Fisher information and Rényi dimensions: A thermodynamical formalism

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Fisher information and Rényi dimensions: A thermodynamical formalism

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The relation between the Fisher information and Rényi dimensions is established: the Fisher information can be expressed as a linear combination of the first and second derivatives of the Rényi dimensions with respect to the Rényi parameter β. The Rényi parameter β is the parameter of the Fisher information. A thermodynamical description based on the Fisher information with β being the inverse temperature is introduced for chaotic systems. The link between the Fisher information and the heat capacity is emphasized, and the Fisher heat capacity is introduced. *Published by AIP Publishing*. [http://dx.doi.org/10.1063/1.4959908]

Thermodynamic formalism of chaotic systems has turned to be very useful in the study of multifractals. It is based on an analogy with the canonical thermodynamics. This paper throws a new light on the thermodynamics of chaos by introducing an alternative formalism established on Fisher information. It is shown that the Fisher information, its Legendre transforms, and the Fisher heat capacity describe the fluctuation and are sensitive to changes of higher order than their analogues of the usual formalism.

I. INTRODUCTION

Fisher information[1] has been receiving a growing interest in physics.[2] Frieden *et al.*[2,3] presented a Legendre transform structure reflecting classical thermodynamics. Fisher temperature was defined, and Fisher thermodynamics was introduced.[2,4,5] In a recent paper,[6] an alternative formulation of the thermodynamics based on Fisher information has been presented. This is equivalent to the traditional thermodynamics with an appropriate Legendre structure.

In this paper, this alternative formulation based on Fisher information is extended to chaotic systems. The thermodynamic formalism of chaotic systems[7–11] has proved to be very powerful for characterization of fractal and multifractal objects. The formalism utilizes the analogy of the statistical mechanics and the chaos theory. Here, it is shown that this analogy can be extended by introducing a Fisher information based thermodynamic description. This study provides a new insight into the chaos theory. In this paper, the relation between the Fisher information and Rényi dimensions is established.

The outline of this paper is as follows: in Sec. II, the dual structure of the alternative formulation of the thermodynamics[6] is summarized. In Sections III and IV, the thermodynamic formalism of chaotic systems is reviewed. Section V presents the Fisher information based thermodynamic description. Section VI is devoted to illustrative examples and discussion.

II. FISHER INFORMATION AND CANONICAL ENSEMBLE

Fisher information is a measure of the ability to estimate a parameter and is a measure of the state of disorder of a system or phenomenon. The Fisher informational functional[1] is defined as

$$I(\theta)=\int p(x|\theta)\left[\frac{\partial \ln p(x|\theta)}{\partial\theta}\right]^2 dx=\int\frac{1}{p(x|\theta)}\left[\frac{\partial p(x|\theta)}{\partial\theta}\right]^2 dx. \quad (1)$$

$p(x|\theta)$ is a probability density function, obeying proper regularity conditions and depending on a parameter θ. In case of discrete probability distribution, the Fisher information takes the form

$$I(\theta)=\sum_i p_i(\theta)\left[\frac{\partial \ln p_i(\theta)}{\partial\theta}\right]^2=\sum_i\frac{1}{p_i(\theta)}\left[\frac{\partial p_i(\theta)}{\partial\theta}\right]^2. \quad (2)$$

Consider a system with fixed volume V and particle number N which is in thermal equilibrium with a reservoir at temperature T. The canonical distribution of a random variable corresponding to the energy $\hat{U}$ has the form

$$p(\hat{U};\beta)=\frac{g(\hat{U})}{Z(\beta)}e^{-\beta\hat{U}}=g(\hat{U})e^{-\Phi(\beta)-\beta\hat{U}}, \quad (3)$$

where $\beta=1/T$ is the inverse temperature and $g(\hat{U})$ is the density of states. (The unit, where the Boltzmann constant is equal to 1, is used.) The relationship between the Massieu function $\Phi(\beta)$ and the partition function $Z(\beta)$ is

$$\Phi(\beta)=\ln(Z(\beta))=-\frac{F}{T}, \quad (4)$$

where F is the Helmholtz free energy. The form of the partition function $Z(\beta)$ is

$$Z(\beta)=\int g(\hat{U})e^{-\beta\hat{U}}d\hat{U}. \quad (5)$$

It is worth writing it for discrete probability distribution

$$Z(\beta)=\sum_r g_r e^{-\beta\hat{U}_r}. \quad (6)$$

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The derivative of Eq. (4) gives the internal energy U

$$-\frac{\partial \Phi(\beta)}{\partial \beta} = \langle \hat{U} \rangle = U. \tag{7}$$

The second derivative with respect to the inverse temperature β is the variance of the energy $\hat{U}$

$$\mathcal{F}_U = \frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = \left(U - \langle \hat{U} \rangle\right)^2. \tag{8}$$

Comparing Eq. (1) with Eq. (8), one can immediately see that the second derivative of $\Phi(\beta)$, that is, the variance of the energy U, is equal to the Fisher information

$$\mathcal{F}_U = \frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = I. \tag{9}$$

Utilizing the relationship between the variance and the heat capacity C, one is led to

$$\mathcal{F}_U = \frac{C(\beta)}{\beta^2} = C(T)T^2. \tag{10}$$

In the entropy representation, the entropy is a fundamental function of the variables $S(U,V,N)$ with $\partial S/\partial U = \beta$. A Legendre transformation gives the Massieu function

$$\Phi = S - \beta U = -\frac{F}{T}, \tag{11}$$

with the fundamental relations

$$\frac{\partial \Phi}{\partial \beta} = -U. \tag{12}$$

We can obtain another function $\mathcal{F}_S$ by Legendre transformation

$$\mathcal{F}_S = \beta \mathcal{F}_U + U, \tag{13}$$

satisfying the fundamental relation

$$\frac{\partial \mathcal{F}_S}{\partial \mathcal{F}_U} = \beta. \tag{14}$$

In the energy representation, the Legendre transform of $\mathcal{F}_{\mathcal{U}}$ takes the form

$$\mathcal{F}_F = \mathcal{F}_U - T\mathcal{F}_S, \tag{15}$$

with

$$\frac{\partial \mathcal{F}_F}{\partial T} = -\mathcal{F}_S. \tag{16}$$

As the Fisher information corresponds to the internal energy U (Eq. (7)) of the usual thermodynamics (Ref. 6), we can now define the "Fisher" heat capacity as the derivative of the Fisher information $\mathcal{F}_U$

$$\frac{\partial \mathcal{F}_U}{\partial T} = \mathcal{C}_F. \tag{17}$$

III. TEMPERATURE IN CHAOS THEORY

In nonlinear dynamics, one can also formally introduce canonical distributions.[7] Let p_i be the observed relative frequencies of events i. For example, such an event can be that an iterate is in the cell i by applying a given map. Then, the so-called escort probability distribution is defined as

$$P_i = \frac{(p_i)^\beta}{\sum_j (p_j)^\beta}. \tag{18}$$

Only events with nonzero probability are taken into account. β is a real parameter. The escort distribution can be rewritten as

$$P_i = e^{-\Phi - \beta b_i}, \tag{19}$$

where $b_i = -\ln p_i$ is the so-called bit-number[7] and $\Phi(\beta) = \ln(Z(\beta))$. The partition function is defined as

$$Z(\beta) = \sum_j e^{-\beta b_j} = \sum_j (p_j)^\beta. \tag{20}$$

We can immediately see the resemblance to canonical formalism of statistical mechanics. The Helmholtz free energy can also be defined as

$$F(\beta) = -T\ln(Z(\beta)) = -\frac{1}{\beta}\ln(Z(\beta)) = -\frac{1}{\beta}\Phi(\beta). \tag{21}$$

It is related to the Rényi information[12]

$$R^{(\beta)} = \frac{1}{1-\beta}\ln \sum p_i^\beta, \quad \beta \neq 1, \tag{22}$$

as

$$F(\beta) = -\frac{1-\beta}{\beta} R^{(\beta)}. \tag{23}$$

Note that the Rényi information leads to the Shannon information[13] in the limit $\beta \to 1$.

IV. THERMODYNAMIC RELATIONS FOR MULTIFRACTALS

Consider a multifractal with a given probability measure and divide the d-dimensional phase space into boxes of equal size. The probability p_i attributed to the box i with side length ϵ can be written as

$$p_i = \epsilon^{\alpha_i}, \tag{24}$$

where $\alpha_i(\epsilon)$ is the crowding index. The escort distributions have the form

$$P_i = e^{-\Phi - \beta b_i}, \tag{25}$$

where

$$b_i = -\ln p_i = -\alpha_i \ln \epsilon \tag{26}$$

and

$$\Phi(\beta) = \ln(Z(\beta)) = \ln\left(\sum_i e^{-\beta b_i}\right). \tag{27}$$

The partition function can be written as

$$Z(\beta) = \sum_j e^{-\beta b_j} = \sum_j (p_j)^{\beta}. \tag{28}$$

It is related to the Rényi information as

$$R^{(\beta)} = \frac{1}{1-\beta}\ln(Z(\beta)) = \frac{1}{1-\beta}\Phi(\beta). \tag{29}$$

To emphasize the thermodynamic analogy, the notation

$$V = -\ln\epsilon \tag{30}$$

is applied.[7] The limit $\epsilon \to 0$ corresponds to V going infinity, that is, the thermodynamic limit. Then, the sum in Eq. (27) can be replaced by an integral

$$\Phi(\beta) = \ln\int_{\alpha_{min}}^{\alpha_{max}} d\alpha\gamma(\alpha)e^{-\beta\alpha V}, \tag{31}$$

where $\gamma(\alpha)$ is the number of boxes having the crowing index α in the range between α and $\alpha + d\alpha$. $\gamma(\alpha)$ can be considered the analogue of the state of density in statistical mechanics as αV corresponds to the energy. The asymptotic scaling behaviour is

$$\gamma(\alpha) \sim \epsilon^{-f(\alpha)}, \tag{32}$$

where $f(\alpha)$ is the so-called spectrum of singularities. In the asymptotic limit the saddle point method can be used, that is, only the maximum value of the integrand has a contribution to the integral in

$$\Phi \sim \ln\int_{\alpha_{min}}^{\alpha_{max}} d\alpha e^{[f(\alpha)-\beta\alpha]V}. \tag{33}$$

Therefore

$$\Phi \sim [f(\alpha) - \beta\alpha]V. \tag{34}$$

From the maximum of the integrand follows that

$$\frac{\partial f(\alpha)}{\partial\alpha} = \beta. \tag{35}$$

Eq. (34) can be rewritten as

$$\Phi \sim fV - \beta b \tag{36}$$

using Eqs. (26) and (30). Taking into account the Legendre transformation in (11), we arrive at

$$\Phi = S - \beta b = S - \beta\alpha V. \tag{37}$$

That is, in the thermodynamic limit, $f(\alpha)$ is the entropy density S/V

$$\lim_{V\to\infty}\frac{S}{V} = f(\alpha). \tag{38}$$

The Legendre transformation of $f(\alpha)$ defines the function $\tau(\beta)$

$$\tau(\beta) = \beta\alpha - f(\alpha), \tag{39}$$

with

$$\frac{\partial\tau}{\partial\beta} = \alpha. \tag{40}$$

From Eqs. (37)–(40), we are led to

$$\lim_{V\to\infty}\frac{\Phi}{V} = -\tau(\beta) \tag{41}$$

in the thermodynamic limit. The function $\tau(\beta)$ can be also expressed with the Rényi dimension $D(\beta)$ as

$$\tau(\beta) = (\beta - 1)D(\beta), \tag{42}$$

where

$$D(\beta) = -\lim_{\epsilon\to 0}\frac{R^{(\beta)}}{\ln\epsilon} = \lim_{\epsilon\to 0}\frac{1}{\ln\epsilon}\frac{1}{\beta-1}\ln(Z(\beta)). \tag{43}$$

It can also be written as

$$D(\beta) = \lim_{V\to\infty}\frac{R^{(\beta)}}{V} = \lim_{V\to\infty}\frac{1}{V}\frac{1}{1-\beta}\ln(Z(\beta)). \tag{44}$$

V. THERMODYNAMICS OF MULTIFRACTALS VIA FISHER INFORMATION

Comparing Eq. (11) and Eqs. (37)–(39), we can notice the

$$\tau \sim -\Phi \tag{45}$$

correspondence. Further the first derivatives (7) and (40) are also in agreement

$$\alpha \sim U, \tag{46}$$

and the second derivatives give the Fisher information Eq. (8) and

$$\mathcal{F}_\alpha = -\frac{d^2\tau(\beta)}{d\beta^2} = -\frac{\partial\alpha}{\partial\beta}. \tag{47}$$

The Legendre transformation leading to $\mathcal{F}_S$ in Eq. (13) here takes the form

$$\mathcal{F}_f = \alpha - \beta\frac{d\alpha}{d\beta} = \alpha + \beta\mathcal{F}_\alpha, \tag{48}$$

satisfying the fundamental relation

$$\frac{\partial\mathcal{F}_f}{\partial\mathcal{F}_\alpha} = \beta. \tag{49}$$

The quantity corresponding to the free energy of the usual thermodynamic formalism now has the form

$$\mathcal{F}_F = \mathcal{F}_\alpha - T\mathcal{F}_f. \tag{50}$$

Table I shows thermodynamic variables and their analogues in multifractals in the usual thermodynamic and the Fisher information representation.

The Fisher information can be related to the Rényi dimension $D(\beta)$ (Eq. (42))

$$\mathcal{F}_\alpha = -\frac{d^2\tau(\beta)}{d\beta^2} = (1-\beta)\frac{d^2D}{d\beta^2} - 2\frac{dD}{d\beta}. \tag{51}$$

The Fisher information, on the other hand, is the variance of the energy U (Eq. (8)), that is, here the Fisher information gives the variance of α

$$\mathcal{F}_\alpha = (\alpha - \langle\alpha\rangle)^2. \tag{52}$$

The analogue of the heat capacity given in Eq. (10) can also be introduced here

$$\mathcal{F}_\alpha = \frac{C(\beta)}{\beta^2} = C(T)T^2. \tag{53}$$

Earlier, specific heat was studied by Schlögl and Schöll[7,14] for $\beta = 1$.

As the Fisher information $\mathcal{F}_\alpha$ corresponds to α (Eq. (7)) of the usual thermodynamics of fractals (see Eq. (40)), we define now the Fisher heat capacity as the derivative of the Fisher information $\mathcal{F}_\alpha$ with respect to T

$$\frac{\partial\mathcal{F}_\alpha}{\partial T} = \mathcal{C}_F. \tag{54}$$

VI. ILLUSTRATIVE EXAMPLE AND DISCUSSION

As a first example, consider the two-scale Cantor set. Here, we have two scaling parameters a_1 and a_2 with probabilities w_1 and w_2, where $a_1 \neq a_2$, $a_1 + a_2 < 1$ and $w_1 + w_2 = 1$. It is well-known[7,15] that

$$\frac{w_1^\beta}{a_1^\tau} + \frac{w_2^\beta}{a_2^\tau} = 1. \tag{55}$$

The multifractals are generally described by the generalized dimension $D(\beta)$ (or the function $\tau(\beta)$) and the function $f(\alpha)$. Eq. (55) implicitly determines τ as a function of β. Fig. 1 presents $\tau(\beta)$ and the Rényi dimension $D(\beta)$ for the parameter values $a_1 = 0.2$, $a_2 = 0.45$, $w_1 = 0.7$, and $w_2 = 0.3$. The first derivative of τ gives α (Eq. (40))

$$\frac{d\tau}{d\beta} = \frac{\sum_{i=1}^{2} w_i^\beta a_i^{-\tau} \ln w_i}{\sum_{i=1}^{2} w_i^\beta a_i^{-\tau} \ln a_i} = \alpha. \tag{56}$$

The function $f(\alpha)$ can be obtained by the Legendre transformation of $\tau(\beta)$ using Eq. (39).

The Fisher information given by the second derivative of τ

$$\mathcal{F}_\alpha = -\frac{\sum_{i=1}^{2} w_i^\beta a_i^{-\tau}[\alpha \ln a_i - \ln w_i]^2}{\sum_{i=1}^{2} w_i^\beta a_i^{-\tau} \ln a_i} \tag{57}$$

is plotted in Fig. 2.

In the usual thermodynamic formalism, multifractals are characterized by the functions $\tau(\beta)$ and $f(\alpha)$. Now, we propose to describe these objects in the Fisher information representation via the Fisher information $\mathcal{F}_\alpha$ and its Legendre transforms. The Fisher information is a fundamental function of $\mathcal{F}_f$ constructed by the Legendre transform of Eq. (48). $\mathcal{F}_f$ is presented on Fig. 3. Another Legendre transform $\mathcal{F}_F$ corresponding to the free energy of the usual thermodynamics is also shown in Fig. 3.

Instead of the Fisher information $\mathcal{F}_f$, the heat capacity $C(\beta)$ (Eq. (53)) can also be used to describe multifractals. Earlier, Schlögl and Schöll[14] showed that the specific heat

TABLE I. Thermodynamic variables and their analogues in multifractals in the usual thermodynamic and the Fisher information representation.

Thermodynamics		Multifractal	
Entropy	Fisher information	Entropy	Fisher information
S	$\mathcal{F}_S$	f	$\mathcal{F}_f$
U	$\mathcal{F}_U$	α	$\mathcal{F}_\alpha$
$\Phi = S - \beta U$	$U = \mathcal{F}_S - \beta\mathcal{F}_U$	$-\tau(\beta) = f(\alpha) - \beta\alpha$	$\alpha = \mathcal{F}_f - \beta\mathcal{F}_\alpha$
$F = -T\Phi$	$\mathcal{F}_F = -TU$	$F = T\tau(\beta)$	$\mathcal{F}_F = -T\alpha$
$= U - TS$	$= \mathcal{F}_U - T\mathcal{F}_S$	$= \alpha - Tf$	$= \mathcal{F}_\alpha - T\mathcal{F}_f$
$S(U)$	$\mathcal{F}_S(\mathcal{F}_U)$	$f(\alpha)$	$\mathcal{F}_f(\mathcal{F}_\alpha)$
$U(S)$	$\mathcal{F}_U(\mathcal{F}_S)$	$\alpha(f)$	$\mathcal{F}_\alpha(\mathcal{F}_f)$
$S = \beta U + \Phi$	$\mathcal{F}_S = \beta\mathcal{F}_U + U$	$f = \beta\alpha - \tau$	$\mathcal{F}_f = \beta\mathcal{F}_\alpha + \alpha$
$U = TS + F$	$\mathcal{F}_U = T\mathcal{F}_S + \mathcal{F}_U$	$\alpha = Tf + F$	$\mathcal{F}_\alpha = T\mathcal{F}_f + \mathcal{F}_F$
$\frac{\partial S}{\partial U} = \beta$	$\frac{\partial\mathcal{F}_S}{\partial\mathcal{F}_U} = \beta$	$\frac{\partial f}{\partial\alpha} = \beta$	$\frac{\partial\mathcal{F}_f}{\partial\mathcal{F}_\alpha} = \beta$
$\frac{\partial U}{\partial S} = T$	$\frac{\partial\mathcal{F}_U}{\partial\mathcal{F}_S} = T$	$\frac{\partial\alpha}{\partial f} = T$	$\frac{\partial\mathcal{F}_\alpha}{\partial\mathcal{F}_f} = T$
$\frac{\partial\Phi}{\partial\beta} = -U$	$\frac{\partial U}{\partial\beta} = -\mathcal{F}_U$	$-\frac{\partial\tau}{\partial\beta} = -\alpha$	$\frac{\partial\alpha}{\partial\beta} = -\mathcal{F}_\alpha$
$\frac{\partial F}{\partial T} = -S$	$\frac{\partial\mathcal{F}_F}{\partial T} = -\mathcal{F}_S$	$\frac{\partial F}{\partial T} = -S$	$\frac{\partial\mathcal{F}_F}{\partial T} = -\mathcal{F}_f$

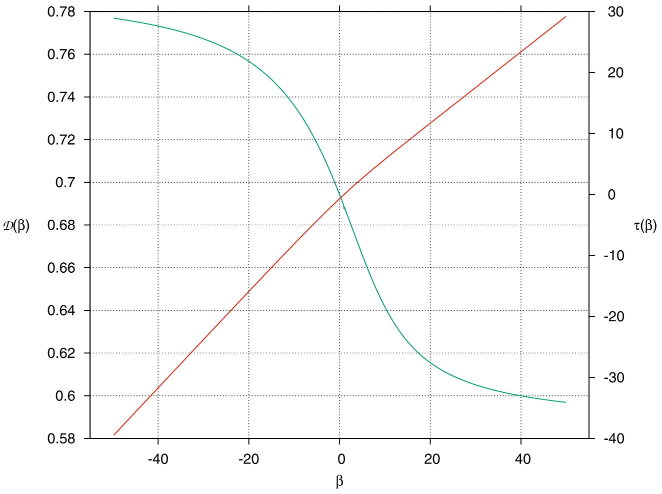

FIG. 1. $\tau(\beta)$ (red) and the Rényi dimension $D(\beta)$ (green) of the two-scale Cantor set as a function of β for the parameter values $a_1 = 0.2$, $a_2 = 0.45$, $w_1 = 0.7$, and $w_2 = 0.3$.

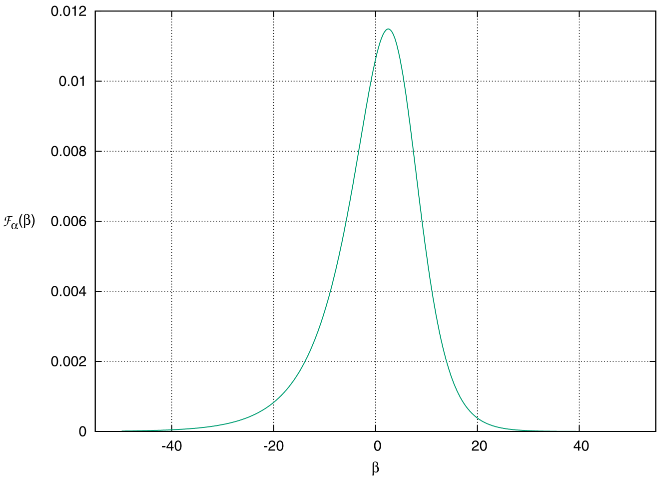

FIG. 2. The Fisher information $\mathcal{F}_\alpha$ of the two-scale Cantor set as a function of β for the parameter values $a_1 = 0.2$, $a_2 = 0.45$, $w_1 = 0.7$, and $w_2 = 0.3$.

C in Eq. (53) for $\beta = 1$ is a characteristic measure in chaos. Fig. 4 presents the heat capacity $C(\beta)$. The Fisher information $\mathcal{F}_\alpha$ or the heat capacity $C(\beta)$ describe the fluctuation in α (Eq. (50)), that is, they are sensitive to changes of higher order than their analogues of the usual formalism.

Using Eq. (54) the Fisher heat capacity takes the form

$$\mathcal{C}_F = -\beta^2 \frac{\sum_{i=1}^{2} w_i^{\beta} a_i^{-\tau}(\alpha \ln a_i - \ln w_i)[(\ln w_i - \alpha \ln a_i)^2 + 3\mathcal{F}_\alpha \ln a_i]}{\sum_{i=1}^{2} w_i^{\beta} a_i^{-\tau} \ln a_i}, \tag{58}$$

and is shown in Fig. 4. The Fisher heat capacity $\mathcal{C}_F$ is even more sensitive on fluctuation than the heat capacity $C(\beta)$.

As a second example, the logistic map is selected

$$x_{n+1} = r x_n (1 - x_n), \quad 0 < x_n < 1 \quad \text{and} \quad 0 < r < 4. \tag{59}$$

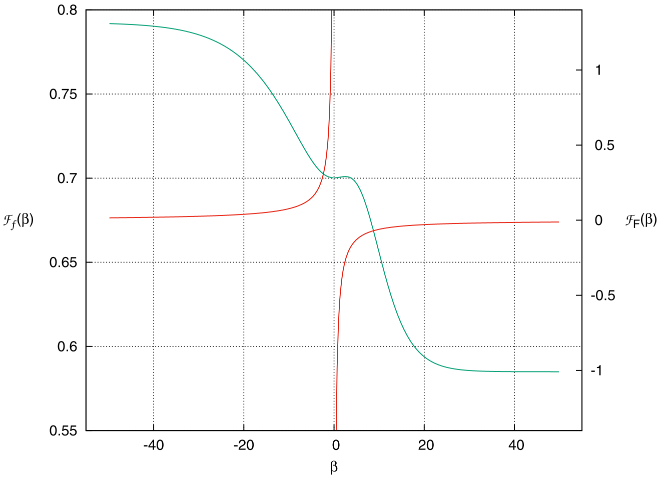

FIG. 3. Legendre transforms $\mathcal{F}_f$ (green) and $\mathcal{F}_F$ (red) of the Fisher information $\mathcal{F}_\alpha$ for the two-scale Cantor set as a function of β for the parameter values $a_1 = 0.2$, $a_2 = 0.45$, $w_1 = 0.7$, and $w_2 = 0.3$.

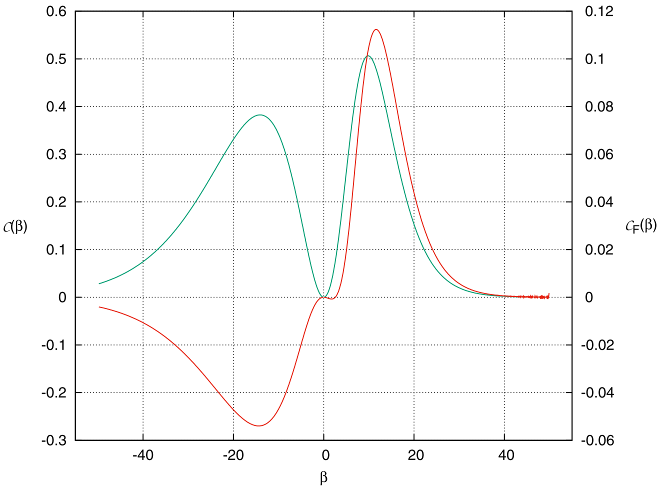

FIG. 4. The heat capacity $C(\beta)$ (green) and the Fisher heat capacity $\mathcal{C}_F$ (red) of the two-scale Cantor set as a function of β for the parameter values $a_1 = 0.2$, $a_2 = 0.45$, $w_1 = 0.7$, and $w_2 = 0.3$.

One can observe period-doubling cascade if the control parameter r is smaller than $r_\infty = 3.56995$. Though most values beyond r_∞ exhibit chaotic behaviour, there are several periodic windows, too. Fig. 5 presents the Fisher information $\mathcal{F}_\alpha$ and its Legendre transforms $\mathcal{F}_f$ at $\beta = 2.25$ as a function of the control parameter r. We can see that the Fisher information $\mathcal{F}_\alpha$ is zero for periodic orbitals and different from zero in case of chaotic behavior. For a period-n orbit, the probabilities are $p_i = 1/n$. Then, we obtain that the Rényi dimension $D(\beta) = \ln n$ does not depend on the β. Then, Eq. (42) leads to

$$\tau(\beta) = (\beta - 1)\ln n. \tag{60}$$

The first and the second derivatives give

$$\alpha = \frac{\partial \tau(\beta)}{\partial \beta} = \ln n \tag{61}$$

and

$$\mathcal{F}_\alpha = -\frac{d^2 \tau(\beta)}{d\beta^2} = 0, \tag{62}$$

respectively. That is, the Fisher information disappears for periodic orbitals. As $\mathcal{F}_\alpha$ measures the fluctuations, one can observe huge peaks at the edge of the periodic windows. A complicated structure reflecting the chaotic nature is exhibited in the chaotic regimes.

The other quantity, $\mathcal{F}_f$ behaves somewhat differently. Eqs. (48), (61), and (62) lead to

$$\mathcal{F}_f = \alpha + \beta \mathcal{F}_\alpha = \alpha = \ln n \tag{63}$$

for a period-n orbit. Fig. 5 shows that $\mathcal{F}_f$ has a step structure for the periodic orbits, that is, the plot of $\mathcal{F}_f$ gives the periods of the orbitals. $\mathcal{F}_f$ takes much higher values in case of chaotic behaviour. Fig. 5 is plotted for $\beta = 2.25$. Other values of β were also studied. Increasing the value of β, the steps in the periodic windows are decreasing in magnitude, while more detailed structure can be observed in the chaotic regions. It can be concluded that both the Fisher information $\mathcal{F}_\alpha$ and its Legendre transforms $\mathcal{F}_f$ can be considered measures of complexity.

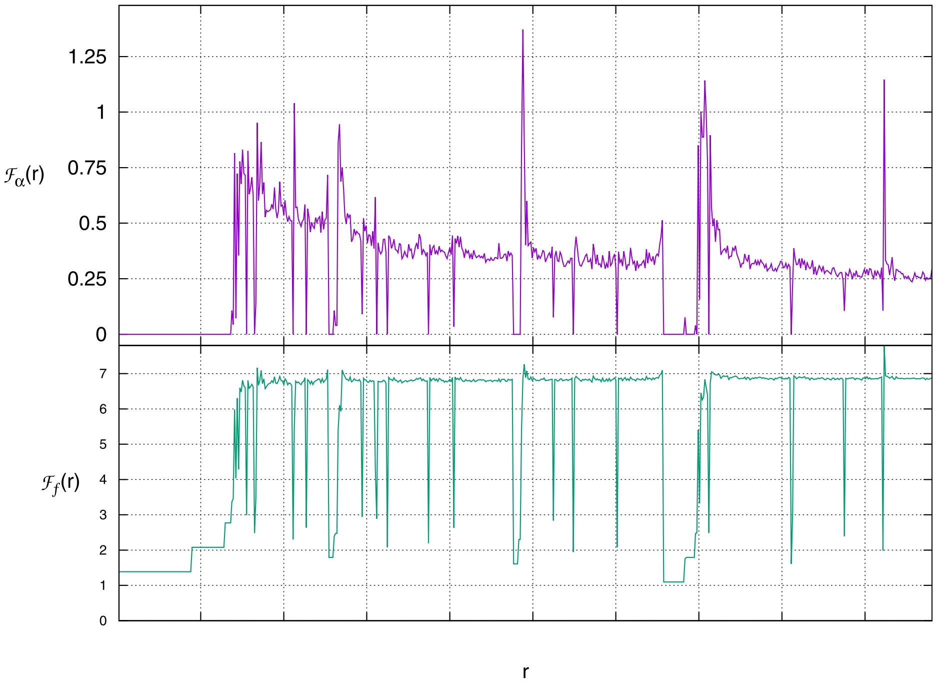

FIG. 5. The Fisher information $\mathcal{F}_\alpha$ (top) and its Legendre transforms $\mathcal{F}_f$ (bottom) at $\beta = 2.25$ for the logistic map as a function of the control parameter r.

We have proposed that the Fisher information based thermodynamical formalism provides an alternative way of characterising chaotic systems. The new quantities: the Fisher information and its Legendre transforms are capable of describing the properties of non-linear systems. Because of their definitions, these quantities are sensitive to correlations of higher degree than the characteristics of the usual formalism.

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