AN EXPONENTIAL INEQUALITY AND STRONG LIMIT THEOREMS FOR CONDITIONAL EXPECTATIONS

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Abstract

An exponential inequality for the tail of the conditional expectation of sums of centered independent random variables is obtained. This inequality is applied to prove analogues of the Law of the Iterated Logarithm and the Strong Law of Large Numbers for conditional expectations. As corollaries we obtain certain strong theorems for the generalized allocation scheme and for the nonuniformly distributed allocation scheme.

1 Introduction

The generalized allocation scheme is widely studied (see [5], [6], [11]). The scheme contains several special cases such as the usual allocation scheme (see [13], [12], [7]) and the random forests (see [5], [11]).

In the papers [1] and [2] we obtained Strong Laws of Large Numbers (SLLN) for random allocations and for random forests, respectively. The proofs of those SLLN's were based on fourth moment inequalities (the moment inequalities were obtained by long direct calculations). Then we presented a unified approach to the above mentioned SLLN's ([3]). In the present paper we continue the study started in [3].

Consider the probability measure $\mathbb{P}$ and the conditional probability measure $\mathbb{P}^A$ with respect to the fixed event A. Let $\mathbb{E}^A$ denote the expectation with respect to $\mathbb{P}^A$. Let μ be the average of N bounded independent random variables and let σ^2 be the average of the variances.

We prove the following exponential inequality for the tails

$$\mathbb{P}^A\left\{\frac{|\mu-\mathbb{E}^A\mu|}{\sqrt{N}}\geq\varepsilon\right\}\leq\frac{\sqrt{2}}{\mathbb{P}(A)}e^{-\frac{\varepsilon^2}{16\sigma^2}}(1+B),$$

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see Theorem 2.1, where B depends on ε and σ^2. The method of the proof of the exponential inequality is the following. We estimate the tail probability by the p-th moment of the sum of certain random variables which we majorize using Khintchine's inequality. Finally, we compute the minimum in p.

Using the above inequality, we prove some analogues of the Law of the Iterated Logarithm and the Strong Law of Large Numbers for the conditional expectations (Section 3).

Then the Law of the Iterated Logarithm and the Strong Laws of Large Numbers is applied to the generalized allocation scheme (Section 4). The random variables $\eta_1, \dots, \eta_N$ satisfy a generalized allocation scheme, if equation (4.1) is satisfied with $\xi_1, \dots, \xi_N$ having distribution (4.3).

Finally, in Section 5, we apply our general results to the allocations of balls into boxes having nonuniform probabilities.

Both in Section 4 and in Section 5 strong theorems are proved for $\mu_s = \mu_{snN}$ being the number of boxes containing s balls when n balls are placed into N boxes. We emphasize that the random variables $\mu_s = \mu_{snN}$ depend on the number of boxes (N) and the number of balls (n), so that $n, N \to \infty$. Therefore we have proved strong limit results for a two-indexed sequence of random variables with indices varying in a sector (Theorem 4.2, Corollary 4.1, Theorem 5.1, Corollary 5.1).

2 The main inequality

In this section we prove exponential inequalities for the conditional tail probability of a sum of independent bounded random variables (Theorem 2.1).

Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote the underlying probability space. Let $A \in \mathcal{A}$ be a fixed event such that $\mathbb{P}(A) > 0$. Recall that the conditional probability $\mathbb{P}^A$ is defined by the formula

$$\mathbb{P}^A(B) = \mathbb{P}(B \cap A)/\mathbb{P}(A), \quad B \in \mathcal{A}.$$

We will denote by $\mathbb{E}^A$ the expectation with respect to the probability measure $\mathbb{P}^A$.

Let τ_i, $1 \le i \le N$, be nondegenerate independent random variables with variances σ_i^2, $1 \le i \le N$, and such that $|\tau_i| \le C < \infty$, $1 \le i \le N$, almost surely. Let $\{\tau_i' : i = 1, \dots, N\}$ be an independent copy of $\{\tau_i : i = 1, \dots, N\}$. Denote s_i the variance of $(\tau_i - \tau_i')^2$. Introduce the notation

$$\sigma^2 = \frac{1}{N} \sum\nolimits_{i=1}^N \sigma_i^2, \qquad s = \frac{1}{N} \sum\nolimits_{i=1}^N s_i.$$

We will consider the random variable

$$\mu = \sum\nolimits_{i=1}^N \tau_i$$

with respect to the probability $\mathbb{P}^A$ and also with respect to the probability $\mathbb{P}$.

The main result of the paper is the following. Let Z denote a standard normal random variable.

Theorem 2.1. *(i) Let $\varepsilon \geq 4\sqrt{2}\sigma$. Then we have*

$$\mathbb{P}^A\left\{\frac{|\mu - \mathbb{E}^A\mu|}{\sqrt{N}} \geq \varepsilon\right\} \leq \frac{\sqrt{2}}{\mathbb{P}(A)} e^{-\frac{\varepsilon^2}{16\sigma^2}}(1+B) \tag{2.1}$$

where

$$B = B(N,\sigma) = \frac{s}{32}\left(\frac{\varepsilon^2}{8\sigma^4\sqrt{N}}\right)^2 f_2\left(\frac{2\varepsilon^2 C^2}{8\sigma^4\sqrt{N}}\right) + \mathrm{O}\left(\frac{8\sigma^2}{\varepsilon^2}\right) \tag{2.2}$$

and

$$f_2(x) = 2\mathbb{E}\left\{Z^2\exp(x|Z|)\right\} - 1. \tag{2.3}$$

(ii) Let $\varepsilon \geq 4\sqrt{2}\sigma$. Then we have

$$\mathbb{P}^A\left\{\frac{|\mu - \mathbb{E}^A\mu|}{\sqrt{N}} \geq \varepsilon\right\} \leq \frac{\sqrt{2}}{\mathbb{P}(A)} e^{-\frac{\varepsilon^2}{16\sigma^2}}(1+B_3) \tag{2.4}$$

where

$$B_3 = \frac{s}{32}\left(\frac{\varepsilon^2}{8\sigma^4\sqrt{N}}\right)^2 + \frac{sC^2}{24}\left(\frac{\varepsilon^2}{8\sigma^4\sqrt{N}}\right)^3 f_3\left(2C^2\frac{\varepsilon^2}{8\sigma^4\sqrt{N}}\right) + \mathrm{O}\left(\frac{8\sigma^2}{\varepsilon^2}\right)$$

and

$$f_3(x) = \mathbb{E}\left\{|Z|^3\exp(x|Z|)\right\}. \tag{2.5}$$

Remark 2.1.

$$f_2(x) = 4\left(e^{x^2/2}(x^2+1)\Phi(x) + x/\sqrt{2\pi}\right) - 1, \tag{2.6}$$

$$f_3(x) = 2e^{x^2/2}(x^3+3x)\Phi(x) + 2(x^2+2)/\sqrt{2\pi}, \tag{2.7}$$

where $\Phi(x) = \int_{-\infty}^{x}(1/\sqrt{2\pi})e^{-t^2/2}dt$ is the standard normal cumulative distribution function.

In the proof of Theorem 2.1 we will use the following lemmas.

Lemma 2.1. *Let $1 \leq p < \infty$ and let S be a random variable. Then we have*

$$\mathbb{E}^A|S - \mathbb{E}^A S|^p \leq 2^p\frac{\mathbb{E}|S - \mathbb{E}S|^p}{\mathbb{P}(A)}. \tag{2.8}$$

Inequality (2.8) was obtained in [3] with 4^p in place of 2^p. The above form of the inequality was proved in [9].

Our study will be based on the following Khintchine type inequality.

Lemma 2.2. *Let $1 \leq p < \infty$. Let ζ_i, $1 \leq i \leq N$, be independent centered random variables. Let $\{\zeta_i' : 1 \leq i \leq N\}$ be an independent copy of $\{\zeta_i : 1 \leq i \leq N\}$. Then we have*

$$\mathbb{E}\left|\sum_{i=1}^N \zeta_i\right|^p \leq (B_p)^p\,\mathbb{E}\left|\sum_{i=1}^N (\zeta_i - \zeta_i')^2\right|^{p/2}, \tag{2.9}$$

where $B_p = 1$ if $0 < p < 2$ and $B_p = (\mathbb{E}|Z|^p)^{1/p}$ if $2 \le p < \infty$ with Z being a Gaussian random variable with mean 0 and variance 1.

If the ζ_i's are symmetric, then we have

$$\mathbb{E}\left|\sum_{i=1}^N \zeta_i\right|^p \le (B_p)^p\,\mathbb{E}\left|\sum_{i=1}^N (\zeta_i)^2\right|^{p/2}. \tag{2.10}$$

Proof. First recall that Khintchine's original inequality ([8]) has the following version (see [4]). Let $0 < p < \infty$ be fixed and let r_i, $i \in \mathbb{N}$, be the Rademacher functions. Let $\mathbb{E}_r$ denote the expectation with respect to $\{r_i\}$. Then, for any $c_i \in \mathbb{R}$, $1 \le i \le N$, we have

$$\left(\mathbb{E}_r\left|\sum_{i=1}^N c_i r_i\right|^p\right)^{1/p} \le B_p\left(\sum_{i=1}^N (c_i)^2\right)^{1/2}, \tag{2.11}$$

where $B_p = 1$, if $0 < p < 2$ and $B_p = (\mathbb{E}|Z|^p)^{1/p}$, if $2 \le p < \infty$.

Now we can assume that $\{r_i : 1 \le i \le N\}$ is independent of $\{\zeta_i' : 1 \le i \le N\}$ and $\{\zeta_i : 1 \le i \le N\}$. Therefore $(\zeta_i - \zeta_i')r_i$ has the same distribution as $\zeta_i - \zeta_i'$. By the Jensen inequality, the independence, and (2.11), we have

$$\mathbb{E}\left|\sum_{i=1}^N (\zeta_i - 0)\right|^p \le \mathbb{E}\left|\sum_{i=1}^N (\zeta_i - \zeta_i')\right|^p =$$

$$= \mathbb{E}\mathbb{E}_r\left|\sum_{i=1}^N (\zeta_i - \zeta_i')r_i\right|^p \le (B_p)^p\mathbb{E}\left|\sum_{i=1}^N (\zeta_i - \zeta_i')^2\right|^{p/2}. \quad \square$$

Lemma 2.3. *Let ζ_i, $1 \le i \le N$, be independent random variables with expectations a_i and variances v_i^2, and such that $|\zeta_i| \le C < \infty$ almost surely, $1 \le i \le N$. Let*

$$a' = \frac{1}{N}\sum_{i=1}^N a_i, \quad a = |a'|, \quad v^2 = \frac{1}{N}\sum_{i=1}^N v_i^2.$$

Assume that $a \ne 0$.

(i) Let $p \ge 2$. Then we have

$$a^p \le \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N \zeta_i\right|^p \le a^p(1 + B) \tag{2.12}$$

where

$$B = \frac{v^2}{2}\frac{p(p-1)}{a^2 N} f_2\left(\frac{p-2}{a\sqrt{N}}2C\right)$$

and $f_2(x)$ is given by (2.3).

(ii) Let $p \ge 3$. Then we have

$$a^p\left(1 + \frac{v^2}{2}\frac{p(p-1)}{(a\sqrt{N})^2} - \frac{v^2}{3}\frac{p(p-1)(p-2)}{(a\sqrt{N})^3}2C f_3\left(\frac{p-3}{a\sqrt{N}}2C\right)\right) \le \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N \zeta_i\right|^p \le$$

$$\le a^p\left(1 + \frac{v^2}{2}\frac{p(p-1)}{(a\sqrt{N})^2} + \frac{v^2}{3}\frac{p(p-1)(p-2)}{(a\sqrt{N})^3}2C f_3\left(\frac{p-3}{a\sqrt{N}}2C\right)\right) \tag{2.13}$$

where $f_3(x)$ is given by (2.5).

Proof. (i) The left hand side of (2.12) follows from Jensen's inequality. Let $\{\zeta_i' : 1 \le i \le N\}$ be an independent copy of $\{\zeta_i : 1 \le i \le N\}$. Since $p \ge 2$, the function $g(x) = |a' + x|^p$ has a continuous second derivative. By Taylor's expansion, it holds that

$$g(x) = a^p + \operatorname{sgn}(a')\frac{pa^{p-1}}{1!}x + \frac{p(p-1)|a' + \theta x|^{p-2}}{2!}x^2,$$

where $\theta \in (-1, 1)$. Therefore, with $x = \sum_{i=1}^N (\zeta_i - a_i)/N$, we obtain

$$\mathbb{E}\left|\frac{1}{N}\sum\nolimits_{i=1}^N \zeta_i\right|^p = a^p + \frac{p(p-1)}{2}\mathbb{E}\left\{\left|a' + \frac{\theta}{N}\sum\nolimits_{i=1}^N(\zeta_i - a_i)\right|^{p-2}\left(\frac{1}{N}\sum\nolimits_{i=1}^N(\zeta_i - a_i)\right)^2\right\} =$$

$$= a^p(1 + B'),$$

where $\theta = \theta(\omega)$ with $-1 \le \theta \le 1$. Since $1 + x \le e^x$, $x \in \mathbb{R}$, moreover

$$\mathbb{E}\left|\frac{1}{N}\sum\nolimits_{i=1}^N(\zeta_i - \zeta_i')^2\right|^{\frac{k+2}{2}} \le \mathbb{E}\left(\frac{1}{N}\sum\nolimits_{i=1}^N(\zeta_i - \zeta_i')^2\right)(2C)^k = 2v^2(2C)^k, \tag{2.14}$$

therefore, using Khintchine's inequality (2.9), we obtain

$$B' \le \frac{p(p-1)}{2}\mathbb{E}\left\{\left(1 + \left|\frac{\sum_{i=1}^N(\zeta_i - a_i)}{Na}\right|\right)^{p-2}\left(\frac{\sum_{i=1}^N(\zeta_i - a_i)}{Na}\right)^2\right\} \le$$

$$\le \frac{p(p-1)}{2}\mathbb{E}\left\{\exp\left((p-2)\left|\frac{\sum_{i=1}^N(\zeta_i - a_i)}{Na}\right|\right)\left(\frac{\sum_{i=1}^N(\zeta_i - a_i)}{Na}\right)^2\right\} =$$

$$= \frac{p(p-1)}{2}\sum_{k=0}^\infty \frac{1}{k!}(p-2)^k\mathbb{E}\left|\frac{\sum_{i=1}^N(\zeta_i - a_i)}{Na}\right|^{k+2} \le$$

$$\le \frac{1}{2}\frac{p(p-1)}{a^2N}\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{p-2}{a\sqrt{N}}\right)^k\mathbb{E}\left|\frac{\sum_{i=1}^N(\zeta_i - a_i)}{\sqrt{N}}\right|^{k+2} \le$$

$$\le \frac{1}{2}\frac{p(p-1)}{a^2N}\left(v^2 + \sum_{k=1}^\infty \frac{1}{k!}\left(\frac{p-2}{a\sqrt{N}}\right)^k\mathbb{E}|Z|^{k+2}\mathbb{E}\left|\frac{\sum_{i=1}^N(\zeta_i - \zeta_i')^2}{N}\right|^{\frac{k+2}{2}}\right) \le$$

$$\le \frac{v^2}{2}\frac{p(p-1)}{a^2N}\left(1 + 2\sum_{k=1}^\infty \frac{1}{k!}\left(\frac{p-2}{a\sqrt{N}}\right)^k\mathbb{E}|Z|^{k+2}(2C)^k\right) =$$

$$= \frac{v^2}{2}\frac{p(p-1)}{a^2N}\left(1 + 2\left(\mathbb{E}\left(Z^2\exp\left(\frac{p-2}{a\sqrt{N}}2C|Z|\right)\right) - 1\right)\right) = \frac{v^2}{2}\frac{p(p-1)}{a^2N}f_2\left(\frac{p-2}{a\sqrt{N}}2C\right) = B.$$

The proof of (i) is complete.

(ii) Now $p \geq 3$. Since $g(x) = |a' + x|^p$ has a continuous third derivative, by Taylor's expansion, we have

$$g(x) = a^p + \frac{\operatorname{sgn}(a')pa^{p-1}}{1!}x + \frac{p(p-1)a^{p-2}}{2!}x^2 + \frac{\operatorname{sgn}(a'+\theta x)p(p-1)(p-2)|a'+\theta x|^{p-3}}{3!}x^3$$

where $\theta \in (-1,1)$. Therefore, with $x = \sum_{i=1}^N (\zeta_i - a_i)/N$, we obtain

$$\mathbb{E}\left|\frac{1}{N}\sum\nolimits_{i=1}^N \zeta_i\right|^p = a^p + \frac{p(p-1)}{2N}a^{p-2}v^2 +$$

$$+\frac{p(p-1)(p-2)}{6}\mathbb{E}\left\{S\left|a' + \theta\frac{1}{N}\sum\nolimits_{i=1}^N(\zeta_i - a_i)\right|^{p-3}\left(\frac{1}{N}\sum\nolimits_{i=1}^N(\zeta_i - a_i)\right)^3\right\} =$$

$$= a^p\left(1 + \frac{p(p-1)}{2(a\sqrt{N})^2}v^2 + B'\right)$$

where $\theta = \theta(\omega)$ with $-1 \leq \theta \leq 1$ and $S = \operatorname{sgn}\left\{a' + \theta\sum_{i=1}^N(\zeta_i - a_i)/N\right\}$. Since $1 + x \leq e^x$, $x \in \mathbb{R}$, by (2.14) and Khintchine's inequality (2.9), we obtain

$$|B'| \leq \frac{p(p-1)(p-2)}{6}\mathbb{E}\left\{\left(1 + \left|\frac{1}{Na}\sum\nolimits_{i=1}^N(\zeta_i - a_i)\right|\right)^{p-3}\left|\frac{1}{Na}\sum\nolimits_{i=1}^N(\zeta_i - a_i)\right|^3\right\} \leq$$

$$\leq \frac{p(p-1)(p-2)}{6}\mathbb{E}\left\{\exp\left((p-3)\left|\frac{1}{Na}\sum\nolimits_{i=1}^N(\zeta_i - a_i)\right|\right)\left|\frac{1}{Na}\sum\nolimits_{i=1}^N(\zeta_i - a_i)\right|^3\right\} \leq$$

$$\leq \frac{p(p-1)(p-2)}{6}\sum_{k=0}^\infty \frac{1}{k!}(p-3)^k\mathbb{E}\left|\frac{\sum_{i=1}^N(\zeta_i - a_i)}{Na}\right|^{k+3} \leq$$

$$\leq \frac{p(p-1)(p-2)}{6(a\sqrt{N})^3}\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{p-3}{a\sqrt{N}}\right)^k\mathbb{E}\left|\frac{\sum_{i=1}^N(\zeta_i - a_i)}{\sqrt{N}}\right|^{k+3} \leq$$

$$\leq \frac{p(p-1)(p-2)}{6(a\sqrt{N})^3}\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{p-3}{a\sqrt{N}}\right)^k\mathbf{E}|Z|^{k+3}\mathbb{E}\left|\frac{\sum_{i=1}^N(\zeta_i - \zeta_i')^2}{N}\right|^{\frac{k+3}{2}} \leq$$

$$\leq \frac{p(p-1)(p-2)}{6(a\sqrt{N})^3}2v^2 2C\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{p-3}{a\sqrt{N}}\right)^k\mathbb{E}|Z|^{k+3}(2C)^k =$$

$$\leq \frac{p(p-1)(p-2)}{6(a\sqrt{N})^3}2v^2 2C\mathbb{E}\left(|Z|^3\exp\left(\frac{p-3}{a\sqrt{N}}2C|Z|\right)\right) =$$

$$= \frac{v^2}{3}\frac{p(p-1)(p-2)}{(a\sqrt{N})^3}2Cf_3\left(\frac{p-3}{a\sqrt{N}}2C\right). \tag{2.15}$$

We have seen that

$$a^p\left(1+\frac{p(p-1)}{2(a\sqrt{N})^2}v^2-|B'|\right)\le \mathbb{E}\left|\frac{1}{N}\sum\nolimits_{i=1}^N\zeta_i\right|^p\le a^p\left(1+\frac{p(p-1)}{2(a\sqrt{N})^2}v^2+|B'|\right).$$

Therefore (2.15) implies (2.13). The proof of (ii) is complete. □

Proof of Theorem 2.1. It is known that $\mathbb{E}|Z|^p=(2^{\frac{p}{2}}/\sqrt{\pi})\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)$. Using [10], we have

$$\mathbb{E}|Z|^p=\frac{2^{\frac{p}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)=\frac{2^{\frac{p}{2}}}{\sqrt{\pi}}e^{-\frac{p}{2}-\frac{1}{2}}\left(\frac{p}{2}+\frac{1}{2}\right)^{\frac{p}{2}+\frac{1}{2}}\sqrt{\frac{2\pi}{\left(\frac{p}{2}+\frac{1}{2}\right)}}\left(1+\mathrm{O}\left(\frac{1}{p}\right)\right)=$$

$$=\sqrt{2}\,2^{\frac{p}{2}}e^{-\frac{p}{2}-\frac{1}{2}}\left(\frac{p}{2}+\frac{1}{2}\right)^{\frac{p}{2}}\left(1+\mathrm{O}\left(\frac{1}{p}\right)\right)=\sqrt{2}e^{-\frac{p}{2}}e^{-\frac{1}{2}}\,(p+1)^{\frac{p}{2}}\left(1+\mathrm{O}\left(\frac{1}{p}\right)\right)=$$

$$=\sqrt{2}e^{-\frac{p}{2}}e^{-\frac{1}{2}}\left(\frac{p+1}{p}\right)^{\frac{p}{2}}p^{\frac{p}{2}}\left(1+\mathrm{O}\left(\frac{1}{p}\right)\right)\le\sqrt{2}e^{-\frac{p}{2}}p^{\frac{p}{2}}\left(1+\mathrm{O}\left(\frac{1}{p}\right)\right).$$

Therefore, using Markov's inequality, Lemma 2.1, Khintchine's inequality (2.9) for $p/2\ge 2$, and (2.12), we obtain

$$\mathbb{P}^A\left\{\frac{|\mu-\mathbb{E}^A\mu|}{\sqrt{N}}\ge\varepsilon\right\}\le\frac{1}{\varepsilon^p}\mathbb{E}^A\left(\frac{|\mu-\mathbb{E}^A\mu|}{\sqrt{N}}\right)^p\le$$

$$\le\frac{2^p}{\varepsilon^p\mathbb{P}(A)}\mathbb{E}\left(\frac{|\mu-\mathbb{E}\mu|}{\sqrt{N}}\right)^p\le\sqrt{2}\frac{2^pe^{-\frac{p}{2}}p^{p/2}}{\varepsilon^p\mathbb{P}(A)}\mathbb{E}\left|\frac{\sum_{i=1}^N(\tau_i-\tau_i')^2}{N}\right|^{p/2}\left(1+\mathrm{O}\left(\frac{1}{p}\right)\right)\le \tag{2.16}$$

$$\le\sqrt{2}\frac{2^pe^{-\frac{p}{2}}p^{p/2}}{\varepsilon^p\mathbb{P}(A)}(2\sigma^2)^{p/2}\left(1+\frac{s}{2}\left(\frac{p}{4\sigma^2\sqrt{N}}\right)^2f_2\left(\frac{p}{4\sigma^2\sqrt{N}}8C^2\right)\right)\left(1+\mathrm{O}\left(\frac{1}{p}\right)\right). \tag{2.17}$$

For $p=\frac{\varepsilon^2}{8\sigma^2}$ from (2.17) we obtain (2.1), i.e.

$$\mathbb{P}^A\left\{\frac{|\mu-\mathbb{E}^A\mu|}{\sqrt{N}}\ge\varepsilon\right\}\le e^{-\frac{\varepsilon^2}{16\sigma^2}}\frac{\sqrt{2}}{\mathbb{P}(A)}\left(1+\frac{s}{8}\left(\frac{\varepsilon^2}{16\sigma^4\sqrt{N}}\right)^2f_2\left(\frac{\varepsilon^2C^2}{4\sigma^4\sqrt{N}}\right)\right)\left(1+\mathrm{O}\left(\frac{8\sigma^2}{\varepsilon^2}\right)\right).$$

To prove (2.4), apply (2.13) for the right hand side of (2.16). Then we obtain

$$\mathbb{P}^A\left\{\frac{|\mu-\mathbb{E}^A\mu|}{\sqrt{N}}\ge\varepsilon\right\}\le\frac{\sqrt{2}}{\mathbb{P}(A)}e^{-p/2}\left(\frac{8p\sigma^2}{\varepsilon^2}\right)^{p/2}\left(1+\frac{p^2s}{8(2\sigma^2\sqrt{N})^2}+\frac{p^3sC^2}{3(2\sigma^2\sqrt{N})^3}f_3\left(\frac{p2C^2}{\sigma^2\sqrt{N}}\right)\right).$$

Then put $p=\frac{\varepsilon^2}{8\sigma^2}$. The proof is complete. □

3 Strong limit theorems

In this section we apply the exponential inequalities of Section 2 to describe the almost sure asymptotic behaviour of sums of certain non-independent random variables.

Assume that all random variables are defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The conditional distribution of the stochastic element η with respect to the fixed event A is defined as $\mathbb{P}(\eta \in B \,|\, A)$, $B \in \mathcal{A}$. We denote it by $(\eta \,|\, A)$.

In this section we study the following model. Consider the strictly increasing sequence of integer numbers N_k, $k \in \mathbb{N}$, and the sequence of the events $A = A(k) \in \mathcal{A}$, $k \in \mathbb{N}$, having property

$$\mathbb{P}(A(k)) \geq C''/N_k^{\beta} \tag{3.1}$$

for some $\beta > 0$, $C'' > 0$. For each fixed k, let ξ_{ki}, $1 \leq i \leq N_k$, be independent random variables with variances σ_{ki}^2, and such that $|\xi_{ki}| \leq C_0 < \infty$ almost surely, $1 \leq i \leq N_k$. Suppose that the family $\{\eta_{ki} \,:\, 1 \leq i \leq N_k\}$ has the same distribution as the conditional distribution $\{(\xi_{ki} \,|\, A(k)) \,:\, 1 \leq i \leq N_k\}$. We emphasize that the random variables $\eta_{k1}, \ldots, \eta_{kN_k}$ can be dependent. Let

$$\sigma_k^2 = \frac{1}{N_k} \sum\nolimits_{i=1}^{N_k} \sigma_{ki}^2.$$

Assume that there exists $0 < C_1 < \infty$ such that

$$\frac{\ln(N_k)}{\sigma_k^2 \sqrt{N_k}} \leq C_1 \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.2}$$

Consider the sums of random variables

$$S_k = \sum\nolimits_{i=1}^{N_k} \eta_{ki}.$$

Now we will prove the following analogue of the Law of the Iterated Logarithm.

Theorem 3.1. *Suppose that* (3.1) *and* (3.2) *are valid for the above defined model. Then we have*

$$\limsup_{k\to\infty} \frac{|S_k - \mathbb{E}S_k|}{\sqrt{N_k \ln(N_k)}\, \sigma_k} \leq 4\sqrt{\beta+1} \quad \textit{almost surely.} \tag{3.3}$$

Proof. Let $S_k' = \sum_{i=1}^{N_k} \xi_{ki}$, $k = 1, 2, \ldots$. By Theorem 2.1 we have

$$\sum_{k=1}^{\infty} \mathbb{P}\left\{ \frac{|S_k - \mathbb{E}S_k|}{\sqrt{N_k \ln(N_k)}\sigma_k} \geq t \right\} = \sum_{k=1}^{\infty} \mathbb{P}\left\{ \frac{|S_k - \mathbb{E}S_k|}{\sqrt{N_k}} \geq \sigma_k \sqrt{\ln(N_k)}t \right\} =$$

$$= \sum_{k=1}^{\infty} \mathbb{P}^{A(k)}\left\{ \frac{|S_k' - \mathbb{E}^{A(k)} S_k'|}{\sqrt{N_k}} \geq \sigma_k \sqrt{\ln(N_k)}t \right\} \leq \sum_{k=1}^{\infty} \frac{\sqrt{2}}{P(A(k))} e^{-\frac{\ln(N_k)t^2\sigma_k^2}{16\sigma_k^2}} (1 + B(N_k, \sigma_k))$$

where $B(N,\sigma)$ is defined by(2.2). Therefore

$$B(N_k,\sigma_k) = \frac{s}{8}\left(\frac{t^2\ln(N_k)}{16\sigma_k^2\sqrt{N_k}}\right) f_2\left(\frac{t^2\ln(N_k)C^2}{4\sigma_k^2\sqrt{N_k}}\right) \mathrm{O}\left(\frac{8}{t^2\ln(N_k)}\right).$$

As $|\xi_{ik}| \le C_0$, therefore s is bounded, so, by (3.2), $\{B(N_k,\sigma_k)\}$ is a bounded sequence. Now, by (3.1),

$$\sum_{k=1}^{\infty}\mathbb{P}\left\{\frac{|S_k-\mathbb{E}S_k|}{\sqrt{N_k\ln(N_k)}\sigma_k} \ge t\right\} \le K\sum_{k=1}^{\infty} N_k^{\beta-t^2/16}.$$

It is finite when $t > 4\sqrt{\beta+1}$. Consequently, by the Borel-Cantelli lemma, we have $\limsup_{k\to\infty}\frac{|S_k-\mathbb{E}S_k|}{\sqrt{N_k\ln(N_k)}\sigma_{0k}} \le t$ almost surely. This implies (3.3). □

As corollaries of Theorem 3.1, we obtain the following laws of large numbers.

Corollary 3.1. *Suppose that* (3.1) *and* (3.2) *are valid for the model defined in this section. Let d_k, $k\in\mathbb{N}$, be a sequence such that $\sigma_k\sqrt{N_k\ln(N_k)}/d_k \to 0$, as $k\to\infty$. Then we have*

$$\lim_{k\to\infty}\frac{S_k-\mathbb{E}S_k}{d_k} = 0 \quad \textit{almost surely.}$$

Proof. Observe that

$$\frac{S_k-\mathbb{E}S_k}{d_k} = \frac{\sigma_k\sqrt{N_k\ln(N_k)}}{d_k}\frac{S_k-\mathbb{E}S_k}{\sigma_k\sqrt{N_k\ln(N_k)}}.$$

By Theorem 3.1, $\frac{S_k-\mathbb{E}S_k}{\sigma_k\sqrt{N_k\ln(N_k)}}$ is an almost surely bounded sequence. This implies the corollary. □

Corollary 3.2. *Suppose that* (3.1) *and* (3.2) *are valid for the model defined in this section. Let d_k, $k\in\mathbb{N}$, be a sequence such that $\sigma_k\sqrt{N_k\ln(N_k)}/d_k \to 0$ and $\mathbb{E}S_k/d_k \to a$, as $k\to\infty$. Then we have*

$$\lim_{k\to\infty}\frac{S_k}{d_k} = a \quad \textit{almost surely.}$$

Proof. Since

$$\frac{S_k}{d_k} = \frac{S_k-\mathbb{E}S_k}{d_k} + \frac{\mathbb{E}S_k}{d_k},$$

the result follows from Corollary 3.1. □

4 Application to the generalized allocation scheme

In the generalized scheme of allocations of particles into cells, the distribution of the cell contents is represented as the conditional distribution of independent random variables under

the condition that their sum is fixed, see [5], [11]. We can describe the generalized allocation scheme as follows. Let $\eta_1, \dots, \eta_N$ be nonnegative integer-valued random variables. They are considered as certain numerical characteristics of the combinatorial structure of n elements consisting of N components such that $\eta_1 + \dots + \eta_N = n$. If there exist independent random variables $\xi_1, \dots, \xi_N$ such that the joint distribution of $\eta_1, \dots, \eta_N$ admits the representation

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \mathbb{P}\{\xi_1 = k_1, \dots, \xi_N = k_N \mid \xi_1 + \dots + \xi_N = n\}, \tag{4.1}$$

where $k_1, \dots, k_N$ are arbitrary nonnegative integers such that $k_1 + k_2 + \dots + k_N = n$, then we say that $\eta_1, \dots, \eta_N$ obey a generalized allocation scheme with parameters n and N, and independent random variables $\xi_1, \dots, \xi_N$. Throughout this section we assume that the random variables $\eta_1, \dots, \eta_N$ and $\xi_1, \dots, \xi_N$ satisfy (4.1). When $\xi_1, \dots, \xi_N$ are i.i.d. Poissonian random variables, then the vector $\eta_1, \dots, \eta_N$ describes the usual allocation of n balls into N boxes. Details of the theory of generalized allocation schemes can be found in [5], [11].

In view of independence of the random variables $\xi_1, \dots, \xi_N$, the study of several questions of the generalized allocation scheme can be reduced to problems of sums of independent random variables.

Let the random variables $\xi_1, \dots, \xi_N$ be identically distributed. Usually (see, e.g., [6]) the random variables $\xi_1, \dots, \xi_N$ are distributed as follows: $q_k = \mathbb{P}\{\xi_1 = k\} = (b_k\theta^k)/(k!B(\theta))$ where $b_0, b_1, \dots$ is a certain sequence of non-negative numbers, and $B(\theta) = \sum_{k=0}^{\infty} b_k\theta^k/k!$.

Therefore, in this section, we consider a sequence of non-negative numbers $b_0, b_1, \dots$ with $b_0 > 0, b_1 > 0$ and assume that the convergence radius R of the series

$$B(\theta) = \sum\nolimits_{k=0}^{\infty} \frac{b_k\theta^k}{k!} \tag{4.2}$$

is positive. Let us introduce the integer-valued random variable $\xi = \xi(\theta)$ (where $\theta > 0$) with distribution

$$q_k = q_k(\theta) = \mathbb{P}\{\xi = k\} = \frac{b_k\theta^k}{k!B(\theta)}, \quad k = 0, 1, 2, \dots. \tag{4.3}$$

By [6], one has

$$m = m(\theta) = \mathbb{E}\xi = \frac{\theta B'(\theta)}{B(\theta)}$$

and

$$\sigma^2 = \sigma^2(\theta) = \mathbb{D}^2\xi = \frac{\theta^2 B''(\theta)}{B(\theta)} + \frac{\theta B'(\theta)}{B(\theta)} - \frac{\theta^2 (B'(\theta))^2}{(B(\theta))^2}. \tag{4.4}$$

The last equality implies that

$$\sigma^2(\theta) = \theta m'(\theta). \tag{4.5}$$

Let $0 < \theta' < \theta'' < R$. If $\sigma^2(\theta) = 0$ for some $\theta \in [\theta', \theta'']$, then the random variable $\xi(\theta)$ is a constant. Since $b_0 > 0, b_1 > 0$, the random variable $\xi(\theta)$ is not a constant. Therefore $\sigma^2(\theta)$, $\theta \in [\theta', \theta'']$ is a positive continuous function. Consequently,

$$0 < C_1 = \inf_{\theta \in [\theta', \theta'']} \sigma^2(\theta) \le \sup_{\theta \in [\theta', \theta'']} \sigma^2(\theta) = C_2 < \infty. \tag{4.6}$$

By (4.5) and (4.6), we have

$$0 < \frac{C_1}{\theta''} = \inf_{\theta\in[\theta',\theta'']} m'(\theta) \le \sup_{\theta\in[\theta',\theta'']} m'(\theta) = \frac{C_2}{\theta'} < \infty.$$

Since $m'(\theta) > 0$, then $m(\theta)$, $\theta \in [\theta', \theta'']$, is a positive, continuous, strictly increasing function. We will denote by m^{-1} the inverse function of m.

We see that the random variable $\xi(\theta)$ has all moments, if $\theta < R$.

Throughout this section, let $\xi_1 = \xi_1(\theta), \dots, \xi_N = \xi_N(\theta)$ be independent copies of $\xi(\theta)$ where $\xi = \xi(\theta)$ has distribution (4.3). Let $\eta_1 = \eta_1(\theta), \dots, \eta_N = \eta_N(\theta)$ satisfy (4.1) with $\xi_1 = \xi_1(\theta), \dots, \xi_N = \xi_N(\theta)$. So $\eta_1, \dots, \eta_N$ also depend on θ. Introduce notation $\alpha = n/N$. Let $\theta_\alpha = m^{-1}(\alpha)$. So in what follows the parameter $\theta = \theta_\alpha = m^{-1}(\alpha)$ will depend on $\alpha = n/N$.

Consider the event

$$A = A_N(n) = \{\omega \in \Omega : \xi_1(\theta_\alpha)(\omega) + \xi_2(\theta_\alpha)(\omega) + \dots + \xi_N(\theta_\alpha)(\omega) = n\}$$

where $\theta_\alpha = m^{-1}(\alpha)$. Let $P_N(n) = \mathbb{P}(A_N(n))$.

Lemma 4.1. *(See [3].) Let $0 < \alpha' < \alpha''$ be such that $m^{-1}(\alpha'') < R$. Let $\alpha = n/N$. Then there exists $N_0 \in \mathbb{N}$ with the following property: if $n, N \in \mathbb{N}$ are such that $N > N_0$ and $\alpha' \le \alpha \le \alpha''$, then we have*

$$P_N(n) > \frac{1}{4\sigma(\theta_\alpha)\sqrt{N}}. \tag{4.7}$$

Let $\sigma^2_{t\alpha} = q_t(\theta_\alpha)(1 - q_t(\theta_\alpha))$. We see that $\sigma^2_{t\alpha}$ is the variance of the indicator $\mathbb{I}_{\{\xi(\theta_\alpha)=t\}}$.

First consider a sequence of generalized allocations of n_k balls into N_k boxes such that $N_k < N_{k+1}$, $k \in \mathbb{N}$. Let $\alpha_k = n_k/N_k$, $\theta_k = m^{-1}(\alpha_k)$, $\sigma^2_{tk} = \sigma^2_{t\alpha_k}$ and

$$\mu_{tk} = \sum\nolimits_{i=1}^{N_k} \mathbb{I}_{\{\eta_i=t\}}.$$

μ_{tk} can be considered as the number of boxes containing t balls. We remark that if $b_t = 0$, then $\mu_{tk} = 0$. So we can concentrate on the case $b_t > 0$. Now we will prove two analogues of the Law of the Iterated Logarithm for μ_{tk}.

Theorem 4.1. *Let $0 < \alpha' < \alpha''$ be such that $m^{-1}(\alpha'') < R$. Suppose that $0 < \alpha' \le \alpha_k \le \alpha''$ for all $k \in \mathbb{N}$. Let t be fixed and assume that $b_t > 0$. Then we have*

$$\limsup_{k\to\infty} \frac{|\mu_{tk} - \mathbb{E}\mu_{tk}|}{\sqrt{N_k \ln(N_k)}\sigma_{tk}} \le 4\sqrt{3/2} \quad \textit{almost surely.} \tag{4.8}$$

Proof. By Lemma 4.1 and relation (4.6), we have

$$P(A_{N_k}(n_k)) > \frac{1}{4\sigma(\theta_k)\sqrt{N_k}} \ge \frac{C''}{N_k^{1/2}}.$$

As $C'' > 0$, condition (3.1) is valid with $\beta = \frac{1}{2}$. We have $\sigma_{tk}^2 = \sigma_{t\alpha_k}^2 = q_t(\theta_{\alpha_k})(1 - q_t(\theta_{\alpha_k}))$. Here

$$q_t(\theta_{\alpha_k}) = \frac{b_t \theta_{\alpha_k}^t}{t! B(\theta_{\alpha_k})} \geq \frac{b_t (\theta')^t}{t! B(\theta'')} > 0$$

as $b_t > 0$. As $b_0 > 0$, $b_1 > 0$, we have $1 - q_t(\theta_{\alpha_k}) > c > 0$. So $\sigma_{tk}^2 \geq c_2 > 0$ for each k, i.e. (3.2) is satisfied. So Theorem 4.1 follows from Theorem 3.1. □

Let μ_t be the number of the random variables $\eta_1, \ldots, \eta_N$ being equal to t, $t = 0, 1, \ldots, n$. That is

$$\mu_t = \mu_{tnN} = \sum\nolimits_{i=1}^{N} \mathbb{I}_{\{\eta_i = t\}}. \tag{4.9}$$

Theorem 4.2. *Let $0 < \alpha' < \alpha''$ be such that $m^{-1}(\alpha'') < R$. Let t be fixed and assume that $b_t > 0$. Then, for μ_{tnN} defined in (4.9), we have*

$$\limsup_{n,N\to\infty, \alpha'<\alpha<\alpha''} \frac{|\mu_{tnN} - \mathbb{E}\mu_{tnN}|}{\sqrt{N \ln(N)} \sigma_{t\alpha}} \leq 2\sqrt{10} \qquad \textit{almost surely.} \tag{4.10}$$

Proof. First we remark that, for any two index sequence $l_{n,N}$, $\limsup_{n,N\to\infty,\alpha'<\alpha<\alpha''} l_{n,N}$ exists and it is unique because of $0 < \alpha' < \alpha'' < \infty$. As $b_t > 0$, and $b_0 > 0$, $b_1 > 0$, we have $\sigma_{t\alpha}^2 \geq c_1 > 0$. Therefore for B from (2.2) we have $B = B(N, \sigma_\alpha) \leq K < \infty$ as $N, n \to \infty$ such that $\alpha' < \alpha < \alpha''$.

Let $z > 2\sqrt{10}$. Then $\frac{z^2}{16} - \frac{1}{2} - 1 > 1$. Therefore, by (2.1), we obtain

$$\sum_{N=N_0+1}^{\infty} \sum_{N\alpha'<n<\alpha''N} \mathbb{P}\left\{ \frac{|\mu_{tnN} - \mathbb{E}\mu_{tnN}|}{\sqrt{N \ln(N)} \sigma_{t\alpha}} \geq z \right\} =$$

$$= \sum_{N=N_0+1}^{\infty} \sum_{N\alpha'<n<\alpha''N} \mathbb{P}\left\{ \frac{|\mu_{tnN} - \mathbb{E}\mu_{tnN}|}{\sqrt{N}} \geq \sigma_{t\alpha} \sqrt{\ln(N)} z \right\} \leq$$

$$\leq \left(\sup_{\{N>N_0, \alpha'<\alpha<\alpha''\}} (1 + B) \right) \sum_{N=N_0+1}^{\infty} 4\sqrt{2} \left((\alpha'' - \alpha') N \right) \sqrt{N} \sqrt{C_2} e^{-\frac{\ln(N) z^2}{16}} \leq$$

$$\leq \left(\sup_{\{N>N_0, \alpha'<\alpha<\alpha''\}} (1 + B) \right) 4\sqrt{2} \sqrt{C_2} (\alpha'' - \alpha') \sum_{N=N_0+1}^{\infty} N^{-\frac{z^2}{16} + \frac{1}{2} + 1} < \infty.$$

Consequently, by the Borel-Cantelli lemma, we have $\limsup_{n,N\to\infty,\alpha'<\alpha<\alpha''} \frac{|\mu_{tnN} - \mathbb{E}\mu_{tnN}|}{\sigma_{t\alpha}\sqrt{N \ln(N)}} \leq z$ almost surely. This implies (4.10). □

Corollary 4.1. *Let $0 < \alpha' < \alpha''$ be such that $m^{-1}(\alpha'') < R$. Let $b_t > 0$. Let $d_N \in \mathbb{R}$, $N \in \mathbb{N}$, be such that $\lim_{N\to\infty} \frac{\sqrt{N \ln(N)}}{d_N} = 0$. Then, for μ_{tnN} defined in (4.9), we have*

$$\lim_{n,N\to\infty, \alpha'<\alpha<\alpha''} \frac{\mu_{tnN} - \mathbb{E}\mu_{tnN}}{d_N} = 0 \quad \textit{almost surely.} \tag{4.11}$$

Proof. As $\sigma_{t\alpha} \leq C_2 < \infty$, (4.11) is an easy consequence of Theorem 4.2. □

5 Applications to the nonuniform allocation scheme

Consider the nonuniform scheme of allocations of n distinguishable balls into N boxes with nonuniform probabilities of boxes. The number of the balls in the boxes can be described with random variables $\eta_1, \ldots, \eta_N$ having polynomial distribution. As we shall consider $N \to \infty$, the distribution depends on N. So, for each N, we have random variables $\eta_1, \eta_2, \ldots, \eta_N$ with the distribution

$$\mathbb{P}\{\eta_1 = k_1, \eta_2 = k_2, \ldots, \eta_N = k_N\} = \frac{n!}{k_1! k_2! \ldots k_N!} (p_{N1})^{k_1} (p_{N2})^{k_2} \ldots (p_{NN})^{k_N} \tag{5.1}$$

where $k_1 + k_2 + \ldots k_N = n$, $p_{Ni} > 0$, $1 \le i \le N$, and $p_{N1} + p_{N2} + \cdots + p_{NN} = 1$.

Then the number of boxes which contain s balls is

$$\mu_{snN} = \sum\nolimits_{i=1}^{N} \mathbb{I}_{\{\eta_i = s\}}. \tag{5.2}$$

Let $\pi = \pi(\lambda)$ be a Poissonian random variable with parameter λ, i.e. $\pi(\lambda, k) = \mathbb{P}(\pi(\lambda) = k) = \lambda^k e^{-\lambda}/k!$. Let $\alpha = \alpha_{n,N} = \frac{n}{N}$. We will assume that $\alpha' \le \alpha_{nN} \le \alpha''$ for some $0 < \alpha' < \alpha'' < \infty$.

For the sequence of functions

$$f_N(t) = N p_{N[Nt]}, \quad t \in [0,1], \quad N \in \mathbb{N},$$

we will consider the condition

$$\lim_{N \to \infty} f_N(t) \to f(t) \tag{5.3}$$

uniformly for $t \in [0,1]$. Here $f(t)$, $t \in [0,1]$, is a bounded nonnegative function such that

$$\int_0^1 f(t) dt = 1.$$

Let σ^2_{nNi} be the variance of $\mathbb{I}_{\{\pi(np_{Ni}) = s\}}$. Consider their arithmetic mean, i.e. let

$$\bar{\sigma}^2_{nN} = \frac{1}{N} \sum_{i=1}^{N} \sigma^2_{nNi} = \frac{1}{N} \sum_{i=1}^{N} \pi(np_{Ni}, s) \left(1 - \pi(np_{Ni}, s)\right) =$$

$$= \int_0^1 \pi(\alpha f_N(t), s) \left(1 - \pi(\alpha f_N(t), s)\right) dt.$$

Theorem 5.1. *Let $0 < \alpha' < \alpha''$. Assume that (5.3) is satisfied. Then, for μ_{snN} defined in (5.2), we have*

$$\limsup_{n,N \to \infty, \alpha' < \alpha < \alpha''} \frac{|\mu_{snN} - \mathbb{E}\mu_{snN}|}{\sqrt{N \ln(N)}\, \bar{\sigma}^2_{nN}} \le 2\sqrt{10} \quad \textit{almost surely.}$$

Proof. The probability (5.1) we can be represented as the conditional probability

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} =$$

$$= \mathbb{P}\{\pi_1(p_{N1}) = k_1, \dots, \pi_N(p_{NN}) = k_N | \pi_1(p_{N1}) + \dots + \pi_N(p_{NN}) = n\} =$$

$$= \mathbb{P}\{\pi_1(np_{N1}) = k_1, \dots, \pi_N(np_{NN}) = k_N | \pi_1(np_{N1}) + \dots + \pi_N(np_{NN}) = n\}$$

where for each fixed N, $\{\pi_i(p_{Ni}) : 1 \le i \le N\}$ and $\{\pi_i(np_{Ni}) : 1 \le i \le N\}$ are two families of independent Poissonian random variables.

We know that $\pi_1(np_{N1}) + \pi_2(np_{N2}) + \dots + \pi_N(np_{NN})$ is a Poissonian random variable with parameter n. Therefore, by the Stirling formula, we have

$$\mathbb{P}(A) = \mathbb{P}\{\pi(n) = n\} = e^{-n}\frac{n^n}{n!} = \frac{1}{\sqrt{2\pi n}}(1 + \mathrm{o}(1)) \ge \frac{1}{\sqrt{2\pi\alpha'' N}}(1 + o(1)).$$

Therefore (3.1) is valid with $\beta = \frac{1}{2}$. From the uniform convergence (5.3) it follows that $\lim_{N\to\infty} |\bar{\sigma}_{nN} - \bar{\sigma}_\alpha| = 0$ uniformly in $\alpha' \le \alpha \le \alpha''$ where

$$\bar{\sigma}_\alpha^2 = \int_0^1 \pi(\alpha f(t), s)\,(1 - \pi(\alpha f(t), s))\,dt.$$

Here $\bar{\sigma}_\alpha$ is a continuous positive function of $\alpha \in [\alpha', \alpha'']$. Therefore $\inf_{\alpha' \le \alpha \le \alpha''} \bar{\sigma}_\alpha^2 = C_4 > 0$. So we can find N_0 such that for all $N > N_0$ and $\alpha' \le \alpha \le \alpha''$ we have $\bar{\sigma}_{nN}^2 \ge \frac{C_4}{2} > 0$. So the proof can be finished as the proof of Theorem 4.2. □

Corollary 5.1. *Let $0 < \alpha' < \alpha''$. Let $b_N \in \mathbb{R}$, $N \in \mathbb{N}$, be such that $\lim_{N\to\infty} \frac{\sqrt{N\ln(N)}}{b_N} = 0$. Suppose that (5.3) is valid. Then, for μ_{snN} defined in (5.2), we have*

$$\lim_{n,N\to\infty, \alpha' < \alpha < \alpha''} \frac{\mu_{snN} - \mathbb{E}\mu_{snN}}{b_N} = 0 \quad \textit{almost surely.} \qquad \square$$

Consider a sequence of allocations of n_k balls into N_k boxes, $k \in \mathbb{N}$, such that $N_k < N_{k+1}$, $k \in \mathbb{N}$. Let $\alpha_k = n_k/N_k$, $\sigma_{sk}^2 = \bar{\sigma}_{n_k N_k}^2$ and

$$\mu_{sk} = \sum\nolimits_{i=1}^{N_k} \mathbb{I}_{\{\eta_i = s\}}. \tag{5.4}$$

Theorem 5.2. *Let $0 < \alpha' < \alpha'' < \infty$. Let (5.3) be valid. Suppose that $0 < \alpha' \le \alpha_k \le \alpha''$ for all $k \in \mathbb{N}$. Then for μ_{snN} defined in (5.4), we have*

$$\limsup_{k\to\infty} \frac{|\mu_{sk} - \mathbb{E}\mu_{sk}|}{\sqrt{N_k \ln(N_k)}\sigma_{sk}} \le 4\sqrt{3/2} \quad \textit{almost surely.}$$

Proof. From the proof of Theorem 5.1 it follows that (3.1) is valid with $\beta = \frac{1}{2}$. Also the proof of Theorem 5.1 implies that (3.2) is valid, too. So Theorem 5.2 follows from Theorem 3.1. □

Theorem 5.3. *Let $0 < \alpha_0 < \infty$. Assume that (5.3) is satisfied. Then, for μ_{snN} defined in (5.2), we have*

$$\frac{1}{N}\mu_{snN} \to \frac{\alpha_0^s}{s!}\int_0^1 (f(t))^s e^{-f(t)\alpha_0}dt$$

almost surely as $n, N \to \infty$ such that $\alpha \to \alpha_0$.

Proof. Let $\alpha' < \alpha_0 < \alpha''$. If $n, N \to \infty$ such that $\alpha \to \alpha_0$, then there exists $N_0 \in \mathbb{N}$ such that $\alpha' < \alpha < \alpha''$ for $n, N > N_0$. By Theorem 5.1, $\frac{\mu_{snN} - \mathbb{E}\mu_{snN}}{N} \to 0$ almost surely, as $n, N \to \infty$ such that $\alpha \to \alpha_0$. Furthermore, we have

$$\frac{1}{N}\mathbb{E}\mu_{snN} = \frac{1}{N}\sum_{k=1}^{N}\frac{n!}{s!(n-s)!}(p_{Nk})^s\,(1-p_{Nk})^{n-s} =$$

$$= \frac{1}{s!}\frac{1}{N}\sum_{k=1}^{N}\frac{n}{N}\cdot\frac{n-1}{N}\cdots\frac{n-s+1}{N}(Np_{Nk})^s\left(1-\frac{Np_{Nk}}{N}\right)^{n-s} = \int_0^1 g_{nN}(f_N(t))dt.$$

Here

$$g_{nN}(f_N(t)) = \frac{1}{s!}\frac{n}{N}\cdot\frac{n-1}{N}\cdots\frac{n-s+1}{N}(f_N(t))^s\left(1-\frac{f_N(t)}{N}\right)^{n-s} \approx$$

$$\approx \frac{\alpha^s}{s!}\,(f_N\,(t))^s\left(1-\frac{f_N\,(t)}{N}\right)^{N\alpha}.$$

So $0 \le g_{nN}(f_N(t) \le C$ for all $t \in [0,1]$. Moreover, $g_{nN}(f_N(t)) \to \frac{(\alpha_0 f(t))^s}{s!}e^{-f(t)\alpha_0}$ as $n, N \to \infty$ such that $\alpha \to \alpha_0$ for all $t \in [0,1]$. Therefore, by the Lebesgue theorem, it holds that

$$\frac{1}{N}\mathbb{E}\mu_{snN} \to \frac{\alpha_0^s}{s!}\int_0^1 (f(t))^s e^{-f(t)\alpha_0}dt,$$

as $n, N \to \infty$ such that $\alpha \to \alpha_0$. Consequently,

$$\frac{1}{N}\mu_{snN} = \frac{1}{N}\mu_{snN} - \frac{1}{N}\mathbb{E}\mu_{snN} + \frac{1}{N}\mathbb{E}\mu_{snN} \to \frac{\alpha_0^s}{s!}\int_0^1 (f(t))^s e^{-f(t)\alpha_0}dt,$$

as $n, N \to \infty$ such that $\alpha \to \alpha_0$. The proof is complete. □

Consider the number of boxes containing less than r balls and consider also the number of boxes containing more than r balls, i.e. let

$$\mu_{<rN} = \mu_{<rnN} = \sum_{i=1}^{N}\mathbb{I}_{\{\eta_i<r\}}, \quad \mu_{>rN} = \mu_{>rnN} = \sum_{i=1}^{N}\mathbb{I}_{\{\eta_i>r\}}.$$

Since $\mu_{<rnN} = \sum_{k=0}^{r-1}\mu_{knN}$ and $\mu_{>rnN} = N - \mu_{<rnN} - \mu_{rnN}$, therefore Theorem 5.3 implies the following.

Corollary 5.2. *Let $0 < \alpha_0 < \infty$. Suppose that* (5.3) *is valid. Then we have*

$$\frac{1}{N}\mu_{<rnN} \to \sum_{k<r} \frac{\alpha_0^k}{k!} \int_0^1 (f(t))^k e^{-f(t)\alpha_0} dt$$

and

$$\frac{1}{N}\mu_{>rnN} \to \sum_{k>r} \frac{\alpha_0^k}{k!} \int_0^1 (f(t))^k e^{-f(t)\alpha_0} dt$$

almost surely, as $n, N \to \infty$ such that $\alpha \to \alpha_0$.

Consider the number of boxes such that the numbers of balls contained by these boxes belong to the subset $\mathbb{N}' \subset \mathbb{N}$, i.e. let

$$\mu_{\mathbb{N}'N} = \mu_{\mathbb{N}'nN} = \sum_{i=1}^{N} \mathbb{I}_{\{\eta_i \in \mathbb{N}'\}}.$$

Corollary 5.3. *Let $0 < \alpha_0 < \infty$. Suppose that* (5.3) *is valid. Then we have*

$$\frac{1}{N}\mu_{\mathbb{N}'nN} \to \sum_{k\in\mathbb{N}'} \frac{\alpha_0^k}{k!} \int_0^1 (f(t))^k e^{-f(t)\alpha_0} dt \tag{5.5}$$

almost surely, as $n, N \to \infty$ such that $\alpha \to \alpha_0$.

Proof. First let $\mathbb{N}'$ be a finite set. Since $\mu_{\mathbb{N}'nN} = \sum_{k\in\mathbb{N}'} \mu_{knN}$, by Theorem 5.3, (5.5) is true. Now, let $\mathbb{N}'$ be an arbitrary set. Denote $\mathbb{N}'_r = \{k \in \mathbb{N}' : k \le r\}$. Then we have

$$\mu_{\mathbb{N}'_r N} \le \mu_{\mathbb{N}'N} \le \mu_{\mathbb{N}'_r N} + \mu_{>rN}.$$

Therefore

$$\sum_{k\in\mathbb{N}'_r} \frac{\alpha_0^k}{k!} \int_0^1 (f(t))^k e^{-f(t)\alpha_0} dt \le \liminf \frac{1}{N}\mu_{\mathbb{N}'N} \le \limsup \frac{1}{N}\mu_{\mathbb{N}'N} \le$$

$$\sum_{k\in\mathbb{N}'_r} \frac{\alpha_0^k}{k!} \int_0^1 (f(t))^k e^{-f(t)\alpha_0} dt + \sum_{k>r} \frac{\alpha_0^k}{k!} \int_0^1 (f(t))^k e^{-f(t)\alpha_0} dt. \tag{5.6}$$

almost surely. Let $r \to \infty$, then (5.6) implies (5.5). The proof is complete. □

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