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HADAMARD–TYPE INEQUALITIES FOR GENERALIZED CONVEX FUNCTIONS

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Abstract. In this paper we investigate (ω_1, ω_2)-convex functions and obtain characterization theorems and Hadamard-type inequalities for them.

1. Introduction

The geometrical meaning of convexity is quite descriptive: each line segment joining two distinct points of a real function's graph (that is defined over a real interval I) passes "above" the graph. Beckenbach [1] generalized this geometric idea by replacing straight lines, (that is, functions of the form $\alpha + \beta x$) by a two parameter family of continuous functions $\mathcal{F}$ such that, for any pairs $(x_1, y_1), (x_2, y_2) \in I \times \mathbb{R}$ with $x_1 \neq x_2$ there exists a unique element $\varphi \in \mathcal{F}$ such that $\varphi(x_i) = y_i$, $i = 1, 2$. Results on the regularity properties of such, so-called *generalized convex functions* and also on their second-order characterizations were obtained by Beckenbach and Bing [2] and can also be found in the papers of Peixoto [14], [16], [15]. In the special case when the two parameter family is the solution-set of a second-order homogeneous linear differential equation, similar results were independently obtained by Bonsall [5]. Generalized convexity was characterized by the support property in the paper [3] by Ben-Tal and Ben-Israel. Stability properties of generalized convexity were investigated by Krzyszkowski [9].

The notion of (ordinary) convexity can also be characterized via the following condition (cf. Popoviciu [17]): a function $f : I \to \mathbb{R}$ is convex if and only if

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} \geqslant 0$$

whenever $x, y, z \in I$ with $x < y < z$.

Replacing the functions 1 and x by two arbitrary functions ω_1 and ω_2, we can introduce the notion of (ω_1, ω_2)-convexity.

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DEFINITION 1. Let $I \subset \mathbb{R}$ be a nonempty interval and $\omega_1, \omega_2 : I \to \mathbb{R}$ be given functions. We say that the function $f : I \to \mathbb{R}$ is (ω_1, ω_2)*-convex* if

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} \geqslant 0 \tag{1}$$

whenever $x < y < z$, $x, y, z \in I$.

Obviously, (ω_1, ω_2)-convexity is the particular case of generalized convexity in the sense of Beckenbach and it is a generalization of standard convexity. Now the generalized lines corresponding to the functions ω_1, ω_2 are their linear combinations. It turns out to be useful to assume (as for the generalized convexity due to Beckenbach) that each generalized line is continuous and every two points of $I \times \mathbb{R}$ with distinct first coordinates can uniquely be connected by a generalized line. Formulating these two regularity properties in mathematical terms, we have the following

DEFINITION 2. Let $I \subset \mathbb{R}$ be a nonempty interval and $\omega_1, \omega_2 : I \to \mathbb{R}$. We say that (ω_1, ω_2) is a *regular pair* on I if ω_1 and ω_2 are continuous functions and

$$\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} \neq 0$$

whenever $x, y \in I$ with $x < y$.

If ω_1 and ω_2 are twice differentiable and their Wronski determinant is non-vanishing then the generalized lines (i.e., the linear combinations of ω_1 and ω_2) can be obtained as the solution set of a homogeneous second-order linear differential equation, which is the setting investigated by Bonsall [5].

The aim of this paper is to obtain various characterizations of (ω_1, ω_2)-convexity *without assuming further regularity properties on them*. Another result offers a transformation that maps generalized lines into straight lines and thus also (ω_1, ω_2)-convex functions to ordinary convex functions. Using these characterizations, we then derive a generalization of Hadamard's inequality [8] known for ordinary convex functions (see the book of Dragomir and Pearce [7] for further references and details and also the paper of Mitrinović and Lacković [11] and [13] for interesting historical remarks). Bonsall [5] also obtained a generalization of Hadamard's inequality that involves the adjoint differential operator with respect to the linear differential operator that vanishes on ω_1 and ω_2. Generalizations of Hadamard's inequality for hihger-order convexity have recently been obtained by the authors [4].

2. Generalized lines

First we investigate some essential properties of regular pairs and generalized lines.

THEOREM 1. *Let (ω_1, ω_2) be a regular pair on I. Then,*

(i) ω_1 and ω_2 have at most one zero in I;

(ii) ω_1 and ω_2 cannot be equal to zero simultaneously;

(iii) the function

$$\Omega(x,y) := \begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}$$

keeps its sign if $x < y$, $x, y \in I$.

Proof. The assertions (i) and (ii) are simple consequences of our definition. For proving the third one, observe that the function Ω is continuous and nowhere zero on the connected set $\{(x,y) \in I \times I : x < y\}$, therefore it keeps its sign according to Bolzano's theorem. □

The part (iii) of Theorem 1 states that the function Ω is either positive or negative if $x < y$. According to this property, a regular pair (ω_1, ω_2) is said to be *positive* if

$$\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} > 0 \tag{2}$$

whenever $x < y$, $x, y \in I$.

The most important property of generalized lines guarantees the existence of a generalized line "parallel" to the x-axis.

THEOREM 2. *Let* (ω_1, ω_2) *be a regular pair on the nonempty interval* I. *Then, there exist* $\alpha, \beta \in \mathbb{R}$ *such that the inequality*

$$\alpha\omega_1(x) + \beta\omega_2(x) > 0 \tag{3}$$

holds for all $x \in I^\circ$, *where* I° *denotes the interior of* I.

Proof. For simplicity, we may assume that (ω_1, ω_2) is a positive regular pair. If ω_1 has no zero in I°, then $\alpha := 1$ or $\alpha := -1$ (according to the sign of ω_1) and $\beta := 0$ fulfills the requirements of the theorem. Suppose that $\omega_1(\xi) = 0$ for some $\xi \in I^\circ$. Then, according to (i) of Theorem 1, we may assume that the inequalities

$$\begin{aligned} \omega_1(x) < 0 \qquad (x < \xi) \\ \omega_1(y) > 0 \qquad (y > \xi) \end{aligned}$$

hold. Let $x, y \in I$ with $x < \xi < y$ be arbitrary. The negativity of $\omega_1(x)$, the positivity of $\omega_2(y)$, and (2) imply that

$$\frac{\omega_2(y)}{\omega_1(y)} < \frac{\omega_2(x)}{\omega_1(x)}.$$

Therefore,

$$\sup_{y>\xi}\left[\frac{\omega_2(y)}{\omega_1(y)}\right] \leqslant \inf_{x<\xi}\left[\frac{\omega_2(x)}{\omega_1(x)}\right] \tag{4}$$

and both of the sides are real numbers. Define the constant α by the formula

$$\alpha := \sup_{y>\xi}\left[\frac{\omega_2(y)}{\omega_1(y)}\right],$$

and choose $\beta := -1$. We will show that

$$\alpha\omega_1(x) - \omega_2(x) > 0 \tag{5}$$

holds for all $x \in I^\circ$. This inequality remains true if $x := \xi$. Indeed, $\omega_1(\xi) = 0$; on the other hand, substituting $x := \xi$ into (2) and applying the positivity of $\omega_1(y)$, we get that $-\omega_2(\xi) > 0$. If $y > \xi$, then the definition of α gives that

$$\alpha \geqslant \frac{\omega_2(y)}{\omega_1(y)};$$

therefore, multiplying both sides by the positive $\omega_1(y)$, we get

$$\alpha\omega_1(y) - \omega_2(y) \geqslant 0.$$

If $x < \xi$, then inequality (4) gives that

$$\alpha \leqslant \frac{\omega_2(x)}{\omega_1(x)};$$

therefore, multiplying both sides by the negative $\omega_1(x)$, it follows that

$$\alpha\omega_1(x) - \omega_2(x) \geqslant 0.$$

Finally, we show that the left hand side of (5) always differs from zero. Assume indirectly that there exists $\eta \in I^\circ$ such that

$$\alpha\omega_1(\eta) - \omega_2(\eta) = 0.$$

Then,

$$\alpha = \frac{\omega_2(\eta)}{\omega_1(\eta)}.$$

If $\xi < \eta$, choose $y \in I$ such that $\eta < y$ hold. Substituting $x := \eta$ and y into (2), then applying the positivity of $\omega_1(\eta)$ and $\omega_1(y)$, we get the inequality

$$\alpha = \frac{\omega_2(\eta)}{\omega_1(\eta)} < \frac{\omega_2(y)}{\omega_1(y)},$$

which contradicts the definition of α. If $\xi > \eta$, then choose $x \in I$ with $x < \eta$. Substituting x and $y := \eta$ into (2), then applying the negativity of $\omega_1(x)$ and $\omega_1(\eta)$, we get the inequality

$$\alpha = \frac{\omega_2(\eta)}{\omega_1(\eta)} > \frac{\omega_2(x)}{\omega_1(x)},$$

which contradicts (4). □

As a consequence of the previous result, a regular pair can always be replaced by a canonical regular pair.

THEOREM 3. *Let* (ω_1, ω_2) *be a regular pair on the nonempty interval* $I \subset \mathbb{R}$. *Then there exists a regular pair* (ω_1^*, ω_2^*) *on* I *that possesses the following properties:*

(i) ω_1^* *is positive on* I°;

(ii) ω_2^*/ω_1^* *is strictly monotonic on* I°

(iii) (ω_1, ω_2)*-convexity is equivalent to* (ω_1^*, ω_2^*)*-convexity.*

Conversely, if the functions $\omega_1, \omega_2 : I \to \mathbb{R}$ *are continuous with the properties*

(i') ω_1 *is positive;*

(ii') ω_2/ω_1 *is strictly monotone increasing (resp. decreasing),*

then (ω_1, ω_2) *is a positive (resp. negative) regular pair on* I.

Proof. The previous Theorem 2 guarantees the existence of real constants α and β such that (3) holds for all $x \in I^\circ$. Define

$$\omega_1^* := \alpha\omega_1 + \beta\omega_2, \qquad \omega_2^* := -\beta\omega_1 + \alpha\omega_2.$$

Then, using the product rule of determinants, for $x, y \in I$ with $x < y$, we have

$$\begin{vmatrix} \omega_1^*(x) & \omega_1^*(y) \\ \omega_2^*(x) & \omega_2^*(y) \end{vmatrix} = \begin{vmatrix} \alpha & \beta \\ -\beta & \alpha \end{vmatrix} \cdot \begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} = (\alpha^2+\beta^2) \begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} \neq 0,$$

therefore, (ω_1^*, ω_2^*) is also a regular pair on I. Using that ω_1^* is positive, one can easily deduce that the positivity (resp. negativity) of the determinant

$$\begin{vmatrix} \omega_1^*(x) & \omega_1^*(y) \\ \omega_2^*(x) & \omega_2^*(y) \end{vmatrix}$$

yields the strictly increasing (resp. decreasing) property of the function ω_2^*/ω_1^* on the interior of I.

To see that (ω_1, ω_2)-convexity is equivalent to (ω_1^*, ω_2^*)-convexity, let $f : I \to \mathbb{R}$ be an arbitrary function and $x < y < z$ be arbitrary elements of I. Then, by the product rule of determinants,

$$\begin{aligned} \begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1^*(x) & \omega_1^*(y) & \omega_1^*(z) \\ \omega_2^*(x) & \omega_2^*(y) & \omega_2^*(z) \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{vmatrix} \cdot \begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} \\ &= (\alpha^2 + \beta^2) \cdot \begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix}, \end{aligned}$$

whence the equivalence of the corresponding convexities follows.

The converse assertion is a simple calculation. □

3. (ω_1, ω_2)-convex functions

Our goal is deriving characterization theorems for (ω_1, ω_2)-convex functions. The first one is the generalization of a classical theorem for convex functions, while the second one formulates a connection between (ω_1, ω_2)-and the usual convexity.

THEOREM 4. *Let* (ω_1, ω_2) *be a positive regular pair on the nonempty interval* I *such that* ω_1 *is positive. The following statements are equivalent:*

(i) $f : I \to \mathbb{R}$ *is* (ω_1, ω_2)*-convex;*

(ii) *for all* $x, y, z \in I:\ x < y < z$ *we have that*

$$\frac{\begin{vmatrix} f(y) & f(z) \\ \omega_1(y) & \omega_1(z) \end{vmatrix}}{\begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix}} \leqslant \frac{\begin{vmatrix} f(x) & f(y) \\ \omega_1(x) & \omega_1(y) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}};$$

(iii) *for all* $x_0 \in I^\circ$ *there exist* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\begin{aligned} \alpha\omega_1(x_0) + \beta\omega_2(x_0) &= f(x_0), \\ \alpha\omega_1(x) + \beta\omega_2(x) &\leqslant f(x) \qquad (x \in I); \end{aligned}$$

(iv) *for all* $n \in \mathbb{N}$, $x_0, x_1, \dots, x_n \in I$ *and* $\lambda_1, \dots, \lambda_n \geqslant 0$ *satisfying the conditions*

$$\sum_{k=1}^{n} \lambda_k\omega_1(x_k) = \omega_1(x_0) \tag{6}$$

$$\sum_{k=1}^{n} \lambda_k\omega_2(x_k) = \omega_2(x_0) \tag{7}$$

we have that

$$f(x_0) \leqslant \sum_{k=1}^{n} \lambda_k f(x_k); \tag{8}$$

(v) *for all* $x_0, x_1, x_2 \in I$ *and* $\lambda_1, \lambda_2 \geqslant 0$ *satisfying the conditions*

$$\lambda_1\omega_j(x_1) + \lambda_2\omega_j(x_2) = \omega_j(x_0) \qquad (j = 1, 2)$$

we have that

$$f(x_0) \leqslant \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

Proof. $(i) \Rightarrow (ii)$. Assume indirectly that (ii) is not true. Using the positivity of the denominators, it follows that there exist $x, y, z \in I$ with $x < y < z$ such that

$$\begin{vmatrix} f(y) & f(z) \\ \omega_1(y) & \omega_1(z) \end{vmatrix} \cdot \begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} > \begin{vmatrix} f(x) & f(y) \\ \omega_1(x) & \omega_1(y) \end{vmatrix} \cdot \begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix},$$

or, after rearranging this inequality,

$$\begin{aligned} f(y)\left[\omega_1(x)\begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix} + \omega_1(z)\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}\right] \\ > \omega_1(y)\left[f(x)\begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix} + f(z)\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}\right]. \end{aligned}$$

Subtracting

$$f(y)\omega_1(y)\begin{vmatrix} \omega_1(x) & \omega_1(z) \\ \omega_2(x) & \omega_2(z) \end{vmatrix}$$

from both sides and applying the expansion theorem "backward", we get that

$$f(y)\begin{vmatrix} \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} > \omega_1(y)\begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix}.$$

The left hand side of this inequality equals zero, while the (ω_1, ω_2)-convexity of f implies that the right hand side is non-negative, which is contradiction.

$(ii) \Rightarrow (iii)$. Choose $x_0 \in I^\circ$ and define the constant β by

$$\beta := \inf_{x>x_0}\left[-\frac{\begin{vmatrix} f(x_0) & f(x) \\ \omega_1(x_0) & \omega_1(x) \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(x) \\ \omega_2(x_0) & \omega_2(x) \end{vmatrix}}\right].$$

According to (ii), if $\xi < x_0 < x$, then the inequality

$$-\frac{\begin{vmatrix} f(\xi) & f(x_0) \\ \omega_1(\xi) & \omega_1(x_0) \end{vmatrix}}{\begin{vmatrix} \omega_1(\xi) & \omega_1(x_0) \\ \omega_2(\xi) & \omega_2(x_0) \end{vmatrix}} \leqslant -\frac{\begin{vmatrix} f(x_0) & f(x) \\ \omega_1(x_0) & \omega_1(x) \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(x) \\ \omega_2(x_0) & \omega_2(x) \end{vmatrix}}$$

holds, whence $\beta > -\infty$. Now define the constant α by

$$\alpha := \frac{f(x_0) - \beta\omega_2(x_0)}{\omega_1(x_0)}.$$

Then we immediately get the equality

$$\alpha\omega_1(x_0) + \beta\omega_2(x_0) = f(x_0).$$

The desired inequality

$$\alpha\omega_1(x) + \beta\omega_2(x) \leqslant f(x)$$

can be rewritten into the equivalent form

$$\beta\begin{vmatrix} \omega_1(x_0) & \omega_1(x) \\ \omega_2(x_0) & \omega_2(x) \end{vmatrix} + \begin{vmatrix} f(x_0) & f(x) \\ \omega_1(x_0) & \omega_1(x) \end{vmatrix} \leqslant 0. \tag{9}$$

The definition of β guarantees that (9) is satisfied if $x_0 < x$. Assume that $x < x_0$ and choose $\xi \in I$ such that $x < x_0 < \xi$ hold. Then, applying (ii), we have the inequality

$$\frac{\begin{vmatrix} f(x_0) & f(\xi) \\ \omega_1(x_0) & \omega_1(\xi) \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(\xi) \\ \omega_2(x_0) & \omega_2(\xi) \end{vmatrix}} \leqslant \frac{\begin{vmatrix} f(x) & f(x_0) \\ \omega_1(x) & \omega_1(x_0) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(x_0) \\ \omega_2(x) & \omega_2(x_0) \end{vmatrix}}.$$

Observe that the denominator of the right hand side is positive, therefore, after rearranging this inequality, we get that

$$-\frac{\begin{vmatrix} f(x_0) & f(\xi) \\ \omega_1(x_0) & \omega_1(\xi) \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(\xi) \\ \omega_2(x_0) & \omega_2(\xi) \end{vmatrix}}\begin{vmatrix} \omega_1(x_0) & \omega_1(x) \\ \omega_2(x_0) & \omega_2(x) \end{vmatrix} + \begin{vmatrix} f(x_0) & f(x) \\ \omega_1(x_0) & \omega_1(x) \end{vmatrix} \leqslant 0,$$

which, and the choice of β immediately implies (9).

$(iii) \Rightarrow (iv)$. First assume that $x_0 = x_1 = \ldots = x_n$. We recall that $\omega_1(x_0)$ and $\omega_2(x_0)$ cannot be equal to zero simultaneously due to Theorem 1; therefore (6) or (7) gives the identity $\sum_{k=1}^{n} \lambda_k = 1$, and the inequality (8) trivially holds.

Now assume that $x_0, x_1, \ldots, x_n$ are distinct points of I such that the equations (6) and (7) are satisfied. We will show that necessarily $x_0 \in I^\circ$ must hold. If $\inf(I) \in I$ and indirectly $x_0 = \inf(I)$, then we have the inequalities

$$\omega_1(x_0)\omega_2(x_k) - \omega_1(x_k)\omega_2(x_0) \geqslant 0$$

for all $k = 1, \ldots, n$ because (ω_1, ω_2) is a positive regular pair on I; furthermore, at least one inequality is strict. Multiplying the kth inequality by the positive λ_k, and summing from 1 to n, we obtain that

$$\omega_1(x_0)\sum_{k=1}^{n} \lambda_k \omega_2(x_k) > \omega_2(x_0)\sum_{k=1}^{n} \lambda_k \omega_1(x_k).$$

But, due to the equations (6) and (7), both sides have the common value $\omega_1(x_0)\omega_2(x_0)$, which is contradiction. An analogous argument gives that the case $x_0 = \sup(I)$ also impossible, therefore $x_0 \in I^\circ$ follows.

Now, according to (iii), choose $\alpha, \beta \in \mathbb{R}$ so that the relations

$$\begin{aligned} \alpha\omega_1(x_0) + \beta\omega_2(x_0) &= f(x_0) \\ \alpha\omega_1(x) + \beta\omega_2(x) &\leqslant f(x) \qquad (x \in I) \end{aligned}$$

be valid. Then, substituting $x = x_k$ into the last inequality and applying the equations (6) and (7), we get that

$$\begin{aligned} \sum_{k=1}^{n} \lambda_k f(x_k) &\geqslant \sum_{k=1}^{n} \lambda_k \alpha\omega_1(x_k) + \sum_{k=1}^{n} \lambda_k \beta\omega_2(x_k) \\ &= \alpha\omega_1(x_0) + \beta\omega_2(x_0) = f(x_0), \end{aligned}$$

which was to be proved.

$(iv) \Rightarrow (v)$. Taking the particular case $n = 2$ in (iv), we get (v).

$(v) \Rightarrow (i)$. Choose $x < y < z \in I$ and define the constants λ_1, λ_2 by the formulas

$$\lambda_1 = \frac{\begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(z) \\ \omega_2(x) & \omega_2(z) \end{vmatrix}}, \qquad \lambda_2 = \frac{\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(z) \\ \omega_2(x) & \omega_2(z) \end{vmatrix}}.$$

Obviously, $\lambda_1, \lambda_2 \geqslant 0$ and, according to Cramer's rule, λ_1, λ_2 satisfy the system of linear equations

$$\begin{aligned} \lambda_1\omega_1(x) + \lambda_2\omega_1(z) &= \omega_1(y) \\ \lambda_1\omega_2(x) + \lambda_2\omega_2(z) &= \omega_2(y). \end{aligned}$$

Substituting λ_1 and λ_2 into the inequality of (v) and multiplying it by the positive base determinant, we get that

$$0 \leqslant f(x)\begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix} - f(y)\begin{vmatrix} \omega_1(x) & \omega_1(z) \\ \omega_2(x) & \omega_2(z) \end{vmatrix} + f(z)\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix};$$

or equivalently,

$$0 \leqslant \begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix},$$

which completes the proof. □

If the base functions ω_1 and ω_2 are twice differentiable with positive Wronski determinant, then Bonsall [5] showed that a twice differentiable function $f : I \to \mathbb{R}$ is (ω_1, ω_2)-convex if and only if

$$\begin{vmatrix} f(x) & f'(x) & f''(x) \\ \omega_1(x) & \omega_1'(x) & \omega_1''(x) \\ \omega_2(x) & \omega_2'(x) & \omega_2''(x) \end{vmatrix} \geqslant 0$$

holds for all $x \in I$. This result can also be deduced from Theorem 4.

Specializing Theorem 4 to the standard setting, we get the classical characterization of convexity (cf. [10, p. 152].

COROLLARY 1. *Let $I \subset \mathbb{R}$ be a nonempty interval. The following statements are equivalent:*

(i) $f : I \to \mathbb{R}$ *is convex;*

(ii) for all $x, y, z \in I$: $x < y < z$ *we have that*

$$\frac{f(y) - f(x)}{y - x} \leqslant \frac{f(z) - f(y)}{z - y};$$

(iii) for all $x_0 \in I^\circ$ *there exist* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\alpha + \beta x_0 = f(x_0), \qquad \alpha + \beta x \leqslant f(x) \qquad (x \in I);$$

(iv) for all $n \in \mathbb{N}$, $x_0, x_1, \ldots, x_n \in I$ *and* $\lambda_1, \ldots, \lambda_n \geqslant 0$ *satisfying the conditions*

$$\sum_{k=1}^{n} \lambda_k = 1, \qquad \sum_{k=1}^{n} \lambda_k x_k = x_0$$

we have that

$$f(x_0) \leqslant \sum_{k=1}^{n} \lambda_k f(x_k);$$

(v) for all $x_0, x_1, x_2 \in I$ *and* $\lambda_1, \lambda_2 \geqslant 0$ *satisfying the conditions*

$$\lambda_1 + \lambda_2 = 1, \qquad \lambda_1 x_1 + \lambda_2 x_2 = x_0.$$

we have that

$$f(x_0) \leqslant \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

Hint. Choose $\omega_1(x) := 1$ and $\omega_2(x) = x$. Then the requirements of Theorem 4 are fulfilled, and we can derive the statements of our corollary from that of Theorem 4. $\square$

THEOREM 5. *Let (ω_1, ω_2) be a positive regular pair on the nonempty open interval I such that ω_1 is positive. The function $f : I \to \mathbb{R}$ is (ω_1, ω_2)-convex if and only if the function $g : \omega_2/\omega_1(I) \to \mathbb{R}$ defined by*

$$g := \frac{f}{\omega_1} \circ \left(\frac{\omega_2}{\omega_1}\right)^{-1}$$

is convex in the standard sense.

Proof. In this case the function ω_2/ω_1 is continuous and strictly monotone increasing, according to Theorem 3. Therefore, the image of the interval I by the function ω_2/ω_1 is a nonempty open interval. Consider the identity

$$\begin{aligned} \begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} &= \omega_1(x)\omega_1(y)\omega_1(z) \begin{vmatrix} (f/\omega_1)(x) & (f/\omega_1)(y) & (f/\omega_1)(z) \\ 1 & 1 & 1 \\ (\omega_2/\omega_1)(x) & (\omega_2/\omega_1)(y) & (\omega_2/\omega_1)(z) \end{vmatrix} \\ &= \omega_1(x)\omega_1(y)\omega_1(z) \begin{vmatrix} g(u) & g(v) & g(w) \\ 1 & 1 & 1 \\ u & v & w \end{vmatrix}, \end{aligned}$$

where

$$u = (\omega_2/\omega_1)(x), \qquad v = (\omega_2/\omega_1)(y), \qquad w = (\omega_2/\omega_1)(z).$$

The positivity of ω_1 forces that both of the sides are simultaneously positive, negative or zero. That is, the function f is (ω_1, ω_2)-convex if and only if the function g is convex in the standard sense. $\square$

Observe that Theorem 5 yields also regularity properties for (ω_1, ω_2)-convex functions. Namely, we have the following important

COROLLARY 2. *Let (ω_1, ω_2) be a regular pair on the nonempty interval I. If the function $f : I \to \mathbb{R}$ is (ω_1, ω_2)-convex, then f is continuous on I°. If I is of the form $[a, b]$, then f is Riemann integrable on I.*

Hint. If the function f is (ω_1, ω_2)-convex on I, then, using the notation of Theorem 5, the function g is convex in the standard sense on $J := \omega_2/\omega_1(I)$. Therefore, by well known regularity properties of convex functions (cf. [10, p. 149] and [18]), g is continuous on J°. On the other hand, we have that

$$f = \omega_1 \cdot g \circ \left(\frac{\omega_2}{\omega_1}\right),$$

and the right hand side is continuous on I° whence the continuity of the function f follows.

For proving the integrability if I is the compact interval $[a, b]$, it is enough to show that f is bounded on $I = [a, b]$. Taking an arbitrary interior point x_0, the inequality in (iii) of Theorem 4 implies that f is bounded from below on I. Putting $x := a$ and $z := b$ into the definition inequality (1) of (ω_1, ω_2)-convexity, we get that f is also bounded by a linear combination of ω_1 and ω_2 from above on I. Hence f is a bounded function, indeed. $\square$

4. Hadamard-inequality for (ω_1, ω_2)-convex functions

The previous corollary enables us to formulate our main result.

THEOREM 6. *Let (ω_1, ω_2) be a positive regular pair on the interval $[a, b]$ such that ω_1 is positive on $]a, b[$. If $f : [a, b] \to \mathbb{R}$ is an (ω_1, ω_2)-convex function, then the inequalities*

$$cf(\xi) \leqslant \int_a^b f(x)dx \leqslant c_1 f(a) + c_2 f(b), \tag{10}$$

hold, where

$$\xi = \left(\frac{\omega_2}{\omega_1}\right)^{-1}\left(\frac{\int_a^b \omega_2(x)dx}{\int_a^b \omega_1(x)dx}\right), \qquad c = \frac{\int_a^b \omega_1(x)dx}{\omega_1(\xi)} \tag{11}$$

and

$$c_1 = \frac{\begin{vmatrix} \int_a^b \omega_1(x)dx & \omega_1(b) \\ \int_a^b \omega_2(x)dx & \omega_2(b) \end{vmatrix}}{\begin{vmatrix} \omega_1(a) & \omega_1(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} \omega_1(a) & \int_a^b \omega_1(x)dx \\ \omega_2(b) & \int_a^b \omega_2(x)dx \end{vmatrix}}{\begin{vmatrix} \omega_1(a) & \omega_1(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix}}. \tag{12}$$

In the proof we will use the consequences of Theorem 4, but a more direct approach can also be followed.

Proof. Define c and ξ by (11). We show that they are constructed such that the left hand side inequality of (10) is exact for $f = \omega_1$ and $f = \omega_2$, respectively. By the definition of (11), we have that

$$\frac{\int_a^b \omega_2(x)dx}{\int_a^b \omega_1(x)dx} = \frac{\omega_2(\xi)}{\omega_1(\xi)}.$$

Using the definition of c and this equation, we get the equations

$$\int_a^b \omega_1(x)dx = c\omega_1(\xi) \tag{13}$$

$$\int_a^b \omega_2(x)dx = c\omega_2(\xi), \tag{14}$$

which show that (10) holds for $f = \omega_i$.

Let $f : [a, b] \to \mathbb{R}$ be an arbitrary (ω_1, ω_2)-convex function.

According to (iii) of Theorem 4, there exist $\alpha, \beta \in \mathbb{R}$ such that the relations

$$\begin{aligned} \alpha\omega_1(\xi) + \beta\omega_2(\xi) &= f(\xi) \\ \alpha\omega_1(x) + \beta\omega_2(x) &\leqslant f(x) \end{aligned}$$

are satisfied for all $x \in [a,b]$. Therefore, due to the formulas (13) and (14), we get that

$$\begin{aligned} \int_a^b f(x)dx &\geqslant \alpha\int_a^b \omega_1(x)dx + \beta\int_a^b \omega_2(x)dx \\ &= c\alpha\omega_1(\xi) + c\beta\omega_2(\xi) = cf(\xi) \end{aligned}$$

which results the left hand side inequality of (10).

To prove the right hand side inequality of (10), observe first that c_1 and c_2 are constructed so that the left hand side inequality of (10) holds with equality for $f = \omega_1$ and $f = \omega_2$.

Substituting $x = a$ and $z = b$ into (1) and developing the determinant by its second column, we get the inequality

$$f(y)\begin{vmatrix} \omega_1(a) & \omega_1(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix} \leqslant \omega_1(y)\begin{vmatrix} f(a) & f(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix} - \omega_2(y)\begin{vmatrix} f(a) & f(b) \\ \omega_1(a) & \omega_1(b) \end{vmatrix}$$

for all $y \in [a,b]$. Thus, with the constants

$$\alpha := \frac{\begin{vmatrix} f(a) & f(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix}}{\begin{vmatrix} \omega_1(a) & \omega_1(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix}}, \qquad \beta := -\frac{\begin{vmatrix} f(a) & f(b) \\ \omega_1(a) & \omega_1(b) \end{vmatrix}}{\begin{vmatrix} \omega_1(a) & \omega_1(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix}},$$

we have that

$$\begin{gathered} f(a) = \alpha\omega_1(a) + \beta\omega_2(a), \qquad f(b) = \alpha\omega_1(b) + \beta\omega_2(b), \\ f(y) \leqslant \alpha\omega_1(y) + \beta\omega_2(y) \quad (y \in I). \end{gathered}$$

Therefore, after integrating the last inequality and using that the right hand side of (10) is exact for $f = \omega_i$, we get that

$$\begin{aligned} \int_a^b f(y)dy &\leqslant \alpha\int_a^b \omega_1(y)dy + \beta\int_a^b \omega_2(y)dy \\ &= c_1\big(\alpha\omega_1(a) + \beta\omega_2(a)\big) + c_2\big(\alpha\omega_1(b) + \beta\omega_2(b)\big) = c_1f(a) + c_2f(b). \end{aligned}$$

Thus the proof of the theorem is complete. □

Now, without the sake of completeness, we list up some Hadamard-type inequalities as applications of our last theorem.

COROLLARY 3. (Hadamard [8]). *If $f : [a,b] \to \mathbb{R}$ is a $(1,x)$-convex function (i.e., convex in the standard sense), then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}.$$

COROLLARY 4. *If* $f : [a, b] \to \mathbb{R}$ *is a* $(\cosh, \sinh)$*-convex function, then*

$$2\sinh\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)dx \leqslant \tanh\left(\frac{b-a}{2}\right)(f(a)+f(b)).$$

COROLLARY 5. *If* $f : [a, b] \subset]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}$ *is a* $(\cos, \sin)$*-convex function, then*

$$2\sin\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)dx \leqslant \tan\left(\frac{b-a}{2}\right)(f(a)+f(b)).$$

COROLLARY 6. *If* $f : [a, b] \to \mathbb{R}$ *is a* $(1, \exp)$*-convex function, then*

$$(b-a)f\left(\log\frac{\exp(b)-\exp(a)}{b-a}\right) \leqslant \int_a^b f(x)dx$$

$$\leqslant \left(\frac{(b-a)\exp(b)}{\exp(b)-\exp(a)} - 1\right) f(a) + \left(1 - \frac{(b-a)\exp(a)}{\exp(b)-\exp(a)}\right) f(b).$$

COROLLARY 7. *If* $f : [a, b] \subset]0, \infty[\to \mathbb{R}$ *is an* (x^p, x^q)*-convex function* *(*$(q-p)(p+1)(q+1) \neq 0$*), then*

$$\left(\frac{b^{p+1}-a^{p+1}}{p+1}\right)^q \left(\frac{q+1}{b^{q+1}-a^{q+1}}\right)^p f\left(\sqrt[q-p]{\frac{(p+1)(b^{q+1}-a^{q+1})}{(q+1)(b^{p+1}-a^{p+1})}}\right) \leqslant \int_a^b f(x)dx$$

$$\leqslant \frac{\frac{(b^{p+1}-a^{p+1})b^q}{p+1} - \frac{(b^{q+1}-a^{q+1})b^p}{q+1}}{a^p b^q - a^q b^p} f(a) + \frac{\frac{(b^{q+1}-a^{q+1})a^p}{q+1} - \frac{(b^{p+1}-a^{p+1})a^q}{q+1}}{a^p b^q - a^q b^p} f(b).$$

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