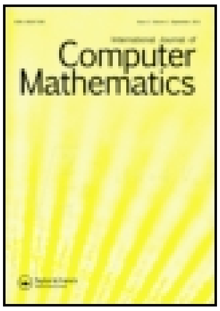

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article

Isoperimetrically Optimal Polygons in the Triangular Grid with Jordan-type Neighbourhood on the Boundary

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The digital spaces have some properties that are not present in the Euclidean space. A digitized circle do not necessarily have the smallest (digital arc length) perimeter of all objects having a given area. In digital geometry various measures of perimeter and area lead to various definitions of digital circles using the digital version(s) of the isoperimetric inequality. Usually the square grid is used as digital plane with either the cityblock or the chessboard neighbourhood relation. In this paper the triangular grid is also used with two types of neighbourhood relation that play importance in Jordan curves. We search for those (digital) objects that have optimal measures and therefore they can be considered as digital circles by our definition. We show that special, (almost) regular hexagons are Pareto optimal, i.e., they fulfill both versions of the isoperimetric inequality: they have maximal area among objects having perimeter at most a given length; and they have minimal perimeter among objects enclosing at most a certain area. The optimal objects can be build in a similar way as the Wang-spiral for the square grid.

Keywords: discrete isoperimetric problem; digital geometry; digital circles; triangular grid; optimal shapes

AMS Subject Classification: 52C05, 68R05, 68U10.
ACM Computing Classification System: G.2.1 Combinatorics; F.2.2 Computations on discrete structures; G.1.6 Optimization; I.3.5 Computational Geometry.

1. Introduction

In the Euclidean geometry various objects have various characteristic properties. A very good example for such a property is the isoperimetric inequality that shows the privileged role of the Euclidean circles. It states that the area enclosed by a closed simple curve is the largest when enclosed by a circle of the same length, with equality occurring only for circles. This infers two corollaries, as two sides of a coin. In one side, among closed simple curves of (at most) a certain length, a circle encloses a maximal area. On the other side, among curves enclosing (at most) a certain area, a circle has minimal length. Therefore (any or both of) these properties can be used to 'define' the circles. In digital spaces the digital versions of such properties can be used. However, the digital objects that defined by one such property do not necessarily satisfy other such property. This fact leads to various definitions of the digital objects that can be used for various purposes.

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In digital geometry, i.e., in grids (regular tessellations of the plane) there are some phenomena which do not occur in the Euclidean plane, and vice versa. For instance, there is a point (moreover there are infinitely many distinct points) between any two distinct points of the Euclidean plane. Opposite to this fact, there are neighbour points (pixels) in digital planes. The two types of basic neighbourhood relation for the square grid are the cityblock and chessboard neighbourhood [27]. Other phenomenon that does not occur in the Euclidean plane is that on the square grid one may have 'lines' that intersect each other without sharing a common point, e.g., check the diagonals of a chessboard. This paradox is connected to the fact that Jordan curve theorem does not work for every digital plane: neither cityblock nor chessboard neighbourhood itself allows an analogue of the Jordan curve theorem [28]. To eliminate this topological deficiency, a combination of the two binary relations can be used: an object should be connected with any of the neighbourhood criterion and the other neighbourhood relation must be used at the background to provide connected background in any case.

There is another important example of non-correspondence of Euclidean and digital concepts [17]: A digitized circle (see [3, 22]) doesn't have the smallest (digital arc length) perimeter of all objects that have a given area. Therefore there is a big difference between digitized and digital circles: the former ones are the possible digital representations of the Euclidean circles, e.g., Gauss digitizations (see [17]); while the latter ones are defined by the digital version of a characteristic property of the circles. In discrete spaces special shaped objects have minimal 'perimeter' for various definitions of the perimeter. In $\mathbb{Z}^n$, Wang and Wang [30] presented such an ordering of grid points that every finite prefix of this sequence forms an object with minimal boundary size for that cardinality. Other related results can be found in [2, 4, 10, 15].

In digital geometry spaces consist of points described by integer coordinate values. The square and cubic grids are well-known and frequently used in applications, since the Cartesian coordinate system fits to them very well. For other grids appropriate methods are needed to define well applicable coordinate systems. The pixels of the hexagonal grid can be addressed with pairs of integers [14]. There is a more elegant solution using coordinate triplets, where the sum of the values is zero in a triplet, reflecting the symmetry of the grid [12]. We note here that the digital topology of the hexagonal grid works well: the usual neighbourhood has no such a disadvantage as the neighbourhood relations of the square grid. Jordan theory works for this grid. The isoperimetrically optimal shapes on this grid are applied in chemistry also [5, 9] due to the related structure of some hydrocarbon moleculae. In [11] a correspondence is established between paths in the rectangular lattice that satisfy certain diagonal constraints and perfect matchings in certain classes of benzenoid graphs. There is a closely related problem in graph theory: the vertex isoperimetric problem. That is to minimize the number of vertices of the outer boundary. The edge isoperimetric problem, that is to minimize the number of outgoing edges, is completely solved for various types of graphs (see, e.g., [6, 13]).

A *grid-polygon* is said to be *optimal* if both of the two constraints, to have maximal area among the grid-polygons having perimeter at most a given length, and to have minimal perimeter among the grid-polygons enclosing at most a certain area, are fulfilled. In discrete space these two constraints are more or less concurrent, and so, usually do not coincide. In [29] these polygons are called Pareto-optimal: in game-theory [26] when the aims of the players concur the optimal solution is the saddle point. In [29] results are presented for the square and for the hexagonal grid.

Similarly to the hexagonal grid, each element of the triangular grid can also be

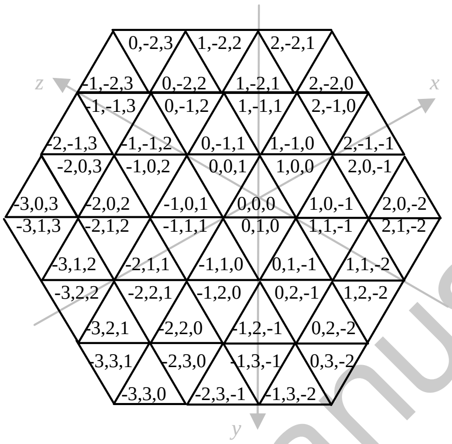


Figure 1. A part of the triangular grid with a symmetric coordinate frame.

described with three coordinate values [20] as we recall in Figure 1. Note that each element of that grid is a triangle which is identified with a pixel, that is, with a grid point having three coordinates. There are two orientations of the used triangles, the sum of coordinate values are zero and one, respectively; and therefore the triangular plane can be identified as two parallel planes of the cubic grid [19]. There are various types of neighbourhood on this grid. Two pixels are 1-neighbours if they share a side. Two pixels are 3-neighbours if they share at least a point on their boundaries (e.g., a corner point). These two types of neighbourhood relations form a Jordan-pair in a similar way as the cityblock and chessboard neighbours on the square grid, therefore they are called Jordan-type neighbourhood relations, see, e.g., [23]. In this paper we are using these two types of neighbourhood relations.

In this paper we recall some results of [1, 29] (regarding isoperimetric inequality in the square grid), moreover we give an alternative (and in some sense simpler) proof of the result based on combinatorics ([25]). Our main results are extensions of these results to the triangular grid with the two types of Jordan neighbourhood.

In an arbitrary grid whose elements are called grid points or pixels, any finite subset of pixels will be called an object (grid polygon, or binary picture in other terms). Based on the obvious intuitive observation that optimal objects are connected and topologically have no holes, we will consider in this paper only connected objects without holes, provided some connectivity concept for the grid. Clearly each object can be interpreted as a grid polygon whose vertices are certain object pixels. Our aim is to find those objects that have maximal area among those that have at most the same perimeter and, at the same time, they have minimal perimeter among those objects that have at least the same area, provided concepts of area and perimeter for the grid. Actually the basic concept of the grids is the pixel. Therefore both the area and the perimeter will be measured by cardinalities of certain sets of pixels.

In the next section we recall results on the square grid with alternative proofs. In Section 3 we present some basic concepts on the triangular grid that are used in Section 4 and 5 where our main results are presented computing the perimeter as the number of 1-neighbour and 3-neighbour pixels of the object, respectively. In Section 6 some concluding remarks close the paper.

2. Preliminary Results: The Square Grid

The square grid is the most usual digital grid. The Cartesian coordinate frame is used to address the pixels (the terms grid point and square are also used). The grid itself can be described by $\mathbb{Z}^2$ and hence, in this section, we use pairs (x, y) to address a pixel where $x, y \in \mathbb{Z}$. There are two types of usual neighbourhood relations. The pixels (x, y) and (x', y') are called 1-neighbours if $|x - x'| + |y - y'| = 1$. They are strict 2-neighbours if $|x - x'| = |y - y'| = 1$. The union of the sets of 1-neighbour pixels and strict 2-neighbour pixels of a square form the set of its 2-neighbours. Their original names, i.e., cityblock (or Manhattan for 1-neighbourhood) and chessboard neighbourhood (for 2-neighbourhod), come from the initial paper on digital geometry by Rosenfeld and Pfaltz [27]. By the number of these neighbour pixels, the terms 4-neighbourhood and 8-neighbourhood are widely used in image processing literature. In cellular automata theory the terms Moore and von Neumann neighbourhood are used [16]. Our preferred terms, the 1-neighbours and 2-neighbours, are meaning the number of coordinate values that may differ in various types of neighbour pixel pairs. In this paper these terms are preferred since they allow simple extensions to higher dimensions and to other grids, as we will use later on the triangular grid.

The *area* of an object can easily be measured by the *number of pixels* (which now are squares) belonging to the object. However, the perimeter depends on the used neighbourhood criterion. Let us see which objects are Pareto optimal, first, using 1-neighbourhood boundary as perimeter.

2.1 *Square grid with cityblock neighbourhood*

In this subsection the perimeter of an object is defined by the 1-neighbourhood relation.

The *perimeter* of an object is the *number of pixels* that belong to the *1-neighbourhood* of an *object* pixel but do not belong to the object. Formally:

The perimeter of an object L on the square grid with 1-neighbourhood is the cardinality of the set

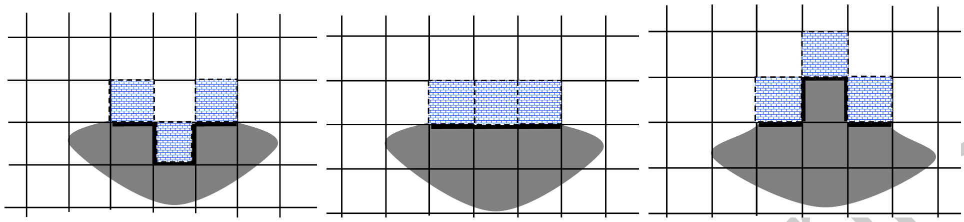
$$\{(x, y) \mid (x, y) \notin L, \exists x', y' \ : \ (x', y') \in L, |x - x'| + |y - y'| = 1\}.$$

The *embedding rectangle* of an object can also be defined: it is a *diamond* object defined by *four stair-type sides*, i.e., diagonal line-segments in the following way. Let $p(p(1), p(2))$ be a/the object pixel for which the value $p(1) + p(2)$ is minimal and let $d_{min} = p(1) + p(2)$. Let $p'(p'(1), p'(2))$ be a/the object pixel for which the value $p'(1) + p'(2)$ is maximal, and let d_{max} denote this sum. Then the stair-type 'lines' consisting of pixels $q(q(1), q(2))$ with $q(1) + q(2) = d_{max}$ and $q(1) + q(2) = d_{min}$ are two of the sides of this diamond. The other two sides are defined by the minimal/maximal value of $p(1) - p(2)$ for the object pixels $p(p(1), p(2))$: let $e_{max} = \max\{p(1) - p(2) \mid (p(1), p(2)) \in L\}$, and similarly $e_{min} = \min\{p(1) - p(2) \mid (p(1), p(2)) \in L\}$.

Then the embedding rectangle consists of the pixels

$$\{(x, y) \mid d_{min} \leq x + y \leq d_{max}, e_{min} \leq x - y \leq e_{max}\}.$$

As one can see in Fig. 2 the perimeter does not increase, while the area is strictly increasing if side parts (a) and (b) are replaced by side part (c). By iteration one can obtain a 'diamond shape' object with four 'stair-type' sides. Moreover at the



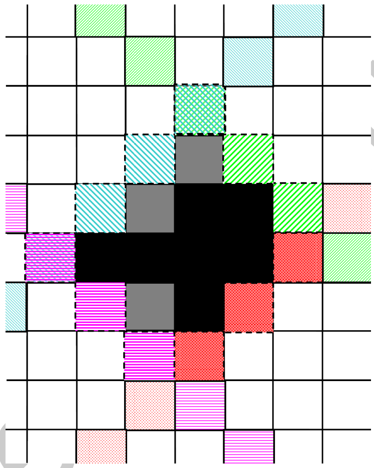


Figure 2. (a)-(c): Excluding 'concavity' and long 'straight' sides. (d): An object (black) and its embedding rectangle (grey) with diagonal line-segments containing the pixels of the perimeter.

'corners' of these sides there are at most two 1-neighbour pixels on the boundary. Therefore, a shape that is not embedding rectangle (diamond) of itself, is definitely not optimal. Its area can be extended to its embedding rectangle without decreasing its perimeter. So, optimal objects could only be these diamonds. They are called *simple shapes* in [29].

To find the optimal objects, we need to find the side-lengths of the optimal diamonds. Let us consider a diamond shape object (see, e.g., Fig. 2 (d): black and grey pixels). The parameters are the distance of the 'parallel' sides and the type of the left corner: Let $\Delta d = d_{max} - d_{min}$, where the stair-type sides of direction $\backslash$ are defined by pixels $r(r(1), r(2))$ with $r(1)+r(2) = d_{max}$ and $r(1)+r(2) = d_{min}$, respectively. Similarly, let $\Delta e = e_{max} - e_{min}$, where the sides of direction $/$ are defined by pixels $r(r(1), r(2))$ with $r(1) - r(2) = e_{max}$ and $r(1) - r(2) = e_{min}$, respectively. Actually, in Fig. 2 (d), the diagonal line-segments with various color contain the pixels of the parameter. They are defined by the pixels $r(r(1), r(2))$ with $r(1) + r(2) = d_{max} + 1$, $r(1) + r(2) = d_{min} - 1$, $r(1) - r(2) = e_{max} + 1$ and $r(1) - r(2) = e_{min} - 1$, respectively.

Furthermore there are 4 possibilities for the left corner (it is at the intersection of lines defined by d_{min} and e_{min}) by the parity of d_{min} and e_{min}: *oo* means both are odd, *oe* means d_{min} is odd and e_{min} is even, and similarly *eo* and *ee* is defined.

By an induction on the sidelengths it can be proven that the perimeter of the diamond is $\Delta d + \Delta e + 4$.

The area of the maximal diamond with these parameters (the area also depends on the left corner and so it may be 1 less as one may prove it by combinatorial case analysis): $\left\lfloor \frac{(\Delta d+1)(\Delta e+1)+1}{2} \right\rfloor$.

Fixing the perimeter there is only one variable and it gives a maximal value for the area with $\Delta d = \Delta e$.

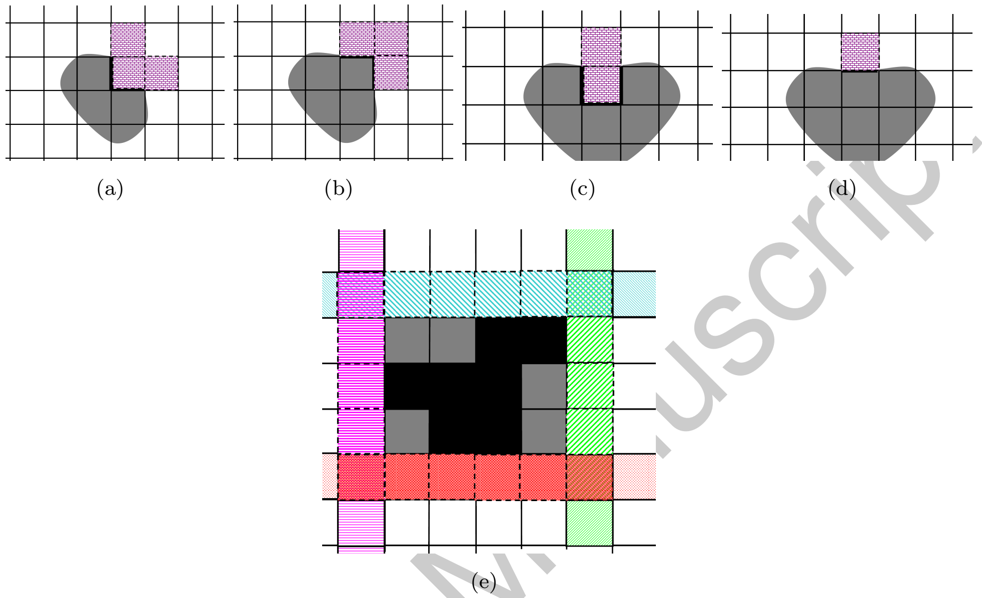


Figure 3. Excluding concavity (a,b) at corners and (c,d) at sides. (e): An object (black) and its embedding rectangle (grey) with the perimeter pixels (various color) and their lanes.

Based on these values one can easily see that the optimal digital shapes are those where Δd and Δe are (only almost if the perimeter is odd) equal. The results are shown in details in [29].

2.2 *Square grid with chessboard neighbourhood*

In this subsection 2-neighbourhood boundary is considered with a similar argument as in the 1-neighbour case. The perimeter of an object L on the square grid with 2-neighbourhood is the cardinality of the set

$$\{(x, y) \mid (x, y) \notin L, \exists x', y' \ : \ (x', y') \in L, |x - x'| \leq 1, |y - y'| \leq 1\}.$$

Now we define the *embedding rectangle* for this case: it is a rectangle given by *straight sides* by the lanes of the minimal/maximal x and y values of the pixels of the object.

Formally, for a finite object L of the square grid, let $x_{max} = \max\{x \mid (x, y) \in L\}$, $x_{min} = \min\{x \mid (x, y) \in L\}$, $y_{max} = \max\{y \mid (x, y) \in L\}$ and $y_{min} = \min\{y \mid (x, y) \in L\}$. Then the embedding rectangle consists of the pixels of the set

$$\{(x, y) \mid x_{min} \leq x \leq x_{max}, y_{min} \leq y \leq y_{max}\}.$$

A *lane* is a set of grid points with a fixed coordinate value, e.g., $\{(2, y)\}$.

From Figure 3 (a,b) it is clear that if the object has a 'corner' which is concave, then the area can be extended without increasing the perimeter. Moreover, from Fig. 3 (c,d) one can see that a similar argument holds when the concave part is not on the corner, but on one of the sides. The area can be extended without increasing the perimeter in these cases too. By a combinatorial way it can easily be proven that there is no other way of concavity to occur.

By iteration, it can be seen that only embedding rectangles (and in this case they are really rectangles) can be optimal. Also since there must be pixels at each value of x and y between the maximal and minimal values (see Figure 3 (e) also),

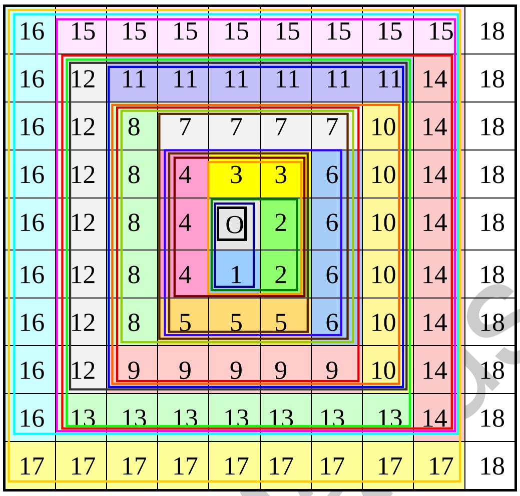


Figure 4. Spiral to build optimal rectangles.

actually there are at least two pixels that can be assigned to each x and y value of the object in its border.

Let us analyse the rectangles and find the optimal ones. The parameters are $x_{min}, x_{max}, y_{min}, y_{max}$, we use the notation
$\Delta x = x_{max} - x_{min} + 1$, and $\Delta y = y_{max} - y_{min} + 1$.

The perimeter of these rectangles is $2(\Delta x + \Delta y) + 4$. Actually the perimeter consists of grid points from the lanes that bounds the rectangle, i.e., lanes defined by $x_{min} - 1, x_{max} + 1, y_{min} - 1, y_{max} + 1$ (see the coloured lanes on Figure 3 (e)).

The area of these rectangles is $\Delta x \Delta y$.

By fixing the perimeter there is only one parameter. By searching for the extremal value (simple derivation): it is at $\Delta x = \Delta y$.

One can easily check that for a perimeter value which is divisible by 4 the equality $\Delta x = \Delta y$ gives an integer solution and so the squares are optimal in the grid.

For perimeter values that are even, but not divisible by 4, in square grid with perimeter based on 2-neighbourhood the optimal possibilities are those when $\Delta x = \Delta y \pm 1$.

We note here that all the optimal shapes can be obtained by a spiral construction as Figure 4 shows some of the first steps of this procedure.

The results of this section can also be found in [29] with a much sophisticated and detailed proof. In the next sections, in the triangular grid we prove our main results.

3. Definitions and Notions for the Triangular Grid

The triangular grid can be described using a subset of $\mathbb{Z}^3$ [18, 19]. One way of doing it is to take the union of the planes having grid points with coordinate sums 0 and 1. We refer the points of these two planes as *even/odd* points of the grid, respectively. These two types of points are exactly the triangles of the grid oriented as $\triangle$ and ∇. In this way, the description of the grid is symmetric by the three coordinate values [20, 21]. All the coordinate axes x, y and z are used in similar roles (see Figure 1 also).

Let $p(p(1), p(2), p(3))$ and $q = (q(1), q(2), q(3))$ be two triangles of the grid. For

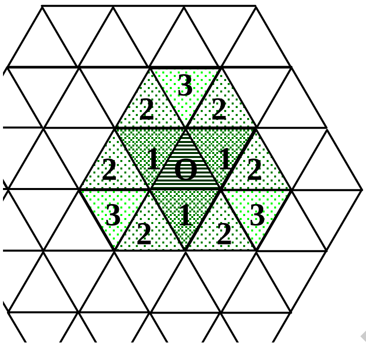


Figure 5. Various neighbourhood relations on the triangular gird (the triangles indicated by $k = 1, 2, 3$ are strict k-neighbours of the triangle marked by 'O'.

$k = 1, 2, 3$, the triangles p and q are *triangular k-neighbours* if

$$|p(i) - q(i)| \leq 1 \quad \text{for} \quad 1 \leq i \leq 3 \qquad \text{and}$$

$$\sum_{i=1}^{3} |p(i) - q(i)| \leq k.$$

In case of equality of the second formula the term *strict* k-neighbourhood is used. See Figure 5. An even grid point $(p(1), p(2), -p(1) - p(2))$ has the following 1-neighbours $(p(1) + 1, p(2), -p(1) - p(2))$, $(p(1), p(2) + 1, -p(1) - p(2))$, $(p(1), p(2), 1 - p(1) - p(2))$. Otherwise, if we consider the triangles where the sum of the coordinates is 1, the neighbour triangles have difference vectors similar to the vectors above but with inverted signs. As we have seen on Figure 5 this closest neighborhood can be extended by adding 6 more strict 2-neighbours. By adding 3 more pixels, the strict 3-neighbours we got an extended neighbourhood consisting a pixel with its twelve neighbours.

In this paper, as it was already mentioned, we use the 1-neighbourhood and 3-neighbourhood relations, they coincide to the terms of cityblock and chessboard neighbourhood of the square grid as follows. The 1-neighbour pixels are side-neighbours, i.e., they share a side as cityblock neighbours do. The triangle pixel is 3-neighbour if they share at least a point (a corner) of their boundaries, similarly to the chessboard neighbour squares.

Let a coordinate value be fixed (e.g., $x = -1$ or $y = 2$); a *lane* is the set of pixels that have this fixed coordinate value, as it is shown in Figure 6.

Analogously, as any object on the square grid has an embedding rectangle, we define the concept of embedding hexagon on the triangular gird:

The *embedding hexagon* consists of the pixels that are in the intersection of those lanes (all three directions) which have at least one pixel of the object. Formally, for an object L:

Let

$$x_{min} = \min_{(p(1),p(2),p(3)) \in L} \{p(1)\}, \quad x_{max} = \max_{(p(1),p(2),p(3)) \in L} \{p(1)\},$$

similarly,

$$y_{min} = \min_{(p(1),p(2),p(3)) \in L} \{p(2)\}, \quad y_{max} = \max_{(p(1),p(2),p(3)) \in L} \{p(2)\},$$

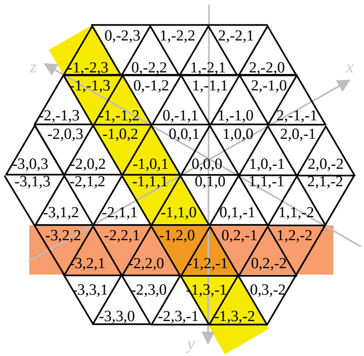


Figure 6. Examples for lanes ($x = -1$ yellow, $y = 2$ orange).

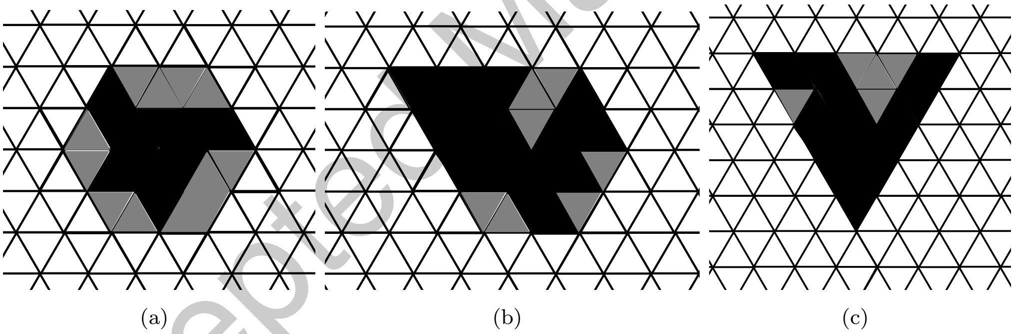


Figure 7. Embedding hexagons (grey) on the triangular grid (of the black object). The examples (b) and (c) are degenerated.

and

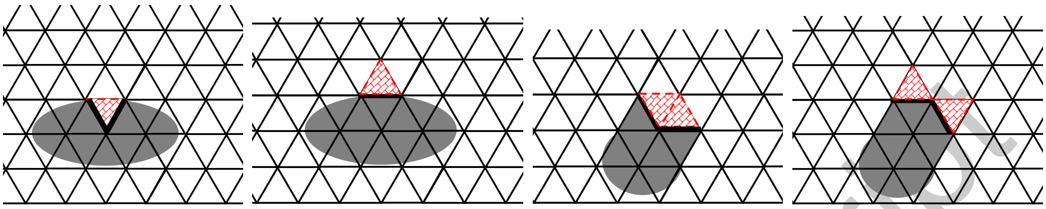
$$z_{min} = \min_{(p(1),p(2),p(3))\in L} \{p(3)\}, \quad z_{max} = \max_{(p(1),p(2),p(3))\in L} \{p(3)\}.$$

Then the embedding hexagon of L is defined as the set of pixels, i.e., an object:

$$\{(x, y, z) \mid x_{min} \le x \le x_{max}, y_{min} \le y \le y_{max}, z_{min} \le z \le z_{max}\}.$$

See Fig. 7 for examples. Any object of the triangular grid has an embedding hexagon. In some cases it can be degenerated with some sides whose length is zero (see Fig. 7 (b,c)).

As we already mentioned 1-neighbourhood and 3-neighbourhood together fulfills the Jordan property: the object should be connected by any of these types of neighbourhood, and then the background is connected by the other types of neighbourhood. In the next sections we will analyse the optimal shapes using these two types of neighbourhood in the boundary (i.e., computing the perimeter). Jordan-type neighbourhood relations play special importance for our purpose, using 1-neighbourhood boundary the perimeter is defined by a 3-neighbourhood connected curve and using 3-neighbourhood boundary the perimeter is given by a 1-neighbourhood connected curve.



(a) (b) (c) (d)

Figure 8. Excluding concavity in the triangular grid. The object in (a) is blown up to (b) by excluding corner with angle $\frac{5\pi}{3}$, while the object in (c) is extended to (d) by changing an angle $\frac{4\pi}{3}$ to $\frac{2\pi}{3}$ without increasing the perimeter.

4. Triangular grid with the closest neighbourhood

In this section we present analogous results that are presented for the square grid about Pareto-optimal polygons using the triangular grid with 1-neighbourhood in the boundary. The results of this section can also be found in [25], but for sake of completeness they are fully presented here.

The area of an object on the triangular grid is the number of its pixels. As we use 1-neighbourhood boundary, the perimeter of an object is the number of triangles outside of the object that are 1-neighbours of some triangles of the object, formally, for an object L, its perimeter is the cardinality of the set

$$\{(x, y, z) \mid (x, y, z) \notin L, \exists x', y', z' \ : \ (x', y', z') \in L, |x - x'| + |y - y'| + |z - z'| = 1\}.$$

First we will prove that optimal shapes, the digital circles, are hexagons, and then we will prove that in optimal polygons the difference of the sidelengths of the hexagon is as small as possible.

4.1 *The shape of optimal circles*

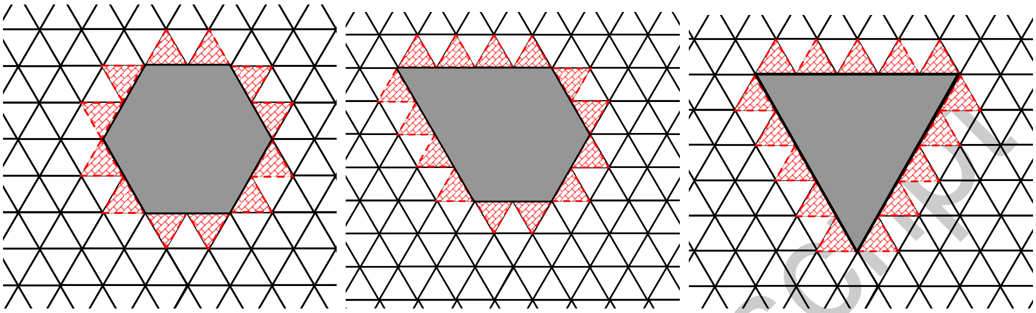
The aim of this subsection is to show that optimal polygons have only straight sides (i.e., sides parallel to sides of the triangles of the grid and there are no 'hilly' and 'sawtooth' sides in the terms of [21]).

In fact, we show that the embedding hexagon B (possibly a degenerated version) of a given object A has at most the same perimeter as the perimeter of A, while the area of B is not less than the area of A. (The area of A and B equal if and only if the objects A and B coincide.)

There are two possible types of connection combination of edges that cannot belong to a (degenerated) embedding hexagon. These two cases can be seen on Fig. 8 (a) and (c): concavity can occur only in these two ways having the angle $\frac{5\pi}{3}$ or $\frac{4\pi}{3}$ between edges. On Fig. 8 (b) and (d) it is shown how these objects can be extended by adding one or two triangles to them, respectively. The area of the object is strictly increasing in such a step by the triangle(s) that was on the boundary before the step and become(s) part of the object, while the perimeter of the object does not increase.

By an appropriate iterative use of the previous local blowing steps the embedding hexagon is obtained. The area is growing in each of these steps, while the perimeter does not increase. Therefore the optimal shapes can be only the embedding hexagons.

Let the sidelengths of a hexagon be a, b, c, d, e, f in this order, then by the geom-



(a) (b) (c)

Figure 9. Embedding hexagons and their perimeter (red color).

etry of the grid the lengths of the sides are not independent, but the equations

$$a + b = d + e, \quad b + c = e + f, \quad \text{and} \quad c + d = f + a \qquad (*)$$

must hold (note that in degenerate cases one or more variables are 0). The perimeter (using 1-neighbourhood) of a hexagon is $P = a + b + c + d + e + f$. Having independent parameters a, b, c, d the area of a hexagon can be computed by the following formula: $A = 2(a + b)(c + d) - a^2 - d^2$.

In the next subsection we determine the optimal embedding hexagons.

4.2 *The side-lengths of optimal polygons*

In the Euclidean plane those hexagons are optimal that have equal sides. An exactly analog result for discrete planes modeled by grids cannot be obtained. In the triangular grid there are six cases for the possible perimeters of the hexagons. The results are presented by these cases.

Theorem 4.1 *The next table shows the perimeter and area of the optimal hexagons on the triangular grid using 1-neighbourhood to compute the perimeter.*

Case	*Perimeter*	*Area*
1	$6n$	$6n^2$
2	$6n+1$	$6n^2 + 2n - 1$
3	$6n+2$	$6n^2 + 4n$
4	$6n+3$	$6n^2 + 6n + 1$
5	$6n+4$	$6n^2 + 8n + 2$
6	$6n+5$	$6n^2 + 12n + 3$

where n is a natural number, and for the cases 4,5,6, n is a natural number or $n = 0$.

Proof The perimeter of a hexagon is P while its area is denoted by A. The lengths of the sides (in order a, b, c, d, e, f) depend on each other by geometry: $a+b = d+e$, $b+c = e+f$, $c+d = f+a$, as we already mentioned $(*)$. The proof goes by cases. There are six possible remainder of the division perimeter/6. Let n be a natural number, and for cases 4,5,6, suppose n to be a natural or $n = 0$.

- case 1: $P = 6n$
  (a) Statement to prove: the object is optimal if all sides are equal.
  $P = 6n$
  $A = 2(2n)(2n) - 2n^2 = 6n^2$

(b) Perimeter is unchanged and some sides are changed so that $P = 6n$ still holds. We can't change the length of only two sides or all six sides because in this case the equations among the sides won't hold.
So we change the length of (at least) two-two sides: $(1 \leq |k| < n)$
$n-k, n, n+k, n-k, n, n+k$
Area:
$A = 2(n-k+n)(n+k+n-k) - (n-k)^2 - (n-k)^2 =$
$= 6n^2 - 2k^2$
As $|k| \geq 1$ this area is smaller than the one in part (a).

- case 2: $P = 6n + 1$
  (a) Statement to prove: We get the largest area when the sides are:
  $n-1, n+1, n, n, n, n+1$
  (By symmetry, the area doesn't change when we rotate the hexagon, i.e., we "move the sides around".)
  $A = 2(n-1+n+1)(n+n) - (n-1)^2 - n^2 =$
  $= 6n^2 + 2n - 1$
  (b) We can change 4 sides so that the equations $(*)$ hold.
    i. Sides: $n-1+k, n+1-k, n, n+k, n-k, n+1$ $(-n < k < n,\ k = 0, 1$ give no change)
    $P = 6n + 1$
    Area:
    $A = 2(n-1+k+n+1-k)(n+n+k) - (n-1+k)^2 - (n+k)^2 =$
    $= 6n^2 + 2n - 1 + 2k - 2k^2$
    We need to show that: $2k - 2k^2 < 0$, it is, iff $k < k^2$.
    For $1 < k$ and $0 > k$ this is always true.
    ii. Sides: $n-1, n+1+k, n-k, n, n+k, n+1-k$ $(1 \leq |k| < n)$
    $P = 6n + 1$
    Area:
    $A = 2(n-1+n+1+k)(n-k+n) - (n-1)^2 - n^2 =$
    $= 6n^2 + 2n - 1 - 2k^2$
    We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
    As $k \neq 0$ this is always true.
    iii. Sides: $n-1-k, n+1, n+k, n-k, n, n+1+k$ $(-n < k < n,\ k \neq 0,$ $k \neq -1$, since $k = 0$ and $k = -1$ do not change the hexagon)
    $P = 6n + 1$
    Area:
    $A = 2(n-1-k+n+1)(n+k+n-k) - (n-1-k)^2 - (n-k)^2 =$
    $= 6n^2 + 2n - 1 - 2k - 2k^2$
    We need to show that: $-2k - 2k^2 < 0$, it is, iff $k + k^2 > 0$.
    This is true for every $k < -1$ and for every $k > 0$.
- case 3: $P = 6n + 2$
  (a) Statement to prove: We get the largest area when the sides are:
  $n, n, n+1, n, n, n+1$
  (The area doesn't change when we "move the sides around".)
  $A = 2(n+n)(n+1+n) - n^2 - n^2 = 6n^2 + 4n$
  (b) We can change 4 sides:
    i. Sides: $n+k, n, n+1-k, n+k, n, n+1-k$ $(-n < k < n,\ k \neq 0,\ k \neq 1,$ $k = 1$ only rotates the hexagon)
    $P = 6n + 2$
    Area:
    $A = 2(n+k+n)(n+1-k+n+k) - (n+k)^2 - (n+k)^2 =$

$= 6n^2 + 4n + 2k - 2k^2$
We need to show that: $2k - 2k^2 < 0$, it is, iff $k < k^2$.
For values $1 < k$ and $0 > k$ this is always true.

ii. Sides: $n-k, n+k, n+1, n-k, n+k, n+1$ $(1 \le |k| < n)$
$P = 6n + 2$
Area:
$A = 2(n-k+n+k)(n+1+n-k) - (n-k)^2 - (n-k)^2 =$
$= 6n^2 + 4n - k^2$
We need to show that: $-k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

iii. Sides: $n, n-k, n+1+k, n, n-k, n+1+k$ $(-n < k < n,\ k \neq 0,\ k \neq -1)$
$P = 6n + 2$
Area:
$A = 2(n+n-k)(n+1+k+n) - n^2 - n^2 =$
$= 6n^2 + 4n - 2k - 2k^2$
We need to show that: $-2k - 2k^2 < 0$, it is, iff $k + k^2 > 0$.
This is always true for values $0 < k$ and for values $k < -1$.

- case 4: $P = 6n + 3$

(a) Statement to prove: We get the largest area when the sides are:
$n, n+1, n, n+1, n, n+1$
(The area doesn't change when we "move the sides around".)
$A = 2(n+n+1)(n+n+1) - n^2 - (n+1)^2 =$
$= 6n^2 + 6n + 1$

(b) We can change 4 sides so that the equations hold.

i. Sides: $n+k, n+1-k, n, n+1+k, n-k, n+1$ $(1 \le |k| < n)$
$P = 6n + 3$
Area:
$A = 2(n+k+n+1-k)(n+n+1+k) - (n+k)^2 - (n+1+k)^2 =$
$= 6n^2 + 6n + 1 - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

ii. Sides: $n, n+1+k, n-k, n+1, n+k, n+1-k$ $(1 \le |k| < n)$
$P = 6n + 3$
Area:
$A = 2(n+n+1+k)(n-k+n+1) - n^2 - (n+1)^2 =$
$= 6n^2 + 6n + 1 - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

iii. Sides: $n-k, n+1, n+k, n+1-k, n, n+1+k$ $(1 \le |k| < n)$
$P = 6n + 3$
Area:
$A = 2(n-k+n+1)(n+k+n+1-k) - (n-k)^2 - (n+1-k)^2 =$
$= 6n^2 + 6n + 1 - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

- case 5: $P = 6n + 4$

(a) Statement to prove: We get the largest area when the sides are:
$n, n+1, n+1, n, n+1, n+1$
(The area doesn't change when we "move the sides around".)
$A = 2(n+n+1)(n+1+n) - n^2 - n^2 =$
$= 6n^2 + 8n + 2$

(b) We can change 4 sides so that the 3 equations hold.

i. Sides: $n+k, n+1-k, n+1, n+k, n+1-k, n+1$ ($-n < k < n$, $k = 0$ and $k = 1$ do not change the size, $k = 1$ only rotates the hexagon)
$P = 6n + 4$
Area:
$A = 2(n + k + n + 1 - k)(n + 1 + n + k) - (n + k)^2 - (n + k)^2 =$
$= 6n^2 + 8n + 2 + 2k - 2k^2$
We need to show that: $2k - 2k^2 < 0$, it is, iff $k < k^2$
As $1 < k$ or $0 > k$ this is always fulfilled.

ii. Sides: $n, n+1+k, n+1-k, n, n+1+k, n+1-k$ ($1 \leq |k| < n$)
$P = 6n + 4$
Area:
$A = 2(n + n + 1 + k)(n + 1 - k + n) - n^2 - n^2 =$
$= 6n^2 + 8n + 2 - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

iii. Sides: $n-k, n+1, n+1+k, n-k, n+1, n+1+k$ ($-n < k < n$, $k \neq -1$ and $k \neq 0$ since these cases do not modify the hexagon)
$P = 6n + 4$
Area:
$A = 2(n - k + n + 1)(n + 1 + k + n - k) - (n - k)^2 - (n - k)^2 =$
$= 6n^2 + 8n + 2 - 2k - 2k^2$
We need to show that: $-2k - 2k^2 < 0$, it is, iff $0 < k + k^2$.
It is always true for $k \geq 1$ and for $k < -1$.

- case 6: $P = 6n + 5$

(a) Statement to prove: We get the largest area when the sides are:
$n, n+1, n+1, n+1, n, n+2$ (The area doesn't change when we "move the sides around".)
$A = 2(n + n + 1)(n + 1 + n + 1) - (n)^2 - (n + 1)^2 =$
$= 6n^2 + 12n + 3$

(b) Again we may change 4 sides so that the equations hold.

i. Sides: $n+k, n+1-k, n+1, n+1+k, n-k, n+2$ ($1 \leq |k| < n+1$)
$P = 6n + 5$
Area:
$A = 2(n+k+n+1-k)(n+1+n+1+k) - (n+k)^2 - (n+1+k)^2 =$
$= 6n^2 + 12n + 3 - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $1 \leq k$ this is always true.

ii. Sides: $n, n+1+k, n+1-k, n+1, n+k, n+2-k$ ($-n < k < n+1$, $k \neq 1$, $k \neq 0$; the case $k = 0$ do not change anything, $k = 1$ only rotates the hexagon)
$P = 6n + 5$
Area:
$A = 2(n + n + 1 + k)(n + 1 - k + n + 1) - n^2 - (n + 1)^2 =$
$= 6n^2 + 12n + 3 + 2k - 2k^2$
We need to show that: $2k - 2k^2 < 0$, it is, iff $k < k^2$.
For values $k > 1$ and $k < 0$ this is always true.

iii. Sides: $n-k, n+1, n+1+k, n+1-k, n, n+2+k$ ($1 \leq |k| < n$)
$P = 6n + 5$
Area:
$A = 2(n-k+n+1)(n+1+k+n+1-k) - (n-k)^2 - (n+1-k)^2 =$
$= 6n^2 + 12n + 3 - 2k^2$
We need to show that: $-2k^2 < 0$. As $0 \neq k$ this is always true.

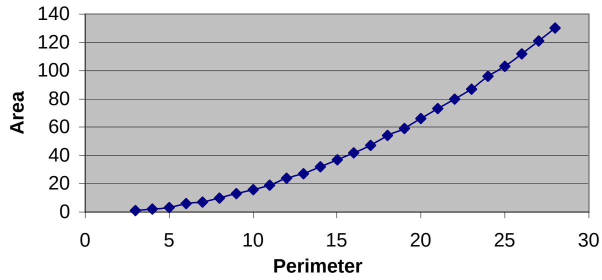


Figure 10. Pareto optimal values on the triangular grid using 1-neighbourhood boundary

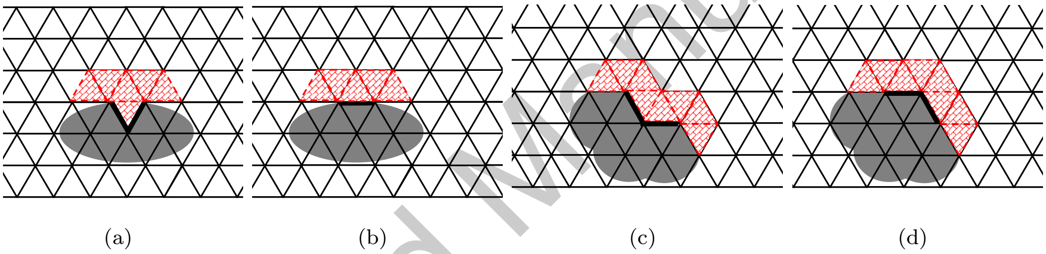


Figure 11. Excluding concavity in the triangular grid with 3-neighbourhood boundary. The object in (a) is extended to object in (b) by excluding corner with angle $\frac{5\pi}{3}$; the object in (c) is extended to (d) by changing an angle $\frac{4\pi}{3}$ to $\frac{2\pi}{3}$ without increasing the perimeter.

All the cases have been considered, the proof is finished. ■

Fig. 10 presents the area of the optimal objects depending on their perimeter. The optimal hexagons approximate the regular hexagons (or in case, the perimeter is divisible by 6, they are the regular hexagons). Increasing the difference of the sides of the hexagon the area is decreasing with a fixed perimeter.

5. Triangular grid with extended neighbourhood

In this section the 3-neighbourhood is considered on the boundary of the objects. The area of a binary image is the number of its triangles in this case, too. As we use 3-neighbourhood boundary, the perimeter of an object is the number of triangles outside of the object that are 3-neighbours of some triangles of the object. For an object L its perimeter is the cardinality of the set

$$\{(x, y, z) \mid (x, y, z) \notin L, \exists x', y', z' \ : \ (x', y', z') \in L, |x-x'| \leq 1, |y-y'| \leq 1, |z-z'| \leq 1\}.$$

5.1 *Shapes of digital circles*

In this subsection we show that optimal shapes are convex in this case, too.

Let us consider the two types of concavity that can occur at objects on the triangular grid. If the object has concavity with an angle $\frac{5\pi}{3}$, then there is a triangle of the perimeter that can be united to the object without increasing the perimeter. (See also Figure 11 (a,b).) If angle $\frac{4\pi}{3}$ occurs at the object, then it can be extended by two triangles (they are at the perimeter originally at that angle) without increasing the perimeter of the object. (See also Figure 11 (c,d).) The object can be extended at all these angles independently, and so, after some (finitely many) iteration of the extension steps the embedding hexagon can be obtained.

Therefore the optimal objects are the (embedding) hexagons themselves, simi-

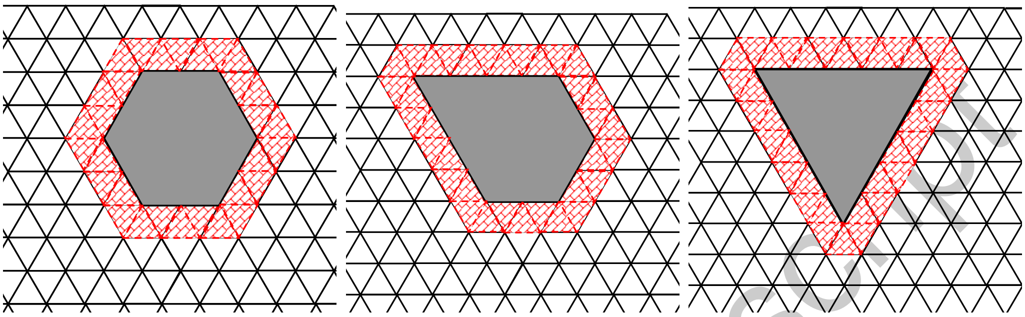


Figure 12. Embedding hexagons and their perimeter using 3-neighbourhood boundary (red color).

larly to the case of smaller neighbourhood we have already seen. Figure 12 shows the pixels that forms the perimeter of some embedding hexagons (including degenerated cases).

We need to find the sidelengths of the optimal hexagons.

5.2 *Optimal size of the hexagons*

In this section we compute the size of the Pareto optimal hexagons.

As it was the case with the 2-neighbourhood in the square grid, the perimeter of the hexagons with 3-neighbourhood in the triangular grid is always even. Actually, if a, b, c, d, e, f denotes the length of the sides (similarly as we used in the previous section), the perimeter of the hexagon is $P = 2(a + b + c + d + e + f) + 6$. This formula works for the degenerated cases as well. The smallest nonempty object (a pixel) has perimeter 12.

Actually, the perimeter of an object can be odd: when a concavity of type (a) of Figure 11 is introduced to a side of hexagon. In this case the area will be one less than the area of the hexagon while the perimeter will be 1 more than the perimeter of the hexagon. In this way one can obtain objects with odd perimeter from perimeter 17 (area: 2). The area cannot be increased when a perimeter is increased by one, opposite to this it must decrease. Therefore objects with odd parameter are not optimal.

THEOREM 5.1 *The next table shows the perimeter and area of the optimal hexagons on the triangular grid using 3-neighbourhood to compute the perimeter.*

Case	*Perimeter*	*Area*
1	$12n$	$6n^2 - 6n + 1$
2	$12n + 2$	$6n^2 - 4n$
3	$12n + 4$	$6n^2 - 2n - 1$
4	$12n + 6$	$6n^2$
5	$12n + 8$	$6n^2 + 2n - 1$
6	$12n + 10$	$6n^2 + 4n$

where n is a natural number.

Proof The perimeter of a hexagon is P while its area is denoted by A. The proof goes by cases. There are six possible remainder of the division perimeter/12 (since the value can only be even). Let n be a natural number.

- case 1: $P = 12n$
  (a) Statement to prove: We get the largest area when the sides are:
  $n, n-1, n, n-1, n, n-1$

(The area doesn't change when we rotate/mirror the hexagon.)
$A = 2(n+n-1)(n+n-1) - n^2 - (n-1)^2 =$
$= 6n^2 - 6n + 1$

(b) We can change 4 sides so that the equations $(*)$ hold.

i. Sides: $n+k, n-1-k, n, n-1+k, n-k, n-1$ $(1 \leq |k| < n)$
$P = 12n$
Area:
$A = 2(n+k+n-1-k)(n+n-1+k) - (n+k)^2 - (n-1+k)^2 =$
$= 6n^2 - 6n + 1 - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

ii. Sides: $n, n-1+k, n-k, n-1, n+k, n-1-k$ $(1 \leq |k| < n)$
$P = 12n$
Area:
$A = 2(n+n-1+k)(n-k+n-1) - n^2 - (n-1)^2 =$
$= 6n^2 - 6n + 1 - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

iii. Sides: $n-k, n-1, n+k, n-1-k, n, n-1+k$ $(1 \leq |k| < n)$
$P = 12n$
Area:
$A = 2(n-k+n-1)(n+k+n-1-k) - (n-k)^2 - (n-1-k)^2 =$
$= 6n^2 - 6n + 1 - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

- case 2: $P = 12n + 2$

(a) Statement to prove: We get the largest area when the sides are:
$n-1, n, n, n-1, n, n$
(The area doesn't change when we "move the sides around".)
$A = 2(n-1+n)(n+n-1) - (n-1)^2 - (n-1)^2 =$
$= 6n^2 - 4n$

(b) We can change 4 sides so that the equations $(*)$ hold.

i. Sides: $n-1+k, n-k, n, n-1+k, n-k, n$ $(-n < k < n,\ k \neq 0$ and $k \neq 1$; since $k = 1$ do not modify the hexagon)
$P = 12n + 2$
Area:
$A = 2(n-1+k+n-k)(n+n-1+k) - (n-1+k)^2 - (n-1+k)^2 =$
$= 6n^2 - 4n + 2k - 2k^2$
We need to show that: $2k - 2k^2 < 0$, it is, iff $k < k^2$
For values $1 < k$ and $k < 0$ this is always true.

ii. Sides: $n-1, n+k, n-k, n-1, n+k, n-k$ $(1 \leq |k| < n)$
$P = 12n + 2$
Area:
$A = 2(n-1+n+k)(n-k+n-1) - (n-1)^2 - (n-1)^2 =$
$= 6n^2 - 4n - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

iii. Sides: $n-1-k, n, n+k, n-1-k, n, n+k$ $(-n < k < n,\ k \neq 0\ k \neq -1$ these cases would not change the size of the hexagon)
$P = 12n + 2$
Area:
$A = 2(n-1-k+n)(n+k+n-1-k) - (n-1-k)^2 - (n-1-k)^2 =$

$= 6n^2 - 4n - 2k^2 - 2k$
We need to show that: $-2k - 2k^2 < 0$, it is, iff $0 < k + k^2$.
It is always true for $k \geq 1$ and for $k < -1$.

- case 3: $P = 12n + 4$
  (a) Statement to prove: We get the largest area when the sides are:
  $n - 1, n, n, n, n - 1, n + 1$ (The area doesn't change when we "move the sides around".)
  $A = 2(n - 1 + n)(n + n) - (n - 1)^2 - n^2 =$
  $= 6n^2 - 2n - 1$
  (b) Again we may change 4 sides so that the equations $(*)$ hold.
  i. Sides: $n - 1 + k, n - k, n, n + k, n - 1 - k, n + 1$ $(1 \leq |k| < n)$
  $P = 12n + 4$
  Area:
  $A = 2(n - 1 + k + n - k)(n + n + k) - (n - 1 + k)^2 - (n + k)^2 =$
  $= 6n^2 - 2n - 1 - 2k^2$
  We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
  As $1 \leq |k|$ this is always true.
  ii. Sides: $n - 1, n + k, n - k, n, n - 1 + k, n + 1 - k$ $(-n < k < n,\ k \neq 0,$ $k \neq -1$, these cases would not change the size of the hexagon)
  $P = 12n + 4$
  Area:
  $A = 2(n - 1 + n + k)(n - k + n) - (n - 1)^2 - n^2 =$
  $= 6n^2 - 2n - 1 - 2k - 2k^2$
  We need to show that: $-2k - 2k^2 < 0$, it is, iff $0 < k + k^2$.
  As $k \geq 1$ or $k < -1$ this is always true.
  iii. Sides: $n - 1 - k, n, n + k, n - k, n - 1, n + 1 + k$ $(-n < k < n,\ k \neq 0,$ $k \neq -1$; $k = -1$ only rotates the hexagon)
  $P = 12n + 4$
  Area:
  $A = 2(n - 1 - k + n)(n + k + n - k) - (n - 1 - k)^2 - (n - k)^2 =$
  $= 6n^2 - 2n - 1 - 2k - 2k^2$
  We need to show that: $-2k - 2k^2 < 0$. As $1 \leq k$ or $0 < k$ this is always true.

- case 4: $P = 12n + 6$
  (a) Statement to prove: the object is optimal if all sides are equal.
  $P = 12n + 6$
  $A = 2(2n)(2n) - 2n^2 = 6n^2$
  (b) We can change the length of (at least) two-two sides: $(1 \leq |k| < n)$
  $n - k, n, n + k, n - k, n, n + k$
  Area:
  $A = 2(n - k + n)(n + k + n - k) - (n - k)^2 - (n - k)^2 =$
  $= 6n^2 - 2k^2$
  As $|k| > 0$ this area is smaller than the one in part (a).

- case 5: $P = 12n + 8$
  (a) Statement to prove: We get the largest area when the sides are:
  $n - 1, n + 1, n, n, n, n + 1$
  (By symmetry, the area doesn't change when we "move the sides a-round".)
  $A = 2(n - 1 + n + 1)(n + n) - (n - 1)^2 - n^2 =$
  $= 6n^2 + 2n - 1$

(b) We can change 4 sides so that the equations on the lengths of the sides hold.

i. Sides: $n-1+k, n+1-k, n, n+k, n-k, n+1$ ($-n < k < n$, $k \neq 0$, $k \neq 1$; $k=1$ and $k=0$ would give no change)
$P = 12n+8$
Area:
$A = 2(n-1+k+n+1-k)(n+n+k) - (n-1+k)^2 - (n+k)^2 =$
$= 6n^2 + 2n - 1 + 2k - 2k^2$
We need to show that: $2k - 2k^2 < 0$, it is, iff $k < k^2$.
As $k < 0$ or $1 < k$, this is always true.

ii. Sides: $n-1, n+1+k, n-k, n, n+k, n+1-k$ ($1 \leq |k| < n$)
$P = 12n+8$
Area:
$A = 2(n-1+n+1+k)(n-k+n) - (n-1)^2 - n^2 =$
$= 6n^2 + 2n - 1 - 2k^2$
We need to show that: $-2k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

iii. Sides: $n-1-k, n+1, n+k, n-k, n, n+1+k$ ($-n < k < n$, $k \neq 0$, $k \neq -1$, these cases would not change the hexagon)
$P = 12n+8$
Area:
$A = 2(n-1-k+n+1)(n+k+n-k) - (n-1-k)^2 - (n-k)^2 =$
$= 6n^2 + 2n - 1 - 2k - 2k^2$
We need to show that: $-2k - 2k^2 < 0$, it is, iff $k + k^2 > 0$.
For values $k \geq 1$ and $k < -1$ this is always true.

- case 6: $P = 12n+10$

(a) Statement to prove: We get the largest area when the sides are:
$n, n, n+1, n, n, n+1$
(The area doesn't change when we "move the sides around".)
$A = 2(n+n)(n+1+n) - n^2 - n^2 = 6n^2 + 4n$

(b) We can change 4 sides:

i. Sides: $n+k, n, n+1-k, n+k, n, n+1-k$ ($-n < k < n$, $k \neq 0$, $k \neq 1$; these cases would not change the hexagon)
$P = 12n+10$
Area:
$A = 2(n+k+n)(n+1-k+n+k) - (n+k)^2 - (n+k)^2 =$
$= 6n^2 + 4n + 2k - 2k^2$
We need to show that: $2k - 2k^2 < 0$, it is, iff $k < k^2$.
For values $k < 0$ and $1 < k$ this is always fulfilled.

ii. Sides: $n-k, n+k, n+1, n-k, n+k, n+1$ ($1 \leq |k| < n$)
$P = 12n+10$
Area:
$A = 2(n-k+n+k)(n+1+n-k) - (n-k)^2 - (n-k)^2 =$
$= 6n^2 + 4n - 2k^2$
We need to show that: $-k^2 < 0$, it is, iff $0 < k^2$.
As $k \neq 0$ this is always true.

iii. Sides: $n, n-k, n+1+k, n, n-k, n+1+k$ ($-n < k < n$, $k \neq 0$, $k \neq -1$; the case $k = -1$ does not modify the hexagon)
$P = 12n+10$
Area:
$A = 2(n+n-k)(n+1+k+n) - n^2 - n^2 =$
$= 6n^2 + 4n - 2k - 2k^2$

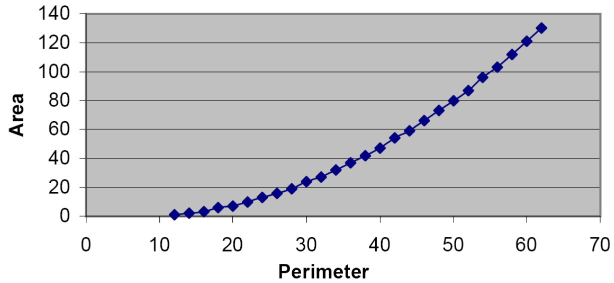


Figure 13. Pareto optimal values on the triangular grid using 3-neighbourhood boundary

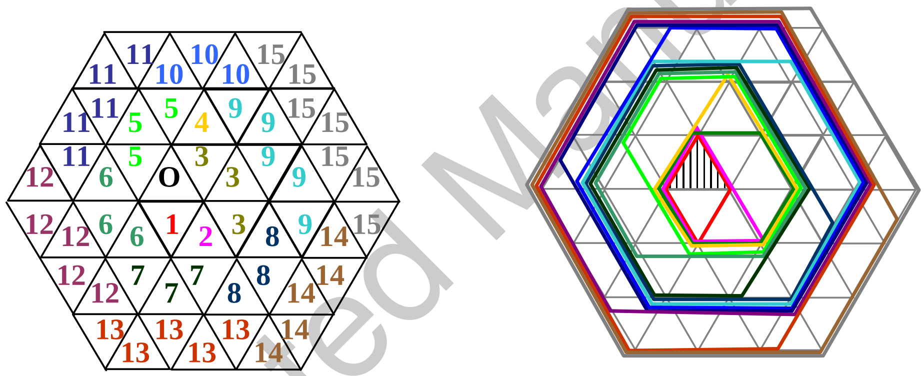


Figure 14. Spiral to build optimal hexagons on the triangular grid.

We need to show that: $-2k - 2k^2 < 0$, it is, iff $k + k^2 > 0$.
For values $k < -1$ and $0 < k$ this is always true.

■

In Figure 13 we show what are the parameters of the optimal objects (of relatively small size).

6. Summary and conclusions

Non-traditional grids are effectively used in image processing and computer graphics. In two dimensions the hexagonal and the triangular grids are the alternatives of the square grid [7, 8, 31–33]. Pareto optimal objects on the triangular grid using both Jordan-type neighbourhood relations are presented. It is proven that they are hexagons in both cases and the lengths of their sides are as close as possible depending on the perimeter. Our formulae presented in the previous sections work for small objects also. It is not surprising that in both cases the optimal hexagons of the continuous case are approximated; increasing the difference of the sides the area is decreasing with a fixed perimeter.

Actually, the optimal shapes can be obtained by a spiral construction, see Figure 14 for the beginning of this procedure.

Our optimal shapes are much closer to the Euclidean optimal shapes (i.e., circles) than the rectangles/squares of the square grid. Our problem is closely connected to the vertex-isoperimetric problem of the triangular grid graph and therefore our results can also be applied on that field. In the other side, our optimal polygons can be viewed as results of an optimization process, therefore these results could be connected and applied in some discrete optimization problems.

The result can be extended by using other neighbourhood structure, i.e., using the nine 2-neighbours; the problem becomes more complex and it is addressed in a forthcoming paper. Another possible extension is to define the perimeter of the objects by the help of a topological coordinate system, where the sides and the corners of the pixels can also be addressed and used (see, e.g., [24]). In this way a topological/geometrical aim can be fulfilled not having the same type of measure for the area and the perimeter.

Acknowledgements

This paper is the extended version of [25]. Some of the results of Sections 2 and 4 can also be found there. Apart from the extension of these sections, entirely new results are provided in Section 5. The authors wish to thank to the reviewers for their comments and remarks that helped to increase the quality of the paper.
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