Finsler connection for general Lagrangian systems

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We give a Finsler non-linear connection by a new simplified definition for not only regular case but also singular case. In regular case, it corresponds to non-linear connection part of Berwald's connection, but our connection is expressed not in line element space but in point-Finsler space. In this view we recognize Finsler metric $L(x, dx)$ as a "non-linear form", which is a natural generalisation of Riemannian metric having original expression, $\sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu}$. Furthermore our formulae provide easier calculation rather than conventional treatments, so we think that they suits to application to physics. Our definition can be used in the singular case of Finsler metric, which correspond to gauged constraint systems in mechanics. Here we give some non trivial examples of constraint systems for exposition of validity of our connection.

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I. INTRODUCTION

Finsler geometry has much potential of application to physics or other mathematical sciences. Usually Finsler metric $L(x, dx)$ on M is defined by a function of TM which satisfies i) *regularity*, ii) *positive homogeneity*, and iii) *strong convexity*, see the standard textbook [2]. In application of Finsler geometry to Lagrange systems, almost all systems do not satisfy the regularity condition, L is not defined on slit tangent bundle $TM^\circ = TM - \{0\}$ but on a sub-bundle $D(L) \subset TM$ depending on L, and on the sub-bundle the Finsler function L is regular. Furthermore some important Lagrange systems in physics, gauge systems does not have strong convexity. Therefore we suppose only the weaker regularity condition and the following positive homogeneity condition of L:

$$L(x, \lambda dx) = \lambda L(x, dx), \quad {}^\forall \lambda > 0. \tag{I.1}$$

Any Lagrangian system of finite degree of freedom can be reformulated in such a Finsler manifold [7, 11] without changing their physical contents, and the action functional is given by integral of the Finsler metric which is made of Lagrangian, then the variational principle becomes geometric and independent of parameterisation, which we will call covariant.

From physicist point of view, in especially thinking of the covariant Lagrangian formulation, we would like to define a non-linear connection, not in the line element space TM° but on the point manifold M [6]. Usual treatments of Finsler connection based on line element space, slit tangent bundle or projected tangent bundle, are rather similar with Hamiltonian formulation. There is a best covariant Hamiltonian formulation using contact manifold [1], which has deep relationship with the covariant Lagrangian formulation. Furthermore, the Hamiltonian formulation is correspondent to projected tangent bundle formulation of Finsler [3] in special cases. For we do not consider Hamiltonian formulation but Lagrangian, we think that the point Finsler viewpoint is the most suitable for it. If we deal only with Euler-Lagrange equation, symmetry of the system and Noether's theorem [8, 9], we don't essentially need Finsler connection. But if we treat autoparallel forms (similar to Hamiltonian equation form) of Euler-Lagrange equation, or we seek new symmetries and conserved currents, our Finsler non-linear connection hugely help us.

Next section we give a generalisation of a linear connection of a vector bundle $E \overset{\pi}{\to} M$ to non-linear one, from a some different kind of view. We define a non-linear connection not to vector fields but to dual (covector) fields, by generalising coefficients of linear connection as functions of x^μ and e^a, where x^μ are coordinates functions of M and e^a are the dual basis of e_a, the basis of section of E. In Section 3, we generalise a linear connection of TM to the non-linear connection preserving Finsler metric L, that is our Finsler non-linear connection. In our point of view, the non-linear connection is defined on $\Gamma(TM)$, that is, for tangent vector fields over M, and leads non-linearly parallel transport preserving Finsler norm. There we show the existence and uniqueness of such a non-linear connection in the case of general Finsler metrics including singular metric. In Section 4, we give a short review of a covariant Lagrangian formulation using Finsler manifold, and give the Euler-Lagrange equation to an auto-parallel form in the general cases. In the last section, we give some examples of Lagrangian systems including non-trivial gauge system. We hope that our non-linear connection will be applied to more several areas.

II. NON-LINEAR GENERALISATION OF LINEAR CONNECTION

Let $\overset{\circ}{\nabla}$ be a linear connection on a vector bundle $E \overset{\pi}{\to} M$, i.e., $\overset{\circ}{\nabla}\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$. Contracting with the tangent vector field over M, we can get the covariant derivative $\overset{\circ}{\nabla}_X\colon \Gamma(E) \to \Gamma(E)$ along the tangent vector field $X = X^\mu \frac{\partial}{\partial x^\mu}$. This $\overset{\circ}{\nabla}$ can also define the covariant derivative on dual vector bundle E^*, $\overset{\circ}{\nabla}_X\colon \Gamma(E^*) \to \Gamma(E^*)$. In coordinates, this can be given by $\overset{\circ}{\nabla}_{\frac{\partial}{\partial x^\mu}} e^a = -\overset{\circ}{\Gamma}{}^a{}_{b\mu}(x) e^b$, where $e^* = \{e^a\}$ is the basis of E^*. We generalise this linear connection $\overset{\circ}{\nabla}$ on vector bundle to non-linear connection ∇ by replacing the component $\overset{\circ}{\Gamma}{}^a{}_{b\mu}(x)$ by $\Gamma^a{}_{b\mu}(x, e^*)$, which is a 0-degree homogeneous function of e^c. That is, the non-linear connection ∇ is defined by,

$$\nabla e^a := -\Gamma^a{}_b(x, e^*) e^b = -dx^\mu \otimes \Gamma^a{}_{b\mu}(x, e^*) e^b, \quad \nabla_{\frac{\partial}{\partial x^\mu}} e^a = -\Gamma^a{}_{b\mu}(x, e^*) e^b. \tag{II.1}$$

For a general section $\rho = \rho_a e^a \in \Gamma(E^*)$,

$$\nabla \rho = d\rho_a \otimes e^a - \rho_a dx^\mu \otimes \Gamma^a{}_{b\mu}(x, e^*) e^b. \tag{II.2}$$

Notice that $\nabla_X \rho \notin \Gamma(E^*)$, however it holds linearity, $\nabla_X(\rho_1 + \rho_2) = \nabla_X \rho_1 + \nabla_X \rho_2$.

The action of ∇ to the section of E is derived as follows. Let $\xi = \xi^i \otimes e_i$ be a smooth section of E, and consider the derivative of $\xi^a = \langle e^a, \xi \rangle$,

$$\begin{aligned} &d\xi^a = \langle \nabla e^a, \xi \rangle + \langle e^a, \nabla \xi \rangle = -\langle \Gamma^a{}_b(x, e) e^b, \xi \rangle + \langle e^a, \nabla \xi \rangle, \\ &\text{i.e. } \nabla \xi = \left(d\xi^a + \Gamma^a{}_b(x, e^*(\xi)) \xi^b dx^\mu \right) \otimes e_a, \qquad \text{(II.3)} \\ &\nabla_X \xi = X^\mu \left(\frac{\partial \xi^a}{\partial x^\mu} + \Gamma^a{}_{b\mu}(x, e^*(\xi)) \xi^b \right) \otimes e_a. \qquad \text{(II.4)} \end{aligned}$$

∇_X maps $\Gamma(E)$ to $\Gamma(E)$, but the linearity does not hold, $\nabla_X(\xi_1 + \xi_2) \neq \nabla_X \xi_1 + \nabla_X \xi_2$.

Considering physical problems, we almost use the derivative of covariant quantities than contravariant quantities, so our definition of non-linear generalisation is useful to application of physics.

III. GENERALISED BERWALD'S NON-LINEAR CONNECTION

Let us consider a Finsler manifold (M, L), where M is a $(n + 1)$-dimensional differentiable manifold and $L(x, dx)$ be a Finsler metric, using coordinates (x^μ) of M, which is a function of x^μ and dx^μ and 1-degree homogeneous function of dx^μ. In our introduction we assume only reguralty on a sub-bundle $D(L) \subset TM$ and homogeneity condition (I.1), and not convexity condition. We define a new Finsler connection, which is a generalisation of Berwald connection, using previous non-linear connection defined on TM which preserves the Finsler 1-form L; $\nabla L = 0$.

Definition III.1. A non-linear Finsler connection ∇ is such that satisfies the following conditions:

$$\nabla dx^\alpha = -N^\alpha{}_\beta dx^\beta = -dx^\mu \otimes N^\alpha{}_{\beta\mu} dx^\beta, \tag{III.1}$$

$$N^\alpha{}_{\beta\mu} - N^\alpha{}_{\mu\beta} = 0, \tag{III.2}$$

$$\frac{\partial N^\alpha{}_{\beta\mu}}{\partial dx^\gamma} - \frac{\partial N^\alpha{}_{\gamma\mu}}{\partial dx^\beta} = 0, \tag{III.3}$$

$$\frac{\partial L}{\partial x^\mu} = p_\alpha N^\alpha{}_{\beta\mu} dx^\beta, \quad p_\alpha = \frac{\partial L}{\partial dx^\alpha}. \tag{III.4}$$

where $N^\alpha{}_{\beta\mu} = N^\alpha{}_{\beta\mu}(x, dx)$ are 0-degree homogeneous functions of $dx = \{dx^\mu\}$.

The last condition means the condition of preserving the Finsler metric L; $0 = \nabla L = \nabla x^\mu \frac{\partial L}{\partial x^\mu} + \nabla dx^\mu \frac{\partial L}{\partial dx^\mu}$, which is a generalisation of covariant derivative to "non-linear form" L.

Definition III.2. We denote $L_{\mu\nu} := \frac{\partial^2 L}{\partial dx^\mu \partial dx^\nu}$. For Finsler metric of our definition, $\mathrm{rank}\,(L_{\mu\nu}) \le n$ is realized, where $\mathrm{dim} M = n + 1$. If $\mathrm{rank}\,(L_{\mu\nu}) = n$, then the Finsler metric L is called *regular*. Otherwise it is called *singular*.

In application to Lagrange systems, we will see later that the Lagrangian of non-constrained systems correspond to regular Finsler metrics, and constrained systems (gauge theories) correspond to singular Finsler metrics. In application to physics, the singular Finsler manifolds are very important for gauge theories which often appear in several areas of physics.

If L is a singular Finsler metric and $\mathrm{rank}\,(L_{\mu\nu}) = n-1-D$, there are D independent functions $v^\mu_I(x, dx)\;(I = 1, 2, \cdots, D)$ satisfying

$$L_{\mu\nu} v^\nu_I = 0, \quad p_\mu v^\mu_I = 0. \tag{III.5}$$

For if v^μ_I satisfy only the former and not the latter $p_\mu v^\mu_I = w_I \neq 0$, we can replace v^μ_I to $v^\mu_I - w_I \frac{dx^\mu}{L}$.

Proposition III.1. Then we can take coordinates which $\det(L_{ab}) \neq 0,\ (a, b = D+1, D+2, \cdots n)$ and prove that

$$\begin{aligned} \ell^\mu_0 = \frac{dx^\mu}{L}, \quad \ell^\mu_I &= v^\mu_I, & \ell^\mu_a &= L \frac{\partial \ell^\mu_0}{\partial dx^a} = \delta^\mu_a - \frac{p_a dx^\mu}{L}, \\ (I &= 1, 2, \cdots D), & (a &= D+1, D+2, \cdots, n), \end{aligned}$$

become independent.

Proof. If $A\ell^\mu_0 + B^I \ell^\mu_I + C^a \ell^\mu_a = 0$, multiplying this by p_μ,

$$A = 0, \quad B^I \ell^\mu_I + C^a \ell^\mu_a = 0,$$

for $p_\mu \ell^\mu_I = p_\mu \ell^\mu_a = 0$. Differentiating the second by $dx^b\;(b = D+1, D+2, \cdots, n)$ and times with p_μ,

$$B^I p_\mu \frac{\partial \ell^\mu_I}{\partial dx^b} + C^a p_\mu \frac{\partial \ell^\mu_a}{\partial dx^b} = 0,$$

and we use the formula from differentiating the latter of (III.5) by dx^b,

$$L_{\mu b} v^\mu_I + p_\mu \frac{\partial v^\mu_I}{\partial dx^b} = p_\mu \frac{\partial \ell^\mu_I}{\partial dx^b} = 0,$$

and so,

$$C^a p_\mu \frac{\partial \ell^\mu_a}{\partial dx^b} = C^a p_\mu \left(-L_{ba} \frac{dx^\mu}{L} - p_a \frac{\ell^\mu_b}{L} \right) = -L_{ba} C^a = 0.$$

Therefore we can get $C^a = 0$ and $B_I \ell^\mu_I = 0$. Finally for we assume that $\ell^\mu_I = v^\mu_I$ are functionally independent, so we get $B_I = 0$. □

It is very helpful for calculation if we define an auxiliary function $G^\mu := \frac{1}{2} N^\mu{}_{\alpha\beta} dx^\alpha dx^\beta$. Straightforward calculation with the use of the homogeneity property of $N^\mu{}_{\alpha\beta}$ and (III.3) gives us,

$$\begin{aligned} \frac{\partial G^\mu}{\partial dx^\alpha} &= N^\mu{}_{\alpha\beta} dx^\beta + \frac{1}{2} \frac{\partial N^\mu{}_{\nu\beta}}{\partial dx^\alpha} dx^\nu dx^\beta = N^\mu{}_{\alpha\beta} dx^\beta, \\ \frac{\partial^2 G^\mu}{\partial dx^\beta \partial dx^\alpha} &= N^\mu{}_{\alpha\beta} + \frac{\partial N^\mu{}_{\alpha\nu}}{\partial dx^\beta} dx^\nu = N^\mu{}_{\alpha\beta}. \end{aligned}$$

Therefore we are able to express the coefficients of connection by G^μ:

$$N^\mu{}_{\alpha\beta}dx^\beta = \frac{\partial G^\mu}{\partial dx^\alpha}, \quad N^\mu{}_{\alpha\beta} = \frac{\partial^2 G^\mu}{\partial dx^\beta \partial dx^\alpha}, \quad \frac{\partial N^\mu{}_{\alpha\beta}}{\partial dx^\gamma} = \frac{\partial^3 G^\mu}{\partial dx^\gamma \partial dx^\beta \partial dx^\alpha}. \tag{III.6}$$

To determine the connection, it is sufficient to consider G^μ instead of $N^\mu{}_{\alpha\beta}$. This will be proved in the following.

Proposition III.2 (Existence)**.** From the definition (III.6) of coefficients of the connection, (III.2) and (III.3) are automatically satisfied. If L is singular with $\det(L_{ab}) \neq 0\,(a, b = D+1, D+2, \ldots, n)$, we can get the coefficients of connection which satisfies (III.4) from the following G^μ;

$$G^\mu = \frac{1}{2}\left(dx^\beta \frac{\partial L}{\partial x^\beta}\right)\ell^\mu_0 + \lambda^I \ell^\mu_I + \lambda^a \ell^\mu_a, \quad M_I = L^{ab} L_{aI} M_b, \tag{III.7}$$

$$\lambda^a = L^{ab} M_b, \quad M_\mu = \frac{1}{2}\left(-\frac{\partial L}{\partial x^\mu} + dx^\rho \frac{\partial^2 L}{\partial dx^\mu \partial x^\rho}\right). \tag{III.8}$$

Where $\ell^\mu_0 = \frac{dx^\mu}{L}$, $\ell^\mu_I = v^\mu_I$, $\ell^\mu_a = L\frac{\partial \ell^\mu}{\partial dx^i}$, $\mu, \beta = 0, 1, 2, \ldots, n$, $I = 1, 2, \ldots, D$, $a, b = D+1, D+2, \ldots, n$, L^{ab} is the inverse of L_{ab} and λ^I are arbitrary function.

Proof. If we multiply (III.4) by dx^β, we get inhomogeneous linear equation of G^μ;

$$dx^\beta \frac{\partial L}{\partial x^\beta} = 2 p_\mu G^\mu. \tag{III.9}$$

which can be solved as,

$$G^\mu = \frac{1}{2}\left(dx^\beta \frac{\partial L}{\partial x^\beta}\right)\frac{dx^\mu}{L} + \lambda^I \ell^\mu_I + \lambda^a \ell^\mu_a, \tag{III.10}$$

using $\ell^\mu_0, \ell^\mu_I, \ell^\mu_a$ and $p_\mu \ell^\mu_0 = 1$, $p_\mu \ell^\mu_I = p_\mu \ell^\mu_a = 0$. Here λ^I and λ^a are still unknown functions of x^μ and dx^μ. Let us determine λ^a from (III.4). Differentiating G^μ by dx^β,

$$\begin{aligned}\frac{\partial G^\mu}{\partial dx^\beta} = \frac{1}{2}\left(\frac{\partial L}{\partial x^\beta} + dx^\nu \frac{\partial^2 L}{\partial dx^\beta \partial x^\nu}\right)\frac{dx^\mu}{L} + \frac{1}{2}\left(dx^\gamma \frac{\partial L}{\partial x^\gamma}\right)\frac{L\delta^\mu_\beta - p_\beta dx^\mu}{L^2} \\ + \frac{\partial \lambda^I}{\partial dx^\beta}\ell^\mu_I + \lambda^I \frac{\partial \ell^\mu_I}{\partial dx^\beta} + \frac{\partial \lambda^a}{\partial dx^\beta}\ell^\mu_a + \lambda^a \frac{\partial \ell^\mu_a}{\partial dx^\beta},\end{aligned} \tag{III.11}$$

and putting this into the *r.h.s.* of (III.4),

$$p_\mu N^\mu{}_{\alpha\beta} dx^\alpha = p_\mu \frac{\partial G^\mu}{\partial dx^\beta} = \frac{1}{2}\left(\frac{\partial L}{\partial x^\beta} + dx^\nu \frac{\partial^2 L}{\partial dx^\beta \partial x^\nu}\right) + \lambda^a p_\mu \frac{\partial \ell^\mu_a}{\partial dx^\beta}. \tag{III.12}$$

The last term of (III.12) becomes

$$p_\mu \frac{\partial \ell^\mu_a}{\partial dx^\beta} = p_\mu \left(-\frac{\delta^\mu_\beta p_a}{L} + \frac{dx^\mu p_a p_\beta}{L^2} - \frac{dx^\mu}{L}\frac{\partial p_a}{\partial dx^\beta}\right) = -\frac{\partial p_a}{\partial dx^\beta} = -L_{\beta a}, \tag{III.13}$$

then (III.4) becomes the following equations for λ^i;

$$L_{\beta a}\lambda^a = M_\beta, \quad M_\beta := \frac{1}{2}\left(-\frac{\partial L}{\partial x^\beta} + dx^\nu \frac{\partial^2 L}{\partial dx^\beta \partial x^\nu}\right). \tag{III.14}$$

First we can solve the following $n-D$-linear equations,

$$L_{ab}\lambda^a = M_b. \tag{III.15}$$

From (III.15) we can determine λ^a by using the inverse matrix L^{ab},

$$\lambda^a = L^{ab}M_b. \tag{III.16}$$

The other equations which can be obtained from (III.14) are,

$$L_{0a}\lambda^a = M_0, \quad L_{Ia}\lambda^a = M_I. \tag{III.17}$$

If we assume the second equation of (III.17), then we can prove that the first equation of (III.17) is satisfied. Because

$$\begin{aligned} dx^0 L_{0a}\lambda^a &= \left(-dx^I L_{Ia} - dx^b L_{ba}\right)\lambda^a = -dx^I M_I - dx^b M_b = -dx^\mu M_\mu + dx^0 M_0 \\ &= -dx^\mu \frac{1}{2}\left(-\frac{\partial L}{\partial x^\mu} + dx^\beta \frac{\partial^2 L}{\partial dx^\mu \partial x^\beta}\right) + dx^0 M_0 \\ &= \frac{1}{2}\left(dx^\mu \frac{\partial L}{\partial x^\mu} - dx^\beta \frac{\partial L}{\partial x^\beta}\right) + dx^0 M_0 = dx^0 M_0. \end{aligned}$$

The second equation of (III.17) should be regarded as constraints of the system. These derivation shows that G^μ exists, and is uniquely determined up to arbitrary λ^I $(I = 1, 2, \ldots, D)$. □

Proposition III.3 (Uniqueness)**.** Up to arbitrary D function λ^I $(I = 1, 2, \ldots, D)$, generalised Berwald connection ${N^\mu}_{\alpha\beta}(x, dx)$ which satisfies (III.2), (III.3) and (III.4) is uniquely obtained by ${N^\mu}_{\alpha\beta} = \frac{\partial^2 G^\mu}{\partial dx^\alpha \partial dx^\beta}$, where G^μ is of the previous proposition.

Proof. From Proposition III.2, G^μ is unique. So we should prove that if ${N^\mu}_{\alpha\beta}$ and $\tilde{N}^\mu_{\alpha\beta}$ satisfy (III.2), (III.3), (III.4) and $G^\mu = \frac{1}{2}{N^\mu}_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{2}\tilde{N}^\mu_{\alpha\beta}dx^\alpha dx^\beta$, then ${N^\mu}_{\alpha\beta} = \tilde{N}^\mu_{\alpha\beta}$.

Let us define ${B^\mu}_{\alpha\beta} = \tilde{N}^\mu_{\alpha\beta} - {N^\mu}_{\alpha\beta}$. Then ${B^\mu}_{\alpha\beta}$ satisfies

$${B^\mu}_{\alpha\beta} = {B^\mu}_{\beta\alpha}, \quad \frac{\partial {B^\mu}_{\alpha\beta}}{\partial dx^\gamma} = \frac{\partial {B^\mu}_{\gamma\beta}}{\partial dx^\alpha}, \quad {B^\mu}_{\alpha\beta}dx^\alpha dx^\beta = 0. \tag{III.18}$$

Differentiating the third equation of (III.18) with respect to dx^γ,

$$\begin{aligned} 0 &= \frac{\partial {B^\mu}_{\alpha\beta}}{\partial dx^\gamma}dx^\alpha dx^\beta + {B^\mu}_{\gamma\beta}dx^\beta + {B^\mu}_{\alpha\gamma}dx^\alpha \\ &= \frac{\partial {B^\mu}_{\gamma\beta}}{\partial dx^\alpha}dx^\alpha dx^\beta + 2{B^\mu}_{\gamma\beta}dx^\beta = 2{B^\mu}_{\gamma\beta}dx^\beta. \end{aligned}$$

In the last equation, we used the fact that ${B^\mu}_{\alpha\beta}$ are 0-degree homogeneous functions with respect to dx^γ, since ${N^\mu}_{\alpha\beta}$ and $\tilde{N}^\mu_{\alpha\beta}$ are 0-degree homogeneous functions. Differentiating again with dx^α, we obtain ${B^\mu}_{\alpha\beta} = 0$ in the same way. □

IV. EULER-LAGRANGE EQUATION

For arbitrary singular Finsler manifold (M, L) which has $\mathrm{rank}(L_{\mu\nu}) = n - D$, the Euler-Lagrange equations are defined by,

$$0 = \frac{\partial L}{\partial x^\gamma} - d\left(\frac{\partial L}{\partial dx^\gamma}\right) = \frac{\partial L}{\partial x^\gamma} - \frac{\partial^2 L}{\partial x^\beta \partial dx^\gamma} dx^\beta - \frac{\partial^2 L}{\partial dx^\beta \partial dx^\gamma} d^2 x^\beta. \tag{IV.1}$$

Here we should think this (IV.1) as $0 = c^* \left\{ \frac{\partial L}{\partial x^\mu} - d\left(\frac{\partial L}{\partial dx^\mu}\right) \right\}$ by using a parameterisation of an oriented curve $c(t) : T \subset \mathbb{R} \to M$, and dx^μ and d^2x^μ as pull-back $c^* dx^\mu = \frac{dx^\mu(t)}{dt} dt$ and $c^* d^2 x^\mu = \frac{d^2 x^\mu(t)}{dt^2} dt^2$. In our paper for avoiding cumbersome symbol c^*, we will drop the pull-back symbols. Please look at the paper [9] for these convenient notations in more details.

Let us express this equation with the previous Finsler non-linear connection. From the condition (III.4), by differentiation by dx^γ we can get

$$\frac{\partial^2 L}{\partial dx^\gamma \partial x^\beta} = \frac{\partial p_\mu}{\partial dx^\gamma} N^\mu{}_{\alpha\beta} dx^\alpha + p_\mu \frac{\partial N^\mu{}_{\alpha\beta}}{\partial dx^\gamma} dx^\alpha + p_\mu N^\mu{}_{\gamma\beta}. \tag{IV.2}$$

Multiplying this by dx^β,

$$\frac{\partial^2 L}{\partial dx^\gamma \partial x^\beta} dx^\beta = \frac{\partial p_\mu}{\partial dx^\gamma} N^\mu{}_{\alpha\beta} dx^\alpha dx^\beta + p_\mu \frac{\partial N^\mu{}_{\alpha\beta}}{\partial dx^\gamma} dx^\alpha dx^\beta + p_\mu N^\mu{}_{\gamma\beta} dx^\beta. \tag{IV.3}$$

Furthermore with (III.2), (III.3) and 0-th homogeneity of $N^\mu{}_{\alpha\beta}$,

$$\begin{aligned} \frac{\partial^2 L}{\partial x^\beta \partial dx^\gamma} dx^\beta &= \frac{\partial p_\mu}{\partial dx^\gamma} N^\mu{}_{\alpha\beta} dx^\alpha dx^\beta + p_\mu \frac{\partial N^\mu{}_{\alpha\gamma}}{\partial dx^\beta} dx^\alpha dx^\beta + p_\mu N^\mu{}_{\beta\gamma} dx^\beta \\ &= \frac{\partial^2 L}{\partial dx^\gamma \partial dx^\mu} N^\mu{}_{\alpha\beta} dx^\alpha dx^\beta + \frac{\partial L}{\partial x^\gamma}. \end{aligned}$$

Then,

$$\frac{\partial L}{\partial x^\gamma} - \frac{\partial^2 L}{\partial x^\beta \partial dx^\gamma} dx^\beta = -\frac{\partial^2 L}{\partial dx^\gamma \partial dx^\mu} N^\mu{}_{\alpha\beta} dx^\alpha dx^\beta. \tag{IV.4}$$

Therefore Euler-Lagrange equation (IV.1) can be written as

$$0 = \frac{\partial^2 L}{\partial dx^\gamma \partial dx^\mu} \left(d^2 x^\mu + N^\mu{}_{\alpha\beta} dx^\alpha dx^\beta \right). \tag{IV.5}$$

Using the previous Finsler non-linear connection, the Euler-Lagrange equation of L is equivalent to the auto-parallel equation

$$\begin{cases} d^2 x^\mu + 2G^\mu(x, dx) = \lambda^0 \ell_0^\mu + \lambda^I \ell_I^\mu, \\ M_I = L_{Ia} L^{ab} M_b \end{cases} \tag{IV.6}$$

where λ^0, λ^I are arbitrary 2-nd homogeneous function with respect to dx^μ. From physical viewpoint, (IV.6) are equations of motion of system constrained on hypersurface which is defined by second equations. The arbitrary function λ^0 is determined by taking a time parameter, and another arbitrary functions of λ^I are determined from consistency with derivatives of the second constraint equations of (IV.6), and the others of λ^I remaine arbitrary. Also in Riemannian space, we can define *Finsler arc length parameter* s which satisfies $L\left(x(s), \frac{dx(s)}{ds}\right) = 1$. Taking the differentiation with respect to s we get a "time fixing condition",

$$0 = \frac{\partial L^*}{\partial x^\mu}\frac{dx^\mu}{ds} + \frac{\partial L^*}{\partial dx^\mu}\frac{d^2x^\mu}{ds^2}, \quad L^* := L\left(x(s), \frac{dx(s)}{ds}\right). \tag{IV.7}$$

Using the parameterised auto-parallel equation (IV.6),

$$\frac{d^2x^\mu}{ds^2} + G^\mu\left(x(s), \frac{dx(s)}{ds}\right) = \xi^0\frac{dx^\mu}{ds} + \xi^I v_I\left(x(s), \frac{dx(s)}{ds}\right), \tag{IV.8}$$

where $\xi^0 := \lambda^0\left(x(s), \frac{dx(s)}{ds}\right)/L^*$ and $\xi^I := \lambda^I\left(x(s), \frac{dx(s)}{ds}\right)$, and parameterised property of G^μ of Finsler non-linear connection (III.9),

$$\frac{\partial L^*}{\partial x^\mu}\frac{dx^\mu}{ds} = 2\frac{\partial L^*}{\partial dx^\mu}G^\mu\left(x(s), \frac{dx(s)}{ds}\right), \tag{IV.9}$$

we can show the $\xi^0 = 0$ by the following,

$$0 = \frac{\partial L^*}{\partial x^\mu}\frac{dx^\mu}{ds} + \frac{\partial L^*}{\partial dx^\mu}\left\{\xi^0\frac{dx^\mu}{ds} + \xi^I v^I - 2G^\mu\right\} = \xi^0. \tag{IV.10}$$

Therefore, choosing the Finsler arc length parameter corresponds to $\lambda^0 = 0$.

V. EXAMPLES

Regular simple case

For the most simple and important case of the potential system, which is a particle motion in three dimensional Euclidian space $\mathbb{R}^3$ influenced potential force, the Finsler metric is given by,

$$L = \frac{m}{2}\frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{dx^0} - V(x^1)dx^0. \tag{V.1}$$

We will calculate the G^μ from this Finsler metric by

$$2G^\mu = \left(dx^\beta\frac{\partial L}{\partial x^\beta}\right)\frac{dx^\mu}{L} + L^{ab}\ell^\mu_a\left(-\frac{\partial L}{\partial x^b} + dx^\rho\frac{\partial^2 L}{\partial dx^b\partial x^\rho}\right), \tag{V.2}$$

where the Greek indices runs as, $\beta, \mu, \rho = 0, 1, 2, 3$ and the Latin indices runs as $a, b = 1, 2, 3$, and we also use summation convention. For this Finsler metric is regular, there is no ℓ_I^μ terms and no constraint equation in (V.2), and its auto-parallel equation becomes

$$d^2x^\mu + 2G^\mu(x, dx) = \lambda \ell_0^\mu, \tag{V.3}$$

with an arbitrary function $\lambda(x, dx)$. Let us calculate and check this.

$$p_0 = -\left\{\frac{m}{2}\sum_{i=1}^{3}\left(\frac{dx^i}{dx^0}\right)^2 + V(x^1, x^2, x^3)\right\}, \quad p_i = m\frac{dx^i}{dx^0}, \quad (i = 1, 2, 3),$$

$$(L_{\mu\nu}) = \begin{pmatrix} m\frac{(dx^1)^2}{(dx^0)^3} & -m\frac{dx^1}{(dx^0)^2} & -m\frac{dx^2}{(dx^0)^2} & -m\frac{dx^3}{(dx^0)^2} \\ -m\frac{dx^1}{(dx^0)^2} & \frac{m}{dx^0} & 0 & 0 \\ -m\frac{dx^2}{(dx^0)^2} & 0 & \frac{m}{dx^0} & 0 \\ -m\frac{dx^3}{(dx^0)^2} & 0 & 0 & \frac{m}{dx^0} \end{pmatrix}, \quad (L^{ab}) = \begin{pmatrix} \frac{dx^0}{m} & 0 & 0 \\ 0 & \frac{dx^0}{m} & 0 \\ 0 & 0 & \frac{dx^0}{m} \end{pmatrix},$$

$$\ell_a^\mu = L\frac{\partial}{\partial dx^a}\left(\frac{dx^\mu}{L}\right) = \delta_a^\mu - \frac{m dx^\mu dx^a}{L dx^0}, \quad L^{ab}\ell_a^\mu = \begin{cases} -\dfrac{dx^0 dx^b}{L} & (\mu = 0), \\ \dfrac{dx^0\delta_b^\mu}{m} - \dfrac{dx^\mu dx^b}{L} & (\mu = 1, 2, 3), \end{cases}$$

$$dx^\beta\frac{\partial L}{\partial x^\beta} = -\frac{\partial V}{\partial x^a}dx^0 dx^a, \quad -\frac{\partial L}{\partial x^b} + dx^\rho\frac{\partial^2 L}{\partial dx^b \partial x^\rho} = \frac{\partial V}{\partial x^b}dx^0 \quad (b = 1, 2, 3),$$

$$2G^\mu = \begin{cases} -2\dfrac{\partial V}{\partial x^b}\dfrac{(dx^0)^2 dx^b}{L} & (\mu = 0), \\ \dfrac{\partial V}{\partial x^b}\left\{\dfrac{(dx^0)^2\delta^{ab}}{m} - 2\dfrac{dx^0 dx^a dx^b}{L}\right\} & (\mu = a = 1, 2, 3). \end{cases}$$

If we take Finsler arc length parameter s, which is defined by

$$1 =: L\left(x^\mu(s), \frac{dx^\mu(s)}{ds}\right) = \frac{m}{2}\sum_{a=1}^{3}\frac{(\dot{x}^a)^2}{\dot{x}^0} - V(x(s))\dot{x}^0, \quad \dot{x}^\mu := \frac{dx^\mu(s)}{ds}, \tag{V.4}$$

and its derivative by s,

$$m\left(\frac{\dot{x}^a}{\dot{x}^0}\right)\ddot{x}^a - \left\{\frac{m}{2}\left(\frac{\dot{x}^a}{\dot{x}^0}\right)^2 + V\right\}\ddot{x}^0 - \dot{x}^0\dot{x}^a\frac{\partial V}{\partial x^a} = 0, \tag{V.5}$$

are time gauge fixing conditions. Using this parameter s,

$$2G^0 = -2\frac{\partial V}{\partial x^b}(\dot{x}^0)^2\dot{x}^b(ds)^2, \quad 2G^a = \frac{\partial V}{\partial x^b}\left\{\frac{(\dot{x}^0)^2\delta^{ab}}{m} - 2\dot{x}^0\dot{x}^a\dot{x}^b\right\}(ds)^2, \tag{V.6}$$

and the auto-parallel equation (V.3) becomes

$$\ddot{x}^0 - 2(\dot{x}^0)^2\dot{x}^b\frac{\partial V}{\partial x^b} = \xi\dot{x}^0, \quad \ddot{x}^a + \frac{(\dot{x}^0)^2}{m}\frac{\partial V}{\partial x^a} - 2\dot{x}^0\dot{x}^a\dot{x}^b\frac{\partial V}{\partial x^b} = \xi\dot{x}^a, \tag{V.7}$$

where we substitute λ to $\xi = \lambda\left(x(s), \frac{dx(s)}{ds}\right)$ which is an arbitrary function of s. By this equation and time gauge fixing conditions (V.4) and (V.5), we can eliminate $\ddot{x}^\mu$ and then we can get $\xi = 0$, which was also showed in previous section in the case of using Finsler arc length parameter.

In this case, the auto-parallel equation (V.7) with $\xi = 0$ can be also derived from time gauge fixing (V.4), (V.5) and Euler-Lagrange equation of $L\,(x(s), \dot{x}(s))$,

$$0 = \frac{d}{ds}\left\{\frac{m}{2}\left(\frac{\dot{x}^a}{\dot{x}^0}\right)^2 + V\right\}, \quad 0 = \dot{x}^0\dot{x}^a\frac{\partial V}{\partial x^a} - \frac{d}{ds}\left(\frac{m\ddot{x}^a}{\dot{x}^0}\right),$$

for this Lagrange system is not gauge system.

If we choose other parametrisation $t = x^0$ and redefine $\dot{x}^\mu := \frac{dx^\mu(t)}{dt}$ and $\ddot{x}^\mu := \frac{d^2x^\mu(t)}{dt^2}$, then

$$2G^0\left(x(t), \frac{dx(t)}{dt}\right) = -\frac{2\dot{x}^a}{L}\frac{\partial V}{\partial x^a}, \quad 2G^a\left(x(t), \frac{dx(t)}{dt}\right) = \left\{\frac{\delta^{ab}}{m} - 2\frac{\dot{x}^a\dot{x}^b}{L}\right\}\frac{\partial V}{\partial x^b}, \quad \text{(V.8)}$$

and the auto-parallel equation (V.3) becomes

$$0 - \frac{2\dot{x}^a}{L}\frac{\partial V}{\partial x^a} = \xi, \quad \ddot{x}^a + \frac{1}{m}\frac{\partial V}{\partial x^a} - 2\frac{\dot{x}^a\dot{x}^b}{L}\frac{\partial V}{\partial x^b} = \xi\dot{x}^a. \quad \text{(V.9)}$$

Therefore the equation corresponds to $\ddot{x}^a + \frac{1}{m}\frac{\partial V}{\partial x^a} = 0$, that is usual form of equation of motion.

Gauge system (2nd class constraint)

In physics, we call the system which has singular Finsler Lagrangian a "gauge system". Let us think a specific example of gauge system given by

$$M = \mathbb{R}^3, \quad L(x, dx) = x^1dx^2 - x^2dx^1 + \left\{(x^1)^2 + (x^2)^2\right\}dx^0. \quad \text{(V.10)}$$

Conjugate momentum p_μ and $(L_{\mu\nu})$ are

$$p_0 = (x^1)^2 + (x^2)^2, \quad p_1 = -x^2, \quad p_2 = x^1, \quad (L_{\mu\nu}) = O, \quad \text{(V.11)}$$

therefore there are two zero-eigen functions of $(L_{\mu\nu})$, $v_1^\mu = \delta_1^\mu$ and $v_2^\mu = \delta_2^\mu$ except for dx^μ. We can get G^μ from the formula (III.7),

$$2G^\mu = \frac{2(x^1dx^1 + x^2dx^2)dx^0dx^\mu}{L} + \lambda^1v_1^\mu + \lambda^2v_2^\mu, \quad \text{(V.12)}$$

where λ^I $(I = 1, 2)$ are arbitrary funciton of x and dx, and there are two constraints:

$$M_1 = -dx^2 - x^1dx^0 = 0, \quad M_2 = dx^1 - x^2dx^0 = 0. \quad \text{(V.13)}$$

Therefore the auto-parallel equation and constraints become

$$\begin{cases} d^2x^0 + \frac{2(x^1dx^1+x^2dx^2)(dx^0)^2}{L} = \lambda dx^0, \\ d^2x^1 + \frac{2(x^1dx^1+x^2dx^2)dx^0dx^1}{L} + \lambda^1 = \lambda dx^1, \\ d^2x^2 + \frac{2(x^1dx^1+x^2dx^2)dx^0dx^2}{L} + \lambda^2 = \lambda dx^2, \\ dx^2 + x^1dx^0 = 0, \\ dx^1 - x^2dx^0 = 0. \end{cases} \tag{V.14}$$

If we take a conventional time parameter $t = x^0$, then $\lambda = 2(x^1dx^1 + x^2dx^2)/L$ and the first equation of (V.14) becomes trivial and the other equations are

$$\begin{cases} \ddot{x}^1 + \xi^1 = 0, \;\; \ddot{x}^2 + \xi^2 = 0, \\ \dot{x}^1 - x^2 = 0, \;\; \dot{x}^2 + x^1 = 0, \end{cases} \tag{V.15}$$

where $\dot{x}^a := \frac{dx^a(t)}{dt}$, $\ddot{x}^a := \frac{d^2x^a(t)}{dt^2}$ and $\xi^I := \lambda^I \left(x(t), \frac{dx(t)}{dt}\right)$ $(I = 1, 2)$ are arbitrary function of t, but ξ^I are determined by consistency of (V.15). By taking differentiation of the second line equations of (V.15) with t, we can determine ξ^I such as $\xi^1 = x^1$, $\xi^2 = x^2$. That is the equations are

$$\begin{cases} \ddot{x}^1 + x^1 = 0, \;\; \ddot{x}^2 + x^2 = 0, \\ \dot{x}^1 - x^2 = 0, \;\; \dot{x}^2 + x^1 = 0, \end{cases} \tag{V.16}$$

and have consistency. This system is Fermi particle model whose dynamics corresponds harmonic oscillator, and last equations (V.16) are equivalent to the Hamiltonian equations handled by Dirac's prescription from Lagrangian system. This model is also gauge systems having 2nd class constraint of Dirac's classification, so the all arbitrary function ξ^I, $(I = 1, 2)$ are determined from consistency condition between auto-parallel equation and constraints. Usually the Dirac's prescription for gauge systems is very bother [4, 10], but using our Finsler non-linear connection it looks like very simple!

Gauge system (Frenkel's model)

Next example of gauge system is quite pathological example, but it is known as Dirac conjecture does not hold [5, 10].

$$M = \mathbb{R}^4, \quad L(x, dx) = \frac{dx^2(dx^3)^2}{(dx^0)^2} - \frac{1}{2}x^1(x^3)^2dx^0. \tag{V.17}$$

This Euler-Lagrange equation becomes

$$\begin{cases} 0 = d\left\{-2\dfrac{dx^2(dx^3)^2}{(dx^0)^3} - \dfrac{1}{2}x^1(x^3)^2\right\}, \\ 0 = \dfrac{1}{2}(x^3)^2 dx^0, \\ 0 = d\left(\dfrac{dx^3}{dx^0}\right)^2, \\ 0 = x^1 x^3 dx^0 + 2d\left\{\dfrac{dx^2 dx^3}{(dx^0)^2}\right\}, \end{cases} \Leftrightarrow \begin{cases} 0 = \dfrac{dx^3}{dx^0}\, d\left(\dfrac{dx^3}{dx^0}\right), \\ 0 = x^1 x^3 dx^0 + 2d\left\{\dfrac{dx^2 dx^3}{(dx^0)^2}\right\}, \\ 0 = x^3. \end{cases} \quad \text{(V.18)}$$

If we take a conventional time parameter $t = x^0$, then

$$\dot{x}^3\ddot{x}^3 = 0, \quad 2\ddot{x}^2\dot{x}^3 + 2\dot{x}^2\ddot{x}^3 + x^1x^3 = 0, \quad x^3 = 0, \quad \text{(V.19)}$$

and these equal to

$$x^1 = \xi^1(t), \quad x^2 = \xi^2(t), \quad x^3 = 0, \quad \text{(V.20)}$$

where ξ^1 and ξ^2 are arbitrary function of t.

We will express this system to auto-parallel form using our Finsler non-linear connection. Conjugate momentum $p_\mu = \frac{\partial L}{\partial dx^\mu}$ are

$$p_0 = -2\frac{dx^2(dx^3)^2}{(dx^0)^3} - \frac{1}{2}x^1(x^3)^2,\ p_1 = 0,\ p_2 = \left(\frac{dx^3}{dx^0}\right)^2,\ p_3 = 2\frac{dx^2dx^3}{(dx^0)^2}, \quad \text{(V.21)}$$

and $(L_{\mu\nu})$ is

$$(L_{\mu\nu}) = \begin{pmatrix} 6\frac{dx^2(dx^3)^2}{(dx^0)^4} & 0 & -2\frac{(dx^3)^2}{(dx^0)^3} & -4\frac{dx^2dx^3}{(dx^0)^3} \\ 0 & 0 & 0 & 0 \\ -2\frac{(dx^3)^2}{(dx^0)^3} & 0 & 0 & 2\frac{dx^3}{(dx^0)^2} \\ -4\frac{dx^2dx^3}{(dx^0)^3} & 0 & 2\frac{dx^3}{(dx^0)^2} & 2\frac{dx^2}{(dx^0)^2} \end{pmatrix}. \quad \text{(V.22)}$$

We can recognise $\mathrm{rank}(L_{\mu\nu}) = 2$ and so there is one zero eigen function v^μ of $(L_{\mu\nu})$ except for dx^μ, and we can take $v_1^\mu = \delta_1^\mu$. Calculations are following

$$\begin{aligned} &\ell_0^\mu = \frac{dx^\mu}{L}, \quad \ell_1^\mu = v_1^\mu = \delta_1^\mu, \quad \ell_2^\mu = \delta_2^\mu - \frac{dx^\mu p_2}{L}, \quad \ell_3^\mu = \delta_3^\mu - \frac{dx^\mu p_3}{L}, \\ &M_0 = \frac{1}{4}(x^3)^2dx^1 + \frac{1}{2}x^1x^3dx^3,\ M_1 = \frac{1}{4}(x^3)^2dx^0,\ M_2 = 0,\ M_3 = \frac{1}{2}x^2x^3dx^0, \\ &(L^{ab}) = \begin{pmatrix} -\frac{(dx^0)^2dx^2}{2(dx^3)^2} & \frac{(dx^0)^2}{2dx^3} \\ \frac{(dx^0)^2}{2dx^3} & 0 \end{pmatrix},\ (\lambda^a) = (L^{ab}G_b) = \begin{pmatrix} x^2x^3\frac{(dx^0)^3}{4dx^3} \\ 0 \end{pmatrix},\ (a,b = 2,3), \\ &dx^\beta\frac{\partial L}{\partial x^\beta} = -\frac{1}{2}(x^3)^2dx^0dx^1 - x^1x^3dx^0dx^3, \quad M_1 = L^{ab}L_{a1}M_b = 0. \end{aligned}$$

With a constraint $M_1 = \frac{1}{3}(x^3)^2 = 0$, G^μ of the non-linear connection are given by

$$\begin{aligned} G^\mu &= \frac{1}{2}\left(dx^\beta \frac{\partial L}{\partial x^\beta}\right)\ell_0^\mu + \lambda^1\ell_1^\mu + \lambda^2\ell_2^\mu + \lambda^3\ell_3^\mu \\ &= \left\{-\frac{1}{4}(x^3)^2 dx^0 dx^1 - \frac{1}{2}x^1x^3 dx^0 dx^3\right\}\frac{dx^\mu}{L} + \lambda^1\delta_1^\mu - \frac{x^2x^3}{4}\frac{(dx^0)^3}{dx^3}\ell_2^\mu \\ &= \lambda^1\delta_1^\mu - \lim_{x^3, dx^3\to 0}\frac{x^2x^3}{4}\frac{(dx^0)^3}{dx^3}\delta_2^\mu. \end{aligned} \tag{V.23}$$

The last term is ambiguity because the limit cannot be defined. Corresponding difficulty also occurs in (L_{ab}). The matrix (L_{ab}) takes a value with the constraint $x^3 = 0$,

$$(L_{ab}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2dx^2}{(dx^0)} \end{pmatrix}, \tag{V.24}$$

but it has no inverse. The total rank of $(L_{\mu\nu})$ becomes one. In other word, there is also another zero eigenvalued function of $(L_{\mu\nu})$: $v_2^\mu = \delta_2^\mu$. Then, G^μ has the following form,

$$G^\mu = \frac{1}{2}\left(dx^\beta \frac{\partial L}{\partial x^\beta}\right)\frac{dx^\mu}{L} + \lambda^1 v_1^\mu + \lambda^2 v_2^\mu + \lambda^3\ell_3^\mu = \lambda^1\delta_1^\mu + \lambda^2\delta_2^\mu, \tag{V.25}$$

where we have used the constraint $x^3 = 0$ in the last equality. Therefore the auto-parallel equation becomes

$$\begin{cases} d^2x^0 = \lambda dx^0, \\ d^2x^1 = \lambda dx^1 + \lambda^1, \\ d^2x^2 = \lambda dx^2 + \lambda^2, \\ d^2x^3 = \lambda dx^3. \end{cases} \tag{V.26}$$

If we take a time parameter $t = x^0$, then $\lambda = 0$ and equation (V.26) becomes consistent with the constraint $x^3 = 0$ and λ^1 and λ^2 are arbitrary function of t, so the above equations are equivallent to (V.20). Even in these non-trivial pathological example, our non-linear connection will be convenient for rewriting Lagrangian form to Hamiltonian form without quite bother Dirac's prescription.

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