Inn(L)_2 = \{g(u_2,u_1,\frac{1}{h_1}u_3,u_1,u_2,u_3); u_i \in \mathbb{R}\}, i = 1,2,3$ for $G_{4,4}$ with $h_2 = \frac{1}{h_1}$. Two arbitrary left transversals to the groups $Inn(L)_1$ and $Inn(L)_2$ in $G_{i,i}$, $i = 2,3,4$, are

$$S = \{g(u,v,w,f_1(u,v,w),f_2(u,v,w),f_3(u,v,w)) : u,v,w \in \mathbb{R}\},$$

$$T = \{g(k,l,m,g_1(k,l,m),g_2(k,l,m),g_3(k,l,m)) : k,l,m \in \mathbb{R}\},$$

where $f_i(u,v,w) : \mathbb{R}^3 \to \mathbb{R}$ and $g_i(k,l,m) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1,2,3$, are continuous functions with $f_i(0,0,0) = g_i(0,0,0) = 0$. For all $s \in S, t \in T$ the condition $s^{-1}t^{-1}st \in Inn(L)_1$ holds if and only if in the cases $G_{2,2}$, $G_{3,3}$ with $h_1 = 1$ and $G_{4,4}$ with $h_2 = h_1$ the equation

$$le^{-h_1m}(1-e^{-h_1w}) + ve^{-h_1w}(e^{-h_1m}-1) =$$

$$g_2(k,l,m)e^{-h_1g_3(k,l,m)}(1-e^{-h_1f_3(u,v,w)})+$$
$$f_2(u,v,w)e^{-h_1f_3(u,v,w)}(e^{-h_1g_3(k,l,m)}-1), \qquad (6)$$

and for $G_{2,2}$ the equation

$$e^{-m}(1-e^{-w})(k-lm) + e^{-w}(e^{-m}-1)(u-vw) + (wl-mv)e^{-w-m} =$$

$$e^{-g_3(k,l,m)}(1-e^{-f_3(u,v,w)})(g_1(k,l,m)-g_2(k,l,m)g_3(k,l,m))+$$
$$e^{-f_3(u,v,w)}(e^{-g_3(k,l,m)}-1)(f_1(u,v,w)-f_2(u,v,w)f_3(u,v,w))+$$

$$(g_2(k,l,m)f_3(u,v,w) - f_2(u,v,w)g_3(k,l,m))e^{-f_3(u,v,w)-g_3(k,l,m)}, \quad (7)$$

for $G_{3,3}$ with $h_1 = 1$ and for $G_{4,4}$ with $h_2 = h_1$ the equation

$$ke^{-m}(1-e^{-w}) + ue^{-w}(e^{-m}-1) =$$

$$g_1(k,l,m)e^{-g_3(k,l,m)}(1-e^{-f_3(u,v,w)}) + f_1(u,v,w)e^{-f_3(u,v,w)}(e^{-g_3(k,l,m)}-1) \quad (8)$$

are satisfied for all $k,l,m,u,v,w \in \mathbb{R}$. The products $s^{-1}t^{-1}st$ are contained in $Inn(L)_2$ if and only if the equations

$$le^{-h_1 m}(1-e^{-h_1 w}) + ve^{-h_1 w}(e^{-h_1 m}-1) =$$

$$g_1(k,l,m)e^{-g_3(k,l,m)}(1-e^{-f_3(u,v,w)}) + f_1(u,v,w)e^{-f_3(u,v,w)}(e^{-g_3(k,l,m)}-1), \quad (9)$$

$$ke^{-m}(1-e^{-w})+ue^{-w}(e^{-m}-1) = g_2(k,l,m)e^{-\frac{1}{h_1}g_3(k,l,m)}(1-e^{-\frac{1}{h_1}f_3(u,v,w)})$$
$$+f_2(u,v,w)e^{-\frac{1}{h_1}f_3(u,v,w)}(e^{-\frac{1}{h_1}g_3(k,l,m)}-1) \quad (10)$$

are satisfied for all $u,v,w,k,l,m \in \mathbb{R}$. Equations (6), respectively (9) are satisfied precisely if one has $f_3(u,v,w) = w$, $f_2(u,v,w) = v$, $g_3(k,l,m) = m$, $g_2(k,l,m) = l$, respectively $f_3(u,v,w) = h_1 w$, $f_1(u,v,w) = v$, $g_3(k,l,m) = h_1 m$, $g_1(k,l,m) = l$. Then $S \cup T$ does not generate the groups $G_{i,i}$, $i = 2,3$, $G_{4,4}$ with $h_2 = h_1$ and $G_{4,4}$ with $h_2 = \frac{1}{h_1}$. By Lemma 7 the Lie algebras $\mathbf{g}_{i,i}$, $i = 2,3$, $\mathbf{g}_{4,4}$ with $h_2 = h_1$ and with $h_2 = \frac{1}{h_1}$, are not the Lie algebras of the groups $Mult(L)$ of L.

Hence it remains to deal with the Lie algebra $\mathbf{g} = \mathbf{l}_2 \oplus \mathbf{l}_2 \oplus \mathbf{l}_2 = \langle f_1, f_2 \rangle \oplus \langle f_3, f_4 \rangle \oplus \langle f_5, f_6 \rangle$ given by the Lie brackets $[f_1, f_2] = f_1$, $[f_3, f_4] = f_3$, $[f_5, f_6] = f_5$. The Lie algebra $\mathbf{g}$ has the 1-dimensional ideals $\mathbf{i}_1 = \langle f_1 \rangle$, $\mathbf{i}_2 = \langle f_3 \rangle$, $\mathbf{i}_3 = \langle f_5 \rangle$. The ideals $\mathbf{s}_1 = \langle f_1, f_2 \rangle$, $\mathbf{s}_2 = \langle f_3, f_4 \rangle$, $\mathbf{s}_3 = \langle f_5, f_6 \rangle$ have the properties $\mathbf{i}_j < \mathbf{s}_j$ and $\mathbf{g}/\mathbf{s}_j$, $j = 1,2,3$, are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. If $\mathbf{g}$ is the Lie algebra of the multiplication group of L, then the orbits $I_j(e)$, $j = 1,2,3$, where I_j is the simply connected Lie group of $\mathbf{i}_j$, are 1-dimensional normal subloops of L such that the factor loops $L/I_j(e)$ are isomorphic to the 2-dimensional non-abelian Lie group $\mathcal{L}_2$ (cf. Proposition 19 (ii)).

For the ideals $\mathbf{a}_1 = \langle f_1, f_3 \rangle$, $\mathbf{a}_2 = \langle f_1, f_5 \rangle$, $\mathbf{a}_3 = \langle f_3, f_5 \rangle$ the factor Lie algebras $\mathbf{g}/\mathbf{a}_j$, $j = 1,2,3$, are not isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. Hence these ideals and the commutator ideal $\mathbf{g}' = \langle f_1, f_3, f_5 \rangle$ satisfy the condition of Proposition 19 d). Therefore the orbits $A_j(e)$ and $G'(e)$, where A_j, respectively G', is the simply connected Lie group of $\mathbf{a}_j$, $j = 1, 2, 3$, respectively $\mathbf{g}'$, are the same 2-dimensional normal subloop M of L. Furthermore, one has $\mathbf{inn}(\mathbf{L}) \cap \mathbf{a}_j = \{0\}$ for all $j = 1, 2, 3$, and $\dim(\mathbf{g}' \cap \mathbf{inn}(\mathbf{L})) = 1$. The commutator subalgebra $\mathbf{inn}(\mathbf{L})'$ of $\mathbf{inn}(\mathbf{L})$ is the intersection $\mathbf{g}' \cap \mathbf{inn}(\mathbf{L})$. As every element of $\mathbf{inn}(\mathbf{L})'$ is contained in one of the ideals $\mathbf{a}_j$ and $\mathbf{inn}(\mathbf{L}) \cap \mathbf{a}_j = \{0\}$ for all $j = 1, 2, 3$, the Lie algebra $\mathbf{inn}(\mathbf{L})$ is abelian. The 5-dimensional ideals of $\mathbf{g}$ are:

$$\mathbf{v}_1 = \langle f_1, f_3, f_5, f_2 + k_1 f_6, f_4 + k_2 f_6 \rangle, \ \mathbf{v}_2 = \langle f_1, f_3, f_5, f_2 + k_3 f_4, f_6 + k_4 f_4 \rangle,$$

$$\mathbf{v}_3 = \langle f_1, f_3, f_5, f_4 + k_5 f_2, f_6 + k_6 f_2 \rangle, \quad k_i \in \mathbb{R}, \ i = 1, \dots, 6.$$

Each 3-dimensional abelian subalgebra of a 5-dimensional ideal $\mathbf{v}_j$, $j = 1, 2, 3$, contains a non-trivial ideal of $\mathbf{g}$. Hence the Lie algebra $\mathbf{g} = \mathbf{l}_2 \oplus \mathbf{l}_2 \oplus \mathbf{l}_2$ is not the Lie algebra of the group $Mult(L)$. □

5 6-dimensional solvable multiplication group having 1-dimensional centre

In this Chapter we determine the 6-dimensional solvable Lie groups with 1-dimensional centre which are the multiplication groups of 3-dimensional topological loops L. In the class of the 6-dimensional indecomposable solvable Lie groups with 5-dimensional nilradical there are 7 families which are the groups $Mult(L)$ of L (cf. Theorem 24). We find that among the 6-dimensional indecomposable solvable Lie groups with 4-dimensional nilradical only three families can be represented as the group $Mult(L)$ of L (cf. Theorem 25). Finally, there are 18 families of 6-dimensional decomposable solvable Lie groups which are the groups $Mult(L)$ of L (cf. Theorem 26). In all these cases the loop L has 1-dimensional centre and nilpotency class 2. Hence Theorem 15 is valid.

First we consider the case that the Lie algebra $\mathbf{mult}(\mathbf{L})$ of the multiplication group of L is a 6-dimensional solvable indecomposable Lie algebra with 5-dimensional nilradical.

Theorem 24. *Let L be a connected simply connected topological proper loop of dimension 3 such that the Lie algebra of its multiplication group $Mult(L)$ is a 6-dimensional solvable indecomposable Lie algebra having 5-dimensional nilradical. Then L has nilpotency class 2 and the following pairs $(\mathbf{g}, \mathbf{k})$ of Lie algebras are Lie algebra $\mathbf{g}$ of the group $Mult(L)$ and the subalgebra $\mathbf{k}$ of the subgroup $Inn(L)$:*
$\mathbf{g}_1 := \mathbf{g}_{6,14}^{a=b=0}$: $[e_2, e_3] = e_1 = [e_5, e_6]$, $[e_4, e_6] = e_4$, $\mathbf{k}_{1,1} = \langle e_2, e_4 + e_1, e_5 \rangle$, $\mathbf{k}_{1,2} = \langle e_3, e_4 + e_1, e_5 \rangle$;
$\mathbf{g}_2 := \mathbf{g}_{6,22}^{a=0}$: $[e_2, e_3] = e_1 = [e_5, e_6]$, $[e_2, e_6] = e_3$, $[e_4, e_6] = e_4$, $\mathbf{k}_2 = \langle e_3, e_4 + e_1, e_5 \rangle$,
$\mathbf{g}_3 := \mathbf{g}_{6,17}^{\delta=1, a=\varepsilon=0}$: $[e_2, e_3] = e_1 = [e_4, e_6]$, $[e_3, e_6] = e_4$, $[e_5, e_6] = e_5$, $\mathbf{k}_{3,1} = \langle e_3, e_4, e_5 + e_1 \rangle$, $\mathbf{k}_{3,2} = \langle e_2, e_4, e_5 + e_1 \rangle$;
$\mathbf{g}_4 := \mathbf{g}_{6,51}^{\varepsilon=\pm 1}$: $[e_1, e_5] = e_2$, $[e_4, e_5] = e_1$, $[e_3, e_6] = e_3$, $[e_4, e_6] = \varepsilon e_2$, $\mathbf{k}_4 = \langle e_1 + a_1 e_2, e_3 + e_2, e_4 \rangle$, $a_1 \in \mathbb{R}$;
$\mathbf{g}_5 := \mathbf{g}_{6,54}^{a=b=0}$: $[e_3, e_5] = e_1 = [e_1, e_6]$, $[e_4, e_5] = e_2$, $[e_3, e_6] = e_3$, $\mathbf{k}_5 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_6 := \mathbf{g}_{6,63}^{a=0}$: $[e_3, e_5] = e_1 = [e_1, e_6]$, $[e_3, e_6] = e_3$, $[e_4, e_5] = e_2 = [e_4, e_6]$, $\mathbf{k}_6 = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle$, $a_2 \in \mathbb{R}$;
$\mathbf{g}_7 := \mathbf{g}_{6,25}^{a=b=0}$: $[e_2, e_3] = e_1 = [e_1, e_6]$, $[e_2, e_6] = e_2$, $[e_4, e_6] = e_5$, $\mathbf{k}_7 = \langle e_1 + e_5, e_2 + \varepsilon e_5, e_4 \rangle$, $\varepsilon = 0, 1$.
The multiplication groups $Mult(L)$ and the inner mapping groups $Inn(L)$ of L are given by the multiplications on pages $46-47$ in cases $i = 1, \cdots, 7$.

Proof. According to Lemma 9 we may assume that L is homeomorphic to $\mathbb{R}^3$. Firstly, we deal with the 6-dimensional solvable Lie algebras such that their nilradical is isomorphic to the direct sum $\mathbf{f}_3 \oplus \mathbb{R}^2$. These are listed in [36], p. 38. The Lie algebra $\mathbf{g}_{6,27}^{a=1,b=\delta=0}$ has the centre $\mathbf{i} = \langle e_5 \rangle$. For all other Lie algebras $\mathbf{g}_{6,i}$, $i = 13, \ldots, 38$, we consider the ideal $\mathbf{i} = \langle e_1 \rangle$. With exception of the Lie algebras $\mathbf{g}_{6,23}^{\delta=0}$, $\mathbf{g}_{6,24}$ there does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,i}$ such that $\mathbf{i} \leq \mathbf{s}$ and the factor Lie algebras $\mathbf{g}_{6,i}/\mathbf{s}$ are isomorphic to a Lie algebra $\mathbf{f}_n$, $n = 4, 5$. Let I be the simply connected Lie group of the ideal $\mathbf{i}$. If the Lie algebras $\mathbf{g}_{6,i}$, $i = 13, \ldots, 38$, are the Lie algebras of the groups $Mult(L)$ of L, then the orbit $I(e)$ is a normal subloop of L isomorphic to $\mathbb{R}$, the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$ and $I(e) = Z(L)$ (cf. Theorem 17 (a) and Proposition 19 a), e)). Hence the simply connected loop L is a central extension of the group $\mathbb{R}$ by the group $\mathbb{R}^2$. This means it has nilpotency class 2. By Proposition 19 a) (i) the Lie algebra $\mathbf{g}_{6,i}$ has a

4-dimensional abelian ideal $\mathbf{p} = \mathbf{z} \oplus \mathbf{k}$, where $\dim(\mathbf{z}) = 1$ and $\mathbf{k}$ is the Lie algebra of the group $Inn(L)$. One has $\mathbf{g}'_{6,i} < \mathbf{p}$. According to Lemma 8 the subalgebra $\mathbf{k}$ does not contain any non-zero ideal of $\mathbf{g}_{6,i}$ and the normalizer $N_{\mathbf{g}_{6,i}}(\mathbf{k})$ of $\mathbf{k}$ in $\mathbf{g}_{6,i}$ is $\mathbf{p}$. Then for the triples $(\mathbf{g}_{6,i}, \mathbf{p}, \mathbf{k})$ we obtain:
(a) The Lie algebras $\mathbf{g}_{6,i}^{a=0}$, $i = 21, 22, 36$, and $\mathbf{g}_{6,j}$, $j = 24, 30$, have the centre $\mathbf{z} = \langle e_1 \rangle$ and $\mathbf{p}$ is $\langle e_1, e_3, e_4, e_5 \rangle$. The subalgebra $\mathbf{k}$ has the form: $\mathbf{k}_{a_1,a_2,a_3} = \langle e_3 + a_1 e_1, e_4 + a_2 e_1, e_5 + a_3 e_1 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that
in the case $\mathbf{g}_{6,21}^{a=0}$: $a_2 \neq 0$, $a_3 \neq 0$ since $\langle e_4 \rangle$ and $\langle e_5 \rangle$ are ideals of $\mathbf{g}_{6,21}^{a=0}$,
in the case $\mathbf{g}_{6,22}^{a=0}$: $a_2 \neq 0$ because $\langle e_4 \rangle$ is an ideal of $\mathbf{g}_{6,22}^{a=0}$,
in the cases $\mathbf{g}_{6,i}$, $i = 24, 30$: $a_3 \neq 0$ since $\langle e_5 \rangle$ is an ideal of $\mathbf{g}_{6,i}$,
in the case $\mathbf{g}_{6,36}^{a=0}$: $a_2 \neq 0$ or $a_3 \neq 0$ because $\langle e_4, e_5 \rangle$ is an ideal of $\mathbf{g}_{6,36}^{a=0}$.
Using the automorphism $\alpha(e_i) = e_i$, $i = 1, 2$, $\alpha(e_3) = e_3 - a_1 e_1$, $\alpha(e_4) = a_2 e_4$, $\alpha(e_5) = a_3 e_5$, $\alpha(e_6) = e_6 - a_1 e_3$ for $\mathbf{g}_{6,21}^{a=0}$, respectively $\alpha(e_5) = e_5 - a_3 e_1$ for $\mathbf{g}_{6,22}^{a=0}$, respectively $\alpha(e_4) = e_4 - a_2 e_1$, $\alpha(e_6) = e_6 + a_2 e_2 - a_1 e_3$ for $\mathbf{g}_{6,24}$, respectively $\alpha(e_j) = a_3 e_j$, $j = 4, 5$, for $\mathbf{g}_{6,30}$, the Lie algebra $\mathbf{k}_{a_1,a_2,a_3}$ reduces to $\mathbf{k} = \langle e_3, e_4 + e_1, e_5 + e_1 \rangle$, respectively $\mathbf{k} = \langle e_3, e_4 + e_1, e_5 \rangle$, respectively $\mathbf{k} = \langle e_3, e_4, e_5 + e_1 \rangle$, respectively $\mathbf{k}_{a_2} = \langle e_3, e_4 + a_2 e_1, e_5 + e_1 \rangle$, $a_2 \in \mathbb{R}$.
Applying the automorphism $\alpha(e_i) = e_i$, $i = 1, 2$, $\alpha(e_3) = e_3 - a_1 e_1$, $\alpha(e_j) = a_2 e_j$, $j = 4, 5$, $\alpha(e_6) = e_6 - a_1 e_3$ for the Lie algebra $\mathbf{g}_{6,36}^{a=0}$, if $a_2 \neq 0$, respectively $\alpha(e_j) = a_3 e_j$, $j = 4, 5$, if $a_2 = 0$ and $a_3 \neq 0$, we can reduce $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_{a_3} = \langle e_3, e_4 + e_1, e_5 + a_3 e_1 \rangle$, $a_3 \in \mathbb{R}$, respectively $\mathbf{k}_{a_1,0,a_3}$ to $\mathbf{k} = \langle e_3, e_4, e_5 + e_1 \rangle$.
(b) The Lie algebras $\mathbf{g}_{6,14}^{a=b=0}$ and $\mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$ have the centre $\mathbf{z} = \langle e_1 \rangle$ and the ideal $\mathbf{p}$ has one of the forms: $\mathbf{p}_{1,k} = \langle e_1, e_2 + k e_3, e_4, e_5 \rangle$, $k \in \mathbb{R}$, and $\mathbf{p}_2 = \langle e_1, e_3, e_4, e_5 \rangle$. With respect to the ideals $\mathbf{p}_{1,k}$, $\mathbf{p}_2$ we obtain the subalgebras $\mathbf{k}_{1,k} = \langle e_2 + k e_3 + a_1 e_1, e_4 + a_2 e_1, e_5 + a_3 e_1 \rangle$, $k \in \mathbb{R}$, $\mathbf{k}_2 = \langle e_3 + a_1 e_1, e_4 + a_2 e_1, e_5 + a_3 e_1 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that for $\mathbf{g}_{6,14}^{a=b=0}$ one has $a_2 \neq 0$, since $\langle e_4 \rangle$ is an ideal of $\mathbf{g}_{6,14}^{a=b=0}$, and for $\mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$ we get $a_3 \neq 0$ because $\langle e_5 \rangle$ is an ideal of $\mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$. The automorphism $\alpha(e_i) = e_i$, $i = 1, 6$, $\alpha(e_4) = a_2 e_4$, $\alpha(e_5) = e_5 - a_3 e_1$, $\alpha(e_2) = e_2 - k e_3 - a_1 e_1$, $\alpha(e_3) = e_3$, respectively $\alpha(e_2) = e_2$, $\alpha(e_3) = e_3 - a_1 e_1$, of $\mathbf{g}_{6,14}^{a=b=0}$ maps the subalgebra $\mathbf{k}_{1,k}$ onto $\mathbf{k} = \langle e_2, e_4 + e_1, e_5 \rangle$, respectively $\mathbf{k}_2$ onto $\mathbf{k} = \langle e_3, e_4 + e_1, e_5 \rangle$. The automorphism $\alpha(e_1) = e_1$, $\alpha(e_4) = e_4 - a_2 e_1$, $\alpha(e_5) = a_3 e_5$, $\alpha(e_6) = e_6 + a_2 e_2$, $\alpha(e_2) = e_2 - a_1 e_1 - k e_3$, $\alpha(e_3) = e_3$, respectively $\alpha(e_3) = e_3 - a_1 e_1$, $\alpha(e_2) = e_2$, of $\mathbf{g}_{6,17}^{\delta=1,a=\varepsilon=0}$

maps the subalgebra $\mathbf{k}_{1,k}$ onto $\mathbf{k} = \langle e_2, e_4, e_5 + e_1 \rangle$, respectively $\mathbf{k}_2$ onto $\mathbf{k} = \langle e_3, e_4, e_5 + e_1 \rangle$.
(c) The Lie algebras $\mathbf{g}_{6,17}^{\delta=\varepsilon=0,a\neq 0}$ and $\mathbf{g}_{6,17}^{\delta=0,a=\varepsilon=1}$ have the centre $\mathbf{z} = \langle e_4 \rangle$ and the ideal $\mathbf{p}$ equals to $\langle e_1, e_2, e_4, e_5 \rangle$. Hence the subalgebra $\mathbf{k}$ has the form $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 e_4, e_2 + a_2 e_4, e_5 + a_3 e_4 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that for $\mathbf{g}_{6,17}^{\delta=\varepsilon=0,a\neq 0}$ we have $a_1 \neq 0$, $a_3 \neq 0$, since it has the ideals $\langle e_1 \rangle$, $\langle e_5 \rangle$, and for $\mathbf{g}_{6,17}^{\delta=0,a=\varepsilon=1}$ one obtains $a_1 \neq 0$ because it has the ideal $\langle e_1 \rangle$. With the automorphism $\alpha(e_i) = a_1 e_i$, $i = 1, 2$, $\alpha(e_j) = e_j$, $j = 3, 4, 6$, $\alpha(e_5) = a_3 e_5$ of $\mathbf{g}_{6,17}^{\delta=\varepsilon=0,a\neq 0}$, respectively $\alpha(e_5) = a_1 e_5$ for $\mathbf{g}_{6,17}^{\delta=0,a=\varepsilon=1}$, we can change the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{a_2} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + e_4 \rangle$, $a_2 \in \mathbb{R}$, respectively $\mathbf{k}_{a_2,a_3} = \langle e_1 + e_4, e_2 + a_2 e_4, e_5 + a_3 e_4 \rangle$, $a_2, a_3 \in \mathbb{R}$.
(d) The Lie algebras $\mathbf{g}_{6,15}^{a=0}$, $\mathbf{g}_{6,16}$, $\mathbf{g}_{6,25}^{a=b=0}$, $\mathbf{g}_{6,27}^{a=1,b=\delta=0}$ have the centre $\mathbf{z} = \langle e_5 \rangle$ and their ideal $\mathbf{p}$ is $\langle e_1, e_2, e_4, e_5 \rangle$. Therefore the subalgebra $\mathbf{k}$ has the form $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 e_5, e_2 + a_2 e_5, e_4 + a_3 e_5 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$. For $\mathbf{g}_{6,15}^{a=0}$ one has $a_1 \neq 0$, $a_3 \neq 0$ since $\langle e_1 \rangle$, $\langle e_4 \rangle$ are ideals of $\mathbf{g}_{6,15}^{a=0}$. For $\mathbf{g}_{6,k}$, $k = 16, 25, 27$, we have $a_1 \neq 0$ because $\langle e_1 \rangle$ is an ideal of $\mathbf{g}_{6,k}$. For $\mathbf{g}_{6,15}^{a=0}$ and $\mathbf{g}_{6,16}$ using the automorphism $\alpha(e_i) = a_1 e_i$, $i = 1, 2, 4$, $\alpha(e_j) = e_j$, $j = 3, 5, 6$, the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ reduces to $\mathbf{k}_{a_2} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 + a_3 e_5 \rangle$. For $\mathbf{g}_{6,25}^{a=b=0}$ applying the automorphism $\alpha(e_i) = e_i$, $i = 5, 6$, $\alpha(e_1) = a_1 e_1$, $\alpha(e_4) = e_4 - a_3 e_5$, $\alpha(e_2) = a_2 e_2$, $\alpha(e_3) = \frac{a_1}{a_2} e_3$, if $a_2 \neq 0$, respectively $\alpha(e_2) = e_2$, $\alpha(e_3) = a_1 e_3$, if $a_2 = 0$, we can change the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k} = \langle e_1 + e_5, e_2 + e_5, e_4 \rangle$, respectively $\mathbf{k}_{a_1,0,a_3}$ to $\mathbf{k} = \langle e_1 + e_5, e_2, e_4 \rangle$. The automorphism $\alpha(e_i) = e_i$, $i = 5, 6$, $\alpha(e_j) = a_1 e_j$, $j = 1, 2$, $\alpha(e_3) = e_3 - a_3 e_4$, $\alpha(e_4) = e_4 - a_3 e_5$ of $\mathbf{g}_{6,27}^{a=1,b=\delta=0}$ maps $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{a_2} = \langle e_1 + e_5, e_2 + a_2 e_5, e_4 \rangle$. Now we consider the exceptional cases: $\mathbf{g}_{6,23}^{\delta=0}$ and $\mathbf{g}_{6,24}$ with its ideal $\mathbf{s}$ such that $\mathbf{g}_{6,24}/\mathbf{s} \cong \mathbf{f}_4$. The Lie algebra $\mathbf{g}_{6,23}^{\delta=0}$ has 2-dimensional centre. Hence it is excluded by Theorem 20.

The Lie algebra $\mathbf{g}_{6,24}$ has the centre $\mathbf{z} = \langle e_1 \rangle$ and the ideals $\mathbf{i}_2 = \langle e_5 \rangle$, $\mathbf{s} = \langle e_1, e_5 \rangle$, $\mathbf{a} = \langle e_1, e_4 \rangle$, $\mathbf{b} = \langle e_1, e_3, e_4 \rangle$, $\mathbf{g}_{6,24}' = \langle e_1, e_3, e_4, e_5 \rangle$. Let Z, I_2, $\mathcal{S}$, A, B, N be the simply connected Lie groups of $\mathbf{z}$, $\mathbf{i}_2$, $\mathbf{s}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{g}_{6,24}'$ in this order. The factor Lie algebra $\mathbf{g}_{6,24}/\mathbf{s}$ is isomorphic to the Lie algebra $\mathbf{f}_4$. If $\mathbf{g}_{6,24}$ is the Lie algebra of the group $Mult(L)$ of L, then from the above discussion it follows that the factor loop $L/Z(e) = L/\mathcal{S}(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. Since $Z(e) = \mathbb{R} = \mathcal{S}(e)$, one has $\dim(\mathbf{s} \cap \mathbf{inn(L)}) = 1$. The orbit $I_2(e)$ is a normal subgroup of L isomorphic to $\mathbb{R}$ (cf. Proposition 19 a). As $\mathbf{i}_2 < \mathbf{s}$ we have $I_2(e) = \mathcal{S}(e)$ and $\mathbf{i}_2 \cap \mathbf{inn(L)} = 0$. For the ideals $\mathbf{a}$, $\mathbf{b}$, $\mathbf{g}'_{6,24}$ the conditions of Proposition 19 b), e) are satisfied. Since $\mathbf{z} < \mathbf{a}$ the orbit $A(e)$ contains the centre $Z(e)$ of L. If $\dim(A(e)) = 1$, then $A(e) = Z(e)$. As the factor Lie algebra $\mathbf{g}_{6,24}/\mathbf{a}$ is not isomorphic to the Lie algebra $\mathbf{f}_4$, the factor loop $L/Z(e)$ cannot be isomorphic to a loop $L_{\mathcal{F}}$. Hence one has $\dim(A(e)) = 2$

According to Proposition 19 b) we obtain that $A(e) = B(e) = N(e) = M$, where M is a 2-dimensional connected normal subloop of L such that $\mathbf{a} \cap \mathbf{inn(L)} = 0$, $\mathbf{b} \cap \mathbf{inn(L)}$ has dimension 1 whereas $\mathbf{g}'_{6,24} \cap \mathbf{inn(L)}$ has dimension 2 and $Z(e) < M$. For the ideal $\mathbf{v}$ in Proposition 19 b) we obtain one of the following forms: $\mathbf{v}_{1,k} = \langle e_1, e_3, e_4, e_5, e_2 + ke_6 \rangle$, $k \in \mathbb{R}$, $\mathbf{v}_2 = \langle e_1, e_3, e_4, e_5, e_6 \rangle$. Hence the Lie algebra $\mathbf{inn(L)}$ has either the generators $b_1 = e_3 + a_1 e_1 + a_2 e_4$, $b_2 = e_5 + b_1 e_1$, $b_{3,k} = e_2 + ke_6 + c_1 e_1 + c_2 e_4$ or $b_1, b_2, b_3 = e_6 + c_1 e_1 + c_2 e_4$, $a_i, b_1, k, c_i \in \mathbb{R}$, $i = 1, 2$, $b_1 \neq 0$. None of the vector spaces $\langle b_1, b_2, b_{3,k} \rangle$, $\langle b_1, b_2, b_3 \rangle$ are 3-dimensional Lie algebras.

Now we deal with the 6-dimensional solvable indecomposable Lie algebras having nilradical isomorphic either to the direct sum $\mathbf{f}_4 \oplus \mathbb{R}$ or to the Lie algebra defined by $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$. These Lie algebras are listed in [36], p. 39, and denoted by $\mathbf{g}_{6,i}$, $i = 39, \dots, 70$. The Lie algebra $\mathbf{g}_{6,70}$ has trivial centre and the unique minimal ideal $\mathbf{i} = \langle e_1, e_2 \rangle$. Let I be the simply connected Lie group of $\mathbf{i}$. By Theorems 12, 16 and Proposition 19 e) the orbit $I(e)$ is a 1-dimensional normal subloop of L such that the factor loop $L/I(e)$ is isomorphic to a loop $L_{\mathcal{F}}$. The factor Lie algebra $\mathbf{g}_{6,70}/\mathbf{i}$ is not isomorphic to the elementary filiform Lie algebra $\mathbf{f}_4$. Hence the Lie algebra $\mathbf{g}_{70}$ is not the group $Mult(L)$ of L.

All other Lie algebras have the ideal $\mathbf{i} = \langle e_2 \rangle$. With the exception of the Lie algebra $\mathbf{g}_{6,52}$ there does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,i}$, $i = 39, \dots, 69$, such that $\mathbf{i} \leq \mathbf{s}$ and the factor Lie algebras $\mathbf{g}_{6,i}/\mathbf{s}$ are isomorphic to a Lie algebra $\mathbf{f}_n$, $n \in \{4, 5\}$. By Proposition 19 a) and e), if $\mathbf{g}_{6,i}$, $i = 39, \dots, 69$, would be the Lie algebra of the group $Mult(L)$ of L, then the simply connected loop L has a 1-dimensional centre $Z(L) = I(e) \cong \mathbb{R}$, where $I = \exp \mathbf{i}$, and the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$. Hence L has nilpotency class 2. According to Proposition 19 a) (i) and c) we seek for Lie algebras $\mathbf{g}_{6,i}$ such that the nilradical of $\mathbf{g}_{6,i}$ contains an ideal $\mathbf{p} = \mathbf{z} \oplus \mathbf{inn(L)} \cong \mathbb{R}^4$ of $\mathbf{g}_{6,i}$ and $\mathbf{g}'_{6,i}$ lies in $\mathbf{p}$. Here $\mathbf{z}$ is the 1-dimensional centre of $\mathbf{g}_{6,i}$. By Lemma 8 the Lie algebra $\mathbf{k}$ does not contain any non-zero ideal of $\mathbf{g}_{6,i}$ and the normalizer $N_{\mathbf{g}_{6,i}}(\mathbf{k})$ of $\mathbf{k}$ in $\mathbf{g}_{6,i}$ is $\mathbf{p}$. The following pairs $(\mathbf{g}_{6,i}, \mathbf{k})$ have

the above properties:
(a) The Lie algebra $\mathbf{g}_{6,49}$ has the centre $\mathbf{z} = \langle e_3 \rangle$ and $\mathbf{p} = \langle e_1, e_2, e_3, e_4 \rangle$. Hence for the subalgebra $\mathbf{k}$ we obtain $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 e_3, e_2 + a_2 e_3, e_4 + a_3 e_3 \rangle$, $a_2 \neq 0$, because $\langle e_2 \rangle$ is an ideal of $\mathbf{g}_{6,49}$ and $a_1, a_3 \in \mathbb{R}$. The automorphism $\alpha(e_i) = a_2 e_i$, $i = 1, 2, 4$, $\alpha(e_j) = e_j$, $j = 3, 5, 6$, maps the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{a_1,a_3} = \langle e_1 + a_1 e_3, e_2 + e_3, e_4 + a_3 e_3 \rangle$.
(b) The Lie algebras $\mathbf{g}_{6,k}$, $k = 51, 52$, $\mathbf{g}^{a=b=0}_{6,54}$, $\mathbf{g}^{a=0}_{6,57}$, $\mathbf{g}_{6,59}$, $\mathbf{g}^{a=0}_{6,63}$ have the centre $\mathbf{z} = \langle e_2 \rangle$ and the ideal $\mathbf{p}$ equals to $\langle e_1, e_2, e_3, e_4 \rangle$. Hence the Lie algebra $\mathbf{k}$ has the form $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 e_2, e_3 + a_2 e_2, e_4 + a_3 e_2 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that $a_2 \neq 0$ for the Lie algebras $\mathbf{g}_{6,k}$, $k = 51, 52$, since $\langle e_3 \rangle$ is their ideal, and $a_1 \neq 0$ for the Lie algebras $\mathbf{g}_{6,k}$, $k = 54, 57, 59, 63$, because $\langle e_1 \rangle$ is their ideal. Applying the automorphism $\alpha(e_i) = e_i$, $i = 1, 2, 5$, $\alpha(e_3) = a_2 e_3$, $\alpha(e_4) = e_4 - a_3 e_2$, $\alpha(e_6) = e_6$ for $\mathbf{g}_{6,51}$, respectively $\alpha(e_6) = e_6 + a_3 e_1$ for $\mathbf{g}_{6,52}$, the subalgebra $\mathbf{k}_{a_1,a_2,a_3}$ reduces to $\mathbf{k}_{a_1} = \langle e_1 + a_1 e_2, e_3 + e_2, e_4 \rangle$. The automorphism $\alpha(e_i) = e_i$, $i = 2, 5, 6$, $\alpha(e_j) = a_1 e_j$, $j = 1, 3$, $\alpha(e_4) = e_4 - a_3 e_2$ for $\mathbf{g}_{6,k}$, $k = 54, 63$, respectively $\alpha(e_6) = e_6 + a_3 e_4$ for $\mathbf{g}_{6,l}$, $l = 57, 59$, maps $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{a_2} = \langle e_1 + e_2, e_3 + a_2 e_2, e_4 \rangle$.
Now we consider the Lie algebra $\mathbf{g}_{6,52}$ with its ideal $\mathbf{s}$ such that the factor Lie algebra $\mathbf{g}_{6,52}/\mathbf{s}$ is isomorphic to $\mathbf{f}_4$. The Lie algebra $\mathbf{g}_{6,52}$ has the centre $\mathbf{z} = \langle e_2 \rangle$ and the ideals $\mathbf{i}_2 = \langle e_3 \rangle$, $\mathbf{s} = \langle e_2, e_3 \rangle$, $\mathbf{a} = \langle e_1, e_2 \rangle$, $\mathbf{b}_1 = \langle e_1, e_2, e_3 \rangle$, $\mathbf{b}_2 = \langle e_1, e_2, e_4 \rangle$, $\mathbf{g}'_{6,52} = \langle e_1, e_2, e_3, e_4 \rangle$. Denote by Z, I_2, $\mathcal{S}$, A, B_i, $i = 1, 2$, and N the simply connected Lie groups of $\mathbf{z}$, $\mathbf{i}_2$, $\mathbf{s}$, $\mathbf{a}$, $\mathbf{b}_i$, $i = 1, 2$, and $\mathbf{g}'_{6,52}$. If $\mathbf{g}_{6,52}$ would be the Lie algebra of the group $Mult(L)$ of L, then the above discussion yields that the factor loop $L/Z(e) = L/\mathcal{S}(e)$ is isomorphic to a loop $L_{\mathcal{F}}$, because $\mathbf{g}_{6,52}/\mathbf{s} \cong \mathbf{f}_4$. As $Z(e) = \mathbb{R} = \mathcal{S}(e)$, we have $\dim(\mathbf{s} \cap \mathbf{inn}(\mathbf{L})) = 1$. Since the orbit $I_2(e)$ is a normal subgroup of L isomorphic to $\mathbb{R}$ and $\mathbf{i}_2 < \mathbf{s}$, we obtain $I_2(e) = \mathcal{S}(e)$ and $\mathbf{i}_2 \cap \mathbf{inn}(\mathbf{L}) = 0$. The ideals $\mathbf{a}$, $\mathbf{b}_i$, $i = 1, 2$, and $\mathbf{g}'_{6,52}$ have the properties as in Proposition 19 b). Since $\mathbf{z} < \mathbf{a}$, one has $Z(e) < A(e)$. If $\dim(A(e)) = 1$, then we get $A(e) = Z(e)$. Since the factor Lie algebra $\mathbf{g}_{6,52}/\mathbf{a}$ is not isomorphic to the Lie algebra $\mathbf{f}_4$, the factor loop $L/Z(e)$ is not isomorphic to a loop $L_{\mathcal{F}}$. Hence we get $\dim(A(e)) = 2$.

If the orbit $A(e)$ is a 2-dimensional connected normal subloop M of L, then one has $B_1(e) = B_2(e) = N(e) = M$ (cf. Proposition 19 b) such that $Z(e) < A(e)$. The ideal $\mathbf{v}$ in Proposition 19 b) has one of the following forms: $\mathbf{v}_{1,k} = \langle e_1, e_2, e_3, e_4, e_5 + ke_6\rangle$, $k \in \mathbb{R}$, $\mathbf{v}_2 = \langle e_1, e_2, e_3, e_4, e_6\rangle$. Therefore the Lie algebra $\mathbf{inn}(\mathbf{L})$ has either the basis elements $b_1 = e_3 + a_1e_2$, $b_2 = e_4 + b_1e_1 + b_2e_2$, $b_{3,k} = e_5 + ke_6 + c_1e_1 + c_2e_2$ or b_1, b_2, $b_3 = e_6 + c_1e_1 + c_2e_2, a_1, b_i, k, c_i \in \mathbb{R}$, $i = 1, 2$, $a_1 \neq 0$. The vector spaces $\langle b_1, b_2, b_{3,k}\rangle$, $\langle b_1, b_2, b_3\rangle$ are not 3-dimensional Lie algebras. Hence the Lie $\mathfrak{g}_{6,52}$ is excluded.

Summarizing the above discussion the following 6-dimensional solvable indecomposable Lie algebras with 5-dimensional nilradical can occur as the Lie algebra $\mathfrak{g}$ of the group $Mult(L)$ of L:

$\mathfrak{g}_1 := \mathfrak{g}_{6,14}^{a=b=0}$, $\mathbf{k}_{1,1} = \langle e_2, e_4 + e_1, e_5\rangle$, $\mathbf{k}_{1,2} = \langle e_3, e_4 + e_1, e_5\rangle$;
$\mathfrak{g}_2 := \mathfrak{g}_{6,22}^{a=0}$, $\mathbf{k}_2 = \langle e_3, e_4 + e_1, e_5\rangle$,
$\mathfrak{g}_3 := \mathfrak{g}_{6,17}^{\delta=1,a=\varepsilon=0}$, $\mathbf{k}_{3,1} = \langle e_3, e_4, e_5 + e_1\rangle$, $\mathbf{k}_{3,2} = \langle e_2, e_4, e_5 + e_1\rangle$;
$\mathfrak{g}_4 := \mathfrak{g}_{6,51}^{\varepsilon=\pm1}$, $\mathbf{k}_4 = \langle e_1 + a_1e_2, e_3 + e_2, e_4\rangle$, $a_1 \in \mathbb{R}$;
$\mathfrak{g}_5 := \mathfrak{g}_{6,54}^{a=b=0}$, $\mathbf{k}_5 = \langle e_1 + e_2, e_3 + a_2e_2, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathfrak{g}_6 := \mathfrak{g}_{6,63}^{a=0}$, $\mathbf{k}_6 = \langle e_1 + e_2, e_3 + a_2e_2, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathfrak{g}_7 := \mathfrak{g}_{6,25}^{a=b=0}$, $\mathbf{k}_7 = \langle e_1 + e_5, e_2 + \varepsilon e_5, e_4\rangle$, $\varepsilon = 0, 1$;
$\mathfrak{g}_8 := \mathfrak{g}_{6,15}^{a=0}$, $\mathbf{k}_8 = \langle e_1 + e_5, e_2 + a_2e_5, e_4 + a_3e_5\rangle$, $a_3 \in \mathbb{R} \setminus \{0\}$, $a_2 \in \mathbb{R}$;
$\mathfrak{g}_9 := \mathfrak{g}_{6,21}^{a=0,0<|b|\leq1}$, $\mathbf{k}_9 = \langle e_3, e_4 + e_1, e_5 + e_1\rangle$;
$\mathfrak{g}_{10} := \mathfrak{g}_{6,24}$, $\mathbf{k}_{10} = \langle e_3, e_4, e_5 + e_1\rangle$;
$\mathfrak{g}_{11} := \mathfrak{g}_{6,30}$, $\mathbf{k}_{11} = \langle e_3, e_4 + a_2e_1, e_5 + e_1\rangle$, $a_2 \in \mathbb{R}$;
$\mathfrak{g}_{12} := \mathfrak{g}_{6,36}^{a=0,b\geq0}$, $\mathbf{k}_{12,1} = \langle e_3, e_4, e_5 + e_1\rangle$, $\mathbf{k}_{12,2} = \langle e_3, e_4 + e_1, e_5 + a_3e_1\rangle$, $a_3 \in \mathbb{R}$;
$\mathfrak{g}_{13} := \mathfrak{g}_{6,16}$, $\mathbf{k}_{13} = \langle e_1 + e_5, e_2 + a_2e_5, e_4 + a_3e_5\rangle$, $a_2, a_3 \in \mathbb{R}$;
$\mathfrak{g}_{14} := \mathfrak{g}_{6,27}^{a=1,b=\delta=0}$, $\mathbf{k}_{14} = \langle e_1 + e_5, e_2 + a_2e_5, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathfrak{g}_{15} := \mathfrak{g}_{6,49}^{\varepsilon=0,\pm1}$, $\mathbf{k}_{15} = \langle e_1 + a_1e_3, e_2 + e_3, e_4 + a_3e_3\rangle$, $a_1, a_3 \in \mathbb{R}$;
$\mathfrak{g}_{16} := \mathfrak{g}_{6,52}^{\varepsilon=0,\pm1}$, $\mathbf{k}_{16} = \langle e_1 + a_1e_2, e_3 + e_2, e_4\rangle$, $a_1 \in \mathbb{R}$;
$\mathfrak{g}_{17} := \mathfrak{g}_{6,57}^{a=0}$, $\mathbf{k}_{17} = \langle e_1 + e_2, e_3 + a_2e_2, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathfrak{g}_{18} := \mathfrak{g}_{6,59}^{\delta=1}$, $\mathbf{k}_{18} = \langle e_1 + e_2, e_3 + a_2e_2, e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathfrak{g}_{19} := \mathfrak{g}_{6,17}^{\delta=\varepsilon=0,a\neq0}$, $\mathbf{k}_{19} = \langle e_1 + e_4, e_2 + a_2e_4, e_5 + e_4\rangle$, $a_2 \in \mathbb{R}$;
$\mathfrak{g}_{20} := \mathfrak{g}_{6,17}^{\delta=0,a=\varepsilon=1}$, $\mathbf{k}_{20} = \langle e_1 + e_4, e_2 + a_2e_4, e_5 + a_3e_4\rangle$, $a_2, a_3 \in \mathbb{R}$.

In this list the Lie algebra $\mathbf{k}$ is the Lie algebra of the group $Inn(L)$. Now we determine a suitable linear reporesentation for the simply connected Lie groups of the Lie algebras $\mathfrak{g}_i$, $i = 1, \cdots, 20$. To obtain this we make the following procedure. In [40] a single matrix M is established depending on six variables such that the span of the matrices engenders the given Lie algebra in the list $\mathbf{g}_i$, $i = 1, \cdots, 20$. To obtain the matrix Lie group G_i of the Lie algebra $\mathbf{g}_i$ we exponentiate the space of matrices spanned by the matrix M. Simplifying the obtained exponential image we get a suitable simple form of a matrix Lie group such that by differentiating and evaluating at the identity its Lie algebra is isomorphic to the Lie algebra $\mathbf{g}_i$. In case of the Lie algebras $\mathbf{g}_j$, $j = 1, 2, 8, 9, 16$, we take in order the exponential image of the matrices:

$$M_1 = \begin{pmatrix} 0 & -s_3 & s_2 & 0 & -s_6 & 2s_1 \\ 0 & 0 & 0 & 0 & 0 & s_2 \\ 0 & 0 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & -s_6 & 0 & s_4 \\ 0 & 0 & 0 & 0 & 0 & 2s_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ s_i \in \mathbb{R}, i = 1, \cdots, 6,$$

$$M_2 = \begin{pmatrix} 0 & -s_3 & s_2 & 0 & -s_6 & 2s_1 \\ 0 & 0 & 0 & 0 & 0 & s_2 \\ 0 & -s_6 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & -s_6 & 0 & s_4 \\ 0 & 0 & 0 & 0 & 0 & 2s_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ s_i \in \mathbb{R}, i = 1, \cdots, 6,$$

$$M_8 = \begin{pmatrix} -s_6 & -s_3 & -s_2 & 0 & 0 & 2s_1 \\ 0 & -s_6 & 0 & 0 & 0 & s_2 \\ 0 & 0 & 0 & 0 & 0 & -s_3 \\ 0 & -s_6 & 0 & -s_6 & 0 & s_4 \\ 0 & 0 & -s_6 & 0 & 0 & -s_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ s_i \in \mathbb{R}, i = 1, \cdots, 6,$$

$$M_9 = \begin{pmatrix} 0 & -s_3 & s_2 & 0 & 0 & 2s_1 \\ 0 & 0 & 0 & 0 & 0 & s_2 \\ 0 & -s_6 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & -s_6 & 0 & s_4 \\ 0 & 0 & 0 & 0 & -bs_6 & s_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ s_i \in \mathbb{R}, i = 1, \cdots, 6,$$

$$M_{16}=\begin{pmatrix} -s_6 & 0 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 2s_5 & -\varepsilon s_6 & \varepsilon s_4 & 2s_2 \\ 0 & 0 & 0 & s_5 & 0 & -s_1 \\ 0 & 0 & 0 & 0 & s_5 & s_4 \\ 0 & 0 & 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ s_i\in\mathbb{R}, \varepsilon=0,\pm 1,$$

$i=1,\cdots,6$. The simply connected Lie groups G_i and its subgroups K_i of the Lie algebras $\mathbf{g}_i$ and its subalgebras $\mathbf{k}_i$, $i=1,\cdots,20$, are isomorphic to the linear groups of matrices the multiplication of which are given in this order by:

$$i=1: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$

$$g(x_1+y_1+x_2y_3-x_3y_2-x_6y_5, x_2+y_2, x_3+y_3, x_4+y_4e^{-x_6}, x_5+y_5, x_6+y_6),$$

$$K_{1,1}=\{g(u_1,u_3,0,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$

$$K_{1,2}=\{g(u_1,0,u_3,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$

$$i=2: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=g(x_1+y_1+x_2y_3-x_3y_2-x_6(y_5+x_2y_2), x_2+y_2, x_3+y_3-x_6y_2, x_4+y_4e^{-x_6}, x_5+y_5, x_6+y_6),$$

$$K_2=\{g(u_1,0,u_3,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$

$$i=3: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=g(x_1+y_1-x_6y_4+(\frac{1}{2}x_6^2+x_3)y_2, x_2+y_2, x_3+y_3, x_4+y_4-x_6y_2, x_5+y_5e^{-x_6}, x_6+y_6),$$

$$K_{3,1}=\{g(u_2,u_3,0,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$

$$K_{3,2}=\{g(u_2,0,u_3,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$

$$i=4: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=g(x_1+y_1+x_5y_4, x_2+y_2+x_5y_1+\varepsilon x_4y_6+\frac{1}{2}x_5^2y_4, x_3+y_3e^{-x_6}, x_4+y_4, x_5+y_5, x_6+y_6),$$

$$K_4=\{g(u_1,a_1u_1+u_2,u_2,u_3,0,0); u_i\in\mathbb{R}, i=1,2,3\}, a_1\in\mathbb{R}, \varepsilon=\pm 1,$$

$$i=5: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$

$$g(x_1+(y_1+x_5y_3)e^{-x_6}, x_2+y_2+x_5y_4, x_3+y_3e^{-x_6}, x_4+y_4, x_5+y_5, x_6+y_6),$$

$$K_5=\{g(u_1,u_1+a_2u_2,u_2,u_3,0,0); u_i\in\mathbb{R}, i=1,2,3\}, a_2\in\mathbb{R},$$

$$i = 6 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + (y_1 + y_3x_5)e^{-x_6}, x_2 + y_2 - (x_5 + x_6)y_4,$$
$$x_3 + y_3e^{-x_6}, x_4 + y_4, x_5 + y_5, x_6 + y_6),$$
$$K_6 = \{g(u_1, u_1 + a_2u_2, u_2, u_3, 0, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R},$$
$$i = 7 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1+(y_1+y_2x_3)e^{-x_6}, x_2+y_2e^{-x_6}, x_3+y_3, x_4+y_4, x_5+y_5-x_4y_6, x_6+y_6),$$
$$K_7 = \{g(u_1, u_2, 0, u_3, u_1 + \varepsilon u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},\ \varepsilon = 0, 1,$$
$$i = 8 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + (y_1 + y_2x_3)e^{-x_6} - y_3x_2, x_2 + y_2e^{-x_6}, x_3 + y_3,$$
$$x_4 + (y_4 - y_2x_6)e^{-x_6}, x_5 + y_5 - x_6y_3, x_6 + y_6),$$
$$K_8 = \{g(u_1, u_2, 0, u_3, u_1 + a_2u_2 + a_3u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$a_3 \in \mathbb{R} \setminus \{0\}, a_2 \in \mathbb{R},$$
$$i = 9 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 + x_2y_3 -$$
$$(x_3 + x_2x_6)y_2, x_2 + y_2, x_3 + y_3 - x_6y_2, x_4 + y_4e^{-x_6}, x_5 + y_5e^{-bx_6}, x_6 + y_6),$$
$$K_9 = \{g(u_1 + u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$i = 10 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 - 2x_6y_4 + (x_6^2 - x_2)y_3 - \left(\frac{1}{3}x_6^3 - x_2x_6 - x_3\right) y_2, x_2 + y_2,$$
$$x_3 + y_3 - x_6y_2, x_4 + y_4 - x_6y_3 + \frac{1}{2}x_6^2y_2, x_5 + y_5e^{-x_6}, x_6 + y_6),$$
$$K_{10} = \{g(u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$
$$i = 11 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 + x_2y_3 - \frac{1}{2}x_2^2y_6, x_2 + y_2, x_3 + y_3 - x_2y_6,$$
$$x_4 + y_4e^{-x_6}, x_5 + y_5e^{-x_6} - x_4y_6, x_6 + y_6),$$
$$K_{11} = \{g(a_2u_1 + u_2, 0, u_3, u_1, u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R},$$
$$i = 12 : g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+y_1-x_2y_3+y_2(x_3+x_2x_6), x_2+y_2, x_3+y_3-x_6y_2,$$
$$x_4+y_4e^{-bx_6}\cos x_6+y_5e^{-bx_6}\sin x_6,$$
$$x_5-y_4e^{-bx_6}\sin x_6+y_5e^{-bx_6}\cos x_6, x_6+y_6),$$
$$K_{12,1}=\{g(u_2,0,u_3,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$K_{12,2}=\{g(u_1+a_3u_2,0,u_3,u_1,u_2,0); u_i\in\mathbb{R}, i=1,2,3\}, a_3\in\mathbb{R},$$
$$i=13: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+[y_1-y_4x_6+y_2(\frac{1}{2}x_6^2+x_3)]e^{-x_6}-x_2y_3, x_2+y_2e^{-x_6},$$
$$x_3+y_3, x_4+(y_4-y_2x_6)e^{-x_6}, x_5+y_5-x_6y_3, x_6+y_6),$$
$$K_{13}=\{g(u_1,u_2,0,u_3,u_1+a_2u_2+a_3u_3,0); u_i\in\mathbb{R}, i=1,2,3\}, a_i\in\mathbb{R},$$
$$i=14: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1e^{-x_6}+x_2y_3, x_2+y_2e^{-x_6}, x_3+y_3,$$
$$x_4+y_4-x_6y_3, x_5+y_5-x_6y_4+\frac{1}{2}x_6^2y_3, x_6+y_6),$$
$$K_{14}=\{g(u_1,u_2,0,u_3,u_1+a_2u_2,0); u_i\in\mathbb{R}, i=1,2,3\}, a_2\in\mathbb{R},$$
$$i=15: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1e^{-x_6}+x_4y_5, x_2+(y_2-2\varepsilon y_4x_6-y_1x_5)e^{-x_6}+(x_1-x_4x_5)y_5,$$
$$x_3+y_3-x_6y_5, x_4+y_4e^{-x_6}, x_5+y_5, x_6+y_6),$$
$$K_{15}=\{g(u_1,u_2,a_1u_1+u_2+a_3u_3,u_3,0,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$a_1,a_3\in\mathbb{R}, \varepsilon=0,\pm1,$$
$$i=16: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=g(x_1+y_1+$$
$$x_5y_4+\frac{1}{2}x_5^2y_6, x_2+y_2+2x_5y_1+(x_5^2-\varepsilon x_6)y_4+(\frac{1}{3}x_5^3+\varepsilon(x_4-x_5x_6))y_6,$$
$$x_3+y_3e^{-x_6}, x_4+y_4+x_5y_6, x_5+y_5, x_6+y_6),$$
$$K_{16}=\{g(u_1,a_1u_1+u_2,u_2,u_3,0,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$a_1\in\mathbb{R}, \varepsilon=0,\pm1,$$
$$i=17: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$

$$g(x_1+(y_1+x_5y_3)e^{-x_6}, x_2+y_2+x_5y_4-\frac{1}{2}x_5^2y_6,$$
$$x_3+y_3e^{-x_6}, x_4+y_4-x_5y_6, x_5+y_5, x_6+y_6),$$
$$K_{17}=\{g(u_1,u_1+a_2u_2,u_2,u_3,0,0); u_i\in\mathbb{R}, i=1,2,3\}, a_2\in\mathbb{R},$$
$$i=18: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1+y_3x_5)e^{-x_6}, x_2+y_2-(x_5+x_6)y_4-\frac{1}{2}(x_5+x_6)^2y_5,$$
$$x_3+y_3e^{-x_6}, x_4+y_4+(x_5+x_6)y_5, x_5+y_5, x_6+y_6),$$
$$K_{18}=\{g(u_1,u_1+a_2u_2,u_2,u_3,0,0); u_i\in\mathbb{R}, i=1,2,3\}, a_2\in\mathbb{R},$$
$$i=19: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=g(x_1+y_1e^{-ax_6}+$$
$$x_3y_2, x_2+y_2, x_3+y_3e^{-ax_6}, x_4+y_4-x_6y_2, x_5+y_5e^{-x_6}, x_6+y_6),$$
$$K_{19}=\{g(u_1,0,u_2,u_1+a_2u_2+u_3,u_3,0); u_i\in\mathbb{R}, i=1,2,3\},$$
$$a_2\in\mathbb{R}, a\in\mathbb{R}\setminus\{0\},$$
$$i=20: g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1-x_6y_5+y_2x_3)e^{-x_6}, x_2+y_2e^{-x_6},$$
$$x_3+y_3, x_4+y_4-x_3y_6, x_5+y_5e^{-x_6}, x_6+y_6),$$
$$K_{20}=\{g(u_1,u_2,0,u_1+a_2u_2+a_3u_3,u_3,0); u_i\in\mathbb{R}, i=1,2,3\}, a_i\in\mathbb{R}.$$

Among these Lie groups only the group G_1 has 2-dimensional commutator subgroup and the groups G_i, $i=2,\cdots,7$, have 3-dimensional commutator subgroup. We show that among the 6-dimensional solvable indecomposable Lie groups with 5-dimensional nilradical precisely these Lie groups G_i, $i=1,\cdots,7$, are the multiplication groups of 3-dimensional connected simply connected topological loops.

If L exists, then there exists its universal covering loop $\tilde{L}$ which is homeomorphic to $\mathbb{R}^3$. We prove that none of the groups G_i, $i=8,\cdots,20$, satisfy the condition of Lemma 7, i.e. there does not exist continuous left transversals S and T to K_i in G_i such that for all $s\in S$ and $t\in T$ one has $s^{-1}t^{-1}st\in K_i$. Hence the groups G_i, $i=8,\cdots,20$, are not the groups $Mult(L)$ of $\tilde{L}$. Since no proper loop $\tilde{L}$ exists, it follows that also no proper loop L exists.

Two arbitrary left transversals to the group K_i in G_i are:
For $i = 9, 10, 11, 12$,

$$S = \{g(u, v, h_1(u, v, w), h_2(u, v, w), h_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$

$$T = \{g(k, l, f_1(k, l, m), f_2(k, l, m), f_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

for $i = 8, 13, 14, 15$,

$$S = \{g(h_1(u, v, w), h_2(u, v, w), u, h_3(u, v, w), v, w); u, v, w \in \mathbb{R}\},$$

$$T = \{g(f_1(k, l, m), f_2(k, l, m), k, f_3(k, l, m), l, m); k, l, m \in \mathbb{R}\},$$

for $i = 16, 17, 18$,

$$S = \{g(h_1(u, v, w), u, h_2(u, v, w), h_3(u, v, w), v, w); u, v, w \in \mathbb{R}\},$$

$$T = \{g(f_1(k, l, m), k, f_2(k, l, m), f_3(k, l, m), l, m); k, l, m \in \mathbb{R}\},$$

for $i = 19$

$$S = \{g(h_1(u, v, w), u, h_2(u, v, w), v, h_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$

$$T = \{g(f_1(k, l, m), k, f_2(k, l, m), l, f_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

for $i = 20$

$$S = \{g(h_1(u, v, w), h_2(u, v, w), u, v, h_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$

$$T = \{g(f_1(k, l, m), f_2(k, l, m), k, l, f_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

where $h_i(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$ and $f_i(k, l, m) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, are continuous functions with $f_i(0, 0, 0) = h_i(0, 0, 0) = 0$. Taking in G_i, $i = 9, 11, 12$, the elements

$$s = g(0, v, h_1(0, v, 0), h_2(0, v, 0), h_3(0, v, 0), 0) \in S,$$

$$t = g(0, 0, f_1(0, 0, m), f_2(0, 0, m), f_3(0, 0, m), m) \in T$$

and in G_{17} the elements

$$s = g(h_1(0, v, 0), 0, h_2(0, v, 0), h_3(0, v, 0), v, 0) \in S,$$

$$t = g(f_1(0,0,m), 0, f_2(0,0,m), f_3(0,0,m), 0, m) \in T$$

one has $s^{-1}t^{-1}st \in K_i$ if and only if
for $i = 9$

$$mv^2 - 2vf_1(0,0,m) = h_2(0,v,0)(1-e^m) + h_3(0,v,0)(1-e^{bm}), \quad (11)$$

for $i = 11$

$$\frac{1}{2}mv^2 + vf_1(0,0,m) + e^m m h_2(0,v,0) =$$
$$(e^m - 1)(h_3(0,v,0) + a_2 h_2(0,v,0)), \quad (12)$$

for $i = 12$ and for $K_{12,1}$

$$2vf_1(0,0,m) - mv^2 = (1 - e^{bm}\cos m)h_3(0,v,0) - e^{bm}\sin m h_2(0,v,0), \quad (13)$$

for $i = 12$ and for $K_{12,2}$

$$2vf_1(0,0,m) - mv^2 = (1 - e^{bm}\cos m)(h_2(0,v,0) + a_3 h_3(0,v,0)) +$$
$$e^{bm}\sin m(h_3(0,v,0) - a_3 h_2(0,v,0)), \quad (14)$$

for $i = 17$

$$-\frac{1}{2}mv^2 - vf_3(0,0,m) + e^m v f_2(0,0,m) =$$
$$(1 - e^m)[h_1(0,v,0) + (a_2 - v)h_2(0,v,0)] \quad (15)$$

is satisfied for all $m, v \in \mathbb{R}$. On the left hand side of equations (11), (12), (13), (14), (15) is the term mv^2 hence there does not exist any function $f_i(0,0,m)$ and $h_i(0,v,0)$, $i = 1,2,3$, satisfying these equations. Taking in G_{10} the elements

$$s = g(0, v, h_1(0,v,w), h_2(0,v,w), h_3(0,v,w), w) \in S,$$

$$t = g(0, 0, f_1(0,0,m), f_2(0,0,m), f_3(0,0,m), m) \in T,$$

respectively in G_{18} the elements

$$s = g(h_1(0,v,w), 0, h_2(0,v,w), h_3(0,v,w), v, w) \in S,$$

$$t = g(f_1(0,0,m), 0, f_2(0,0,m), f_3(0,0,m), 0, m) \in T,$$

respectively in G_{16} the elements

$$s = g(h_1(0,v,0), 0, h_2(0,v,0), h_3(0,v,0), v, 0) \in S,$$

$$t = g(f_1(0,l,m), 0, f_2(0,l,m), f_3(0,l,m), l, m) \in T$$

we obtain that $s^{-1}t^{-1}st \in K_i$ if and only if in case $i = 10$ the equation

$$e^w(1-e^m)h_3(0,v,w) + e^m(e^w-1)f_3(0,0,m) =$$

$$(w^2+2v+2mw)f_1(0,0,m) + 2wf_2(0,0,m) - (m^2+2wm)h_1(0,v,w) -$$

$$2mh_2(0,v,w) - m^2wv - w^2mv - mv^2 - \frac{1}{3}vm^3, \qquad (16)$$

respectively in case $i = 18$ the equation

$$e^m(e^w-1)(f_1(0,0,m) + a_2f_2(0,0,m)) +$$

$$e^w(1-e^m)[h_1(0,v,w) + (a_2-v)h_2(0,v,w)] = e^{m+w}vf_2(0,0,m) +$$

$$(w+v)f_3(0,0,m) - mh_3(0,v,w) + v^2m + \frac{1}{2}m^2v + wvm, \qquad (17)$$

respectively in case $i = 16$ the equation

$$-\frac{1}{3}v^3m - v^2lm - l^2vm - \frac{1}{2}a_1v^2m - \varepsilon m^2v - a_1vlm =$$

$$(1-e^m)h_2(0,v,0) - 2lh_1(0,v,0) +$$

$$(l^2+2vl+a_1l+2\varepsilon m)h_3(0,v,0) + 2vf_1(0,l,m) - (v^2+2vl+a_1v)f_3(0,l,m) \qquad (18)$$

holds for all $m, l, v, w \in \mathbb{R}$. Substituting into (16)

$$f_2(0,0,m) = f_2'(0,0,m) - mf_1(0,0,m),$$

$$h_2(0,v,w) = h_2'(0,v,w) - wh_1(0,v,w),$$

respectively into (17)

$$f_1(0,0,m) = f_1'(0,0,m) - a_2f_2(0,0,m),$$

$$h_1(0,v,w) = h_1'(0,v,w) + (v-a_2)h_2(0,v,w),$$

respectively into (18)

$$h_1(0,v,0) = h'_1(0,v,0) + (v + \frac{a_1}{2})h_3(0,v,0),$$

$$f_1(0,l,m) = f'_1(0,l,m) + (l + \frac{a_1}{2})f_3(0,l,m),$$

we get in case $i = 10$

$$e^w(1-e^m)h_3(0,v,w) + e^m(e^w-1)f_3(0,0,m) =$$

$$(w^2+2v)f_1(0,0,m) - m^2h_1(0,v,w)+$$

$$2wf'_2(0,0,m) - 2mh'_2(0,v,w) - m^2wv - w^2mv - mv^2 - \frac{1}{3}vm^3, \quad (19)$$

respectively in case $i = 18$

$$e^m(e^w-1)f'_1(0,0,m) - e^{m+w}vf_2(0,0,m) + e^w(1-e^m)h'_1(0,v,w) =$$

$$(w+v)f_3(0,0,m) - mh_3(0,v,w) + v^2m + \frac{1}{2}m^2v + wvm, \qquad (20)$$

respectively in case $i = 16$

$$(1-e^m)h_2(0,v,0) + (l^2+2\varepsilon m)h_3(0,v,0)-$$

$$v^2f_3(0,l,m) - 2lh'_1(0,v,0) + 2vf'_1(0,l,m) =$$

$$-\frac{1}{3}v^3m - v^2lm - l^2vm - \frac{1}{2}a_1v^2m - \varepsilon m^2v - a_1vlm. \qquad (21)$$

Since on the right hand side of (19), respectively (20), respectively (21), there is the term $-\frac{1}{3}vm^3$, respectively $\frac{1}{2}m^2v$, respectively $-\frac{1}{3}v^3m$, there is no function $f_i(0,0,m)$ and $h_i(0,v,w)$, $i = 1,2,3$, respectively $f_i(0,l,m)$, $i = 1,3$, and $h_j(0,v,0)$, $j = 1,2,3$, satisfying equation (19), respectively (20), respectively (21).
Taking in G_i, $i = 8, 13, 14$, the elements

$$s = g(h_1(0,0,w), h_2(0,0,w), 0, h_3(0,0,w), 0, w) \in S,$$

$$t = g(f_1(k,0,m), f_2(k,0,m), k, f_3(k,0,m), 0, m) \in T,$$

respectively in G_{19} the elements

$$s = g(h_1(0,0,w), 0, h_2(0,0,w), 0, h_3(0,0,w), w) \in S,$$

$$t = g(f_1(k,0,m), k, f_2(k,0,m), 0, f_3(k,0,m), m) \in T,$$

respectively in G_{20} the elements

$$s = g(h_1(0,0,w), h_2(0,0,w), 0, 0, h_3(0,0,w), w) \in S,$$

$$t = g(f_1(k,0,m), f_2(k,0,m), k, 0, f_3(k,0,m), m) \in T$$

we have $s^{-1}t^{-1}st \in K_i$ precisely if for $i = 8$ the equation

$$wk = e^w(1-e^m)[(a_2 + a_3 w)h_2(0,0,w) + a_3 h_3(0,0,w) + h_1(0,0,w)]+$$
$$e^m(e^w - 1)[(a_3 m + a_2 - k)f_2(k,0,m) + a_3 f_3(k,0,m) + f_1(k,0,m)]+$$
$$e^{m+w}[a_3 w f_2(k,0,m) + (2k - a_3 m)h_2(0,0,w)], \tag{22}$$

for $i = 13$ the equation

$$wk = e^w(1-e^m)[(\frac{1}{2}w^2 + a_2 + a_3 w)h_2(0,0,w) + (a_3 + w)h_3(0,0,w)+$$
$$h_1(0,0,w)] + e^m(e^w - 1)[(\frac{1}{2}m^2 - k + a_3 m + a_2)f_2(k,0,m)+$$
$$(m+a_3)f_3(k,0,m) + f_1(k,0,m)] + e^{m+w}[((m+a_3)w + \frac{1}{2}w^2)f_2(k,0,m)+$$
$$(2k - \frac{1}{2}m^2 - (w+a_3)m)h_2(0,0,w)] + e^{m+w}(wf_3(k,0,m) - mh_3(0,0,w)), \tag{23}$$

for $i = 14$ the equation

$$\frac{1}{2}w^2 k + mwk + wf_3(k,0,m) - mh_3(0,0,w) =$$
$$e^w(1-e^m)(h_1(0,0,w) + a_2 h_2(0,0,w))+$$
$$e^m(e^w - 1)(f_1(k,0,m) + a_2 f_2(k,0,m)) - e^{m+w}kh_2(0,0,w), \tag{24}$$

for $i = 19$ the equation

$$wk = e^w(1-e^m)h_3(0,0,w) - e^m(1-e^w)f_3(k,0,m)-$$

$$e^{a(m+w)}kh_2(0,0,w) + e^{aw}(1-e^{am})(h_1(0,0,w)+$$
$$a_2h_2(0,0,w)) - e^{am}(1-e^{aw})(f_1(k,0,m) + a_2f_2(k,0,m)), \qquad (25)$$

for $i = 20$ the equation

$$-wk = e^w(1-e^m)(h_1(0,0,w) + a_2h_2(0,0,w) + (w+a_3)h_3(0,0,w))+$$
$$e^m(1-e^w)((k-a_2)f_2(k,0,m) - f_1(k,0,m) - (m+a_3)f_3(k,0,m))+$$
$$e^{m+w}(kh_2(0,0,w) - mh_3(0,0,w) + wf_3(k,0,m)) \qquad (26)$$

is satisfied for all $k, m, w \in \mathbb{R}$, $a_2, a_3 \in \mathbb{R}$. Putting into (22)

$$h_1(0,0,w) = h_1'(0,0,w) - (a_3w + a_2)h_2(0,0,w) - a_3h_3(0,0,w),$$

$$f_1(k,0,m) = f_1'(k,0,m) + (k - a_3m - a_2)f_2(k,0,m) - a_3f_3(k,0,m),$$

respectively into (23)

$$h_1(0,0,w) = h_1'(0,0,w)-$$

$$(\frac{1}{2}w^2 + a_3w + a_2)h_2(0,0,w) - (a_3+w)h_3(0,0,w),$$

$$f_1(k,0,m) = f_1'(k,0,m)+$$

$$(k - \frac{1}{2}m^2 - a_3m - a_2)f_2(k,0,m) - (m+a_3)f_3(k,0,m),$$

$$f_3(k,0,m) = f_3'(k,0,m) - (m+a_3)f_2(k,0,m),$$

$$h_3(0,0,w) = h_3'(0,0,w) - (w+a_3)h_2(0,0,w),$$

respectively into (24)

$$h_1(0,0,w) = h_1'(0,0,w) - a_2h_2(0,0,w),$$

$$f_3(k,0,m) = f_3'(k,0,m) - mk,$$

$$f_1(k,0,m) = f_1'(k,0,m) - a_2f_2(k,0,m),$$

respectively into (25)

$$h_1(0,0,w) = h_1'(0,0,w) - a_2h_2(0,0,w),$$

$$f_1(k,0,m) = f_1'(k,0,m) - a_2f_2(k,0,m),$$

respectively into (26)

$$h_1(0,0,w) = h_1'(0,0,w) - a_2 h_2(0,0,w) - (w+a_3)h_3(0,0,w),$$

$$f_1(k,0,m) = f_1'(k,0,m) + (k-a_2)f_2(k,0,m) - (m+a_3)f_3(k,0,m)$$

in order equations (22), (23), (24), (25), (26) reduce in case $i=8$ to

$$wk = e^w(1-e^m)h_1'(0,0,w) + e^m(e^w-1)f_1'(k,0,m) +$$
$$e^{m+w}[a_3 w f_2(k,0,m) + (2k - a_3 m)h_2(0,0,w)], \tag{27}$$

in case $i=13$ to

$$wk = e^w(1-e^m)h_1'(0,0,w) +$$
$$e^m(e^w-1)f_1'(k,0,m) + e^{m+w}[\frac{1}{2}w^2 f_2(k,0,m) +$$
$$(2k - \frac{1}{2}m^2)h_2(0,0,w) + wf_3'(k,0,m) - mh_3'(0,0,w)], \tag{28}$$

in case $i=14$ to

$$\frac{1}{2}w^2 k + wf_3'(k,0,m) - mh_3(0,0,w) = e^w(1-e^m)h_1'(0,0,w) +$$
$$e^m(e^w-1)f_1'(k,0,m) - e^{m+w}kh_2(0,0,w), \tag{29}$$

in case $i=19$ to

$$wk = e^w(1-e^m)h_3(0,0,w) - e^m(1-e^w)f_3(k,0,m) - e^{a(m+w)}kh_2(0,0,w)$$
$$+e^{aw}(1-e^{am})h_1'(0,0,w) - e^{am}(1-e^{aw})f_1'(k,0,m), \tag{30}$$

and in case $i=20$ to

$$-wk = e^w(1-e^m)h_1'(0,0,w) + e^m(e^w-1)f_1'(k,0,m) +$$
$$e^{m+w}(kh_2(0,0,w) - mh_3(0,0,w) + wf_3(k,0,m)). \tag{31}$$

Since on the left hand side of (27), (28), (30), (31), respectively of (29), is the term wk, respectively $\frac{1}{2}w^2k$, there is no function $f_i(k,0,m)$, $h_i(0,0,w)$, $i=1,2,3$, such that equations (27), (28), (30), (31), respectively (29), are satisfied.

Taking in G_{15} the elements

$$s = g(h_1(0,0,w), h_2(0,0,w), 0, h_3(0,0,w), 0, w) \in S,$$

$$t = g(f_1(0,l,m), f_2(0,l,m), 0, f_3(0,l,m), l, m) \in T$$

the product $s^{-1}t^{-1}st$ lies in K_{15} if and only if the equation

$$wl = e^w(1-e^m)[h_2(0,0,w) + (a_3 + 2w\varepsilon)h_3(0,0,w) + a_1h_1(0,0,w)] +$$
$$e^m(e^w - 1)[f_2(0,l,m) + (l + a_1)f_1(0,l,m) + (a_3 + 2m\varepsilon)f_3(0,l,m)] +$$
$$e^{m+w}[2w\varepsilon f_3(0,l,m) - 2lh_1(0,0,w) - (l^2 + 2m\varepsilon + a_1l)h_3(0,0,w)] \quad (32)$$

is satisfied for all $m, l, w \in \mathbb{R}$. Substituting into (32)

$$h_1(0,0,w) = h_1'(0,0,w) - \frac{a_1}{2}h_3(0,0,w),$$

$$h_2(0,0,w) = h_2'(0,0,w) - a_1h_1(0,0,w) - (a_3 + 2w\varepsilon)h_3(0,0,w),$$
$$f_2(0,l,m) = f_2'(0,l,m) - (l + a_1)f_1(0,l,m) - (a_3 + 2m\varepsilon)f_3(0,l,m),$$

we obtain

$$wl = e^w(1-e^m)h_2'(0,0,w) + e^m(e^w - 1)f_2'(0,l,m) +$$

$$e^{m+w}[2w\varepsilon f_3(0,l,m) - 2lh_1'(0,0,w) - (l^2 + 2m\varepsilon)h_3(0,0,w)]. \quad (33)$$

On the left hand side of equation (33) is the term wl hence there is no function $f_i(0,l,m)$, $i = 2, 3$, and $h_j(0,0,w)$, $j = 1, 2, 3$ such that equation (33) holds.

The sets

$$S_1 = \{g(k, 1-e^m, l, me^{-m}, 2l, m); k, l, m \in \mathbb{R}\},$$

$$T_1 = \{g(u, w, v, 2ve^{-w}, 1-e^w, w); u, v, w \in \mathbb{R}\},$$

respectively

$$S_2 = \{g(k, l, 1-e^m, me^{-m}, -2l, m); k, l, m \in \mathbb{R}\},$$

$$T_2 = \{g(u, v, w, -2ve^{-w}, 1-e^w, w); u, v, w \in \mathbb{R}\}$$

are $K_{1,1}$-, respectively $K_{1,2}$-connected left transversals in G_1. The sets

$$S = \{g(k, l, l, me^{-m}, l^2 - 1 + e^m, m); k, l, m \in \mathbb{R}\},$$

$$T = \{g(u, v, v, -we^{-w}, v^2 + 1 - e^w, w); u, v, w \in \mathbb{R}\}$$

are K_2-connected left transversals in G_2. The sets

$$S_1 = \{g(k, \frac{1}{2}m^2 - l, l, e^m - 1 - m(\frac{1}{2}m^2 - l), me^{-m}, m); k, l, m \in \mathbb{R}\},$$

$$T_1 = \{g(u, \frac{1}{2}w^2 - v, v, 1 - e^w - w(\frac{1}{2}w^2 - v), -we^{-w}, w); u, v, w \in \mathbb{R}\},$$

respectively

$$S_2 = \{g(k, l, \frac{1}{2}m^2 + e^m - 1, -lm + m, le^{-m}, m); k, l, m \in \mathbb{R}\},$$

$$T_2 = \{g(u, v, \frac{1}{2}w^2 - e^w + 1, -vw + w, -ve^{-w}, w); u, v, w \in \mathbb{R}\},$$

are $K_{3,1}$-, respectively $K_{3,2}$-connected left transversals in G_3. The sets

$$S = \{g((l+a_1)(1-e^m)+l, k, -e^{-m}(\frac{1}{2}l^2+\varepsilon m), 1-e^m, l, m); k, l, m \in \mathbb{R}\},$$

$$T = \{g((v+a_1)(e^w-1)+v, u, e^{-w}(\frac{1}{2}v^2+\varepsilon w), e^w-1, v, w); u, v, w \in \mathbb{R}\}$$

are K_4-connected left transversals in G_4. The sets

$$S = \{g(le^{-k}(a_2 - l + 1), m, -le^{-k}, 1 - le^k - e^k, l, k); k, l, m \in \mathbb{R}\},$$

$$T = \{g(ve^{-u}(v - 1 - a_2), w, ve^{-u}, ve^u + e^u - 1, v, u); u, v, w \in \mathbb{R}\}$$

are K_5-connected left transversals in G_5. The sets

$$S = \{g((l - a_2)l + (l + m)e^{-m}, k, l, e^m - 1, l, m); k, l, m \in \mathbb{R}\},$$

$$T = \{g((v - a_2)v - (v + w)e^{-w}, u, v, 1 - e^w, v, w); u, v, w \in \mathbb{R}\}$$

are K_6-connected left transversals in G_6. The sets

$$S = \{g((\varepsilon - k)me^{-m}, -me^{-m}, k, -ke^m, l, m), k, l, m \in \mathbb{R}\},$$

$$T = \{g((u-\varepsilon)we^{-w}, we^{-w}, u, ue^{w}, v, w), u, v, w \in \mathbb{R}\},$$

are K_7-connected left transversals in G_7. For all $i = 1, \cdots, 7$, the sets S_1, T_1, respectively S_2, T_2, generate the group G_i. According to Lemma 7 the pairs (G_i, K_i), $i = 1, \cdots, 7$, are multiplication groups and inner mapping groups of L which proves the assertion. $\square$

In the next theorem we consider the case that the group $Mult(L)$ of a 3-dimensional connected simply connected topological proper loop L has 4-dimensional nilradical.

Theorem 25. *Let L be a 3-dimensional connected simply connected topological proper loop such that its multiplication group $Mult(L)$ is a 6-dimensional solvable indecomposable Lie group having 4-dimensional nilradical. Then L has nilpotency class 2 and the following Lie groups are the multiplication groups $Mult(L)$ of L and the following subgroups are the inner mapping groups $Inn(L)$ of L:*
1) $Mult(L)_1$ is given by the multiplication

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1e^{x_5}\cos(x_6) - y_2e^{x_5}\sin(x_6), x_2 + y_2e^{x_5}\cos(x_6) + y_1e^{x_5}\sin(x_6),$$
$$x_3 + y_3, x_4 + y_4 + (ax_6 + x_5)y_3, x_5 + y_5, x_6 + y_6), a \in \mathbb{R},$$

$Inn(L)_1$ is the subgroup

$$\{g(u_1, u_2, u_3, \varepsilon_1u_1 + \varepsilon_2u_2 + \varepsilon_3u_3, 0, 0); u_1, u_2, u_3 \in \mathbb{R}\},$$

$\varepsilon_k \in \{0, 1\}$, $k = 1, 2, 3$, such that $\varepsilon_1^2 + \varepsilon_2^2 \neq 0$.
2) The multiplication group $Mult(L)_2$ is defined by

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1e^{x_5+ax_6},$$
$$x_2 + y_2e^{x_6}, x_3 + y_3, x_4 + y_4 + x_5y_3, x_5 + y_5, x_6 + y_6), a \in \mathbb{R}\setminus\{0\},$$

$Inn(L)_2$ is $\{g(u_1, u_2, u_3, u_1 + u_2 + \varepsilon u_3, 0, 0); u_1, u_2, u_3 \in \mathbb{R}\}$, $\varepsilon = 0, 1$.
3) The multiplication group $Mult(L)_3$ is given by

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1, x_2 + y_2 + x_5y_1, x_3 + y_3e^{x_6}\cos(x_5) - y_4e^{x_6}\sin(x_5),$$

$$x_4 + y_4 e^{x_6} \cos(x_5) + y_3 e^{x_6} \sin(x_5), x_5 + y_5, x_6 + y_6),$$

$Inn(L)_3$ *is the subgroup*

$$\{g(u_1, \varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3, u_2, u_3, 0, 0); u_1, u_2, u_3 \in \mathbb{R}\},$$

$\varepsilon_k \in \{0, 1\}, k = 1, 2, 3$, *such that* $\varepsilon_2^2 + \varepsilon_3^2 \neq 0$.

Proof. We may assume that L is homeomorphic to $\mathbb{R}^3$ (see Lemma 9). According to Proposition 21 it remains to deal with the 6-dimensional solvable indecomposable Lie algebras $N_{6,i}$, $i = 20, \dots, 27$, with abelian nilradical and 1-dimensional centre (cf. Table II. in [41], p. 1348). Foremost, by Theorem 17 (a) we have to prove that there is a normal subgroup $N \cong \mathbb{R}$ of L such that the factor loop L/N is isomorphic to $\mathbb{R}^2$. The Lie algebra $N_{6,20}^{a,b}$, $a^2 + b^2 \neq 0$, have the ideals $\mathbf{i}_1 = \langle n_1 \rangle$, $\mathbf{i}_2 = \langle n_2 \rangle$, $\mathbf{i}_3 = \langle n_3 \rangle$, $\mathbf{i}_4 = \langle n_4 \rangle$. If $N_{6,20}^{a,b}$ is the Lie algebra of the group $Mult(L)$ of L, then the orbits $I_k(e)$, $k \in \{1, 2, 3, 4\}$, are normal subgroups of L isomorphic to $\mathbb{R}$ (cf. Lemma 6). The Lie algebra $N_{6,20}^{a,b}$ has no factor Lie algebra isomorphic to an elementary filiform Lie algebra. Hence the factor loops $L/I_k(e)$, $k \in \{1, 2, 3, 4\}$, are isomorphic either to $\mathcal{L}_2$ or to $\mathbb{R}^2$ (cf. Proposition 19 a). If all factor loops $L/I_k(e)$, $k \in \{1, 2, 3, 4\}$, are isomorphic to $\mathcal{L}_2$, then by Proposition 19 (ii) there are 2-dimensional ideals $\mathbf{s}_k$, $k \in \{1, 2, 3, 4\}$ such that $\mathbf{i}_k < \mathbf{s}_k$ and the factor Lie algebras $N_{6,20}^{a,b}/\mathbf{s}_k$ are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. For the ideal $\mathbf{s}_1 = \mathbf{s}_2 = \langle n_1, n_2 \rangle$ one has $N_{6,20}^{a,b}/\mathbf{s}_k \cong \mathbf{l}_2 \oplus \mathbf{l}_2$, $k = 1, 2$. The factor Lie algebra $N_{6,20}^{a,b}/\langle n_1, n_3 \rangle$ is isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ if and only if $a = 0$ and $N_{6,20}^{a,b}/\langle n_1, n_4 \rangle$ is isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ precisely if $b = 0$. This contradiction to $a^2 + b^2 \neq 0$ yields that at least one of the factor loops $L/I_k(e)$, $k \in \{1, 2, 3, 4\}$, is isomorphic to $\mathbb{R}^2$. For such $k \in \{1, 2, 3, 4\}$ the orbit $I_k(e)$ is the requested normal subgroup N of L in Theorem 17 (a).
The Lie algebras $N_{6,21}^{a}$, $N_{6,22}^{\varepsilon,a}$, $N_{6,24}$, $N_{6,25}^{a,b}$, $N_{6,26}^{a}$, $N_{6,27}^{\varepsilon}$ have the ideal $\mathbf{i} = \langle n_2 \rangle$ and the unique 1-dimensional ideal of the Lie algebra $N_{6,23}^{a,\varepsilon}$ is its centre $\mathbf{i} = \langle n_4 \rangle$. There does not exist any ideal $\mathbf{s}$ of these Lie algebras $N_{6,i}$ containing $\mathbf{i}$ such that the factor Lie algebras $N_{6,i}/\mathbf{s}$ are isomorphic either to $\mathbf{l}_2 \oplus \mathbf{l}_2$ or to $\mathbf{f}_4$. If $N_{6,i}$, $i = 21, \dots, 27$, is the Lie algebra of the group $Mult(L)$ of L, then the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$ (cf. Proposition 19 (i)). Hence L has nilpotent of class 2. According to Proposition 19 a) (i) and d) the Lie algebra $\mathbf{g}$ of the group $Mult(L)$ has abelian

nilradical $\mathbf{n}_{\text{rad}} = \mathbf{z} \oplus \mathbf{inn}(\mathbf{L})$, where $dim(\mathbf{z}) = 1$. Hence $\mathbf{inn}(\mathbf{L})$ is a 3-dimensional abelian subalgebra of g which does not contain any non-zero ideal of g and the normalizer $N_{\mathbf{g}}(\mathbf{inn}(\mathbf{L}))$ coincides with $\mathbf{n}_{\text{rad}}$ (cf. Lemma 8). From now on we use the following notation:
(a) $\mathbf{g}_1 := N^{a,b}_{6,20}$, $\mathbf{k}_1 = \langle n_2 + n_1, n_3 + n_1, n_4 + n_1 \rangle$.
(b) $\mathbf{g}_2 := N^{a}_{6,21}$, $\mathbf{g}_3 := N_{6,24}$, $\mathbf{k}_2 = \mathbf{k}_3 = \langle n_2 + n_1, n_3 + \varepsilon_1 n_1, n_4 + n_1 \rangle$, $\varepsilon_1 = 0, 1$.
(c) $\mathbf{g}_4 := N^{a,b}_{6,25}$, $\mathbf{g}_5 := N^{a}_{6,26}$, $\mathbf{k}_4 = \mathbf{k}_5 = \langle n_2 + n_1, n_3 + \varepsilon_1 n_1, n_4 + \varepsilon_2 n_1 \rangle$, $\varepsilon_i = 0, 1$, $i = 1, 2$, such that at least one of $\{\varepsilon_1, \varepsilon_2\}$ is different from 0.
(d) $\mathbf{g}_6 := N^{\varepsilon}_{6,27}$, $\mathbf{k}_6 = \langle n_1 + \varepsilon_1 n_2, n_3 + \varepsilon_2 n_2, n_4 + \varepsilon_3 n_2 \rangle$, $\varepsilon_i = 0, 1$, $i = 1, 2, 3$, such that at least one of $\{\varepsilon_2, \varepsilon_3\}$ differs from 0.
(e) $\mathbf{g}_7 := N^{a,\varepsilon}_{6,23}$, $\mathbf{k}_7 = \langle n_1 + \varepsilon_1 n_4, n_2 + \varepsilon_2 n_4, n_3 + \varepsilon_3 n_4 \rangle$, $\varepsilon_i = 0, 1$, $i = 1, 2, 3$, such that at least one of $\{\varepsilon_1, \varepsilon_2\}$ is different from 0.
(f) $\mathbf{g}_8 := N^{a,\varepsilon}_{6,22}$, $\mathbf{k}_3 = \langle n_1 + n_4, n_2 + n_4, n_3 + \varepsilon_1 n_4 \rangle$, $\varepsilon_1 = 0, 1$. We compute the 3-dimensional abelian subalgebras $\mathbf{k}$ of $\mathbf{g}_i$, $i = 1, \cdots, 8$, which are the Lie algebra $\mathbf{inn}(L)$ of L. The Lie algebras $\mathbf{g}_i$, $i = 1, \ldots, 5$, have the centre $\mathbf{z} = \langle n_1 \rangle$. For these Lie algebras the subalgebra $\mathbf{k}$ has the form

$$\mathbf{k}_{a_2,a_3,a_4} = \langle n_2 + a_2 n_1, n_3 + a_3 n_1, n_4 + a_4 n_1 \rangle,$$

such that in the case $\mathbf{g}_1$: $a_2 a_3 a_4 \neq 0$, since $\langle n_2 \rangle$, $\langle n_3 \rangle$, $\langle n_4 \rangle$ are ideals of $\mathbf{g}_1$, in the cases $\mathbf{g}_2$, $\mathbf{g}_3$: $a_2 a_4 \neq 0$ because $\langle n_4 \rangle$ and $\langle n_2 \rangle$ are ideals of $\mathbf{g}_i$, $i = 2, 3$,
in the cases $\mathbf{g}_4$, $\mathbf{g}_5$: $a_2 \neq 0$ and at least one of the constants $\{a_3, a_4\}$ differs from 0 since $\langle n_2 \rangle$ and $\langle n_3, n_4 \rangle$ are ideals of $\mathbf{g}_i$, $i = 4, 5$. For the Lie algebras $\mathbf{g}_i$, $i = 1, \ldots, 5$, using the automorphism $\alpha(n_1) = n_1$, $\alpha(x_i) = x_i$, $i = 1, 2$, $\alpha(n_2) = a_2 n_2$, $\alpha(n_i) = a_i n_i$, $i = 3, 4$, if $a_i \neq 0$, otherwise $\alpha(n_i) = n_i$, we can change $\mathbf{k}_{a_2,a_3,a_4}$ onto $\mathbf{k} = \langle n_2 + n_1, n_3 + \varepsilon_1 n_1, n_4 + \varepsilon_2 n_1 \rangle$, such that ε_1, respectively ε_2 is equal to 0 or 1, according whether a_3, respectively a_4, is 0 or $\neq 0$.

The Lie algebra $\mathbf{g}_6$ has the centre $\mathbf{z} = \langle n_2 \rangle$ and hence for the subalgebra $\mathbf{k}$ one has $\mathbf{k}_{a_1,a_3,a_4} = \langle n_1 + a_1 n_2, n_3 + a_3 n_2, n_4 + a_4 n_2 \rangle$, such that $a_3 \neq 0$ or $a_4 \neq 0$ because $\langle n_3, n_4 \rangle$ is an ideal of $\mathbf{g}_6$. Using the automorphism $\alpha(n_2) = n_2$, $\alpha(x_i) = x_i$, $i = 1, 2$, $\alpha(n_i) = a_i n_i$, if $a_i \neq 0$, otherwise $\alpha(n_i) = n_i$, $i = 1, 3, 4$, we can reduce the Lie algebra $\mathbf{k}_{a_1,a_3,a_4}$ to $\mathbf{k} = \langle n_1 + \varepsilon_1 n_2, n_3 + \varepsilon_2 n_2, n_4 + \varepsilon_3 n_2 \rangle$, $\varepsilon_i = 0, 1$, $i = 1, 2, 3$, such that at least one of $\{\varepsilon_2, \varepsilon_3\}$ is different from 0.

The centre of the Lie algebras $\mathbf{g}_i$, $i = 7, 8$, is $\langle n_4 \rangle$. For the subalgebra $\mathbf{k}$ of $\mathbf{g}_i$, $i = 7, 8$, we obtain

$$\mathbf{k}_{a_1,a_2,a_3} = \langle n_1 + a_1 n_4, n_2 + a_2 n_4, n_3 + a_3 n_4 \rangle,$$

such that in the case $\mathbf{g}_7$: $a_1 \neq 0$ or $a_2 \neq 0$, since $\langle n_1, n_2 \rangle$ is an ideal of $\mathbf{g}_7$, in the case $\mathbf{g}_8$: $a_1 a_2 \neq 0$ because $\langle n_1 \rangle$ and $\langle n_2 \rangle$ are ideals of $\mathbf{g}_8$. For $\mathbf{g}_i$, $i = 7, 8$, using the automorphism $\alpha(n_4) = n_4$, $\alpha(x_i) = x_i$, $i = 1, 2$, $\alpha(n_i) = a_i n_i$, if $a_i \neq 0$, otherwise $\alpha(n_i) = n_i$, $i = 1, 2, 3$, we can change $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k} = \langle n_1 + \varepsilon_1 n_4, n_2 + \varepsilon_2 n_4, n_3 + \varepsilon_3 n_4 \rangle$, such that ε_i is equal to 0 or 1, according whether $a_i = 0$ or $a_i \neq 0$, $i = 1, 2, 3$.

The linear representations of the simply connected Lie groups G_i of $\mathbf{g}_i$ are given in this order by

$$i = 1: \; g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 + x_6 y_5, x_2 + y_2 e^{a x_5 + b x_6}, x_3 + y_3 e^{x_6}, x_4 + y_4 e^{x_5}, x_5 + y_5, x_6 + y_6),$$
$$i = 2: \; g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 + x_6 y_5, x_2 + y_2 e^{x_5 + a x_6}, x_3 + y_3 e^{x_6}, x_4 + y_4 e^{x_6} + x_3 y_5, x_5 + y_5, x_6 + y_6),$$
$$i = 3: \; g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 + x_5 y_6, x_2 + y_2 e^{x_6}, x_3 + y_3 e^{x_5}, x_4 + y_4 e^{x_5} + x_5 e^{x_5} y_3, x_5 + y_5, x_6 + y_6),$$
$$i = 4: \; g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 - x_6 y_5, x_2 + y_2 e^{a x_5 + b x_6}, y_3 + x_3 \cos(y_5) e^{y_6} - x_4 \sin(y_5) e^{y_6},$$
$$y_4 + x_4 \cos(y_5) e^{y_6} + x_3 \sin(y_5) e^{y_6}, x_5 + y_5, x_6 + y_6),$$
$$i = 5: \; g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1 + x_6 y_5, x_2 + y_2 e^{x_6}, x_3 + y_3 \cos(x_5) e^{a x_5} + y_4 \sin(x_5) e^{a x_5},$$
$$x_4 + y_4 \cos(x_5) e^{a x_5} - y_3 \sin(x_5) e^{a x_5}, x_5 + y_5, x_6 + y_6),$$
$$i = 6: \varepsilon = 0: \; g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$
$$g(x_1 + y_1, x_2 + y_2 + x_5 y_1, x_3 + y_3 e^{x_6} \cos(x_5) - y_4 e^{x_6} \sin(x_5),$$
$$x_4 + y_4 e^{x_6} \cos(x_5) + y_3 e^{x_6} \sin(x_5), x_5 + y_5, x_6 + y_6),$$
$$i = 6: \varepsilon = 1: g(x_1, x_2, x_3, x_4, x_5, x_6) g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+y_1+x_5y_6, x_2+y_2+x_5y_1+\frac{1}{2}x_5^2y_6, y_3+x_3e^{y_6}\cos(y_5)-x_4e^{y_6}\sin(y_5),$$
$$y_4+x_4e^{y_6}\cos(y_5)+x_3e^{y_6}\sin(y_5), x_5+y_5, x_6+y_6),$$
$$i=7: \varepsilon=0: \ g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1e^{x_5}\cos(x_6)-y_2e^{x_5}\sin(x_6), x_2+y_2e^{x_5}\cos(x_6)+y_1e^{x_5}\sin(x_6),$$
$$x_3+y_3, x_4+y_4+(ax_6+x_5)y_3, x_5+y_5, x_6+y_6),$$
$$i=7: \varepsilon=1, a=0: \ g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1e^{x_5}\cos(x_6)+y_2e^{x_5}\sin(x_6), x_2+y_2e^{x_5}\cos(x_6)-y_1e^{x_5}\sin(x_6),$$
$$x_3+y_3+x_5y_6, x_4+y_4+x_5y_3+\frac{1}{2}x_5^2y_6, x_5+y_5, x_6+y_6),$$
$$i=7: \varepsilon=1, a\neq 0: \ g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1e^{x_5}\cos(x_6)+y_2e^{x_5}\sin(x_6), x_2+y_2e^{x_5}\cos(x_6)-y_1e^{x_5}\sin(x_6),$$
$$x_3+y_3+(ax_6+x_5)y_5, x_4+y_4+(ax_6+x_5)y_3+\frac{1}{2}(ax_6+x_5)^2y_5,$$
$$x_5+y_5, x_6+y_6),$$
$$i=8: \varepsilon=0: \ g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1e^{x_5+ax_6}, x_2+y_2e^{x_6}, x_3+y_3, x_4+y_4+x_5y_3, x_5+y_5, x_6+y_6),$$
$$i=8: \varepsilon=1: \ g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1e^{x_5+ax_6}, x_2+y_2e^{x_6}, x_3+y_3+x_5y_6, x_4+y_4+x_5y_3+\frac{1}{2}x_5^2y_6,$$
$$x_5+y_5, x_6+y_6)$$

(cf. [35], pp. 16-21). Using these linear representations the Lie groups of the Lie algebras $\mathbf{k}_i$ are
for $i=1$: $Inn(L)=\{g(u_1+u_2+u_3, u_1, u_2, u_3, 0, 0); u_j \in \mathbb{R}\}$, $j=1,2,3$,
for $i=2,3$: $Inn(L)=\{g(u_1+\varepsilon u_2+u_3, u_1, u_2, u_3, 0, 0); u_j \in \mathbb{R}\}$, $j=1,2,3$, $\varepsilon=0,1$,
for $i=4,5$: $Inn(L)=\{g(u_1+\varepsilon_2u_2+\varepsilon_3u_3, u_1, u_2, u_3, 0, 0); u_j \in \mathbb{R}\}$, $j=1,2,3$, $\varepsilon_k=0,1$, $k=2,3$ such that at least one of $\{\varepsilon_2, \varepsilon_3\}$ is different from 0,
for $i=6$: $Inn(L)=\{g(u_1, \varepsilon_1u_1+\varepsilon_2u_2+\varepsilon_3u_3, u_2, u_3, 0, 0); u_j \in \mathbb{R}\}$, $j=1,2,3$, $\varepsilon_k=0,1$, $k=1,2,3$, such that at least one of $\{\varepsilon_2, \varepsilon_3\}$ is

different from 0,
for $i = 7$: $Inn(L) = \{g(u_1, u_2, u_3, \varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3, 0, 0); u_j \in \mathbb{R}\}$, $j = 1, 2, 3$, $\varepsilon_k = 0, 1$, $k = 1, 2, 3$, such that at least one of $\{\varepsilon_1, \varepsilon_2\}$ differs from 0,
for $i = 8$: $Inn(L) = \{g(u_1, u_2, u_3, u_1 + u_2 + \varepsilon u_3, 0, 0); u_j \in \mathbb{R}\}$, $j = 1, 2, 3$, $\varepsilon = 0, 1$.
Two arbitrary left transversals to the group $Inn(L)$ in G_i, $i = 1, \ldots, 5$, are

$$S = \{g(k, f_1(k, l, m), f_2(k, l, m), f_3(k, l, m), l, m), k, l, m \in \mathbb{R}\},$$

$$T = \{g(u, h_1(u, v, w), h_2(u, v, w), h_3(u, v, w), v, w), u, v, w \in \mathbb{R}\},$$

those to the group $Inn(L)$ in G_6 are

$$S = \{g(f_1(k, l, m), k, f_2(k, l, m), f_3(k, l, m), l, m), k, l, m \in \mathbb{R}\},$$

$$T = \{g(h_1(u, v, w), u, h_2(u, v, w), h_3(u, v, w), v, w), u, v, w \in \mathbb{R}\},$$

those to the group $Inn(L)$ in G_i, $i = 7, 8$, are

$$S = \{g(f_1(k, l, m), f_2(k, l, m), f_3(k, l, m), k, l, m), k, l, m \in \mathbb{R}\},$$

$$T = \{g(h_1(u, v, w), h_2(u, v, w), h_3(u, v, w), u, v, w), u, v, w \in \mathbb{R}\},$$

where $f_i(k, l, m) : \mathbb{R}^3 \to \mathbb{R}$ and $h_i(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, are continuous functions with $f_i(0, 0, 0) = h_i(0, 0, 0) = 0$. We prove that none of the groups G_i, $i = 1, \ldots, 5$, and $G_j^{\varepsilon=1}$, $j = 6, 7, 8$, satisfy the condition that for all $s \in S$ and $t \in T$ one has $s^{-1}t^{-1}st \in Inn(L)$. By Lemma 7 these groups are not multiplication groups of L. Taking the elements

$$s = g(0, f_1(0, 0, m), f_2(0, 0, m), f_3(0, 0, m), 0, m) \in S,$$

$$t = g(0, h_1(0, v, 0), h_2(0, v, 0), h_3(0, v, 0), v, 0) \in T$$

in G_i, $i = 1, 3, 4, 5$, the products $s^{-1}t^{-1}st$ are contained in $Inn(L)$ if and only if the equation

$$\begin{aligned} i = 1: \ vm &= (1 - e^{-m})h_2(0, v, 0) + (e^{-v} - 1)f_3(0, 0, m) + \\ &h_1(0, v, 0)e^{-av}(1 - e^{-bm}) + f_1(0, 0, m)e^{-bm}(e^{-av} - 1), \qquad (34) \\ i = 3: \ -vm &= (1 - e^{-m})h_1(0, v, 0) - ve^{-v}f_2(0, 0, m) + \end{aligned}$$

$$(f_3(0,0,m)+\varepsilon f_2(0,0,m))(e^{-v}-1), \tag{35}$$

$$i=4:\; -vm = h_1(0,v,0)e^{-av}(1-e^{-bm}) + f_1(0,0,m)e^{-bm}(e^{-av}-1)+$$
$$(1-e^{m})(\varepsilon_1 h_2(0,v,0)+\varepsilon_2 h_3(0,v,0))+$$
$$(\cos v-1)(\varepsilon_1 f_2(0,0,m)+\varepsilon_2 f_3(0,0,m))+$$
$$\sin v(\varepsilon_2 f_2(0,0,m)-\varepsilon_1 f_3(0,0,m)) \tag{36}$$

$$i=5:\; vm = \varepsilon_1 e^{-av}[f_2(0,0,m)(\cos(v)-e^{av})-\sin(v)f_3(0,0,m)]+$$
$$\varepsilon_2 e^{-av}[\sin(v)f_2(0,0,m)+f_3(0,0,m)(\cos(v)-e^{av})]+h_1(0,v,0)(1-e^{-m}) \tag{37}$$

holds for all $m,v\in\mathbb{R}$. On the left hand side of (34), (35), (36), (37) there is the term vm hence there is no function $h_i(0,v,0)$, $f_i(0,0,m)$, $i=1,2,3$, satisfying equations (34), (35), (36), (37).
Taking the elements $s=g(0,f_1(0,0,m),f_2(0,0,m),f_3(0,0,m),0,m)\in S$, $t=g(0,h_1(0,v,w),h_2(0,v,w),h_3(0,v,w),v,w)\in T$ of G_2 the products $s^{-1}t^{-1}st$ are contained in $Inn(L)$ if and only if the equation

$$mv = e^{-m-w}f_2(0,0,m)v + h_1(0,v,w)e^{-aw-v}(1-e^{-am})+$$
$$f_1(0,0,m)e^{-am}(e^{-aw-v}-1)+(h_3(0,v,w)+\varepsilon h_2(0,v,w))e^{-w}(1-e^{-m})+$$
$$(f_3(0,0,m)+\varepsilon f_2(0,0,m))e^{-m}(e^{-w}-1) \tag{38}$$

holds for all $m,v,w\in\mathbb{R}$. The left hand side is mv. But there does not exist any function $h_i(0,v,w)$, $f_i(0,0,m)$, $i=1,2,3$, satisfying equation (38).
Taking the elements $s=g(f_1(0,0,m),0,f_2(0,0,m),f_3(0,0,m),0,m)\in S$, $t=g(h_1(0,v,0),0,h_2(0,v,0),h_3(0,v,0),v,0)\in T$ of $G_6^{\varepsilon=1}$, respectively the elements

$$s=g(f_1(0,0,m),f_2(0,0,m),f_3(0,0,m),0,0,m)\in S,$$

$$t=g(h_1(0,v,0),h_2(0,v,0),h_3(0,v,0),0,v,0)\in T$$

of $G_7^{\varepsilon=1,a=0}$ and of $G_8^{\varepsilon=1}$ the products $s^{-1}t^{-1}st$ are contained in $Inn(L)$ if and only if in case $G_6^{\varepsilon=1}$ the equation

$$\frac{1}{2}v^2m - vf_1(0,0,m) = (1-e^{m})(\varepsilon_2 h_2(0,v,0)+\varepsilon_3 h_3(0,v,0))-\varepsilon_1 vm+$$
$$(\cos(v)-1)(\varepsilon_2 f_2(0,0,m)+\varepsilon_3 f_3(0,0,m))+$$

$$\sin(v)(\varepsilon_3 f_2(0,0,m) - \varepsilon_2 f_3(0,0,m)), \tag{39}$$

respectively in case $G_7^{\varepsilon=1,a=0}$ the equation

$$\frac{1}{2}v^2 m - v f_3(0,0,m) =$$

$$(f_1(0,0,m) - h_1(0,v,0))e^{-v}(\varepsilon_1 \cos(m) + \varepsilon_2 \sin(m)) +$$
$$(f_2(0,0,m) - h_2(0,v,0))e^{-v}(\varepsilon_2 \cos(m) - \varepsilon_1 \sin(m)) +$$
$$\sin(m)(\varepsilon_1 f_2(0,0,m) - \varepsilon_2 f_1(0,0,m)) -$$
$$\cos(m)(\varepsilon_1 f_1(0,0,m) + \varepsilon_2 f_2(0,0,m)) +$$
$$e^{-v}(\varepsilon_1 h_1(0,v,0) + \varepsilon_2 h_2(0,v,0)) - \varepsilon_3 vm, \tag{40}$$

respectively in case $G_8^{\varepsilon=1}$ the equation

$$\frac{1}{2}v^2 m - v f_3(0,0,m) = h_1(0,v,0)e^{-v}(1 - e^{-am}) + h_2(0,v,0)(1 - e^{-m}) +$$

$$f_1(0,0,m)e^{-am}(e^{-v} - 1) - \varepsilon_1 vm \tag{41}$$

holds for all $m, v \in \mathbb{R}$. On the left hand side of equations (39), (40), (41) there is the term $\frac{1}{2}v^2 m$. Hence there does not exist any function $h_i(0,v,0)$, $f_i(0,0,m)$, $i = 1,2,3$, satisfying equations (39), (40), (41).
The products $s^{-1}t^{-1}st$ with

$$s = g(f_1(0,l,m), f_2(0,l,m), f_3(0,l,m), 0, l, m)$$

$$t = g(h_1(0,v,0), h_2(0,v,0), h_3(0,v,0), 0, v, 0)$$

in $G_7^{a\neq 0,\varepsilon=1}$ are contained in $Inn(L)$ if and only if the equation

$$\frac{1}{2}(vl^2 - v^2 l - a^2 vm^2) + (am + l)h_3(0,v,0) - v f_3(0,l,m) - amv^2 = \varepsilon_3 vam +$$

$$(f_1(0,l,m) - h_1(0,v,0))e^{-v-l}(\varepsilon_1 \cos m + \varepsilon_2 \sin m) + e^{-v}\varepsilon_1 h_1(0,v,0) +$$
$$(f_2(0,l,m) - h_2(0,v,0))e^{-v-l}(\varepsilon_2 \cos m - \varepsilon_1 \sin m) + e^{-v}\varepsilon_2 h_2(0,v,0) +$$
$$f_2(0,l,m)e^{-l}(\varepsilon_1 \sin m + \varepsilon_2 \cos m) - f_1(0,l,m)e^{-l}(\varepsilon_2 \sin m + \varepsilon_1 \cos m) \tag{42}$$

holds for all $l, m, v \in \mathbb{R}$, where $\varepsilon_i \in \{0,1\}$, $i = 1,2,3$, such that $\varepsilon_1 \neq 0$ or $\varepsilon_2 \neq 0$. Since on the left hand side of (42) is the term $-\frac{1}{2}v^2 l$ and $a \neq 0$ there is no function $f_i(0,l,m)$, $h_i(0,v,0)$, $i = 1,2,3$, such that equation (42) holds.
The set

$$S = T =$$

$$\{g(\frac{1}{\sigma}(e^l(l+am)(\varepsilon_1 cos(m) - \varepsilon_2 sin(m)) + sin(m))(\varepsilon_1 sin(m) + \varepsilon_2 cos(m)),$$

$$\frac{1}{\sigma}(e^l(l+am)(\varepsilon_1 sin(m) + \varepsilon_2 cos(m)) - sin(m)(\varepsilon_1 cos(m) - \varepsilon_2 cos(m)),$$

$$e^{-l}cos(m) - 1, k, l, m); k, l, m \in \mathbb{R}\},$$

where $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ and $\sigma = \varepsilon_1^2 + \varepsilon_2^2 \neq 0$, is an $Inn(L)_7$-connected left transversal in $G_7^{\varepsilon=0}$ with the property $G_7^{\varepsilon=0} = \langle S \rangle$.
The set

$$S = \{g(0, le^m, 1 - e^{-l-am}, k, l, m); k, l, m \in \mathbb{R}\},$$

$$T = \{g(-ve^{v+aw}, 0, e^{-w} - 1, u, v, w); u, v, w \in \mathbb{R}\}$$

are $Inn(L)_8$-connected left transversals in the group $G_8^{\varepsilon=0}$ such that $S \cup T$ generates $G_8^{\varepsilon=0}$.
The sets

$$S = T = \{g(e^{-m}cos(l) - 1, k,$$

$$\frac{1}{\delta}(le^m(\varepsilon_2 cos(l) - \varepsilon_3 sin(l)) + sin(l))(\varepsilon_3 cos(l) + \varepsilon_2 sin(l)),$$

$$\frac{1}{\delta}(le^m(\varepsilon_2 sin(l) + \varepsilon_3 cos(l)) + sin(l)(\varepsilon_3 sin(l) - \varepsilon_2 cos(l)), l, m);$$

$$k, l, m \in \mathbb{R}\},$$

with $\varepsilon_2, \varepsilon_3 \in \{0,1\}$ and $\delta = \varepsilon_2^2 + \varepsilon_3^2 \neq 0$ is an $Inn(L)_6$-connected left transversal in $G_6^{\varepsilon=0}$ which generates the group $G_6^{\varepsilon=0}$. This shows the assertion. □

Now we study the 6-dimensional decomposable solvable Lie groups with 1-dimensional centre which are the groups $Mult(L)$ of 3-dimensional connected simply connected topological loops L.

Theorem 26. *Let L be a connected topological loop of dimension 3 such that its multiplication group $Mult(L)$ is a 6-dimensional decomposable solvable Lie group with 1-dimensional centre. Then L has nilpotency class 2. Moreover, the following Lie algebra pairs can occur as the Lie algebra $\mathbf{g}$ of the group $Mult(L)$ and the subalgebra $\mathbf{k}$ of the subgroup $Inn(L)$:*
If $\mathbf{g}$ has the form $\mathbf{g} = \mathbb{R} \oplus \mathbf{h} = \langle f_1 \rangle \oplus \langle e_1, e_2, e_3, e_4, e_5 \rangle$, where $\mathbf{h}$ is a 5-dimensional solvable indecomposable Lie algebra with trivial centre, then one has:

- $\mathbf{g}_1 = \mathbb{R} \oplus \mathbf{g}_{5,19}^{\alpha=0,\beta\neq 0}$: $[e_2, e_3] = e_1$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = \beta e_4$, $\mathbf{k}_{1,\epsilon} = \langle e_1 + f_1, e_2 + \epsilon f_1, e_4 + f_1 \rangle$, $\epsilon = 0, 1$,

- $\mathbf{g}_2 = \mathbb{R} \oplus \mathbf{g}_{5,20}^{\alpha=0}$: $[e_2, e_3] = e_1$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = e_1 + e_4$, $\mathbf{k}_{2,\epsilon} = \langle e_1 + f_1, e_2 + \epsilon f_1, e_4 + a_3 f_1 \rangle$, $a_3 \in \mathbb{R}$, $\epsilon = 0, 1$,

- $\mathbf{g}_3 = \mathbb{R} \oplus \mathbf{g}_{5,27}$: $[e_2, e_3] = e_1$, $[e_1, e_5] = e_1$, $[e_3, e_5] = e_3 + e_4$, $[e_4, e_5] = e_1 + e_4$, $\mathbf{k}_3 = \langle e_1 + f_1, e_3, e_4 + a_3 f_1 \rangle$, $a_3 \in \mathbb{R}$,

- $\mathbf{g}_4 = \mathbb{R} \oplus \mathbf{g}_{5,28}^{\alpha=0}$: $[e_2, e_3] = e_1$, $[e_1, e_5] = e_1$, $[e_3, e_5] = e_3 + e_4$, $[e_4, e_5] = e_4$, $\mathbf{k}_4 = \langle e_1 + a_1 f_1, e_3, e_4 + f_1 \rangle$, $a_1 \in \mathbb{R} \backslash \{0\}$,

- $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,32}$: $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$, $[e_3, e_5] = h e_1 + e_3$, $\mathbf{k}_5 = \langle e_1 + f_1, e_2 + a_2 f_1, e_3 \rangle$, $h, a_2 \in \mathbb{R}$,

- $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,33}$: $[e_1, e_4] = e_1$, $[e_3, e_4] = \beta e_3$, $[e_2, e_5] = e_2$, $[e_3, e_5] = \gamma e_3$, $\beta^2 + \gamma^2 \neq 0$, $\mathbf{k}_6 = \langle e_1 + f_1, e_2 + f_1, e_3 + f_1 \rangle$,

- $\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,34}$: $[e_1, e_4] = \alpha e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$, $[e_1, e_5] = e_1$, $[e_3, e_5] = e_2$, $\mathbf{k}_7 = \langle e_1 + f_1, e_2 + f_1, e_3 + a_3 f_1 \rangle$, $\alpha, a_3 \in \mathbb{R}$,

- $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,35}$: $[e_1, e_4] = h e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$, $[e_1, e_5] = \alpha e_1$, $[e_2, e_5] = -e_3$, $[e_3, e_5] = e_2$, $h^2 + \alpha^2 \neq 0$, $\mathbf{k}_{8,1} = \langle e_1 + f_1, e_2 + f_1, e_3 + a_3 f_1 \rangle$, $a_3 \in \mathbb{R}$, $\mathbf{k}_{8,2} = \langle e_1 + f_1, e_2, e_3 + f_1 \rangle$.

If $\mathbf{g}$ is the Lie algebra $\mathbf{l}_2 \oplus \mathbf{n} = \langle f_1, f_2 \rangle \oplus \langle e_1, e_2, e_3, e_4 \rangle$, where $\mathbf{n}$ is a 4-dimensional solvable Lie algebra with 1-dimensional centre $\langle e_1 \rangle$, then we have:

- $\mathbf{g}_9 = \mathbf{l}_2 \oplus \mathbf{g}_{4,1}$: $[f_1, f_2] = f_1$, $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $\mathbf{k}_9 = \langle f_1 + e_1, e_2 + a_2 e_1, e_3 \rangle$, $a_2 \in \mathbb{R}$,

- $\mathbf{g}_{10} = \mathbf{l}_2 \oplus \mathbf{g}_{4,3}$: $[f_1, f_2] = f_1$, $[e_1, e_4] = e_1$, $[e_3, e_4] = e_2$, $\mathbf{k}_{10} = \langle f_1 + e_2, e_1 + e_2, e_3 \rangle$.

If $\mathbf{g}$ is one of the following Lie algebras $\mathbf{f}_3 \oplus \mathbf{g}_{3,i}$ and $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}$, $i = 2, 3, 4, 5$, where the centre of $\mathbf{f}_3 = \langle e_1, e_2, e_3 \rangle$ is $\langle e_1 \rangle$ and $\mathbf{g}_{3,i} = \langle e_4, e_5, e_6 \rangle$ is a 3-dimensional solvable Lie algebra with trivial centre, then one has:

- $\mathbf{g}_{11} = \mathbf{f}_3 \oplus \mathbf{g}_{3,2}$: $[e_2, e_3] = e_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = e_4 + e_5$, $\mathbf{k}_{11,1} = \langle e_2, e_4 + e_1, e_5 \rangle$, $\mathbf{k}_{11,2} = \langle e_3, e_4 + e_1, e_5 \rangle$,

- $\mathbf{g}_{12} = \mathbf{f}_3 \oplus \mathbf{g}_{3,3}$: $[e_2, e_3] = e_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = e_5$, $\mathbf{k}_{12,1} = \langle e_2, e_4 + e_1, e_5 + e_1 \rangle$, $\mathbf{k}_{12,2} = \langle e_3, e_4 + e_1, e_5 + e_1 \rangle$,

- $\mathbf{g}_{13} = \mathbf{f}_3 \oplus \mathbf{g}_{3,4}$: $[e_2, e_3] = e_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = he_5$, $-1 \leq h < 1$, $h \neq 0$, $\mathbf{k}_{13,1} = \langle e_2, e_4 + e_1, e_5 + e_1 \rangle$, $\mathbf{k}_{13,2} = \langle e_3, e_4 + e_1, e_5 + e_1 \rangle$,

- $\mathbf{g}_{14} = \mathbf{f}_3 \oplus \mathbf{g}_{3,5}$: $[e_2, e_3] = e_1$, $[e_4, e_6] = pe_4 - e_5$, $[e_5, e_6] = e_4 + pe_5$, $p \geq 0$, $\mathbf{k}_{14,1} = \langle e_2, e_4 + e_1, e_5 + a_3 e_1 \rangle$, $\mathbf{k}_{14,2} = \langle e_3, e_4 + e_1, e_5 + a_3 e_1 \rangle$, $a_3 \in \mathbb{R} \backslash \{0\}$, $\mathbf{k}_{14,3} = \langle e_2, e_4, e_5 + e_1 \rangle$, $\mathbf{k}_{14,4} = \langle e_3, e_4, e_5 + e_1 \rangle$,

- $\mathbf{g}_{15} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,2}$: $[f_1, f_2] = f_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = e_4 + e_5$, $\mathbf{k}_{15} = \langle f_1 + e_3, e_4 + e_3, e_5 \rangle$,

- $\mathbf{g}_{16} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,3}$: $[f_1, f_2] = f_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = e_5$, $\mathbf{k}_{16} = \langle f_1 + e_3, e_4 + e_3, e_5 + e_3 \rangle$,

- $\mathbf{g}_{17} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,4}$: $[f_1, f_2] = f_1$, $[e_4, e_6] = e_4$, $[e_5, e_6] = he_5$, $-1 \leq h < 1$, $h \neq 0$, $\mathbf{k}_{17} = \langle f_1 + e_3, e_4 + e_3, e_5 + e_3 \rangle$,

- $\mathbf{g}_{18} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,5}$: $[f_1, f_2] = f_1$, $[e_4, e_6] = pe_4 - e_5$, $[e_5, e_6] = e_4 + pe_5$, $p \geq 0$, $\mathbf{k}_{18,1} = \langle f_1 + e_3, e_4 + e_3, e_5 + a_3 e_3 \rangle$, $a_3 \in \mathbb{R}$, $\mathbf{k}_{18,2} = \langle f_1 + e_3, e_4, e_5 + e_3 \rangle$.

Proof. By Lemma 9 we may assume that the loop L is simply connected and hence it is homeomorphic to $\mathbb{R}^3$. Every 6-dimensional decomposable solvable Lie algebra with 1-dimensional centre has one of the following forms: $\mathbb{R} \oplus \mathbf{h}$, $\mathbf{l}_2 \oplus \mathbf{n}$, $\mathbf{f}_3 \oplus \mathbf{g}_{3,i}$, and $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}$, where $\mathbf{h}$, $\mathbf{n}$, $\mathbf{g}_{3,i}$ are described in the assertion. For $\mathbf{h}$ we have the following possibilities: $\mathbf{g}_{5,i}$, $i = 7, 9, 11, 12, 13, 16, 17, 18, 21, 23, 24, 27, 31, 32, 33, 34, 35, 36, 37$ and $\mathbf{g}_{5,15}^{\gamma \neq 0}$, $\mathbf{g}_{5,j}^{\alpha = 0}$, $j = 19, 20, 28$, $\mathbf{g}_{5,k}^{p \neq 0}$, $k = 25, 26$, $\mathbf{g}_{5,30}^{h \neq -2}$. For $\mathbf{n}$ one has the following Lie algebras $\mathbf{g}_{4,i}$, $i = 1, 3$, $\mathbf{g}_{4,8}^{h=-1}$, $\mathbf{g}_{4,9}^{p=0}$ and for the 3-dimensional solvable Lie algebras with trivial centre we have $\mathbf{g}_{3,i}$, $i = 2, 3, 4, 5$ (cf. [24], §4, §5, and [25], §10, p. 105-106).

To prove the first assertion we have to show that L has a normal subloop N isomorphic to $\mathbb{R}$ such that the factor loop L/N is isomorphic to $\mathbb{R}^2$ (cf. Theorems 12 and 17). Assume firstly that the Lie algebra g of the group $Mult(L)$ of L has the form $\mathbb{R} \oplus \mathbf{h} = \langle f_1 \rangle \oplus \langle e_1, e_2, e_3, e_4, e_5 \rangle$. If $\mathbf{h} \neq \mathbf{g}_{5,i}$, $i = 33, 34$, then there does not exist any ideal $\mathbf{s}$ containing the centre $\mathbf{z} = \langle f_1 \rangle$ of $\mathbf{g}$ such that the factor Lie algebras $(\mathbb{R} \oplus \mathbf{h})/\mathbf{s}$ are isomorphic to $\mathbf{f}_n$, $n = 4, 5$ or to $\mathbf{l}_2 \oplus \mathbf{l}_2$. According to Proposition 19 a) the factor loop $L/Z(e)$, where $Z = \exp(\mathbf{z})$, is isomorphic to $\mathbb{R}^2$ and the orbit $Z(e)$ is the normal subloop N.

The Lie algebras $\mathbb{R} \oplus \mathbf{g}_{5,i}$, $i = 33, 34$, have no factor Lie algebras isomorphic to $\mathbf{f}_n$, $n = 4, 5$. The Lie algebra $\mathbb{R} \oplus \mathbf{g}_{5,34}$ has the ideal $\mathbf{i} = \langle e_1 \rangle$. None of the factor Lie algebras $\mathbb{R} \oplus \mathbf{g}_{5,34}/\mathbf{s}$, where $\mathbf{s}$ is any ideal containing $\mathbf{i}$, is isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. Therefore the orbit $I(e)$, where I is the simply connected Lie group of $\mathbf{i}$, can choose as the normal subloop N.

The Lie algebra $\mathbb{R} \oplus \mathbf{g}_{5,33}$, $\beta^2 + \gamma^2 \neq 0$, have the ideals $\mathbf{i}_1 = \langle f_1 \rangle$, $\mathbf{i}_2 = \langle e_1 \rangle$, $\mathbf{i}_3 = \langle e_2 \rangle$, $\mathbf{i}_4 = \langle e_3 \rangle$. If $\mathbb{R} \oplus \mathbf{g}_{5,33}$ is the Lie algebra of the group $Mult(L)$ of L, then the orbits $I_j(e)$, $j \in \{1, 2, 3, 4\}$, are normal subgroups of L isomorphic to $\mathbb{R}$. The factor loops $L/I_j(e)$, $j \in \{1, 2, 3, 4\}$, are isomorphic either to $\mathcal{L}_2$ or to $\mathbb{R}^2$ (cf. Proposition 19 a). If all factor loops $L/I_j(e)$, $j \in \{1, 2, 3, 4\}$, are isomorphic to $\mathcal{L}_2$, then by Proposition 19 (ii) there are 2-dimensional ideals $\mathbf{s}_j$, $j \in \{1, 2, 3, 4\}$, such that $\mathbf{i}_j < \mathbf{s}_j$ and the factor Lie algebras $\mathbb{R} \oplus \mathbf{g}_{5,33}/\mathbf{s}_j$ are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. For the ideal $\mathbf{s}_1 = \mathbf{s}_4 = \langle f_1, e_3 \rangle$ one has $\mathbb{R} \oplus \mathbf{g}_{5,33}/\mathbf{s}_l \cong \mathbf{l}_2 \oplus \mathbf{l}_2$, $l = 1, 4$. The factor Lie algebra $\mathbb{R} \oplus \mathbf{g}_{5,33}/\langle f_1, e_1 \rangle$ is isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ if and only if $\gamma = 0$ and $\mathbb{R} \oplus \mathbf{g}_{5,33}/\langle f_1, e_2 \rangle$ is isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ precisely if $\beta = 0$. This contradiction to $\beta^2 + \gamma^2 \neq 0$ yields that at least one of the factor loops $L/I_j(e)$, $j \in \{1, 2, 3, 4\}$, is isomorphic to $\mathbb{R}^2$. For such $j \in \{1, 2, 3, 4\}$ the orbit $I_j(e)$ is the requested normal subgroup N of L. Hence L has nilpotency class 2.

By Proposition 19 (i) the Lie algebra $\mathbb{R} \oplus \mathbf{h}$ has a 4-dimensional abelian ideal $\mathbf{p} = \mathbf{z} \oplus \mathbf{k}$, where $\mathbf{z} = \langle f_1 \rangle$ and $\mathbf{k}$ is the Lie algebra of the group $Inn(L)$ and $\mathbf{p}$ contains the commutator subalgebra of $\mathbb{R} \oplus \mathbf{h}$. According to Lemma 8 the subalgebra $\mathbf{k}$ does not contain any non-zero ideal of $\mathbf{g}$ and the

normalizer $N_{\mathbf{g}}(\mathbf{k})$ of $\mathbf{k}$ in $\mathbf{g}$ is $\mathbf{p}$. The commutator subalgebra of $\mathbb{R}\oplus\mathbf{h}$ coincides with the commutator subalgebra $\mathbf{h}'$ of $\mathbf{h}$. The intersection of $\mathbf{z}$ and $\mathbf{h}'$ is trivial. Since $\mathbf{h}' < \mathbf{p}$ the Lie algebra $\mathbf{h}$ has a 3-dimensional abelian commutator subalgebra. Then for the triples $(\mathbf{g}, \mathbf{p}, \mathbf{k})$ we obtain:
(a) For the Lie algebras $\mathbb{R}\oplus\mathbf{g}_{5,j}^{\alpha=0}$, $j = 19, 20$, we have $\mathbf{p} = \langle f_1, e_1, e_2, e_4\rangle$. The subalgebra $\mathbf{k}$ has the form: $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 f_1, e_2 + a_2 f_1, e_4 + a_3 f_1\rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that for $j = 19$ one has $a_1 \neq 0, a_3 \neq 0$ since $\langle e_1\rangle$ and $\langle e_4\rangle$ are ideals of $\mathbb{R}\oplus\mathbf{g}_{5,19}^{\alpha=0}$,
for $j = 20$ we have $a_1 \neq 0$ since $\langle e_1\rangle$ is an ideal of $\mathbb{R}\oplus\mathbf{g}_{5,20}^{\alpha=0}$. Applying the automorphism $\phi(f_1) = f_1$, $\phi(e_1) = a_1 e_1$, $\phi(e_2) = e_2$, $\phi(e_3) = a_1 e_3$, $\phi(e_4) = a_3 e_4$, $\phi(e_5) = e_5$ for the Lie algebra $\mathbb{R}\oplus\mathbf{g}_{5,19}^{\alpha=0}$, if $a_2 = 0$, respectively $\phi(e_2) = a_2 e_2$, $\phi(e_3) = \frac{a_1}{a_2} e_3$, if $a_2 \neq 0$, we can reduce $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_{1,\epsilon}$, where ϵ equals to 0, respectively to 1. Using the automorphism $\phi(f_1) = \frac{1}{a_1} f_1$, $\phi(e_i) = e_i$, $i = 1, 2, 3, 4, 5$, for the Lie algebra $\mathbb{R}\oplus\mathbf{g}_{5,20}^{\alpha=0}$, if $a_2 = 0$, respectively $\phi(f_1) = f_1$, $\phi(e_j) = a_1 e_j$, $j = 1, 4$, $\phi(e_2) = a_2 e_2$, $\phi(e_3) = \frac{a_1}{a_2} e_3$, $\phi(e_5) = e_5$ if $a_2 \neq 0$ the Lie algebra $\mathbf{k}_{a_1,a_2,a_3}$ reduces to $\mathbf{k}_{2,\epsilon}$, where ϵ is equal to 0, respectively to 1.
(b) For the Lie algebras $\mathbb{R}\oplus\mathbf{g}_{5,27}$ and $\mathbb{R}\oplus\mathbf{g}_{5,28}^{\alpha=0}$ we have $\mathbf{p} = \langle f_1, e_1, e_3, e_4\rangle$, the subalgebra $\mathbf{k}$ has the form: $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 f_1, e_3 + a_2 f_1, e_4 + a_3 f_1\rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that if $j = 27$ one has $a_1 \neq 0$, since $\langle e_1\rangle$ is an ideal of $\mathbb{R}\oplus\mathbf{g}_{5,27}$,
if $j = 28$ one has $a_1 a_3 \neq 0$ since $\langle e_1\rangle$ and $\langle e_4\rangle$ are ideals of $\mathbb{R}\oplus\mathbf{g}_{5,28}^{\alpha=0}$.

Using the automorphism $\phi(f_1) = f_1$, $\phi(e_i) = a_1 e_i$, $i = 1, 4$, $\phi(e_j) = e_j$, $j = 2, 5$, $\phi(e_3) = a_1 e_3 + a_2 e_1$ for $\mathbb{R}\oplus\mathbf{g}_{5,27}$, respectively $\phi(e_1) = a_1 a_3 e_1$, $\phi(e_2) = a_1 e_2$, $\phi(e_3) = a_3 e_3 + a_2 e_4$, $\phi(e_4) = a_3 e_4$ for $\mathbb{R}\oplus\mathbf{g}_{5,28}^{\alpha=0}$ we can reduce $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_3$, respectively to $\mathbf{k}_4$ in the assertion.
(c) For the Lie algebras $\mathbb{R}\oplus\mathbf{g}_{5,i}$, $i = 32, 33, 34, 35$, we get that the ideal $\mathbf{p} = \langle f_1, e_1, e_2, e_3\rangle$, the subalgebra $\mathbf{k}$ has the form: $\mathbf{k}_{a_1,a_2,a_3} = \langle e_1 + a_1 f_1, e_2 + a_2 f_1, e_3 + a_3 f_1\rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that if $j = 32$ we have $a_1 \neq 0$ since $\langle e_1\rangle$ is an ideal of $\mathbb{R}\oplus\mathbf{g}_{5,32}$,
if $j = 33$ we have $a_1 a_2 a_3 \neq 0$ since $\langle e_1\rangle$, $\langle e_2\rangle$ and $\langle e_3\rangle$ are ideals of $\mathbb{R}\oplus\mathbf{g}_{5,33}$,
if $j = 34$ we have $a_1 a_2 \neq 0$ since $\langle e_1\rangle$, $\langle e_2\rangle$ are ideals of $\mathbb{R}\oplus\mathbf{g}_{5,34}$,
if $j = 35$ we have $a_1 \neq 0$ and at least one of $\{a_2, a_3\}$ is different from 0 since $\langle e_1\rangle$ and $\langle e_2, e_3\rangle$ are ideals of $\mathbb{R}\oplus\mathbf{g}_{5,35}$.

The automorphism $\phi(f_1) = f_1$, $\phi(e_i) = a_1 e_i$, $i = 1, 2$, $\phi(e_3) = a_1 e_3 +$

a_3e_1 and $\phi(e_j) = e_j$, $j = 4, 5$ for $\mathbb{R} \oplus \mathbf{g}_{5,32}$, respectively $\phi(e_2) = a_2e_2$, $\phi(e_3) = a_3e_3$ for $\mathbb{R}\oplus\mathbf{g}_{5,33}$, respectively $\phi(e_s) = a_2e_s$, $s = 2, 3$, for $\mathbb{R}\oplus\mathbf{g}_{5,34}$ reduces the Lie algebra $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_5$, respectively to $\mathbf{k}_6$, respectively to $\mathbf{k}_7$ in the assertion. Applying the automorphism $\phi(f_1) = f_1$, $\phi(e_1) = a_1e_1$, $\phi(e_i) = a_2e_i$, $i = 2, 3$ and $\phi(e_j) = e_j$, $j = 4, 5$, for the Lie algebra $\mathbb{R} \oplus \mathbf{g}_{5,35}$ if $a_1a_2 \neq 0$, respectively $\phi(e_s) = a_3e_s$, $s = 2, 3$, if $a_1a_3 \neq 0$ and $a_2 = 0$ we can reduce $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_{8,1}$, respectively $\mathbf{k}_{a_1,0,a_3}$ to $\mathbf{k}_{8,2}$ in the assertion.

Secondly, assume that the Lie algebra of the group $Mult(L)$ of L has the shape: $\mathbf{l}_2\oplus\mathbf{n} = \langle f_1, f_2\rangle\oplus\langle e_1, e_2, e_3, e_4\rangle$ as in the assertion. If $\mathbf{n} \neq \mathbf{g}_{4,1}$, then there does not exist any ideal $\mathbf{s}$ containing the ideal $\mathbf{i} = \langle f_1\rangle$ such that the factor Lie algebras $(\mathbf{l}_2\oplus\mathbf{n})/\mathbf{s}$ are isomorphic to $\mathbf{f}_n$, $n = 4, 5$ or to $\mathbf{l}_2\oplus\mathbf{l}_2$. The Lie algebra $\mathbf{l}_2 \oplus \mathbf{g}_{4,1}$ has the centre $\mathbf{i} = \langle e_1\rangle$, but it has no factor Lie algebra isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. None of the factor Lie algebras $\mathbf{l}_2 \oplus \mathbf{g}_{4,1}/\mathbf{s}$, where $\mathbf{s}$ is any ideal containing $\mathbf{i}$, are isomorphic to $\mathbf{f}_n$, $n = 4, 5$. Hence in both cases the orbit $I(e)$, where I is the simply connected Lie group of $\mathbf{i}$, is a normal subgroup of L isomorphic to $\mathbb{R}$ and the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$ (cf. Proposition 19 a). According to Lemma 10 a) and Theorem 17 in both cases the orbit $I(e)$ coincides with the centre $Z(L)$ of L and L has nilpotency class 2.

Moreover, the Lie algebra $\mathbf{l}_2 \oplus \mathbf{n}$ has a 4-dimensional abelian ideal $\mathbf{p} = \mathbf{z}\oplus\mathbf{k}$, where $\mathbf{z}$ is the 1-dimensional centre of $\mathbf{l}_2\oplus\mathbf{n}$ and $\mathbf{k}$ is the Lie algebra of the group $Inn(L)$, such that $\mathbf{p}$ contains the commutator subalgebra of $\mathbf{l}_2 \oplus \mathbf{n}$. The commutator subalgebra of $\mathbf{l}_2 \oplus \mathbf{g}_{4,i}$, $i = 1, 3, 8, 9$, is the direct sum $\langle f_1\rangle \oplus \mathbf{g}'_{4,i}$, where $\mathbf{g}'_{4,i}$ is the commutator subalgebra of $\mathbf{g}_{4,i}$. Since the commutator subalgebras $\mathbf{g}'_{4,j}$, $j = 8, 9$, are not abelian, the Lie algebras $\mathbf{l}_2 \oplus \mathbf{g}_{4,j}$, $j = 8, 9$, are excluded. Hence we have to deal with the Lie algebras $\mathbf{l}_2 \oplus \mathbf{g}_{4,k}$, $k = 1, 3$.

(d) The Lie algebra $\mathbf{l}_2 \oplus \mathbf{g}_{4,1}$ has the centre $\mathbf{z} = \langle e_1\rangle$ and one has $\mathbf{p} = \langle f_1, e_1, e_2, e_3\rangle$. The subalgebra $\mathbf{k}$ has the form: $\mathbf{k}_{a_1,a_2,a_3} = \langle f_1 + a_1e_1, e_2 + a_2e_1, e_3 + a_3e_1\rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that $a_1 \neq 0$ since $\langle f_1\rangle$ is an ideal of $\mathbf{l}_2 \oplus \mathbf{g}_{4,1}$. The centre of the Lie algebra $\mathbf{l}_2 \oplus \mathbf{g}_{4,3}$ is $\mathbf{z} = \langle e_2\rangle$ and the ideal $\mathbf{p}$ is again $\langle f_1, e_1, e_2, e_3\rangle$. The subalgebra $\mathbf{k}$ has the form: $\mathbf{k}_{a_1,a_2,a_3} = \langle f_1 + a_1e_2, e_1 + a_2e_2, e_3 + a_3e_2\rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$ such that $a_1 \neq 0$ and $a_2 \neq 0$ since $\langle f_1\rangle$ and $\langle e_1\rangle$ are ideals of $\mathbf{l}_2 \oplus \mathbf{g}_{4,3}$. The automorphism $\phi(f_1) = a_1f_1$, $\phi(f_2) = f_2$, $\phi(e_3) = e_3 - a_3e_1$, $\phi(e_i) = e_i$,

$i = 1, 2, 4$, of $\mathbf{l}_2 \oplus \mathbf{g}_{4,1}$, respectively $\phi(e_1) = a_2 e_1$, $\phi(e_3) = e_3 - a_3 e_2$ of $\mathbf{l}_2 \oplus \mathbf{g}_{4,3}$, reduces the Lie algebra $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_9$, respectively to $\mathbf{k}_{10}$, in the assertion.

Finally, for the Lie algebras $\mathbf{f}_3 \oplus \mathbf{g}_{3,i} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle$, respectively $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i} = \langle f_1, f_2 \rangle \oplus \langle e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle$, $i = 2, 3, 4, 5$, there does not exist any ideal $\mathbf{s}_1$, respectively $\mathbf{s}_2$, containing the ideal $\mathbf{i}_1 = \langle e_1 \rangle$, respectively $\mathbf{i}_2 = \langle f_1 \rangle$, such that the factor Lie algebras $\mathbf{f}_3 \oplus \mathbf{g}_{3,i}/\mathbf{s}_1$, respectively $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}/\mathbf{s}_2$, are isomorphic to $\mathbf{f}_n$, $n = 4, 5$, or to $\mathbf{l}_2 \oplus \mathbf{l}_2$. Hence if $\mathbf{f}_3 \oplus \mathbf{g}_{3,i}$ or $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}$, $i = 2, 3, 4, 5$, is the Lie algebra of the group $Mult(L)$ of L, then the orbits $I_i(e)$, $i = 1, 2$, are the centre of L such that the factor loops $L/I_i(e)$, $i = 1, 2$, are isomorphic to $\mathbb{R}^2$ (cf. Lemma 10 a) and Theorem 17). Hence L is centrally nilpotent of class 2.

According to Proposition 19 (i) we have to find an ideal $\mathbf{p} = \mathbf{z} \oplus \mathbf{k} \cong \mathbb{R}^4$ of the Lie algebras $\mathbf{f}_3 \oplus \mathbf{g}_{3,i}$ and $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}$, $i = 2, 3, 4, 5$, where $\mathbf{z}$ is their 1-dimensional centre, $\mathbf{p}$ contains their commutator subalgebra and $\mathbf{k}$ is the Lie algebra of the group $Inn(L)$ satisfying the assertion of Lemma 8.

(e) The Lie algebras $\mathbf{f}_3 \oplus \mathbf{g}_{3,i}$, $i = 2, 3, 4, 5$, have the centre $\mathbf{z} = \langle e_1 \rangle$ and the ideal $\mathbf{p}$ has one of the forms : $\mathbf{p}_r = \langle e_1, e_2 + r e_3, e_4, e_5 \rangle$, $r \in \mathbb{R}$, and $\mathbf{p} = \langle e_1, e_3, e_4, e_5 \rangle$. With respect to the ideals $\mathbf{p}_r$, $\mathbf{p}$ we obtain the subalgebras $\mathbf{k}_r = \langle e_2 + r e_3 + a_1 e_1, e_4 + a_2 e_1, e_5 + a_3 e_1 \rangle$, $\mathbf{k}_{a_1,a_2,a_3} = \langle e_3 + a_1 e_1, e_4 + a_2 e_1, e_5 + a_3 e_1 \rangle$, $r, a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that in the case $\mathbf{f}_3 \oplus \mathbf{g}_{3,2}$ one has $a_2 \neq 0$ since $\langle e_4 \rangle$ is an ideal of $\mathbf{f}_3 \oplus \mathbf{g}_{3,2}$,
in the cases $\mathbf{f}_3 \oplus \mathbf{g}_{3,i}$, $i = 3, 4$, we have $a_2 a_3 \neq 0$ since $\langle e_4 \rangle$, $\langle e_5 \rangle$ are ideals of $\mathbf{f}_3 \oplus \mathbf{g}_{3,i}$,
in the case $\mathbf{f}_3 \oplus \mathbf{g}_{3,5}$ one has $a_2 \neq 0$ or $a_3 \neq 0$ since $\langle e_4, e_5 \rangle$ is an ideal of $\mathbf{f}_3 \oplus \mathbf{g}_{3,5}$. The automorphism $\phi(e_2) = e_2 - r e_3 - a_1 e_1$, $\phi(e_4) = a_2 e_4$, $\phi(e_5) = a_2 e_5 + a_3 e_4$ and $\phi(e_j) = e_j$, $j = 1, 3, 6$, respectively $\phi(e_2) = e_2$, $\phi(e_3) = e_3 - a_1 e_1$, of $\mathbf{f}_3 \oplus \mathbf{g}_{3,2}$ maps the subalgebra $\mathbf{k}_r$ onto $\mathbf{k}_{11,1}$, respectively $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{11,2}$. The automorphism $\phi(e_2) = e_2 - r e_3 - a_1 e_1$, $\phi(e_4) = a_2 e_4$, $\phi(e_5) = a_3 e_5$ and $\phi(e_j) = e_j$, $j = 1, 3, 6$, respectively $\phi(e_2) = e_2$ and $\phi(e_3) = e_3 - a_1 e_1$, of $\mathbf{g}_{3,1} \oplus \mathbf{g}_{3,i}$, $i = 3, 4$, maps the subalgebra $\mathbf{k}_r$ onto $\mathbf{k}_{12,1} = \mathbf{k}_{13,1}$, respectively $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k}_{12,2} = \mathbf{k}_{13,2}$, in the assertion. For the Lie algebra $\mathbf{f}_3 \oplus \mathbf{g}_{3,5}$ the automorphisms $\phi(e_2) = e_2 - r e_3 - a_1 e_1$, $\phi(e_j) = a_2 e_j$, $j = 4, 5$, $\phi(e_i) = e_i$, $i = 1, 3, 6$, respectively $\phi(e_2) = e_2$, $\phi(e_3) = e_3 - a_1 e_1$, if $a_2 \neq 0$, reduce $\mathbf{k}_r$ to $\mathbf{k}_{14,1}$, respectively $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_{14,2}$. Moreover, if $a_3 \neq 0$ and $a_2 = 0$, then the automorphisms

$\phi(e_2) = e_2 - re_3 - a_1e_1$, $\phi(e_j) = a_3e_j$, $j = 4, 5$, $\phi(e_i) = e_i$, $i = 1, 3, 6$, respectively $\phi(e_2) = e_2$, $\phi(e_3) = e_3 - a_1e_1$, change the Lie algebra $\mathbf{k}_r$ to $\mathbf{k}_{14,3}$, respectively $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_{14,4}$, in the assertion.

The centre of the Lie algebras $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}$ with $i = 2, 3, 4, 5$, is $\mathbf{z} = \langle e_3 \rangle$ and their ideal $\mathbf{p}$ is $\langle f_1, e_4, e_5, e_3 \rangle$. The subalgebra $\mathbf{k}$ has the form: $\mathbf{k}_{a_1,a_2,a_3} = \langle f_1 + a_1e_3, e_4 + a_2e_3, e_5 + a_3e_3 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that in the case $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,2}$ we have $a_1a_2 \neq 0$ since $\langle f_1 \rangle$ and $\langle e_4 \rangle$ are ideals of $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,2}$,
in the cases $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}$, $i = 3, 4$, one has $a_1a_2a_3 \neq 0$ since $\langle f_1 \rangle$, $\langle e_4 \rangle$ and $\langle e_5 \rangle$ are ideals of $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}$,
in the case $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,5}$ we have $a_1 \neq 0$ and at least one of $\{a_2, a_3\}$ is different from 0 since $\langle f_1 \rangle$ and $\langle e_4, e_5 \rangle$ are ideals of $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,5}$. Using the automorphism $\phi(f_1) = a_1f_1$, $\phi(f_2) = f_2$, $\phi(e_4) = a_2e_4$, $\phi(e_5) = a_2e_5 + a_3e_4$ and $\phi(e_j) = e_j$, $j = 3, 6$, for $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,2}$, respectively $\phi(e_5) = a_3e_5$ for $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,i}$, $i = 3, 4$, the Lie algebra $\mathbf{k}_{a_1,a_2,a_3}$ reduces to $\mathbf{k}_{15}$, respectively to $\mathbf{k}_{16} = \mathbf{k}_{17}$. Applying the automorphism $\phi(f_1) = a_1f_1$, $\phi(f_2) = f_2$, $\phi(e_4) = a_2e_4$, $\phi(e_5) = a_2e_5$ and $\phi(e_j) = e_j$, $j = 3, 6$, for $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,5}$, if $a_1a_2 \neq 0$, respectively $\phi(e_4) = a_3e_4$ and $\phi(e_5) = a_3e_5$, if $a_1a_3 \neq 0$ and $a_2 = 0$, we can reduce $\mathbf{k}_{a_1,a_2,a_3}$ to $\mathbf{k}_{18,1}$, respectively $\mathbf{k}_{a_1,0,a_3}$ to $\mathbf{k}_{18,2}$. This proves the assertion. □

Using ([39], §4) we obtain:

Lemma 27. *The simply connected Lie group G_i and its subgroup K_i of the Lie algebra $\mathbf{g}_i$ and its subalgebra $\mathbf{k}_i$, $i = 1, ..., 18$, given in Theorem 26 is isomorphic to the linear group of matrices the multiplication of which is given by:*
for $i = 1$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + (y_1 - x_3y_2)e^{x_5},$$

$$x_2 + y_2e^{x_5}, (x_3 + y_3)e^{x_5+y_5}, x_4 + y_4e^{bx_5}, x_5 + y_5, x_6 + y_6),$$

$$K_{1,\epsilon} = \{g(u_1, u_2, 0, u_3, 0, u_1 + \epsilon u_2 + u_3); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

$$b \in \mathbb{R}\backslash\{0\}, \epsilon = 0, 1,$$

for $i = 2$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+(y_1-x_3y_2+x_5y_4)e^{x_5}, x_2+y_2e^{x_5},$$
$$(x_3+y_3)e^{x_5+y_5}, x_4+y_4e^{x_5}, x_5+y_5, x_6+y_6),$$
$$K_{2,\epsilon}=\{g(u_1,u_2,0,u_3,0,u_1+\epsilon u_2+a_3u_3); u_i\in\mathbb{R}, i=1,2,3\},$$
$$\epsilon=0,1, a_3\in\mathbb{R},$$

for $i=3$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1+x_5y_4+\frac{1}{2}(2x_2+x_5^2)y_3)e^{x_5}, (x_2+y_2+x_5y_5+\frac{1}{2}y_5^2+\frac{1}{2}x_5^2)e^{x_5+y_5},$$
$$x_3+y_3e^{x_5}, x_4+(y_4+x_5y_3)e^{x_5}, x_5+y_5, x_6+y_6),$$
$$K_3=\{g(u_1,0,u_2,u_3,0,u_1+a_3u_3); u_i\in\mathbb{R}, i=1,2,3\}, a_3\in\mathbb{R},$$

for $i=4$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=g(x_1+(y_1+x_2y_3)e^{x_5},$$
$$(x_2+y_2)e^{x_5+y_5}, x_3+y_3e^{x_5}, x_4+(y_4+x_5y_3)e^{x_5}, x_5+y_5, x_6+y_6),$$
$$K_4=\{g(u_1,0,u_2,u_3,0,a_1u_1+u_3); u_i\in\mathbb{R}, i=1,2,3\}, a_1\in\mathbb{R}\backslash\{0\},$$

for $i=5$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1+x_4y_2+ax_5y_3+\frac{1}{2}x_4^2y_3)e^{x_5}, x_2+(y_2+x_4y_3)e^{x_5},$$
$$x_3+y_3e^{x_5}, (x_4+y_4)e^{x_5+y_5}, x_5+y_5, x_6+y_6), a\in\mathbb{R},$$
$$K_5=\{g(u_1,u_2,u_3,0,0,u_1+a_2u_2); u_i\in\mathbb{R}, i=1,2,3\}, a_2\in\mathbb{R},$$

for $i=6$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1e^{x_4}, x_2+y_2e^{x_5}, x_3+y_3e^{ax_5+bx_4}, x_4+y_4, x_5+y_5, x_6+y_6),$$
$$K_6=\{g(u_1,u_2,u_3,0,0,u_1+u_2+u_3); u_i\in\mathbb{R}, i=1,2,3\}, a^2+b^2\neq 0,$$

for $i = 7$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1e^{ax_4+x_5},$$

$$x_2 + (y_2 + x_5y_3)e^{x_4}, x_3 + y_3e^{x_4}, x_4 + y_4e^{ax_4+x_5}, (x_5 + y_5)e^{x_4+y_4}, x_6 + y_6),$$

$$K_7 = \{g(u_1, u_2, u_3, 0, 0, u_1 + u_2 + a_3u_3); u_i \in \mathbb{R}, i = 1, 2, 3\}, a, a_3 \in \mathbb{R},$$

for $i = 8$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1e^{ax_5+bx_4},$$

$$x_2 + (y_2cos(x_5) - y_3sin(x_5))e^{x_4}, x_3 + (y_3cos(x_5) + y_2sin(x_5))e^{x_4},$$

$$x_4 + y_4, x_5 + y_5, x_6 + y_6), a^2 + b^2 \neq 0,$$

$$K_{8,1} = \{g(u_1, u_2, u_3, 0, 0, u_1 + u_2 + a_3u_3); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_3 \in \mathbb{R},$$

$$K_{8,2} = \{g(u_1, u_2, u_3, 0, 0, u_1 + u_3); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 9$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+y_1+x_4y_2+\frac{1}{2}x_4^2y_3, x_2+y_2+x_4y_3, x_3+y_3, x_4+y_4, x_5+y_5e^{x_6}, x_6+y_6),$$

$$K_9 = \{g(u_1 + a_2u_2, u_2, u_3, 0, u_1, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2 \in \mathbb{R},$$

for $i = 10$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1e^{x_4}, x_2 + y_2 + x_4y_3, x_3 + y_3, x_4 + y_4, x_5 + y_5e^{x_6}, x_6 + y_6),$$

$$K_{10} = \{g(u_1, u_1 + u_3, u_2, 0, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 11$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 + x_2y_3,$$

$$x_2 + y_2, x_3 + y_3, x_4 + (y_4 + x_6y_5)e^{x_6}, x_5 + y_5e^{x_6}, x_6 + y_6),$$

$$K_{11,1} = \{g(u_2, u_1, 0, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

$$K_{11,2} = \{g(u_2, 0, u_1, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 12$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1 + x_2y_3, x_2 + y_2, x_3 + y_3, x_4 + y_4e^{x_6}, x_5 + y_5e^{x_6}, x_6 + y_6),$$

$$K_{12,1} = \{g(u_2 + u_3, u_1, 0, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

$$K_{12,2} = \{g(u_2 + u_3, 0, u_1, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 13$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1 + x_2y_3,$$

$$x_2 + y_2, x_3 + y_3, x_4 + y_4e^{x_6}, x_5 + y_5e^{hx_6}, x_6 + y_6), -1 \leq h < 1, h \neq 0,$$

$$K_{13,1} = \{g(u_2 + u_3, u_1, 0, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

$$K_{13,2} = \{g(u_2 + u_3, 0, u_1, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 14$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1 + x_2y_3, x_2 + y_2, x_3 + y_3, x_4 + (y_4cos(x_6) + y_5sin(x_6))e^{px_6},$$

$$x_5 + (y_5cos(x_6) - y_4sin(x_6))e^{px_6}, x_6 + y_6), p \geq 0,$$

$$K_{14,1} = \{g(u_2 + a_3u_3, u_1, 0, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3, \}, a_3 \in \mathbb{R}\backslash\{0\},$$

$$K_{14,2} = \{g(u_2 + a_3u_3, 0, u_1, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3, \}, a_3 \in \mathbb{R}\backslash\{0\},$$

$$K_{14,3} = \{g(u_3, u_1, 0, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

$$K_{14,4} = \{g(u_3, 0, u_1, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 15$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1e^{x_2}, x_2 + y_2, x_3 + y_3e^{x_2}, x_4 + (y_4 + x_6y_5)e^{x_6}, x_5 + y_5e^{x_6}, x_6 + y_6),$$

$$K_{15} = \{g(u_1, 0, u_1 + u_2, u_2, u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 16$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+y_1e^{x_2},x_2+y_2,x_3+y_3e^{x_2},x_4+y_4e^{x_6},x_5+y_5e^{x_6},x_6+y_6),$$
$$K_{16}=\{g(u_1,0,u_1+u_2+u_3,u_2,u_3,0);u_i\in\mathbb{R},i=1,2,3\},$$
for $i=17$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=g(x_1+y_1e^{x_2},$$

$$x_2+y_2,x_3+y_3e^{x_2},x_4+y_4e^{x_6},x_5+y_5e^{hx_6},x_6+y_6),-1\le h<1,h\ne 0,$$
$$K_{17}=\{g(u_1,0,u_1+u_2+u_3,u_2,u_3,0);u_i\in\mathbb{R},i=1,2,3\},$$
for $i=18$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$

$$g(x_1+y_1e^{x_2},x_2+y_2,x_3+y_3e^{x_2},x_4+(y_4cos(x_6)+y_5sin(x_6))e^{px_6},$$
$$x_5+(y_5cos(x_6)-y_4sin(x_6))e^{px_6},x_6+y_6),p\ge 0,$$
$$K_{18,1}=\{g(u_1,0,u_1+u_2+a_3u_3,u_2,u_3,0);u_i\in\mathbb{R},i=1,2,3\},a_3\in\mathbb{R},$$
$$K_{18,2}=\{g(u_1,0,u_1+u_3,u_2,u_3,0);u_i\in\mathbb{R},i=1,2,3\}.$$

Proposition 28. *There does not exist any* 3*-dimensional connected topological proper loop* L *such that the Lie algebra* **g** *of the group* $Mult(L)$ *is one of the Lie algebras* $\mathbf{g}_i$*,* $i=14,18$*, with* $p=0$*.*

Proof. We may assume that L is simply connected and hence it is homeomorphic to $\mathbb{R}^3$ (cf. Lemma 9). We prove that none of the groups G_i, $i=14,18$, with $p=0$ allow the existence of continuous left transversals S and T to K_i in G_i such that for all $s\in S$ and $t\in T$ one has $s^{-1}t^{-1}st\in K_i$ and $S\cup T$ generates G_i. Hence Lemma 7 yields that the groups G_i, $i=14,18$, with $p=0$ are not the multiplication group of a loop L. This proves the assertion. Two arbitrary left transversals to the subgroups $K_{14,i}$, $i=1,3$, in G_{14} are:

$$S=\{g(u,f_1(u,v,w),v,f_2(u,v,w),f_3(u,v,w),w);u,v,w\in\mathbb{R}\},$$

$$T=\{g(k,g_1(k,l,m),l,g_2(k,l,m),g_3(k,l,m),m);k,l,m\in\mathbb{R}\},$$

those to the subgroups $K_{14,j}$, $j=2,4$, in G_{14} are:

$$S=\{g(u,v,f_1(u,v,w),f_2(u,v,w),f_3(u,v,w),w);u,v,w\in\mathbb{R}\},$$

$$T = \{g(k, l, g_1(k, l, m), g_2(k, l, m), g_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

and those to the subgroups $K_{18,j}$, $j = 1, 2$, in G_{18} are:

$$S = \{g(f_1(u, v, w), u, v, f_2(u, v, w), f_3(u, v, w), w); u, v, w \in \mathbb{R}\},$$

$$T = \{g(g_1(k, l, m), k, l, g_2(k, l, m), g_3(k, l, m), m); k, l, m \in \mathbb{R}\},$$

where $f_i(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$ and $g_i(k, l, m) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, are continuous functions with $f_i(0, 0, 0) = g_i(0, 0, 0) = 0$. The products $s^{-1}t^{-1}st$, $s \in S$, $t \in T$, are elements in $K_{14,1}$, respectively in $K_{14,2}$, respectively in $K_{18,1}$, if and only if

$$\begin{gathered}(cos(m) - 1)(f_2(u, v, w)(cos(w) + a_3 sin(w)))+\\(cos(m) - 1)(f_3(u, v, w)(a_3 cos(w) - sin(w)))+\\(cos(w) - 1)(g_3(k, l, m)(sin(m) - a_3 cos(m)))-\\(cos(w) - 1)(g_2(k, l, m)(cos(m) + a_3 sin(m)))-\\sin(m)(f_2(u, v, w)(sin(w) - a_3 cos(w))+\\f_3(u, v, w)(cos(w) + a_3 sin(w)))+\\sin(w)(g_3(k, l, m)(cos(m)+a_3 sin(m))+g_2(k, l, m)(sin(m)-a_3 cos(m)))\\= f_1(u, v, w)l - g_1(k, l, m)v,\end{gathered} \tag{43}$$

respectively

$$= g_1(k, l, m)v - f_1(u, v, w)l, \tag{44}$$

respectively

$$= e^{-u}(1 - e^{-k})(f_1(u, v, w) - v) - e^{-k}(1 - e^{-u})(g_1(k, l, m) - l), \tag{45}$$

are satisfied for all $k, l, m, u, v, w \in \mathbb{R}$, with $a_3 \in \mathbb{R}$. Moreover, the products $s^{-1}t^{-1}st$, $s \in S$, $t \in T$, are elements in $K_{14,3}$, respectively in $K_{14,4}$, respectively in $K_{18,2}$, precisely if

$$\begin{gathered}(cos(m) - 1)(f_2(u, v, w)sin(w) + f_3(u, v, w)cos(w))-\\(cos(w) - 1)(g_2(k, l, m)sin(m) + g_3(k, l, m)cos(m))+\\sin(m)(f_2(u, v, w)cos(w) - f_3(u, v, w)sin(w))+\end{gathered}$$

$$sin(w)(g_3(k,l,m)sin(m) - g_2(k,l,m)cos(m))$$
$$= f_1(u,v,w)l - g_1(k,l,m)v, \tag{46}$$

respectively

$$= g_1(k,l,m)v - f_1(u,v,w)l, \tag{47}$$

respectively

$$= e^{-u}(1-e^{-k})(f_1(u,v,w)-v) - e^{-k}(1-e^{-u})(g_1(k,l,m)-l) \tag{48}$$

hold for all $k,l,m,u,v,w \in \mathbb{R}$. The equations (43), (44), (45), (46), (47) and (48) are satisfied precisely if their left hand side as well as their right hand side are zero. The right hand side of these equations is zero if and only if $f_1(u,v,w) = v$ and $g_1(k,l,m) = l$. In that case the set $S \cup T$ does not generate G_{14}, respectively G_{18}. □

Theorem 29. *Let* L *be a connected simply connected topological proper loop of dimension* 3 *having a* 6*-dimensional solvable decomposable Lie group with* 1*-dimensional centre as its multiplication group. Then the pairs of the Lie groups* (G_i, K_i)*,* $i = 1, \cdots, 18$*, given in Lemma 27, such that for* $i = 14, 18$*, one has* $p \neq 0$*, are the multiplication groups* $Mult(L)$ *and the inner mapping groups* $Inn(L)$ *of* L.

Proof. Taking into account Theorem 26 and Proposition 28 it remains to find for each group G_i, $i = 1, \cdots, 18$, in Lemma 27, such that for $i = 14, 18$, one has $p \neq 0$, K_i-connected left transversals S_i, T_i (cf. Lemma 7). The sets

$$S_{1,0} = \{g(1 - e^v - ue^v(1-e^{-bv}), e^v(1-e^{-bv}),$$
$$u, ue^{bv-v}, v, w); u, v, w \in \mathbb{R}\},$$
$$T_{1,0} = \{g(1 - e^l - ke^l(e^{-bl}-1), e^l(e^{-bl}-1),$$
$$k, -ke^{bl-l}, l, m); k, l, m \in \mathbb{R}\},$$

respectively

$$S_{1,1} = \{g(1 - e^v(2 + u - e^{-bv}(1+u)), e^v(1-e^{-bv}),$$
$$u, ue^{bv-v}, v, w); u, v, w \in \mathbb{R}\},$$

$$T_{1,1} = \{g(1 - e^l(e^{-bl} + ke^{-bl} - k), e^l(e^{-bl} - 1),$$
$$k, -ke^{bl-l}, l, m); k, l, m \in \mathbb{R}\},$$

are $K_{1,0}$-, respectively $K_{1,1}$-connected left transversals, in $G_1^{b\neq 0}$. The set

$$S_{2,0} = T_{2,0} = \{g(v^2 - u^2 - a_3 v + e^v - 1, u, u, v, v, w); u, v, w \in \mathbb{R}\},$$

respectively

$$S_{2,1} = T_{2,1} = \{g(v^2 - u^2 - u - a_3 v + e^v - 1, u, u, v, v, w); u, v, w \in \mathbb{R}\},$$

$a_3 \in \mathbb{R}$, are $K_{2,0}$, respectively $K_{2,1}$-connected left transversals, in G_2. The set

$$S_3 = T_3 = \{g(e^v - 1 + (v - a_3)v(1 + u + a_3 v - \frac{1}{2}v^2) + (u + a_3 v - \frac{1}{2}v^2)^2,$$

$$u, u + a_3 v - \frac{1}{2}v^2, v(1 + u + a_3 v - \frac{1}{2}v^2), v, w); u, v, w \in \mathbb{R}\}, a_3 \in \mathbb{R},$$

respectively $S_4 = T_4 =$

$$\{g(-w, u, a_1 u + v, 1 - e^v + a_1 w + (a_1 u + v)^2, v, w); u, v, w \in \mathbb{R}\}, a_1 \neq 0,$$

are K_3-connected, respectively K_4-connected left transversals, in G_3, respectively in G_4. The set

$$S_5 = T_5 = \{g(1 - e^v + 2auv - a_2(\frac{1}{2}u^2 + av - ua_2), av + \frac{1}{2}u^2 - ua_2,$$

$$u, u, v, w); u, v, w \in \mathbb{R}\}, a_2 \in \mathbb{R},$$

is K_5-connected left transversal in G_5^a. The sets

$$S_6 = \{g(e^u - e^{u-av-bu}, e^v - e^{v-u}, e^{av+bu-v} - e^{av+bu}, u, v, w); u, v, w \in \mathbb{R}\},$$

$$T_6 = \{g(e^k - e^{k-l}, e^{l-al-bk} - e^l, e^{al+bk} - e^{al+bk-k}, k, l, m); k, l, m \in \mathbb{R}\}$$

are K_6-connected left transversals in $G_6^{a^2+b^2\neq 0}$. The set

$$S_7 = T_7 = \{g(ve^{au+v-u}, 1 - e^u - (a_3 - v)(e^u - e^{u-au-v}),$$

$$e^u - e^{u-au-v}, u, v, w); u, v, w \in \mathbb{R}\}, a_3 \in \mathbb{R},$$

is K_7-connected left transversal in G_7^a. The set

$$S_{8,1} = T_{8,1} = \{g(e^{av+bu-u}sin(v), \frac{1}{\xi}(e^u(1-e^{-av-bu})(sin(v)+a_3cos(v))+$$

$$(e^u - cos(v))(cos(v) - a_3sin(v)), \frac{1}{\xi}((e^u - cos(v))(sin(v)+a_3cos(v))$$
$$-(e^u - e^{u-av-bu})(cos(v) - a_3sin(v)), u, v, w);$$
$$u, v, w \in \mathbb{R}\}, a_3 \in \mathbb{R}, \xi = 1 + a_3^2,$$

respectively

$$S_{8,2} = T_{8,2} = \{g(esin(v), (e^u - e^{u-av-bu})cos(v) - (e^u - cos(v))sin(v),$$
$$(e^u - e^{u-av-bu})sin(v) + (e^u - cos(v))cos(v), u, v, w); u, v, w \in \mathbb{R}\},$$

are $K_{8,1}$-, respectively $K_{8,2}$-connected left transversals, in $G_8^{a^2+b^2\neq 0}$. The sets

$$S_9 = \{g(u, v + (1-e^{-w})(a_2+v), 1-e^{-w}, v, -\frac{1}{2}v^2e^w, w), u, v, w \in \mathbb{R}\},$$

$$T_9 = \{g(k, l + (e^{-m}-1)(a_2+l), e^{-m}-1, l, \frac{1}{2}l^2e^m, m), k, l, m \in \mathbb{R}\},$$

$a_2 \in \mathbb{R}$, respectively

$$S_{10} = \{g(ve^v, u, 1-e^{-w}, v, e^w - e^{w-v}, w), u, v, w \in \mathbb{R}\},$$

$$T_{10} = \{g(e^l - e^{l-m}, k, e^{-l}-1, l, -le^m, m), k, l, m \in \mathbb{R}\},$$

are K_9-connected, respectively K_{10}-connected left transversals, in G_9, respectively in G_{10}. The sets

$$S_{11,1} = \{g(u, -we^{-w}, v, e^w + vwe^w - 1, ve^w, w), u, v, w \in \mathbb{R}\},$$

$$T_{11,1} = \{g(k, me^{-m}, l, e^m - mle^m - 1, -le^m, m), k, l, m \in \mathbb{R}\},$$

respectively

$$S_{11,2} = \{g(u, v, we^{-w}, e^w + vwe^w - 1, ve^w, w), u, v, w \in \mathbb{R}\},$$

$$T_{11,2} = \{g(k, l, -me^{-m}, e^m - mle^m - 1, -le^m, m), k, l, m \in \mathbb{R}\},$$

are $K_{11,1}$-, respectively $K_{11,2}$-connected left transversals, in G_{11}. The sets

$$S_{12,1} = \{g(u, e^{-w} - 1, v, ve^w - u, u, w), u, v, w \in \mathbb{R}\},$$

$$T_{12,1} = \{g(k, 1 - e^{-m}, l, -le^m - k, k, m), k, l, m \in \mathbb{R}\},$$

respectively

$$S_{12,2} = \{g(u, v, 1 - e^{-w}, ve^w - u, u, w), u, v, w \in \mathbb{R}\},$$

$$T_{12,2} = \{g(k, l, e^{-m} - 1, -le^m - k, k, m), k, l, m \in \mathbb{R}\},$$

are $K_{12,1}$-, respectively $K_{12,2}$-connected left transversals, in G_{12}. The sets

$$S_{13,1} = \{g(u, 1 - e^w, v, -ve^w, e^{-w} - e^{-2w}, w); u, v, w \in \mathbb{R}\},$$

$$T_{13,1} = \{g(k, e^{-m} - 1, l, e^m - e^{2m}, le^{-m}, m); k, l, m \in \mathbb{R}\},$$

respectively

$$S_{13,2} = \{g(u, v, e^w - 1, -ve^w, e^{-w} - e^{-2w}, w); u, v, w \in \mathbb{R}\},$$

$$T_{13,2} = \{g(k, l, 1 - e^{-m}, e^m - e^{2m}, le^{-m}, m); k, l, m \in \mathbb{R}\},$$

are $K_{13,1}$-, respectively $K_{13,2}$-connected left transversals, in $G_{13}^{h=-1}$ and the sets

$$S_{13,3} = \{g(u, 1 - e^{-w}, v, e^w - e^{w-hw}, -ve^{hw}, w); u, v, w \in \mathbb{R}\},$$

$$T_{13,3} = \{g(k, e^{-hm} - 1, l, le^m, e^{hm} - e^{hm-m}, m); k, l, m \in \mathbb{R}\},$$

respectively

$$S_{13,4} = \{g(u, v, e^{-w} - 1, e^w - e^{w-hw}, -ve^{hw}, w); u, v, w \in \mathbb{R}\},$$

$$T_{13,4} = \{g(k, l, 1 - e^{-hm}, le^m, e^{hm} - e^{hm-m}, m); k, l, m \in \mathbb{R}\},$$

are $K_{13,1}$-, respectively $K_{13,2}$-connected left transversals, in $G_{13}^{-1<h<1}$. The set

$$S_{14,1} = T_{14,1} = \{g(u, e^{-pw} sin(w), v,$$

$$\frac{1}{\sigma}(e^{pw}v(sin(w)-a_3cos(w))+(cos(w)-e^{pw})(cos(w)+a_3sin(w))),$$

$$\frac{1}{\sigma}(e^{pw}v(a_3sin(w)+cos(w))+(cos(w)-e^{pw})(a_3cos(w)-sin(w))),w);$$

$$u,v,w\in\mathbb{R}\},\sigma=a_3^2+1,$$

respectively

$$S_{14,2}=T_{14,2}=\{g(u,v,-e^{-pw}sin(w),$$

$$\frac{1}{\sigma}(e^{pw}v(sin(w)-a_3cos(w))+(cos(w)-e^{pw})(cos(w)+a_3sin(w))),$$

$$\frac{1}{\sigma}(e^{pw}v(a_3sin(w)+cos(w))+(cos(w)-e^{pw})(a_3cos(w)-sin(w))),w);$$

$$u,v,w\in\mathbb{R}\},a_3\neq 0,\sigma=a_3^2+1,$$

are $K_{14,1}$-, respectively $K_{14,2}$-connected left transversals, in $G_{14}^{p\neq 0}$, and the set

$$S_{14,3}=T_{14,3}=\{g(u,e^{-pw}sin(w),v,sin(w)(cos(w)-e^{pw})-$$

$$e^{pw}vcos(w),e^{pw}vsin(w)+cos(w)(cos(w)-e^{pw}),w);u,v,w\in\mathbb{R}\},$$

respectively

$$S_{14,4}=T_{14,4}=\{g(u,v,-e^{-pw}sin(w),sin(w)(cos(w)-e^{pw})-$$

$$e^{pw}vcos(w),e^{pw}vsin(w)+cos(w)(cos(w)-e^{pw}),w);u,v,w\in\mathbb{R}\},$$

are $K_{14,3}$-, respectively $K_{14,4}$-connected left transversals, in $G_{14}^{p\neq 0}$. The sets

$$S_{15}=\{g(e^{u-w}-e^u+v,u,v,e^w-e^{w-u}+w^2,w,w),u,v,w\in\mathbb{R}\},$$

$$T_{15}=\{g(e^k-e^{k-m}+l,k,l,e^{m-k}-e^m+m^2,m,m),k,l,m\in\mathbb{R}\},$$

are K_{15}-connected left transversals in G_{15}. The set

$$S_{16}=T_{16}=\{g(e^u+v-1,u,v,e^w-u-1,u,w),u,v,w\in\mathbb{R}\}$$

is K_{16}-connected left transversal in G_{16}. The sets

$$S_{17}=\{g(e^{u-hw}-e^u+v,u,v,e^w-e^{w-u},e^{hw}-e^{hw-w},w);u,v,w\in\mathbb{R}\},$$

$T_{17} = \{g(e^k - e^{k-m} + l, k, l, e^m - e^{m-hm}, e^{hm-k} - e^{hm}, m); k, l, m \in \mathbb{R}\}$, are K_{17}-connected left transversals in $G_{17}^{-1 \le h < 1}$. The set

$$S_{18,1} = T_{18,1} = \{g(e^u + v - 1, u, v,$$

$$\frac{1}{\xi}((cos(w) - e^{pw})(a_3 sin(w) + cos(w)) - sin(w)(a_3 cos(w) - sin(w)),$$

$$\frac{1}{\xi}((cos(w) - e^{pw})(a_3 cos(w) - sin(w)) + sin(w)(a_3 sin(w) + cos(w)), w);$$

$$u, v, w \in \mathbb{R}\}, a_3 \in \mathbb{R}, \xi = 1 + a_3^2,$$

respectively

$$S_{18,2} = T_{18,2} = \{g(e^u + v - 1, u, v, sin(w)(e^{pw} - cos(w)) - cos(w)sin(w),$$

$$sin(w)^2 + cos(w)(e^{pw} - cos(w)), w); u, v, w \in \mathbb{R}\},$$

are $K_{18,1}$-, respectively $K_{18,2}$-connected left transversals, in $G_{18}^{p \neq 0}$.

For all $i = 1, \cdots, 18$, the set $S_i \cup T_i$ generates the group G_i. By Lemma 7 the assertion is proved. □

6 6-dimensional solvable multiplication group having 2-dimensional centre

In this Chapter we determine the at most 6-dimensional solvable Lie groups with 2-dimensional centre which can be represented as the multiplication groups $Mult(L)$ of 3-dimensional connected simply connected topological proper loops L. These Lie groups are decomposable (cf. Theorem 20) and the corresponding loops have a 2-dimensional centre $Z(L)$ isomorphic to $\mathbb{R}^2$ such that the factor loop $L/Z(L)$ is isomorphic to $\mathbb{R}$. These loops are centrally nilpotent of class 2.

Theorem 30. *Let L be a connected simply connected topological proper loop of dimension 3 such that its multiplication group is an at most 6-dimensional decomposable nilpotent Lie group. Then the loop L is centrally nilpotent of class 2 and the groups $\mathbb{R} \times \mathcal{F}_4$, $\mathbb{R} \times \mathcal{F}_5$ are the multiplication groups of L.*

Proof. Each nilpotent Lie group has a centre of dimension ≥ 1. If the group $Mult(L)$ is decomposable and nilpotent, then it has a 2-dimensional centre and the loop L has nilpotency class 2 (cf. Lemma 10, a), b)). According to the list of the Lie algebras in [24], §5, and [25], p. 100, the Lie algebra of the group $Mult(L)$ is either the direct sum $\mathbf{f}_3 \oplus \mathbf{f}_3$ or $\mathbb{R} \oplus \mathbf{f}_n$, $n = 4, 5$, or $\mathbb{R} \oplus \mathbf{g}_{5,i}$, $i = 4, 5, 6$. By Lemma 10 c) the Lie algebra of $Mult(L)$ has a 5-dimensional abelian ideal containing its centre and its commutator subalgebra. Since there does not exist any such ideal for the Lie algebras $\mathbf{f}_3 \oplus \mathbf{f}_3$ and $\mathbb{R} \oplus \mathbf{g}_{5,i}$, $i = 4, 5, 6$, these Lie algebras are excluded. Now the assertion follows from Proposition 5.1. in [10] , pp. 400-406. □

Theorem 31. *Let L be a 3-dimensional connected simply connected topological loop which has a 6-dimensional solvable non-nilpotent Lie algebra with 2-dimensional centre as the Lie algebra $\mathbf{g}$ of its multiplication group. Then L has nilpotency class 2 and the following Lie algebra pairs $(\mathbf{g}, \mathbf{k})$ are the Lie algebra $\mathbf{g}$ of the group $Mult(L)$ and the subalgebra $\mathbf{k}$ of the subgroup $Inn(L)$:*
If $\mathbf{g}_i = \mathbb{R}^2 \oplus \mathbf{n}_i = \langle f_1, f_2 \rangle \oplus \langle e_1, \cdots, e_4 \rangle$, $i = 1, \cdots, 4$, where $\mathbf{n}$ is a 4-dimensional solvable indecomposable Lie algebra with trivial centre, then one has

- $\mathbf{n}_1 = \mathbf{g}_{4,2}^{\alpha \neq 0}$*:* $[e_1, e_4] = \alpha e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$, $\mathbf{k}_1 = \langle e_1 + f_1, e_2 + f_1, e_3 \rangle$,

- $\mathbf{n}_2 = \mathbf{g}_{4,4}$*:* $[e_1, e_4] = e_1, [e_2, e_4] = e_1 + e_2, [e_3, e_4] = e_2 + e_3$, $\mathbf{k}_2 = \langle e_1 + f_1, e_2 + a_2 f_1, e_3 + a_3 f_1 \rangle$, $a_2, a_3 \in \mathbb{R}$,

- $\mathbf{n}_3 = \mathbf{g}_{4,5}^{-1 \leq \gamma \leq \beta \leq 1, \gamma\beta \neq 0}$*:* $[e_1, e_4] = e_1, [e_2, e_4] = \beta e_2, [e_3, e_4] = \gamma e_3$, $\mathbf{k}_3 = \langle e_1 + f_1, e_2 + f_1, e_3 + f_1 \rangle$,

- $\mathbf{n}_4 = \mathbf{g}_{4,6}^{p \geq 0, \alpha \neq 0}$*:* $[e_1, e_4] = \alpha e_1, [e_2, e_4] = p e_2 - e_3, [e_3, e_4] = e_2 + p e_3$, $\mathbf{k}_{4,1} = \langle e_1 + f_1, e_2 + f_1, e_3 + a_3 f_1 \rangle$, $a_3 \in \mathbb{R}$, $\mathbf{k}_{4,2} = \langle e_1 + f_1, e_2, e_3 + f_1 \rangle$.

If $\mathbf{g}_j = \mathbb{R} \oplus \mathbf{h}_j = \langle f_1 \rangle \oplus \langle e_1, e_2, e_3, e_4, e_5 \rangle$, where $\mathbf{h}_j$, $j = 5, \cdots, 8$, is a 5-dimensional solvable indecomposable Lie algebra with 1-dimensional centre, then we have

- $\mathbf{h}_5 = \mathbf{g}_{5,8}^{0 < |\gamma| \leq 1}$*:* $[e_2, e_5] = e_1, [e_3, e_5] = e_3, [e_4, e_5] = \gamma e_4$, $\mathbf{k}_{5,\epsilon} = \langle e_2 + \epsilon f_1, e_3 + e_1, e_4 + e_1 \rangle$, $\epsilon = 0, 1$,

- $\mathbf{h}_6 = \mathbf{g}_{5,10}$: $[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_4$, $\mathbf{k}_{6,\epsilon} = \langle e_2, e_3 + \epsilon f_1, e_4 + e_1 \rangle$, $\epsilon = 0, 1$, $\mathbf{k}_{6,2} = \langle e_2 + b_1 f_1, e_3 + b_2 f_1, e_4 + f_1 + a e_1 \rangle$, $b_1, b_2 \in \mathbb{R}$, $a \neq 0$,

- $\mathbf{h}_7 = \mathbf{g}_{5,14}^{p \neq 0}$: $[e_2, e_5] = e_1, [e_3, e_5] = pe_3 - e_4, [e_4, e_5] = e_3 + pe_4$, $\mathbf{k}_{7,\epsilon} = \langle e_2 + \epsilon f_1, e_3 + e_1, e_4 + a_3 e_1 \rangle$, $\epsilon = 0, 1$, $a_3 \in \mathbb{R}$, $\mathbf{k}_{7,\delta} = \langle e_2 + \delta f_1, e_3, e_4 + e_1 \rangle$, $\delta = 0, 1$,

- $\mathbf{h}_8 = \mathbf{g}_{5,15}^{\gamma=0}$: $[e_1, e_5] = e_1, [e_2, e_5] = e_1 + e_2, [e_4, e_5] = e_3$, $\mathbf{k}_{8,\epsilon} = \langle e_1 + e_3, e_2, e_4 + \epsilon f_1 \rangle$, $\epsilon = 0, 1$.

Proof. By Lemma 9 we may assume that the loop L is simply connected and hence it is homeomorphic to $\mathbb{R}^3$. As the centre of group $Mult(L)$ of L has dimension 2, the loop L has nilpotency class 2 (cf. Lemma 10 a), b)). By Theorem 20 the group $Mult(L)$ is decomposable. Hence for the Lie algebra of $Mult(L)$ we have the following possibilities: $\mathbb{R}^2 \oplus \mathbf{n}$, $\mathbb{R} \oplus \mathbf{h}$, $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{f}_3$, and $\mathbf{l}_2 \oplus \mathbf{l}_2 \oplus \mathbb{R}^2$, where $\mathbf{n}$ and $\mathbf{h}$ are characterized in the assertion. By [24], §5, $\mathbf{n}$ is one of the following Lie algebras $\mathbf{g}_{4,i}$, $i = 2, 4, 5, 6, 7, 10$, $\mathbf{g}_{4,8}^{h \neq -1}$, $\mathbf{g}_{4,9}^{p \neq 0}$. Moreover, the Lie algebras $\mathbf{g}_{5,j}$, $j = 8, 10, 22, 29, 38, 39$, $\mathbf{g}_{5,14}^{p \neq 0}$, $\mathbf{g}_{5,15}^{\gamma=0}$, $\mathbf{g}_{5,19}^{\alpha=-1}$, $\mathbf{g}_{5,20}^{\alpha=-1}$, $\mathbf{g}_{5,28}^{\alpha=-1}$, $\mathbf{g}_{5,25}^{p=0}$, $\mathbf{g}_{5,26}^{p=0}$ and $\mathbf{g}_{5,30}^{h=-2}$ can consider as $\mathbf{h}$ (cf. [25], §10, p. 105-106).

If these Lie algebras would be the Lie algebra of the multiplication group of L, then they have a 5-dimensional abelian ideal containing their commutator ideal and their centre (cf. Lemma 10 c)). Since the Lie algebras $\mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{f}_3$, $\mathbf{l}_2 \oplus \mathbf{l}_2 \oplus \mathbb{R}^2$, $\mathbb{R}^2 \oplus \mathbf{g}_{4,j}$, $j = 7, 10$, $\mathbb{R}^2 \oplus \mathbf{g}_{4,8}^{h \neq -1}$, $\mathbb{R}^2 \oplus \mathbf{g}_{4,9}^{p \neq 0}$, $\mathbb{R} \oplus \mathbf{g}_{5,r}^{\alpha=-1}$, $r = 19, 20, 28$, $\mathbb{R} \oplus \mathbf{g}_{5,l}^{p=0}$, $l = 25, 26$, $\mathbb{R} \oplus \mathbf{g}_{5,30}^{h=-2}$, $\mathbb{R} \oplus \mathbf{g}_{5,p}$, $p = 22, 29, 38, 39$, do not contain any 5-dimensional abelian ideal, these Lie algebras are not the Lie algebra of the group $Mult(L)$ of L. Hence it remain to deal with the Lie algebras $\mathbf{g}_i$, $i = 1, \cdots, 8$, in the assertion. None of the Lie algebras $\mathbf{g}_i$, $i = 1, \cdots, 8$, have a factor Lie algebra isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$.

The 1-dimensional central subalgebras of $\mathbf{g}_i$, $i = 1, 2, 3, 4$, are $\mathbf{i}_1 = \langle f_2 \rangle$ and $\mathbf{i}_2 = \langle f_1 + a f_2 \rangle$, $a \in \mathbb{R}$, those of $\mathbf{g}_j$, $j = 5, 6, 7$, are $\mathbf{i}_3 = \langle f_1 + b e_1 \rangle$, $b \in \mathbb{R}$, and $\mathbf{i}_4 = \langle e_1 \rangle$, whereas those of $\mathbf{g}_8$ are $\mathbf{i}_5 = \langle f_1 + c e_3 \rangle$, $c \in \mathbb{R}$, and $\mathbf{i}_6 = \langle e_3 \rangle$. With the exception of the Lie algebra $\mathbf{g}_6$ for every ideal s of the Lie algebras $\mathbf{g}_i$, $i = 1, \cdots, 8$, such that s contains a 1-dimensional central subalgebra of $\mathbf{g}_i$, the factor Lie algebras $\mathbf{g}_i / \mathrm{s}$ are not isomorphic to $\mathbf{f}_4$. The

Lie algebra $\mathbf{g}_6$ has the ideal $\mathbf{s} = \langle f_1 + be_1, e_4 \rangle$ containing $\mathbf{i}_3$ such that the factor Lie algebra $\mathbf{g}_6/\mathbf{s}$ is isomorphic to $\mathbf{f}_4$.

According to Lemma 10 d) the simply connected Lie groups G_i of $\mathbf{g}_i$, $i = 1, \cdots, 8$, has a 1-dimensional connected central subgroup C such that the orbit $C(e)$ is isomorphic to $\mathbb{R}$ and the factor loop $L/C(e)$ is isomorphic to $\mathbb{R}^2$. By Proposition 19 (i) the Lie algebras $\mathbf{g}_i$, $i = 1, \cdots, 8$, have a 4-dimensional abelian ideal $\mathbf{p} = \mathbf{c} \oplus \mathbf{k}$, where $\mathbf{c}$ is a 1-dimensional central subalgebra of $\mathbf{g}_i$ and $\mathbf{k}$ is the Lie algebra of the group $Inn(L)$ of L such that $\mathbf{g}_i' < \mathbf{p}$ and $\mathbf{k}$ has the properties as in Lemma 8. Then for the triples $(\mathbf{g}_i, \mathbf{p}, \mathbf{k})$ we obtain:

(a) For the Lie algebras $\mathbf{g}_i$, $i = 1, 2, 3, 4$, the ideal $\mathbf{p}$ has one of the following forms $\mathbf{p}_a = \langle f_1 + af_2, e_1, e_2, e_3 \rangle$, $a \in \mathbb{R}$ and $\widetilde{\mathbf{p}} = \langle f_2, e_1, e_2, e_3 \rangle$. Hence for the subalgebras $\mathbf{k}$ one has $\mathbf{k}_a = \langle e_1 + a_1(f_1 + af_2), e_2 + a_2(f_1 + af_2), e_3 + a_3(f_1 + af_2) \rangle$, $a \in \mathbb{R}$ and $\widetilde{\mathbf{k}} = \langle e_1 + a_1 f_2, e_2 + a_2 f_2, e_3 + a_3 f_2 \rangle$, where $a_j \in \mathbb{R}$, $j = 1, 2, 3$. Using the automorphism $\phi(f_1) = f_2$, $\phi(f_2) = f_1 + af_2$, $\phi(e_i) = e_i$, $i = 1, 2, 3, 4$, the Lie algebra $\widetilde{\mathbf{k}}$ reduces to $\mathbf{k}_a$. So it remains to consider the subalgebra $\mathbf{k}_a$ of $\mathbf{g}_i$, $i = 1, 2, 3, 4$, such that
if $i = 1$, then: $a_1 a_2 \neq 0$ since $\langle e_1 \rangle$ and $\langle e_2 \rangle$ are ideals of $\mathbf{g}_1$,
if $i = 2$, then: $a_1 \neq 0$ because $\langle e_1 \rangle$ is an ideal of $\mathbf{g}_2$,
if $i = 3$, then: $a_1 a_2 a_3 \neq 0$ since $\langle e_1 \rangle$, $\langle e_2 \rangle$ and $\langle e_3 \rangle$ are ideals of $\mathbf{g}_3$,
if $i = 4$, then: $a_1 \neq 0$ and at least one of $\{a_2, a_3\}$ is different from 0 because $\langle e_1 \rangle$ and $\langle e_2, e_3 \rangle$ are ideals of $\mathbf{g}_4$.
Using the automorphism $\phi(f_1) = f_1 - af_2$, $\phi(f_2) = f_2$, $\phi(e_1) = a_1 e_1$, $\phi(e_2) = a_2 e_2$, $\phi(e_3) = a_2 e_3 + a_3 e_2$ and $\phi(e_4) = e_4$ for $\mathbf{g}_1$, respectively $\phi(e_j) = a_1 e_j$, $j = 2, 3$, for $\mathbf{g}_2$, respectively $\phi(e_3) = a_3 e_3$ for $\mathbf{g}_3$, the Lie algebra $\mathbf{k}_a$ reduces to $\mathbf{k}_1$, respectively $\mathbf{k}_2$, $a_2, a_3 \in \mathbb{R}$, respectively $\mathbf{k}_3$, in the assertion. Applying the automorphism $\phi(f_1) = f_1 - af_2$, $\phi(f_2) = f_2$, $\phi(e_1) = a_1 e_1$, $\phi(e_j) = a_2 e_j$, $j = 2, 3$ and $\phi(e_4) = e_4$ for $\mathbf{g}_4$, if $a_2 \neq 0$, respectively $\phi(e_j) = a_3 e_j$, $j = 2, 3$, if $a_2 = 0$ and $a_3 \neq 0$, we can reduce $\mathbf{k}_a$ to $\mathbf{k}_{4,1}$, $a_3 \in \mathbb{R}$, respectively to $\mathbf{k}_{4,2}$, in the assertion.

(b) For the Lie algebras $\mathbf{g}_j$, $j = 5, 7$, the ideal $\mathbf{p}$ has one of the following shapes $\mathbf{p}_a = \langle e_1, f_1 + ae_2, e_3, e_4 \rangle$, $a \in \mathbb{R}\backslash\{0\}$, $\widetilde{\mathbf{p}} = \langle e_1, e_2, e_3, e_4 \rangle$. Hence the subalgebras $\mathbf{k}$ are $\mathbf{k}_a = \langle f_1 + ae_2 + a_1 e_1, e_3 + a_2 e_1, e_4 + a_3 e_1 \rangle$, $a \in \mathbb{R}\backslash\{0\}$, and $\widetilde{\mathbf{k}} = \langle e_2 + a_1 e_1, e_3 + a_2 e_1, e_4 + a_3 e_1 \rangle$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that
for $\mathbf{g}_5$: $a_2 a_3 \neq 0$ since $\langle e_3 \rangle$, $\langle e_4 \rangle$ are ideals of $\mathbf{g}_5$, and

for $\mathbf{g}_7$: $a_2 \neq 0$ or $a_3 \neq 0$ because $\langle e_3, e_4 \rangle$ is an ideal of $\mathbf{g}_7$.
The automorphism $\phi(f_1) = f_1$, $\phi(e_i) = e_i$, $i = 1, 5$, $\phi(e_2) = e_2 - a_1 e_1$, $\phi(e_3) = a_2 e_3$ and $\phi(e_4) = a_3 e_4$, respectively $\phi(f_1) = a f_1 - a_1 e_1$, $\phi(e_2) = e_2$, of $\mathbf{g}_5$ map the subalgebra $\widetilde{\mathbf{k}}$ onto $\mathbf{k}_{5,0}$, respectively the subalgebra $\mathbf{k}_a$ onto $\mathbf{k}_{5,1}$, in the assertion. If $a_2 \neq 0$, then the automorphism $\phi(f_1) = f_1$, $\phi(e_i) = e_i$, $i = 1, 5$, $\phi(e_2) = e_2 - a_1 e_1$, $\phi(e_j) = a_2 e_j$, $j = 3, 4$, respectively $\phi(f_1) = a f_1 - a_1 e_1$, $\phi(e_2) = e_2$, of the Lie algebra $\mathbf{g}_7$ reduces the subalgebra $\widetilde{\mathbf{k}}$ to $\mathbf{k}_{7,\epsilon}$ with $\epsilon = 0$, respectively the subalgebra $\mathbf{k}_a$ to $\mathbf{k}_{7,\epsilon}$ with $\epsilon = 1$, in the assertion. If $a_2 = 0$ and $a_3 \neq 0$, then using the automorphism $\phi(f_1) = f_1$, $\phi(e_i) = e_i$, $i = 1, 5$, $\phi(e_2) = e_2 - a_1 e_1$, $\phi(e_j) = a_3 e_j$, $j = 3, 4$, respectively $\phi(f_1) = a f_1 - a_1 e_1$, $\phi(e_2) = e_2$, of $\mathbf{g}_7$ we can change the subalgebra $\widetilde{\mathbf{k}}$ to $\mathbf{k}_{7,\delta}$ with $\delta = 0$, respectively the subalgebra $\mathbf{k}_a$ to $\mathbf{k}_{7,\delta}$ such that $\delta = 1$, in the assertion.

(c) For the Lie algebra $\mathbf{g}_8$ the ideal $\mathbf{p}$ has one of the following forms $\widetilde{\mathbf{p}} = \langle e_1, e_2, e_3, e_4 \rangle$, $\mathbf{p}_a = \langle e_1, e_2, e_3, f_1 + a e_4 \rangle$, $a \in \mathbb{R} \backslash \{0\}$. Therefore for the subalgebras $\mathbf{k}$ one has $\widetilde{\mathbf{k}} = \langle e_1 + a_1 e_3, e_2 + a_2 e_3, e_4 + a_3 e_3 \rangle$, $\mathbf{k}_a = \langle e_1 + a_1 e_3, e_2 + a_2 e_3, f_1 + a e_4 + a_3 e_3 \rangle$, $a \in \mathbb{R} \backslash \{0\}$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$, such that $a_1 \neq 0$ since $\langle e_1 \rangle$ is an ideal of $\mathbf{g}_8$. The automorphism $\phi(f_1) = f_1$, $\phi(e_i) = e_i$, $i = 3, 5$, $\phi(e_1) = a_1 e_1$, $\phi(e_2) = a_1 e_2 - a_2 e_1$, and $\phi(e_4) = e_4 - a_3 e_3$, respectively $\phi(f_1) = a f_1 - a_3 e_3$, $\phi(e_4) = e_4$, map the subalgebra $\widetilde{\mathbf{k}}$ onto $\mathbf{k}_{8,0}$, respectively $\mathbf{k}_a$ onto $\mathbf{k}_{8,1}$, of the assertion.

(d) If the Lie algebra $\mathbf{g}_6$ is the Lie algebra of the group $Mult(L)$ of L, then the factor loop $L/I_4(e)$, where $I_4 = \exp(\mathbf{i}_4)$, is isomorphic to $\mathbb{R}^2$. Hence the Lie algebra $\mathbf{k}$ of the group $Inn(L)$ of L is a subalgebra of the ideal $\mathbf{p}$ having one of the following forms $\widetilde{\mathbf{p}} = \langle e_1, e_2, e_4, e_3 \rangle$, $\mathbf{p}_a = \langle e_1, e_2, e_4, f_1 + a e_3 \rangle$, $a \in \mathbb{R} \backslash \{0\}$. Therefore we obtain the subalgebras $\widetilde{\mathbf{k}} = \langle e_2 + a_1 e_1, e_3 + a_2 e_1, e_4 + a_3 e_1 \rangle$, $\mathbf{k}_a = \langle e_2 + a_1 e_1, f_1 + a e_3 + a_2 e_1, e_4 + a_3 e_1 \rangle$, where $a \in \mathbb{R} \backslash \{0\}$, $a_i \in \mathbb{R}$, $i = 1, 2$, and $a_3 \neq 0$, since $\langle e_4 \rangle$ is an ideal of $\mathbf{g}_6$. With the automorphism $\phi(f_1) = f_1$, $\phi(e_i) = e_i$, $i = 1, 5$, $\phi(e_2) = e_2 - a_1 e_1$, $\phi(e_3) = e_3 - a_1 e_2 - a_2 e_1$ and $\phi(e_4) = a_3 e_4$, respectively $\phi(f_1) = a f_1 - a_2 e_1$, $\phi(e_3) = e_3 - a_1 e_2$, we can change the subalgebra $\widetilde{\mathbf{k}}$ onto $\mathbf{k}_{6,0}$, respectively $\mathbf{k}_a$ onto $\mathbf{k}_{6,1}$, in the assertion.

Since for the ideal $\mathbf{s} = \langle f_1 + b e_1, e_4 \rangle$, $b \in \mathbb{R}$, of $\mathbf{g}_6$, the factor Lie algebra $\mathbf{g}_6/\mathbf{s}$ is isomorphic to $\mathbf{f}_4$, the factor loop $L/I_3(e)$, where $I_3 = \exp(\mathbf{i}_3)$, is isomorphic to a loop $L_{\mathcal{F}}$. The orbit $\mathcal{S}(e)$, where $\mathcal{S} = exp(s)$, coincides with $I_3(e)$ (cf. Lemma 10 d (ii)). Hence the Lie algebra $\mathbf{k}$ contains the basis element $e_4 + a_3(f_1 + ae_1)$, $a_3 \in \mathbb{R}\setminus\{0\}$. Since $\mathbf{k}$ is a 3-dimensional subalgebra of the 5-dimensional abelian ideal $\mathbf{v} = \langle f_1, e_1, e_2, e_3, e_4\rangle$, it has the form $\mathbf{k} = \langle e_2+b_1f_1+a_1e_1, e_3+b_2f_1+a_2e_1, e_4+a_3(f_1+ae_1)\rangle$, $a, a_i, b_i \in \mathbb{R}$, $i = 1, 2, 3$, $aa_3 \neq 0$. Using the automorphism $\phi(f_1) = f_1$, $\phi(e_i) = e_i$, $i = 1, 5$, $\phi(e_2) = e_2 - a_1e_1$, $\phi(e_3) = e_3 - a_1e_2 - a_2e_1$ and $\phi(e_4) = a_3e_4$, the subalgebra $\mathbf{k}$ reduces to $\mathbf{k}_{6,2} = \langle e_2 + b_1f_1, e_3 + b_2f_1, e_4 + f_1 + ae_1\rangle$. This proves the assertion. □

Applying ([39], §4) we obtain:

Lemma 32. *The linear representation of the simply connected Lie group G_i and its subgroup K_i of the Lie algebra $\mathbf{g}_i$ and its subalgebra $\mathbf{k}_i$, $i = 1, ..., 8$, in Theorem 31 is given by the multiplication: for $i = 1$*

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1+y_1e^{ax_4}, x_2+(y_2+x_4y_3)e^{x_4}, x_3+y_3e^{x_4}, x_4+y_4, x_5+y_5, x_6+y_6), a \neq 0,$$

$$K_1 = \{g(u_1, u_2, u_3, 0, u_1 + u_2, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

for $i = 2$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + (y_1 + x_4y_2 + \frac{1}{2}x_4^2y_3)e^{x_4}, x_2 + (y_2 + x_4y_3)e^{x_4},$$
$$x_3 + y_3e^{x_4}, x_4 + y_4, x_5 + y_5, x_6 + y_6),$$

$$K_2 = \{g(u_1, u_2, u_3, 0, u_1+a_2u_2+a_3u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\}, a_2, a_3 \in \mathbb{R},$$

for $i = 3$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) =$$

$$g(x_1 + y_1e^{x_4}, x_2 + y_2e^{ax_4}, x_3 + y_3e^{bx_4}, x_4 + y_4, x_5 + y_5, x_6 + y_6),$$

$$K_3 = \{g(u_1, u_2, u_3, 0, u_1 + u_2 + u_3, 0); u_i \in \mathbb{R}, i = 1, 2, 3\},$$

$-1 \leq a \leq b \leq 1, ab \neq 0,$

for $i = 4$

$$g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) = g(x_1 + y_1e^{ax_4},$$

$$x_2+(y_2cos(x_4)+y_3sin(x_4))e^{bx_4}, x_3+(y_3cos(x_4)-y_2sin(x_4))e^{bx_4},$$
$$x_4+y_4, x_5+y_5, x_6+y_6), a\neq 0, b\geq 0,$$
$$K_{4,1}=\{g(u_1,u_2,u_3,0,u_1+u_2+a_3u_3,0); u_i\in\mathbb{R}, i=1,2,3\}, a_3\in\mathbb{R},$$
$$K_{4,2}=\{g(u_1,u_2,u_3,0,u_1+u_3,0); u_i\in\mathbb{R}, i=1,2,3\},$$

for $i=5$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1+x_5y_2, x_2+y_2, x_3+y_3e^{x_5}, x_4+y_4e^{cx_5}, x_5+y_5, x_6+y_6), 0<|c|\leq 1,$$
$$K_{5,\epsilon}=\{g(u_2+u_3,u_1,u_2,u_3,0,\epsilon u_1); u_i\in\mathbb{R}, i=1,2,3\}, \epsilon=0,1,$$

for $i=6$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1+x_5y_2+\frac{1}{2}x_5^2y_3, x_2+y_2+x_5y_3, x_3+y_3, x_4+y_4e^{x_5}, x_5+y_5, x_6+y_6),$$
$$K_{6,\epsilon}=\{g(u_3,u_1,u_2,u_3,0,\epsilon u_2); u_i\in\mathbb{R}, i=1,2,3\}, \epsilon=0,1,$$
$$K_{6,2}=\{g(au_3,u_1,u_2,u_3,0,b_1u_1+b_2u_2+u_3); u_i\in\mathbb{R}, i=1,2,3\},$$

$b_1, b_2\in\mathbb{R}, a\neq 0$,

for $i=7$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+y_1+x_2y_5, x_2+y_2, x_3+(y_3cos(x_5)-y_4sin(x_5))e^{px_5},$$
$$x_4+(y_4cos(x_5)+y_3sin(x_5))e^{px_5}, x_5+y_5, x_6+y_6), p\neq 0,$$
$$K_{7,\epsilon}=\{g(u_2+a_3u_3,u_1,u_2,u_3,0,\epsilon u_1); u_i\in\mathbb{R}, i=1,2,3\}, \epsilon=0,1, a_3\in\mathbb{R},$$
$$K_{7,\delta}=\{g(u_3,u_1,u_2,u_3,0,\varepsilon u_1); u_i\in\mathbb{R}, i=1,2,3\}, \delta=0,1,$$

for $i=8$

$$g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)=$$
$$g(x_1+(y_1+y_2x_5)e^{x_5}, x_2+y_2e^{x_5}, x_3+y_3+x_5y_4, x_4+y_4, x_5+y_5, x_6+y_6),$$
$$K_{8,\epsilon}=\{g(u_1,u_2,u_1,u_3,0,\epsilon u_3); u_i\in\mathbb{R}, i=1,2,3\}, \epsilon=0,1,$$

Proposition 33. *There does not exist any 3-dimensional connected topological proper loop L having $\mathbf{g}_6$ as the Lie algebra of its multiplication group and the Lie algebra $\mathbf{k}_{6,2}$ as the Lie algebra of its inner mapping group.*

Proof. We may assume that L is simply connected. Therefore it is homeomorphic to $\mathbb{R}^3$ (cf. Lemma 9). We show that the Lie group G_6 does not allow continuous left transversals S and T to the subgroup $K_{6,2}$ such that for all $s \in S$, $t \in T$ one has $s^{-1}t^{-1}st \in K_{6,2}$ and the set $S \cup T$ generates G_6.

Two arbitrary left transversals to the group $K_{6,2}$ in G_6 are:

$$S = \{g(u, h_1(u, v, w), h_2(u, v, w), h_3(u, v, w), v, w); u, v, w \in \mathbb{R}\},$$

$$T = \{g(k, g_1(k, l, m), g_2(k, l, m), g_3(k, l, m), l, m); k, l, m \in \mathbb{R}\},$$

where $h_i(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$ and $g_i(k, l, m) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, are continuous functions with $h_i(0, 0, 0) = g_i(0, 0, 0) = 0$. The products $s^{-1}t^{-1}st$, $s \in S$, $t \in T$, are elements of $K_{6,2}$ if and only if the equations

$$a(g_3(k, l, m)e^{-l}(1 - e^{-v}) - h_3(u, v, w)e^{-v}(1 - e^{-l})) =$$
$$vg_1(k, l, m) - vlg_2(k, l, m) - lh_1(u, v, w) + lvh_2(u, v, w)+$$
$$\frac{1}{2}l^2h_2(u, v, w) - \frac{1}{2}v^2g_2(k, l, m), \tag{49}$$

$$g_3(k, l, m)e^{-l}(1 - e^{-v}) - h_3(u, v, w)e^{-v}(1 - e^{-l}) =$$
$$b_1lh_2(u, v, w) - b_1vg_2(k, l, m) \tag{50}$$

are satisfied for all $k, l, m, u, v, w \in \mathbb{R}$. Applying equation (50) equation (49) reduces to

$$vg_1(k, l, m) + ab_1vg_2(k, l, m) - vlg_2(k, l, m) - \frac{1}{2}v^2g_2(k, l, m) =$$

$$lh_1(u, v, w) + ab_1lh_2(u, v, w) - lvh_2(u, v, w) - \frac{1}{2}l^2h_2(u, v, w). \tag{51}$$

Using the new functions

$$g_1'(k, l, m) = g_1(k, l, m) + ab_1g_2(k, l, m) - lg_2(k, l, m),$$

$$h_1'(u,v,w) = h_1(u,v,w) + ab_1 h_2(u,v,w) - v h_2(u,v,w)$$

equation (51) reduces to

$$v g_1'(k,l,m) - \frac{1}{2} v^2 g_2(k,l,m) = l h_1'(u,v,w) - \frac{1}{2} l^2 h_2(u,v,w). \quad (52)$$

Equation (52) holds precisely if the functions $g_1'(k,l,m)$ and $g_2(k,l,m)$, respectively $h_1'(u,v,w)$ and $h_2(u,v,w)$, are polynomials of l, respectively of v, with order at most 2. Using this, equation (50) is satisfied if and only if its left hand side and its right hand side are 0. This holds precisely if one has $g_3(l) = c(e^l - 1)$ and $h_3(v) = c(e^v - 1)$, where c is a real constant. In this case the set $S \cup T$ does not generate the group G_6. Hence by Lemma 7 the group G_6 and the subgroup $K_{6,2}$ are not the multiplication group and the inner mapping group of L. This proves the assertion. □

Theorem 34. *Let L be a connected simply connected topological proper loop of dimension 3 such that its multiplication group is a 6-dimensional solvable non-nilpotent Lie group having 2-dimensional centre. Then the pairs of the Lie groups (G_i, K_i), $i = 1, \cdots, 8$, given in Lemma 32 are the multiplication groups $Mult(L)$ and the inner mapping groups $Inn(L)$ of L with the only exception $(G_6, K_{6,2})$.*

Proof. By Theorem 31 the pairs (G_i, K_i), $i = 1, \cdots, 8$, in Lemma 32 can occur as the group $Mult(L)$ and the subgroup $Inn(L)$ of L. According to Proposition 33 the pair $(G_6, K_{6,2})$ is excluded. In all other cases we give continuous left transversals S_i, T_i to the subgroup K_i, $i = 1, \cdots, 8$, which fulfill the requirements of Lemma 7.

Appropriate K_1-connected left transversals in the group G_1^a are: for $a < -1$ and for $a > 1$ the sets

$$S_{1,1} = \{g(e^{au}(e^{-u} - 1), e^u(1 - e^{-au}) + u^2, u, u, v, w); u, v, w \in \mathbb{R}\},$$

$$T_{1,1} = \{g(e^{ak}(1 - e^{-k}), k^2 - e^k(1 - e^{-ak}), k, k, l, m); k, l, m \in \mathbb{R}\},$$

for $0 < a < 1$ and for $-1 < a < 0$ the sets

$$S_{1,2} = \{g(-ue^{au-u}, 1 - e^u + ue^u(1 - e^{-au}),$$

$$e^u - e^{-au+u}, u, v, w); u, v, w \in \mathbb{R}\},$$

$T_{1,2} = \{g(ke^{ak-k}, 1-e^k+ke^k(e^{-ak}-1), e^{-ak+k}-e^k, k, l, m); k, l, m \in \mathbb{R}\}$,

for $a = 1$ the sets

$$S_{1,3} = \{g(w, e^u - 1 - w + u^2, u, u, v, w); u, v, w \in \mathbb{R}\},$$

$$T_{1,3} = \{g(l^2, e^k - 1 - l^2 + k^2, k, k, l, m); k, l, m \in \mathbb{R}\},$$

for $a = -1$ the sets

$$S_{1,4} = \{g(ue^{-2u}, e^u - 1 - ue^u + ue^{2u}, e^{2u} - e^u, u, v, w); u, v, w \in \mathbb{R}\},$$

$$T_{1,4} = \{g(-ke^{-2k}, e^k - 1 + ke^k - ke^{2k}, e^k - e^{2k}, k, l, m); k, l, m \in \mathbb{R}\}.$$

Appropriate K_2-connected left transversals in G_2 are the sets

$$S_2 = \{g(e^u - 1 - u^3 + \frac{3}{2}a_2u^2 + u(a_3 - a_2^2),$$

$$a_2u - \frac{3}{2}u^2, -u, u, v, w); u, v, w \in \mathbb{R}\},$$

$$T_2 = \{g(e^k - 1 + k^3 - \frac{3}{2}k^2a_2 + k(a_2^2 - a_3),$$

$$\frac{3}{2}k^2 - a_2k, k, k, l, m); k, l, m \in \mathbb{R}\}, a_2, a_3 \in \mathbb{R}.$$

Appropriate K_3-connected left transversals in $G_3^{a,b}$ are: for $-1 \leq a = b \leq 1$ the sets

$$S_{3,1} = \{g(e^u(e^{-au} - 1), e^{au}(1 - e^{-u}) - w, w, u, v, w); u, v, w \in \mathbb{R}\},$$

$$T_{3,1} = \{g(e^k(1 - e^{-ak}), e^{ak}(e^{-k} - 1) - m, m, k, l, m); k, l, m \in \mathbb{R}\},$$

for $-1 \leq a < b \leq 1$ the sets

$$S_{3,2} = \{g(e^{u-au} - e^{u-bu}, e^{au} - e^{au-u}, e^{bu-u} - e^{bu}, u, v, w); u, v, w \in \mathbb{R}\},$$

$$T_{3,2} = \{g(e^{k-bk} - e^{k-ak}, e^{ak-k} - e^{ak}, e^{bk} - e^{bk-k}, k, l, m); k, l, m \in \mathbb{R}\},$$

where $ab \neq 0$. An appropriate $K_{4,1}$-connected left transversal in $G_4^{a,b}$ is the set

$$S_{4,1} = T_{4,1} = \{g(e^{au-bu}sin(u),$$

$$\frac{1}{a_3^2+1}((e^{bu}-e^{bu-au})(sin(u)-a_3cos(u))+$$
$$(e^{bu}-cos(u))(cos(u)+a_3sin(u))),$$
$$\frac{1}{a_3^2+1}((e^{bu}-e^{bu-au})(a_3sin(u)+cos(u))+$$
$$(e^{bu}-cos(u))(a_3cos(u)-sin(u))),$$
$$u,v,w);u,v,w\in\mathbb{R}\},a_3\in\mathbb{R}.$$

An appropriate $K_{4,2}$-connected left transversal in $G_4^{a,b}$ is the set

$$S_{4,2}=T_{4,2}=\{g(e^{au-bu}sin(u),$$
$$(e^{bu-au}-e^{bu})cos(u)+sin(u)(e^{bu}-cos(u)),$$
$$cos(u)(e^{bu}-cos(u))-(e^{bu-au}-e^{bu})sin(u),u,v,w);u,v,w\in\mathbb{R}\},$$

$a\neq 0$, $b\geq 0$. Appropriate $K_{5,\epsilon}$-connected left transversals in G_5^c with $\epsilon=0,1$ are: for $c=1$ the sets

$$S_5^{c=1}=\{g(u,1-e^{-v},u,ve^v-u,v,w);u,v,w\in\mathbb{R}\},$$
$$T_5^{c=1}=\{g(k,e^{-l}-1,k,-le^l-k,l,m);k,l,m\in\mathbb{R}\},$$

for $c\neq 1$ the sets

$$S_5^{c\neq 1}=\{g(u,e^{-v}-e^{-cv},-ve^v,ve^{cv},v,w);u,v,w\in\mathbb{R}\},$$
$$T_5^{c\neq 1}=\{g(k,e^{-cl}-e^{-l},le^l,-le^{cl},l,m);k,l,m\in\mathbb{R}\}.$$

Appropriate $K_{6,\epsilon}$-connected left transversals in G_6, where $\epsilon=0,1$, are the sets

$$S_6=\{g(u,1-v^2-e^{-v},-v,\frac{1}{2}v^2e^v,v,w);u,v,w\in\mathbb{R}\},$$
$$T_6=\{g(k,l+\frac{1}{2}l^2-le^{-l},1-e^{-l},-le^l,l,m);k,l,m\in\mathbb{R}\}.$$

An appropriate $K_{7,\epsilon}$-connected left transversal in $G_7^{p\neq 0}$ with $\epsilon=0,1$ is the set

$$S_7=T_7=\{g(u,e^{-pv}sin(v),$$

$$\frac{1}{a_3^2+1}(e^{pv}v(sin(v)+a_3cos(v))+(e^{pv}-cos(v))(cos(v)-a_3sin(v))),$$

$$\frac{1}{a_3^2+1}(e^{pv}v(a_3sin(v)-cos(v))+(e^{pv}-cos(v))(a_3cos(v)+sin(v))),$$

$$v,w);u,v,w\in\mathbb{R}\},a_3\in\mathbb{R}.$$

An appropriate $K_{7,\delta}$-connected left transversal in $G_7^{p\neq 0}$, where $\delta=0,1$, is the set

$$S_7=T_7=\{g(u,e^{-pv}sin(v),e^{pv}vcos(v)+sin(v)(e^{pv}-cos(v)),$$

$$e^{pv}vsin(v)-cos(v)(e^{pv}-cos(v)),v,w);u,v,w\in\mathbb{R}\}.$$

Appropriate $K_{8,\epsilon}$-connected left transversals in G_8 with $\epsilon=0,1$ are the sets

$$S_8=\{g(ve^v+v^2,v,u,1-e^{-v},v,w);u,v,w\in\mathbb{R}\},$$

$$T_8=\{g(l^2-le^l,l,k,e^{-l}-1,l,m);k,l,m\in\mathbb{R}\}.$$

Hence the assertion follows from Lemma 7. □

Corollary 35. *All solvable decomposable Lie groups of dimension* 6 *which are the groups* $Mult(L)$ *of* 3*-dimensional connected topological loops have* 1*- or* 2*-dimensional centre and* 3*-dimensional commutator subgroup.*

Proof. If L has 1-dimensional centre, then the assertion follows from Theorem 26. If L has 2-dimensional centre, then Theorem 31 yields the assertion. □

Corollary 36. *Each solvable Lie group of dimension* 6 *which is realized as the multiplication group* $Mult(L)$ *of a* 3*-dimensional connected topological proper loop* L *has* 1*- or* 2*-dimensional centre and* 2*- or* 3*-dimensional commutator subgroup.*

Proof. If L has a 6-dimensional solvable indecomposable Lie group as its multiplication group, then the assertion is proved in Theorems 24, 25. If L has a 6-dimensional solvable decomposable Lie group as its multiplication group, then Corollary 35 gives the assertion. □

7 Summary

In this dissertation we consider connected topological proper loops L such that their multiplication groups $Mult(L)$ are solvable Lie groups. In [29] topological loops L having a Lie group as the group G_ℓ generated by all left translations of L are consistently studied. The multiplication group $Mult(L)$ of a loop L is the group generated by all left and right translations of L. The stabilizer of the identity element $e \in L$ in the group $Mult(L)$ is called the inner mapping group $Inn(L)$ of L. One of our aims is to prove relations between nilpotence and solvability for topological loops and the associated multiplication groups. Another goal of our investigation is to determine the at most 6-dimensional solvable Lie groups which can be realized as the group $Mult(L)$ of a connected simply connected topological loop L having dimension 3. In Chapter 1 we collect notions, tools and results, which we use in the dissertation. For our study the concepts of solvability and central nilpotency for loops play an important role. Normal subloops and factor loops of a loop L are defined analogously as in group theory. A normal subloop N of L is said to be central in L and abelian in L if $[N, L]_L = \{e\}$ and $[N, N]_L = \{e\}$, respectively. The centre $Z(L)$ of a loop L is the normal subloop of L consisting of all elements $z \in L$ which satisfy the identities $zx = xz,\ zx \cdot y = z \cdot xy,\ x \cdot yz = xy \cdot z,\ xz \cdot y = x \cdot zy$ for all $x, y \in L$. A loop L is classically solvable if there is a series $\{e\} = L_0 \leq L_1 \leq \ldots \leq L_n = L$ of subloops of L such that L_{i-1} is normal in L_i and the factor loop L_i/L_{i-1} is an abelian group for all $i = 1, 2, \cdots, n$. A loop L is called congruence solvable if there exists a chain $\{e\} = L_0 \leq L_1 \leq \ldots \leq L_n = L$ of normal subloops of L such that every factor loop L_i/L_{i-1} is abelian in L/L_{i-1}. If we put $Z_0 = \{e\}$, $Z_1 = Z(L)$ and $Z_i/Z_{i-1} = Z(L/Z_{i-1})$, then we obtain a series of normal subloops of L. If Z_{n-1} is a proper subloop of L but $Z_n = L$, then we say that L is centrally nilpotent of class n. The centrally nilpotent loops are congruence solvable. Every congruence solvable loop is classically solvable. From the nilpotency of the group $Mult(L)$ it follows that the loop L is centrally nilpotent (see [4]). In this case the inner mapping group $Inn(L)$ of L is commutative. For finite loops it was proved in [42] that the solvability of the group $Mult(L)$ forces the classical solvability of the loop L. Chapter 2 is devoted to investigate the two solvability properties for 3-dimensional

connected simply connected topological loops. Our main results are:

Theorem 37. *If L is a 3-dimensional connected simply connected topological loop such that its multiplication group is a solvable Lie group, then L is classically solvable. The loop L has a 1-dimensional normal subgroup N isomorphic to $\mathbb{R}$. For each 1-dimensional normal subgroup N there exists a normal series $\{e\} = L_0 \leq N = L_1 \leq M = L_2 \leq L = L_3$ of L such that every factor loop L_i/L_{i-1}, $i = 1, 2, 3$, is the group $\mathbb{R}$. Moreover, the loops M and L/N are isomorphic either to a 2-dimensional simply connected Lie group or to a loop $L_{\mathcal{F}}$.*

Congruence solvable loops are loops obtained by iterated abelian extensions (see [38]).

Theorem 38. *Let L be a 3-dimensional connected simply connected topological proper loop with a solvable Lie multiplication group. The loop L is congruence solvable if and only if L has one of the following properties:*

- *the centre of L has dimension 1 or 2,*
- *L has discrete centre and L is an abelian extension of a normal subgroup $N \cong \mathbb{R}$ by the factor loop L/N isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$.*

The following example shows that the class of congruence solvable loops is a proper subclass of the class of classical solvable loops also for topological loops. For finite loops it was illustrated in Exercise 10 in [18] and Construction 9.1 and Example 9.3 in [37].

Example 2. *Let $(Q, \cdot, 1)$ be a topological loop of dimension n having a normal subloop Q_1 such that the factor loop Q/Q_1 is isomorphic to the group $\mathbb{R}$. Let $\phi : (Q, \cdot) \to (\mathbb{R}, +)$ be a homomorphism. We consider a one-parameter family of loops $\Gamma_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(a, b) \mapsto \Gamma_t(a, b) = a *_t b$, $t \in \mathbb{R}$, such that $\Gamma_0(a, b) = a + b$ and Γ_t is not commutative for some $t \in \mathbb{R}$. Suppose that for all $t \in \mathbb{R}$ the loops Γ_t have the same identity element 0. We denote by $\Delta_t(a, b) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the right division map $(a, b) \mapsto \Delta_t(a, b) = a/_t b$, $t \in \mathbb{R}$, of the loop Γ_t. For the loops Γ_t, $t \neq 0$, we can take loops defined by the sharply transitive section $\sigma_t : \widetilde{PSL_2}(\mathbb{R})/\mathcal{L}_2 \to \widetilde{PSL_2}(\mathbb{R})$ determined by the functions $f(u) = \exp[\frac{1}{6} \sin^2 t \cos u(\cos u - 1)]$*

and $g(u) = (f(u)^{-1} - f(u)) \cot u$ *(see Proposition 18.15 and its proof in [29], pp. 244-245). All loops* Γ_t*,* $t \neq 0$*, are proper and hence they are not commutative (cf. Corollary 18.19. in [29], p. 248). The multiplication*

$$(x, a) \circ (y, b) = (x \cdot y, \Gamma_{\phi(x \cdot y)}(a, b))$$

on $Q \times \mathbb{R}$ *defines a loop* L_ϕ *which is an extension of the group* $\mathbb{R}$ *by the loop* Q*. The loop* L_ϕ *has the identity element* $(1, 0)$ *since one has* $(1, 0) \circ (y, b) = (y, \Gamma_{\phi(y)}(0, b)) = (y, b) = (y, b) \circ (1, 0)$*. Hence the loop* L_ϕ *is an Albert extension of the group* $\mathbb{R}$ *by the loop* $(Q, \cdot)$ *given by the one-parameter family* Γ_t *of the loop multiplications on* $\mathbb{R}$ *(see [28], p. 4). Let* x *be an element of* Q *such that* $\phi(x) \neq 0$*. For the inner map* $T(x, a) = \rho^{-1}(x, a)\lambda(x, a)$ *we obtain* $T(x, a)(1, c) = ((x, a) \circ (1, c))/(x, a) = (x, \Gamma_{\phi(x)}(a, c))/(x, a) = (1, \Delta_{\phi(x)}(\Gamma_{\phi(x)}(a, c), a))$*. This expression is not independent of* $a \in \mathbb{R}$ *because the loop* $\Gamma_{\phi(x)}$ *is not commutative. Hence the normal subgroup* $\mathbb{R}$ *is not abelian in the loop* L_ϕ *(see Proof of Theorem 4.1 in [38], p. 377). In particular, if the loop* $(Q, \cdot)$ *is the group* $\mathcal{L}_2$ *or a loop* $L_{\mathcal{F}}$*, then this construction yields a* 3*-dimensional connected topological loop, which is a non-abelian extension of the group* $\mathbb{R}$ *by the loop* $(Q, \cdot)$*.*

Applying the classification of the solvable Lie algebras of dimension ≤ 6 (cf. [25], [36], [41]) in Chapters 3, 4, 5, 6 we deal with the determination of the connected simply connected 6-dimensional solvable Lie groups G and their subgroups K which are the multiplication groups $Mult(L)$ and the inner mapping groups $Inn(L)$ of 3-dimensional connected simply connected topological loops L. Taking into account this restriction for the dimension of the group $Mult(L)$ we obtain:

Theorem 39. *If* L *is a connected topological proper loop of dimension* ≤ 3 *such that its multiplication group* $Mult(L)$ *is an at most* 6*-dimensional solvable Lie group, then* L *has nilpotency class* 2*.*

To prove Theorem 39, we describe the structure of the 3-dimensional connected simply connected topological loops and their groups $Mult(L)$, if $Mult(L)$ is a solvable Lie group. In Theorem 40 the group $Mult(L)$ has discrete centre.

Theorem 40. *Let* L *be a proper connected simply connected topological loop of dimension* 3 *having a solvable Lie group with discrete centre as* *its multiplication group $Mult(L)$. The loop L is classically solvable. It has a connected normal subgroup N isomorphic to $\mathbb{R}$ and the factor loop L/N is isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$. One has dim$(Mult(L)) \leq 6$ and the group $Mult(L)$ has a normal subgroup $\mathcal{S}$ containing $Mult(N) \cong \mathbb{R}$ such that the factor group $Mult(L)/\mathcal{S}$ is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$, if $L/N \cong \mathcal{L}_2$, or to a group $\mathcal{F}_n$, $n \geq 4$, if $L/N \cong L_{\mathcal{F}}$. For each normal subgroup N of L the loop L has a normal subloop M isomorphic either to $\mathbb{R}^2$ or to $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$ such that $N < M$ and L/M is isomorphic to $\mathbb{R}$. The group $Mult(L)$ contains a normal subgroup V such that $Mult(L)/V \cong \mathbb{R}$ and the orbit $V(e)$ is the loop M. The inner mapping group $Inn(L)$ of L, the multiplication group $Mult(M)$ of M and the commutator subgroup of $Mult(L)$ are subgroups of V. The normalizer $N_{Mult(L)}(Inn(L))$ equals to the group $Inn(L)$.*

In Theorem 41 the centre of the group $Mult(L)$ has dimension 1.

Theorem 41. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group $Mult(L)$ is a solvable Lie group with 1-dimensional centre Z. Then the loop L is congruence solvable. The orbit $K(e)$, where K is a 1-dimensional connected normal subgroup of $Mult(L)$, is a normal subgroup of L isomorphic to $\mathbb{R}$. Moreover, one of the following possibilities holds:*
(a) If the factor loop $L/K(e)$ is isomorphic to $\mathbb{R}^2$, then L has nilpotency class 2 and the orbit $K(e)$ coincides with the centre $Z(L)$ of L. The connected simply connected group $Mult(L)$ is a semidirect product of the abelian normal subgroup $P = Z \times Inn(L)$ by a group $Q \cong \mathbb{R}^2$ and the orbit $P(e)$ is $Z(L)$.
(b) If the factor loop $L/K(e)$ is isomorphic either to the group $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$, then $Mult(L)$ has a normal subgroup $\mathcal{S}$ containing K such that the orbits $\mathcal{S}(e)$ and $K(e)$ coincide. The factor group $Mult(L)/\mathcal{S}$ is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$, if $L/K(e) \cong \mathcal{L}_2$, or to a Lie group $\mathcal{F}_n$, $n \geq 4$, if $L/K(e) \cong L_{\mathcal{F}}$. In particular, if $K(e) = Z(L)$ and $L/Z(L)$ is isomorphic to a loop $L_{\mathcal{F}}$, then L is centrally nilpotent of class 3.
The loop L contains a 2-dimensional normal subloop M with $K(e) < M$ and the group $Mult(L)$ has a normal subgroup V as in Theorem 40.

In Theorem 42 we consider the case that the centre of $Mult(L)$ has dimension 2.

Theorem 42. *If L is a proper connected simply connected topological loop of dimension 3 such that its multiplication group $Mult(L)$ is a solvable Lie group with 2-dimensional centre Z, then L has nilpotency class 2. The group $Mult(L)$ is a semidirect product of the normal subgroup $V = Z \times Inn(L) \cong \mathbb{R}^{m-1}$ by a group $Q \cong \mathbb{R}$, where $\mathbb{R}^2 = Z \cong Z(L)$ and $m = dim(Mult(L))$. For every 1-dimensional connected subgroup N of Z the orbit $N(e)$ is a connected central subgroup of L and the factor loop $L/N(e)$ is isomorphic either to $\mathbb{R}^2$ or to a loop $L_{\mathcal{F}}$. In particular, if the group $Mult(L)$ is indecomposable, then one has $L/N(e) \cong L_{\mathcal{F}}$.*
If $L/N(e) \cong \mathbb{R}^2$, then Theorem 41 (a) holds. If $L/N(e) \cong L_{\mathcal{F}}$, then the group $Mult(L)$ contains a normal subgroup $\mathcal{S}$ with $N < \mathcal{S}$. The factor group $Mult(L)/\mathcal{S}$ is isomorphic to a Lie group $\mathcal{F}_n$ with $n \geq 4$.

To classify the groups $Mult(L)$ and $Inn(L)$ of 3-dimensional connected simply connected topological loops L we proceed in the following way. The steps of the procedure are based on the result of [33].
1. step: For each 6-dimensional solvable Lie algebra g we have to find a suitable linear representation of the corresponding connected simply connected Lie group G.
2. step: As $dim(L) = 3$ we determine those 3-dimensional Lie subgroups K of G which have no non-trivial normal subgroup of G and satisfy the condition that the normalizer $N_G(K)$ is the direct product $K \times Z$, where Z is the centre of G (cf. Lemma 8).
3. step: We have to find left transversals S and T to K in G such that for all $s \in S$ and $t \in T$ one has $s^{-1}t^{-1}st \in K$ and G is generated by $S \cup T$ (cf. Lemma 7).
3.1. Since the transversals S and T are continuous, they are determined by 3 continuous real functions of 3 variables. The condition that the products $s^{-1}t^{-1}st$, $s \in S$ and $t \in T$, are in K is formulated by functional equations. Solving these functional equations we obtain the possible forms of the left transversals S and T. The left transversals S and T are the set $\Lambda(L)$ of all left translations and the set $P(L)$ of all right translations of L, respectively. These sets play an important role for the construction of the loop multiplication using the group G_ℓ, respectively G_r (cf. [29], p. 17-18).
3.2. We check whether the set $S \cup T$ generates the group G. If this is the case, then G is the multiplication group $Mult(L)$ of a loop L and K is the inner mapping group of L.

In Chapter 4 we prove that some classes of 6-dimensional solvable Lie algebras are not the Lie algebras of the multiplication groups of 3-dimensional connected topological loops. These Lie algebras have one of the following properties:

- they have discrete centre,
- they are indecomposable and have 2-dimensional centre,
- they are indecomposable and have 4-dimensional non-abelian nilradical,
- they are indecomposable and their nilradical is either $\mathbb{R}^5$ or a 5-dimensional indecomposable nilpotent Lie algebra with exception of the Lie algebra $[e_3, e_5] = e_1$, $[e_4, e_5] = e_2$.

Every 6-dimensional indecomposable solvable Lie algebra has 4- or 5-dimensional nilradical. In the class C_1 of the 6-dimensional solvable indecomposable Lie algebras g having 5-dimensional nilradical only those can be represented as the Lie algebra of the group $Mult(L)$ of a 3-dimensional connected simply connected topological loop L whose nilradical is isomorphic either to the direct sum of the 3-dimensional Heisenberg Lie algebra and $\mathbb{R}^2$ or to the direct sum of the 4-dimensional elementary filiform Lie algebra and $\mathbb{R}$ or to the 5-dimensional indecomposable Lie algebra such that its 2-dimensional centre coincides with its commutator ideal. We prove that there are seven families of Lie algebras in C_1 which are the Lie algebras of the groups $Mult(L)$ of L. Among the 40 classes of the 6-dimensional indecomposable solvable Lie algebras with 4-dimensional nilradical only three families can be realized as the Lie algebra of the group $Mult(L)$ of L. All of them have 1-dimensional centre and abelian nilradical. They have 3-dimensional abelian commutator subalgebras and their nilradical has an abelian complement in the Lie algebra g. In the class C_2 of the 6-dimensional solvable decomposable Lie algebras g having 1-dimensional centre there is 18 families of Lie algebras which are the Lie algebras for the groups $Mult(L)$ of 3-dimensional connected simply connected topological loops L. These Lie algebras have 3-dimensional commutator subalgebra. In the class C_3 of the at most 6-dimensional solvable Lie algebras having a 2-dimensional centre nine families can be represented as the Lie algebra the group $Mult(L)$ of L. These Lie algebras are decomposable. Among them, the 6-dimensional Lie algebras have 3-dimensional commutator subalgebra. In the following theorems we give the list of the solvable Lie algebras $\mathbf{g}$ of dimension ≤ 6 which are the Lie algebras of the groups $Mult(L)$ of L. To formulate these theorems we use the notation given in [24], [26], [36], [41].

Theorem 43. *Let L be a connected simply connected topological proper loop of dimension 3 such that the Lie algebra of its multiplication group $Mult(L)$ is a 6-dimensional solvable Lie algebra $\mathbf{g}$ having 1-dimensional centre. Then L is centrally nilpotent of class 2. Moreover, for the Lie algebra $\mathbf{g}$ we obtain:*

- *If $\mathbf{g}$ is an indecomposable Lie algebra having 5-dimensional nilradical, then the Lie algebra $\mathbf{g}$ is one of the following: $\mathbf{g}_1 = \mathbf{g}_{6,14}^{a=0=b}$, $\mathbf{g}_2 = \mathbf{g}_{6,22}^{a=0}$, $\mathbf{g}_3 = \mathbf{g}_{6,17}^{\delta=1,a=0=\varepsilon}$, $\mathbf{g}_4 = \mathbf{g}_{6,51}^{\varepsilon=\pm 1}$, $\mathbf{g}_5 = \mathbf{g}_{6,54}^{a=0=b}$, $\mathbf{g}_6 = \mathbf{g}_{6,63}^{a=0}$, $\mathbf{g}_7 = \mathbf{g}_{6,25}^{a=0=b}$.*
- *If $\mathbf{g}$ is an indecomposable Lie algebra with 4-dimensional nilradical, then for the Lie algebra $\mathbf{g}$ we get one of the following: $\mathbf{g}_1 = N_{6,23}^a$, $a \in \mathbb{R}$, $\mathbf{g}_2 = N_{6,22}^a$, $a \in \mathbb{R}\backslash\{0\}$, $\mathbf{g}_3 = N_{6,27}$.*
- *If $\mathbf{g}$ is a decomposable Lie algebra, then for the Lie algebra $\mathbf{g}$ we have one of the following: $\mathbf{g}_1 = \mathbb{R} \oplus \mathbf{g}_{5,19}^{\alpha=0,\beta\neq 0}$, $\mathbf{g}_2 = \mathbb{R} \oplus \mathbf{g}_{5,20}^{\alpha=0}$, $\mathbf{g}_3 = \mathbb{R} \oplus \mathbf{g}_{5,27}$, $\mathbf{g}_4 = \mathbb{R} \oplus \mathbf{g}_{5,28}^{\alpha=0}$, $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,32}$, $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,33}$, $\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,34}$, $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,35}$, $\mathbf{g}_9 = \mathbf{l}_2 \oplus \mathbf{g}_{4,1}$, $\mathbf{g}_{10} = \mathbf{l}_2 \oplus \mathbf{g}_{4,3}$, $\mathbf{g}_{11} = \mathbf{f}_3 \oplus \mathbf{g}_{3,2}$, $\mathbf{g}_{12} = \mathbf{f}_3 \oplus \mathbf{g}_{3,3}$, $\mathbf{g}_{13} = \mathbf{f}_3 \oplus \mathbf{g}_{3,4}$, $\mathbf{g}_{14} = \mathbf{f}_3 \oplus \mathbf{g}_{3,5}^{p>0}$, $\mathbf{g}_{15} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,2}$, $\mathbf{g}_{16} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,3}$, $\mathbf{g}_{17} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,4}$, $\mathbf{g}_{18} = \mathbf{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,5}^{p>0}$.*

Theorem 44. *Let L be a 3-dimensional connected simply connected topological proper loop having an at most 6-dimensional solvable Lie algebra $\mathbf{g}$ with 2-dimensional centre as the Lie algebra of the multiplication group $Mult(L)$ of L. Then L is centrally nilpotent of class 2 and for the Lie algebra $\mathbf{g}$ we have the following possibilities:*

1 *The nilpotent Lie algebras: $\mathbb{R} \oplus \mathbf{f}_4$, $\mathbb{R} \oplus \mathbf{f}_5$.*

2 *The solvable, non-nilpotent Lie algebras:* $\mathbf{g}_1 = \mathbb{R}^2 \oplus \mathbf{g}_{4,2}^{\alpha \neq 0}$, $\mathbf{g}_2 = \mathbb{R}^2 \oplus \mathbf{g}_{4,4}$, $\mathbf{g}_3 = \mathbb{R}^2 \oplus \mathbf{g}_{4,5}^{-1 \leq \gamma \leq \beta \leq 1, \gamma\beta \neq 0}$, $\mathbf{g}_4 = \mathbb{R}^2 \oplus \mathbf{g}_{4,6}^{p \geq 0, \alpha \neq 0}$, $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,8}^{0<|\gamma| \leq 1}$, $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,10}$, $\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,14}^{p \neq 0}$, $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,15}^{\gamma = 0}$.

In the dissertation we find linear representations of the associated connected simply connected Lie groups G. These groups are multiplication groups $Mult(L)$ of 3-dimensional connected simply connected topological loops L. We determine also in all cases the inner mapping groups $Inn(L)$ of L.

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