1949

Generalized Open Sets, Minimality and Connectedness Properties in Relator Spaces

Thesis for the Degree of Doctor of Philosophy (PhD)

by

Muwafaq Mahdi Salih

Supervisor : Dr. Árpád Száz

UNIVERSITY OF DEBRECEN

Doctoral Council of Natural Sciences and Information Technology

Doctoral School of Mathematical and Computational Sciences

Debrecen, 2022

Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences of the University of Debrecen in order to obtain a PhD Degree in Natural Sciences from the University of Debrecen.

I declare that the results published in this thesis are not reported in any other PhD theses.

Debrecen, 07. 03. 2022.

. . . .

Muwafaq M. Salih
doctoral candidate

Hereby I confirm that Muwafaq Mahdi Salih candidate conducted his studies with my supervision within the Mathematical Analysis Program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen between 2017 and 2022. The independent studies and research work of the candidate significantly contributed to the results published in this thesis. I also declare that the results published in the thesis are not reported in any other PhD theses.

I support the acceptance of the dissertation.

Debrecen, 07. 03. 2022.

. . . .

Árpád Száz
supervisor

Generalized Open Sets, Minimality and Connectedness Properties in Relator Spaces

Dissertation submitted in partial fulfilment of the requirements for the doctoral (PhD) degree in mathematics and computing.

Written by: Muwafaq Mahdi Salih certified mathematician

Prepared in the framework of the Mathematical Analysis, Functional Equations and Inequalities program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen.

Dissertation advisor: Dr. Árpád Száz

The official opponents of the dissertation:

Dr.	....
Dr.	....

The evaluation committee:

chairperson:	Dr.	....
members:	Dr.	....
	Dr.	....
	Dr.	....
	Dr.	....

The date of the dissertation defence: .... 2022.

Acknowledgments

While I was discomposed to complete this dissertation as being the most challenging academic step in my life so far, several persons inspired me and paved my way to achieving this mission. Those who I want to give them a warm thank and indicate to their excellent assistance.

At the outset, I wish to express my sincere and deepest gratitude to my supervisor Dr. Árpád Száz for careful guidance, unlimited favors, and continuous motivation throughout my studies at the University of Debrecen. He first suggested the topic of Relator Spaces and guided my first, not easy steps in this direction. Without his advice and unique support, this thesis would never have become a reality. It was a great pleasure for me to work with him and be under his supervision. I am really indebted to him more than he knows.

I am also extremely thankful to the head of Doctoral School of Mathematical and Computational Sciences, Prof. Dr. Páles Zsolt for his usual help, guidance and continuous support during my years of study.

I would also like to acknowledge all the members of the Institute of Mathematics especially, members of the Department of Analysis, and my special gratitude goes to the head of the department Dr. Zoltán Boros for all his efforts to success my work.

I am indebted to my parents for providing me with unfailing support and continuous encouragement throughout my years of study.

I indeed owe gigantic thanks to a very special person in my life, my lovely wife Aveen for her pure, continued and unfailing love, usual support, encouragement, patience, and understanding throughout the years. Thanks for making my dreams come true. Also, to the three most beautiful things in my life, my lovely children: Neven, Mustafa, and Laveen, you have provided me with the needed motivation to live happily and keep me always meditating for a brighter future.

Also, I would like to thank all of those people unnamed for their moral support and those who helped me directly and indirectly to complete my PhD work.

Last but not least, I shall gratefully acknowledge the Tempus Public Foundation "Stipendium Hungaricum Scholarship Programme" and to the Ministry of Higher Education and Scientific Research of Kurdistan Region of Iraq for providing me this opportunity, and supporting me during my years of study.

Contents

Introduction

By Thron [143, p. 18], topological spaces were first suggested by Tietze [144] and Alexandroff [3]. They were later standardized by Bourbaki [12], Kelley [61] and Engelking [39]. (For some historical facts, see also Folland [41].)

If $\mathcal{T}$ is a family of subsets of a set X such that $\mathcal{T}$ is closed under finite intersections and arbitrary unions, then the family $\mathcal{T}$ is called a *topology* on X, and the ordered pair $X(\mathcal{T}) = (X, \mathcal{T})$ is called a *topological space*.

The members of $\mathcal{T}$ are called the *open subsets* of X. While, the members of $\mathcal{F} = \{A^c : A \in \mathcal{T}\}$, where $A^c = X \setminus A$, are called the *closed subsets* of X. And, the members of $\mathcal{T} \cap \mathcal{F}$ are called the *clopen subsets* of X.

Since, $\emptyset = \bigcup \emptyset$ and $X = \bigcap \emptyset$, we necessary have $\{\emptyset, X\} \subseteq \mathcal{T} \cap \mathcal{F}$. If in particular $\mathcal{T} = \{\emptyset, X\}$, then $\mathcal{T}$ is called *minimal (indiscrete)*. While, if $\mathcal{T} \cap \mathcal{F} = \{\emptyset, X\}$, then $\mathcal{T}$ is called *connected* [143, p. 31].

For a subset A of $X(\mathcal{T})$, the sets $A^\circ = \operatorname{int}(A) = \bigcup (\mathcal{T} \cap \mathcal{P}(A))$,

$$A^- = \operatorname{cl}(A) = \operatorname{int}(A^c)^c \qquad \text{and} \qquad A^\dagger = \operatorname{res}(A) = \operatorname{cl}(A) \setminus A$$

are called the *interior, closure and residue of* A, respectively.

Thus, $-$ is a *Kuratowski closure operation* on $\mathcal{P}(X)$. That is, $\emptyset^- = \emptyset$, and $-$ is *extensive, idempotent and additive* in the sense that, for any $A, B \subseteq X$, we have $A \subseteq A^-$, $A^{--} = A^-$ and $(A \cup B)^- = A^- \cup B^-$.

In particular, the members of the family

$$\mathcal{N} = \{ A \subseteq X : \ A^{-\circ} = \emptyset \}$$

are called the *rare (or nowhere dense) subsets* of $X(\mathcal{T})$.

According to Száz [118, 122, 124], the members of the family

$$\mathcal{E} = \{ A \subseteq X : \ \exists\ U \in \mathcal{T} \setminus \{\emptyset\} : \ U \subseteq A \}$$

may be naturally called the *fat subsets* of X.

Hence, it is clear that $\mathcal{E} \neq \emptyset$ if and only if $X \neq \emptyset$. Moreover, $\mathcal{E}$ is a *proper stack* on X in the sense that $\emptyset \notin \mathcal{E}$ and $\mathcal{E}$ is *ascending* in X. That is, if $A \in \mathcal{E}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{E}$ also holds.

Moreover, it can be easily seen that

$$\mathcal{D} = \{ A \subseteq X : \ A^c \notin \mathcal{E} \} = \{ A \subseteq X : \ \forall\ B \in \mathcal{E} : \ A \cap B \neq \emptyset \}.$$

Thus, $\mathcal{D}$ is just the family of all *dense subsets* of X.

For instance, if $A \subseteq X$ such that there exists $B \in \mathcal{E}$ such that $A \cap B = \emptyset$, then $B \subseteq A^c$. Hence, by using that $\mathcal{E}$ is ascending, we can infer that $A^c \in \mathcal{E}$. Therefore, $A^c \notin \mathcal{E}$ implies that $A \cap B \neq \emptyset$ for all $B \in \mathcal{E}$.

In 1922, a subset A of a closure space $X(-)$ was called *regular open* by Kuratowski [64] if $A = A^{-\circ}$. With suitable operations, the family of regular open subsets forms a complete *Boolean agebra* [46, p. 66].

In 1982, a subset A of $X(\mathcal{T})$ was called *preopen* by Mashhour et al. [86] if $A \subseteq A^{-\circ}$. However, by Dontchev [33], preopen sets, under various names, were much earlier studied by several mathematicians.

In 1964, Corson and Michael [15] called a subset A of $X(\mathcal{T})$ *locally dense* if it is a dense subset of some $V \in \mathcal{T}$, i. e., $A \subseteq V \subseteq A^{-}$. Moreover, they noted that this property is equivalent to the inclusion $A \subseteq A^{-\circ}$.

This equivalence was later also stated by Jun et al. [58]. Moreover, Ganster [42] proved that A is preopen if and only if there exist $V \in \mathcal{T}$ and $B \in \mathcal{N}$ such that $A = V \cap B$. (See also Dontchev [33].)

In 1963, a subset A of $X(\mathcal{T})$ was called *semi-open* by Levine [71] if there exists $V \in \mathcal{T}$ such that $V \subseteq A \subseteq V^{-}$. First of all, he showed that the set A is semi-open if and only if $A \subseteq A^{\circ -}$.

Moreover, he also proved that if A is a semi-open subset of $X(\mathcal{T})$, then there exist $V \in \mathcal{T}$ and $B \in \mathcal{N}$ such that $A = V \cup B$ and $V \cap B = \emptyset$. In addition, he also noted that the converse statement is false.

Levine's statement closely resembles to a famous theorem of Hyers [53] which says that an ε–approximately additive function of one Banach space to another is the sum of an additive function and an ε–small function.

Analogously to the paper of Hyers, Levine's paper has also attracted the interest of a surprisingly great number of mathematicians. For instance, by the Google Scholar, until the date of writing this Dissertation, it has been cited by 3080 works.

Moreover, the above statement of Levine was improved by Dlaska et al. [29] who observed that a subset A of $X(\mathcal{T})$ is semi-open if and only if there exist $V \in \mathcal{T}$ and $B \subseteq \operatorname{res}(V)$ such that $A = V \cup B$.

The latter observation was later reformulated, in a more convenient form, by Duszyński and Noiri [34] who noted that a subset A of $X(\mathcal{T})$ is semi-open if and only if there exists $B \subseteq \operatorname{res}(A^{\circ})$ such that $A = A^{\circ} \cup B$.

In particular, in 1965 and 1971, Njåstad [89] and Isomichi [55], not being aware of the paper of Levine, studied semi-open sets under the names β*-sets* and *subcondensed sets*, respectively.

Moreover, Njåstad called a subset A of $X(\mathcal{T})$ an α*–set* if $A \subseteq A^{\circ-\circ}$. And, among others, he proved that the set A is an α–set if and only if there exist $V \in \mathcal{T}$ and $B \in \mathcal{N}$ such that $A = V \setminus B$.

On the other hand, in 1983 the subset A was called β*–open* by Abd El-Monsef et al. [1] if $A \subseteq A^{-\circ-}$. While, in 1986, Andrijević [5] used the term *semi-preopen* instead of β–open.

Actually, Andrijević called a subset A of $X(\mathcal{T})$ to be semi-preopen if there exists a preopen subset V of $X(\mathcal{T})$ such that $V \subseteq A \subseteq V^{-}$. And, he showed that this is equivalent to the inclusion $A \subseteq A^{-\circ-}$.

In 1996, a subset A of $X(\mathcal{T})$ was called b*–open* by Andrijević [8] if $A \subseteq A^{\circ-} \cup A^{-\circ}$. He proved that A is b–open if and only if there exist a preopen subset B and a semi-open subset C of $X(\mathcal{T})$ such that $A = B \cup C$.

Now, having in mind the *poset* (partially ordered set) $\mathcal{P}(X)$ of all subsets of X, a topology $\mathcal{T}$ on X may be naturally called *minimal* and *maximal*, instead of *indiscrete* and *discrete*, if $\mathcal{T} = \{\emptyset, X\}$ and $\mathcal{T} = \mathcal{P}(X)$, respectively,

Moreover, by the celebrated Riesz-Lennes-Hausdorff definition of connectedness [143, 147], the topology $\mathcal{T}$ may be naturally called *connected* if $\mathcal{T} \cap \mathcal{F} = \{\emptyset, X\}$. That is, the family of clopen sets is minimal.

On the other hand, by Steen and Seebach [112, p. 29], the topology $\mathcal{T}$ may be naturally called *hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{T} \setminus \{\emptyset\}$. That is, the family $\mathcal{T} \setminus \{\emptyset\}$ is closed under pairwise (binary) intersections.

Hyperconnected topologies were formerly studied by Bourbaki [13, p. 119] and Levine [75] under the names irreducible and dense topologies. It is noteworthy that $\mathcal{T}$ is hyperconnected if and only if $\mathcal{T} \setminus \{\emptyset\} \subseteq \mathcal{D}$, or equivalently $\mathcal{E} \subseteq \mathcal{D}$.

Also by Steen and Seebach [112, p. 29], the topology $\mathcal{T}$ may be naturally called *ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F} \setminus \{\emptyset\}$. Ultraconnected topologies were formerly studied by Levine [73] under the name strongly connected topologies.

Following Kelley [61, p. 76], a topology $\mathcal{T}$ on X may be naturally called a *door topology* if every subset of X is either open or closed. That is, $\mathcal{P}(X) = \mathcal{T} \cup \mathcal{F}$. Thus, unlike a door, a subset of X can be both open and closed.

While, according to Levine [74], a topology $\mathcal{T}$ on X may be naturally called a *superset topology* if every subset of X which contains a nonvoid member of $\mathcal{T}$ is also in $\mathcal{T}$. That is, $\mathcal{E} \subseteq \mathcal{T}$ in our former notation.

Now, following Dontchev [31], a connected superset topology $\mathcal{T}$ on X may be naturally called *superconnected*. The importance of this notion lies in the fact that a topology $\mathcal{T}$ on X is superconnected if and only if $\mathcal{E} = \mathcal{T} \setminus \{\emptyset\}$.

Moreover, by Bourbaki [12, p. 139] and Hewitt [51], a topology $\mathcal{T}$ on X may be naturally called *submaximal* and *resolvable* if $\mathcal{D} \subseteq \mathcal{T}$ and $\mathcal{D} \not\subseteq \mathcal{E}$, respectively. Namely, $\mathcal{D} \not\subseteq \mathcal{E}$ if and only if $A^c \in \mathcal{D}$ for some $A \in \mathcal{D}$.

Instead of open sets, Hausdorff [49], Kuratowski [65], Weil [146], Tukey [145], Efremovič and Švarc [35, 36], Kowalsky [63], Császár [19], Doičinov [30], Herrlich [50] and others [109, 54, 14, 88] offered some more powerful tools.

For instance, from the works of Davis [28], Pervin [95] and Hunsaker and Lindgren [52], it should have been completely clear that topologies, closures and proximities should not be studied without generalized uniformities.

Considering several papers and some books on generalized uniformities and their induced structures, Száz in [114, 122, 123] offered *relators* (families of relations) as the most suitable basic term on which analysis should be based on.

In the sequel, following a terminology introduced by Száz [114, 122], a family $\mathcal{R}$ of relations on a set X will be called a *relator* on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ will be called a *relator space*.

Thus, relator spaces are generalizations of not only ordered sets [27] and uniform spaces [40], but also topological, closure and proximity spaces [87]. However, to include context spaces [44] a further generalization is needed [122, 123].

For instance, according to [126], each generalized topology $\mathcal{T}$ on X can be easily derived from the family $\mathcal{R}_{\mathcal{T}}$ of all Pervin's preorder relations $R_V = V^2 \cup (V^c \times X)$ with $V \in \mathcal{T}$. Thus, generalized topologies need not be studied separately.

Motivated by corresponding definitions of the various generalized open sets in topological spaces, we introduce and study ten kinds of generalized topologically open sets in relator spaces.

Moreover, having in mind the various connectedness properties considered in topological spaces we introduce and study several reasonable connectedness properties of relator spaces.

The dissertation consists of three chapters. In the first chapter, we shall briefly lay out the necessary prerequisites on relations and relators.

In the second chapter, we shall show that several well-known, basic facts on generalized open sets can be much better treated in relator spaces than in generalized topological and closure spaces.

In the third chapter, we shall introduce several reasonable minimality (well-chainedness) and connectedness properties of relators, and investigate their interrelationships.

Finally, at the end of the dissertation, great possibilities for some further, more general investigations are suggested.

Chapter 1. Relator Spaces

1.1 A Few Basic Facts on Relations

A subset F of a product set $X \times Y$ is called a *relation on* X *to* Y. In particular, a relation on X to itself is called a *relation on* X. And, $\Delta_X = \{(x, x) : x \in X\}$ is called the *identity relation of* X.

If F is a relation on X to Y, then by the above definitions we may also state that F is a relation on $X \cup Y$. However, for several purposes, the latter view of the relation F would be quite unnatural.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{ y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup \{F(x) : x \in A\}$ are called the *images or neighbourhoods of* x *and* A *under* F, respectively.

If $(x, y) \in F$, then instead of $y \in F(x)$, we may also write $x\,F\,y$. However, instead of $F[A]$, we cannot write $F(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

Now, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X]$ may be called the *domain* and *range* of F, respectively. If in particular $D_F = X$, then we may say that F is a *relation of* X *to* Y, or that F is a *non-partial relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation on* X. While, a function $*$ of X^2 to X is called a *binary operation on* X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$, respectively.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection function* of F if $f(x) \in F(x)$ for all $x \in D_F$. By using the Axiom of Choice, it can be shown that every relation is the union of its selection functions.

For a relation F on X to Y, we may naturally define two *set-valued functions* φ_F of X to $\mathcal{P}(Y)$ and Φ_F of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_F(x) = F(x)$ for all $x \in X$ and $\Phi_F(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more general objects than relations on X to Y. In [131, 137, 138], they were briefly called *corelations* on X to Y.

However, a relation on $\mathcal{P}(X)$ to Y should be rather called a *super relation* on X to Y, and a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ should be rather called a *hyper relation* on X to Y. Thus, closures (proximities) [143] are super (hyper) relations.

If F is a relation on X to Y, then one can easily see that $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the images $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement* F^c and the *inverse* F^{-1} can be defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$ and $F^{-1}(y) = \{x \in X : \ y \in F(x)\}$ for all $y \in Y$. Thus, it can be easily seen that $F^c = (X \times Y) \setminus F$.

Moreover, if in addition G is a relation on Y to Z, then the *composition* $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, it can be easily seen that $(G \circ F)[A] = G\big[F[A]\big] = \bigcup_{y \in F[A]} G(y)$ for all $A \subseteq X$.

While, if G is a relation on Z to W, then the *box product* $F \boxtimes G$ can be defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, it can be shown that $(F \boxtimes G)[R] = G \circ R \circ F^{-1}$ for all $R \subseteq X \times Z$ [129].

Hence, by taking $R = \{(x, z)\}$, and $R = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

Now, a relation R on X may be briefly defined to be *reflexive* on X if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$, and *total* if $X^2 \subseteq R \cup R^{-1}$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For any relation R on X, we may also naturally define $R^0 = \Delta_X$ and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also naturally define $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, R^∞ is the smallest preorder relation on X containing R [47].

For $A \subseteq X$, *Pervin's relation* $R_A = A^2 \cup (A^c \times X)$, with $A^2 = A \times A$, is an important preorder on X. While, for a *pseudometric* d on X, *Weil's surrounding* $B_r = \{(x, y) \in X^2 : \ d(x, y) < r\}$, with $r > 0$, is an important tolerance on X [146].

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap \big(A^c\big)^2$ is already an equivalence relation on X. And, more generally if $\mathcal{A}$ is a *cover (partition)* of X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

As an important generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker-Lindgren relation* $R_{(A, B)} = (A \times B) \cap (A^c \times Y)$ [52]. Namely, thus we evidently have $R_A = R_{(A, A)}$.

The Pervin relations R_A and the Hunsaker-Lindgren relations $R_{(A, B)}$ were actually first used by Davis [28] and Császár [19, pp. 42 and 351] in some less explicit and convenient forms, respectively.

1.2 A Few Basic Facts on Relators

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator on* X *to* Y, and the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space*. For the origins of this notion, see [114, 123].

If in particular $\mathcal{R}$ is a relator on X to itself, then $\mathcal{R}$ is simply called a *relator on* X. Thus, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ instead of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$.

Relator spaces of this simpler homogeneous type are already substantial generalizations of the various *ordered sets* [27] and *uniform spaces* [40]. However, they are insufficient for some important purposes. (See, for instance, [44] and [122].)

A relator $\mathcal{R}$ on X to Y, or the relator space $(X, Y)(\mathcal{R})$, is called *simple* if $\mathcal{R} = \{R\}$ for some relation R on X to Y. Simple relator spaces of the forms $(X, Y)(R)$ and $X(R)$ were called *formal contexts* and *gosets* in [44] and [134], respectively.

Moreover, a relator $\mathcal{R}$ on X, or the relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive on X. Thus, we may also naturally speak of *preorder, tolerance, and equivalence relators*.

For instance, for a family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{R_A : A \in \mathcal{A}\}$, where $R_A = A^2 \cup (A^c \times X)$, is an important preorder relator on X. Such relators were first used by Pervin [95] and Levine [77].

While, for a family $\mathcal{D}$ of *pseudo-metrics* on X, the family $\mathcal{R}_{\mathcal{D}} = \{B_r^d : r > 0, d \in \mathcal{D}\}$, where $B_r^d = \{(x, y) : d(x, y) < r\}$, is an important tolerance relator on X. Such relators were first considered by Weil [146].

Moreover, if $\mathfrak{S}$ is a family of *covers (partitions)* of X, then the family $\mathcal{R}_{\mathfrak{S}} = \{S_{\mathcal{A}} : \mathcal{A} \in \mathfrak{S}\}$, where $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$, is an important tolerance (equivalence) relator on X. Equivalence relators were first investigated by Levine [76].

If $\star$ is a unary operation for relations on X to Y, then for any relator $\mathcal{R}$ on X to Y we may naturally define $\mathcal{R}^{\star} = \{R^{\star} : R \in \mathcal{R}\}$. However, this plausible notation may cause some confusions whenever, for instance, $\star = c$.

In particular, for any relator $\mathcal{R}$ on X, we may naturally define $\mathcal{R}^{\infty} = \{R^{\infty} : R \in \mathcal{R}\}$. Moreover, we may also naturally define $\mathcal{R}^{\partial} = \{S \subseteq X^2 : S^{\infty} \in \mathcal{R}\}$. These operations were first introduced by Mala [80, 82] and Pataki [93, 94].

While, if $*$ is a binary operation for relations, then for any two relators $\mathcal{R}$ and $\mathcal{S}$ we may naturally define $\mathcal{R} * \mathcal{S} = \{R * S : R \in \mathcal{R}, S \in \mathcal{S}\}$. However, this plausible notation may again cause some confusions whenever, for instance, $* = \cap$.

Therefore, in general we shall rather write $\mathcal{R} \wedge \mathcal{S} = \{R \cap S : R \in \mathcal{R}, S \in \mathcal{S}\}$. Moreover, for instance, we shall also write $\mathcal{R} \triangle \mathcal{R}^{-1} = \{R \cap R^{-1} : R \in \mathcal{R}\}$. Note that thus $\mathcal{R} \triangle \mathcal{R}^{-1}$ is a symmetric relator such that $\mathcal{R} \triangle \mathcal{R}^{-1} \subseteq \mathcal{R} \wedge \mathcal{R}^{-1}$.

A function $\square$ of the family of all relators on X to Y is called a *direct (indirect) unary operation for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^{\square} = \square(\mathcal{R})$ is a relator on X to Y (on Y to X).

For instance, c and -1 are "involutive operations" for relators. While, ∞ and ∂ are "projection operations" for relators. Moreover, the operation $\square = c$, ∞ or ∂ is inversion compatible in the sense that $\mathcal{R}^{\square\, -1} = \mathcal{R}^{-1\, \square}$.

More generally, a function $\mathfrak{F}$ of the family of all relators on X to Y is called a *structure for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ is in a power set depending only on X and Y.

For instance, if $\operatorname{cl}_{\mathcal{R}}(B) = \bigcap \{R^{-1}[B] : R \in \mathcal{R}\}$ for every relator $\mathcal{R}$ on X to Y and $B \subseteq Y$, then the function $\mathfrak{F}$, defined by $\mathfrak{F}(\mathcal{R}) = \operatorname{cl}_{\mathcal{R}}$, is a structure for relators such that $\mathfrak{F}(\mathcal{R}) \subseteq \mathcal{P}(Y) \times X$, and thus $\mathfrak{F}(\mathcal{R}) \in \mathcal{P}\big(\mathcal{P}(Y) \times X\big)$.

A structure $\mathfrak{F}$ for relators is called *increasing* if $\mathcal{R} \subseteq \mathcal{S}$ implies $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and S on X to Y. And, $\mathfrak{F}$ is called *quasi-increasing* if $R \in \mathcal{R}$ implies $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X to Y. Note that here $\mathfrak{F}_R = \mathfrak{F}_{\{R\}}$.

Moreover, the structure $\mathfrak{F}$ is called *union-preserving* if $\mathfrak{F}_{\bigcup_{i \in I} \mathcal{R}_i} = \bigcup_{i \in I} \mathfrak{F}_{\mathcal{R}_i}$ for any family $(\mathcal{R}_i)_{i \in I}$ of relators on X to Y. It can be shown that $\mathfrak{F}$ is union-preserving if and only if $\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_R$ for every relator $\mathcal{R}$ on X to Y [131].

In particular, an increasing operation $\square$ for relators on X to Y is called a *projection or modification operation* for relators if it is idempotent in the sense that $\mathcal{R}^{\square\square} = \mathcal{R}^{\square}$ holds for any relator $\mathcal{R}$ on X to Y.

Moreover, a modification operation $\square$ for relators on X to Y is called a *closure or refinement operation* for relators if it is extensive in the sense that $\mathcal{R} \subseteq \mathcal{R}^{\square}$ holds for any relator $\mathcal{R}$ on X to Y.

By using Pataki connections [93, 139], several closure operations can be derived from union-preserving structures. However, more generally, one can find first the Galois adjoint $\mathfrak{G}$ of such a structure $\mathfrak{F}$, and then take $\square_{\mathfrak{F}} = \mathfrak{G} \circ \mathfrak{F}$ [125].

Now, for some operation $\square$ for relators, a relator $\mathcal{R}$ on X to Y may be naturally called $\square$*–fine* if $\mathcal{R}^{\square} = \mathcal{R}$. And, for some structure $\mathfrak{F}$ for relators, two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y may be naturally called $\mathfrak{F}$*–equivalent* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.

Moreover, for some structure $\mathfrak{F}$ for relators, a relator $\mathcal{R}$ on X to Y may, for instance, be naturally called $\mathfrak{F}$*–simple* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_R$ for some relation R on X to Y. Thus, in particular singleton relators have to be actually called *properly simple*.

1.3 Structures Derived from Relators

Notation 1.3.1. Throughout the sequel, for the readers convenience and the requirements of most of the forthcoming sections, **we shall assume that** $\mathcal{R}$ **is a relator on** X, not a relator on X to Y.

Definition 1.3.2. For any $A, B \subseteq X$ and $x, y \in X$, we define

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subseteq B$ for some $R \in \mathcal{R}$;

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$;

(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$;

(4) $x \in \sigma_{\mathcal{R}}(y)$ if $x \in \operatorname{int}_{\mathcal{R}}(\{y\})$;

(5) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$;

(6) $x \in \rho_{\mathcal{R}}(y)$ if $x \in \operatorname{cl}_{\mathcal{R}}(\{y\})$;

(7) $B \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$;

(8) $B \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(B) = X$.

Remark 1.3.3. The relations $\operatorname{Int}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$ are called *the proximal, topological and infinitesimal interiors* generated by $\mathcal{R}$, respectively. While, the members of the families, $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are called the *fat and dense subsets* of the relator space $X(\mathcal{R})$, respectively.

The origins of the relations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ go back to Efremović's proximity δ [35] and Smirnov's strong inclusion $\Subset$ [111], respectively. While, the notations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$, and family $\mathcal{E}_{\mathcal{R}}$, together with its dual $\mathcal{D}_{\mathcal{R}}$, were first used by Száz [114, 116, 118, 124].

The following theorem shows that the big closure and interior relations are equivalent tools in a relator space.

Theorem 1.3.4. *For any* $A \subseteq X$, *we have*

(1) $\operatorname{Cl}_{\mathcal{R}}(A) = \operatorname{Int}_{\mathcal{R}}(A^c)^c$; (2) $\operatorname{Int}_{\mathcal{R}}(A) = \operatorname{Cl}_{\mathcal{R}}(A^c)^c$.

Proof. For any $B \subseteq X$, we have

$$\begin{aligned} B \in \operatorname{Cl}_{\mathcal{R}}(A) &\iff \forall\ R \in \mathcal{R}:\ R[B] \cap A \neq \emptyset \iff \forall\ R \in \mathcal{R}:\ R[B] \not\subseteq A^c \\ &\iff B \notin \operatorname{Int}_{\mathcal{R}}(A^c) \iff B \in \operatorname{Int}_{\mathcal{R}}(A^c)^c. \end{aligned}$$

Therefore, assertion (1) is true. Now, assertion (2) can be derived from (1) by using complementations.

Remark 1.3.5. By using the notation $\mathcal{C}_X(A) = X \setminus A$, assertion (1) can be expressed in the more concise form that $\operatorname{Cl}_{\mathcal{R}} = \big(\operatorname{Int}_{\mathcal{R}} \circ \mathcal{C}_X\big)^c = \big(\operatorname{Int}_{\mathcal{R}}\big)^c \circ \mathcal{C}_X$.

From Theorem 1.3.4, we can easily derive the following

Theorem 1.3.6. *For any* $A \subseteq X$, *we have*
(1) $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{int}_{\mathcal{R}}(A^c)^c$; (2) $\operatorname{int}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(A^c)^c$.

Remark 1.3.7. By using the notations $A^- = \operatorname{cl}_{\mathcal{R}}(A)$ and $A^\circ = \operatorname{int}_{\mathcal{R}}(A)$, assertion (1) can be expressed in the more concise form that $- = c \circ c$, i. e., $-c = c\circ$.

The small closure and interior are usually much weaker tools than the big ones. Namely, in general, we can only prove the following

Theorem 1.3.8. *For any* $A, B \subseteq X$, *we have*
(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ *implies that* $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$;
(2) $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *implies that* $A \in \operatorname{Cl}_{\mathcal{R}}(B)$.

Concerning closures and interiors, we can also prove the following two theorems which show that, despite their equivalences, closures are sometimes more convenient tools than interiors.

Theorem 1.3.9. *We have*
(1) $\operatorname{Cl}_{\mathcal{R}^{-1}} = \operatorname{Cl}_{\mathcal{R}}^{-1}$; (2) $\operatorname{Int}_{\mathcal{R}^{-1}} = \mathcal{C}_X \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}_X$.

Theorem 1.3.10. *For any* $A \subseteq X$, *we have*
(1) $\operatorname{cl}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} R^{-1}[A]$; (2) $\operatorname{int}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} R^{-1}[A^c]^c$.

Proof. By Definition 1.3.2, for any $x \in X$ we have

$$\begin{aligned} x \in \operatorname{cl}_{\mathcal{R}}(A) &\iff \forall\ R \in \mathcal{R}:\ R[\{x\}] \cap A \neq \emptyset \\ &\iff \forall\ R \in \mathcal{R}:\ x \in R^{-1}[A] \iff x \in \bigcap_{R \in \mathcal{R}} R^{-1}[A]. \end{aligned}$$

Therefore, assertion (1) is true. Hence, by using Theorem 1.3.6, we can easily see that assertion (2) is also true.

From the particular case $A = \{y\}$ of this theorem, we can easily derive

Corollary 1.3.11. *We have* $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = \big(\bigcap \mathcal{R}\big)^{-1}$.

Moreover, by using the particular case $\mathcal{R} = \{R\}$ of Theorem 1.3.10, we can prove

Theorem 1.3.12. *For any* $R \in \mathcal{R}$ *and* $A, B \subseteq X$, *we have*

$$\mathrm{cl}_{R^{-1}}(A) \subseteq B \iff A \subseteq \mathrm{int}_R(B).$$

Proof. By Theorem 1.3.10 and partial converse of Theorem 1.3.8, we have

$$\mathrm{cl}_{R^{-1}}(A) \subseteq B \iff R[A] \subseteq B \iff A \in \mathrm{Int}_R(B) \iff A \subseteq \mathrm{int}_R(B).$$

Remark 1.3.13. This shows that the mappings $A \mapsto \mathrm{cl}_{R^{-1}}(A)$ and $B \mapsto \mathrm{int}_R(B)$, where $A, B \subseteq X$, establish a Galois connection between the posets $\mathcal{P}(X)$ and itself.

The above important closure-interior Galois connection, used first in [136], is not independent from the well-known upper and lower bound one [128].

The following two closely related theorems show that the fat and dense sets are also equivalent tools in a relator space.

Theorem 1.3.14. *For any* $A \subseteq X$, *we have*
(1) $A \in \mathcal{D}_{\mathcal{R}} \iff A^c \notin \mathcal{E}_{\mathcal{R}}$; (2) $A \in \mathcal{E}_{\mathcal{R}} \iff A^c \notin \mathcal{D}_{\mathcal{R}}$.

Proof. To prove (1), note that, by Definition 1.3.2 and Theorem 1.3.6, we have

$$\begin{aligned} A \in \mathcal{D}_{\mathcal{R}} \iff \mathrm{cl}_{\mathcal{R}}(A) = X &\iff \mathrm{int}_{\mathcal{R}}(A^c)^c = X \\ &\iff \mathrm{int}_{\mathcal{R}}(A^c) = \emptyset \iff A^c \notin \mathcal{E}_{\mathcal{R}}. \end{aligned}$$

Theorem 1.3.15. *For any* $A \subseteq X$, *we have*
(1) $A \in \mathcal{D}_{\mathcal{R}}$ *if and only if* $A \cap E \neq \emptyset$ *for all* $E \in \mathcal{E}_{\mathcal{R}}$;
(2) $A \in \mathcal{E}_{\mathcal{R}}$ *if and only if* $A \cap D \neq \emptyset$ *for all* $D \in \mathcal{D}_{\mathcal{R}}$.

Proof. This theorem can, in principle, be derived from Theorem 1.3.14. However, it can be more easily proved with the help of Theorem 1.3.17.

Namely, assume that $A \in \mathcal{D}_{\mathcal{R}}$, then for any $x \in X$ and $R \in \mathcal{R}$ we have $R(x) \cap A \neq \emptyset$. Moreover, if $E \in \mathcal{E}_{\mathcal{R}}$, then there exists $x_0 \in X$ and $R_0 \in R$ such that $R_0(x_0) \subseteq E$. Therefore, $\emptyset \neq R_0(x_0) \cap A \subseteq E \cap A$, and thus $A \cap E \neq \emptyset$.

Remark 1.3.16. By the corresponding definitions, we have $R(x) \in \mathcal{E}_{\mathcal{R}}$ and thus also $R(x)^c \notin \mathcal{D}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$.

While, by using the notation $\mathcal{U}_{\mathcal{R}}(x) = \mathrm{int}_{\mathcal{R}}^{-1}(x) = \{B \subseteq X : \ x \in \mathrm{int}_{\mathcal{R}}(B)\}$, we can note that $\mathcal{E}_{\mathcal{R}} = \bigcup_{x \in X} \mathcal{U}_{\mathcal{R}}(x)$.

Now, by using the corresponding properties of the relations $\mathrm{cl}_{\mathcal{R}}$ and $\mathrm{int}_{\mathcal{R}}$, we can easily establish the following theorem.

Theorem 1.3.17. *For any* $A \subseteq X$, *we have*
(1) $A \in \mathcal{E}_{\mathcal{R}}$ *if and only if* $R(x) \subseteq A$ *for some* $x \in X$ *and* $R \in \mathcal{R}$;
(2) $A \in \mathcal{D}_{\mathcal{R}}$ *if and only if* $R(x) \cap A \neq \emptyset$ *for all* $x \in X$ *and* $R \in \mathcal{R}$.

By using Definition 1.3.2, we may naturally introduce several further important definitions. For instance, we may also naturally have the following

Definition 1.3.18. For any $A \subseteq X$, we define
(1) $\operatorname{bnd}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(A) \setminus \operatorname{int}_{\mathcal{R}}(A)$;
(2) $\operatorname{res}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(A) \setminus A$; (3) $\operatorname{bor}_{\mathcal{R}}(A) = A \setminus \operatorname{int}_{\mathcal{R}}(A)$.

Remark 1.3.19. Somewhat differently, the *border, boundary and residue of a set* in neighbourhood and closure spaces were also introduced by Hausdorff [49] and Kuratowski [64, pp. 4–5]. (See also Elez and Papaz [38] for a recent treatment.)

If in particular $\mathcal{R}$ is reflexive, then by Definition 1.3.2, for any $A \subseteq X$, we have $\operatorname{int}_{\mathcal{R}}(A) \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}}(A)$. Therefore,
$$\operatorname{bnd}_{\mathcal{R}}(A) = \operatorname{res}_{\mathcal{R}}(A) \cup \operatorname{bor}_{\mathcal{R}}(A) = \operatorname{res}_{\mathcal{R}}(A) \cup \operatorname{res}_{\mathcal{R}}(A^c).$$
Namely, by using Definition 1.3.18 and Remark 1.3.7, we can easily see that
$$\operatorname{res}_{\mathcal{R}}(A^c) = A^{c-} \setminus A^c = A^{c-} \cap A^{cc} = A^{\circ c} \cap A = A \setminus A^{\circ} = \operatorname{bor}_{\mathcal{R}}(A).$$
Note that if in particular $A \in \mathcal{T}_{\mathcal{R}}$ in the sense that $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$, then $\operatorname{bor}_{\mathcal{R}}(A) = \emptyset$. Therefore, in this particular case, by the above equality, we state that $\operatorname{bnd}_{\mathcal{R}}(A) = \operatorname{res}_{\mathcal{R}}(A)$.

1.4 Further Structures Derived from Relators

By using Definition 1.3.2, we may also naturally introduce the following

Definition 1.4.1. For any $A \subseteq X$, we also define

(1) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$; (2) $A \in \tau\!\!\!-_{\mathcal{R}}$ if $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$;
(3) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$; (4) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$;
(5) $A \in \mathcal{N}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$; (6) $A \in \mathcal{M}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{D}_{\mathcal{R}}$.

Remark 1.4.2. The members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$ are called the *proximally open, topologically open and rare (or nowhere dense) subsets* of the relator space $X(\mathcal{R})$, respectively.

The family $\tau_{\mathcal{R}}$ was first introduced by Száz in [116, 118]. While, the practical notation $\tau\!\!\!-_{\mathcal{R}}$ was suggested by János Kurdics who first noticed that "connectedness" is a particular case of "well-chainedness". (See [66, 68, 94, 107].)

By using the corresponding results of Section 1.3, we can easily establish the following theorems.

Theorem 1.4.3. *For any* $A \subseteq X$, *we have*
(1) $A \in \tau\!\!\!-_{\mathcal{R}} \iff A^c \in \tau_{\mathcal{R}}$; (2) $A \in \tau_{\mathcal{R}} \iff A^c \in \tau\!\!\!-_{\mathcal{R}}$.

Theorem 1.4.4. *We have*

(1) $\mathscr{F}_{\mathcal{R}} = \mathscr{T}_{\mathcal{R}^{-1}}$; (2) $\mathscr{T}_{\mathcal{R}} = \mathscr{F}_{\mathcal{R}^{-1}}$.

Theorem 1.4.5. *If $\mathcal{R}$ is nonvoid, then*

(1) $\{ \emptyset , X \} \subseteq \mathscr{T}_{\mathcal{R}}$; (2) $\{ \emptyset , X \} \subseteq \mathscr{F}_{\mathcal{R}}$.

Remark 1.4.6. Conversely, it can be shown that if $\mathcal{A}$ is a family of subsets of X containing $\emptyset$ and X, then there exists a nonvoid, preorder relator $\mathcal{R}$ such that $\mathcal{A} = \mathscr{T}_{\mathcal{R}}$. (See again [126] .) Thus, minimal structures should not also be studied without generalized uniformities.

Theorem 1.4.7. *For any $A \subseteq X$, we have*

(1) $A \in \mathcal{T}_{\mathcal{R}}$ *if and only if for each $x \in A$ there exists $R \in \mathcal{R}$ such that $R(x) \subseteq A$;*

(2) $A \in \mathcal{F}_{\mathcal{R}}$ *if and only if for each $x \in A^c$ there exists $R \in \mathcal{R}$ such that $A \cap R(x) = \emptyset$.*

Theorem 1.4.8. *For any $A \subseteq X$, we have*

(1) $A \in \mathcal{F}_{\mathcal{R}} \iff A^c \in \mathcal{T}_{\mathcal{R}}$; (2) $A \in \mathcal{T}_{\mathcal{R}} \iff A^c \in \mathcal{F}_{\mathcal{R}}$.

Corollary 1.4.9. *If $A \subseteq X$ and $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \cap V = \emptyset$, then $\operatorname{cl}_{\mathcal{R}}(A) \cap V = \emptyset$ also hold.*

Proof. By Theorem 1.4.3, we have $V^c \in \mathcal{F}_{\mathcal{R}}$. Thus, by Definition 1.4.1, we also have $V^{c-} \subseteq V^c$. Hence, by using the increasingness of the operation $-$, we can already see that $A \cap V = \emptyset \implies A \subseteq V^c \implies A^- \subseteq V^{c-} \implies A^- \subseteq V^c \implies A^- \cap V = \emptyset$.

Remark 1.4.10. Note that if $\mathcal{R}$ is reflexive, then $A \subseteq A^-$ for any $A \subseteq X$. Therefore, $A^- \cap V = \emptyset$ trivially implies $A \cap V = \emptyset$ for any $A, V \subseteq X$.

Theorem 1.4.11. *We have*

(1) $\mathscr{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$; (2) $\mathscr{F}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Remark 1.4.12. In particular, we have

(1) $\mathscr{T}_R = \mathcal{T}_R$; (2) $\mathscr{F}_R = \mathcal{F}_R$.

Theorem 1.4.13. *We have*

(1) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$; (2) $\mathcal{D}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{X\}$.

Remark 1.4.14. Hence, by using global complementations, we can easily infer that $\mathcal{F}_{\mathcal{R}} \subseteq \big(\mathcal{D}_{\mathcal{R}}\big)^c \cup \{X\}$ and $\mathcal{D}_{\mathcal{R}} \subseteq \big(\mathcal{F}_{\mathcal{R}}\big)^c \cup \{X\}$.

Theorem 1.4.15. *For any $A \subseteq X$ we have*

(1) $A \in \mathcal{E}_{\mathcal{R}}$ *if $V \subseteq A$ for some $V \in \mathcal{T}_R \setminus \{\emptyset\}$;*

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *if $A \setminus W \neq \emptyset$ for all $W \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$.*

Remark 1.4.16. The fat sets are frequently more convenient tools than the topologically open ones. For instance, if $\leq$ is a relation on X, then $\mathcal{T}_{\leq}$ and $\mathcal{E}_{\leq}$ are the families of all ascending and residual subsets of the goset $X(\leq)$, respectively.

Moreover, if in particular $X = \mathbb{R}$ and $R(x) = \{ x - 1 \} \cup [x , +\infty [$ for all $x \in X$, then R is a reflexive relation on X such that $\mathcal{T}_R = \{ \emptyset , X \}$, but $\mathcal{E}_R$ is quite a large family. Namely, the supersets of each $R(x)$ are also contained in $\mathcal{E}_R$.

However, the importance of fat and dense sets lies mainly in the following

Definition 1.4.17. If φ and ψ are functions of a relator space $\Gamma(\mathcal{U})$ to X, then by using the notation $(\varphi, \psi)(\gamma) = \big(\varphi(\gamma),\ \psi(\gamma)\big)$ for all $\gamma \in \Gamma$, we may also naturally define

(1) $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$ if $(\varphi, \psi)^{-1}[R] \in \mathcal{E}_{\mathcal{U}}$ for all $R \in \mathcal{R}$,
(2) $\varphi \in \mathrm{Adh}_{\mathcal{R}}(\psi)$ if $(\varphi, \psi)^{-1}[R] \in \mathcal{D}_{\mathcal{U}}$ for all $R \in \mathcal{R}$.

Moreover, for any $x \in X$, we may also naturally define

(3) $x \in \mathrm{lim}_{\mathcal{R}}(\psi)$ if $x_{\Gamma} \in \mathrm{Lim}_{\mathcal{R}}(\psi)$,
(4) $x \in \mathrm{adh}_{\mathcal{R}}(\psi)$ if $x_{\Gamma} \in \mathrm{Adh}_{\mathcal{R}}(\psi)$,

where x_{Γ} is a function of Γ to X such that $x_{\Gamma}(\gamma) = x$ for all $\gamma \in \Gamma$.

Remark 1.4.18. Fortunately, the small limit and adherece relations are equivalent to the small closure and interior ones.

However, the big limit and adherence relations, suggested by Efremović and Švarc [36], are usually stronger tools than the big closure and interior ones.

In this respect, it seems convenient to only mention here the following

Theorem 1.4.19. *For any* $A, B \subseteq X$, *the following assertions are equivalent:*

(1) $A \in \mathrm{Cl}_{\mathcal{R}}(B)$*;*
(2) *there exist functions* φ *and* ψ *of the poset* $\mathcal{R}(\supseteq)$ *to* A *and* B *respectively, such that* $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$*;*
(3) *there exist functions* φ *and* ψ *of a relator space* $\Gamma(\mathcal{U})$ *to* A *and* B*, respectively, such that* $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$.

Proof. For instance, if (1) holds, then for each $R \in \mathcal{R}$, we have $R[A] \cap B \neq \emptyset$. Therefore, there exist $\varphi(R) \in A$ and $\psi(R) \in B$ such that $\psi(R) \in R\big(\varphi(R)\big)$. Hence, we can already infer that $(\varphi, \psi)(R) = \big(\varphi(R),\ \psi(R)\big) \in R$, and thus also $R \in (\varphi, \psi)^{-1}[R]$.

Therefore, if $R \in \mathcal{R}$, then for any $S \in \mathcal{R}$, with $R \supseteq S$, we have

$$S \in (\varphi, \psi)^{-1}[S] \subseteq (\varphi, \psi)^{-1}[R].$$

This shows that $(\varphi, \psi)^{-1}[R]$ is a fat subset of $\mathcal{R}(\supseteq)$, and thus $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$.

Remark 1.4.20. According to [123], for any $A, B \subseteq X$, we may also naturally define

(1) $A \in \mathrm{Lb}_{\mathcal{R}}(B)$ and $B \in \mathrm{Ub}_{\mathcal{R}}(A)$ if $A \times B \subseteq R$ for some $R \in \mathcal{R}$;
(2) $\mathrm{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \mathrm{Lb}_{\mathcal{R}}(A)$;
(3) $\mathrm{Sup}_{\mathcal{R}}(A) = \mathrm{Min}_{\mathcal{R}}\big(\mathrm{Ub}_{\mathcal{R}}(A)\big)$.

However, the above algebraic structures are not independent of the former topological ones. Namely, by using appropriate complements, it can be easily shown that

$$\mathrm{Lb}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}_X \qquad \text{and} \qquad \mathrm{Int}_{\mathcal{R}} = \mathrm{Lb}_{\mathcal{R}^c} \circ \mathcal{C}_X .$$

Therefore, in contrast to a common belief, some algebraic and topological structures are just as closely related to each other, by the above equalities, as the exponential and the trigonometric functions are so by the celebrated Euler formulas.

1.5 Closure Operations on Relators

Similar operations for relators have formerly been studied by Kenyon [62], Nakano–Nakano [88], Száz [119, 121] and Pataki [93].

Definition 1.5.1. The relators

$$\mathcal{R}^{*} = \big\{ S \subseteq X^2 : \quad \exists\ R \in \mathcal{R} : \quad R \subseteq S \big\};$$
$$\mathcal{R}^{\#} = \big\{ S \subseteq X^2 : \quad \forall\ A \subseteq X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subseteq S[A] \big\};$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X^2 : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subseteq S(x) \big\};$$
$$\mathcal{R}^{\vartriangle} = \big\{ S \subseteq X^2 : \quad \forall\ x \in X : \quad \exists\ u \in X : \quad \exists\ R \in \mathcal{R} : \quad R(u) \subseteq S(x) \big\}$$

are called the *uniform, proximal, topological and paratopological closures (or refinements)* of the relator $\mathcal{R}$, respectively.

Remark 1.5.2. Thus, we evidently have $\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge} \subseteq \mathcal{R}^{\vartriangle}$. Moreover, we can also easily prove that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*\infty} \subseteq \mathcal{R}^{\infty *} \subseteq \mathcal{R}^{*}$.

Remark 1.5.3. However, it is now more important to note that, because of Definition 1.3.2, we also have

$$\mathcal{R}^{\#} = \big\{ S \subseteq X^2 : \quad \forall\ A \subseteq X : \quad A \in \operatorname{Int}_{\mathcal{R}}\big(S[A] \big) \big\},$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X^2 : \quad \forall\ x \in X : \quad x \in \operatorname{int}_{\mathcal{R}}\big(S(x) \big) \big\},$$
$$\mathcal{R}^{\vartriangle} = \big\{ S \subseteq X^2 : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \big\}.$$

Morover, by using Pataki connections [93, 139, 107], the following equivalences and their corollaries can be proved in a unified way.

Theorem 1.5.4. *$\#$, $\wedge$ and $\vartriangle$ are closure operations for relators on X such that, for any relator $\mathcal{S}$ on X, we have*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\#} \iff \mathcal{S}^{\#} \subseteq \mathcal{R}^{\#} \iff \operatorname{Int}_{\mathcal{S}} \subseteq \operatorname{Int}_{\mathcal{R}} \iff \operatorname{Cl}_{\mathcal{R}} \subseteq \operatorname{Cl}_{\mathcal{S}}$*;*

(2) $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge} \iff \operatorname{int}_{\mathcal{S}} \subseteq \operatorname{int}_{\mathcal{R}} \iff \operatorname{cl}_{\mathcal{R}} \subseteq \operatorname{cl}_{\mathcal{S}}$*;*

(3) $\mathcal{S} \subseteq \mathcal{R}^{\vartriangle} \iff \mathcal{S}^{\vartriangle} \subseteq \mathcal{R}^{\vartriangle} \iff \mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \iff \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$.

Corollary 1.5.5. *We have*

(1) *$\mathcal{S} = \mathcal{R}^{\#}$ is the largest relator on X such that $\operatorname{Int}_{\mathcal{S}} \subseteq \operatorname{Int}_{\mathcal{R}}$ $\big(\operatorname{Int}_{\mathcal{S}} = \operatorname{Int}_{\mathcal{R}}\big)$, or equivalently $\operatorname{Cl}_{\mathcal{R}} \subseteq \operatorname{Cl}_{\mathcal{S}}$ $\big(\operatorname{Cl}_{\mathcal{S}} = \operatorname{Cl}_{\mathcal{R}}\big)$;*

(2) *$\mathcal{S} = \mathcal{R}^{\wedge}$ is the largest relator on X such that $\operatorname{int}_{\mathcal{S}} \subseteq \operatorname{int}_{\mathcal{R}}$ $\big(\operatorname{int}_{\mathcal{S}} = \operatorname{int}_{\mathcal{R}}\big)$, or equivalently $\operatorname{cl}_{\mathcal{R}} \subseteq \operatorname{cl}_{\mathcal{S}}$ $\big(\operatorname{cl}_{\mathcal{S}} = \operatorname{cl}_{\mathcal{R}}\big)$;*

(3) *$\mathcal{S} = \mathcal{R}^{\vartriangle}$ is the largest relator on X such that $\mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}}$ $\big(\mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{R}}\big)$, or equivalently $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$ $\big(\mathcal{D}_{\mathcal{S}} = \mathcal{D}_{\mathcal{R}}\big)$.*

Remark 1.5.6. To prove some similar statements for the operation $*$, the structures $\operatorname{Lim}_{\mathcal{R}}$ and $\operatorname{Adh}_{\mathcal{R}}$ have to be used [114].

Moreover, for instance, to investigate the structures $\operatorname{Lb}_{\mathcal{R}}$ and $\operatorname{Ub}_{\mathcal{R}}$ the compound operation $\circledast = c \# c$ is needed [132].

Concerning the above basic closure operations, we can prove the following two theorems.

Theorem 1.5.7. *We have*

(1) $\mathcal{R}^{\#} = \mathcal{R}^{\Diamond\,\#} = \mathcal{R}^{\#\,\Diamond}$ *with* $\Diamond = *$ *and* $\#$*;*
(2) $\mathcal{R}^{\wedge} = \mathcal{R}^{\Diamond\,\wedge} = \mathcal{R}^{\wedge\,\Diamond}$ *with* $\Diamond = *, \#$ *and* $\wedge$*;*
(3) $\mathcal{R}^{\triangle} = \mathcal{R}^{\Diamond\,\triangle} = \mathcal{R}^{\triangle\,\Diamond}$ *with* $\Diamond = *, \#, \wedge$ *and* $\triangle$*.*

Proof. To prove (1), note that, by Remark 1.5.2 and the closure properties, we have $\mathcal{R}^{\#} \subseteq \mathcal{R}^{\#\,*} \subseteq \mathcal{R}^{\#\,\#} = \mathcal{R}^{\#}$ and $\mathcal{R}^{\#} \subseteq \mathcal{R}^{*\,\#} \subseteq \mathcal{R}^{\#\,\#} = \mathcal{R}^{\#}$.

Theorem 1.5.8. *We have*
(1) $\mathcal{R}^{*\,-1} = \mathcal{R}^{-1\,*}$*;* (2) $\mathcal{R}^{\#\,-1} = \mathcal{R}^{-1\,\#}$.

Proof. To prove (2), note that, by Theorems 1.3.9 and 1.5.4 we have

$$\mathrm{Cl}_{\mathcal{R}^{\#\,-1}} = \mathrm{Cl}_{\mathcal{R}^{\#}}^{-1} = \mathrm{Cl}_{\mathcal{R}}^{-1} = \mathrm{Cl}_{\mathcal{R}^{-1}}$$

and thus in particular $\mathrm{Cl}_{\mathcal{R}}^{-1} \subseteq \mathrm{Cl}_{\mathcal{R}^{\#\,-1}}$. Hence, by using Theorem 1.5.4, we can infer that $\mathcal{R}^{\#\,-1} \subseteq \mathcal{R}^{-1\,\#}$. Now, by writing $\mathcal{R}^{-1}$ in place of $\mathcal{R}$, we can see that $\mathcal{R}^{-1\,\#\,-1} \subseteq \mathcal{R}^{\#}$, and thus $\mathcal{R}^{-1\,\#} \subseteq \mathcal{R}^{\#\,-1}$. Therefore, (2) is also true.

Remark 1.5.9. For instance, the elementwise operations c and ∞ are also inversion compatible. Moreover, the operation ∂ is also inversion compatible.

However, unfortunately, the operations $\wedge$ and $\triangle$ are not inversion compatible. Therefore, in addition to Definition 1.5.1, we must also have the following

Definition 1.5.10. We define

$$\mathcal{R}^{\vee} = \mathcal{R}^{\wedge\,-1} \qquad \text{and} \qquad \mathcal{R}^{\triangledown} = \mathcal{R}^{\triangle\,-1}.$$

Remark 1.5.11. The latter operations have very curious properties. For instance, if $\mathcal{R}$ is nonvoid, then $\mathcal{R}^{\vee\,\wedge} = \{\rho_{\mathcal{R}}\}^{\wedge}$ [84].

Thus, in particular, the relator $\mathcal{R}^{\vee}$ is topologically simple in the sense that it is topologically equivalent to a singleton relator.

1.6 Some Further Theorems on the Operations $\wedge$ and $\triangle$

A preliminary form of the following theorem was already proved in [114].

Theorem 1.6.1. *If $\mathcal{R}$ is nonvoid, then for any $B \subseteq X$ we have:*
(1) $\mathrm{Int}_{\mathcal{R}^{\wedge}}(B) = \mathcal{P}\big(\mathrm{int}_{\mathcal{R}}(B)\big)$*;* (2) $\mathrm{Cl}_{\mathcal{R}^{\wedge}}(B) = \mathcal{P}\big(\mathrm{cl}_{\mathcal{R}}(B)^{c}\big)^{c}$.

Proof. If $A \in \mathrm{Int}_{\mathcal{R}^{\wedge}}(B)$, then by Definition 1.3.2 and Theorem 1.5.4 we have

$$A \subseteq \mathrm{int}_{\mathcal{R}^{\wedge}}(B) = \mathrm{int}_{\mathcal{R}}(B),$$

and thus $A \in \mathcal{P}\big(\mathrm{int}_{\mathcal{R}}(B)\big)$. Therefore, $\mathrm{Int}_{\mathcal{R}^{\wedge}}(B) \subseteq \mathcal{P}\big(\mathrm{int}_{\mathcal{R}}(B)\big)$.

While, if $A \in \mathcal{P}\big(\mathrm{int}_{\mathcal{R}}(B)\big)$, then $A \subseteq \mathrm{int}_{\mathcal{R}}(B)$. Therefore, for each $x \in A$, there exists $R_x \in \mathcal{R}$ such that $R_x(x) \subseteq B$. Now, by defining

$$S(x) = R_x(x) \quad \text{for all} \quad x \in A \qquad \text{and} \qquad S(x) = X \quad \text{for all} \quad x \in A^c,$$

we can easily see that $S \in \mathcal{R}^\wedge$ such that $S[A] \subseteq B$. Therefore, we also have $A \in \operatorname{Int}_{\mathcal{R}^\wedge}(B)$. Consequently, $\mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big) \subseteq \operatorname{Int}_{\mathcal{R}^\wedge}(B)$, and thus (1) also holds.

Now, by using Theorems 1.3.4 and 1.3.6, we can also easily see that

$$\operatorname{Cl}_{\mathcal{R}^\wedge}(B) = \operatorname{Int}_{\mathcal{R}^\wedge}(B^c)^c = \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B^c)\big)^c = \mathcal{P}\big(\operatorname{cl}_{\mathcal{R}}(B)^c\big)^c.$$

Remark 1.6.2. Thus, for any $A \subseteq X$, we have

$$A \in \operatorname{Cl}_{\mathcal{R}^\wedge}(B) \iff A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset.$$

From Theorem 1.6.1, by using Definition 1.4.1, we can immediately derived

Corollary 1.6.3. *If $\mathcal{R}$ is nonvoid, then*
(1) $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$; (2) $\tau\!\!\!\!-_{\mathcal{R}^\wedge} = \mathcal{F}_{\mathcal{R}}$.

Remark 1.6.4. Hence, since $\tau_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \tau_R = \bigcup_{R \in \mathcal{R}} \mathcal{T}_R$, we can infer that $\mathcal{T}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}^\wedge} \mathcal{T}_R$.

Unfortunately, in contrast to the structures Int, int, $\mathcal{E}$ and τ, the increasing structure $\mathcal{T}$ is already not union-preserving.

Example 1.6.5. If $\operatorname{card}(X) > 2$, and $R_i = \{x_i\}^2 \cup \big(X \setminus \{x_i\}\big)^2$ for all $i = 1, 2$, then $\mathcal{R} = \{R_1, R_2\}$ is an equivalence relator on X such that

$$\{x_1, x_2\} \in \mathcal{T}_{\mathcal{R}} \setminus \big(\mathcal{T}_{R_1} \cup \mathcal{T}_{R_2}\big), \qquad \text{and thus} \qquad \mathcal{T}_{\mathcal{R}} \not\subseteq \mathcal{T}_{R_1} \cup \mathcal{T}_{R_2}.$$

From Corollary 1.6.3, by using Theorem 1.5.7, we can also derive

Corollary 1.6.6. *If $\mathcal{R}$ is nonvoid, then*
(1) $\tau_{\mathcal{R}^\vartriangle} = \mathcal{T}_{\mathcal{R}^\vartriangle}$; (2) $\tau\!\!\!\!-_{\mathcal{R}^\vartriangle} = \mathcal{F}_{\mathcal{R}^\vartriangle}$.

Concerning the operation $\vartriangle$, we can also prove the following

Theorem 1.6.7. *If $\mathcal{R}$ is nonvoid, then for any $B \subseteq X$ we have*
(1) $\operatorname{Int}_{\mathcal{R}^\vartriangle}(B) = \{\emptyset\}$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{Int}_{\mathcal{R}^\vartriangle}(B) = \mathcal{P}(X)$ *if* $B \in \mathcal{E}_{\mathcal{R}}$;
(2) $\operatorname{Cl}_{\mathcal{R}^\vartriangle}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^\vartriangle}(B) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $B \in \mathcal{D}_{\mathcal{R}}$.

Proof. If $A \in \operatorname{Int}_{\mathcal{R}^\vartriangle}(B)$, then there exists $S \in \mathcal{R}^\vartriangle$ such that $S[A] \subseteq B$. Therefore, if $A \neq \emptyset$, then there exists $x \in X$ such that $S(x) \subseteq B$. Hence, since $S(x) \in \mathcal{E}_{\mathcal{R}}$, it follows that $B \in \mathcal{E}_{\mathcal{R}}$. Therefore, the first part of (1) is true.

To prove the second part of (1), it is enough to note only that if $B \in \mathcal{E}_{\mathcal{R}}$, then $R = X \times B \in \mathcal{R}^\vartriangle$ such that $R[A] \subseteq B$, and thus $A \in \operatorname{Int}_{\mathcal{R}^\vartriangle}(B)$ for all $A \subseteq X$.

Assertion (2) can again be derived from (1) by using Theorem 1.3.4.

From this theorem, by Definition 1.3.2, it is clear that in particular we also have

Corollary 1.6.8. *If $\mathcal{R}$ is nonvoid, then for any $B \subseteq X$*
(1) $\operatorname{cl}_{\mathcal{R}^\vartriangle}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{cl}_{\mathcal{R}^\vartriangle}(B) = X$ *if* $B \in \mathcal{D}_{\mathcal{R}}$;
(2) $\operatorname{int}_{\mathcal{R}^\vartriangle}(B) = \emptyset$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{int}_{\mathcal{R}^\vartriangle}(B) = X$ *if* $B \in \mathcal{E}_{\mathcal{R}}$.

Hence, by using Definitions 1.3.2 and 1.4.1, we can immediately derive

Corollary 1.6.9. *We have*

(1) $\mathcal{T}_{\mathcal{R}^{\triangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$; (2) $\mathcal{F}_{\mathcal{R}^{\triangle}} = \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}$.

Remark 1.6.10. Note that if in particular $\mathcal{R} = \emptyset$, then $\mathcal{E}_{\mathcal{R}} = \emptyset$. Moreover, $\mathcal{R}^{\triangle} = \emptyset$ if $X \neq \emptyset$, and $\mathcal{R}^{\triangle} = \{\emptyset\}$ if $X = \emptyset$. Therefore, $\mathcal{T}_{\mathcal{R}^{\triangle}} = \{\emptyset\}$, and thus (1) is still true.

Now, since $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ if $\mathcal{R}$ is non-partial, we can also state

Corollary 1.6.11. *If $\mathcal{R}$ is non-partial, then*

(1) $\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\}$, (2) $\mathcal{D}_{\mathcal{R}} = \big(\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}^{\triangle}}\big) \cup \{X\}$.

1.7 Projection Operations on Relators

By using the basic properties of the operation ∞ , in addition to a particular case of Theorem 1.5.4, we can also prove the following

Theorem 1.7.1. *∞ is a closure operation for relations on X such that, for any two relations R and S on X, we have*

$$S \subseteq R^{\infty} \iff S^{\infty} \subseteq R^{\infty} \iff \tau_R \subseteq \tau_S \iff \tau\!\!\!-_R \subseteq \tau\!\!\!-_S .$$

Proof. To prove that $\tau_R \subseteq \tau_S \iff S \subseteq R^{\infty}$, note that if $x \in X$, then because of the inclusion $R \subseteq R^{\infty}$ and the transitivity of R^{∞} we have

$$R\,[\,R^{\infty}(x)\,] \subseteq R^{\infty}\,[\,R^{\infty}(x)\,] = \big(R^{\infty} \circ R^{\infty}\big)(x) \subseteq R^{\infty}(x) .$$

Thus, by the definition of $\tau_{\mathcal{R}}$, we have $R^{\infty}(x) \in \tau_{\mathcal{R}}$. Now, if $\tau_{\mathcal{R}} \subseteq \tau_S$ holds, then we can see that $R^{\infty}(x) \in \tau_S$, and thus $S\,[\,R^{\infty}(x)\,] \subseteq R^{\infty}(x)$. Hence, by using the reflexivity of R^{∞} , we can already infer that $S(x) \subseteq R^{\infty}(x)$. Therefore, $S \subseteq R^{\infty}$ also holds.

While, if $A \in \tau_R$, then by the definition of $\tau_{\mathcal{R}}$ we have $R\,[\,A\,] \subseteq A$. Hence, by induction, we can see that $R^n\,[\,A\,] \subseteq A$ for all $n \in \mathbb{N}$. Now, since $R^0\,[\,A\,] = \Delta_X\,[\,A\,] = A$ also holds, we can already state that

$$R^{\infty}\,[\,A\,] = \Big(\bigcup_{n=0}^{\infty} R^n\Big)[\,A\,] = \bigcup_{n=0}^{\infty} R^n\,[\,A\,] \subseteq \bigcup_{n=0}^{\infty} A = A .$$

Therefore, if $S \subseteq R^{\infty}$ holds, then we have $S\,[A\,] \subseteq R^{\infty}\,[\,A\,] \subseteq A$, and thus $A \in \tau_S$ also holds.

Now, analogously to Corollary 1.5.5, we can also state

Corollary 1.7.2. *For any relation R on X, $S = R^{\infty}$ is the largest relation on X such that $\tau_R \subseteq \tau_S$ $\big(\tau_R = \tau_S\big)$, or equivalently $\tau\!\!\!-_R \subseteq \tau\!\!\!-_S$ $\big(\tau\!\!\!-_R = \tau\!\!\!-_S\big)$.*

Remark 1.7.3. Preliminary forms of the above theorem and its corollary were first proved by Mala [80] .

Moreover, he also proved that $R^{\infty}(x) = \bigcap\,\{\,A \in \tau_R : \; x \in A\,\}$ for all $x \in X$, and thus $R^{\infty} = \bigcap\,\{\,R_A : \; A \in \tau_R\,\}$.

By using Theorem 1.7.1, as an analogue of Theorem 1.5.4, we can also prove

Theorem 1.7.4. *$\#\,\partial$ is a closure operation for relators on X such that, for any relator $\mathcal{S}$ on X, we have*
$$\mathcal{S} \subseteq \mathcal{R}^{\#\partial} \iff \mathcal{S}^{\#\partial} \subseteq \mathcal{R}^{\#\partial} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \tau\!\!\!-_{\mathcal{S}} \subseteq \tau\!\!\!-_{\mathcal{R}}\,.$$

Thus, analogously to Corollary 1.5.5, we can also state

Corollary 1.7.5. *$\mathcal{S} = \mathcal{R}^{\#\partial}$ is the largest relator on X such that $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $\big(\tau_{\mathcal{S}} = \tau_{\mathcal{R}}\big)$, or equivalently $\tau\!\!\!-_{\mathcal{S}} \subseteq \tau\!\!\!-_{\mathcal{R}}$ $\big(\tau\!\!\!-_{\mathcal{S}} = \tau\!\!\!-_{\mathcal{R}}\big)$.*

By using the Galois property of the operations ∞ and ∂, Theorem 1.7.4 can be reformulated in the following more convenient form.

Theorem 1.7.6. *$\#\,\infty$ is a projection operation for relators on X such that, for any relator $\mathcal{S}$ on X, we have*
$$\mathcal{S}^{\infty} \subseteq \mathcal{R}^{\#} \iff \mathcal{S}^{\#\infty} \subseteq \mathcal{R}^{\#\infty} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \tau\!\!\!-_{\mathcal{S}} \subseteq \tau\!\!\!-_{\mathcal{R}}.$$

Remark 1.7.7. Moreover, it can be easily shown that the inclusions $\mathcal{S}^{\infty} \subseteq \mathcal{R}^{\#}$, $\mathcal{S}^{\#\infty} \subseteq \mathcal{R}^{\#}$ and $\mathcal{S}^{\infty\#} \subseteq \mathcal{R}^{\infty\#}$ are also equivalent.

Now, analogously to our former corollaries, we can also state

Corollary 1.7.8. *$\mathcal{S} = \mathcal{R}^{\#\infty}$ is the largest preorder relator on X such that $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $\big(\tau_{\mathcal{S}} = \tau_{\mathcal{R}}\big)$, or equivalently $\tau\!\!\!-_{\mathcal{S}} \subseteq \tau\!\!\!-_{\mathcal{R}}$ $\big(\tau\!\!\!-_{\mathcal{S}} = \tau\!\!\!-_{\mathcal{R}}\big)$.*

Remark 1.7.9. The advantage of the projection operation $\#\,\infty$ over the closure operation $\#\,\partial$ lies mainly in the fact that, in contrast to $\#\,\partial$, it is *stable* in the sense $\{X^2\}^{\#\infty} = \{X^2\}$.

Since the structure $\mathcal{T}$ is not union-preserving, by using some parts of the theory of Pataki connections [93, 139, 107], we can only prove the following

Theorem 1.7.10. *$\wedge\,\partial$ is a preclosure operation for relators such that, for any relator $\mathcal{S}$ on X, we have*
$$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}} \implies \mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge\partial} \implies \mathcal{S}^{\wedge\partial} \subseteq \mathcal{R}^{\wedge\partial}.$$

Remark 1.7.11. If $\operatorname{card}(X) > 2$, then by using the equivalence relator $\mathcal{R} = \{X^2\}$ Mala [80, Example 5.3] proved that there does not exist a largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$.

Moreover, Pataki [93, Example 7.2] proved that $\mathcal{T}_{\mathcal{R}^{\wedge\partial}} \not\subseteq \mathcal{T}_{\mathcal{R}}$ and $\wedge\,\partial$ is not idempotent. (Actually, it can be proved that $\mathcal{R}^{\wedge\partial\wedge} \not\subseteq \mathcal{R}^{\wedge\partial}$ also holds [125, Example 10.11].)

Fortunately, as an analogue of Theorem 1.7.6, we can also prove

Theorem 1.7.12. *$\wedge\,\infty$ is a projection operation for relators on X such that, for any two nonvoid relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have*
$$\mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge\infty} \iff \mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}}\,.$$

Thus, in particular, we can also state

Corollary 1.7.13. *If* $\mathcal{R}$ *is nonvoid, then* $\mathcal{S} = \mathcal{R}^{\wedge\infty}$ *is the largest preorder relator on* X *such that* $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}}$ $\big(\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}\big)$*, or equivalently* $\mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}}$ $\big(\mathcal{F}_{\mathcal{S}} = \mathcal{F}_{\mathcal{R}}\big)$.

Remark 1.7.14. In the light of the several disadvantages of the structure $\mathcal{T}$, it is rather curious that most of the works in general topology and abstract analysis have been based on open sets suggested by Tietze [144] and Alexandroff [3], and standardized by Bourbaki [12] and Kelley [61]. (See Thron [143, p. 18].)

Moreover, it also a very striking fact that, despite the results of Davis [28], Pervin [96], Hunsaker and Lindgren [52] and Száz [118, 126], generalized proximities and closures, minimal structures, generalized topologies and stacks (ascending systems) are still intensively investigated by a great number of mathematicians without using generalized uniformities.

1.8 Reflexive, Non-Partial and Non-Degenerated Relators

Definition 1.8.1. The relator $\mathcal{R}$ is called *reflexive* if each member R of $\mathcal{R}$ is a reflexive relation on X.

Remark 1.8.2. Thus, the following assertions are equivalent:

(1) $\mathcal{R}$ is reflexive;

(2) $x \in R(x)$ for all $x \in X$ and $R \in \mathcal{R}$;

(3) $A \subseteq R[A]$ for all $A \subseteq X$ and $R \in \mathcal{R}$.

The importance of reflexive relators is also apparent from the following two theorems.

Theorem 1.8.3. *The following assertions are equivalent:*

(1) $\rho_{\mathcal{R}}$ *is reflexive;*

(2) $\mathcal{R}$ *is reflexive;*

(3) $A \subseteq \operatorname{cl}_{\mathcal{R}}(A)$ *for all* $A \subseteq X$*;*

(4) $\operatorname{int}_{\mathcal{R}}(A) \subseteq A$ *for all* $A \subseteq X$.

Proof. To prove the equivalence of (1) and (2), recall that by Corollary 1.3.11 we have $\rho_{\mathcal{R}} = \big(\bigcap \mathcal{R}\big)^{-1}$.

Remark 1.8.4. The relator $\mathcal{R}$ is reflexive if and only if $A^{\circ} \subseteq A$ $(A \subseteq A^{-})$ for all $A \subseteq X$.

Therefore, if $\mathcal{R}$ is reflexive, then for any $A \subseteq X$ we have $A \in \mathcal{T}_{\mathcal{R}}$ $(A \in \mathcal{F}_{\mathcal{R}})$ if and only if $A^{\circ} = A$ $(A^{-} = A)$.

Theorem 1.8.5. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is reflexive;*

(2) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ *implies* $A \subseteq B$ *for all* $A, B \subseteq X$*;*

(3) $A \cap B \neq \emptyset$ *implies* $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq X$.

Remark 1.8.6. In addition to the above two theorems , it is also worth mentioning that if $\mathcal{R}$ is reflexive, then

(1) $\operatorname{Int}_{\mathcal{R}}$ is a transitive relation on $\mathcal{P}(X)$;

(2) $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ implies $\mathcal{P}(X) = \operatorname{Cl}_{\mathcal{R}}(A)^{c} \cup \operatorname{Cl}_{\mathcal{R}}^{-1}(B)$;

(3) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{bor}_{\mathcal{R}}(A)\big) = \emptyset$ and $\operatorname{int}_{\mathcal{R}}\big(\operatorname{res}_{\mathcal{R}}(A)\big) = \emptyset$ for all $A \subseteq X$.

Thus, for instance, for any $A \subseteq X$ we have $\operatorname{res}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$ if and only if $A \in \mathcal{F}_{\mathcal{R}}$.

Analogously to Definition 1.8.1, we may also naturally have the following

Definition 1.8.7. The relator $\mathcal{R}$ is called *non-partial* if each member R of $\mathcal{R}$ is a non-partial relation on X.

Remark 1.8.8. Thus, the following assertions are equivalent:

(1) $\mathcal{R}$ is non-partial;

(2) $R^{-1}[X] = X$ for all $R \in \mathcal{R}$; (3) $R(x) \neq \emptyset$ for all $x \in X$ and $R \in \mathcal{R}$.

The importance of non-partial relators is apparent from the following

Theorem 1.8.9. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is non-partial;*

(2) $\emptyset \notin \mathcal{E}_{\mathcal{R}}$; (3) $\mathcal{D}_{\mathcal{R}} \neq \emptyset$; (4) $X \in \mathcal{D}_{\mathcal{R}}$; (5) $\mathcal{E}_{\mathcal{R}} \neq \mathcal{P}(X)$.

Sometimes, we also need the following localized form of Definition 1.8.7.

Definition 1.8.10. The relator $\mathcal{R}$ is called *locally non-partial* if for each $x \in X$ there exists $R \in \mathcal{R}$ such that for any $y \in R(x)$ and $S \in \mathcal{R}$ we have $S(y) \neq \emptyset$.

Remark 1.8.11. Thus, if either $X = \emptyset$ or $\mathcal{R}$ is nonvoid and non-partial, then $\mathcal{R}$ is locally non-partial.

Moreover, by using the corresponding definitions, we can also easily prove

Theorem 1.8.12. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is locally non-partial;* (2) $X = \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(X)\big)$.

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then for each $x \in X$ there exists $R \in \mathcal{R}$ such that for any $y \in R(x)$ and for any $S \in \mathcal{R}$ we have $S(y) \cap X = S(y) \neq \emptyset$, and thus $y \in \operatorname{cl}_{\mathcal{R}}(X)$. Therefore, for each $x \in X$ there exists $R \in \mathcal{R}$ such that $R(x) \subseteq \operatorname{cl}_{\mathcal{R}}(X)$, and thus $x \in \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(X)\big)$. Hence, we can already see that $X \subseteq \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(X)\big)$, and thus (2) also holds.

In addition to Definition 1.8.7, it is also worth introducing the following

Definition 1.8.13. The relator $\mathcal{R}$ is called *non-degerated* if $X \neq \emptyset$ and $\mathcal{R} \neq \emptyset$.

Thus, analogously to Theorem 1.8.9, we can also easily establish the following

Theorem 1.8.14. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is non-degenerated;*

(2) $\emptyset \notin \mathcal{D}_{\mathcal{R}}$; (3) $\mathcal{E}_{\mathcal{R}} \neq \emptyset$; (4) $X \in \mathcal{E}_{\mathcal{R}}$; (5) $\mathcal{D}_{\mathcal{R}} \neq \mathcal{P}(X)$.

Remark 1.8.15. In addition to Theorems 1.8.9 and 1.8.14, it is also worth mentioning that if $\mathcal{R}$ is paratopologically simple in the sense that $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_S$, or equivalently $\mathcal{R}^{\vartriangle} = \{S\}^{\vartriangle}$, for some relation S on X, then the stack $\mathcal{E}_{\mathcal{R}}$ has a base $\mathcal{B}$ with $\operatorname{card}(\mathcal{B}) \leq \operatorname{card}(X)$. (See [92, Theorem 5.9] of Pataki.)

The existence of a non-paratopologically simple (actually finite equivalence) relator, proved first by Pataki [92, Example 5.11], shows that in our definitions of the relations $\operatorname{Lim}_{\mathcal{R}}$ and $\operatorname{Adh}_{\mathcal{R}}$ we cannot restrict ourselves to functions of gosets (generalized ordered sets) without a substantial loss of generality.

1.9 Topological and Quasi-Topological Relators

The following improvement of [115, Definition 2.1] was first considered in [116].

Definition 1.9.1. The relator $\mathcal{R}$ is called:

(1) *quasi-topological* if $x \in \operatorname{int}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(R(x))\big)$ for all $x \in X$ and $R \in \mathcal{R}$;

(2) *topological* if for any $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$, such that $x \in V \subseteq R(x)$.

The appropriateness of these definitions is already quite obvious from the following four theorems.

Theorem 1.9.2. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topological;*

(2) $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{T}_{\mathcal{R}}$ *for all* $x \in X$ *and* $R \in \mathcal{R}$*;*

(3) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}$ $\big(\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}\big)$ *for all* $A \subseteq X$.

Theorem 1.9.3. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological;* (2) $\mathcal{R}$ *is reflexive and quasi-topological.*

Remark 1.9.4. By Theorem 1.9.2, the relator $\mathcal{R}$ may be called *weakly (strongly) quasi-topological* if $\rho_{\mathcal{R}}(x) \in \mathcal{F}_{\mathcal{R}}$ $\big(R(x) \in \mathcal{T}_{\mathcal{R}}\big)$ for all $x \in X$ and $R \in \mathcal{R}$.

Moreover, by Theorem 1.9.3, the relator $\mathcal{R}$ may be called *weakly (strongly) toplogical* if it is reflexive and weakly (strongly) quasi-topological.

The following theorem shows that in a topological relator space $X(\mathcal{R})$, the relation $\operatorname{cl}_{\mathcal{R}}$ $(\operatorname{int}_{\mathcal{R}})$ and the family $\mathcal{F}_{\mathcal{R}}$ $(\mathcal{T}_{\mathcal{R}})$ are equivalent tools.

Theorem 1.9.5. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological;*

(2) $\operatorname{int}_{\mathcal{R}}(A) = \bigcup \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$ *for all* $A \subseteq X$*;*

(3) $\operatorname{cl}_{\mathcal{R}}(A) = \bigcap \mathcal{F}_{\mathcal{R}} \cap \mathcal{P}^{-1}(A)$ *for all* $A \subseteq X$.

Now, as an immediate consequence of this theorem, we can also state

Corollary 1.9.6. *If* $\mathcal{R}$ *is topological, then for any* $A \subset X$*, we have*

(1) $A \in \mathcal{E}_{\mathcal{R}}$ *if and only if there exists* $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ *such that* $V \subseteq A$*;*

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *if and only if for all* $W \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$ *we have* $A \setminus W \neq \emptyset$.

However, it is now more important to note that we can also prove the following

Theorem 1.9.7. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological;*

(2) $\mathcal{R}$ *is topologically equivalent to* $\mathcal{R}^{\wedge\infty}$*;*

(3) $\mathcal{R}$ *is topologically equivalent to a preorder relator on* X.

Proof. To prove the implication (1) $\Longrightarrow$ (3), note that if (1) holds, then by Definition 1.9.1, for any $x \in X$ and $R \in \mathcal{R}$, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$. Hence, by considering the Pervin relator

$$\mathcal{S} = \mathcal{R}_{\mathcal{T}_{\mathcal{R}}} = \big\{ R_V : \quad V \in \mathcal{T}_{\mathcal{R}} \big\}, \qquad \text{with} \qquad R_V = V^2 \cup (V^c \times X),$$

we can show that $\operatorname{int}_{\mathcal{R}}(A) = \operatorname{int}_{\mathcal{S}}(A)$ for all $A \subseteq X$, and thus (3) also holds.

For this, we have to note that

$$R_V(x) = V \quad \text{if} \quad x \in V \qquad \text{and} \qquad R_V(x) = X \quad \text{if} \quad x \in V^c.$$

In addition to Theorem 1.9.2, it is also worth proving the following

Theorem 1.9.8. *The following assertions are equivalent:*
(1) $\mathcal{R}$ *is quasi-topological;* (2) $\mathcal{R} \subseteq \big(\mathcal{R}^{\wedge} \circ \mathcal{R}\big)^{\wedge}$; (3) $\mathcal{R}^{\wedge} \subseteq \big(\mathcal{R}^{\wedge} \circ \mathcal{R}^{\wedge}\big)^{\wedge}$.

Remark 1.9.9. By [115], the relator $\mathcal{R}$ may be naturally called topologically transitive if, for each $x \in X$ and $R \in \mathcal{R}$ there exist $S, T \in \mathcal{R}$ such that $T\,[\,S\,(x)] \subseteq R\,(x)$.

This property can also be reformulated in the more concise form that $\mathcal{R} \subseteq \big(\mathcal{R} \circ \mathcal{R}\big)^{\wedge}$. Thus, the equivalence of (1) and (3) can be expressed by saying that $\mathcal{R}$ is quasi-topological if and only if $\mathcal{R}^{\wedge}$ is topologically transitive.

1.10 Proximal and Quasi-Proximal Relators

Analogously to Definition 1.9.1, we may also naturally have the following

Definition 1.10.1. The relator $\mathcal{R}$ is called

(1) *quasi-proximal* if $A \in \operatorname{Int}_{\mathcal{R}}\big[\,\tau_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{R}}\big(\,R[\,A\,]\,\big)\,\big]$ for all $A \subseteq X$ and $R \in \mathcal{R}$;

(2) *proximal* if for any $A \subseteq X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[A]$.

Remark 1.10.2. Thus, the relator $\mathcal{R}$ is quasi-proximal if and only if, for any $A \subseteq X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $A \in \operatorname{Int}_{\mathcal{R}}(V)$ and $V \in \operatorname{Int}_{\mathcal{R}}\big(\,R\,[\,A\,]\,\big)$.

Now, by using the corresponding definitions, we can also easily prove the following analogues of Theorems 1.9.3 and 1.9.5.

Theorem 1.10.3. *The following assertions are equivalent:*
(1) $\mathcal{R}$ *is proximal;* (2) $\mathcal{R}$ *is reflexive and quasi-proximal.*

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then for any $A \subseteq X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R\,[\,A\,]$. Hence, by taking $A = \{x\}$ for $x \in X$, we can see that $\mathcal{R}$ is reflexive.

Moreover, since $V \in \tau_{\mathcal{R}}$, we can also note that $V \in \operatorname{Int}_{\mathcal{R}}(V)$. Hence, by using that $A \subseteq V$ and $V \subseteq R\,[\,A\,]$, we can already infer that $A \in \operatorname{Int}_{\mathcal{R}}(V)$ and $V \in \operatorname{Int}_{\mathcal{R}}\big(\,R\,[\,A\,]\,\big)$. Therefore, by Remark 1.10.2, $\mathcal{R}$ is quasi-proximal., and thus (2) also holds.

Remark 1.10.4. Note that if $\mathcal{R}$ is only a weakly proximal relator in the sense that, for any $x \in X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$, then because of the inclusion $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$ we can already state that $\mathcal{R}$ is topological.

Theorem 1.10.5. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximal;*

(2) $\operatorname{Int}_{\mathcal{R}}(A) = \mathcal{P}\big[\tau_{\mathcal{R}} \cap \mathcal{P}(A)\big]$ *for all* $A \subseteq X$*;*

(3) $\operatorname{Cl}_{\mathcal{R}}(A) = \bigcap \big\{ \mathcal{P}(W^c)^c : \ W \in \tau_{\mathcal{R}} \cap \mathcal{P}^{-1}(A) \big\}$ *for all* $A \subseteq X$.

Proof. Note that if $A \subseteq X$ and $B \in \mathcal{P}\big[\tau_{\mathcal{R}} \cap \mathcal{P}(A)\big]$, then there exists $V \in \tau_{\mathcal{R}}$ such that $B \in \mathcal{P}(V)$ and $V \in \mathcal{P}(A)$, and thus $B \subseteq V \subseteq A$. Hence, by using that $V \in \operatorname{Int}_{\mathcal{R}}(V)$ we can already infer that $B \in \operatorname{Int}_{\mathcal{R}}(A)$. Thus, the inclusion $\mathcal{P}\big[\tau_{\mathcal{R}} \cap \mathcal{P}(A)\big] \subseteq \operatorname{Int}_{\mathcal{R}}(A)$ is always true.

Therefore, to obtain (1), it is enough to assume only the converse inclusion. For this, note that if $A \subseteq X$ and $R \in \mathcal{R}$, then because of $R[A] \subseteq R[A]$, we always have $A \in \operatorname{Int}_{\mathcal{R}}\big(R[A]\big)$. Therefore, if $\operatorname{Int}_{\mathcal{R}}\big(R[A]\big) \subseteq \mathcal{P}\big[\tau_{\mathcal{R}} \cap \mathcal{P}\big(R[A]\big)\big]$, then we also have $A \in \mathcal{P}\big[\tau_{\mathcal{R}} \cap \mathcal{P}\big(R[A]\big)\big]$. Thus, there exists $V \in \tau_{\mathcal{R}}$ such that $A \in \mathcal{P}(V)$ and $V \in \mathcal{P}\big(R[A]\big)$, and thus $A \subseteq V \subseteq R[A]$.

Remark 1.10.6. Note that $\mathcal{P}(A) = \operatorname{Int}_{\Delta_X}(A)$ for all $A \subseteq X$. Therefore, instead of (2) we may write that $\operatorname{Int}_{\mathcal{R}}(A) = \operatorname{Int}_{\Delta_X}\big[\tau_{\mathcal{R}} \cap \operatorname{Int}_{\Delta_X}(A)\big]$ for all $A \subseteq X$.

However, it is now more important to note that we also have the following

Theorem 1.10.7. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximal;*

(2) $\mathcal{R}$ *is proximally equivalent to* $\mathcal{R}^{\infty}$ *or* $\mathcal{R}^{\#\infty}$*;*

(3) $\mathcal{R}$ *is proximally equivalent to a preorder relator on* X.

In principle, each theorem on topological and quasi-topological relators can be immediately derived from a corresponding theorem on proximal and quasi-proximal relators by using the following two theorems.

Theorem 1.10.8. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topological;* (2) $\mathcal{R}^{\wedge}$ *is quasi-proximal.*

Proof. To prove the implication (1) $\Longrightarrow$ (2), assume that (1) holds, and moreover $A \subseteq X$ and $S \in \mathcal{R}^{\wedge}$. Define $V = \operatorname{int}_{\mathcal{R}}\big(S[A]\big)$. Then, if $\mathcal{R} \neq \emptyset$, by Theorem 1.9.2 and Corollary 1.6.3, we have $V \in \mathcal{T}_{\mathcal{R}} = \tau_{\mathcal{R}^{\wedge}}$. Moreover, since $V \subseteq \operatorname{int}_{\mathcal{R}}\big(S[A]\big)$, by Theorem 1.6.1 we also have $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)$. Therefore, $V \in \tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)$.

Furthermore, since $S[A] \subseteq S[A]$, we can also note that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)$. Hence, by Theorem 1.6.1, we can infer that $A \subseteq \operatorname{int}_{\mathcal{R}}\big(S[A]\big) = V$. Moreover, since $V \in \tau_{\mathcal{R}^{\wedge}}$, we can also note that $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Hence, since $A \subseteq V$, we can infer that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Therefore, since $V \in \tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)$, we also have

$$A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big[\tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)\big].$$

This shows that (1) implies (2) whenever $\mathcal{R} \neq \emptyset$. However, if $\mathcal{R} = \emptyset$, then it can be easily seen that $\mathcal{R}$ is topological and $\mathcal{R}^{\wedge}$ is proximal.

Remark 1.10.9. If assertion (2) holds, then $\mathcal{R}^{\wedge}$ is semi-proximal in the sense that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big[\operatorname{Int}_{\mathcal{R}^{\wedge}}\big(S[A]\big)\big]$ for all $A \subseteq X$ and $S \in \mathcal{R}^{\wedge}$.

Moreover, if in particular $\{x\} \in \operatorname{Int}_{\mathcal{R}^{\wedge}}\big[\operatorname{Int}_{\mathcal{R}^{\wedge}}\big(R(x)\big)\big]$ for all $x \in X$ and $R \in \mathcal{R}$, then we can already prove that assertion (1) also holds.

From Theorem 1.10.8, by using Theorems 1.9.3 and 1.10.3, we can easily derive

Theorem 1.10.10. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological;* (2) $\mathcal{R}^{\wedge}$ *is proximal.*

Remark 1.10.11. By the corresponding definitions, it is clear that the relator $\mathcal{R}^{\wedge}$ is reflexive if and only if $\mathcal{R}$ is reflexive.

However, if $\mathcal{R} \not\subseteq \{X^2\}$, then there exists $R \in \mathcal{R}$ such that $R \neq X^2$. Therefore, there exist $x, y \in X$ such that $x \notin R(y)$. Thus, $S = \big(\{x\} \times R(y)\big) \cup \big(\{x\}^c \times X\big)$ is a non-reflexive relation on X such that $S \in \mathcal{R}^{\vartriangle}$. Therefore, $\mathcal{R}^{\vartriangle}$ cannot be reflexive.

Note that if in particular either $\mathcal{R} = \emptyset$ or $\mathcal{R} = \{X^2\}$, then $\mathcal{R}^{\vartriangle}$ is also reflexive.

1.11 A Few Basic Facts on Filtered Relators

Intersection properties of relators were also first investigated in [115, 116].

Definition 1.11.1. The relator $\mathcal{R}$ is called

(1) *properly filtered* if for any $R, S \in \mathcal{R}$ we have $R \cap S \in \mathcal{R}$;
(2) *uniformly filtered* if for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subseteq R \cap S$;
(3) *proximally filtered* if for any $A \subseteq X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T[A] \subseteq R[A] \cap S[A]$;
(4) *topologically filtered* if for any $x \in X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T(x) \subseteq R(x) \cap S(x)$.

Remark 1.11.2. By using the binary operation $\wedge$ and the unary operations $*$, $\#$ and $\wedge$, the above properties can be reformulated in some more concise forms.

For instance, we can see that $\mathcal{R}$ is topologically filtered if and only if any one of the properties $\mathcal{R} \wedge \mathcal{R} \subseteq \mathcal{R}^{\wedge}$, $(\mathcal{R} \wedge \mathcal{R})^{\wedge} = \mathcal{R}^{\wedge}$ and $\mathcal{R}^{\wedge} \wedge \mathcal{R}^{\wedge} = \mathcal{R}^{\wedge}$ holds.

However, in general, we only have $(R \cap S)[A] \subseteq R[A] \cap S[A]$. Therefore, the corresponding proximal filteredness properties are, unfortunataly, not equivalent.

Despite this, we can easily prove the following theorem which shows the appropriateness of the above proximal filteredness property.

Theorem 1.11.3. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximally filtered;*
(2) $\operatorname{Cl}_{\mathcal{R}}(A \cup B) = \operatorname{Cl}_{\mathcal{R}}(A) \cup \operatorname{Cl}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq X$;
(3) $\operatorname{Int}_{\mathcal{R}}(A \cap B) = \operatorname{Int}_{\mathcal{R}}(A) \cap \operatorname{Int}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq X$.

Proof. To prove the implication (3) $\Longrightarrow$ (1), note that if $A \subseteq X$ and $R, S \in \mathcal{R}$, then by the definition of $\operatorname{Int}_{\mathcal{R}}$ we trivially have $A \in \operatorname{Int}_{\mathcal{R}}\big(R[A]\big)$ and $A \in \operatorname{Int}_{\mathcal{R}}\big(S[A]\big)$. Therefore, if (3) holds, then we also have $A \in \operatorname{Int}_{\mathcal{R}}\big(R[A] \cap S[A]\big)$. Thus, by the definition of $\operatorname{Int}_{\mathcal{R}}$, there exists $T \in \mathcal{R}$ such that $T[A] \subseteq R[A] \cap S[A]$.

Now, as an immediate consequence of this theorem, we can also state

Corollary 1.11.4. *If $\mathcal{R}$ is proximally filtered, then the families $\digamma_{\mathcal{R}}$ and $\tau_{\mathcal{R}}$ are closed under binary unions and intersections, respectively.*

From Theorem 1.11.3, we can also easily derive the following

Theorem 1.11.5. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically filtered;*

(2) $\operatorname{cl}_{\mathcal{R}}(A \cup B) = \operatorname{cl}_{\mathcal{R}}(A) \cup \operatorname{cl}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq X$;

(3) $\operatorname{int}_{\mathcal{R}}(A \cap B) = \operatorname{int}_{\mathcal{R}}(A) \cap \operatorname{int}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq X$.

Thus, in particular, we can also state the following

Corollary 1.11.6. *If $\mathcal{R}$ is topologically filtered, then the families $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ are closed under binary unions and intersections, respectively.*

The following example shows that, for a non-topological relator $\mathcal{R}$, the converse of the above corollary need not be true.

Example 1.11.7. If $X = \{1, 2, 3\}$ and R_i is relation on X, for each $i = 1, 2$, such that

$$R_i(1) = \{1, i+1\} \qquad \text{and} \qquad R_i(2) = R_i(3) = \{2, 3\},$$

then $\mathcal{R} = \big\{R_1, R_2\big\}$ is a reflexive relator on X such that $\mathcal{T}_{\mathcal{R}}$ is closed under arbitrary intersections, but $\mathcal{R}$ is still not topologically filtered.

By the corresponding definitions, it is clear that $\mathcal{T}_{\mathcal{R}} = \big\{\emptyset, \{2, 3\}, X\big\}$. Moreover, we can note that $R_i(1) \not\subseteq R_1(1) \cap R_2(1)$ for each $i = 1, 2$, and thus by Definition 1.11.1 the relator $\mathcal{R}$ is not topologically filtered.

In addition to Theorem 1.11.5, we can also prove the following generalization of [70, Lemma 7] of Levine. The following two results were first considered in [99].

Theorem 1.11.8. *If $\mathcal{R}$ is topologically filtered, $A, B \subseteq X$ and there exists $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$ such that $A \subseteq V$ and $B \subseteq V^c$, then*

$$\operatorname{int}_{\mathcal{R}}(A \cup B) = \operatorname{int}_{\mathcal{R}}(A) \cup \operatorname{int}_{\mathcal{R}}(B).$$

Proof. Because of the increasingness of $\operatorname{int}_{\mathcal{R}}$, we evidently have

$$\operatorname{int}_{\mathcal{R}}(A) \cup \operatorname{int}_{\mathcal{R}}(B) \subseteq \operatorname{int}_{\mathcal{R}}(A \cup B).$$

Therefore, we need actually prove the converse inclusion. For this, note that if $x \in \operatorname{int}_{\mathcal{R}}(A \cup B)$, then by the definition of the operation $\operatorname{int}_{\mathcal{R}}$, there exists $R \in \mathcal{R}$ such that $R(x) \subseteq A \cup B$.

Now, if $x \in V$, then by the definition of $\mathcal{T}_{\mathcal{R}}$, we can see that there exists $S \in \mathcal{R}$ such that $S(x) \subseteq V$. Moreover, since $\mathcal{R}$ is topologically filtered, there exists $T \in \mathcal{R}$ such that $T(x) \subseteq R(x) \cap S(x)$. Hence, we can already infer that

$$T(x) \subseteq R(x) \cap S(x) \subseteq (A \cap B) \cap V = (A \cap V) \cup (B \cap V) = A \cup \emptyset = A\,.$$

Therefore, if $x \in V$, then $x \in \operatorname{int}_{\mathcal{R}}(A)$.

A quite similar argument shows that if $x \in V^{c}$, then $x \in \operatorname{int}_{\mathcal{R}}(B)$. Therefore, in both cases, we have $x \in \operatorname{int}_{\mathcal{R}}(A) \cup \operatorname{int}_{\mathcal{R}}(B)$. Thus, the inclusion $\operatorname{int}_{\mathcal{R}}(A \cup B) \subseteq \operatorname{int}_{\mathcal{R}}(A) \cup \operatorname{int}_{\mathcal{R}}(B)$ is also true.

Remark 1.11.9. More difficult conditions for the equality $\operatorname{cl}_{\mathcal{R}}(A \cap B) = \operatorname{cl}_{\mathcal{R}}(A) \cap \operatorname{cl}_{\mathcal{R}}(B)$ to hold were given by Gottschalk [48] and Jung and Nam [59, 60].

Concerning the latter problem, we shall only mention here the following

Theorem 1.11.10. *If $\mathcal{R}$ is nonvoid and reflexive such that $\operatorname{cl}_{\mathcal{R}}(A \cap B) = \operatorname{cl}_{\mathcal{R}}(A) \cap \operatorname{cl}_{\mathcal{R}}(B)$ for all $A, B \subseteq X$, then $\mathcal{R}^{\wedge\infty} = \mathcal{P}(X^2)^{\infty}$.*

Proof. For this, it is enough to prove only that $\mathcal{T}_{\mathcal{R}} = \mathcal{P}(X)$. Namely, in this case we have $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\Delta_X}$. Hence, by using Theorem 1.7.12 and the corresponding definitions, we can already infer that $\mathcal{R}^{\wedge\infty} = \{\Delta_X\}^{\wedge\infty} = \{\Delta_X\}^{*\infty} = \mathcal{P}(X^2)^{\infty}$.

To prove the equality $\mathcal{T}_{\mathcal{R}} = \mathcal{P}(X)$, note that if this not true, then there exists $A \subseteq X$ such that $A \notin \mathcal{T}_{\mathcal{R}}$, and thus $B = A^c \notin \mathcal{F}_{\mathcal{R}}$. Therefore, $\operatorname{cl}_{\mathcal{R}}(B) \not\subseteq B$, and thus there exists $x \in \operatorname{cl}_{\mathcal{R}}(B)$ such that $X \notin B$. Hence, by using the assumptions of the theorem, we can easily arrive at the contradiction that $x \in \operatorname{cl}_{\mathcal{R}}(\{x\}) \cap \operatorname{cl}_{\mathcal{R}}(B) = \operatorname{cl}_{\mathcal{R}}(\{x\} \cap B) = \operatorname{cl}_{\mathcal{R}}(\emptyset) = \emptyset$.

1.12 A Few Basic Facts on Quasi-Filtered Relators

Since $R \subseteq R^{\infty}$ for every relation R on X, in addition to Definition 1.11.1, we may also naturally introduce the following

Definition 1.12.1. The relator $\mathcal{R}$ is called

(1) *quasi-uniformly filtered* if for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subseteq R^{\infty} \cap S^{\infty}$;

(2) *quasi-proximally filtered* if for any $A \subseteq X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T[A] \subseteq R^{\infty}[A] \cap S^{\infty}[A]$;

(3) *quasi-topologically filtered* if for any $x \in X$ and $R, S \in \mathcal{R}^{\wedge}$ there exists $T \in \mathcal{R}$ such that $T(x) \subseteq R^{\infty}(x) \cap S^{\infty}(x)$.

Remark 1.12.2. Analogously to Remark 1.11.2, the above quasi-filteredness properties can also be reformulated in some more concise forms.

For instance, we can see that $\mathcal{R}$ is quasi-topologically filtered if and only if $\mathcal{R}^{\wedge\infty} \wedge \mathcal{R}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge}$, $(\mathcal{R}^{\wedge\infty} \wedge \mathcal{R}^{\wedge\infty})^{\wedge\infty} = \mathcal{R}^{\wedge\infty}$ or $\mathcal{R}^{\wedge\infty} \wedge \mathcal{R}^{\wedge\infty} = \mathcal{R}^{\wedge\infty}$.

However, it is now more important to note that, by using some former results, we can also prove the following two theorems which show the appropriateness of the above quasi-proximal and quasi-topological filteredness properties.

Theorem 1.12.3. *The following assertions are equivalent :*

(1) $\mathcal{R}$ *is quasi-proximally filtered ;*

(2) $\tau_{\mathcal{R}}$ *is closed under binary unions ;* (3) $\tau_{\mathcal{R}}$ *is closed under binary intersections .*

Theorem 1.12.4. *The following assertions are equivalent :*

(1) $\mathcal{R}$ *is quasi-topologically filtered ;*

(2) $\mathcal{F}_{\mathcal{R}}$ *is closed under binary unions ;* (3) $\mathcal{T}_{\mathcal{R}}$ *is closed under binary intersections .*

Remark 1.12.5. In this respect it is also worth mentioning that the family $\mathcal{E}_{\mathcal{R}}$ is closed under binary intersections if and only if $\mathcal{R}$ is *quasi-directed* in the sense that for any $x,\ y \in X$ and $R,\ S \in \mathcal{R}$ we have $R(x) \cap S(y) \in \mathcal{E}_{\mathcal{R}}$.

From the above two theorems, by using Corollaries 1.11.4 and 1.11.6, we can derive

Corollary 1.12.6. *If* $\mathcal{R}$ *is proximally (topologically) filtered, then* $\mathcal{R}$ *is also quasi-proximally (quasi-topologically) filtered.*

Now, by using Theorem 1.12.3, we can also easily prove the following

Theorem 1.12.7. *If* $\mathcal{R}$ *is quasi-proximally filtered and proximal, then* $\mathcal{R}$ *is proximally filtered.*

Proof. Suppose that $A \subseteq X$ and $R,\ S \in \mathcal{R}$. Then, by Definition 1.10.1, there exist $U,\ V \in \tau_{\mathcal{R}}$ such that $A \subseteq U \subseteq R[A]$ and $A \subseteq V \subseteq S[A]$.

Moreover, by Theorem 1.12.3, we can state that $U \cap V \in \tau_{\mathcal{R}}$. Therefore, by the definition of $\tau_{\mathcal{R}}$, there exists $T \in \mathcal{R}$ such that $T[U \cap V] \subseteq U \cap V$. Hence, we can already see that $T[A] \subseteq T[U \cap V] \subseteq U \cap V \subseteq R[A] \cap S[A]$.

Moreover, by using Theorem 1.12.4, we can quite similarly prove the following

Theorem 1.12.8. *If* $\mathcal{R}$ *is quasi-topologically filtered and topological, then* $\mathcal{R}$ *is topologically filtered.*

Remark 1.12.9. Our former Example 1.11.7 shows that even a quasi-proximally filtered, reflexive relator need not be topologically filtered.

Namely, if X and $\mathcal{R}$ are as in Example 1.11.7, then by the corresponding definitions it is clear that $\tau_{\mathcal{R}} = \{\emptyset,\ \{2,\ 3\},\ X\}$, and thus by Theorem 1.12.3 the relator $\mathcal{R}$ is quasi-proximally filtered.

1.13 Some Further Theorems on Topologically Filtered Relators

In our papers [101, 102], by using the arguments of Kuratowski [65, pp. 39, 45], we have proved some more particular theorems on the relation $\mathrm{cl}_{\mathcal{R}}$.

To make the forthcoming proofs much shorter and more readable, we shall use the following convenient notations

$$A^{-} = \mathrm{cl}_{\mathcal{R}}(A), \qquad A^{\circ} = \mathrm{int}_{\mathcal{R}}(A) \qquad \text{and} \qquad A^{\dagger} = \mathrm{res}_{\mathcal{R}}(A).$$

Theorem 1.13.1. *If $\mathcal{R}$ is topologically filtered, then for any $A, B \subseteq X$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A) \setminus \mathrm{cl}_{\mathcal{R}}(B) = \mathrm{cl}_{\mathcal{R}}(A \setminus B) \setminus \mathrm{cl}_{\mathcal{R}}(B).$$

Proof. By using Theorem 1.11.5, we can see that

$$A^- \cup B^- = (A \cup B)^- = \big((A \setminus B) \cup B\big)^- = (A \setminus B)^- \cup B^-.$$

Hence, we can already infer that

$$A^- \setminus B^- = \big(A^- \cup B^-\big) \setminus B^- = \big((A \setminus B)^- \cup B^-\big) \setminus B^- = (A \setminus B)^- \setminus B^-.$$

Thus, in particular, we can also state the following

Corollary 1.13.2. *If $\mathcal{R}$ is topologically filtered, then for any $A, B \subseteq X$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A) \setminus \mathrm{cl}_{\mathcal{R}}(B) \subseteq \mathrm{cl}_{\mathcal{R}}(A \setminus B).$$

The importance of topologically filtered relators is also apparent from

Theorem 1.13.3. *If $\mathcal{R}$ is topologically filtered, then for any $A, B \subseteq X$ we have*

(1) $\mathrm{cl}_{\mathcal{R}}(A) \cap \mathrm{int}_{\mathcal{R}}(B) \subseteq \mathrm{cl}_{\mathcal{R}}(A \cap B)$;

(2) $\mathrm{int}_{\mathcal{R}}(A \cup B) \subseteq \mathrm{int}_{\mathcal{R}}(A) \cup \mathrm{cl}_{\mathcal{R}}(B)$.

Proof. Assume that $x \in A^- \cap B^\circ$ and $R \in \mathcal{R}$. Then, since $x \in B^\circ$, there exists $S \in \mathcal{R}$ such that $S(x) \subseteq B$. Moreover, since $\mathcal{R}$ is topologically filtered, there exists $T \in \mathcal{R}$ such that $T(x) \subseteq R(x) \cap S(x)$. Furthermore, since $x \in A^-$, there exists $y \in A$ such that $y \in T(x)$. Hence, we can already infer that $y \in A \cap T(x) \subseteq A \cap S(x) \subseteq A \cap B$ and $y \in T(x) \subseteq R(x)$. Therefore, $R(x) \cap (A \cap B) \neq \emptyset$ for all $R \in \mathcal{R}$, and thus $x \in (A \cap B)^-$ also holds. This proves that $A^- \cap B^\circ \subseteq (A \cap B)^-$, and thus assertion (1) is true.

Now, by applying assertion (1) to the sets A^c and B^c and using De Morgan's laws and Theorem 1.3.6, we can easily see that assertion (2) is also true.

From this theorem, and by Definition 1.4.1, we can immediately derive

Corollary 1.13.4. *If $\mathcal{R}$ is topologically filtered, then*

(1) $\mathrm{cl}_{\mathcal{R}}(A) \cap B \subseteq \mathrm{cl}_{\mathcal{R}}(A \cap B)$ *for all* $A \subseteq X$ *and* $B \in \mathcal{T}_{\mathcal{R}}$;

(2) $\mathrm{int}_{\mathcal{R}}(A \cup B) \subseteq \mathrm{int}_{\mathcal{R}}(A) \cup B$ *for all* $A \subseteq X$ *and* $B \in \mathcal{F}_{\mathcal{R}}$.

Remark 1.13.5. The important inclusion $A^- \cap B \subseteq (A \cap B)^-$, with B being open, was first revealed by Kuratowski [65, p. 45].

Later, Császár [20, 21, 22, 23, 24, 25] and Sivagami [110] assumed it as an axiom for an increasing set-to-set function γ.

Now, as some improvements of the above theorem and its corollary, we can also prove the following theorem and its corollary.

Theorem 1.13.6. *If $\mathcal{R}$ is topologically filtered, then for any $A, B \subseteq X$ we have*

(1) $\mathrm{cl}_{\mathcal{R}}(A) \cap \mathrm{int}_{\mathcal{R}}(B) = \mathrm{cl}_{\mathcal{R}}(A \cap B) \cap \mathrm{int}_{\mathcal{R}}(B)$;

(2) $\mathrm{int}_{\mathcal{R}}(A \cup B) \cup \mathrm{cl}_{\mathcal{R}}(B) = \mathrm{int}_{\mathcal{R}}(A) \cup \mathrm{cl}_{\mathcal{R}}(B)$.

Proof. To prove (1), note that, by Theorem 1.13.3, we have $A^- \cap B^\circ \subseteq (A \cap B)^-$, and thus also $A^- \cap B^\circ = A^- \cap B^\circ \cap B^\circ \subseteq (A \cap B)^- \cap B^\circ$.

On the other hand, by using the increasingness of $-$, we can see that $(A \cap B)^- \subseteq A^-$, and thus also $(A \cap B)^- \cap B^\circ = (A \cap B)^- \cap B^\circ \cap B^\circ \subseteq A^- \cap B^\circ$.

Corollary 1.13.7. *If $\mathcal{R}$ is topologically filtered, then*

(1) $\mathrm{cl}_{\mathcal{R}}(A) \cap B = \mathrm{cl}_{\mathcal{R}}(A \cap B) \cap B$ *for all* $A \subseteq X$ *and* $B \in \mathcal{T}_{\mathcal{R}}$;

(2) $\mathrm{int}_{\mathcal{R}}(A \cup B) \cup B = \mathrm{int}_{\mathcal{R}}(A) \cup B$ *for all* $A \subseteq X$ *and* $B \in \mathcal{F}_{\mathcal{R}}$.

Proof. To derive assertion (1) from that of Theorem 1.13.6, note that if $B \in \mathcal{T}_{\mathcal{R}}$, then by Definition 1.4.1 we have $B \subseteq B^\circ$, and thus also $B^\circ \cap B = B$.

However, Theorem 1.13.6 and its corollary are less important than Theorem 1.13.3 and its corollary. Namely, for instance, by using Corollary 1.13.4 and our former theorems on topological relators, we can already prove the following

Theorem 1.13.8. *If $\mathcal{R}$ is topological and topologically filtered, then*

(1) $\mathrm{cl}_{\mathcal{R}}(A \cap B) = \mathrm{cl}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A) \cap B\big)$ *for all* $A \subseteq X$ *and* $B \in \mathcal{T}_{\mathcal{R}}$;

(2) $\mathrm{int}_{\mathcal{R}}(A \cup B) = \mathrm{int}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A) \cup B\big)$ *for all* $A \subseteq X$ *and* $B \in \mathcal{F}_{\mathcal{R}}$.

Proof. To prove (1), note that if $A \subseteq X$ and $B \in \mathcal{T}_{\mathcal{R}}$, then by Corollary 1.13.4 we have $A^- \cap B \subseteq (A \cap B)^-$. Hence, by using Theorem 1.9.2, we can infer

$$(A^- \cap B)^- \subseteq (A \cap B)^{--} \subseteq (A \cap B)^-.$$

On the other hand, by Theorems 1.9.3 and 1.8.3, we have $A \subseteq A^-$, and thus also $A \cap B \subseteq A^- \cap B$. Hence, we can infer that $(A \cap B)^- \subseteq \big(A^- \cap B\big)^-$. Therefore, the corresponding equality is also true.

From this theorem, by using the definitions of the families $\mathcal{D}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$, we can immediately derive the following

Corollary 1.13.9. *If $\mathcal{R}$ is topological and topologically filtered, then*

(1) $\mathrm{cl}_{\mathcal{R}}(A \cap B) = \mathrm{cl}_{\mathcal{R}}(B)$ *for all* $A \in \mathcal{D}_{\mathcal{R}}$ *and* $B \in \mathcal{T}_{\mathcal{R}}$;

(2) $\mathrm{int}_{\mathcal{R}}(A \cup B) = \mathrm{int}_{\mathcal{R}}(A)$ *for all* $A \in \mathcal{E}^c_{\mathcal{R}}$ *and* $B \in \mathcal{F}_{\mathcal{R}}$.

Now, by modifying an argument of Levine [72], we can also prove

Theorem 1.13.10. *If $\mathcal{R}$ is nonvoid and topological, and $A \subseteq X$, then*

(1) $\mathrm{cl}_{\mathcal{R}}(A \cap B) = \mathrm{cl}_{\mathcal{R}}(B)$ *for all* $B \in \mathcal{T}_{\mathcal{R}}$ *implies that* $A \in \mathcal{D}_{\mathcal{R}}$;

(2) $\mathrm{int}_{\mathcal{R}}(A \cup B) = \mathrm{int}_{\mathcal{R}}(A)$ *for all* $B \in \mathcal{F}_{\mathcal{R}}$ *implies that* $A \notin \mathcal{E}_{\mathcal{R}}$.

Proof. For instance, if $A \notin \mathcal{D}_{\mathcal{R}}$, then there exists $x \in X$ such that $x \notin A^-$. Thus, there exists $R \in \mathcal{R}$ such that $A \cap R(x) = \emptyset$. Moreover, since $\mathcal{R}$ is topological, there exists $B \in \mathcal{T}_{\mathcal{R}}$ such that $x \in B \subseteq R(x)$. Thus, we also have $A \cap B = \emptyset$.

Hence, by using the assumptions of (1), we can infer that $B^- = (A \cap B)^- = \emptyset^- = \emptyset$. On the other hand, from $x \in B$, we can now infer that $x \in \{x\}^- \subseteq B^-$, and thus $B^- \neq \emptyset$. This contradiction proves (1).

Remark 1.13.11. If $\mathcal{R}$ is nonvoid and reflexive, and $A \subseteq X$ such that $\mathrm{cl}_{\mathcal{R}}\big(A \cap R(x)\big) = \mathrm{cl}_{\mathcal{R}}\big(R(x)\big)$ for all $x \in X$ and $R \in \mathcal{R}$, then we can even more easily prove that $A \in \mathcal{D}_{\mathcal{R}}$.

1.14 Some More Particular Theorems on Topologically Filtered Relators

The importance of Corollary 1.13.4 is also apparent from the following

Theorem 1.14.1. *If $\mathcal{R}$ is quasi-topological and topologically filtered, then for any $A, B \in \mathcal{N}_{\mathcal{R}}$ we have $A \cup B \in \mathcal{N}_{\mathcal{R}}$.*

Proof. By Theorem 1.9.2, we have $B^- \in \mathcal{F}_{\mathcal{R}}$. Hence, by using Theorem 1.11.5, Corollary 1.13.4 and the definition of $\mathcal{N}_{\mathcal{R}}$, we can see that

$$(A \cup B)^{-\circ} = (A^- \cup B^-)^\circ \subseteq A^{-\circ} \cup B^- = \emptyset \cup B^- = B^-.$$

Moreover, by Theorem 1.9.2, we have $(A \cup B)^{-\circ} \in \mathcal{T}_{\mathcal{R}}$. Hence, by using the increasingness of $\circ$ and the definitions of $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$, we can see that

$$(A \cup B)^{-\circ} \subseteq (A \cup B)^{-\circ\circ} \subseteq B^{-\circ} = \emptyset.$$

Therefore, $(A \cup B)^{-\circ} = \emptyset$, and thus $A \cup B \in \mathcal{N}_{\mathcal{R}}$ also holds.

Now, by using this theorem, we can also easily establish the following

Corollary 1.14.2. *If $\mathcal{R}$ is nonvoid, non-partial, quasi-topological and topologically filtered, then $\mathcal{N}_{\mathcal{R}}$ is an ideal on X.*

Proof. By the definition of $\mathcal{N}_{\mathcal{R}}$ and the increasingness of $-\circ$, it is clear that $\mathcal{N}_{\mathcal{R}}$ is always descending. Moreover, since $\mathcal{R}$ is nonvoid and non-partial, we can also note that $\emptyset^{-\circ} = \emptyset^\circ = \emptyset$. Therefore, $\emptyset \in \mathcal{N}_{\mathcal{R}}$, and thus $\mathcal{N}_{\mathcal{R}} \neq \emptyset$. Furthermore, from Theorem 1.14.1, we know that $\mathcal{N}_{\mathcal{R}}$ is closed under pairwise unions.

Remark 1.14.3. Note that if $\mathcal{R}$ is locally non-partial, then by Theorem 1.8.12 we have $X^{-\circ} = X$. Therefore, if $X \neq \emptyset$, then we can also state that $X \notin \mathcal{N}_{\mathcal{R}}$, and thus $\mathcal{N}_{\mathcal{R}} \neq \mathcal{P}(X)$.

While, if $\mathcal{R}$ is quasi-topological and $A \in \mathcal{N}_{\mathcal{R}}$, then by using Theorem 1.9.2 and the increasingness of $\circ$ we can also see that $A^{--\circ} \subseteq A^{-\circ} = \emptyset$. Therefore, $A^{--\circ} = \emptyset$, and thus $A^- \in \mathcal{N}_{\mathcal{R}}$ also holds.

The importance of topologically filtered relators is also apparent from

Theorem 1.14.4. *If $\mathcal{R}$ is topological and topologically filtered, then for any $A \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\mathrm{res}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}.$$

Proof. By Theorem 1.4.8, we have $U^c \in \mathcal{F}_{\mathcal{R}}$. Moreover, by Theorems 1.9.3 and 1.9.2, we also have $U^- \in \mathcal{F}_{\mathcal{R}}$. Hence, by using Corollary 1.11.6, we can already infer that

$$U^\dagger = U^- \setminus U = U^- \cap U^c \in \mathcal{F}_{\mathcal{R}}.$$

Moreover, by using Theorems 1.11.5 and 1.8.3 and the increasingness of $-$, we can see that

$$U^{\dagger\circ} = \big(U^- \setminus U\big)^\circ = \big(U^- \cap U^c\big)^\circ = U^{-\circ} \cap U^{c\circ} = U^{-\circ} \cap U^{-c} \subseteq U^- \cap U^{-c} = \emptyset,$$

and thus $\big(U^- \setminus U\big)^\circ = \emptyset$. Therefore, $U^\dagger \notin \mathcal{E}_{\mathcal{R}}$, and thus $U^\dagger \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$.

By this theorem, it is clear that in particular we also have

Corollary 1.14.5. *If $\mathcal{R}$ is topological and topologically filtered, then* $\operatorname{res}_{\mathcal{R}}(A) \in \mathcal{N}_{\mathcal{R}}$ *for all* $A \in \mathcal{T}_{\mathcal{R}}$.

Remark 1.14.6. Note that if $\mathcal{R}$ is topological and $U \in \mathcal{T}_{\mathcal{R}}$, then by Theorems 1.9.3, 1.9.2 and 1.8.3 we have $U = U^{\circ}$. Therefore, under the notation $U^{\ddagger} = \operatorname{bnd}_{\mathcal{R}}(U)$, we also have $U^{\dagger} = U^{-} \setminus U = U^{-} \setminus U^{\circ} = U^{\ddagger}$.

Moreover, in Theorem 1.14.4 and Corollary 1.14.5, it is also enough to assume only that $\mathcal{R}$ is quasi-topologically filtered and topological. Namely, in this case, $\mathcal{R}$ is already topologically filtered by Theorem 1.12.8.

1.15 A Few Basic Facts on Simple Relators

Definition 1.15.1. The relator $\mathcal{R}$ is called *properly simple* if it is a singleton relator. That is, there exists a relation R on X such that $\mathcal{R} = \{R\}$.

Much more generally, the relator $\mathcal{R}$ is called $\mathfrak{F}$–*simple*, for some structure $\mathfrak{F}$ for relators on X, if it is $\mathfrak{F}$–equivalent to a singleton relator. That is, there exists a relation R on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{R}$.

Thus, by Theorems 1.5.4 and 1.7.12, for instance, we can at once state the following two theorems.

Theorem 1.15.2. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\wedge$–*simple;* (2) $\mathcal{R}$ *is* cl–*simple;* (3) $\mathcal{R}$ *is* int–*simple*.

Theorem 1.15.3. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\wedge\infty$–*simple;* (2) $\mathcal{R}$ *is* $\mathcal{F}$–*simple;* (3) $\mathcal{R}$ *is* $\mathcal{T}$–*simple*.

Remark 1.15.4. Thus, for instance, the relator $\mathcal{R}$ may be naturally called *topologically simple* if it is $\wedge$–simple. Moreover, the relator $\mathcal{R}$ may be naturally called *quasi-topologically simple* if it is $\wedge\infty$–simple.

Now, we can also easily prove the following

Theorem 1.15.5. *Under the notation* $R = \bigcap \mathcal{R}$, *the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topologically simple;* (2) $R \in \mathcal{R}^{\wedge}$*;* (3) $\mathcal{R}^{\wedge} = \{R\}^{\wedge}$.

Proof. If (1) holds, then by the corresponding definitions there exists a relation S on X such that $\mathcal{R}^{\wedge} = \{S\}^{\wedge}$. Hence, by Corollary 1.3.11 and Theorem 1.5.4, we can infer that

$$R^{-1}(y) = \rho_{\mathcal{R}}(y) = \operatorname{cl}_{\mathcal{R}}(\{y\}) = \operatorname{cl}_{S}(\{y\}) = \rho_{S}(y) = S^{-1}(y)$$

for all $y \in X$. Therefore, $R^{-1} = S^{-1}$, and thus also $R = S$. Consequently, we have $\mathcal{R}^{\wedge} = \{S\}^{\wedge} = \{R\}^{\wedge}$, and thus (3) also holds.

Now, since, (3) trivially implies (1), we need only show that (2) and (3) are also equivalent. For this, note that by Definition 1.5.1 we always have $R \in \{R\}^{\wedge}$ and $\mathcal{R} \subseteq \{R\}^{\wedge}$. Hence, by the corresponding properties of $\wedge$, it is clear that $\mathcal{R}^{\wedge} \subseteq \{R\}^{\wedge\wedge}$, and thus $\mathcal{R}^{\wedge} \subseteq \{R\}^{\wedge}$. Moreover, we can also note that

$$(2) \implies \{R\} \subseteq \mathcal{R}^{\wedge} \implies \{R\}^{\wedge} \subseteq \mathcal{R}^{\wedge\wedge} \implies \{R\}^{\wedge} \subseteq \mathcal{R}^{\wedge}.$$

Remark 1.15.6. Note that, by Definition 1.5.1, for any relation R we actually have $\{R\}^{\wedge} = \{R\}^{\#} = \{R\}^{*}$.

While, for any relation S on X, we have $S \in \{R\}^{\vartriangle}$ if and only if for each $x \in X$ there exists $\varphi(x) \in X$ such that $R\big(\varphi(x)\big) \subseteq S(x)$. That is, there exists a function φ of X to itself such that $R \circ \varphi \subseteq S$. Therefore, $\{R\}^{\vartriangle} = \big(R \circ X^{X}\big)^{*}$.

In addition to Theorem 1.15.5, we can also easily prove the following

Theorem 1.15.7. *If $\mathcal{R}$ is nonvoid, then under the notation $R = \bigcap \mathcal{R}$, we have*

$$\mathcal{R}^{\vee\wedge} = \big\{R^{-1}\big\}^{\wedge}.$$

Proof. By Definitions 1.3.2 and 1.5.10 and Theorems 1.3.9, 1.6.1 and 1.3.10 and Corollary 1.3.11, for any $y \in X$ and $A \subseteq X$, we have

$$\begin{aligned} y \in \mathrm{cl}_{\mathcal{R}^{\vee}}(A) &\iff \{y\} \in \mathrm{Cl}_{\mathcal{R}^{\vee}}(A) \iff \{y\} \in \mathrm{Cl}_{\mathcal{R}^{\wedge -1}}(A) \iff \{y\} \in \mathrm{Cl}_{\mathcal{R}^{\wedge}}^{-1}(A) \\ &\iff A \in \mathrm{Cl}_{\mathcal{R}^{\wedge}}\big(\{y\}\big) \iff A \cap \mathrm{cl}_{\mathcal{R}}\big(\{y\}\big) \neq \emptyset \iff A \cap \rho_{\mathcal{R}}(y) \neq \emptyset \\ &\iff A \cap R^{-1}(y) \neq \emptyset \iff y \in R[A] \iff y \in \mathrm{cl}_{R^{-1}}(A). \end{aligned}$$

Therefore, $\mathrm{cl}_{\mathcal{R}^{\vee}} = \mathrm{cl}_{R^{-1}}$, and thus by Theorem 1.5.4 the required equality is true.

Now, as an immediate consequence of this theorem, we can also state

Corollary 1.15.8. *If $\mathcal{R}$ is nonvoid, then $\mathcal{R}^{\vee}$ is topologically simple.*

Remark 1.15.9. Note that if in particular $\mathcal{R} = \emptyset$, but $X \neq \emptyset$, then by Definition 1.5.1 we have $\mathcal{R}^{\wedge} = \emptyset$, and thus also $\mathcal{R}^{\vee} = \mathcal{R}^{\wedge -1} = \emptyset$ which cannot be topologically simple.

1.16 A Few Basic Facts on Symmetric Relators

In contrast to the reflexivity property of the relator $\mathcal{R}$, we may naturally introduce a great abundance of important symmetry and transitivity properties of $\mathcal{R}$ [116, 117].

For instance, the relator $\mathcal{R}$ may be naturally called *strongly symmetric* if each member of $\mathcal{R}$ is symmetric. Moreover, the relator $\mathcal{R}$ may be naturally called *weakly symmetric* if the relation $R = \bigcap \mathcal{R}$ is symmetric.

However, it is now more important to note that we may also naturally have

Definition 1.16.1. The relator $\mathcal{R}$ is called

(1) *properly symmetric* if $R \in \mathcal{R}$ implies $R^{-1} \in \mathcal{R}$;

(2) *uniformly symmetric* if for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S \subseteq R^{-1}$;

(3) *proximally symmetric* if for each $A \subseteq X$ and $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S[A] \subseteq R^{-1}[A]$;
(4) *topologically symmetric* if for each $x \in X$ and $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(x) \subseteq R^{-1}(x)$.

Remark 1.16.2. By using the operations -1, $*$, $\#$ and $\wedge$, the above properties can be reformulated in more concise forms.

For instance, by using some basic properties of $\#$, we can prove

Theorem 1.16.3. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximally symmetric;* (2) $\mathcal{R}^{-1} \subseteq \mathcal{R}^{\#}$; (3) $\mathcal{R}^{-1\#} = \mathcal{R}^{\#}$;
(4) $\mathcal{R}^{\#-1} \subseteq \mathcal{R}^{\#}$; (5) $\mathcal{R}^{\#-1} = \mathcal{R}^{\#}$.

Thus, in particular we can also state the following

Corollary 1.16.4. *The following assertions are equivalent:*

(1) $\mathcal{R}^{\#}$ *is properly symmetric;* (2) $\mathcal{R}$ *is proximally symmetric;*
(3) $\mathcal{R}^{-1}$ *is proximally symmetric;* (4) $\mathcal{R}$ *and* $\mathcal{R}^{-1}$ *are proximally equivalent.*

Remark 1.16.5. Concerning topological symmetry, we can only prove that $\mathcal{R}$ is topologically symmetric if and only if $\mathcal{R}^{-1} \subseteq \mathcal{R}^{\wedge}$, or equivalently $\mathcal{R}^{-1\wedge} \subseteq \mathcal{R}^{\wedge}$.

Hence, it is clear that, topological symmetry is already not a genuine symmetry property. Therefore, in addition to Definition 1.16.1, we must also have

Definition 1.16.6. The relator $\mathcal{R}$ is called
(1) *properly topologically symmetric* if $\mathcal{R}^{\wedge}$ is properly symmetric;
(2) *topologically bisymmetric* if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are topologically symmetric.

Remark 1.16.7. Thus, by Remark 1.16.5, we can see that $\mathcal{R}$ is topologically bisymmetric if and only if $\mathcal{R}^{\wedge} = \mathcal{R}^{-1\wedge}$. That is, $\mathcal{R}$ and $\mathcal{R}^{-1}$ are topologically equivalent.

Moreover, by using Definitions 1.16.6 and 1.5.10 and Theorem 1.15.7, we can also prove the following two theorems.

Theorem 1.16.8. *The following assertions are equivalent:*
(1) $\mathcal{R}$ *is properly topologically symmetric;*
(2) $\mathcal{R}^{\vee} \subseteq \mathcal{R}^{\wedge}$; (3) $\mathcal{R}^{\vee} = \mathcal{R}^{\wedge}$; (4) $\mathcal{R}^{\vee\wedge} = \mathcal{R}^{\wedge}$; (5) $\mathcal{R}^{\vee\vee} = \mathcal{R}^{\vee}$.

Theorem 1.16.9. *If $\mathcal{R}$ is nonvoid, then under the notation $R = \bigcap \mathcal{R}$ the following assertions are equivalent:*
(1) $\mathcal{R}$ *is properly topologically symmetric;* (2) $R^{-1} \in \mathcal{R}^{\wedge}$; (3) $\mathcal{R}^{\wedge} = \{R^{-1}\}^{\wedge}$.

Proof. If (1) holds, then by Theorems 1.16.8 and 1.15.7 we have $\mathcal{R}^{\wedge} = \mathcal{R}^{\vee\wedge} = \{R^{-1}\}^{\wedge}$, and thus (3) holds. While, if (3) holds, then

$$\mathcal{R}^{\wedge-1} = \{R^{-1}\}^{\wedge-1} = \{R^{-1}\}^{*-1} = \{R^{-1}\}^{-1*} = \{R\}^{*} = \{R\}^{\wedge},$$

and thus (1) also holds. The equivalence of (2) and (3) is even more obvious.

From this theorem, we can easily derive the following

Corollary 1.16.10. *If $\mathcal{R}$ is nonvoid, then the following assertions are equivalent :*

(1) *$\mathcal{R}$ is properly topologically symmetric ;*

(2) *$\mathcal{R}$ is topologically simple and weakly symmetric.*

Proof. To derive the symmetry of $R = \bigcap \mathcal{R}$ from (1), note that if (1) holds, then by Corollary 1.3.11, Definition 1.3.2 and Theorems 1.16.9 and 1.5.4, we have

$$R^{-1} = \rho_{\mathcal{R}} = \rho_{\mathcal{R}^{\wedge}} = \rho_{\{R^{-1}\}^{\wedge}} = \rho_{R^{-1}} = R.$$

Remark 1.16.11. According to Remark 1.16.5, the relator $\mathcal{R}$ may be naturally called *paratopologically symmetric* if $\mathcal{R}^{-1} \subseteq \mathcal{R}^{\vartriangle}$, or equivalently $\mathcal{R}^{-1\,\vartriangle} \subseteq \mathcal{R}^{\vartriangle}$.

Moreover, by Definition 1.16.6, the relator $\mathcal{R}$ may, for instance, be naturally called *properly paratopologically symmetric* if the relator $\mathcal{R}^{\vartriangle}$ is properly symmetric.

On the other hand, by Remark 1.16.5, the relator $\mathcal{R}$ may also be naturally called *quasi-topologically symmetric* if $\mathcal{R}^{-1\wedge\infty} \subseteq \mathcal{R}^{\wedge}$, or equivalently $\mathcal{R}^{-1\wedge\infty} \subseteq \mathcal{R}^{\wedge\infty}$.

By Theorems 1.5.4 and 1.7.12, we can at once state the following two theorems.

Theorem 1.16.12. *The following assertions are equivalent :*

(1) *$\mathcal{R}$ is topologically symmetric ;* (2) $\operatorname{cl}_{\mathcal{R}} \subseteq \operatorname{cl}_{\mathcal{R}^{-1}}$ *;* (3) $\operatorname{int}_{\mathcal{R}^{-1}} \subseteq \operatorname{int}_{\mathcal{R}}$.

Theorem 1.16.13. *If $\mathcal{R}$ is nonvoid, then the following assertions are equivalent :*

(1) *$\mathcal{R}$ is quasi-topologically symmetric ;* (2) $\mathcal{T}_{\mathcal{R}^{-1}} \subseteq \mathcal{T}_{\mathcal{R}}$ *;* (3) $\mathcal{F}_{\mathcal{R}^{-1}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Remark 1.16.14. Thus, in particular, we can also state that $\mathcal{R}$ is quasi-topologically bisymmetric if and only if $\mathcal{T}_{\mathcal{R}^{-1}} = \mathcal{T}_{\mathcal{R}}$, or equivalently $\mathcal{F}_{\mathcal{R}^{-1}} = \mathcal{F}_{\mathcal{R}}$.

However, it is now more important to note that, by using our former results, we can also prove the following three theorems.

Theorem 1.16.15. *If $\mathcal{R}$ is nonvoid, then under the notation $R = \bigcap \mathcal{R}$ the following assertions are equivalent :*

(1) *$\mathcal{R}$ is properly topologically symmetric ;*

(2) $\operatorname{cl}_{\mathcal{R}}(A) = R[A]$ *for all* $A \subseteq X$*;* (3) $\operatorname{int}_{\mathcal{R}}(A) = R[A^c]^c$ *for all* $A \subseteq X$.

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then by Theorem 1.15.7 we have $\mathcal{R}^{\wedge} = \{R^{-1}\}^{\wedge}$. Hence, by using Theorems 1.5.4 and 1.3.10 we can infer that $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{R^{-1}}(A) = R[A]$ for all $A \subseteq X$. Therefore, (2) also holds.

Theorem 1.16.16. *If $\mathcal{R}$ is nonvoid, then the following assertions are equivalent :*

(1) *$\mathcal{R}$ is properly topologically symmetric ;*

(2) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ *implies* $B^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$ *for all* $A, B \subseteq X$*;*

(3) $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *implies* $B \cap \operatorname{cl}_{\mathcal{R}}(A) \neq \emptyset$ *for all* $A, B \subseteq X$.

Proof. To prove the implication (1) $\Longrightarrow$ (3), note that if (1) holds and $R = \bigcap \mathcal{R}$, then by Theorem 1.16.15 and Corollary 1.16.10

$$A \cap \mathrm{cl}_{\mathcal{R}}(B) \neq \emptyset \implies A \cap R[B] \neq \emptyset \implies R^{-1}[A] \cap B \neq \emptyset$$
$$\implies R[A] \cap B \neq \emptyset \implies \mathrm{cl}_{\mathcal{R}}(A) \cap B \neq \emptyset$$

for all $A, B \subseteq X$. Therefore, (3) also holds.

Theorem 1.16.17. *If $\mathcal{R}$ is nonvoid, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is properly topologically symmetric;*

(2) $A \subseteq \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big)$ *for all* $A \subseteq X$*;* (3) $\mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big) \subseteq A$ *for all* $A \subseteq X$.

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if $A \subseteq X$, then by Theorem 1.3.6 we have $\mathrm{cl}_{\mathcal{R}}(A)^c = \mathrm{int}_{\mathcal{R}}(A^c)$, and thus also $\mathrm{cl}_{\mathcal{R}}(A)^c \subseteq \mathrm{int}_{\mathcal{R}}(A^c)$. Therefore, if (1) holds, then by Theorem 1.16.16 we also have $A \subseteq \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big)$. Thus, (2) also holds.

Remark 1.16.18. The latter three theorems can be naturally generalized to topologically simple relators. For instance, it can be shown that a nonvoid relator is topologically simple if and only if under the notation $\mathcal{S} = \mathcal{R}^{-1}$ or $\mathcal{R}^{\vee}$ we have $A \subseteq \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{S}}(A)\big)$ for all $A \subseteq X$.

1.17 Some Basic Facts on the Elementwise Unions of Relators

Notation 1.17.1. In this and the next section, in addition to Notation 1.3.1, we shall assume that $\mathcal{S}$ **is also a relator on** X.

Definition 1.17.2. The relator

$$\mathcal{R} \vee \mathcal{S} = \big\{ R \cup S : \quad R \in \mathcal{R}, \quad S \in \mathcal{S} \big\}$$

is called the *elementwise union* of the relators $\mathcal{R}$ and $\mathcal{S}$.

Remark 1.17.3. If somewhat more generally $\mathcal{R} = \big(R_i\big)_{i \in I}$ and $\mathcal{S} = \big(S_i\big)_{i \in I}$, where R_i and S_i are relations on X, then we may also naturally define $\mathcal{R} \triangledown \mathcal{S} = \big(R_i \cup S_i\big)_{i \in I}$.

Thus, in particular for any relator $\mathcal{R}$, we may also naturally write

$$\mathcal{R} \triangledown \mathcal{R}^{-1} = \big\{ R \cup R^{-1} : \ R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R} \vee \mathcal{R}^{-1} = \big\{ R \cup S^{-1} : \ R, S \in \mathcal{R} \big\}.$$

The importance of the relator $\mathcal{R} \vee \mathcal{S}$ is already apparent from the following

Theorem 1.17.4. *We have*

(1) $\mathrm{Int}_{\mathcal{R} \vee \mathcal{S}} = \mathrm{Int}_{\mathcal{R}} \cap \mathrm{Int}_{\mathcal{S}}$*;* (2) $\mathrm{Cl}_{\mathcal{R} \vee \mathcal{S}} = \mathrm{Cl}_{\mathcal{R}} \cup \mathrm{Cl}_{\mathcal{S}}$.

Proof. If $B \subseteq X$ and $A \in \mathrm{Int}_{\mathcal{R} \vee \mathcal{S}}(B)$, then there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that $(R \cup S)[A] \subseteq B$. Hence, by using that $(R \cup S)[A] = R[A] \cup S[A]$, we can already infer that $R[A] \cup S[A] \subseteq B$, and thus $R[A] \subseteq B$ and $S[A] \subseteq B$. Therefore, $A \in \mathrm{Int}_{\mathcal{R}}(B)$ and $A \in \mathrm{Int}_{\mathcal{S}}(B)$, and thus $A \in \mathrm{Int}_{\mathcal{R}}(B) \cap \mathrm{Int}_{\mathcal{S}}(B) = \big(\mathrm{Int}_{\mathcal{R}} \cap \mathrm{Int}_{\mathcal{S}}\big)(B)$.

This shows that $\mathrm{Int}_{\mathcal{R} \vee \mathcal{S}}(B) \subseteq \big(\mathrm{Int}_{\mathcal{R}} \cap \mathrm{Int}_{\mathcal{S}}\big)(B)$ for all $B \subseteq Y$, and thus $\mathrm{Int}_{\mathcal{R} \vee \mathcal{S}} \subseteq \mathrm{Int}_{\mathcal{R}} \cap \mathrm{Int}_{\mathcal{S}}$ also holds.

The converse inclusion can be proved quite similarly. Moreover, assertion (2) can be derived from (1) by using 1.3.4.

Now, as an immediate consequence of this theorem we can also state

Corollary 1.17.5. *We have*

(1) $\tau_{\mathcal{R}\vee\mathcal{S}} = \tau_{\mathcal{R}} \cap \tau_{\mathcal{S}}$; (2) $\tau\!\!\!-_{\mathcal{R}\vee\mathcal{S}} = \tau\!\!\!-_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{S}}$.

Proof. To prove (1), note that for any $A \subseteq X$, we have

$$\begin{aligned} A \in \tau_{\mathcal{R}\vee\mathcal{S}} &\iff A \in \operatorname{Int}_{\mathcal{R}\vee\mathcal{S}}(A) \iff A \in \big(\operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}}\big)(A) \\ &\iff A \in \operatorname{Int}_{\mathcal{R}}(A) \cap \operatorname{Int}_{\mathcal{S}}(A) \iff A \in \operatorname{Int}_{\mathcal{R}}(A)\,, \quad A \in \operatorname{Int}_{\mathcal{S}}(A) \\ &\iff A \in \tau_{\mathcal{R}}\,, \quad A \in \tau_{\mathcal{S}} \iff A \in \tau_{\mathcal{R}} \cap \tau_{\mathcal{S}}\,. \end{aligned}$$

Hence, by Theorem 1.4.4, it is clear that in particular we also have

Corollary 1.17.6. *We have*

(1) $\tau_{\mathcal{R}\vee\mathcal{S}^{-1}} = \tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{S}}$; (2) $\tau\!\!\!-_{\mathcal{R}\vee\mathcal{S}^{-1}} = \tau\!\!\!-_{\mathcal{R}} \cap \tau_{\mathcal{S}}$;

From Theorem 1.17.4, we can also immediately derive

Theorem 1.17.7. *We have*

(1) $\operatorname{int}_{\mathcal{R}\vee\mathcal{S}} = \operatorname{int}_{\mathcal{R}} \cap \operatorname{int}_{\mathcal{S}}$; (2) $\operatorname{cl}_{\mathcal{R}\vee\mathcal{S}} = \operatorname{cl}_{\mathcal{R}} \cup \operatorname{cl}_{\mathcal{S}}$.

Now, as an immediate consequence of this theorem, we can also state

Corollary 1.17.8. *We have*

(1) $\mathcal{T}_{\mathcal{R}\vee\mathcal{S}} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{S}}$; (2) $\mathcal{F}_{\mathcal{R}\vee\mathcal{S}} = \mathcal{F}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{S}}$.

However, an analogue of Corollary 1.17.6 cannot be stated. Moreover, by using Theorem 1.17.7, we can only prove

Corollary 1.17.9. *We have*

(1) $\mathcal{E}_{\mathcal{R}\vee\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \cap \mathcal{E}_{\mathcal{S}}$; (2) $\mathcal{D}_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{S}} \subseteq \mathcal{D}_{\mathcal{R}\vee\mathcal{S}}$.

Remark 1.17.10. Analogously to Definition 1.17.2 we may also naturally consider the *elementwise intersection*

$$\mathcal{R} \wedge \mathcal{S} = \big\{ R \cap S : \ R \in \mathcal{R}, S \in \mathcal{S} \big\}.$$

Thus, the relator $\mathcal{R}$ may, for instance, be naturally called *uniformly filtered* if $\mathcal{R} \wedge \mathcal{R} \subseteq \mathcal{R}^*$. That is, for any $R,\ S \in \mathcal{R}$ there exists $U \in \mathcal{R}$ such that $U \subseteq R \cap S$.

Thus, it can be shown that $\mathcal{R}$ is uniformly filtered if and only if $\mathcal{R}$ and $\mathcal{R} \wedge \mathcal{R}$ are *uniformly equivalent* in the sense that $\mathcal{R}^* = \big(\mathcal{R} \wedge \mathcal{R}\big)^*$. Or equivalently, $\mathcal{R}^*$ is *properly filtered* in the sense that $\mathcal{R}^* \wedge \mathcal{R}^* \subseteq \mathcal{R}^*$, or equivalently $\mathcal{R}^* \wedge \mathcal{R}^* = \mathcal{R}^*$.

Now, by using the above definition, we can also easily prove the following

Theorem 1.17.11. *If $\mathcal{R}$ is uniformly filtered, then for any $\square \in \{\,*\,, \#\,, \wedge\,, \triangle\,\}$, we have*

$$\big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^{\square} = \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^{\square}.$$

Proof. By the corresponding definitions, we have $\mathcal{R} \triangledown \mathcal{R}^{-1} \subseteq \mathcal{R} \vee \mathcal{R}^{-1}$, and thus $\left(\mathcal{R} \triangledown \mathcal{R}^{-1}\right)^{*} \subseteq \left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^{*}$. Furthermore, if $V \in \left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^{*}$, then there exist $R, S \in \mathcal{R}$ such that $R \cup S^{-1} \subseteq V$. Moreover, since $\mathcal{R}$ is uniformly filtered, there exists $U \in \mathcal{R}$ such that $U \subseteq R \cap S$. Hence, we can already see that $U \cup U^{-1} \subseteq R \cup S^{-1} \subseteq V$, and thus $V \in \left(\mathcal{R} \triangledown \mathcal{R}^{-1}\right)^{*}$. Therefore, $\left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^{*} \subseteq \left(\mathcal{R} \triangledown \mathcal{R}^{-1}\right)^{*}$, and thus the corresponding equality is also true. Hence, since $* \square = \square$, it is clear that the required equality is also true.

Thus, for instance, we can also state the following

Corollary 1.17.12. *If $\mathcal{R}$ is uniformly filtered, then*

(1) $\tau_{\mathcal{R} \triangledown \mathcal{R}^{-1}} = \tau_{\mathcal{R} \vee \mathcal{R}^{-1}}$; (2) $\mathcal{\tau}_{\mathcal{R} \triangledown \mathcal{R}^{-1}} = \mathcal{\tau}_{\mathcal{R} \vee \mathcal{R}^{-1}}$;

(3) $\mathcal{T}_{\mathcal{R} \triangledown \mathcal{R}^{-1}} = \mathcal{T}_{\mathcal{R} \vee \mathcal{R}^{-1}}$; (4) $\mathcal{F}_{\mathcal{R} \triangledown \mathcal{R}^{-1}} = \mathcal{F}_{\mathcal{R} \vee \mathcal{R}^{-1}}$.

Remark 1.17.13. Analogously to Remark 1.17.10, the relator $\mathcal{R}$ may be naturally called *topologically filtered* if the relator $\mathcal{R}^{\wedge}$ is properly filtered. However, since in general $R[A] \cap S[A] \not\subseteq (R \cap S)[A]$, to define "proximally filtered" we have two natural possibilities [116].

Moreover, for instance, the relator $\mathcal{R}$ may be naturally called *quasi-topologically filtered* if the relator $\mathcal{R}^{\wedge\infty}$ is properly filtered. Namely, it can be shown that $\mathcal{R}$ is quasi-topologically filtered if and only if the family $\mathcal{T}_{\mathcal{R}}$ is closed under binary intersections.

1.18 Further Results on the Elementwise Unions of Relators

Concerning the relator $\mathcal{R} \vee \mathcal{S}$, we can also easily prove the following

Theorem 1.18.1. *If $\square \in \{*, \#, \wedge\}$, then*

$$\left(\mathcal{R} \vee \mathcal{S}\right)^{\square} = \mathcal{R}^{\square} \cap \mathcal{S}^{\square}.$$

Proof. We shall only prove the particular case $\square = \#$ of the above equality. For this, note that if $V \in (\mathcal{R} \vee \mathcal{S})^{\#}$, then for each $A \subseteq X$ there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that $(R \cup S)[A] \subseteq V[A]$. Hence, by using that $(R \cup S)[A] = R[A] \cup S[A]$, we can already infer that $R[A] \cup S[A] \subseteq V[A]$, and thus $R[A] \subseteq V[A]$ and $S[A] \subseteq V[A]$. Therefore, $V \in \mathcal{R}^{\#}$ and $V \in \mathcal{S}^{\#}$, and thus $V \in \mathcal{R}^{\#} \cap \mathcal{S}^{\#}$.

On the other hand, if $V \in \mathcal{R}^{\#} \cap \mathcal{S}^{\#}$, then $V \in \mathcal{R}^{\#}$ and $V \in \mathcal{S}^{\#}$. Therefore, for each $A \subseteq X$, there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that $R[A] \subseteq V[A]$ and $S[A] \subseteq V[A]$. Hence, it follows that $(R \cup S)[A] = R[A] \cup S[A] \subseteq V[A]$, and thus $V \in (\mathcal{R} \vee \mathcal{S})^{\#}$.

Remark 1.18.2. By using a similar argument, concerning the operation $\vartriangle$, we can only prove

$$\left(\mathcal{R} \vee \mathcal{S}\right)^{\vartriangle} \subseteq \mathcal{R}^{\vartriangle} \cap \mathcal{S}^{\vartriangle}.$$

From Theorem 1.18.1, we can easily derive the following

Corollary 1.18.3. *If $\square \in \{*, \#, \wedge\}$, then*

(1) $\left(\mathcal{R} \vee \mathcal{S}\right)^{\square} = \left(\mathcal{R}^{\square} \vee \mathcal{S}^{\square}\right)^{\square}$; (2) $\mathcal{R}^{\square} \cap \mathcal{S}^{\square} = \left(\mathcal{R}^{\square} \cap \mathcal{S}^{\square}\right)^{\square}$.

Proof. By Theorem 1.18.1 and the idempotency of $\square$, it is clear that
$(\mathcal{R}\vee\mathcal{S})^{\square}=\mathcal{R}^{\square}\cap\mathcal{S}^{\square}=\mathcal{R}^{\square\square}\cap\mathcal{S}^{\square\square}=\big(\mathcal{R}^{\square}\vee\mathcal{S}^{\square}\big)^{\square}$.

Remark 1.18.4. From assertion (1), it is clear that

$$(\mathcal{R}\vee\mathcal{S})^{\wedge}\subseteq\big(\mathcal{R}^{\#}\vee\mathcal{S}^{\#}\big)^{\wedge}\subseteq\big(\mathcal{R}^{\wedge}\vee\mathcal{S}^{\wedge}\big)^{\wedge}=(\mathcal{R}\vee\mathcal{S})^{\wedge},$$

and thus in particular $(\mathcal{R}\vee\mathcal{S})^{\wedge}=\big(\mathcal{R}^{\#}\vee\mathcal{S}^{\#}\big)^{\wedge}$ is also true.

While, from assertion (2), we can at once see that the relator $\mathcal{R}^{\square}\cap\mathcal{S}^{\square}$ is always $\square$*–fine*. Moreover, if $\mathcal{R}$ and $\mathcal{S}$ are $\square$–fine, then $\mathcal{R}\cap\mathcal{S}$ is also $\square$–fine.

In addition to Theorem 1.18.1, we can also easily prove the following

Theorem 1.18.5. *If* $\square\in\{*,\#,\wedge\}$, *then the following assertions are equivalent:*

(1) $\mathcal{R}\vee\mathcal{S}\subseteq\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$; (2) $\big(\mathcal{R}\vee\mathcal{S}\big)^{\square}\subseteq\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$;

(3) $\big(\mathcal{R}\vee\mathcal{S}\big)^{\square}=\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$; (4) $\mathcal{R}^{\square}\cap\mathcal{S}^{\square}\subseteq\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$;

(5) $\mathcal{R}^{\square}\cap\mathcal{S}^{\square}=\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$.

Proof. Since $\square$ is a closure operation for relators, it is clear that assertions (1) and (2) are equivalent.

Moreover, we can note that $\mathcal{R}\cap\mathcal{S}\subseteq\mathcal{R}\vee\mathcal{S}$, and thus $(\mathcal{R}\cap\mathcal{S})^{\square}\subseteq(\mathcal{R}\vee\mathcal{S})^{\square}$. Therefore, assertions (2) and (3) are equivalent.

On the other hand, by Theorem 1.18.1, it is clear that the equivalences (2) $\Longleftrightarrow$ (4) and (3) $\Longleftrightarrow$ (5) are also true.

Now, combining Theorems 1.17.4 and 1.18.5, we can also easily establish

Theorem 1.18.6. *The following assertions are equivalent:*

(1) $\mathcal{R}\vee\mathcal{S}\subseteq\big(\mathcal{R}\cap\mathcal{S}\big)^{\#}$; (2) $\operatorname{Int}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Int}_{\mathcal{R}}\cap\operatorname{Int}_{\mathcal{S}}$; (3) $\operatorname{Cl}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Cl}_{\mathcal{R}}\cup\operatorname{Cl}_{\mathcal{S}}$.

Proof. If assertion (1) holds, then by Theorem 1.18.5 we also have $(\mathcal{R}\cap\mathcal{S})^{\#}=(\mathcal{R}\vee\mathcal{S})^{\#}$. Hence, by Theorem 1.5.4, it follows that $\operatorname{Int}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Int}_{\mathcal{R}\vee\mathcal{S}}$. Therefore, by Theorem 1.17.4, assertion (2) also holds.

On the other hand, if assertion (2) holds, then by Theorem 1.17.4 we also have $\operatorname{Int}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Int}_{\mathcal{R}\vee\mathcal{S}}$. Hence, again by Theorem 1.5.4, it follows that $(\mathcal{R}\cap\mathcal{S})^{\#}=(\mathcal{R}\vee\mathcal{S})^{\#}$. Therefore, in particular, assertion (1) also holds.

Finally, to complete the proof, we note that the equivalence of assertions (2) and (3) can be easily proved with the help of Theorem 1.3.4.

Analogously to this theorem, we can also prove the following

Theorem 1.18.7. *The following assertions are equivalent:*

(1) $\mathcal{R}\vee\mathcal{S}\subseteq\big(\mathcal{R}\cap\mathcal{S}\big)^{\wedge}$; (2) $\operatorname{int}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{int}_{\mathcal{R}}\cap\operatorname{int}_{\mathcal{S}}$; (3) $\operatorname{cl}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{cl}_{\mathcal{R}}\cup\operatorname{cl}_{\mathcal{S}}$.

Chapter 2. Generalized Topologically Open Sets in Relator Spaces

2.1 Some Generalized Topologically Open Sets

Notation 2.1.1. *In the sequel, to shorten the subsequent proofs, we shall again use the notations*

$$A^{-} = \mathrm{cl}_{\mathcal{R}}(A), \qquad A^{\circ} = \mathrm{int}_{\mathcal{R}}(A) \qquad \textit{and} \qquad A^{\dagger} = \mathrm{res}_{\mathcal{R}}(A).$$

Parts (2) and (3) of the following definition have been suggested by [71, Theorem 1 and Definition 1] of Levine.

For the motivations of parts (1) and (4), see Mashhour et al. [86, p. 47] and Jun et al. [58, Lemma 4.21].

Definition 2.1.2. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically*

(1) *preopen* if $A \subseteq \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big)$;

(2) *semi-open* if $A \subseteq \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)$;

(3) *quasi-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(V)$;

(4) *pseudo-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq \mathrm{cl}_{\mathcal{R}}(A)$.

And, the families of all such subsets A of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = p$, s, q and ps, respectively.

Remark 2.1.3. The inclusions $A \subseteq A^{-\circ}$ and $A \subseteq A^{\circ -}$ mean only that the set A is open with respect to the composite operations $-\circ$ and $\circ -$, respectively.

While, the inclusions $V \subseteq A \subseteq V^{-}$ and $A \subseteq V \subseteq A^{-}$ mean that A is near to V from above and below, or can be approximated by V from below and above.

The next simple example shows that, even in a very particular case, the families $\mathcal{T}_{\mathcal{R}}^{q}$ and $\mathcal{T}_{\mathcal{R}}^{ps}$ may be strictly smaller than the families $\mathcal{T}_{\mathcal{R}}^{s}$ and $\mathcal{T}_{\mathcal{R}}^{p}$, respectively.

Example 2.1.4. If $X = \{1, 2\}$ and R is a relation on X such that

$$R(1) = \{2\} \qquad \text{and} \qquad R(2) = \{1\}$$

then R is an injective function of X onto itself, with $R^{-1} = R$, such that

(1) $\mathcal{T}_{R} = \mathcal{T}_{R}^{q} = \mathcal{T}_{R}^{ps} = \{\emptyset, X\}$; (2) $\mathcal{T}_{R}^{s} = \mathcal{T}_{R}^{p} = \mathcal{P}(X)$.

To check this, note that now, for any $A \subseteq X$, we have $R[A^{c}] = R[A]^{c}$. Therefore, by Theorems 1.3.10 and 1.3.6, we can state that

$$A^{-} = R^{-1}[A] = R[A] \qquad \text{and} \qquad A^{\circ} = A^{c-c} = R[A^{c}]^{c} = R[A]^{cc} = R[A].$$

Thus, by the corresponding definitions, we have $\mathcal{T}_{R} = \{\emptyset, X\}$ and $\mathcal{T}_{R}^{q} = \mathcal{T}_{R}^{ps} = \{\emptyset, X\}$. Moreover, for any $A \subseteq X$, we also have

$$A^{\circ-} = R\big[R[A]\big] = (R \circ R)[A] = \big(R \circ R^{-1}\big)[A] = \Delta_X[A] = A,$$

and quite similarly $A^{-\circ} = A$. Therefore, $\mathcal{T}_R^s = \mathcal{T}_R^p = \mathcal{P}(X)$ also holds.

However, the appropriateness of Definition 2.1.2 can only be completely clear from the subsequent generalizations of the corresponding topological results.

Theorem 2.1.5. *We have*
(1) $\mathcal{T}_{\mathcal{R}}^q \subseteq \mathcal{T}_{\mathcal{R}}^s$; (2) $\mathcal{T}_{\mathcal{R}}^{ps} \subseteq \mathcal{T}_{\mathcal{R}}^p$.

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^q$, then by Definition 2.1.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq V^-$. Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}$ and the increasingness of $\circ$, we can infer that $V \subseteq V^\circ \subseteq A^\circ$. Now, by using the increasingness of $-$, we can also see that $A \subseteq V^- \subseteq A^{\circ-}$. Therefore, by Definition 2.1.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^s$.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{ps}$, then by Definition 2.1.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^-$. Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}$ and the increasingness of $\circ$, we can infer that $A \subseteq V \subseteq V^\circ \subseteq A^{-\circ}$. Therefore, by Definition 2.1.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^p$.

Theorem 2.1.6. *If $\mathcal{R}$ is reflexive, then* $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^q \cap \mathcal{T}_{\mathcal{R}}^{ps}$.

Proof. If $A \in \mathcal{T}_{\mathcal{R}}$, then by taking $V = A$ we have $V \in \mathcal{T}_{\mathcal{R}}$. And, by using Theorem 1.8.3, we can see that $V = A \subseteq A^- = V^-$ and $A = V \subseteq V^- = A^-$. Therefore, by Definition 2.1.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^q$ and $A \in \mathcal{T}_{\mathcal{R}}^{ps}$, and thus also $A \in \mathcal{T}_{\mathcal{R}}^q \cap \mathcal{T}_{\mathcal{R}}^{ps}$.

From this theorem, by using Theorem 2.1.5, we can immediately derive

Corollary 2.1.7. *If $\mathcal{R}$ is reflexive, then* $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p$.

2.2 Further Basic Properties of Generalized Topologically Open Sets

In addition to Theorem 2.1.5, we can also prove the following

Theorem 2.2.1. *If $\mathcal{R}$ is topological, then*
(1) $\mathcal{T}_{\mathcal{R}}^q = \mathcal{T}_{\mathcal{R}}^s$; (2) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^p$.

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^s$, then by Definition 2.1.2, we have $A \subseteq A^{\circ-}$. Hence, by taking $V = A^\circ$, we get $A \subseteq V^-$. Moreover, by using Theorems 1.9.3, 1.8.3 and 1.9.2, we can see that $V = A^\circ \subseteq A$ and $V = A^\circ \in \mathcal{T}_{\mathcal{R}}$. Thus, by Definition 2.1.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^q$. This proves that $\mathcal{T}_{\mathcal{R}}^s \subseteq \mathcal{T}_{\mathcal{R}}^q$.

While, if $A \in \mathcal{T}_{\mathcal{R}}^p$, then by Definition 2.1.2 we have $A \subseteq A^{-\circ}$. Hence, by taking $V = A^{-\circ}$, we get $A \subseteq V$. Moreover, by using Theorems 1.9.3, 1.8.3 and 1.9.2, we can see that $V = A^{-\circ} \subseteq A^-$ and $V = A^{-\circ} \in \mathcal{T}_{\mathcal{R}}$. Thus, by Definition 2.1.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^{ps}$. This proves that $\mathcal{T}_{\mathcal{R}}^p \subseteq \mathcal{T}_{\mathcal{R}}^{ps}$.

Now, by Theorem 2.1.5, we can see that assertions (1) and (2) are also true.

By using our former results on topological relators, we can also prove

Theorem 2.2.2. *If $\mathcal{R}$ is topological, then for any $A \subseteq X$, the following assertions are equivalent*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$;

(2) $\mathrm{cl}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)$;

(3) $\mathrm{cl}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(V)$ *for some* $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$.

Proof. If (1) holds, then by Definition 2.1.2 we have $A \subseteq A^{\circ -}$. Hence, by using the increasingness of $-$, we can infer that $A^{-} \subseteq A^{\circ --}$. Moreover, from Theorems 1.9.3 and 1.9.2, we can see that $A^{\circ -} \in \mathcal{F}_{\mathcal{R}}$. Thus, by the definition of $\mathcal{F}_{\mathcal{R}}$, we also have $A^{\circ --} \subseteq A^{\circ -}$. Therefore, $A^{-} \subseteq A^{\circ -}$.

On the other hand, by Theorems 1.9.3 and 1.8.3, we have $A^{\circ} \subseteq A$. Hence, by using the increasingness of $-$, we can infer that $A^{\circ -} \subseteq A^{-}$. Therefore, we actually have $A^{-} = A^{\circ -}$, and thus assertion (2) also holds.

While, if (2) holds, i. e., $A^{-} = A^{\circ -}$, then by taking $V = A^{\circ}$ we get $A^{-} = V^{-}$. Moreover, by Theorems 1.9.3, 1.8.3 and 1.9.2, we can see that $V \subseteq A$ and $V \in \mathcal{T}_{\mathcal{R}}$, and thus $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$. Therefore, (3) also holds.

Finally, if (3) holds, then there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A$ and $A^{-} = V^{-}$. Hence, by using Theorems 1.9.3 and 1.8.3, we can see that $A \subseteq A^{-} = V^{-}$. Therefore, by Definition 2.1.2, we have $A \in \mathcal{T}_{\mathcal{R}}^{q}$. Thus, by Theorem 2.1.5, assertion (1) also holds.

Remark 2.2.3. Note that if $\mathcal{R}$ is topological, then the family $\mathcal{T}_{\mathcal{R}}$ need not be closed under finite intersections.

Therefore, Theorem 2.2.2 is a generalization of the corresponding observations of Njastad [89, p.961], Isomichi [55, Theorem 2], Noiri [90, Lemma 2] and Pipitone and Russo [97, Lemma 2.2].

Njåstad and Isomichi, not being aware of Levine's paper [71], investigated semi-open sets under the names "β–sets" and "subcondensed sets", respectively. While, Noiri, Pipitone and Russo already used Levine's terminology.

Theorem 2.2.2 is also a certain generalization of [4, Proposition 3.2] of Al-shami since, for every supra topology (generalized topology) $\mathcal{T}$ on X, there exists a nonvoid preorder relator $\mathcal{R}$ such that $\mathcal{T} = \mathcal{T}_{\mathcal{R}}$ [127].

Now, as an immediate consequence of Theorem 2.2.2, we can also state

Corollary 2.2.4. *If $\mathcal{R}$ is topological, then for any $A \subseteq X$, the following assertions are equivalent:*

(1) $A \in \mathcal{F}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{R}}^{s}$;

(2) $A = \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)$;

(3) $A = \mathrm{cl}_{\mathcal{R}}(V)$ *for some* $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$.

2.3 Another Important Property of Topologically Semi-Open and Quasi-Open Sets

By using similar arguments as in the proof of Theorem 2.2.2, we can also prove the following two generalizations of [71, Theorem 3] of Levine.

Theorem 2.3.1. *If $\mathcal{R}$ is quasi-topological,*

$$A \in \mathcal{T}_{\mathcal{R}}^{s} \qquad \textit{and} \qquad A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{s}$ also holds.

Proof. By Definition 2.1.2, we have $A \subseteq A^{\circ -}$. Hence, by using the increasingness of $-$, we can infer that $A^{-} \subseteq A^{\circ - -}$. Moreover, by Theorem 1.9.2, we now also have $A^{\circ -} \in \mathcal{F}_{\mathcal{R}}$. Hence, by using the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $A^{\circ - -} \subseteq A^{\circ -}$. Therefore, we also have $A^{-} \subseteq A^{\circ -}$.

On the other hand, because of the inclusion $A \subseteq B$ and the increasingness properties of $\circ$ and $-$, we also have $A^{\circ -} \subseteq B^{\circ -}$. Therefore, $A^{-} \subseteq B^{\circ -}$ also holds. Hence, because of $B \subseteq A^{-}$, we can already see that $B \subseteq B^{\circ -}$. Thus, by Definition 2.1.2, the required assertion is also true.

Theorem 2.3.2. *If $\mathcal{R}$ is quasi-topological,*

$$A \in \mathcal{T}_{\mathcal{R}}^{q} \qquad \textit{and} \qquad A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{q}$ also holds.

Proof. By Definition 2.1.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq V^{-}$. Hence, since $A \subseteq B$, it is clear that $V \subseteq B$. Moreover, by using the increasingness of $-$, we can also see that $A^{-} \subseteq V^{- -}$.

On the other hand, by Theorem 1.9.2, we now also have $V^{-} \in \mathcal{F}_{\mathcal{R}}$. Hence, by the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $V^{- -} \subseteq V^{-}$. Therefore, $A^{-} \subseteq V^{-}$ also holds. Hence, because of $B \subseteq A^{-}$, we can already see that $B \subseteq V^{-}$ also holds. Thus, by Definition 2.1.2, the required assertion is also true.

Remark 2.3.3. Note that, in the above proof, we have only used a property apparently weaker than the quasi-topologicalness of $\mathcal{R}$.

Moreover, we have proved a little more than what was stated. Namely, that the "approximating set" V used for the set B does not depend on B.

Now, as an immediate consequence of Theorem 2.3.1, we can also state

Theorem 2.3.4. *If $\mathcal{R}$ is topological, then for every $A \in \mathcal{T}_{\mathcal{R}}^{s}$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^{s}.$$

Proof. To derive this from Theorem 2.3.1, note that now by Theorems 1.9.3 the relator $\mathcal{R}$ is reflexive and quasi-topological. Moreover, by Theorem 1.8.3, we have $A \subseteq A^{-}$, and thus also $A \subseteq A^{-} \subseteq A^{-}$. Therefore, by Theorem 2.3.1, we also have $A^{-} \in \mathcal{T}_{\mathcal{R}}^{s}$.

From this theorem, by using Corollary 2.1.7, we can immediately derive

Corollary 2.3.5. *If $\mathcal{R}$ is topological, then for every $V \in \mathcal{T}_{\mathcal{R}}$ we have*
$$\mathrm{cl}_{\mathcal{R}}(V) \in \mathcal{T}_{\mathcal{R}}^{s}.$$

Hence, by using Theorems 1.9.2 and 1.9.3, we can immediately derive

Corollary 2.3.6. *If $\mathcal{R}$ is topological, then for every $A \subseteq X$ we have*
$$\mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big) \in \mathcal{T}_{\mathcal{R}}^{s}.$$

Now, by using Theorems 2.1.6 and 2.3.1, we can also prove the following generalization of [71, Theorem 5] of Levine.

Theorem 2.3.7. *If $\mathcal{R}$ is topological, then $\mathcal{A} = \mathcal{T}_{\mathcal{R}}^{s}$ is the smallest family of subsets of X such that*
(1) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$; (2) $A \in \mathcal{A}$ *and* $A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A)$ *imply* $B \in \mathcal{A}$.

Proof. To prove the stated minimality property of $\mathcal{T}_{\mathcal{R}}^{s}$, note that if $A \in \mathcal{T}_{\mathcal{R}}^{s}$, then by Theorem 2.2.1 we also have $A \in \mathcal{T}_{\mathcal{R}}^{q}$. Thus, Definition 2.1.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq V^{-}$.
Therefore, if $\mathcal{A}$ is a family of subsets of X such that (1) holds, then we have $V \in \mathcal{A}$. Moreover, if (2) also holds, then we also have $A \in \mathcal{A}$. Therefore, $\mathcal{T}_{\mathcal{R}}^{s} \subseteq \mathcal{A}$ even if $\mathcal{R}$ is not assumed to be topological.

Moreover, we can also prove the following generalization of [71, Lemma 2] of Levine.

Theorem 2.3.8. *If $\mathcal{R}$ is topological, then*
$$\mathcal{T}_{\mathcal{R}} = \big\{ \mathrm{int}_{\mathcal{R}}(A) : \quad A \in \mathcal{T}_{\mathcal{R}}^{s} \big\}.$$

Proof. Now, by Theorem 1.9.3, $\mathcal{R}$ is reflexive and quasi-topological. Thus, if in particular $V \in \mathcal{T}_{\mathcal{R}}$, then by Corollary 2.1.7 we have also have $V \in \mathcal{T}_{\mathcal{R}}^{s}$. Moreover, by the definition of $\mathcal{T}_{\mathcal{R}}$ and Theorem 1.8.3, we also have $V \subseteq V^{\circ}$ and $V^{\circ} \subseteq V$, and thus also $V = V^{\circ}$. Therefore, $V \in \big(\mathcal{T}_{\mathcal{R}}^{s}\big)^{\circ}$, and thus $\mathcal{T}_{\mathcal{R}} \subseteq \big(\mathcal{T}_{\mathcal{R}}^{s}\big)^{\circ}$ also holds.

On the other hand, by using Theorem 1.9.2, we have $A^{\circ} \in \mathcal{T}_{\mathcal{R}}$ for all $A \subseteq X$. Thus, $\mathcal{P}(X)^{\circ} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus in particular $\big(\mathcal{T}_{\mathcal{R}}^{s}\big)^{\circ} \subseteq \mathcal{T}_{\mathcal{R}}$ also holds. Therefore, the required equality $\mathcal{T}_{\mathcal{R}} = \big(\mathcal{T}_{\mathcal{R}}^{s}\big)^{\circ}$ is also true.

Remark 2.3.9. Note that, because of Theorem 2.2.1, here we may write $\mathcal{T}_{\mathcal{R}}^{q}$ in place of $\mathcal{T}_{\mathcal{R}}^{s}$.

2.4 Another Important Property of Topologically Preopen and Pseudo-Open Sets

Analogously to Theorems 2.3.1 and 2.3.2, we can also prove the following two theorems.

Theorem 2.4.1. *If $\mathcal{R}$ is quasi-topological,*
$$A \in \mathcal{T}_{\mathcal{R}}^{p} \qquad \text{and} \qquad B \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(B),$$
then $B \in \mathcal{T}_{\mathcal{R}}^{p}$ also holds.

Proof. By Definition 2.1.2, we have $A \subseteq A^{-\circ}$. Hence, by using that $B \subseteq A$, we can see that $B \subseteq A^{-\circ}$. Moreover, from the inclusion $A \subseteq B^{-}$, by using the increasingness of $-$ and Theorem 1.9.2, we can infer that $A^{-} \subseteq B^{--} \subseteq B^{-}$. Hence, by using the increasingness of $\circ$, we can infer that $A^{-\circ} \subseteq B^{-\circ}$. Therefore, because of $B \subseteq A^{-\circ}$, we also have $B \subseteq B^{-\circ}$. Hence, by Definition 2.1.2, we can see that $B \in \mathcal{T}_{\mathcal{R}}^{p}$ also holds.

Theorem 2.4.2. *If $\mathcal{R}$ is quasi-topological,*

$$A \in \mathcal{T}_{\mathcal{R}}^{ps} \qquad \text{and} \qquad B \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(B),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{ps}$ also holds.

Proof. By Definition 2.1.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^{-}$. Hence, by using that $B \subseteq A$, we can see that $B \subseteq V$. Moreover, from the inclusion $A \subseteq B^{-}$, by using the increasingness of $-$ and Theorem 1.9.2, we can infer that $A^{-} \subseteq B^{--} \subseteq B^{-}$. Hence, by using the inclusion $V \subseteq A^{-}$, we can see that $V \subseteq B^{-}$. Therefore, we actually have $B \subseteq V \subseteq B^{-}$. Hence, by Definition 2.1.2, we can see that $B \in \mathcal{T}_{\mathcal{R}}^{ps}$ also holds.

Remark 2.4.3. Again, we have proved a little more than what was stated. Namely, that the "approximating set" V used for the set B does not depend on B.

Unfortunately, some analogues of Theorem 2.3.4 and its corollaries do not seem possible. However, instead of them, we can prove the following two theorems.

Theorem 2.4.4. *If $\mathcal{R}$ is nonvoid, then $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{p}$.*

Proof. If $A \in \mathcal{D}_{\mathcal{R}}$, then by the definition of $\mathcal{D}_{\mathcal{R}}$ we have $A^{-} = X$. Moreover, since $\mathcal{R} \neq \emptyset$, by the definition of $\circ$, we also have $X^{\circ} = X$. Therefore, we actually have $A^{-\circ} = X^{\circ} = X$. Thus, the required inclusion $A \subseteq A^{-\circ}$ trivially holds.

Theorem 2.4.5. *If $\mathcal{R}$ is topologically filtered and $A = V \cap B$ for some $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{D}_{\mathcal{R}}$, then $A \in \mathcal{T}_{\mathcal{R}}^{ps}$.*

Proof. In this case, we have $A \subseteq V$. Moreover, by the definitions of $\mathcal{D}_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and Theorem 1.13.3, we also have $V = X \cap V \subseteq B^{-} \cap V^{\circ} \subseteq \big(B \cap V \big)^{-} = A^{-}$. Therefore, we actually have $A \subseteq V \subseteq A^{-}$. Thus, by Definition 2.1.2, we have $A \in \mathcal{T}_{\mathcal{R}}^{ps}$.

From this theorem, by taking $V = X$ whenever $\mathcal{R} \neq \emptyset$, we can get

Corollary 2.4.6. *If $\mathcal{R}$ is nonvoid and topologically filtered, then $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{ps}$.*

Now, as some close analogues of Theorems 2.3.7 and 2.3.8, we can also prove the following two theorems.

Theorem 2.4.7. *If $\mathcal{R}$ is topological, then $\mathcal{A} = \mathcal{T}_{\mathcal{R}}^{p}$ is the smallest family of subsets of X such that*

(1) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$;

(2) *$A \in \mathcal{A}$ and $A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A)$ imply $B \in \mathcal{A}$.*

Proof. To prove the stated minimality property of $\mathcal{T}_{\mathcal{R}}^{p}$, note that if $A \in \mathcal{T}_{\mathcal{R}}^{p}$, then by Theorem 2.2.1we also have $A \in \mathcal{T}_{\mathcal{R}}^{ps}$. Thus, Definition 2.1.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^{-}$. Therefore, if $\mathcal{A}$ is a family of subsets of X such that (1) holds, then we have $V \in \mathcal{A}$. Moreover, if (2) also holds, then we also have $A \in \mathcal{A}$. Therefore, $\mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{A}$.

Theorem 2.4.8. *If $\mathcal{R}$ is topological, then*

$$\mathcal{T}_{\mathcal{R}} = \big\{ \operatorname{int}_{\mathcal{R}}(A) : \quad A \in \mathcal{T}_{\mathcal{R}}^{p} \big\}.$$

Proof. Now, by Theorem 1.9.3, $\mathcal{R}$ is reflexive and quasi-topological. Thus, if in particular $V \in \mathcal{T}_{\mathcal{R}}$, then by Corollary 2.1.7 we have also have $V \in \mathcal{T}_{\mathcal{R}}^{p}$.

Moreover, by the definition of $\mathcal{T}_{\mathcal{R}}$ and Theorem 1.8.3, we also have $V \subseteq V^{\circ}$ and $V^{\circ} \subseteq V$, and thus also $V = V^{\circ}$. Therefore, $V \in \big(\mathcal{T}_{\mathcal{R}}^{p}\big)^{\circ}$, and thus $\mathcal{T}_{\mathcal{R}} \subseteq \big(\mathcal{T}_{\mathcal{R}}^{p}\big)^{\circ}$ also holds.

On the other hand, by Theorem 1.9.2, we have $A^{\circ} \in \mathcal{T}_{\mathcal{R}}$ for all $A \subseteq X$. Thus, $\mathcal{P}(X)^{\circ} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus in particular $\big(\mathcal{T}_{\mathcal{R}}^{p}\big)^{\circ} \subseteq \mathcal{T}_{\mathcal{R}}$ also holds. Therefore, the required equality $\mathcal{T}_{\mathcal{R}} = \big(\mathcal{T}_{\mathcal{R}}^{p}\big)^{\circ}$ is also true.

Remark 2.4.9. Note that, because of Theorem 2.2.1, here we may write $\mathcal{T}_{\mathcal{R}}^{ps}$ in place of $\mathcal{T}_{\mathcal{R}}^{p}$.

2.5 The Duals of the Families $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = s,\, p,\, q$ and ps

To introduce the corresponding generalized topologically closed sets, we shall use the following plausible notation.

Definition 2.5.1. For any $\kappa = s,\, p,\, q$ and ps, we define

$$\mathcal{F}_{\mathcal{R}}^{\kappa} = \big\{ A \subseteq X : \quad A^{c} \in \mathcal{T}_{\mathcal{R}}^{\kappa} \big\}.$$

Thus, by using Theorem 1.3.6 and Remark 1.3.7, we can prove the following theorems

Theorem 2.5.2. *For any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{F}_{\mathcal{R}}^{q}$ *;*

(2) *there exists* $W \in \mathcal{F}_{\mathcal{R}}$ *such that* $\operatorname{int}_{\mathcal{R}}(W) \subseteq A \subseteq W$.

Proof. To prove that (1) $\Longrightarrow$ (2), note that if (1) holds, then $A^{c} \in \mathcal{T}_{\mathcal{R}}^{q}$. Thus, by Definition 2.1.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A^{c} \subseteq V^{-}$. Hence, by using that $c\circ = -c$, we can infer that $V^{c\circ} = V^{-c} \subseteq A \subseteq V^{c}$. Thus, by taking $W = V^{c}$, we can see that $W \in \mathcal{F}_{\mathcal{R}}$ such that $W^{\circ} \subseteq A \subseteq W$, and thus, by Definition 2.1.2, assertion (2) also holds.

Theorem 2.5.3. *For any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{F}_{\mathcal{R}}^{ps}$ *;*

(2) *there exists* $W \in \mathcal{F}_{\mathcal{R}}$ *such that* $\operatorname{int}_{\mathcal{R}}(A) \subseteq W \subseteq A$.

Proof. To prove that (2) $\Longrightarrow$ (1), note that if (2) holds, then there exists $W \in \mathcal{F}_{\mathcal{R}}$ such that $A^{\circ} \subseteq W \subseteq A$. Hence, by using $\circ c = c-$, we can infer that $A^{c} \subseteq W^{c} \subseteq A^{\circ c} = A^{c-}$. Thus, by taking $V = W^{c}$, we can see that $V \in \mathcal{T}_{\mathcal{R}}$ such that $A^{c} \subseteq V \subseteq A^{c-}$. Therefore, by Definition 2.1.2, we have $A^{c} \in \mathcal{T}_{\mathcal{R}}^{ps}$, and thus (1) also holds.

Theorem 2.5.4. *For any $A \subseteq X$, the following assertions are equivalent:*

(1) $A \in \mathcal{F}_{\mathcal{R}}^{s}$ *;* (2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \subseteq A$.

Proof. By the corresponding definitions, we have

$$(1) \iff A^c \in \mathcal{T}_{\mathcal{R}}^{s} \iff A^c \subseteq A^{c\circ -} \iff A^{c\circ - c} \subseteq A.$$

Moreover, by using $-c = c\circ$ and $c\circ c = -$, we can see that $A^{c\circ - c} = A^{c\circ c\circ} = A^{-\circ}$. Therefore, we actually have $(1) \iff A^{-\circ} \subseteq A \iff (2)$.

Theorem 2.5.5. *For any* $A \subseteq X$, *the following assertions are equivalent:*
(1) $A \in \mathcal{F}_{\mathcal{R}}^{p}$; (2) $\operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \subseteq A$.

Proof. By the corresponding definitions, we have

$$(1) \iff A^c \in \mathcal{T}_{\mathcal{R}}^{p} \iff A^c \subseteq A^{c-\circ} \iff A^{c-\circ c} \subseteq A.$$

Moreover, by using $c- = \circ c$ and $c\circ c = -$, we can see that $A^{c-\circ c} = A^{\circ c\circ c} = A^{\circ -}$. Therefore, we actually have $(1) \iff A^{\circ -} \subseteq A \iff (2)$.

Now, by using Theorem 2.2.2, we can also easily establish the following

Theorem 2.5.6. *If* $\mathcal{R}$ *is topological, then for any* $A \subseteq X$ *the following assertions are equivalent:*
(1) $A \in \mathcal{F}_{\mathcal{R}}^{s}$;
(2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \subseteq \operatorname{int}_{\mathcal{R}}(A)$; (3) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) = \operatorname{int}_{\mathcal{R}}(A)$;
(4) *there exists* $W \in \mathcal{F}_{\mathcal{R}}$ *such that* $A \subseteq W$ *and* $\operatorname{int}_{\mathcal{R}}(A) = \operatorname{int}_{\mathcal{R}}(W)$.

2.6 Topologically Regular Open Sets

Regular open sets were first introduced by Kuratowski [64] with reference to a paper of Henri Lebesgue. However, their importance became completely clear only after the considerations of Stone [113].

Following Kuratowski's definition, in our papers [101, 102], we have also introduced the following

Definition 2.6.1. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically regular open* if

$$A = \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big).$$

Also, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{r}$.

Thus, in contrast to the topological case, $\mathcal{T}_{\mathcal{R}}^{r}$ need not be a subfamily of $\mathcal{T}_{\mathcal{R}}$. To show this, we can use the following

Example 2.6.2. If $X = \{1, 2\}$ and R is a relation on X such that $R(1) = \{2\}$ and $R(2) = \{1\}$, then $\mathcal{R} = \{R\}$ is a symmetric relator on X such that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$ and $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{P}(X)$.

Of course, by Theorem 1.9.2, we evidently have the following

Theorem 2.6.3. *If* $\mathcal{R}$ *is quasi-topological, then* $\mathcal{T}_{\mathcal{R}}^{r} \subseteq \mathcal{T}_{\mathcal{R}}$.

Moreover, by using Theorem 1.8.3, we can easily establish the following

Theorem 2.6.4. *If* $\mathcal{R}$ *is reflexive, then* $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{r}$.

Proof. If $A \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$, then $A \in \mathcal{T}_{\mathcal{R}}$ and $A \in \mathcal{F}_{\mathcal{R}}$. Thus, by Definition 1.4.1 and Theorem 1.8.3, we have $A^{\circ} = A$ and $A = A^{-}$. Therefore, $A^{\circ} = A^{-\circ} = A$, and thus by Definition 2.6.1 we also have $A \in \mathcal{T}_{\mathcal{R}}^{r}$.

Thus, in particular by Theorem 1.9.3 we can also state the following

Corollary 2.6.5. *If* $\mathcal{R}$ *is topological, then* $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{r} \subseteq \mathcal{T}_{\mathcal{R}}$.

The appropriateness of Definition 2.6.1 is also apparent from the following generalization of a statement of Dontchev [33, p. 4].

Theorem 2.6.6. *We have* $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{p} \cap \mathcal{F}_{\mathcal{R}}^{s}$.

Proof. Namely, by the corresponding definitions and Theorem 2.5.4,

$$A \in \mathcal{T}_{\mathcal{R}}^{r} \iff A = A^{-\circ} \iff A \subseteq A^{-\circ}, \quad A^{-\circ} \subseteq A$$
$$\iff A \in \mathcal{T}_{\mathcal{R}}^{p}, \quad A \in \mathcal{F}_{\mathcal{R}}^{s} \iff A \in \mathcal{T}_{\mathcal{R}}^{p} \cap \mathcal{F}_{\mathcal{R}}^{s}.$$

Remark 2.6.7. Now, if $\mathcal{R}$ is reflexive, then by Corollary 2.1.7 and Theorem 2.6.6, we can also state that $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{s} \subseteq \mathcal{T}_{\mathcal{R}}^{r}$.

Thus, by Definition 1.4.1 and Theorem 2.5.4, we can also state

Corollary 2.6.8. *If* $\mathcal{R}$ *is reflexive and* $A \subseteq X$ *such that*

$$\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \subseteq A \subseteq \operatorname{int}_{\mathcal{R}}(A),$$

then $A \in \mathcal{T}_{\mathcal{R}}^{r}$.

Now, by using our former results, we can also easily prove

Theorem 2.6.9. *If* $\mathcal{R}$ *is topological, then* $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{s}$.

Proof. By Theorem 1.9.3, the relator $\mathcal{R}$ is reflexive and quasi-topological. Thus, by Corollary 2.1.7 and Theorem 2.6.3, we have $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{p}$ and $\mathcal{T}_{\mathcal{R}}^{r} \subseteq \mathcal{T}_{\mathcal{R}}$. Hence, by using Theorem 2.6.6, we can already infer that $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{R}}^{p} \cap \mathcal{F}_{\mathcal{R}}^{s} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{s}$.

From this theorem, by using Definition 1.4.1 and Theorem 2.5.4, we can obtain

Corollary 2.6.10. *If* $\mathcal{R}$ *is topological, then for any* $A \subseteq X$ *the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{r}$; (2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \subseteq A \subseteq \operatorname{int}_{\mathcal{R}}(A)$.

From Theorems 2.6.9 and 2.6.6, by using Theorem 2.2.1, we can also derive

Theorem 2.6.11. *If* $\mathcal{R}$ *is topological, then*

(1) $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{q}$ (2) $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{ps} \cap \mathcal{F}_{\mathcal{R}}^{q}$.

Proof. Namely, by Theorem 2.2.1, we have not only $\mathcal{T}_{\mathcal{R}}^{p}=\mathcal{T}_{\mathcal{R}}^{ps}$, but also
$$A\in\mathcal{F}_{\mathcal{R}}^{s}\iff A^{c}\in\mathcal{T}_{\mathcal{R}}^{s}\iff A^{c}\in\mathcal{T}_{\mathcal{R}}^{q}\iff A\in\mathcal{F}_{\mathcal{R}}^{q},$$
and thus $\mathcal{F}_{\mathcal{R}}^{s}=\mathcal{F}_{\mathcal{R}}^{q}$.

Remark 2.6.12. Counterparts of Theorem 2.6.6 were also proved by Ekici [37, Theorem 8] and Jamunarani et al. [57, Theorem 2.2] by using the weak structures of Császár [26] and the generalized weak structures of Ávila and Molina [11].

2.7 Some Further Theorems on the Family $\mathcal{T}_{\mathcal{R}}^{r}$

By using Theorem 2.6.9, we can also prove the following generalization of a statement of Kuratowski [64].

Theorem 2.7.1. *If $\mathcal{R}$ is topological, then for any $A\in\mathcal{T}_{\mathcal{R}}^{s}$ we have*
$$\operatorname{cl}_{\mathcal{R}}(A)^{c}\in\mathcal{T}_{\mathcal{R}}^{r}.$$

Proof. By Theorems 1.9.3 and 1.9.2, we have $A^{-}\in\mathcal{F}_{\mathcal{R}}$. Hence, by Theorem 1.4.8, we infer that $A^{-c}\in\mathcal{T}_{\mathcal{R}}$. Moreover, by Theorem 2.3.4, we have $A^{-}\in\mathcal{T}_{\mathcal{R}}^{s}$, and thus $A^{-c}\in\mathcal{F}_{\mathcal{R}}^{s}$. Hence, by Theorem 2.6.9, we can see that $A^{-c}\in\mathcal{T}_{\mathcal{R}}^{r}$.

From this theorem, by using that $\mathcal{T}_{\mathcal{R}}\subseteq\mathcal{T}_{\mathcal{R}}^{s}$ whenever $\mathcal{R}$ is reflexive, we can easily derive the following

Corollary 2.7.2. *If $\mathcal{R}$ is topological and $\mathcal{A}=\mathcal{T}_{\mathcal{R}}$ or $\mathcal{T}_{\mathcal{R}}^{s}$, then*
$$\mathcal{T}_{\mathcal{R}}^{r}=\left\{\operatorname{cl}_{\mathcal{R}}(A)^{c}:\quad A\in\mathcal{A}\right\}.$$

Proof. Namely, if for instance $B\in\mathcal{T}_{\mathcal{R}}^{r}$, then by choosing $A=B^{c\circ}$, we can see that $A\in\mathcal{T}_{\mathcal{R}}$, and thus also $A\in\mathcal{T}_{\mathcal{R}}^{s}$, such that $A^{-c}=B^{c\circ-c}=B^{c\circ c\circ}=B^{-\circ}=B$.

Remark 2.7.3. Following an observation of Halmos [46, p. 61], it is also worth noticing that, if $\mathcal{R}$ is topological, then we have $\mathcal{T}_{\mathcal{R}}=\left\{\operatorname{cl}_{\mathcal{R}}(A)^{c}:\quad A\subseteq X\right\}$.

Namely, if for instance $V\in\mathcal{T}_{\mathcal{R}}$, then by choosing $A=V^{c}$, we can see that $A\in\mathcal{F}_{\mathcal{R}}$, and thus $A=A^{-}$. Therefore, $V=A^{c}=A^{-c}$ even if $\mathcal{R}$ is assumed to be only reflexive.

From Theorem 2.7.1, by Theorem 1.3.6, we can see that $A^{-\circ}=A^{-c-c}\in\mathcal{F}_{\mathcal{R}}^{r}$ for all $A\in\mathcal{T}_{\mathcal{R}}^{s}$. However, this fact is of no importance for us. Namely, by using Theorem 2.6.9, we can prove a better statement.

Theorem 2.7.4. *If $\mathcal{R}$ is topological, then for any $A\subseteq X$ we have*
$$\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)\in\mathcal{T}_{\mathcal{R}}^{r}.$$

Proof. By Theorems 1.9.3 and 1.9.2, we have $A^{-\circ}\in\mathcal{T}_{\mathcal{R}}$. Moreover, quite similarly, we also have $A^{c\circ}\in\mathcal{T}_{\mathcal{R}}$. Hence, by using Corollary 2.1.7 and Theorem 2.3.4, we can infer that $A^{c\circ-}\in\mathcal{T}_{\mathcal{R}}^{s}$, and thus $A^{c\circ-c}\in\mathcal{F}_{\mathcal{R}}^{s}$. However, by using the equalities $c\circ=-c$ and $c-c=\circ$, we can see that $A^{c\circ-c}=A^{-c-c}=A^{-\circ}$. Therefore, we actually have $A^{-\circ}\in\mathcal{F}_{\mathcal{R}}^{s}$. Hence, by Theorem 2.6.9, we can already see that $A^{-\circ}\in\mathcal{T}_{\mathcal{R}}^{r}$.

Remark 2.7.5. The topological counterparts of Theorems 2.7.1 and 2.7.4 are usually proved directly, by using only the corresponding properties of the operations $-$ and $\circ$.

Now, by using Theorem 2.7.4, we can also easily establish the following

Corollary 2.7.6. *If $\mathcal{R}$ is topological, then*

$$\mathcal{T}_{\mathcal{R}}^{r} = \big\{ \operatorname{int}_{\mathcal{R}}(A) : \quad A \in \mathcal{F}_{\mathcal{R}} \big\} = \big\{ \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) : \quad A \subseteq X \big\}.$$

Remark 2.7.7. Hence, it is clear that Stone's definition [113, p. 376] of a regular open set coincides with that of Kuratowski [64, p. 9].

Finally, we note that, analogously to Theorem 2.5.5, we can also prove

Theorem 2.7.8. *For any $A \subseteq X$, the following assertions are equivalent:*
(1) $A \in \mathcal{F}_{\mathcal{R}}^{r}$; (2) $A = \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)$.

Remark 2.7.9. Several further properties of the family $\mathcal{F}_{\mathcal{R}}^{r}$ can be directly derived from those of the family $\mathcal{T}_{\mathcal{R}}^{r}$.

2.8 Characterizations of Topologically Semi-Open and Quasi-Open Sets

As a generalization of [34, Lemma 1] of Duszyński and Noiri, we can prove the following

Theorem 2.8.1. *If $\mathcal{R}$ is reflexive, then for any $A \subseteq X$ the following assertions are equivalent:*
(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$;
(2) *there exists $B \subseteq X$ such that* $A = \operatorname{int}_{\mathcal{R}}(A) \cup B$ *and* $B \subseteq \operatorname{res}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)$.

Proof. By Theorem 1.8.3, we have $A^{\circ} \subseteq A$. Moreover, if (1) holds, then by Definition 2.1.2 we have $A \subseteq A^{\circ -}$. Hence, by defining $B = A \setminus A^{\circ}$, we can already see that

$$A = A^{\circ} \cup \big(A \setminus A^{\circ} \big) = A^{\circ} \cup B \qquad \text{and} \qquad B = A \setminus A^{\circ} \subseteq A^{\circ -} \setminus A^{\circ} = A^{\circ \dagger}.$$

Therefore, (2) also holds.

By Theorem 1.8.3, we also have $A^{\circ} \subseteq A^{\circ -}$. Therefore, if (2) holds, then we have

$$A = A^{\circ} \cup B \subseteq A^{\circ} \cup A^{\circ \dagger} = A^{\circ} \cup \big(A^{\circ -} \setminus A^{\circ} \big) = A^{\circ -}.$$

Thus, by Definition 2.1.2, assertion (1) also holds.

Remark 2.8.2. Note that if B is as in (2), then $A = A^{\circ} \cup B$. Moreover, since $B \subseteq A^{\circ \dagger} = A^{\circ -} \setminus A^{\circ}$, we have $A^{\circ} \cap B = \emptyset$.

Furthermore, by Theorem 1.8.3, we have $A^{\circ\circ} \subseteq A^{\circ}$. Therefore, by using the notation $A^{\ddagger} = \operatorname{bnd}_{\mathcal{R}}(A)$, we can also state that $B \subseteq A^{\circ \dagger} = A^{\circ -} \setminus A^{\circ} \subseteq A^{\circ -} \setminus A^{\circ\circ} = A^{\circ \ddagger}$.

Now, by using our former results, we can also easily prove the following generalization of an observation of Dlaska, Ergun and Ganster [29, p. 1163].

Theorem 2.8.3. *If $\mathcal{R}$ is topological, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$;

(2) *there exist* $V \in \mathcal{T}_{\mathcal{R}}$ *and* $B \subseteq X$ *such that* $A = V \cup B$ *and* $B \subseteq \operatorname{res}_{\mathcal{R}}(V)$.

Proof. If (1) holds, then by Theorem 2.8.1, there exists $B \subseteq X$ such that $A = A^{\circ} \cup B$ and $B \subseteq A^{\circ\dagger}$. Moreover, by Theorems 1.9.3 and 1.9.2, we have $A^{\circ} \in \mathcal{T}_{\mathcal{R}}$. Hence, by taking $V = A^{\circ}$, we can see that (2) also holds.

On the other hand, by Theorems 1.9.3 and 1.8.3, we have $V \subseteq V^{-}$ for any $V \subseteq X$. Therefore, if (2) holds, then we have not only $V \subseteq A$ but also

$$A = V \cup B \subseteq V \cup V^{\dagger} = V \cup \left(V^{-} \setminus V \right) = V^{-}.$$

Hence, by Definition 2.1.2, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{q}$. Thus, by Theorem 2.2.1, assertion (1) also holds.

Remark 2.8.4. Note that if in particular $\mathcal{R}$ is topologically filtered and topological, then by Corollary 1.14.5 we have $V^{\dagger} \in \mathcal{N}_{\mathcal{R}}$. Hence, since $B \subseteq V^{\dagger}$, it is clear that $B \in \mathcal{N}_{\mathcal{R}}$ also holds.

Therefore, analogously to [71, Theorem 7] of Levine, we can also state the following stability type theorem.

Theorem 2.8.5. *If $\mathcal{R}$ is topologically filtered, topological and $A \in \mathcal{T}_{\mathcal{R}}^{s}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \cup B$ and $V \cap B = \emptyset$.*

The following obvious reformulation of [71, Example 2] of Levine shows that the converse of this theorem is false.

Example 2.8.6. Define $X = \mathbb{R}$ and

$$R_n = \left\{ (x, y) \in X^2 : \quad d(x, y) < n^{-1} \right\}$$

for all $n \in \mathbb{N}$.

Then, $\mathcal{R} = \{ R_n \}_{n=1}^{\infty}$ is a properly filtered, strongly topological relator on X such that, under the notations

$$V =]\,0,\, 1\,[\qquad \text{and} \qquad B = \{2\},$$

we have $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \cup B \notin \mathcal{T}_{\mathcal{R}}^{s}$.

To check the required properties, note that

(1) $R_n = R_n \cap R_m$ if $n, m \in \mathbb{N}$ such that $m \leq n$;

(2) $R_n(x) =]\,x - n^{-1},\ x + n^{-1}\,[$ for all $n \in \mathbb{N}$ and $x \in X$;

(3) for each $n \in \mathbb{N}$, $x \in X$ and $y \in R_n(x)$, we have $R_m(y) \subseteq R_n(x)$ if $m \in \mathbb{N}$ such that $m^{-1} < n^{-1} - d(x, y)$.

By using (2), we can easily see that $A^{\circ} = V$ and $A^{\circ -} = V^{-} = [\,0,\, 1\,]$, and thus $A \not\subseteq A^{\circ -}$. Therefore, by Definition 2.1.2, $A \notin \mathcal{T}_{\mathcal{R}}^{s}$. Although, $A = V \cup B$, $V \cap B = \emptyset$, $V \in \mathcal{T}_{\mathcal{R}}$ and $B^{-\circ} = B^{\circ} = \emptyset$.

While, by using assertion (3) and the property $\sup(\mathbb{N}) = +\infty$, we can easily see that $R_n(x) \in \mathcal{T}_{\mathcal{R}}$ for all $n \in \mathbb{N}$ and $x \in X$. Therefore, $\mathcal{R}$ is strongly quasi-topological.

Remark 2.8.7. Assertion (3) can also be easily derived from (2). However, it does not seem to be a consequence of the very strong reflexivity, symmetry and transitivity properties of $\mathcal{R}$.

2.9 Characterizations of Topologically Preopen and Pseudo-Open Sets

As a conterpart of [42, Proposition 2.1] of Ganster, we can also prove

Theorem 2.9.1. *If $\mathcal{R}$ is topologically filtered and topological, then for any $A \subseteq X$ the following assertions are equivalent :*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{p}$;
(2) *there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$;*
(3) *there exist $V \in \mathcal{T}_{\mathcal{R}}^{r}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$.*

Proof. If (1) holds, then by Definition 2.1.2 we have $A \subseteq A^{-\circ}$. Hence, we can infer that

$$A = A^{-\circ} \setminus \big(A^{-\circ} \setminus A\big) = A^{-\circ} \cap \big(A^{-\circ} \cap A^{c}\big)^{c} = A^{-\circ} \cap \big(A^{-\circ c} \cup A\big).$$

Now, by defining $V = A^{-\circ}$ and $B = A^{-\circ c} \cup A$, we can state that $A = V \cap B$. And, by Theorem 2.7.4, we can state that $V = A^{-\circ} \in \mathcal{T}_{\mathcal{R}}^{r}$.

Moreover, by using Theorem 1.11.5, we can see that $B^{-} = \big(A^{-\circ c} \cup A\big)^{-} = A^{-\circ c -} \cup A^{-}$. And, by using the equality $\circ\, c = c\, -$ and Theorems 1.9.3 and 1.8.3, we can also see that $A^{-\circ c -} = A^{-c--} \supseteq A^{-c-} \supseteq A^{-c}$. Therefore,

$$B^{-} = A^{-\circ c -} \cup A^{-} \supseteq A^{-c} \cup A^{-} = X,$$

and thus $B^{-} = X$. Hence, by the definition of $\mathcal{D}_{\mathcal{R}}$, we can see that $B \in \mathcal{D}_{\mathcal{R}}$, and thus (3) also holds.

From Theorem 2.6.9, we know that $\mathcal{T}_{\mathcal{R}}^{r} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, assertion (2) is an immediate consequence of (3). Moreover, from Theorems 2.4.5 and 2.1.5, we can see that (2) implies (1) even if the relator $\mathcal{R}$ is assumed to be only topologically filtered.

In addition to the above theorem, we can also prove the following

Theorem 2.9.2. *If $\mathcal{R}$ is topologically filtered and topological, then for any $A \subseteq X$ the following assertions are equivalent :*
(1) $A \in \mathcal{T}_{\mathcal{R}}^{p}$;
(2) *there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V$ and $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(V)$;*
(3) *there exists $V \in \mathcal{T}_{\mathcal{R}}^{r}$ such that $A \subseteq V$ and $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(V)$.*

Proof. If (1) holds, then by Theorem 2.9.1 there exist $V \in \mathcal{T}_{\mathcal{R}}^{r}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$. Thus, in particular $A \subseteq V$. Hence, by using the increasingness $-$, we can infer that $A^{-} \subseteq V^{-}$.

Moreover, by Theorem 2.6.9, we have $\mathcal{T}_{\mathcal{R}}^{r} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus $V \in \mathcal{T}_{\mathcal{R}}$. Therefore, by the definitions of $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$, and Theorem 1.13.3, we can also state that $V = V \cap X = V \cap B^{-} \subseteq V^{\circ} \cap B^{-} \subseteq (V \cap B)^{-} = A^{-}$. Hence, by using the increasingness of $-$, Theorems 1.9.3 and 1.9.2 and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $V^{-} \subseteq A^{--} \subseteq A^{-}$. Therefore, we actually have $A^{-} = V^{-}$, and thus (3) also holds.

Again, by the inclusion $\mathcal{T}_{\mathcal{R}}^{r} \subseteq \mathcal{T}_{\mathcal{R}}$, it is clear that (2) is an immediate consequence of (3). Therefore, we need only show that (2) also implies (1).

For this, note that if (2) holds, then by Theorems 1.9.3 and 1.8.3, in addition to $A \subseteq V$, we also have $V \subseteq V^{-} = A^{-}$. Therefore, by Definition 2.1.2, we have $A \in \mathcal{T}_{\mathcal{R}}^{ps}$. Hence, by Theorem 2.1.5, we can see that (1) also holds.

Remark 2.9.3. Note that, by Theorem 2.2.1, in the above two theorems we may again write $\mathcal{T}_{\mathcal{R}}^{ps}$ in place of $\mathcal{T}_{\mathcal{R}}^{p}$.

2.10 A Further Important Property of Topologically Semi-open and Preopen Sets

By using Theorem 1.13.3, we can also prove the following improvement of [91, Lemma 2.5] of Noiri.

Theorem 2.10.1. *If $\mathcal{R}$ is topologically filtered and quasi-topological, and*

$$A \in \mathcal{T}_{\mathcal{R}}^{s}, \qquad B \in \mathcal{T}_{\mathcal{R}}^{p}, \qquad \operatorname{int}_{\mathcal{R}}(A) \subseteq C \subseteq \operatorname{cl}_{\mathcal{R}}(A), \qquad B \subseteq D \subseteq X,$$

then

$$\operatorname{cl}_{\mathcal{R}}(A) \cap B = \operatorname{cl}_{\mathcal{R}}(C \cap D) \cap B.$$

Proof. By Definition 2.1.2, we have $A \subseteq A^{\circ -}$ and $B \subseteq B^{-\circ}$. Hence, by using the increasingness of the operation $-$, Theorem 1.9.2 and the definition $\mathcal{F}_{\mathcal{R}}$, we can infer that $A^{-} \subseteq A^{\circ --} \subseteq A^{\circ -}$. Moreover, by Theorems 1.13.3, 1.9.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that

$$A^{-} \cap B \subseteq A^{\circ -} \cap B^{-\circ} \subseteq \left(A^{\circ} \cap B^{-}\right)^{-} \subseteq \left(A^{\circ\circ} \cap B^{-}\right)^{-}$$
$$\subseteq \left(A^{\circ} \cap B\right)^{--} \subseteq \left(A^{\circ} \cap B\right)^{-} \subseteq \left(C \cap D\right)^{-},$$

and thus $A^{-} \cap B \subseteq \left(C \cap D\right)^{-} \cap B$.

Moreover, by using the corresponding properties of $-$ and the assumption $C \subseteq A^{-}$, we can also see that $(C \cap D)^{-} \subseteq C^{-} \subseteq A^{--} \subseteq A^{-}$, and thus $(C \cap D)^{-} \cap B \subseteq A^{-} \cap B$. Therefore, the required equality is also true.

From this theorem, by choosing C and D appropriately, we can immediately derive a great number equalities for the set $A^{-} \cap B$. For instance, we can at once state the following

Corollary 2.10.2. *If $\mathcal{R}$ is topologically filtered and quasi-topological, and $A \in \mathcal{T}_{\mathcal{R}}^{s}$ and $B \in \mathcal{T}_{\mathcal{R}}^{p}$, then*

$$\operatorname{cl}_{\mathcal{R}}(A) \cap B = \operatorname{cl}_{\mathcal{R}}\left(\operatorname{cl}_{\mathcal{R}}\left(\operatorname{int}_{\mathcal{R}}(A)\right) \cap \operatorname{int}_{\mathcal{R}}\left(\operatorname{cl}_{\mathcal{R}}(B)\right)\right) \cap B.$$

Theorem 2.10.1 strongly suggests that some of the equalities stated by Dontchev [33, p. 4], without proofs and references, cannot be true.

To see that this is really the case, we can use the following

Example 2.10.3. If X and $\mathcal{R}$ are as in Example 2.8.6, and

$$A = [\,0\,,\,1\,] \qquad \text{and} \qquad B =]\,0\,,\,1\,[\,\cap \mathbb{Q}\,,$$

then $\mathcal{R}$ is a properly filtered, strongly topological relator on X, and $A \in \mathcal{T}_{\mathcal{R}}^{\,s}$ and $B \in \mathcal{T}_{\mathcal{R}}^{\,p}$ such that

$$\mathrm{cl}_{\mathcal{R}}(A) \cap B =]\,0\,,\,1\,[\,\cap \mathbb{Q} \qquad \text{and} \qquad \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A) \cap B\big) = [\,0\,,\,1\,]\,,$$

and thus even $\mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A) \cap B\big) \not\subseteq \mathrm{cl}_{\mathcal{R}}(A) \cap B\,.$

Note that the inclusions $A \in \mathcal{T}_{\mathcal{R}}^{\,s}$ and $B \in \mathcal{T}_{\mathcal{R}}^{\,p}$ are immediate consequences of Corollary 2.3.5 and Theorems 2.4.5 and 2.1.5.

2.11 Some Further Generalized Topologically Open Sets

Parts (1) and (2) of the following definition have been suggested by Njåstad [89] and Abd El-Monsef et al. [1].

While, for some motivations of parts (3) and (4), see Theorems 2.3.1 and 2.4.1 and [5, Definition 2.1] of Andrijević.

Definition 2.11.1. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically*

(1) α*–open* if $A \subseteq \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)\big)\,;$

(2) β*–open* if $A \subseteq \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big)\big)\,;$

(3) γ*–open* if there exists $V \in \mathcal{T}_{\mathcal{R}}^{\,s}$ such that $A \subseteq V \subseteq \mathrm{cl}_{\mathcal{R}}(A)\,;$

(4) δ*–open* if there exists $V \in \mathcal{T}_{\mathcal{R}}^{\,p}$ such that $V \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(V)\,.$

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\,\kappa}$ with $\kappa = \alpha,\ \beta,\ \gamma$ and δ, respectively.

Remark 2.11.2. Note that if $\mathcal{R}$ is not topological, then by using the families $\mathcal{T}_{\mathcal{R}}^{\,q}$ and $\mathcal{T}_{\mathcal{R}}^{\,ps}$ instead of $\mathcal{T}_{\mathcal{R}}^{\,s}$ and $\mathcal{T}_{\mathcal{R}}^{\,p}$, respectively, we can get some stronger forms of generalized topologically open sets.

Now, by using Definition 2.11.1 and the increasingness of $\circ$ and $-$, we can easily prove the following

Theorem 2.11.3. *We have*

$$\mathcal{T}_{\mathcal{R}}^{\,\gamma} \cup \mathcal{T}_{\mathcal{R}}^{\,\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\,\beta}\,.$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^{\,\gamma}$, then by Definition 2.11.1 there exists $V \in \mathcal{T}_{\mathcal{R}}^{\,s}$ such that $A \subseteq V \subseteq A^{-}$. Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}^{\,s}$ and the increasingness of $\circ -$, we can see that $A \subseteq V \subseteq V^{\circ -} \subseteq A^{- \circ -}$. Therefore, by Definition 2.11.1, we also have $A \in \mathcal{T}_{\mathcal{R}}^{\,\beta}$.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{\,\delta}$, then by Definition 2.11.1 there exists $V \in \mathcal{T}_{\mathcal{R}}^{\,p}$ such that $V \subseteq A \subseteq V^{-}$. Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}^{\,p}$ and the increasingness of $-$ and $- \circ -$, we can see that $A \subseteq V^{-} \subseteq V^{- \circ -} \subseteq A^{- \circ -}$. Therefore, by Definition 2.11.1, we also have $A \in \mathcal{T}_{\mathcal{R}}^{\,\beta}$.

Moreover, by Definition 2.11.1, Theorem 1.8.3 and Corollary 2.1.7, it is clear that we also have the following

Theorem 2.11.4. *If $\mathcal{R}$ is reflexive, then*
(1) $\mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{ps} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma}$; (2) $\mathcal{T}_{\mathcal{R}}^{p} \cup \mathcal{T}_{\mathcal{R}}^{q} \subseteq \mathcal{T}_{\mathcal{R}}^{\delta}$.

Proof. To prove (1), note that if $A \in \mathcal{T}_{\mathcal{R}}^{s}$, then by taking $V = A$ we evidently have $V \in \mathcal{T}_{\mathcal{R}}^{s}$ such that $A \subseteq V$. Moreover, by Theorem 1.8.3, we can also see that $V \subseteq V^{-} = A^{-}$. Thus, by Definition 2.11.1, $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ also holds.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{ps}$, then by Definition 2.1.2 there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^{-}$. Moreover, by Corollary 2.1.7, we have $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{s}$, and thus in particular $V \in \mathcal{T}_{\mathcal{R}}^{s}$. Therefore, by Definition 2.11.1, $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ also holds.

Now, in addition to Theorem 2.1.6 and Corollary 2.1.7, we can also prove

Theorem 2.11.5. *If $\mathcal{R}$ is reflexive, then $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\kappa}$ also holds with $\kappa = \alpha,\ \beta,\ \gamma$ and δ.*

Proof. By Corollary 2.1.7 and Theorems 2.11.4 and 2.11.3, we have

$$\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma} \cap \mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma} \cup \mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}.$$

Therefore, we need only show that $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha}$ also holds.

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}$, then by Theorem 1.8.3 and the definition of $\mathcal{T}_{\mathcal{R}}$, we have $A \subseteq A^{-}$ and $A \subseteq A^{\circ}$. Hence, by using the increasingness of $\circ$ and $-\circ$, we can infer that $A^{\circ} \subseteq A^{-\circ}$ and $A^{-\circ} \subseteq A^{\circ-\circ}$. Therefore, $A \subseteq A^{\circ-\circ}$. Hence, by Definition 2.11.1, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$.

Concerning reflexive relators, we can also easily prove the following

Theorem 2.11.6. *If $\mathcal{R}$ is reflexive, then*
(1) $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$; (2) $\mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$.

Proof. From Theorems 2.11.4 and 2.11.4, we can see that $\mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma} \cup \mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$, and thus in particular (2) also holds. Therefore, we need only prove (1).

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$, then by Definition 2.11.1 we have $A \subseteq A^{\circ-\circ}$. Moreover, from Theorem 1.8.3, we can see $A^{\circ-\circ} \subseteq A^{\circ-}$. Therefore, we also have $A \subseteq A^{\circ-}$. Hence, by Definition 2.1.2, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{s}$.

Moreover, by Theorem 1.8.3, we also have $A^{\circ} \subseteq A$. Hence, by using the increasingness of the operation $-\circ$, we can infer that $A^{\circ-\circ} \subseteq A^{-\circ}$. Therefore, we also have $A \subseteq A^{-\circ}$. Hence, by Definition 2.1.2, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{p}$ also holds. Therefore, we actually have $A \in \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$, and thus (1) is true.

Remark 2.11.7. Assertions (1) and (2) actually rely on the fact that if $\mathcal{R}$ is reflexive, then for any $A \subseteq X$ we have $A^{\circ-\circ} \subseteq A^{\circ-} \cap A^{-\circ}$ and $A^{\circ-} \cup A^{-\circ} \subseteq A^{-\circ-}$.

2.12 Characterizations of Topologically α–Open and β–Open Sets

Now, as a straightforward generalization of [91, Lemma 2.1] of Noiri and [104, Theorem 3] of Reilly and Vamanamurthy, we can also prove the following

Theorem 2.12.1. *If $\mathcal{R}$ is topological, then*
$$\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}.$$

Proof. From Theorem 2.11.6, we know that $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$ even if the relator $\mathcal{R}$ is assumed to be only reflexive.

Moreover, if $A \in \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$, i. e., $A \in \mathcal{T}_{\mathcal{R}}^{s}$ and $A \in \mathcal{T}_{\mathcal{R}}^{p}$, then by Definition 2.1.2, we have $A \subseteq A^{\circ -}$ and $A \subseteq A^{- \circ}$. Hence, by using the increasingness of the operations $-$ and $\circ$, Theorems 1.9.3 and 1.9.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $A^{-} \subseteq A^{\circ - -} \subseteq A^{\circ -}$ and $A^{- \circ} \subseteq A^{\circ - \circ}$. Therefore, we also have $A \subseteq A^{\circ - \circ}$. Hence, by Definition 2.11.1, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$ also holds even if $\mathcal{R}$ is assumed to be only quasi-topological.

Moreover, as an improvement of [89, Proposition 4] of Njåstad, we can prove

Theorem 2.12.2. *If $\mathcal{R}$ is topologically filtered and topological, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$;

(2) *there exist* $V \in \mathcal{T}_{\mathcal{R}}$ *and* $B \in \mathcal{N}_{\mathcal{R}}$ *such that* $A = V \setminus B$;

(3) *there exist* $V \in \mathcal{T}_{\mathcal{R}}$ *and* $B \subseteq \operatorname{res}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)$ *such that* $A = V \setminus B$.

Proof. If (1) holds, then by Definition 2.11.1 we have $A \subseteq A^{\circ - \circ}$, and thus
$$A = A^{\circ - \circ} \setminus \big(A^{\circ - \circ} \setminus A \big).$$
Hence, by defining $V = A^{\circ - \circ}$ and $B = A^{\circ - \circ} \setminus A$, we can obtain $A = V \setminus B$. Moreover, by Theorems 1.9.3, 1.9.2 and 1.8.3, we can state that $V \in \mathcal{T}_{\mathcal{R}}$ and $B \subseteq A^{\circ -} \setminus A^{\circ} = A^{\circ \dagger}$. Therefore, (3) also holds.

To prove the implication (3) $\Longrightarrow$ (2), it is enough to note only that, by Theorems 1.9.3 and 1.9.2, we have $A^{\circ} \in \mathcal{T}_{\mathcal{R}}$. Thus, by Corollary 1.14.5, we have $A^{\circ \dagger} \in \mathcal{N}_{\mathcal{R}}$. Therefore, if $B \subseteq A^{\circ \dagger}$, then we also have $B \in \mathcal{N}_{\mathcal{R}}$.

Finally, if (2) holds, then by Theorems 1.11.5, 1.9.3, 1.9.2 and 1.8.3 we can see that
$$A^{\circ} = \big(V \setminus B \big)^{\circ} = \big(V \cap B^{c} \big)^{\circ} = V^{\circ} \cap B^{c \circ} = V \cap B^{c \circ}.$$
Hence, by definition of $\mathcal{T}_{\mathcal{R}}$ and by using Theorem 1.13.3, we can infer that $V \cap B^{c \circ -} \subseteq V^{\circ} \cap B^{c \circ -} \subseteq \big(V \cap B^{c \circ} \big)^{-} = A^{\circ -}$. Moreover, by using that $c \circ = - c$ and $c - = \circ c$, we can see that
$$B^{c \circ -} = B^{- c -} = B^{- \circ c} = \emptyset^{c} = X.$$
Therefore, we actually have $V \subseteq A^{\circ -}$, and hence also $A \subseteq V = V^{\circ} \subseteq A^{\circ - \circ}$. Thus, by Definition 2.11.1, assertion (1) also holds.

Remark 2.12.3. If $\mathcal{R}$ is topological, then by Theorems 1.9.3, 1.9.2 and 1.8.3, for any $A \subseteq X$, we have $A^{\circ \circ} = A^{\circ}$.

Therefore, by using the notation $A^{\ddagger} = \operatorname{bnd}_{\mathcal{R}}(A)$, we can also state that
$$A^{\circ \dagger} = A^{\circ -} \setminus A^{\circ} = A^{\circ -} \setminus A^{\circ \circ} = A^{\circ \ddagger}.$$

Now, by using the above theorem, we can also prove the following improvement of [89, Corollary] of Njåstad.

Corollary 2.12.4. *If $\mathcal{R}$ is nonvoid, topologically filtered and topological, then the following assertions are equivalent:*

(1) $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}$; (2) $\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}$; (3) $\mathcal{N}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$; (4) $\mathcal{N}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$.

Proof. From Theorem 2.11.5, we can see that $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha}$. Therefore, (1) and (2) are equivalent even if $\mathcal{R}$ is only reflexive.

Moreover, if $A \in \mathcal{N}_{\mathcal{R}}$, then by using Theorem 1.8.3, the increasingnes of $\circ$, and the definitions of $\mathcal{D}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$, we can see that $A^{\circ} \subseteq A^{-\circ} = \emptyset$, and thus $A^{\circ} = \emptyset$, i.e., $A \notin \mathcal{E}_{\mathcal{R}}$. Therefore, (3) and (4) are also equivalent even if $\mathcal{R}$ is only reflexive.

From Theorem 1.9.3, we know that a topological relator is reflexive. Therefore, to complete the proof, we need only show that now (1) and (3) are also equivalent. For this, note that now $X \in \mathcal{T}_{\mathcal{R}}$ since $\mathcal{R} \neq \emptyset$. Therefore, if $B \in \mathcal{N}_{\mathcal{R}}$, then by Theorem 2.12.2 we also have $B^{c} = X \setminus B \in \mathcal{T}_{\mathcal{R}}^{\alpha}$. Hence, if (1) holds, we can infer that $B^{c} \in \mathcal{T}_{\mathcal{R}}$, and thus $B \in \mathcal{F}_{\mathcal{R}}$. Therefore, (1) implies (3).

On the other hand, if $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$, then by Theorem 2.12.2 there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \setminus B$. Moreover, if (3) holds, we can also state that $B \in \mathcal{F}_{\mathcal{R}}$, and thus $B^{c} \in \mathcal{T}_{\mathcal{R}}$. Hence, by using Corollary 1.11.6, we can infer that $A = V \setminus B = V \cap B^{c} \in \mathcal{T}_{\mathcal{R}}$. Therefore, (1) also holds.

Now, by using the plausible notation $\mathcal{F}_{\mathcal{R}}^{r} = \left\{ A^{c} : \ A \in \mathcal{T}_{\mathcal{R}}^{r} \right\}$, as a partial counterpart of [37, Theorem 26] of Ekici and [57, Theorem 3.7] of Jamunarani et al., we can also prove the following

Theorem 2.12.5. *If $\mathcal{R}$ is topological, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$; (2) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^{s}$; (3) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^{q}$; (4) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}^{r}$.

Proof. From Theorem 2.2.1, we know that (2) and (3) are equivalent. Therefore, we need only prove the equivalence of (1), (2) and (4).

For this, note that if (1) holds, then by Definition 2.11.1 we have $A \subseteq A^{-\circ-}$. Hence, by using the increasingness of $-$, Theorems 1.9.3 and 1.9.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $A^{-} \subseteq A^{-\circ--} \subseteq A^{-\circ-}$. Therefore, by Definition 2.1.2, assertion (2) also holds.

While, if (2) holds, then by using Theorems 1.9.3, 1.8.3, 1.9.2 and 2.2.2, we can see that $A^{-} = A^{--} = A^{-\circ-}$. Hence, by using that $-c = c\circ$ and $\circ c = c-$, we can infer that

$$A^{-c} = A^{-\circ-c} = A^{-\circ c\circ} = A^{-c-\circ} .$$

Thus, by Definition 2.6.1, we have $A^{c} \in \mathcal{T}_{\mathcal{R}}^{r}$, and thus (4) also holds.

Finally, if (4) holds, then $A^{c} \in \mathcal{T}_{\mathcal{R}}^{r}$. Therefore, by Definition 2.6.1, we have $A^{-c} = A^{-c-\circ}$. Hence, by using a similar argument as above, we can infer that $A^{-} = A^{-\circ-}$. Moreover, by Theorems 1.9.3 and 1.8.3, we also have $A \subseteq A^{-}$. Therefore, $A \subseteq A^{-\circ-}$, and thus by Definition 2.11.1 assertion (1) also holds.

Remark 2.12.6. From this theorem, by using our former results on the families $\mathcal{T}_{\mathcal{R}}^{s}$ and $\mathcal{T}_{\mathcal{R}}^{r}$, we can derive several properties of the family $\mathcal{T}_{\mathcal{R}}^{\beta}$.

For instance, from Theorem 2.12.5, by using Theorems 2.2.2 and 2.8.3, we can immediately derive the following

Theorem 2.12.7. *If $\mathcal{R}$ is topological, then for any $A \subseteq X$ the following assertions are equivalent:*
(1) $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ *;*
(2) *there exists* $V \in \mathcal{T}_{\mathcal{R}}$ *such that* $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(V)$ *;*
(3) *there exist* $V \in \mathcal{T}_{\mathcal{R}}$ *and* $B \subseteq X$ *such that* $\operatorname{cl}_{\mathcal{R}}(A) = V \cup B$ *and* $B \subseteq \operatorname{res}_{\mathcal{R}}(V)$.

Hence, we can easily derive the following counterpart of [37, Theorem 27] of Ekici and [57, Theorem 3.8] of Jamunarani et al.

Corollary 2.12.8. *If $\mathcal{R}$ is topological, $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ and $A \subseteq B \subseteq \operatorname{cl}_{\mathcal{R}}(A)$, then $B \in \mathcal{T}_{\mathcal{R}}^{\beta}$ also holds.*

Proof. By using the increasingness of $-$, Theorems 1.9.3 and 1.9.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can see that $A^{-} \subseteq B^{-} \subseteq A^{--} \subseteq A^{-}$, and thus $A^{-} = B^{-}$. Hence, by Theorem 2.12.7, it is clear that the required assertion is also true.

Moreover, from Theorem 2.12.5, by using Theorem 2.8.5, we can also easily derive

Theorem 2.12.9. *If $\mathcal{R}$ is topologically filtered, topological and $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that*
$$\operatorname{cl}_{\mathcal{R}}(A) = V \cup B \qquad \text{and} \qquad V \cap B = \emptyset .$$

Now, analogously to [37, Theorem 23] of Ekici and [57, Theorem 3.5] of Jamunarani et al., we can also prove the following

Theorem 2.12.10. *If $\mathcal{R}$ is topological and $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}^{s}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$.*

Proof. Define $V = A^{-}$ and $B = A \cup A^{-c}$. Then, by Theorem 2.12.5, we have $V \in \mathcal{T}_{\mathcal{R}}^{s}$. Moreover, by using the increasingness of $-$ and Theorems 1.9.3 and 1.8.3, we can see that
$$B^{-} = \big(A \cup A^{-c}\big)^{-} \subseteq A^{-} \cup A^{-c-} \subseteq A^{-} \cup A^{-c} = X .$$
Therefore, $B^{-} = X$, and thus $B \in \mathcal{D}_{\mathcal{R}}$.

Moreover, by using Theorems 1.9.3 and 1.8.3, we can also see that
$$V \cap B = A^{-} \cap \big(A \cup A^{-c}\big) = \big(A^{-} \cap A\big) \cup \big(A^{-} \cap A^{-c}\big) = A \cup \emptyset = A .$$
Therefore, the required assertion is also true.

The following example shows that the converse of this theorem is false.

Example 2.12.11. If $X = \{1, 2, 3, 4\}$ and R_i is a relation on X, for every $i = 1, 2, 3$, such that
$R_1(1) = \{1\}$, $\quad R_1(2) = R_1(3) = R_1(4) = X$;
$R_2(1) = R_2(2) = \{1, 2\}$, $\quad R_2(3) = R_2(4) = X$;
$R_3(1) = X$, $\quad R_3(2) = R_3(3) = \{2, 3\}$, $\quad R_3(4) = X$;

then $\mathcal{R} = \{R_1, R_2, R_3\}$ is a preorder relator on X such that, under the notations $V = \{2, 3\}$ and $B = \{1, 3, 4\}$, we have $V \in \mathcal{T}_{\mathcal{R}}^{s}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B \notin \mathcal{T}_{\mathcal{R}}^{\beta}$.

To check this, note that R_1, R_2 and R_3 are just the Pervin preorders generated by the sets $\{1\}$, $\{1, 2\}$ and $\{2, 3\}$, respectively. Moreover, we have
$A^{-\circ-} = \{3\}^{-\circ-} = \{3, 4\}^{\circ-} = \emptyset^{-} = \emptyset$, $V^{\circ-} = \{2, 3\}^{\circ-} = \{2, 3\}^{-} = \{2, 3, 4\}$ and $B^{-} = \{1, 3, 4\}^{-} = X$.

2.13 Some Further Results on Topologically α–Open and β–Open Sets

By using our former results and the plausible notation $\mathcal{F}_{\mathcal{R}}^{\beta} = \{ A^{c} : A \in \mathcal{T}_{\mathcal{R}}^{\beta} \}$, we can also easily prove the following counterpart of [37, Theorem 7] of Ekici and [57, Theorem 2.1] of Jamunarani et al.

Theorem 2.13.1. *If $\mathcal{R}$ is topological, then $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta}$.*

Proof. By Theorem 2.6.9, we have $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{s}$. Moreover, by Theorems 2.11.5 and 2.11.6, we have $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha}$ and $\mathcal{T}_{\mathcal{R}}^{s} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$, and thus $\mathcal{F}_{\mathcal{R}}^{s} \subseteq \mathcal{F}_{\mathcal{R}}^{\beta}$. Therefore, $\mathcal{T}_{\mathcal{R}}^{r} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta}$.

On the other hand, if $A \in \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta}$, then we have $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$ and $A \in \mathcal{F}_{\mathcal{R}}^{\beta}$, and thus also $A^{c} \in \mathcal{T}_{\mathcal{R}}^{\beta}$. Hence, by using Definition 2.11.1, we can infer that $A \subseteq A^{\circ-\circ}$ and $A^{c} \subseteq A^{c-\circ-}$, and thus also $A^{c-\circ-c} \subseteq A$. Moreover, by using the equalities $c- = \circ c$, $-c = c\circ$ and $- = c \circ c$, we can see that $A^{c-\circ-c} = A^{\circ c \circ - c} = A^{\circ c \circ c \circ} = A^{\circ-\circ}$. Therefore, $A^{\circ-\circ} \subseteq A$, and thus $A = A^{\circ-\circ}$ also holds. Hence, by using Theorem 2.7.4, we can already infer that $A \in \mathcal{T}_{\mathcal{R}}^{r}$. Therefore, $\mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta} \subseteq \mathcal{T}_{\mathcal{R}}^{r}$, and thus the required equality is also true.

Now, by using Theorem 2.11.3 and 2.12.5, we can also easily prove the following two counterparts of [5, Theorem 2.4] of Andrijević.

Theorem 2.13.2. *If $\mathcal{R}$ is topological, then $\mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{T}_{\mathcal{R}}^{\beta}$.*

Proof. By Theorem 2.11.3, we always have $\mathcal{T}_{\mathcal{R}}^{\gamma} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$. Therefore, we need only prove that now $\mathcal{T}_{\mathcal{R}}^{\beta} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma}$ also holds. For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then by Theorem 2.12.5 we have $A^{-} \in \mathcal{T}_{\mathcal{R}}^{s}$. Hence, by defining $V = A^{-}$, we can note that $V \in \mathcal{T}_{\mathcal{R}}^{s}$ such that $V \subseteq A^{-}$. Moreover, by Theorems 1.9.3 and 1.8.3, it is clear that $A \subseteq V$ is also true. Therefore, by Definition 2.11.1 we also have $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$.

Theorem 2.13.3. *If $\mathcal{R}$ is topologically filtered and topological, then $\mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{T}_{\mathcal{R}}^{\beta}$.*

Proof. By Theorem 2.11.3, we always have $\mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$. Therefore, we need only prove that now $\mathcal{T}_{\mathcal{R}}^{\beta} \subseteq \mathcal{T}_{\mathcal{R}}^{\delta}$ also holds.

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then by Theorem 2.12.5 we have $A^{-} \in \mathcal{T}_{\mathcal{R}}^{s}$. Hence, by using Theorems 1.9.3, 1.8.3, 1.9.2 and 2.2.2, we can again infer that $A^{-} = A^{--} = A^{-\circ-}$. Now, by defining $V = A \cap A^{-\circ}$, we can note that $V \subseteq A$. Moreover, by using Theorems 1.13.3, 1.9.3, 1.9.2 and 1.8.3, we can see that

$$V^{-} = \left(A \cap A^{-\circ} \right)^{-} \supseteq A^{-} \cap A^{-\circ\circ} = A^{-} \cap A^{-\circ} = A^{-\circ}.$$

Hence, we can infer that $V^{-\circ} \supseteq A^{-\circ\circ} = A^{-\circ} \supseteq A \cap A^{-\circ} = V$. Thus, by Definition 2.1.2, we also have $V \in \mathcal{T}_{\mathcal{R}}^{p}$.

Moreover, we can now also note that $A \subseteq A^{-} = A^{-\circ-} \subseteq V^{--} = V^{-}$. Therefore, by Definition 2.11.1, we also have $A \in \mathcal{T}_{\mathcal{R}}^{\delta}$.

Unfortunately, concerning the relationship of the families $\mathcal{T}_{\mathcal{R}}^{\alpha}$ and $\mathcal{T}_{\mathcal{R}}^{\gamma}$, we can only prove the following

Theorem 2.13.4. *If $\mathcal{R}$ is reflexive, then $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma} \cap \mathcal{T}_{\mathcal{R}}^{\delta}$.*

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$, then by Theorem 2.11.6 we also have $A \in \mathcal{T}_{\mathcal{R}}^{s}$ and $A \in \mathcal{T}_{\mathcal{R}}^{p}$. Hence, by Theorem 2.11.4, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ and $A \in \mathcal{T}_{\mathcal{R}}^{\delta}$ also hold. Therefore, the required inclusion is also true.

Remark 2.13.5. Later, we shall see that the corresponding equality need not be true. Moreover, $\mathcal{T}_{\mathcal{R}}^{\delta}$ may also be a proper subset of $\mathcal{T}_{\mathcal{R}}^{\gamma}$.

2.14 Topologically a–Open and b–Open Sets

In topological spaces, β–open sets were actually called semi-preopen by Andrijević [5]. Later, this terminology was also used by Ganster and Andrijević [43] and Dontchev [32].

In a subsequent paper [8], Andrijević also introduced the notion of a b–open subset of a topological space. Motivated by his definition, we may also naturally introduce the following

Definition 2.14.1. A subset A of a relator space $X(\mathcal{R})$ will be called *topologically*

(1) *a–open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\left(\operatorname{int}_{\mathcal{R}}(A)\right) \cap \operatorname{int}_{\mathcal{R}}\left(\operatorname{cl}_{\mathcal{R}}(A)\right)$;

(2) *b–open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\left(\operatorname{int}_{\mathcal{R}}(A)\right) \cup \operatorname{int}_{\mathcal{R}}\left(\operatorname{cl}_{\mathcal{R}}(A)\right)$.

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = a$ and b, respectively.

Thus, analogously to Theorem 2.11.6, we evidently have the following

Theorem 2.14.2. *We have*

(1) $\mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$; (2) $\mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{b}$.

Moreover, by using Theorems 2.11.6 and 1.8.3 and Definition 2.11.1, we can also prove

Theorem 2.14.3. *If $\mathcal{R}$ is reflexive, then*

(1) $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{a}$; (2) $\mathcal{T}_{\mathcal{R}}^{b} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$.

Proof. Since (1) follows immediately from Theorems 2.11.6 and 2.14.2, we need only prove (2). For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{b}$, then by Definition 2.14.1 and Remark 2.11.7, we have $A \subseteq A^{\circ-} \cup A^{-\circ} \subseteq A^{-\circ-}$. Thus, by Definition 2.11.1, $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ also holds.

Remark 2.14.4. Note that, by Theorem 2.14.2, we always have $\mathcal{T}_{\mathcal{R}}^{a} \subseteq \mathcal{T}_{\mathcal{R}}^{b}$. Thus, if $\mathcal{R}$ is reflexive, then by Theorem 2.14.3 we also have $\mathcal{T}_{\mathcal{R}}^{a} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$.

Now, in accordance with [8, Remark 1] of Adrijević , we can also prove

Theorem 2.14.5. *If $\mathcal{R}$ is topologically filtered and topological, then for any $A \subseteq X$ the following assertions are equivalent :*
(1) $A \in \mathcal{T}_{\mathcal{R}}^{b}$;
(2) *there exist* $B \in \mathcal{T}_{\mathcal{R}}^{s}$ *and* $C \in \mathcal{T}_{\mathcal{R}}^{p}$ *such that* $A = B \cup C$.

Proof. If (2) holds then by Definition 2.1.2 we have $B \subseteq B^{\circ -}$ and $C \subseteq C^{- \circ}$. Moreover, we can see that $B \subseteq A$ and $C \subseteq A$. Hence, by using the increasingness of $\circ -$ and $- \circ$, we can infer that $B^{\circ -} \subseteq A^{\circ -}$ and $C^{- \circ} \subseteq A^{- \circ}$. Therefore, we have $A = B \cup C \subseteq A^{\circ -} \cup A^{- \circ}$. Thus, by Definition 2.14.1, assertion (1) also holds even if $\mathcal{R}$ is not assumed to have any particular properties.

Conversely, if (1) holds, then by Definition 2.14.1, we have $A \subseteq A^{\circ -} \cup A^{- \circ}$. Hence, we can infer that $A = A \cap \left(A^{\circ -} \cup A^{- \circ} \right) = \left(A \cap A^{\circ -} \right) \cup \left(A \cap A^{- \circ} \right)$. Thus, by defining $B = A \cap A^{\circ -}$ and $C = A \cap A^{- \circ}$, we can at once state that $A = B \cup C$.

Now, by using Theorems 1.11.5, 1.9.3, 1.9.2 and 1.8.3, we can also see that

$$B^{\circ} = \left(A \cap A^{\circ -} \right)^{\circ} = A^{\circ} \cap A^{\circ - \circ} \supseteq A^{\circ} \cap A^{\circ \circ} = A^{\circ} \cap A^{\circ} = A^{\circ},$$

and thus $B^{\circ -} \supseteq A^{\circ -} \supseteq A \cap A^{\circ -} = B$. Therefore, by Definition 2.1.2, we have $B \in \mathcal{T}_{\mathcal{R}}^{s}$.

Moreover, by using Theorems 1.13.3, 1.11.5, 1.9.3, 1.9.2 and 1.8.3, we can see that

$$C^{-} = \left(A \cap A^{- \circ} \right)^{-} \supseteq A^{-} \cap A^{- \circ \circ} = A^{-} \cap A^{- \circ} = A^{- \circ},$$

and thus $C^{- \circ} \supseteq A^{- \circ \circ} = A^{- \circ} \supseteq A \cap A^{- \circ} = C$. Therefore, by Definition 2.1.2, we also have $C \in \mathcal{T}_{\mathcal{R}}^{p}$.

Remark 2.14.6. The above proof shows that if $\mathcal{R}$ is topologically filtered and topological, then for any $A \subseteq X$ we have $A \cap A^{\circ -} \in \mathcal{T}_{\mathcal{R}}^{s}$ and $A \cap A^{- \circ} \in \mathcal{T}_{\mathcal{R}}^{p}$.

2.15 The Duals of the Families $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = \alpha, \beta, \gamma, \delta, a$ and b

Analogously to Definition 2.5.1 we also introduce some further corresponding generalized topologically closed sets, we shall also use the following plausible notation.

Definition 2.15.1. For any $\kappa = \alpha, \beta, \gamma, \delta, a$ and b, we define

$$\mathcal{F}_{\mathcal{R}}^{\kappa} = \left\{ A \subseteq X : \quad A^{c} \in \mathcal{T}_{\mathcal{R}}^{\kappa} \right\}.$$

Analogously to Theorems 2.5.3 and 2.5.2, we can also prove the following two theorems.

Theorem 2.15.2. *For any $A \subseteq X$ the following assertions are equivalent :*
(1) $A \in \mathcal{F}_{\mathcal{R}}^{\gamma}$;
(2) *there exists* $W \in \mathcal{F}_{\mathcal{R}}^{s}$ *such that* $\operatorname{int}_{\mathcal{R}}(A) \subseteq W \subseteq A$.

Theorem 2.15.3. *For any $A \subseteq X$ the following assertions are equivalent:*
(1) $A \in \mathcal{F}_{\mathcal{R}}^{\delta}$;
(2) *there exists* $W \in \mathcal{F}_{\mathcal{R}}^{p}$ *such that* $\operatorname{int}_{\mathcal{R}}(W) \subseteq A \subseteq W$.

Proof. To prove that (1) $\Longrightarrow$ (2), note that if (1) holds, then $A^c \in \mathcal{T}_{\mathcal{R}}^{\delta}$. Thus, by Definition 2.11.1, there exists $V \in \mathcal{T}_{\mathcal{R}}^{p}$ such that $V \subseteq A^c \subseteq V^-$. Hence, by using that $c\circ = -c$, we can infer that $V^{c\circ} = V^{-c} \subseteq A \subseteq V^c$. Thus, by taking $W = V^c$, we can see that $W \in \mathcal{F}_{\mathcal{R}}^{p}$ such that $W^\circ \subseteq A \subseteq W$, and thus (2) also holds.

Moreover, analogously to Theorems 2.5.4 and 2.5.5, we can also prove the following two theorems.

Theorem 2.15.4. *For any $A \subseteq X$, the following assertions are equivalent:*
(1) $A \in \mathcal{F}_{\mathcal{R}}^{\alpha}$; (2) $\operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(\operatorname{cl}_{\mathcal{R}}(A))\big) \subseteq A$.

Theorem 2.15.5. *For any $A \subseteq X$ the following assertions are equivalent:*
(1) $A \in \mathcal{F}_{\mathcal{R}}^{\beta}$; (2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A))\big) \subseteq A$.

Proof. By the corresponding definitions, we have
$$(1) \iff A^c \in \mathcal{T}_{\mathcal{R}}^{\beta} \iff A^c \subseteq A^{c-\circ-} \iff A^{c-\circ-c} \subseteq A.$$
Moreover, by using the equalities $c- = \circ c$, $-c = c\circ$ and $c \circ c = -$, we can see that
$$A^{c-\circ-c} = A^{\circ c\circ-c} = A^{\circ c\circ c\circ} = A^{\circ-\circ}.$$
Therefore, we actually have $(1) \iff A^{\circ-\circ} \subseteq A \iff (2)$.

Now, by using Theorem 2.14.2, we can also easily prove the following

Theorem 2.15.6. *We have* $\mathcal{F}_{\mathcal{R}}^{a} = \mathcal{F}_{\mathcal{R}}^{s} \cap \mathcal{F}_{\mathcal{R}}^{p}$.

Proof. By the corresponding definitions and Theorem 2.14.2, for any $A \subseteq X$, we have
$$A \in \mathcal{F}_{\mathcal{R}}^{a} \iff A^c \in \mathcal{T}_{\mathcal{R}}^{a} \iff A^c \in \mathcal{T}_{\mathcal{R}}^{s}, \quad A^c \in \mathcal{T}_{\mathcal{R}}^{p}$$
$$\iff A \in \mathcal{F}_{\mathcal{R}}^{s}, \quad A \in \mathcal{F}_{\mathcal{R}}^{p} \iff A \in \mathcal{F}_{\mathcal{R}}^{s} \cap \mathcal{F}_{\mathcal{R}}^{p}.$$
Therefore, the required equality is true.

Hence, by using Theorems 2.5.3 and 2.5.4, we can immediately derive

Corollary 2.15.7. *For any $A \subseteq X$, the following assertions are equivalent:*
(1) $A \in \mathcal{F}_{\mathcal{R}}^{a}$; (2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \cup \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \subseteq A$.

The latter statement can be proved directly, by using only the corresponding definitions.

Moreover, by using a direct argument, we can also easily prove the following counterpart of this corollary.

Theorem 2.15.8. *For any $A \subseteq X$, the following assertions are equivalent:*
(1) $A \in \mathcal{F}_{\mathcal{R}}^{b}$; (2) $\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big) \cap \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \subseteq A$.

Proof. By the corresponding definitions, we have

$(1) \iff A^c \in \mathcal{T}_{\mathcal{R}}^{b} \iff A^c \subseteq A^{c\circ -} \cup A^{c-\circ} \iff \left(A^{c\circ -} \cup A^{c-\circ}\right)^c \subseteq A\,.$

Moreover, by using De Morgan's law and the equalities established in the proofs of Theorems 2.5.3 and 2.5.4, we can see that $\left(A^{c\circ -} \cup A^{c-\circ}\right)^c = A^{c\circ - c} \cap A^{c-\circ c} = A^{-\circ} \cap A^{\circ -}$. Therefore, we actually have $(1) \iff A^{-\circ} \cap A^{\circ -} \subseteq A \iff (2)$.

2.16 Generalized Topologically Open Sets Derived from Topologically Simple Relators

By Definition 2.1.2 and Theorem 1.16.17, we can at once state the following

Theorem 2.16.1. *If $\mathcal{R}$ is nonvoid, then the following assertions are equivalent:*
(1) $\mathcal{T}_{\mathcal{R}}^{p} = \mathcal{P}(X)$;
(2) *$\mathcal{R}$ is properly topologically symmetric.*

Hence, by using Theorems 2.6.6 and 2.14.2, we can immediately derive

Corollary 2.16.2. *If $\mathcal{R}$ is nonvoid and properly topologically symmetric, then*
(1) $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{F}_{\mathcal{R}}^{s}$; (2) $\mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{s}$; (3) $\mathcal{T}_{\mathcal{R}}^{b} = \mathcal{P}(X)$.

Proof. Namely, by Theorems 2.6.6 and 2.14.2 and 2.16.1, we have

$$\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{p} \cap \mathcal{F}_{\mathcal{R}}^{s} = \mathcal{P}(X) \cap \mathcal{F}_{\mathcal{R}}^{s} = \mathcal{F}_{\mathcal{R}}^{s} \qquad \text{and} \qquad \mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p} = \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{P}(X) = \mathcal{T}_{\mathcal{R}}^{s}\,.$$

Moreover, by Theorems 2.14.2 and 2.16.1, we have $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{P}(X) = \mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{b}$, and thus $\mathcal{T}_{\mathcal{R}}^{b} = \mathcal{P}(X)$ also holds.

Remark 2.16.3. By using equality (1), we can also easily see that

$$A \in \mathcal{F}_{\mathcal{R}}^{r} \iff A^c \in \mathcal{T}_{\mathcal{R}}^{r} \iff A^c \in \mathcal{F}_{\mathcal{R}}^{s} \iff A \in \mathcal{T}_{\mathcal{R}}^{s}\,.$$

Therefore, in addition to Corollary 2.16.2, we can also state that $\mathcal{F}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{s}$.

Now, in addition to Corollary 2.2.4, we can also state the following

Corollary 2.16.4. *If $\mathcal{R}$ is nonvoid and properly topologically symmetric, then for any $A \subseteq X$, the following assertions are equivalent:*
(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$; (2) $A = \operatorname{cl}_{\mathcal{R}}\left(\operatorname{int}_{\mathcal{R}}(A)\right)$.

Proof. By Remark 2.16.3 and the equalities $c\,- = \circ\, c$ and $c \circ c = -$, we have

$$A \in \mathcal{T}_{\mathcal{R}}^{s} \iff A \in \mathcal{F}_{\mathcal{R}}^{r} \iff A^c \in \mathcal{T}_{\mathcal{R}}^{r} \iff A^c = A^{c-\circ}$$
$$\iff A = A^{c-\circ c} \iff A = A^{\circ c \circ c} \iff A = A^{\circ -}.$$

Moreover, from Theorem 2.16.1, by Theorems 2.2.1 and 2.12.1, we can also derive

Corollary 2.16.5. *If $\mathcal{R}$ is nonvoid, properly topologically symmetric and topological, then*
(1) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{P}(X)$; (2) $\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}^{s}$.

From Corollary 1.16.10, we know that $\mathcal{R}$ is properly topologically symmetric if and only if it is topologically simple and weakly symmetric.

Therefore, as a straightforward generalization of properly symmetric relators, we may also naturally consider here topologically simple relators.

Theorem 2.16.6. *If $\mathcal{R}$ is topologically simple and $R = \bigcap \mathcal{R}$, then for any $A \subseteq X$ we have*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$ *if and only if* $A \subseteq R^{-1}\big[R^{-1}[A^c]^c \big]$;

(2) $A \in \mathcal{T}_{\mathcal{R}}^{p}$ *if and only if* $A \subseteq R^{-1}\big[R^{-1}[A]^c \big]^c$;

(3) $A \in \mathcal{T}_{\mathcal{R}}^{q}$ *if and only if there exists* $V \subseteq X$ *such that* $R[V] \subseteq V \subseteq A \subseteq R^{-1}[V]$;

(4) $A \in \mathcal{T}_{\mathcal{R}}^{ps}$ *if and only if there exists* $V \subseteq X$ *such that* $A \cup R[V] \subseteq V \subseteq R^{-1}[A]$.

Proof. By Theorem 1.15.5, we have $\mathcal{R}^{\wedge} = \{R\}^{\wedge}$. Hence, by Theorems 1.5.4 and 1.3.10, we can see that $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{R}(A) = R^{-1}[A]$ and $\operatorname{int}_{\mathcal{R}}(A) = \operatorname{int}_{R}(A) = R^{-1}[A^c]^c$ for all $A \subseteq X$. Moreover, by the corresponding definitions and Theorem 1.5.4, we can see that

$$A \in \mathcal{T}_{\mathcal{R}} \iff A \subseteq \operatorname{int}_{\mathcal{R}}(A) \iff A \subseteq \operatorname{int}_{R}(A) \iff R[A] \subseteq A .$$

Hence, by Definition 2.1.2, it is clear that required assertions are true.

Remark 2.16.7. By Definitions 2.6.1 and 2.11.1, in addition to assertions (1)–(4), we can, for instance, also state that

(a) $A \in \mathcal{T}_{\mathcal{R}}^{r}$ if and only if $A = R^{-1}\big[R^{-1}[A]^c \big]^c$;

(b) $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ if and only if there exists $V \subseteq X$ such that

$$A \subseteq V \subseteq R^{-1}[A] \cap R^{-1}\big[R^{-1}[A^c]^c \big] .$$

By using our former results on the various refinements of relators, we can also easily prove the following

Theorem 2.16.8. *For any $\square = *$, $\#$ or $\wedge$ and $\kappa = s, p, q, ps, r, \alpha, \beta, \gamma, \delta, a$ or b, we have $\mathcal{T}_{\mathcal{R}}^{\kappa} = \mathcal{T}_{\mathcal{R}^{\square}}^{\kappa}$.*

Proof. By Theorems 1.5.7 and 1.5.4, we have $\mathcal{R}^{\wedge} = \mathcal{R}^{\square\wedge}$. Thus, by Theorem 1.5.4 and the definition of $\mathcal{T}_{\mathcal{R}}$, we have $\operatorname{cl}_{\mathcal{R}} = \operatorname{cl}_{\mathcal{R}^{\square}}$, $\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^{\square}}$ and $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$. Hence, by Definitions 2.1.2, 2.6.1, 2.11.1 and 2.14.1, it is clear that the required equality is also true.

Remark 2.16.9. By [89, 17, 18, 16, 9, 6, 7], for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, the possible consequences and equivalents of the inclusion $\mathcal{T}_{\mathcal{R}}^{\kappa} \subseteq \mathcal{T}_{\mathcal{S}}^{\kappa}$ should also be investigated.

2.17 Generalized Topologically Open Sets Derived from the Paratopological Refinements of Relators

In addition to Theorem 2.16.8, we can also prove the following

Theorem 2.17.1. *If $\mathcal{R}$ is non-degenerated, then for any $A \subseteq X$ the following assertions are equivalent :*

(1) $A \in \mathcal{T}_{\mathcal{R}^{\vartriangle}}^{s} \setminus \{\emptyset\}$; (2) $A \in \mathcal{E}_{\mathcal{R}}$ *and* $\mathcal{R}$ *is non-partial .*

Proof. By Definition 2.1.2, (1) is equivalent to the statement (a) $A \subseteq \operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big)$ and $A \neq \emptyset$. Therefore, if (1) holds, then in particular we have (b) $\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big) \neq \emptyset$.

Next, we show that (b) implies (2). For this, note that if $A \notin \mathcal{E}_{\mathcal{R}}$, then by Corollary 1.6.8 we have $\operatorname{int}_{\mathcal{R}^\vartriangle}(A) = \emptyset$. Therefore, if (b) holds, then $\operatorname{cl}_{\mathcal{R}^\vartriangle}(\emptyset) = \operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big) \neq \emptyset$. Hence, by using Corollary 1.6.8, we can infer that $\emptyset \in \mathcal{D}_{\mathcal{R}}$. Thus, by Theorem 1.8.14, $\mathcal{R}$ cannot be non-degenerated. This contradiction proves that (b) implies $A \in \mathcal{E}_{\mathcal{R}}$.

However, if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 1.6.8 we have $\operatorname{int}_{\mathcal{R}^\vartriangle}(A) = X$. Therefore, if (b) holds, then $\operatorname{cl}_{\mathcal{R}^\vartriangle}(X) = \operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big) \neq \emptyset$. Hence, by using Corollary 1.6.8, we can infer that $X \in \mathcal{D}_{\mathcal{R}}$. Thus, by Theorem 1.8.9, $\mathcal{R}$ is non-partial. Consequently, (b) implies (2), and thus (1) also implies (2).

In the sequel, we shall show that (2) implies that (c) $\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big) = X$ and $A \neq \emptyset$. For this, note that if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 1.6.8, we have $\operatorname{int}_{\mathcal{R}^\vartriangle}(A) = X$. Moreover, if $\mathcal{R}$ is non-partial, then by Theorem 1.8.9 we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ and $X \in \mathcal{D}_{\mathcal{R}}$. Thus, in particular $A \neq \emptyset$. Moreover, by Corollary 1.6.8, we have $\operatorname{cl}_{\mathcal{R}^\vartriangle}(X) = X$. Therefore, if (2) holds, then in addition to $A \neq \emptyset$ we also have $\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big) = \operatorname{cl}_{\mathcal{R}^\vartriangle}(X) = X$, and thus (c) also holds. Hence, since (2) $\Longrightarrow$ (c) $\Longrightarrow$ (a) $\Longrightarrow$ (1), we can see that (2) also implies (1).

Now, by using the above theorem, we can also easily establish the following

Corollary 2.17.2. *If $\mathcal{R}$ is non-partial and non-degenerated, then*

$$\mathcal{T}^{s}_{\mathcal{R}^\vartriangle} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \qquad \textit{with} \qquad \emptyset \notin \mathcal{E}_{\mathcal{R}}.$$

Proof. To prove this, note that by Theorem 2.17.1 we have $\mathcal{T}^{s}_{\mathcal{R}^\vartriangle} \setminus \{\emptyset\} = \mathcal{E}_{\mathcal{R}}$. Moreover, by Definition 2.1.2, we have $\emptyset \in \mathcal{T}^{s}_{\mathcal{R}}$ for any relator $\mathcal{R}$. And, by and Theorem 1.8.9, we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ for any non-partial relator $\mathcal{R}$.

Concerning the family $\mathcal{T}^{p}_{\mathcal{R}^\vartriangle}$, instead of an analogue of Theorem 2.17.1, we can only prove the following counterpart of Corollary 2.17.2.

Theorem 2.17.3. *If $\mathcal{R}$ is non-partial and non-degenerated, then*

$$\mathcal{T}^{p}_{\mathcal{R}^\vartriangle} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \qquad \textit{with} \qquad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Proof. If $A \in \mathcal{T}^{p}_{\mathcal{R}^\vartriangle} \setminus \{\emptyset\}$, then by Definition 2.1.2 we have
(a) $A \subseteq \operatorname{int}_{\mathcal{R}^\vartriangle}\big(\operatorname{cl}_{\mathcal{R}^\vartriangle}(A)\big)$ and $A \neq \emptyset$.
Thus, in particular we have
(b) $\operatorname{int}_{\mathcal{R}^\vartriangle}\big(\operatorname{cl}_{\mathcal{R}^\vartriangle}(A)\big) \neq \emptyset$.

Next, we show that (b) already implies $A \in \mathcal{D}_{\mathcal{R}}$. For this, note that if $A \notin \mathcal{D}_{\mathcal{R}}$, then by Corollary 1.6.8 we have $\operatorname{cl}_{\mathcal{R}^\vartriangle}(A) = \emptyset$. Therefore, if (b) holds, then we have

$$\operatorname{int}_{\mathcal{R}^\vartriangle}(\emptyset) = \operatorname{int}_{\mathcal{R}^\vartriangle}\big(\operatorname{cl}_{\mathcal{R}^\vartriangle}(A)\big) \neq \emptyset.$$

Hence, by using Corollary 1.6.8, we can infer that $\emptyset \in \mathcal{E}_{\mathcal{R}}$. Thus, by Theorem 1.8.9, the relator $\mathcal{R}$ cannot be non-partial. This contradiction proves that (b) implies $A \in \mathcal{D}_{\mathcal{R}}$, and thus $A \in \mathcal{T}^{p}_{\mathcal{R}^\vartriangle} \setminus \{\emptyset\}$ also implies $A \in \mathcal{D}_{\mathcal{R}}$.

In the sequel, we shall show that if $A \in \mathcal{D}_{\mathcal{R}}$, then
(c) $\operatorname{int}_{\mathcal{R}^\vartriangle}\big(\operatorname{cl}_{\mathcal{R}^\vartriangle}(A)\big) = X$ and $A \neq \emptyset$.

For this, note that if $A \in \mathcal{D}_{\mathcal{R}}$, then by Corollary 1.6.8 we have $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(A) = X$. Moreover, by Theorems 1.8.14 and 1.8.9, we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$ and $X \in \mathcal{D}_{\mathcal{R}}$. Thus, in particular $A \neq \emptyset$. Moreover, by Corollary 1.6.8, we have $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(X) = X$. Therefore, in addition to $A \neq \emptyset$, we also have $\operatorname{int}_{\mathcal{R}^{\vartriangle}}\big(\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(A)\big) = \operatorname{int}_{\mathcal{R}^{\vartriangle}}(X) = X$, and thus (c) also holds. Hence, since $A \in \mathcal{D}_{\mathcal{R}} \implies$ (c) $\implies$ (a) $\implies A \in \mathcal{T}^{p}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$, we can see that $A \in \mathcal{D}_{\mathcal{R}}$ also implies $A \in \mathcal{T}^{p}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$.

Thus, we have proved that $\mathcal{T}^{p}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\} = \mathcal{D}_{\mathcal{R}}$. Therefore, to obtain the required assertion, we need only note that, by Definition 2.1.2, we have $\emptyset \in \mathcal{T}^{p}_{\mathcal{R}}$ for any relator $\mathcal{R}$. Moreover, by Theorem 1.8.14 we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$ for any non-degenerated relator $\mathcal{R}$.

From Theorem 2.17.3 and Corollary 2.17.2, by using Theorem 2.6.6, we can derive

Theorem 2.17.4. *If $\mathcal{R}$ is non-partial and non-degenerated, then* $\mathcal{T}^{r}_{\mathcal{R}^{\vartriangle}} = \{\emptyset, X\}$.

Proof. If $A \in \mathcal{T}^{r}_{\mathcal{R}^{\vartriangle}}$, then by Theorem 2.6.6 we have $A \in \mathcal{T}^{p}_{\mathcal{R}^{\vartriangle}}$ and $A \in \mathcal{F}^{s}_{\mathcal{R}^{\vartriangle}}$, and thus $A^{c} \in \mathcal{T}^{s}_{\mathcal{R}^{\vartriangle}}$. Hence, by using Theorem 2.17.3 and Corollary 2.17.2, we can infer that $\big(A \in \mathcal{D}_{\mathcal{R}}$ or $A = \emptyset \big)$ and $\big(A^{c} \in \mathcal{E}_{\mathcal{R}}$ or $A^{c} = \emptyset \big)$, and thus $A \notin \mathcal{D}_{\mathcal{R}}$ or $A = X$. Therefore, if $A \in \mathcal{D}_{\mathcal{R}}$, then only $A = X$ can hold. Hence, we can see that if $A \in \mathcal{T}^{r}_{\mathcal{R}^{\vartriangle}}$, then either $A = X$ or $A = \emptyset$ holds. Thus, $\mathcal{T}^{r}_{\mathcal{R}^{\vartriangle}} \subseteq \{\emptyset, X\}$.

On the other hand, from Theorems 2.17.3 and 1.8.9, we can see that $\emptyset, X \in \mathcal{T}^{p}_{\mathcal{R}^{\vartriangle}}$. Moreover, from Corollary 2.17.2 and Theorem 1.8.14, we can see that $\emptyset, X \in \mathcal{T}^{s}_{\mathcal{R}^{\vartriangle}}$, and thus also $\emptyset, X \in \mathcal{F}^{s}_{\mathcal{R}^{\vartriangle}}$. Therefore, by Theorem 2.6.6, we also have $\emptyset, X \in \mathcal{T}^{r}_{\mathcal{R}^{\vartriangle}}$, and thus also $\{\emptyset, X\} \subseteq \mathcal{T}^{r}_{\mathcal{R}^{\vartriangle}}$.

In addition to Theorem 2.17.1, we can also prove the following

Theorem 2.17.5. *If $\mathcal{R}$ is non-degenerated, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}^{q}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$; (2) *there exists $V \in \mathcal{E}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}}$ such that $V \subseteq A$.*

Proof. If (1) holds, then $A \neq \emptyset$. Moreover, by Definition 2.1.2, there exists $V \subseteq X$ such that $V \in \mathcal{T}_{\mathcal{R}^{\vartriangle}}$ and $V \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}^{\vartriangle}}(V)$. Thus, in particular $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(V) \neq \emptyset$. Hence, by using Corollary 1.6.8, we can infer that $V \in \mathcal{D}_{\mathcal{R}}$. Moreover, from Theorem 1.8.14, we can see that $\emptyset \notin \mathcal{D}_{\mathcal{R}}$, and thus $V \neq \emptyset$. Furthermore, from Corollary 1.6.9, we can see that $\mathcal{T}_{\mathcal{R}^{\vartriangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$. Therefore, we necessarily have $V \in \mathcal{E}_{\mathcal{R}}$, and thus (2) also holds.

On the other hand, if (2) holds, then we have both $V \in \mathcal{E}_{\mathcal{R}}$ and $V \in \mathcal{D}_{\mathcal{R}}$. Hence, by using Corollary 1.6.9, we can also see that $V \in \mathcal{T}_{\mathcal{R}^{\vartriangle}}$. Moreover, from Theorem 1.8.14, we can see that $V \neq \emptyset$, and thus $A \neq \emptyset$. Furthermore, from Corollary 1.6.8, we can see that $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(V) = X$, and thus $A \subseteq \operatorname{cl}_{\mathcal{R}^{\vartriangle}}(V)$ trivially holds. Therefore, by Definition 2.1.2, assertion (1) also holds.

Now, by using the above theorem, we can also easily establish the following

Corollary 2.17.6. *If $\mathcal{R}$ is non-degenerated, then*

$$\mathcal{T}^{q}_{\mathcal{R}^{\vartriangle}} = \mathcal{E}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \qquad \textit{with} \qquad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Proof. To derive this from Theorem 2.17.5, note that $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are ascending families of subsets of X. Therefore, if assertion (2) in Theorem 2.17.5 holds, then we necessarily have $A \in \mathcal{E}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}}$. Moreover, by Definition 2.1.2, we have $\emptyset \in \mathcal{T}_{\mathcal{R}}^{q}$. And, by Theorem 1.8.14, we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$.

Analogously, to Theorem 2.17.5, we can also prove the following

Theorem 2.17.7. *If $\mathcal{R}$ is non-degenerated, then for any $A \subseteq X$ the following assertions are equivalent:*
(1) $A \in \mathcal{T}_{\mathcal{R}^{\vartriangle}}^{ps} \setminus \{\emptyset\}$;
(2) $A \in \mathcal{D}_{\mathcal{R}}$ *and there exists* $V \in \mathcal{E}_{\mathcal{R}}$ *such that* $A \subseteq V$.

Proof. If (1) holds, then $A \neq \emptyset$. Moreover, by Definition 2.1.2, there exists $V \subseteq X$ such that $V \in \mathcal{T}_{\mathcal{R}^{\vartriangle}}$ and $A \subseteq V \subseteq \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(A)$. Thus, in particular $V \neq \emptyset$ and $\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(A) \neq \emptyset$. Hence, by using Corollary 1.6.8, we can infer that $A \in \mathcal{D}_{\mathcal{R}}$. Moreover, from Corollary 1.6.9, we can see that $\mathcal{T}_{\mathcal{R}^{\vartriangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$. Therefore, $V \in \mathcal{E}_{\mathcal{R}}$, and thus (2) also holds.

On the other hand, if (2) holds, then by Theorem 1.8.14 and Corollary 1.6.8 we have $A \neq \emptyset$ and $\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(A) = X$. Thus, we trivially have $V \subseteq \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(A)$. Moreover, from Corollary 1.6.9, we can see that $V \in \mathcal{T}_{\mathcal{R}^{\vartriangle}}$. Therefore, by Definition 2.1.2, assertion (1) also holds.

Now, by using the above theorem, we can also easily establish the following

Corollary 2.17.8. *If $\mathcal{R}$ is non-degenerated, then*
$$\mathcal{T}_{\mathcal{R}^{\vartriangle}}^{ps} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \qquad \textit{with} \qquad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Proof. To derive this from Theorem 2.17.7, note that, by Definition 2.1.2, for any $\mathcal{R}$ we have $\emptyset \in \mathcal{T}_{\mathcal{R}}^{ps}$. Moreover, by Theorem 1.8.14, we have $X \in \mathcal{E}_{\mathcal{R}}$. Therefore, in assertion (2) of Theorem 2.17.7, we can take $V = X$. Furthermore, by Theorem 1.8.14, we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$.

Remark 2.17.9. If $\mathcal{R}$ is non-partial and non-degenerated, then by Theorem 2.17.3 and Corollary 2.17.8 we have $\mathcal{T}_{\mathcal{R}^{\vartriangle}}^{p} = \mathcal{T}_{\mathcal{R}^{\vartriangle}}^{ps}$.

However, the following theorem shows that, for non-degenerated $\mathcal{R}$, the above equality need not be true.

Theorem 2.17.10. *If $\mathcal{R}$ is partial, then*
$$\mathcal{T}_{\mathcal{R}^{\vartriangle}}^{ps} = \{\emptyset\} \qquad \textit{and} \qquad \mathcal{T}_{\mathcal{R}^{\vartriangle}}^{p} = \mathcal{P}(X).$$

Proof. By Theorem 1.8.9, we have $\emptyset \in \mathcal{E}_{\mathcal{R}}$ and $X \notin \mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{D}_{\mathcal{R}}$ is an ascending family of subsets of X, we can see that, for any $A \subseteq X$, we also have $A \notin \mathcal{D}_{\mathcal{R}}$. Now, by using Corollary 1.6.8, we can see that $\mathrm{int}_{\mathcal{R}^{\vartriangle}}\big(\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(A)\big) = \mathrm{int}_{\mathcal{R}^{\vartriangle}}(\emptyset) = X$. Thus, by Definition 2.1.2, we have $\mathcal{T}_{\mathcal{R}^{\vartriangle}}^{p} = \mathcal{P}(X)$. Moreover, by Corollary 2.17.8, we can also see that $\mathcal{T}_{\mathcal{R}^{\vartriangle}}^{ps} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} = \emptyset \cup \{\emptyset\} = \{\emptyset\}$. Namely, since $\mathcal{R}$ is a partial relator, there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) = \emptyset$. Thus, in particular $X \neq \emptyset$ and $\mathcal{R} \neq \emptyset$. Therefore, $\mathcal{R}$ is a non-degenerated relator.

Moreover, by using Theorems 1.8.9 and 1.8.14 and Corollary 1.6.8, we can also prove

Theorem 2.17.11. *If $\mathcal{R}$ is non-partial and non-degenerated, then*

$$\mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \qquad \text{with} \qquad \emptyset \notin \mathcal{E}_{\mathcal{R}} .$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha}$, then by Definition 2.11.1 we have $A \subseteq \operatorname{int}_{\mathcal{R}^\vartriangle}\big(\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big)\big)$. Hence, if $A \neq \emptyset$, we can infer that $\operatorname{int}_{\mathcal{R}^\vartriangle}\big(\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big)\big) \neq \emptyset$. Therefore, by Corollary 1.6.8, we necessarily have $\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big) \in \mathcal{E}_{\mathcal{R}}$. Moreover, from Theorem 1.8.9, we can see that $\emptyset \notin \mathcal{E}_{\mathcal{R}}$, and thus $\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big) \neq \emptyset$. Hence, by using Corollary 1.6.8, we can infer that $\operatorname{int}_{\mathcal{R}^\vartriangle}(A) \in \mathcal{D}_{\mathcal{R}}$. Moreover, from Theorem 1.8.14, we can see that $\emptyset \notin \mathcal{D}_{\mathcal{R}}$, and thus $\operatorname{int}_{\mathcal{R}^\vartriangle}(A) \neq \emptyset$. Hence, by using Corollary 1.6.8, we can already infer that $A \in \mathcal{E}_{\mathcal{R}}$. Thus, we have proved that $\mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, and hence $\mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha} \subseteq \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$.

On the other hand, if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 1.6.8 we have $\operatorname{int}_{\mathcal{R}^\vartriangle}(A) = X$. Moreover, from Theorem 1.8.9, we can see that $X \in \mathcal{D}_{\mathcal{R}}$. Hence, by using Corollary 1.6.8, we can infer that $\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big) = \operatorname{cl}_{\mathcal{R}^\vartriangle}(X) = X$.
Moreover, from Theorem 1.8.14, we can see that $X \in \mathcal{E}_{\mathcal{R}}$. Hence, by using Corollary 1.6.8, we can infer that $\operatorname{int}_{\mathcal{R}^\vartriangle}\big(\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big)\big) = \operatorname{int}_{\mathcal{R}^\vartriangle}(X) = X$.
Therefore, $A \subseteq \operatorname{int}_{\mathcal{R}^\vartriangle}\big(\operatorname{cl}_{\mathcal{R}^\vartriangle}\big(\operatorname{int}_{\mathcal{R}^\vartriangle}(A)\big)\big)$, and thus $A \in \mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha}$ trivially holds. Thus, we have proved that $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha}$. Hence, since $\emptyset \in \mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha}$ trivially holds, we can see that $\mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha}$ is also true.

Now, quite similarly, we can also prove the following

Theorem 2.17.12. *If $\mathcal{R}$ is non-partial and non-degenerated, then*

$$\mathcal{T}_{\mathcal{R}^\vartriangle}^{\beta} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \qquad \text{with} \qquad \emptyset \notin \mathcal{D}_{\mathcal{R}} .$$

Remark 2.17.13. Thus, in contrast to Theorem 2.11.6, we usually have $\mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha} \not\subseteq \mathcal{T}_{\mathcal{R}^\vartriangle}^{\beta}$. This, in accordance with Remark 1.10.11, also shows that the relator $\mathcal{R}^\vartriangle$ is not reflexive in general.

In Definition 3.12.1, the relator $\mathcal{R}$ will be called *hyperconnected* if $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$. That is, the identity function Δ_X of X is *fatness reversing* in the sense of [141, Definition 12.3].

Such relators have several remarkable properties. For instance, it can be easily shown that $\mathcal{R}$ is hyperconnected if and only if $R(x) \cap S(y) \neq \emptyset$ for all $x, y \in X$ and $R, S \in \mathcal{R}$.

Thus, in particular, a hyperconnected relator is non-partial. Therefore, if $\mathcal{R}$ is non-degenerated and hyperconnected, then by Theorems 2.17.11 and 2.17.12 we already have $\mathcal{T}_{\mathcal{R}^\vartriangle}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}^\vartriangle}^{\beta}$.

Now, by using Corollaries 1.6.8 and 2.17.2, we can also prove the following

Theorem 2.17.14. *If $\mathcal{R}$ is non-partial and non-degenerated, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}^\vartriangle}^{\gamma} \setminus \{\emptyset\}$;

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *and there exists* $V \in \mathcal{E}_{\mathcal{R}}$ *such that* $A \subseteq V$.

Proof. If (1) holds, then $A \neq \emptyset$. Moreover, by Definition 2.11.1, there exists $V \subseteq X$ such that $V \in \mathcal{T}_{\mathcal{R}^\vartriangle}^{s}$ and $A \subseteq V \subseteq \operatorname{cl}_{\mathcal{R}^\vartriangle}(A)$. Thus, in particular, $V \neq \emptyset$ and $\operatorname{cl}_{\mathcal{R}^\vartriangle}(A) \neq \emptyset$ also hold. Hence, by using Corollaries 2.17.2 and 1.6.8, we can already infer that $V \in \mathcal{E}_{\mathcal{R}}$ and $A \in \mathcal{D}_{\mathcal{R}}$. Therefore, (2) also holds.

Conversely, if (2) holds, then by using Corollaries 2.17.2 and 1.6.8 we can easily see that $V \in \mathcal{T}^{s}_{\mathcal{R}^\vartriangle}$ and $\operatorname{cl}_{\mathcal{R}^\vartriangle}(A) = X$. Thus, in particular $V \subseteq \operatorname{cl}_{\mathcal{R}^\vartriangle}(A)$ trivially holds. Hence, by Definition 2.11.1, we can see that $\mathcal{T}^{\gamma}_{\mathcal{R}^\vartriangle}$. Moreover, by Theorem 1.8.14, we can also note that $A \neq \emptyset$. Thus, (1) also holds.

Hence, analogously to Corollary 2.17.8, we can also derive

Corollary 2.17.15. *If $\mathcal{R}$ is non-partial and non-degenerated, then*

$$\mathcal{T}^{\gamma}_{\mathcal{R}^\vartriangle} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \qquad \textit{with} \qquad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Concerning the family $\mathcal{T}^{\delta}_{\mathcal{R}}$, instead of an analogue of Theorem 2.17.14, we can only prove the following counterpart of Corollary 2.17.15.

Theorem 2.17.16. *If $\mathcal{R}$ is non-partial and non-degenerated, then*

$$\mathcal{T}^{\delta}_{\mathcal{R}^\vartriangle} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \qquad \textit{with} \qquad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Proof. If $A \in \mathcal{T}^{\delta}_{\mathcal{R}^\vartriangle}$, then by Definition 2.11.1 there exists $V \subseteq X$ such that $V \in \mathcal{T}^{p}_{\mathcal{R}^\vartriangle}$ and $V \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}^\vartriangle}(V)$. Hence, if $A \neq \emptyset$, then we can infer that $\operatorname{cl}_{\mathcal{R}^\vartriangle}(V) \neq \emptyset$. Thus, by Corollary 1.6.8, we necessarily have $V \in \mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{D}_{\mathcal{R}}$ is ascending, it is clear that $A \in \mathcal{D}_{\mathcal{R}}$ also holds. This already shows that $\mathcal{T}^{\delta}_{\mathcal{R}^\vartriangle} \subseteq \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\}$.

On the other hand, if $A \in \mathcal{D}_{\mathcal{R}}$, then by using Theorem 2.17.3 and Corollary 1.6.8, we can see that $A \in \mathcal{T}^{p}_{\mathcal{R}^\vartriangle}$ and $\operatorname{cl}_{\mathcal{R}^\vartriangle}(V) = X$. Hence, by taking $V = A$, we can see that $V \in \mathcal{T}^{p}_{\mathcal{R}^\vartriangle}$ and $V \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}^\vartriangle}(V)$. Thus, by Definition 2.11.1, we also have $A \in \mathcal{T}^{\delta}_{\mathcal{R}^\vartriangle}$. Hence, since $\emptyset \in \mathcal{T}_{\mathcal{R}}$ is always true, we can already infer that $\mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \subseteq \mathcal{T}^{\delta}_{\mathcal{R}^\vartriangle}$.

2.18 A Few Illustrating Examples and a Diagram

The following example will show that, for a non-reflexive relator $\mathcal{R}$, the six inclusions derivable from Theorems 2.1.6 and 2.11.5 need not be true.

Example 2.18.1. If $X = \{1, 2, 3\}$, and R_1 and R_2 are relations on X such that

$R_1(1) = \{1\}$, $R_1(2) = \{1\}$, $R_1(3) = \{2, 3\}$;
$R_2(1) = \{1\}$, $R_2(2) = \{2, 3\}$, $R_2(3) = \{1, 2\}$;

then $\mathcal{R} = \{R_1, R_2\}$ is a non-partial relator on X such that:

(1) $\mathcal{T}^{r}_{\mathcal{R}} = \{\emptyset, X\}$;
(2) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}^{q}_{\mathcal{R}} \cup \{\{2, 3\}\}$;
(3) $\mathcal{T}^{q}_{\mathcal{R}} = \mathcal{T}^{ps}_{\mathcal{R}} = \{\emptyset, \{1\}, \{1, 2\}, X\}$;
(4) $\mathcal{T}^{s}_{\mathcal{R}} = \mathcal{T}^{p}_{\mathcal{R}} = \mathcal{T}^{a}_{\mathcal{R}} = \mathcal{T}^{b}_{\mathcal{R}} = \mathcal{T}^{\alpha}_{\mathcal{R}} = \mathcal{T}^{\beta}_{\mathcal{R}} = \mathcal{T}^{\gamma}_{\mathcal{R}} = \mathcal{T}^{\delta}_{\mathcal{R}} = \mathcal{T}^{q}_{\mathcal{R}} \cup \{\{1, 3\}\}$.

To check this, recall that for any $x \in X$ and $A \subseteq X$ we have

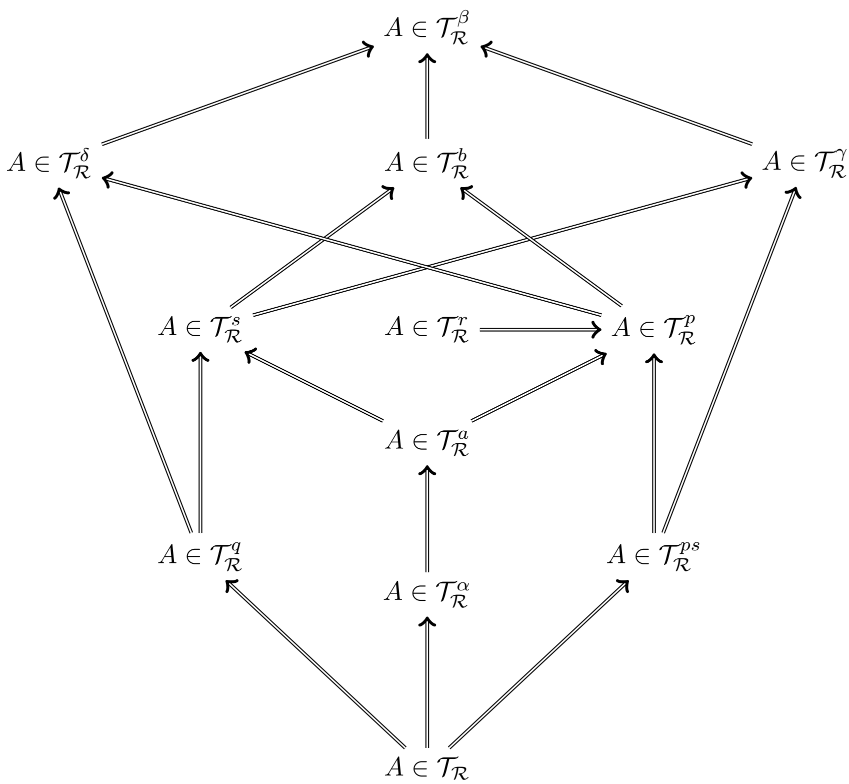
$$x \in A^\circ \iff \exists\, R \in \mathcal{R}: \ R(x) \subseteq A \quad \big(x \in A^- \iff \forall\, R \in \mathcal{R}: \ R(x) \cap A \neq \emptyset \big).$$

Therefore, concerning the relevant operations on subsets of X, we can state:

$\mathcal{P}(X)$	$\circ$	$-$	$\circ-$	$-\circ$	$\circ-\circ$	$-\circ-$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\{1,2\}$	$\{1\}$	X	$\{1,2\}$	X	X
$\{2\}$	$\emptyset$	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1,2\}$	X	X	X	X	X	X
$\{1,3\}$	$\{1,2\}$	X	X	X	X	X
$\{2,3\}$	$\{2,3\}$	$\{3\}$	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$
X	X	X	X	X	X	X

By using Theorems 2.1.5, 2.1.6, 2.6.6, 2.11.3, 2.11.4, 2.11.5, 2.14.2 and 2.14.3, we can easily establish the following

Diagram 2.18.2. For a reflexive relator $\mathcal{R}$, the following implications hold:



Remark 2.18.3. Note that, by Theorems 2.1.5, 2.6.6, 2.11.3 and 2.14.2, nine from the above implications do not require the relator $\mathcal{R}$ to be reflexive.

The following simple example will show only that eight implications in Diagram 2.18.2 are not reversible.

Example 2.18.4. If $X = \{1, 2, 3\}$ and R is a relation on X such that

$$R(1) = \{1, 2\} \qquad \text{and} \qquad R(2) = R(3) = X\,,$$

then $\mathcal{R} = \{R\}$ is a reflexive relator on X such that:
(1) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{q} = \{\emptyset, X\}$;
(2) $\mathcal{T}_{\mathcal{R}}^{s} = \mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{\alpha} = \{\emptyset, \{1, 2\}, X\}$;
(3) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^{p} = \mathcal{T}_{\mathcal{R}}^{b} = \mathcal{T}_{\mathcal{R}}^{\beta} = \mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{P}(X) \setminus \{\{3\}\}$.

To check this, note that now we have:

$\mathcal{P}(X)$	$\circ$	$-$	$\circ-$	$-\circ$	$\circ-\circ$	$-\circ-$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{2\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{3\}$	$\emptyset$	$\{2,3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1,2\}$	$\{1\}$	X	X	X	X	X
$\{1,3\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{2,3\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
X	X	X	X	X	X	X

The following, more difficult example will already show that sixteen implications in Diagram 2.18.2 are not reversible.

Example 2.18.5. If $X = \{1, 2, 3, 4\}$ and R_1 and R_2 are relations on X such that

$$R_1(1) = R_1(2) = \{1, 2, 3\}\,, \qquad R_1(3) = R_1(4) = \{1, 3, 4\}\,;$$

$$R_2(1) = \{1, 2, 3\}\,, \qquad R_2(2) = \{1, 2\}\,, \qquad R_2(3) = R_2(4) = \{3, 4\}\,;$$

then $\mathcal{R} = \{R_1, R_2\}$ is a reflexive relator on X such that:

(1) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^{\alpha} = \{\emptyset, \{3, 4\}, X\}$;
(2) $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cup \{\{2\}\}$;
(3) $\mathcal{T}_{\mathcal{R}}^{q} = \mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}} \cup \{\{1, 3, 4\}\}$;
(4) $\mathcal{T}_{\mathcal{R}}^{s} = \mathcal{T}_{\mathcal{R}}^{q} \cup \{\{1, 2\}\}$;
(5) $\mathcal{T}_{\mathcal{R}}^{p} = \mathcal{P}(X) \setminus \{\{1\}, \{1, 2\}\}$;
(6) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^{p} \setminus \{\{2\}\}$;
(7) $\mathcal{T}_{\mathcal{R}}^{b} = \mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{P}(X) \setminus \{\{1\}\}$;
(8) $\mathcal{T}_{\mathcal{R}}^{\beta} = \mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{P}(X)$.

To check this, note that now we have:

$\mathcal{P}(X)$	$\circ$	$-$	$\circ-$	$-\circ$	$\circ-\circ$	$-\circ-$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\emptyset$	$\{1,2\}$	$\emptyset$	$\{2\}$	$\emptyset$	$\{1,2\}$
$\{2\}$	$\emptyset$	$\{1,2\}$	$\emptyset$	$\{2\}$	$\emptyset$	$\{1,2\}$
$\{3\}$	$\emptyset$	$\{1,3,4\}$	$\emptyset$	$\{3,4\}$	$\emptyset$	$\{1,3,4\}$
$\{4\}$	$\emptyset$	$\{3,4\}$	$\emptyset$	$\{3,4\}$	$\emptyset$	$\{1,3,4\}$
$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{1,2\}$	$\{2\}$	$\{2\}$	$\{1,2\}$
$\{1,3\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{1,4\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{2,3\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{2,4\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{3,4\}$	$\{3,4\}$	$\{1,3,4\}$	$\{1,3,4\}$	$\{3,4\}$	$\{3,4\}$	$\{1,3,4\}$
$\{1,2,3\}$	$\{1,2\}$	X	$\{1,2\}$	X	$\{2\}$	X
$\{1,2,4\}$	$\{2\}$	X	$\{1,2\}$	X	$\{2\}$	X
$\{1,3,4\}$	$\{3,4\}$	X	$\{1,3,4\}$	X	$\{3,4\}$	X
$\{2,3,4\}$	$\{3,4\}$	X	$\{1,3,4\}$	X	$\{3,4\}$	X
X	X	X	X	X	X	X

Remark 2.18.6. Examples 2.18.4 and 2.18.5, together, already show that seventeen implications in Diagram 2.18.2 are not reversible.

However, unfortunately, they cannot be used to show that the remaining implication $A \in \mathcal{T}_{\mathcal{R}}^{\gamma} \implies A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ is also not reversible.

Note that, by Theorem 2.11.3, the above implication does not require the relator $\mathcal{R}$ to be reflexive. Moreover, if $\mathcal{R}$ is topological, then by Theorem 2.13.2 the reverse implication is also true.

Chapter 3. Minimality and Connectedness Properties in Relator Spaces

3.1 Quasi-Proximally and Quasi-Topologically Minimal Relators

Analogously to the definition of a minimal topology, we may naturally introduce

Definition 3.1.1. The relator $\mathcal{R}$ will be called

(1) *quasi-proximally minimal* if $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$;

(2) *quasi-topologically minimal* if $\mathcal{T}_{\mathcal{R}} \subseteq \{\emptyset, X\}$.

Remark 3.1.2. If in particular $\mathcal{R} \neq \emptyset$, then by Theorems 1.4.5 and 1.4.11 we have $\{\emptyset, X\} \subseteq \tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, in this case, we may write equalities instead of inclusions in the above definition.

The use of the term quasi-proximally and quasi-topologically instead of proximally and topologically is only motivated by the fact that the families $\tau_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ are, in general, much weaker tools than the relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$.

Now, as an immediate consequence of Definition 3.1.1, we can state

Theorem 3.1.3. *If $\mathcal{R}$ is quasi-topologically minimal, then $\mathcal{R}$ is quasi- proximally minimal.*

Proof. By Theorem 1.4.11, we have $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$ for any relator $\mathcal{R}$. Moreover, if $\mathcal{R}$ is quasi-topologically minimal, then we also have $\mathcal{T}_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Therefore, in this case $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$ also holds. Thus, $\mathcal{R}$ is quasi-proximally minimal.

Moreover, by using Definition 3.1.1, we can also easily prove the following

Theorem 3.1.4. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topologically minimal;*

(2) *$\mathcal{R}^{\wedge}$ is quasi-proximally minimal.*

Proof. Note that if $\mathcal{R} \neq \emptyset$, then by Corollary 1.6.3 we have $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$. Therefore, $\tau_{\mathcal{R}^{\wedge}} \subseteq \{\emptyset, X\}$ if and only if $\mathcal{T}_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Thus, assertions (1) and (2) are equivalent.
While, if $\mathcal{R} = \emptyset$, then it is clear that $\mathcal{T}_{\mathcal{R}} = \{\emptyset\}$. Therefore, assertion (1) holds. Moreover, we can also note that

(a) if $X \neq \emptyset$, then $\mathcal{R}^{\wedge} = \emptyset$, and thus $\tau_{\mathcal{R}^{\wedge}} = \emptyset$;

(b) if $X = \emptyset$, then $\mathcal{R}^{\wedge} = \{\emptyset\}$, and thus $\tau_{\mathcal{R}^{\wedge}} = \{\emptyset\}$.

Therefore, assertion (2) also holds.

Consequently, if $\mathcal{R} = \emptyset$, then assertions (1) and (2) trivially hold. Thus, in particular, they are equivalent.

Remark 3.1.5. Note that $\mathcal{R} \subseteq \mathcal{R}^{\wedge}$, and thus $\tau_{\mathcal{R}} \subseteq \tau_{\mathcal{R}^{\wedge}}$. Therefore, the quasi-proximal minimality of $\mathcal{R}^{\wedge}$ implies that of $\mathcal{R}$. Thus, Theorem 3.1.3 can be derived from Theorem 3.1.4.

Now, as an immediate consequence of Theorem 3.1.4, we can also state

Corollary 3.1.6. *If $\mathcal{R}$ is topologically fine, then $\mathcal{R}$ is quasi-proximally minimal if and only if $\mathcal{R}$ is quasi-topologically minimal.*

Proof. Here, we have $\mathcal{R}^{\wedge} = \mathcal{R}$. Therefore, by Theorem 3.1.4 , the equivalence is true.

In addition to this corollary, it is also worth proving the following

Theorem 3.1.7. *If $\mathcal{R}$ is proximally simple, then $\mathcal{R}$ is quasi-proximally minimal if and only if $\mathcal{R}$ is quasi-topologically minimal.*

Proof. Now, there exists a relation S that $\mathcal{R}^{\#} = \{S\}^{\#}$, and thus also $\mathcal{R}^{\wedge} = \{S\}^{\wedge}$. Hence, by using Corollary 1.5.5 and Remark 1.4.12, we can see that

$$\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\#}} = \tau_{\{S\}^{\#}} = \tau_{\{S\}} = \mathcal{T}_{\{S\}} = \mathcal{T}_{\{S\}^{\wedge}} = \mathcal{T}_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}.$$

Therefore, by Definition 3.1.1, the required assertion is true.

Concerning quasi-minimal relators, we can also easily prove the following two theorems.

Theorem 3.1.8. *The relator $\mathcal{R}$ is quasi-proximally minimal if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$ and $\mathcal{R}^{-1}$ is quasi-proximally minimal.*

Proof. Recall that $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\square}}$ for all $\square \in \{\infty, *, \#\}$, and $\tau_{\mathcal{R}^{-1}} = \bar{\tau}_{\mathcal{R}} = \{A^c : A \in \tau_{\mathcal{R}}\}$. Therefore, by Definition 3.1.1, the required assertion is true.

Remark 3.1.9. From Theorem 3.1.8, we can see that the relator $\mathcal{R}$ is quasi-proximally minimal if and only if any one of the relators $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally minimal.

Theorem 3.1.10. *The relator $\mathcal{R}$ is quasi-topologically minimal if and only if any one of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\wedge\infty}$ is quasi-topologically minimal.*

Proof. Recall that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$ for all $\square \in \{*, \#, \wedge, \wedge\infty\}$. Therefore, by Definition 3.1.1, the required assertion is true.

Remark 3.1.11. Note that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{\wedge}$, and thus $\mathcal{T}_{\mathcal{R}^{\infty}} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, if $\mathcal{R}$ is quasi-topologically minimal, then $\mathcal{R}^{\infty}$ is also quasi-topologically minimal.

3.2 The Main Characterizations of Quasi-Minimal Relators

From Theorem 3.1.4, we can see that the properties of the quasi-topologically minimal relators can, in priciple, be immediately derived from those of the quasi-proximally minimal ones.

Therefore, it is of major importance to prove the following basic characterization theorem of quasi-proximally minimal relators.

Theorem 3.2.1. *The following assertions are equivalent :*
(1) $\mathcal{R}$ *is quasi-proximally minimal ;*
(2) $\mathcal{R} \subseteq \{X^2\}^{\partial}$; (3) $\mathcal{R}^{\infty} \subseteq \{X^2\}$; (4) $\mathcal{R}^{\#} \subseteq \{X^2\}^{\partial}$; (5) $\mathcal{R}^{\# \infty} \subseteq \{X^2\}$.

Proof. By taking $\mathcal{S} = \{X^2\}$, we can note that $\tau_{\mathcal{S}} = \{\emptyset, X\}$ and $\mathcal{S} = \mathcal{S}^{\#}$. Moreover, by using Theorem 1.7.4 and the Galois property of the operations ∞ and ∂, we can see that

$$\tau_{\mathcal{R}} \subseteq \{\emptyset, X\} \iff \tau_{\mathcal{R}} \subseteq \tau_{\mathcal{S}} \iff \mathcal{R} \subseteq \mathcal{S}^{\# \partial}$$
$$\iff \mathcal{R}^{\infty} \subseteq \mathcal{S}^{\#} \iff \mathcal{R}^{\infty} \subseteq \mathcal{S} \iff \mathcal{R} \subseteq \mathcal{S}^{\partial}.$$

Therefore, assertions (1), (2) and (3) are equivalent.

Now, by using Theorem 3.1.8 and the above equivalences, we can see that assertions (1), (4) and (5) are also equivalent.

Remark 3.2.2. Note that, by Theorem 3.1.8, instead of $\#$ we may also write ∞, $*$ or -1 in the assertions (4) and (5) of the above theorem.

Detailed reformulations of assertion (3) of Theorem 3.2.1 give the following

Corollary 3.2.3. *The following assertions are equivalent :*
(1) $\mathcal{R}$ *is quasi-proximally minimal ;*
(2) *for each* $R \in \mathcal{R}$ *we have* $R^{\infty} = X^2$ *;*
(3) *for each* $R \in \mathcal{R}$ *and* $a, b \in X$ $\exists\, n \in \{0\} \cup \mathbb{N}$ *such that* $(a, b) \in R^n$ *;*
(4) *for each* $R \in \mathcal{R}$ *and* $a, b \in X$ $\exists\, n \in \{0\} \cup \mathbb{N}$ *and a family* $(x_i)_{i=0}^{n}$ *in* X *such that* $x_0 = a$, $x_n = b$ *and* $(x_{i-1}, x_i) \in R$ *for all* $i = 1, 2, \ldots, n$.

Proof. To derive this from Theorem 3.2.1, recall that $\mathcal{R}^{\infty} = \{R^{\infty} : R \in \mathcal{R}\}$ with $R^{\infty} = \bigcup_{n=0}^{\infty} R^n$, where $R^n = \Delta_X$ if $n = 0$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$.

Remark 3.2.4. From the equivalence of assertions (1) and (4) in this corollary, we can see that, for Euclidean and metric spaces, our quasi-proximal minimalness coincides with the well-chainedness (chain-connectedness) studied by G. Cantor in 1883. (See Thron [143, p. 29], and also Wilder [147].)

While, from the equivalence of assertions (1) and (3) in Theorem 3.2.1, we can see that, for uniformities and nonvoid relators, our quasi-proximal minimalness coincides with the *well-chainedness* and *proper well-chainedness* studied mainly by Levine [78] and Kurdics, Pataki and Száz [66, 67, 68, 94].

Now, as an immediate consequence of Theorems 3.1.4 and 3.2.1, we can also state

Theorem 3.2.5. *The following assertions are equivalent :*
(1) $\mathcal{R}$ *is quasi-topologically minimal ;* (2) $\mathcal{R}^{\wedge} \subseteq \{X^2\}^{\partial}$ *;* (3) $\mathcal{R}^{\wedge \infty} \subseteq \{X^2\}$.

Remark 3.2.6. By Theorems 3.2.1 and 3.2.5, the relator $\mathcal{R}$ may be naturally called $\square$–*minimal*, for some unary operation $\square$ for relators, if $\mathcal{R}^{\square} \subseteq \{X^2\}$.

Moreover, in particular the relator $\mathcal{R}$ may be naturally called *quasi–*$\square$*–minimal*, for some unary operation $\square$ for relators on X, if it is $\square\infty$–minimal. That is, $\mathcal{R}^{\square \infty} \subseteq \{X^2\}$.

3.3 Further Characterizations of Quasi-Minimal Relators

A simple reformulation of Definition 3.1.1 gives the following

Theorem 3.3.1. *The following assertions hold:*

(1) $\mathcal{R}$ *is quasi-proximally minimal if and only if* $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$*;*

(2) $\mathcal{R}$ *is quasi-topologically minimal if and only if* $\mathcal{F}_{\mathcal{R}} \subseteq \{\emptyset, X\}$*.*

Proof. By Theorem 1.4.3, for any $A \subseteq X$, we have $A \in \tau_{\mathcal{R}}$ if and only if $A^c \in \tau_{\mathcal{R}}$. Hence, it is clear that $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$ if and if $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Therefore, assertion (1) is true. Assertion (2) can be proved quite similarly by using Theorem 1.4.8.

Concerning quasi-proximally minimal relators, we can also easily prove

Theorem 3.3.2. *If* $\emptyset \notin \mathcal{R}$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally minimal;*

(2) $X = A \cup B$ *implies* $A \in \mathrm{Cl}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq X$ *with* $A, B \neq \emptyset$*;*

(3) $A \in \mathrm{Int}_{\mathcal{R}}(B)$ *implies* $B \not\subseteq A$ *for all* $A, B \subseteq X$ *with* $A \neq \emptyset$ *and* $B \neq X$*.*

Proof. If (1) does not hold, then $\tau_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$. Therefore, there exists $A \in \tau_{\mathcal{R}}$ such that $A \neq \emptyset$ and $A \neq X$. Hence, since $A \subseteq A$ and $A \in \mathrm{Int}_{\mathcal{R}}(A)$, it is clear that (3) does not also hold. Therefore, (3) implies (1).

Conversely, if (3) does not hold, then there exist $A, B \subseteq X$ such that $A \neq \emptyset$, $B \neq X$, $B \subseteq A$ and $A \in \mathrm{Int}_{\mathcal{R}}(B)$. Hence, by the definition of the relation $\mathrm{Int}_{\mathcal{R}}$, it is clear that we also have $A \in \mathrm{Int}_{\mathcal{R}}(A)$ and $B \in \mathrm{Int}_{\mathcal{R}}(B)$, and thus $A, B \in \tau_{\mathcal{R}}$. Now, we can already see that $\tau_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$, and thus (1) does not also hold. Therefore, (1) also implies (3).

Namely, if $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$, then because of $A, B \in \tau_{\mathcal{R}}$ we also have $A, B \in \{\emptyset, X\}$. Hence, since $A \neq \emptyset$ and $B \neq X$, we can infer that $A = X$ and $B = \emptyset$. Therefore, because of $A \in \mathrm{Int}_{\mathcal{R}}(B)$, we actually have $X \in \mathrm{Int}_{\mathcal{R}}(\emptyset)$. Thus, there exists $R \in \mathcal{R}$ such that $R[X] \subseteq \emptyset$, and thus $R[X] = \emptyset$. This implies that $R = \emptyset$, and thus $\emptyset \in \mathcal{R}$. And, this is a contradiction.

Now, to complete the proof, it remains only to show that (2) and (3) are also equivalent. For this, note that if for instance (2) does not hold, then there exist $A, B \subseteq X$ such that $A, B \neq \emptyset$, $X = A \cup B$ and $A \notin \mathrm{Cl}_{\mathcal{R}}(B)$. Hence, by using that $\mathrm{Cl}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \mathrm{Int}_{\mathcal{R}}(B^c)$, we can infer that $A \in \mathrm{Int}_{\mathcal{R}}(B^c)$. Moreover, since $B \neq \emptyset$ and $X = A \cup B$, we can also note that $B^c \neq X$ and $B^c \subseteq A$. Therefore, assertion (3) does not also hold. This shows that (3) implies (2).

Remark 3.3.3. Note that the implications (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (1) do not require the extra condition on $\mathcal{R}$ that $\emptyset \notin \mathcal{R}$.

Moreover, if $\mathcal{R}$ is a quasi-proximally minimal relator such that $\emptyset \in \mathcal{R}$, then by the definition of ∞ and Theorem 3.2.1 we necessarily have $\Delta_X = \emptyset^{\infty} \in \mathcal{R}^{\infty} \subseteq \{X^2\}$, and thus $\Delta_X = X^2$. Therefore, X is either the empty set or a singleton. Consequently, in Theorem 3.3.2, instead of $\emptyset \notin \mathcal{R}$ we may also naturally assume that $\mathrm{card}(X) > 1$.

Theorem 3.3.4. *If* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*
(1) $\mathcal{R}$ *is quasi-topologically minimal;*
(2) $X = A \cup B$ *implies* $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *for all* $A, B \subseteq X$ *with* $A, B \neq \emptyset$;
(3) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ *implies* $B \not\subseteq A$ *for all* $A, B \subseteq X$ *with* $A \neq \emptyset$ *and* $B \neq X$.

Proof. If $\mathcal{R} \neq \emptyset$, then by Theorems 3.1.4, 3.3.2 and 1.6.1 it is clear that the following assertions are equivalent:
(a) $\mathcal{R}$ is quasi-topologically minimal;
(b) $\mathcal{R}^{\wedge}$ is quasi-proximally minimal;
(c) $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$ implies $B \not\subseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$;
(d) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ implies $B \not\subseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$.
Therefore, in this case, assertions (1) and (3) are equivalent.

While, if $\mathcal{R} = \emptyset$, then it is clear that $\mathcal{T}_{\mathcal{R}} = \{\emptyset\}$, and thus (1) trivially holds. Moreover, in this case, we can note that $\operatorname{int}_{\mathcal{R}}(B) = \emptyset$. Therefore, if $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$, then $A = \emptyset$. Thus, (3) also trivially holds.

Now, it remains only to show that (2) and (3) are also equivalent. For this, one can recall that $\operatorname{cl}_{\mathcal{R}}(B) = X \setminus \operatorname{int}_{\mathcal{R}}(B^c)$ for all $B \subseteq X$. Therefore, a similar argument as in the proof of Theorem 3.3.2 can be applied.

Remark 3.3.5. Note that in this theorem, instead of $\operatorname{card}(X) > 1$ we may assume that $\emptyset \notin \mathcal{R}^{\wedge}$. That is, there exists $x \in X$ such that for any $R \in \mathcal{R}$ we have $R(x) \not\subseteq \emptyset$, and thus $R(x) \neq \emptyset$.

3.4 Paratopologically Minimal Relators

Analogously to the definition of a minimal topology, a stack (ascending family) $\mathcal{A}$ of subsets of a set X may be naturally called minimal if $\mathcal{A} \subseteq \{X\}$.

Therefore, having in mind the family $\mathcal{E}_{\mathcal{R}}$ of all fat sets generated by a relator $\mathcal{R}$, we may also naturally introduce the following

Definition 3.4.1. The relator $\mathcal{R}$ will be called *paratopologically minimal* if

$$\mathcal{E}_{\mathcal{R}} \subseteq \{X\}.$$

Remark 3.4.2. Note that if $\mathcal{R}$ is non-degenerated in the sense that both X and $\mathcal{R}$ are non-void, then by Theorem 1.8.14 we have $X \in \mathcal{E}_{\mathcal{R}}$. Therefore, in this case, we may write equality instead of inclusion in Definition 3.4.1.

The following theorems will show that paratopological minimalness is a much stronger property than quasi-topological minimalness.

Theorem 3.4.3. *If* $\mathcal{R}$ *is paratopologically minimal, then* $\mathcal{R}$ *is both quasi-proximally and quasi-topologically minimal.*

Proof. By Theorem 1.4.13, we have $\mathcal{T}_\mathcal{R} \setminus \{\emptyset\} \subseteq \mathcal{E}_\mathcal{R}$ for any $\mathcal{R}$. Moreover, if $\mathcal{R}$ is paratopologically minimal, then we also have $\mathcal{E}_\mathcal{R} \subseteq \{X\}$. Therefore, in this case $\mathcal{T}_\mathcal{R} \setminus \{\emptyset\} \subseteq \{X\}$ also holds. Hence, we can infer that $\mathcal{T}_\mathcal{R} \subseteq \{\emptyset, X\}$. Therefore, $\mathcal{R}$ is quasi-topologically minimal. Now, by Theorem 3.1.3, we can see that $\mathcal{R}$ is quasi-proximally minimal too.

From Theorem 3.4.3, we can easily derive the following stronger statement.

Corollary 3.4.4. *If $\mathcal{R}$ is paratopologically minimal, then $\mathcal{R}^\vartriangle$ is also both quasi-proximally and quasi-topologically minimal.*

Proof. By Corollary 1.5.5, we have $\mathcal{E}_{\mathcal{R}^\vartriangle} = \mathcal{E}_\mathcal{R}$ for any $\mathcal{R}$. Therefore, if in particular $\mathcal{R}$ is paratopologically minimal, then the relator $\mathcal{R}^\vartriangle$ is also paratopologically minimal. Thus, by Theorem 3.4.3, it has the required quasi-minimalness properties.

Now, in addition to Corollary 3.4.4, we can also easily prove the following

Theorem 3.4.5. *If $\mathcal{R}$ is non-partial, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is paratopologically minimal;*

(2) *$\mathcal{R}^\vartriangle$ is quasi-proximally minimal;*

(3) *$\mathcal{R}^\vartriangle$ is quasi-topologically minimal.*

Proof. From Corollary 3.4.4, we know that (1) implies (2). Moreover, from Theorem 1.5.7, we know that $\mathcal{R}^{\vartriangle\wedge} = \mathcal{R}^\vartriangle$. Therefore, by Corollary 3.1.6, assertions (2) and (3) are equivalent.

Thus, we need only prove that (3) also implies (1). For this, note that if (3) holds, then by Corollary 1.6.11 and Definition 3.1.1 we have $\mathcal{E}_\mathcal{R} = \mathcal{T}_{\mathcal{R}^\vartriangle} \setminus \{\emptyset\} \subseteq \{\emptyset, X\} \setminus \{\emptyset\} = \{X\}$. Therefore, by Definition 3.4.1, $\mathcal{R}$ is paratopologically minimal.

Now, combining Theorems 3.2.1 and 3.4.5, we can also state

Theorem 3.4.6. *If $\mathcal{R}$ is non-partial, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is paratopologically minimal;*

(2) $\mathcal{R}^\vartriangle \subseteq \{X^2\}^\partial$; (3) $\mathcal{R}^{\vartriangle\infty} \subseteq \{X^2\}$.

Remark 3.4.7. Note that the implications (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) in the above two theorems do not require the relator $\mathcal{R}$ to be non-partial.

Moreover, by Theorem 3.4.6, a non-partial relator $\mathcal{R}$ is paratopologically minimal if and only if it is quasi–$\vartriangle$–minimal in the sense of Remark 3.2.6.

3.5 Further Characterizations of Paratopologically Minimal Relators

By using Definition 3.4.1, we can also easily prove the following

Theorem 3.5.1. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is paratopologically minimal;* (2) $\mathcal{R} \subseteq \{X^2\}$.

Proof. If $R \in \mathcal{R}$, then by Theorem 1.3.17 we have $R(x) \in \mathcal{E}_{\mathcal{R}}$ for all $x \in X$. Hence, if (1) holds, i.e., $\mathcal{E}_{\mathcal{R}} \subseteq \{X\}$, we can infer that $R(x) = X$ for all $x \in X$, and thus $R = X^2$. This shows that either $\mathcal{R} = \emptyset$ or $\mathcal{R} = \{X^2\}$. Therefore, (2) also holds.

Conversely, if (2) holds, then either $\mathcal{R} = \emptyset$ or $\mathcal{R} = \{X^2\}$. Now, if $\mathcal{R} = \emptyset$, then by Theorem 1.3.17 we can see that $\mathcal{E}_{\mathcal{R}} = \emptyset$. While, if $\mathcal{R} = \{X^2\}$, then we can note that $\mathcal{E}_{\mathcal{R}} = \emptyset$ if $X = \emptyset$ and $\mathcal{E}_{\mathcal{R}} = \{X\}$ if $X \neq \emptyset$. Therefore, in both cases, $\mathcal{E}_{\mathcal{R}} \subseteq \{X\}$, and thus (1) also holds.

Remark 3.5.2. By this theorem and Remark 3.2.6, the relator $\mathcal{R}$ is paratopologically minimal if and only if it is $\Box$–minimal with $\Box$ being the identity operation for relators.

Now, analogously to Theorems 3.1.8 and 3.1.10, we can also easily prove

Theorem 3.5.3. *The relator $\mathcal{R}$ is paratopologically minimal if and only if any one of the relators $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$, $\mathcal{R}^\vartriangle$ and $\mathcal{R}^{-1}$ is paratopologically minimal.*

Proof. By Theorem 1.5.7 and Corollary 1.5.5, we have $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^\Box}$ for all $\Box \in \{*, \#, \wedge, \vartriangle\}$. Therefore, by Definition 3.4.1, the paratopological minimalness of $\mathcal{R}$ is equivalent to that of $\mathcal{R}^\Box$.

Moreover, we evidently have $\mathcal{R} \subseteq \{X^2\}$ if and only if $\mathcal{R}^{-1} \subseteq \{X^2\}$. Therefore, by Theorem 3.5.1, the paratopological minimalness of $\mathcal{R}$ is also equivalent to that of $\mathcal{R}^{-1}$.

Remark 3.5.4. Note that $\mathcal{R}^\infty \subseteq \mathcal{R}^\vartriangle$, and thus $\mathcal{E}_{\mathcal{R}^\infty} \subseteq \mathcal{E}_{\mathcal{R}}$. Therefore, if $\mathcal{R}$ is paratopologically minimal, then $\mathcal{R}^\infty$ is also paratopologically minimal.

Moreover, as some useful reformulations of Definition 3.4.1, we can also easily prove the following two theorems.

Theorem 3.5.5. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is paratopologically minimal;* (2) $\mathcal{P}(X) \setminus \{\emptyset\} \subseteq \mathcal{D}_{\mathcal{R}}$.

Proof. If (1) holds, then $\mathcal{E}_{\mathcal{R}} \subseteq \{X\}$. Hence, by using Theorem 1.3.14, we can see that

$$B \neq \emptyset \implies B^c \neq Y \implies B^c \notin \mathcal{E}_{\mathcal{R}} \implies B \in \mathcal{D}_{\mathcal{R}}$$

for all $B \subseteq Y$. Therefore, (2) also holds.

Conversely, if (2) holds, then by using Theorem 1.3.14 we can similarly see that

$$B \in \mathcal{E}_{\mathcal{R}} \implies B^c \notin \mathcal{D}_{\mathcal{R}} \implies B^c = \emptyset \implies B = X.$$

Therefore, $\mathcal{E}_{\mathcal{R}} \subseteq \{X\}$, and thus (1) also holds.

Theorem 3.5.6. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is paratopologically minimal;*

(2) $\operatorname{Int}_{\mathcal{R}}(B) \subseteq \{\emptyset\}$ *for all* $B \subseteq X$ *with* $B \neq X$;

(3) $\mathcal{P}(X) \setminus \{\emptyset\} \subseteq \operatorname{Cl}_{\mathcal{R}}(B)$ *for all* $B \subseteq X$ *with* $B \neq \emptyset$.

Proof. If $S = X^2$, then we can note that $\{S\}^\# = \{\mathcal{S}\}$. Moreover, by using Theorems 3.5.1 and 1.5.4, we can see that

$$(1) \iff \mathcal{R} \subseteq \{S\} \iff \mathcal{R} \subseteq \{S\}^\# \iff \operatorname{Int}_{\mathcal{R}} \subseteq \operatorname{Int}_S.$$

On the other hand, we can also note that $S[A]=\emptyset$ if $A=\emptyset$ and $S[A]=X$ if $A\neq\emptyset$ for all $A\subseteq X$. Hence, by using Definition 1.3.2, we can see that $\operatorname{Int}_S(B)=\{\emptyset\}$ if $B\neq X$ and $\operatorname{Int}_S(B)=\mathcal{P}(X)$ if $B=X$ for all $B\subseteq X$. Therefore,

$$(1) \iff \operatorname{Int}_{\mathcal{R}}\subseteq\operatorname{Int}_S \iff \forall\, B\subseteq X: \ \operatorname{Int}_{\mathcal{R}}(B)\subseteq\operatorname{Int}_S(B)$$
$$\iff \forall\, B\in\mathcal{P}(X)\setminus\{X\}: \ \operatorname{Int}_{\mathcal{R}}(B)\subseteq\{\emptyset\} \iff (2).$$

The equivalence of assertions (2) and (3) can be proved with the help of Theorem 1.3.4.

Now, as an immediate consequence of this theorem, we can also state

Theorem 3.5.7. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically minimal;*

(2) $\operatorname{int}_{\mathcal{R}}(B)=\emptyset$ *for all* $B\subseteq X$ *with* $B\neq X$*;*

(3) $X=\operatorname{cl}_{\mathcal{R}}(B)$ *for all* $B\subseteq X$ *with* $B\neq\emptyset$.

Proof. To prove the equivalence of assertions (2) of Theorems 3.5.6 and 3.5.7, note that for any $B\subseteq X$, we have $\operatorname{int}_{\mathcal{R}}(B)=\emptyset \iff \operatorname{Int}_{\mathcal{R}}(B)\subseteq\{\emptyset\}$.

From this theorem, it is clear that in particular we also have

Corollary 3.5.8. *If in particular* $\mathcal{R}$ *is paratopologically minimal, then* $\mathcal{T}_{\mathcal{R}}\setminus\{X\}\subseteq\{\emptyset\}$, *and thus also* $\tau_{\mathcal{R}}\setminus\{X\}\subseteq\{\emptyset\}$.

Remark 3.5.9. Analogously to the various minimal relators, the corresponding maximal relators can also be naturally introduced.

However, these are certainly less important than the corresponding minimal ones which are generalizations of well-chained (chain-connected) uniformities.

3.6 Quasi-Proximally and Quasi-Topologically Connected Relators

Analogously to the definition of a connected topology, we may naturally introduce the following

Definition 3.6.1. The relator $\mathcal{R}$ will be called

(1) *quasi-proximally connected* if $\tau_{\mathcal{R}}\cap\tau\!\!\!-_{\mathcal{R}}\subseteq\{\emptyset,\,X\}$;

(2) *quasi-topologically connected* if $\mathcal{T}_{\mathcal{R}}\cap\mathcal{F}_{\mathcal{R}}\subseteq\{\emptyset,\,X\}$.

Remark 3.6.2. If in particular $\mathcal{R}\neq\emptyset$, then by Theorems 1.4.5 and 1.4.11 we have

$$\{\emptyset,\,X\}\subseteq\tau_{\mathcal{R}}\cap\tau\!\!\!-_{\mathcal{R}}\subseteq\mathcal{T}_{\mathcal{R}}\cap\mathcal{F}_{\mathcal{R}}.$$

Therefore, in this case, we may write equalities instead of inclusions in the above definition.

By using Definitions 3.1.1 and 3.6.1, we can easily establish the following

Theorem 3.6.3. *If* $\mathcal{R}$ *is quasi-proximally (quasi-topologically) minimal, then* $\mathcal{R}$ *is quasi-proximally (quasi-topologically) connected.*

Proof. If $\mathcal{R}$ is quasi-proximally minimal, then by Definition 3.1.1 $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Thus, in particular we also have $\tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Therefore, by Definition 3.6.1, $\mathcal{R}$ is quasi-proximally connected.

This proves the first statement of the theorem. The second statement can be proved quite similarly.

Now, as an immediate consequence of Theorems 3.4.3 and 3.6.3, we can also state

Corollary 3.6.4. *If $\mathcal{R}$ is paratopologically minimal, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically connected.*

Moreover, analogously to the corresponding results of Section 3.1, we can also prove the following statements.

Theorem 3.6.5. *If $\mathcal{R}$ is quasi-topologically connected, then $\mathcal{R}$ is quasi-proximally connected.*

Proof. By Theorem 1.4.11 and Definition 3.6.1, we have $\tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Therefore, by Definition 3.6.1, $\mathcal{R}$ is quasi-proximally connected.

Theorem 3.6.6. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topologically connected;*

(2) *$\mathcal{R}^{\wedge}$ is quasi-proximally connected.*

Proof. If $\mathcal{R} \neq \emptyset$, then by Corollary 1.6.3 we have $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$ and $\tau\!\!\!-_{\mathcal{R}^{\wedge}} = \mathcal{F}_{\mathcal{R}}$, and thus also $\tau_{\mathcal{R}^{\wedge}} \cap \tau\!\!\!-_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$. Therefore, by Definition 3.6.1, assertions (1) and (2) are equivalent.

While, if $\mathcal{R} = \emptyset$, then from the proof of Theorem 3.1.4 we know that $\mathcal{R}$ is quasi-topologically minimal and $\mathcal{R}^{\wedge}$ is quasi-proximally minimal. Hence, by using Theorem 3.6.3, we can infer that $\mathcal{R}$ is quasi-topologically connected and $\mathcal{R}^{\wedge}$ is quasi-proximally connected.

Corollary 3.6.7. *If $\mathcal{R}$ is topologically fine, then $\mathcal{R}$ is quasi-proximally connected if and only if $\mathcal{R}$ is quasi-topologically connected.*

Theorem 3.6.8. *If $\mathcal{R}$ is proximally simple, then $\mathcal{R}$ is quasi-proximally connected if and only if $\mathcal{R}$ is quasi-topologically connected.*

Proof. From the proof of Theorem 3.1.7, we know that $\tau_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}$. Hence, it is clear that $\tau\!\!\!-_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}}$, and thus also $\tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$. Therefore, by Definition 3.6.1, the required assertion is also true.

Theorem 3.6.9. *The relator $\mathcal{R}$ is quasi-proximally connected if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$ and $\mathcal{R}^{-1}$ is quasi-proximally connected.*

Proof. Recall that, for any $\square \in \{\infty, *, \#\}$, we have $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\square}}$. Hence, it is clear that $\tau\!\!\!-_{\mathcal{R}} = \tau\!\!\!-_{\mathcal{R}^{\square}}$, and thus also $\tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}} = \tau_{\mathcal{R}^{\square}} \cap \tau\!\!\!-_{\mathcal{R}^{\square}}$.

Moreover, we also have $\tau_{\mathcal{R}^{-1}} = \tau\!\!\!-_{\mathcal{R}}$, and thus also $\tau_{\mathcal{R}^{-1}} \cap \tau\!\!\!-_{\mathcal{R}^{-1}} = \tau\!\!\!-_{\mathcal{R}} \cap \tau_{\mathcal{R}} = \tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}}$. Hence, by Definition 3.6.1, it is clear that the required assertion is true.

Remark 3.6.10. From this theorem, for instance, we can see that $\mathcal{R}$ is quasi-proximally connected if and only if any one of the relators $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally connected.

Theorem 3.6.11. *The relator $\mathcal{R}$ is quasi-topologically connected if and only if any one of the relators $\mathcal{R}^*$, $\mathcal{R}^{\#}$ and $\mathcal{R}^{\wedge}$ is quasi-topologically connected.*

Proof. Recall that, for any $\square \in \{*, \#, \wedge\}$, we have $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$. Hence, it is clear that $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\square}}$, and thus also $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}} \cap \mathcal{F}_{\mathcal{R}^{\square}}$. Therefore, by Definition 3.6.1, the required assertion is true.

Remark 3.6.12. From Remark 3.1.11, we know that $\mathcal{T}_{\mathcal{R}^{\infty}} \subseteq \mathcal{T}_{\mathcal{R}}$. Hence, it follows that $\mathcal{F}_{\mathcal{R}^{\infty}} \subseteq \mathcal{F}_{\mathcal{R}}$, and thus also $\mathcal{T}_{\mathcal{R}^{\infty}} \cap \mathcal{F}_{\mathcal{R}^{\infty}} \subseteq \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$. Therefore, if $\mathcal{R}$ is quasi-topologically connected, then $\mathcal{R}^{\infty}$ is also quasi-topologically connected.

3.7 The Main Characterizations of Quasi-Connected Relators

From Theorem 3.6.6, we can see that the properties of quasi-topologically connected relators can, in principle, be immediately derived from those of the quasi-proximally connected ones.

Therefore, it is of major importance to note that, by using the relator

$$\mathcal{R} \vee \mathcal{R}^{-1} = \{ R \cup S^{-1} : \quad R, S \in \mathcal{R} \},$$

we can also prove the following

Theorem 3.7.1. *The following assertions are equivalent :*

(1) *$\mathcal{R}$ is quasi-proximally connected ;*

(2) *$\mathcal{R} \vee \mathcal{R}^{-1}$ is quasi-proximally minimal.*

Proof. By Corollary 1.17.6, we have $\tau_{\mathcal{R} \vee \mathcal{R}^{-1}} = \tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}}$. Thus, by Definitions 3.1.1 and 3.6.1, assertions (1) and (2) are equivalent.

Now, as an immediate consequence of Theorems 3.6.6 and 3.7.1, we can also state

Theorem 3.7.2. *The following assertions are equivalent :*

(1) *$\mathcal{R}$ is quasi-topologically connected ;* (2) *$\mathcal{R}^{\wedge} \vee \mathcal{R}^{\vee}$ is quasi-proximally minimal.*

Proof. From Theorems 3.6.6 and 3.7.1, we can see that

$$\begin{aligned} \mathcal{R} \text{ is quasi-topologically connected } &\iff \mathcal{R}^{\wedge} \text{ quasi-proximally connected} \\ &\iff \mathcal{R}^{\wedge} \vee \left(\mathcal{R}^{\wedge}\right)^{-1} \text{ quasi-proximally minimal.} \end{aligned}$$

Thus, since $\mathcal{R}^{\vee}$ is defined by $\left(\mathcal{R}^{\wedge}\right)^{-1}$, assertions (1) and (2) are also equivalent.

Remark 3.7.3. The latter two theorems show that the properties of the quasi-proximally and quasi-topologically connected relators can, in principle, be also immediately derived from those of the quasi-proximally minimal ones.

The fact that minimalness is a more important notion than connectedness was first established by Kurdics, Pataki and Száz [66, 68, 94] by using well-chainedness instead of minimalness and the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ instead of $\mathcal{R} \vee \mathcal{R}^{-1}$.

Now, from Theorem 3.7.1, by using Theorem 3.2.1, we can easily derive

Theorem 3.7.4. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally connected;*

(2) $\mathcal{R} \vee \mathcal{R}^{-1} \subseteq \{X^2\}^\partial$; (3) $\left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^\infty \subseteq \{X^2\}$;

(4) $\mathcal{R}^\# \cap \left(\mathcal{R}^\#\right)^{-1} \subseteq \{X^2\}^\partial$; (5) $\left(\mathcal{R}^\# \cap \left(\mathcal{R}^\#\right)^{-1}\right)^\infty \subseteq \{X^2\}$.

Proof. To obtain assertions (4) and (5), instead of the equalities

$$\left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^\# = \mathcal{R}^\# \cap \left(\mathcal{R}^{-1}\right)^\# = \mathcal{R}^\# \cap \left(\mathcal{R}^\#\right)^{-1},$$

it is better to use Theorem 3.6.9 and the equivalence of assertions (1), (2) and (3).

Moreover, from Theorem 3.7.2, by using Theorem 3.2.1, we can similarly derive

Theorem 3.7.5. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topologically connected;*

(2) $\mathcal{R}^\wedge \vee \mathcal{R}^\vee \subseteq \{X^2\}^\partial$; (3) $\left(\mathcal{R}^\wedge \vee \mathcal{R}^\vee\right)^\infty \subseteq \{X^2\}$.

Remark 3.7.6. By Theorems 3.7.1, 3.1.8 and 3.7.2, the relator $\mathcal{R}$ may be naturally called quasi-$\square$–connected, for some unary operation $\square$ for relators on X, if the relator $\mathcal{R}^\square \vee \left(\mathcal{R}^\square\right)^{-1}$ is quasi-proximally minimal.

3.8 Further Characterizations of Quasi-Connected Relators

Now, in addition to Theorems 3.7.1 and 3.7.4, we can also prove the following

Theorem 3.8.1. *If $\emptyset \notin \mathcal{R}$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally connected;*

(2) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ *and* $B^c \in \operatorname{Int}_{\mathcal{R}}(A^c)$ *imply* $B \not\subseteq A$ *for all* $A, B \subseteq X$ *with* $A \neq \emptyset$ *and* $B \neq X$;

(3) $X = A \cup B$ *implies that either* $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ *or* $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ *for all* $A, B \subseteq X$ *with* $A, B \neq \emptyset$.

Proof. Clearly, $\emptyset \notin \mathcal{R}$ implies $\emptyset \notin \mathcal{R} \cup \mathcal{R}^{-1}$. Thus, by using Theorems 3.7.1, 3.3.2, 1.17.4 and 1.3.9, we can see that the following assertions are equivalent:

(a) $\mathcal{R}$ is quasi-proximally connected;

(b) $\mathcal{R} \vee \mathcal{R}^{-1}$ is quasi-proximally minimal;

(c) $X = A \cup B$ implies $A \in \operatorname{Cl}_{\mathcal{R} \vee \mathcal{R}^{-1}}(B)$ for all $A, B \subseteq X$ with $A, B \neq \emptyset$;

(d) $X = A \cup B$ implies that either $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ or $A \in \operatorname{Cl}_{\mathcal{R}^{-1}}(B)$ for all $A, B \subseteq X$ with $A, B \neq \emptyset$;

(d) $X = A \cup B$ implies that either $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ or $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ for all $A, B \subseteq X$ with $A, B \neq \emptyset$.

Therefore, assertions (1) and (3) are equivalent.

Now, it remains only to show that (2) and (3) are also equivalent. For this, note that if for instance (2) does not hold, then there exist $A, B \subseteq X$ such that $A \neq \emptyset$, $B \neq X$, $B \subseteq A$, $A \in \operatorname{Int}_{\mathcal{R}}(B)$ and $B^c \in \operatorname{Int}_{\mathcal{R}}(A^c)$. Hence, by using that $\operatorname{Int}_{\mathcal{R}}(B) = \mathcal{P}(X) \setminus \operatorname{Cl}_{\mathcal{R}}(B^c)$, we can infer that $A \notin \operatorname{Cl}_{\mathcal{R}}(B^c)$ and $B^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$.

Moreover, we can also note $B^c \neq \emptyset$ and $X = A \cup B^c$. Therefore, (3) does not also hold. This shows that (3) implies (2).

Remark 3.8.2. By Remark 3.3.3, the implications (3) $\Longleftrightarrow$ (2) $\Longrightarrow$ (1) do not require the extra condition that $\emptyset \notin \mathcal{R}$.

Moreover, analogously to Theorem 3.3.4, we can also prove the following

Theorem 3.8.3. *If* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topologically connected;*

(2) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ *and* $B^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$ *imply* $B \not\subseteq A$ *for all* $A, B \subseteq X$ *with* $A \neq \emptyset$ *and* $B \neq X$*;*

(3) $X = A \cup B$ *implies that either* $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *or* $\operatorname{cl}_{\mathcal{R}}(A) \cap B \neq \emptyset$ *for all* $A, B \subseteq X$ *with* $A, B \neq \emptyset$.

Proof. If $\mathcal{R} \neq \emptyset$, then by Theorems 3.6.6, 3.8.1 and 1.6.1 it is clear that the following assertions are equivalent:

(a) $\mathcal{R}$ is quasi-topologically connected;

(b) $\mathcal{R}^{\wedge}$ is quasi-proximally connected;

(c) $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(B)$ and $B^c \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(A^c)$ imply $B \not\subseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$;

(d) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ and $B^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$ imply $B \not\subseteq A$ for all $A, B \subseteq X$ with $A \neq \emptyset$ and $B \neq X$.

Therefore, in this case, assertions (1) and (2) are equivalent.

While, if $\mathcal{R} = \emptyset$, then it is clear that $\mathcal{T}_{\mathcal{R}} = \{\emptyset\}$, and thus (1) trivially holds. Moreover, in this case, we can note that $\operatorname{int}_{\mathcal{R}}(B) = \emptyset$ for all $B \subseteq Y$. Therefore, if $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ and $B^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$, then $A = \emptyset$ and $B^c = \emptyset$, i.e., $B = X$. Thus, (2) also trivially holds.

Now, it remains only to show that (2) and (3) are also equivalent. For this, one can note that $\operatorname{cl}_{\mathcal{R}}(B) = X \setminus \operatorname{int}_{\mathcal{R}}\big(B^c\big)$ for all $B \subseteq X$. Therefore, a similar argument as in the proof of Theorem 3.3.1 can be applied.

3.9 Relationships Between Quasi-Connectedness and Mild Continuity

Concerning quasi-proximally connected relators, we can also prove the following

Theorem 3.9.1. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally connected;*

(2) $f^{-1} \circ f \notin \mathcal{R}^{\#}$ *for any function* f *of* X *onto* $\{0, 1\}$*;*

(3) $f^{-1} \circ f \notin \mathcal{R}^{\#\infty}$ *for any function* f *of* X *onto* $\{0, 1\}$.

Proof. If (2) does not hold, then there exists a function f of X onto $\{0, 1\}$ such that $f^{-1} \circ f \in \mathcal{R}^{\#}$. Define $A = f^{-1}(0)$ and $E = f^{-1} \circ f$.

Then, since $A = \{x \in X : \ f(x) = 0\}$, it is clear that A is a proper, nonvoid subset of X such that $A^c = \{x \in X : \ f(x) = 1\} = f^{-1}(1)$. Moreover, we can note that E is a relation on X such that

$$E[A] = \left(f^{-1} \circ f\right)[f^{-1}(0)] = f^{-1}\left[f\left[f^{-1}(0)\right]\right] = f^{-1}(0) = A,$$

and quite similarly

$$E[A^c] = \left(f^{-1} \circ f\right)[f^{-1}(1)] = f^{-1}\left[f\left[f^{-1}(1)\right]\right] = f^{-1}(1) = A^c.$$

On the other hand, since $f^{-1} \circ f \in \mathcal{R}^{\#}$, we can also state that there exist $R, S \in \mathcal{R}$ such that $R[A] \subseteq E[A]$ and $S[A^c] \subseteq E[A^c]$.

Hence, since $E[A] = A$ and $E[A^c] = A^c$, we can see that $R[A] \subseteq A$ and $S[A^c] \subseteq A^c$. Therefore, $A, A^c \in \tau_{\mathcal{R}}$, and thus $A \in \tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}}$. Hence, it is clear that $\tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$, and thus (1) does not also hold. Consequently, (1) implies (2).

Conversely, if (1) does not hold, then there exists a proper, nonvoid subset A of X such that $A \in \tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}}$, and thus $A, A^c \in \tau_{\mathcal{R}}$. Therefore, there exist $R, S \in \mathcal{R}$ such that $R[A] \subseteq A$ and $S[A^c] \subseteq A^c$. Now, by defining $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in A^c$, we can see that f is a function of X onto $\{0, 1\}$. Moreover, we can show that $f^{-1} \circ f \in \mathcal{R}^{\#}$, and thus (2) does not also hold. Consequently, (2) also implies (1).

Namely, if $E = f^{-1} \circ f$, then for any $x, y \in X$ we have

$$\begin{aligned}(x, y) \in E \iff y \in E(x) &\iff y \in \left(f^{-1} \circ f\right)(x) \\ &\iff y \in f^{-1}\left(f(x)\right) \iff f(y) = f(x).\end{aligned}$$

Therefore, if $V \subseteq X$, then for any $y \in X$ we have

$$y \in E[V] \iff \exists\ x \in V : \ y \in E(x) \iff \exists\ x \in V : \ f(x) = f(y),$$

and thus

$$E[V] = \begin{cases} \emptyset & \text{if} \quad V = \emptyset, \\ A & \text{if} \quad \emptyset \neq V \subseteq A, \\ A^c & \text{if} \quad \emptyset \neq V \subseteq A^c, \\ X & \text{if} \quad A \cap V \neq \emptyset, \ \ A^c \cap V \neq \emptyset. \end{cases}$$

Hence, we can see that $R[V] = \emptyset = E[V]$ if $V = \emptyset$, $R[A] \subseteq A = E[V]$ if $\emptyset \neq V \subseteq A$ and $S[A^c] \subseteq A^c = E[V]$ if $\emptyset \neq V \subseteq A^c$, and $S[V] \subseteq X = E[V]$ if $A \cap V \neq \emptyset$ and $A^c \cap V \neq \emptyset$. Therefore, $E \in \mathcal{R}^{\#}$, and thus $f^{-1} \circ f \in \mathcal{R}^{\#}$.

Now, to complete the proof, it remains to prove only that assertions (2) and (3) are also equivalent. For this, note that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*}$, and thus in particular $\mathcal{R}^{\#\infty} \subseteq \mathcal{R}^{\#*} = \mathcal{R}^{\#}$. Therefore, $E \notin \mathcal{R}^{\#}$ implies $E \notin \mathcal{R}^{\#\infty}$, and thus (2) implies (3).

Moreover, for any $x, y \in X$, we have $(x, y) \in E$ if and only if $f(x) = f(y)$. Therefore, E is an equivalence relation on X, and thus in particular $E^{\infty} = E$. Hence, it is clear that $E \in \mathcal{R}^{\#}$ implies $E = E^{\infty} \in \mathcal{R}^{\#\infty}$. Therefore, $E \notin \mathcal{R}^{\#\infty}$ implies $E \notin \mathcal{R}^{\#}$, and thus (3) also implies (2).

From this theorem, by using Theorem 3.6.6, we can immediately derive

Theorem 3.9.2. *The following assertions are equivalent :*

(1) *$\mathcal{R}$ is quasi-topologically connected ;*

(2) $f^{-1} \circ f \notin \mathcal{R}^{\wedge}$ *for any function* f *of* X *onto* $\{0, 1\}$ *;*

(3) $f^{-1} \circ f \notin \mathcal{R}^{\wedge\infty}$ *for any function* f *of* X *onto* $\{0, 1\}$.

Proof. By Theorem 3.6.6, $\mathcal{R}$ is quasi-topologically connected if and only if $\mathcal{R}^{\wedge}$ is quasi-proximally connected. That is, by Theorem 3.9.1, $f^{-1} \circ f \notin \mathcal{R}^{\wedge\#}$, resp. $f^{-1} \circ f \notin \mathcal{R}^{\wedge\#\infty}$ for any function f of X onto $\{0, 1\}$. Hence, by using that $\mathcal{R}^{\wedge\#} = \mathcal{R}^{\wedge}$, we can already see that assertions (1), (2) and (3) are also equivalent.

Remark 3.9.3. Because of Theorems 3.9.1 and 3.9.2, the relator $\mathcal{R}$ may be naturally called $\Box$–*connected*, for some unary operation $\Box$ for relators on X, if $f^{-1} \circ f \notin \mathcal{R}^{\Box}$ for any function f of X onto $\{0, 1\}$. Moreover, in particular the relator $\mathcal{R}$ may be naturally called quasi–$\Box$–connected if it is $\Box\infty$–connected.

Hence, by noticing that $f^{-1} \circ f = f^{-1} \circ \Delta_{\{0,1\}} \circ f$, we can see that the relator $\mathcal{R}$ is $\Box$–connected (quasi–$\Box$–connected) if and only if the constant functions of X to $\{0, 1\}$ can be mildly $\Box$–continuous (quasi– $\Box$–continuous) with respect to the relators $\mathcal{R}$ and $\{\Delta_{\{0,1\}}\}$. (Concerning continuity properties, see [135] .)

3.10 Quasi-Hyperconnected Relators

Analogously to the definition of a hyperconnected topology, we may also naturally introduce the following

Definition 3.10.1. The relator $\mathcal{R}$ will be called

(1) *quasi-proximally hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$;

(2) *quasi-topologically hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.

Remark 3.10.2. Thus, $\mathcal{R}$ is quasi-proximally (quasi-topologically) hyperconnected if and only if the family $\tau_{\mathcal{R}} \setminus \{\emptyset\}$ $\big(\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}\big)$ has a certain pairwise intersection property.

Theorem 3.10.3. *If $\mathcal{R}$ is quasi-proximally (quasi-topologically) minimal, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) hyperconnected.*

Proof. If $\mathcal{R}$ is quasi-proximally minimal, then $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$. Hence, we can infer that

$$\tau_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \{\emptyset, X\} \setminus \{\emptyset\} = \begin{cases} \emptyset & \text{if} \quad X = \emptyset\,, \\ \{X\} & \text{if} \quad X \neq \emptyset\,. \end{cases}$$

Therefore, if $A, B \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$, then we necessarily have $X \neq \emptyset$, and moreover $A \cap B = X \cap X = X$. Thus, $\mathcal{R}$ is quasi-proximally hyperconnected.

This proves the first statement of the theorem. The second statement can be proved quite similarly.

From this theorem, by using Theorem 3.4.3, we can immediately derive

Corollary 3.10.4. *If $\mathcal{R}$ is paratopologically minimal, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically hyperconnected.*

Concerning quasi-hyperconnected relators, we can also easily prove the following

Theorem 3.10.5. *If $\mathcal{R}$ is quasi-proximally (quasi-topologically) hyperconnected, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) connected.*

Proof. If $\mathcal{R}$ is not quasi-proximally connected, then $\tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$. Thus, there exists $A \subseteq X$ such that $A \in \tau_{\mathcal{R}}$ and $A \in \tau\!\!\!-_{\mathcal{R}}$, but $A \neq \emptyset$ and $A \neq X$. Hence, by using Theorem 1.4.3, we can infer that $A^c \in \tau_{\mathcal{R}}$ and $A^c \neq \emptyset$. Therefore, $A, A^c \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$ such that $A \cap A^c = \emptyset$. Thus, $\mathcal{R}$ cannot be quasi-proximally hyperconnected.

This proves the first statement of the theorem. The second statement can be proved quite similarly.

Remark 3.10.6. This theorem shows that Theorem 3.6.3 and Corollary 3.6.4 are actually consequences of Theorem 3.10.3 and Corollary 3.10.4.

Now, analogously to Theorems 3.6.5 and 3.6.6, we can also easily prove the following two theorems.

Theorem 3.10.7. *If $\mathcal{R}$ is quasi-topologically hyperconnected, then $\mathcal{R}$ is quasi-proximally hyperconnected.*

Proof. By Theorem 1.4.11, we have $\tau_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ has the binary intersection property, then $\tau_{\mathcal{R}} \setminus \{\emptyset\}$ also has this property. Thus, by Definition 3.10.1, the required assertion is true.

Theorem 3.10.8. *The following assertions are equivalent :*

(1) *$\mathcal{R}$ is quasi-topologically hyperconnected ;*

(2) *$\mathcal{R}^{\wedge}$ is quasi-proximally hyperconnected .*

Proof. If $\mathcal{R} \neq \emptyset$, then by Corollary 1.6.3 we have $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$, and thus also $\tau_{\mathcal{R}^{\wedge}} \setminus \{\emptyset\} = \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, by Definition 3.10.1, assertions (1) and (2) are equivalent.

While, if $\mathcal{R} = \emptyset$, then from the proof of Theorem 3.1.4 we know that $\mathcal{R}$ is quasi-topologically minimal and $\mathcal{R}^{\wedge}$ is quasi-proximally minimal. Hence, by Theorem 3.10.3, we can see that $\mathcal{R}$ is quasi-topologically hyperconnected and $\mathcal{R}^{\wedge}$ is quasi-proximally hyperconnected.

Moreover, analogously to Theorems 3.6.9 and 3.6.11, we can also prove the following two theorems.

Theorem 3.10.9. *The relator $\mathcal{R}$ is quasi-proximally hyperconnected if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$ and $\mathcal{R}^{\#}$ is quasi-proximally hyperconnected.*

Remark 3.10.10. From this theorem, for instance, we can see that $\mathcal{R}$ is quasi-proximally hyperconnected if and only if any one of the relators $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally hyperconnected.

Theorem 3.10.11. *The relator $\mathcal{R}$ is quasi-topologically hyerconnected if and only if any one of the relators $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$ and $\mathcal{R}^{\wedge\infty}$ is quasi-topologically hyperconnected.*

Remark 3.10.12. From Remark 3.1.11, we know that $\mathcal{T}_{\mathcal{R}^\infty} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus $\mathcal{T}_{\mathcal{R}^\infty} \setminus \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if $\mathcal{R}$ is quasi-topologically hyperconnected, then $\mathcal{R}^\infty$ is also quasi-topologically hyperconnected.

From Definition 3.10.1, by using Theorems 1.4.3 and 1.4.8, we can also easily derive the following two theorems.

Theorem 3.10.13. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-proximally hyperconnected;*

(2) *$A \cup B \neq X$ for all $A, B \in \tau_{\mathcal{R}} \setminus \{X\}$;*

(3) *$A \setminus B \neq \emptyset$ for all $A \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \tau_{\mathcal{R}} \setminus \{X\}$.*

Theorem 3.10.14. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topologically hyperconnected;*

(2) *$A \cup B \neq X$ for all $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$;*

(3) *$A \setminus B \neq \emptyset$ for all $A \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$.*

Proof. For instance, if $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$, then by Theorem 1.4.8 we evidently have $A^c, B^c \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if (1) holds, then $A^c \cap B^c \neq \emptyset$ also holds. Hence, since $(A \cup B)^c = A^c \cap B^c$, we can infer that $(A \cup B)^c \neq \emptyset$, and thus $A \cup B \neq X$. Therefore, (1) implies (2).

3.11 Quasi-Ultraconnected Relators

Analogously to the definition of an ultraconnected topology, we may also naturally introduce the following

Definition 3.11.1. The relator $\mathcal{R}$ will be called

(1) *quasi-proximally ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$;

(2) *quasi-topologically ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$.

Remark 3.11.2. Thus, $\mathcal{R}$ is quasi-proximally (quasi-topologically) hyperconnected if and only if the family $\tau_{\mathcal{R}} \setminus \{\emptyset\}$ $\big(\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}\big)$ has a certain pairwise intersection property.

Now, analogously to our former statements on hyperconnected relators, we can also easily prove the following assertions.

Theorem 3.11.3. *If $\mathcal{R}$ is quasi-proximally (quasi-topologically) minimal, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) ultraconnected.*

Corollary 3.11.4. *If $\mathcal{R}$ is paratopologically minimal, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically ultraconected.*

Theorem 3.11.5. *If $\mathcal{R}$ is quasi-proximally (quasi-topologically) ultraconnected, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) connected.*

Proof. If $\mathcal{R}$ is not quasi-proximally connected, then $\tau_{\mathcal{R}} \cap \tau\!\!\!-_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$. Thus, there exists $A \subseteq X$ such that $A \in \tau_{\mathcal{R}}$ and $A \in \tau\!\!\!-_{\mathcal{R}}$, but $A \neq \emptyset$ and $A \neq X$. Hence, by using Theorem 1.4.3, we can infer that $A^c \in \tau\!\!\!-_{\mathcal{R}}$ and $A^c \neq \emptyset$. Therefore, $A, A^c \in \tau\!\!\!-_{\mathcal{R}} \setminus \{\emptyset\}$ such that $A \cap A^c = \emptyset$. Thus, $\mathcal{R}$ cannot be quasi-proximally ultraconnected.

This proves the first statement of the theorem. The second statement can be proved quite similarly.

Theorem 3.11.6. *If $\mathcal{R}$ is quasi-topologically ultraconnected, then $\mathcal{R}$ is quasi-proximally ultraconnected.*

Theorem 3.11.7. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topologically ultraconnected;*

(2) *$\mathcal{R}^{\wedge}$ is quasi-proximally ultraconnected.*

Theorem 3.11.8. *The relator $\mathcal{R}$ is quasi-proximally ultraconnected if and only if any one of the relators $\mathcal{R}^{\infty}$, $\mathcal{R}^{*}$ and $\mathcal{R}^{\#}$ is quasi-proximally ultraconnected.*

Remark 3.11.9. From this theorem, we can see that $\mathcal{R}$ is quasi-proximally connected if, for instance, any one of the relators and $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\infty\#}$ is quasi-proximally ultraconnected.

Theorem 3.11.10. *The relator $\mathcal{R}$ is quasi-topologically ultraconnected if and only if any one of the relators $\mathcal{R}^{*}$, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\wedge\infty}$ is quasi-topologically ultraconnected.*

Remark 3.11.11. From Remark 3.1.11, we know that $\mathcal{T}_{\mathcal{R}^{\infty}} \subseteq \mathcal{T}_{\mathcal{R}}$. Hence, we can infer that $\mathcal{F}_{\mathcal{R}^{\infty}} \subseteq \mathcal{F}_{\mathcal{R}}$, and thus $\mathcal{F}_{\mathcal{R}^{\infty}} \setminus \{\emptyset\} \subseteq \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$. Therefore, if $\mathcal{R}$ is quasi-topologically ultraconnected, then $\mathcal{R}^{\infty}$ is also quasi-topologically ultraconnected.

Theorem 3.11.12. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-proximally ultraconnected;*

(2) *$A \cup B \neq X$ for all $A, B \in \tau_{\mathcal{R}} \setminus \{X\}$;*

(3) *$A \setminus B \neq \emptyset$ for all $A \in \tau\!\!\!-_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \tau_{\mathcal{R}} \setminus \{X\}$.*

Theorem 3.11.13. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topologically ultraconnected;*

(2) *$A \cup B \neq X$ for all $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{X\}$;*

(3) *$A \setminus B \neq \emptyset$ for all $A \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \mathcal{T}_{\mathcal{R}} \setminus \{X\}$.*

Proof. For instance, if (1) holds and $A \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ and $B \in \mathcal{T}_{\mathcal{R}} \setminus \{X\}$, then because of $B^c \in \mathcal{F} \setminus \{\emptyset\}$, we have $A \setminus B = A \cap B^c \neq \emptyset$. Therefore, (1) implies (3).

Remark 3.11.14. This theorem shows that our quasi-topologically ultraconnectedness also extends the *strong connectedness* of Levine [73] studied also by Leuschen and Sims [69].

Namely, it can be easily seen that assertion (2) of Theorem 3.11.13 can be reformulated in the form that $X = A \cup B$, together with $A, B \in \mathcal{T}_{\mathcal{R}}$, implies that either $A = X$ or $B = X$.

Now, in addition to the above theorems, we can also easily prove the following

Theorem 3.11.15. *The following assertions are equivalent :*

(1) $\mathcal{R}$ *is quasi-proximally ultraconnected ;*

(2) $\mathcal{R}^{-1}$ *is quasi-proximally hyperconnected .*

Proof. By Theorem 1.4.4, we have $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$, and thus also $\tau_{\mathcal{R}} \setminus \{\emptyset\} = \tau_{\mathcal{R}^{-1}} \setminus \{\emptyset\}$ for any relator $\mathcal{R}$. Therefore, $\tau_{\mathcal{R}} \setminus \{\emptyset\}$ has the binary intersection property if and only if $\tau_{\mathcal{R}^{-1}} \setminus \{\emptyset\}$ has this property. Thus, by Definition 3.11.1, assertions (1) and (2) are equivalent.

Remark 3.11.16. This theorem shows that, in contrast to the independence of quasi-topological ulraconnectedness and quasi-topological hyperconnedtedness [112, p. 29], the quasi-proximal ultraconnectedness is not completely independent of the quasi-proximal hyperconnectedness.

3.12 Hyperconnected Relators

Because of a reformulation of the definition of a hyperconnected topology mentioned earlier in the introduction, we may also naturally introduce the following

Definition 3.12.1. The relator $\mathcal{R}$ will be called *hyperconnected* if

$$\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}} .$$

Remark 3.12.2. This property can be expressed in a more instructive form that the identity function Δ_X of X is *fatness reversing*.

Therefore, some of the forthcoming results can be greatly generalized according to the ideas of a former paper [141] of Száz.

Theorem 3.12.3. *If $\mathcal{R}$ is hyperconnected, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically hyperconnected.*

Proof. By Theorem 1.4.13 and Definition 3.12.1, we have $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$. Therefore, if $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$, then we have both $A \in \mathcal{E}_{\mathcal{R}}$ and $A \neq \emptyset$. Hence, by using Theorem 1.3.15, we can infer that $A \cap B \neq \emptyset$. Thus, by Definition 3.10.1, $\mathcal{R}$ is quasi-topologically hyperconnected. Now, by Theorem 3.10.7, we can state that $\mathcal{R}$ is also quasi-proximally hyperconnected.

From this theorem, by using Theorem 3.10.5, we can immediately derive

Corollary 3.12.4. *If $\mathcal{R}$ is hyperconnected, then $\mathcal{R}$ is both quasi-proximally and quasi-topologically connected.*

However, as a certain converse to the above results, we can only prove

Theorem 3.12.5. *If $\mathcal{R}$ is paratopologically minimal on a nonvoid set X, then $\mathcal{R}$ is hyperconnected.*

Proof. By Theorem 3.5.5, we have $\mathcal{P}(X)\setminus\{\emptyset\}\subseteq\mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{E}_{\mathcal{R}}\subseteq\mathcal{P}(X)$, it is clear that $\mathcal{E}_{\mathcal{R}}\setminus\{\emptyset\}\subseteq\mathcal{D}_{\mathcal{R}}$ also holds. Moreover, since $X\neq\emptyset$, we can note that $\mathcal{P}(X)\setminus\{\emptyset\}\neq\emptyset$, and thus also $\mathcal{D}_{\mathcal{R}}\neq\emptyset$. Hence, by Theorem 1.8.9, we can see that $\emptyset\notin\mathcal{E}_{\mathcal{R}}$, and thus $\mathcal{E}_{\mathcal{R}}=\mathcal{E}_{\mathcal{R}}\setminus\{\emptyset\}$. Therefore, we actually have $\mathcal{E}_{\mathcal{R}}\subseteq\mathcal{D}_{\mathcal{R}}$, and thus $\mathcal{R}$ is hyperconnected.

Remark 3.12.6. Note that if in particular $\mathcal{R}$ is a relator to $\emptyset$, then because of $\mathcal{R}\subseteq\mathcal{P}(X^2)=\mathcal{P}(\emptyset)=\{\emptyset\}$ we have either $\mathcal{R}=\emptyset$ or $\mathcal{R}=\{\emptyset\}$.

Hence, by Theorems 1.3.17 and 1.3.14, we can see that either $\mathcal{E}_{\mathcal{R}}=\emptyset$ and $\mathcal{D}_{\mathcal{R}}=\{\emptyset\}$, or $\mathcal{E}_{\mathcal{R}}=\{\emptyset\}$ and $\mathcal{D}_{\mathcal{R}}=\emptyset$. Therefore, in the latter case $\mathcal{R}$ is not hyperconnected despite that in both cases it is paratopologically minimal.

By using the corresponding definitions, we can also easily prove the following

Theorem 3.12.7. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is hyperconnected;*

(2) $R(x)\in\mathcal{D}_{\mathcal{R}}$ *for all* $x\in X$ *and* $R\in\mathcal{R}$*;*

(3) $R(x)\cap S(y)\neq\emptyset$ *for all* $x,\,y\in X$ *and* $R,\,S\in\mathcal{R}$.

Proof. Since by Remark 1.3.16 we have $R(x)\in\mathcal{E}_{\mathcal{R}}$ for all $x\in X$ and $R\in\mathcal{R}$, it is clear that assertion (1), i. e., the inclusion $\mathcal{E}_{\mathcal{R}}\subseteq\mathcal{D}_{\mathcal{R}}$, implies (2).

On the other hand, if $A\in\mathcal{E}_{\mathcal{R}}$, then there exists $x\in X$ and $R\in\mathcal{R}$, such that $R(x)\subseteq A$. Moreover, if (2) holds, then we have $R(x)\in\mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{D}_{\mathcal{R}}$ is ascending, we can already infer that $A\in\mathcal{D}_{\mathcal{R}}$ also holds. This shows that $\mathcal{E}_{\mathcal{R}}\subseteq\mathcal{D}_{\mathcal{R}}$, and thus (1) also holds. Therefore, (2) also implies (1).

The equivalence of (2) and (3) can be proved most directly by noticing that, for any $x\in X$ and $R\in\mathcal{R}$, we have

$$R(x)\in\mathcal{D}_{\mathcal{R}}\iff X\subseteq\operatorname{cl}_{\mathcal{R}}\big(R(x)\big)\iff\forall\,y\in X:\ \forall\,S\in\mathcal{R}:\ \ S(y)\cap R(x)\neq\emptyset\,.$$

Remark 3.12.8. According to [116], the relator $\mathcal{R}$ may be called *semi-directed* if (3) holds. Thus, a relator is hyperconnected if and only if it is semi-directed.

Moreover, the relator $\mathcal{R}$ may be called *quasi-directed* if $R(x)\cap S(y)\in\mathcal{E}_{\mathcal{R}}$ holds for all $x,\,y\in X$ and $R,\,S\in\mathcal{R}$. Thus, a non-partial, quasi-directed relator is semi-directed.

From Theorem 3.12.7, we can also immediately derive

Corollary 3.12.9. *If* $\mathcal{R}$ *is hyperconnected, then* $\mathcal{R}$ *is non-partial.*

Proof. By Theorem 3.12.7, we have $R(x)=R(x)\cap R(x)\neq\emptyset$ for all $x\in X$ and $R\in\mathcal{R}$.

Remark 3.12.10. Moreover, for instance if we have $R=\big\{(x,\,y)\in\mathbb{R}^2:\ \ x\leq y\big\}$ and $S=\big\{(x,\,y)\in\mathbb{R}^2:\ \ |x-y|<r\big\}$ for some $r>0$, then by using Theorem 3.12.7 we can also at once see that $\mathcal{R}=\{R\}$ is hyperconnected, but $\mathcal{S}=\{S\}$ is not hyperconnected.

However, it is now more important to note that, by using Theorem 3.12.7 and the plausible notation $\mathcal{R}^{-1}\circ\mathcal{R}=\big\{S^{-1}\circ R:\ \ R,\,S\in\mathcal{R}\big\}$, we can also easily prove some more instructive characterizations of hyperconnected relators.

Theorem 3.12.11. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is hyperconnected;* (2) $X^2 = S^{-1} \circ R$ *for all* $R, S \in \mathcal{R}$*;*

(3) $\mathcal{R}^{-1} \circ \mathcal{R} \subseteq \{X^2\}$*;* (4) $X^2 = \bigcap \mathcal{R}^{-1} \circ \mathcal{R}$.

Proof. Note that, for any $x, y \in X$ and $R, S \in \mathcal{R}$, we have

$$R(x) \cap S(y) \neq \emptyset \iff y \in S^{-1}[R(x)] \iff y \in \big(S^{-1} \circ R\big)(x) \iff (x, y) \in S^{-1} \circ R.$$

Therefore, by Theorem 3.12.7, assertions (1) and (2) are equivalent.

Thus, to complete the proof, it remains only to note that (3) and (4) are only concise reformulations of (2).

Remark 3.12.12. By using the equality $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1}$, assertion (4) can be written in the shorter form that $X^2 = \rho_{\mathcal{R}^{-1} \circ \mathcal{R}}$.

Moreover, by using the cross product of relations [130], assertion (4) can also reformulated in the shorter form that $\Delta_X \in \mathcal{E}_{\mathcal{R} \boxtimes \mathcal{R}}$.

3.13 Further Characterizations of Hyperconnected Relators

Now, analogously to Theorem 3.10.11, we can also easily prove

Theorem 3.13.1. *The relator* $\mathcal{R}$ *is hyperconnected if and only if any one of the relators* $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$ *and* $\mathcal{R}^\vartriangle$ *is hyperconnected.*

Proof. By Theorem 1.5.7 and Corollary 1.5.5, we have $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^\square}$ and $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^\square}$ for all $\square \in \{*, \#, \wedge, \vartriangle\}$. Therefore, by Definition 3.12.1, the required assertion is also true.

However, it is now more important to note that by using Corollary 1.6.9, we can also prove the following

Theorem 3.13.2. *If* $\mathcal{R}$ *is non-partial, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is hyperconnected;*

(2) $\mathcal{R}^\vartriangle$ *is quasi-proximally connected;*

(3) $\mathcal{R}^\vartriangle$ *is quasi-topologically connected.*

Proof. By Definition 3.6.1, assertion (3) is equivalent to the inclusion

(a) $\mathcal{T}_{\mathcal{R}^\vartriangle} \cap \mathcal{F}_{\mathcal{R}^\vartriangle} \subseteq \{\emptyset, X\}$.

Moreover, by Corollary 1.6.9, we can see that inclusion (a) is equivalent to the inclusion

(b) $\big(\mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}\big) \cap \big(\big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}\big) \subseteq \{\emptyset, X\}$.

However, this inclusion can easily be seen to be equivalent to the simplified inclusions

(c) $\mathcal{E}_{\mathcal{R}} \setminus \mathcal{D}_{\mathcal{R}} \subseteq \{\emptyset, X\}$, (d) $\mathcal{E}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{D}_{\mathcal{R}} \cup \{X\}$.

Namely, because of $\mathcal{E}_{\mathcal{R}} \setminus \mathcal{D}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}} \cap \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big)$, assertion (b) trivially implies (c). Moreover, if (b) does not hold, then there exists $A \subseteq X$ such that

$$A \in \big(\mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}\big) \cap \big(\big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}\big),$$

but $A \notin \{\emptyset, X\}$. This implies that $A \in \mathcal{E}_\mathcal{R} \cup \{\emptyset\}$ and $A \in \big(\mathcal{P}(X) \setminus \mathcal{D}_\mathcal{R}\big) \cup \{X\}$, but $A \neq \emptyset$ and $A \neq X$. Hence, we can already infer that $A \in \mathcal{E}_\mathcal{R} \setminus \mathcal{D}_\mathcal{R}$,but $A \notin \{\emptyset, X\}$. Therefore, (c) does not also hold. This shows that (c) also implies (b). Therefore, assertions (b) and (c) are equivalent.

The equivalence of assertions (c) and (d) can be proved even more easily. Namely, if (d) does not hold, then there exists $A \subseteq X$ such that $A \in \mathcal{E}_\mathcal{R} \setminus \{\emptyset\}$, but $A \notin \mathcal{D}_\mathcal{R} \cup \{X\}$. This, implies that $A \in \mathcal{E}_\mathcal{R}$ and $A \neq \emptyset$, but $A \notin \mathcal{D}_\mathcal{R}$ and $A \neq X$. Hence, we can infer that $A \in \mathcal{E}_\mathcal{R} \setminus \mathcal{D}_\mathcal{R}$, but $A \notin \{\emptyset, X\}$.

Therefore, (c) does not also hold. This shows that (c) implies (d). The converse implication can be proved quite similarly.

Now, to complete the proof, it is enough to note only that, since $\mathcal{R}$ is non-partial, we have $\emptyset \notin \mathcal{E}_\mathcal{R}$ and $X \in \mathcal{D}_\mathcal{R}$. Therefore, inclusion (d) is equivalent to the more simple inclusion $\mathcal{E}_\mathcal{R} \subseteq \mathcal{D}_\mathcal{R}$. Thus, assertion (3) is equivalent to (1).

Moreover, by Corollary 3.6.7, assertions (2) and (3) are also equivalent. Namely, the relator $\mathcal{R}^\vartriangle$ is topologically fine in the sense that $\mathcal{R}^{\vartriangle\wedge} = \mathcal{R}^\vartriangle$.

Remark 3.13.3. This theorem shows that the properties of non-partial hyperconnected relators can, in principle, be immediately derived from those of the quasi-proximally connected ones.

For instance, from our former Theorems 3.7.1 and 3.7.4, by using Theorem 3.13.2, we can immediately derive the following

Theorem 3.13.4. *If $\mathcal{R}$ is non-partial, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is hyperconnected;*

(2) *$\mathcal{R}^\vartriangle \vee \mathcal{R}^\triangledown$ is quasi-proximally minimal;*

(3) *$\mathcal{R}^\vartriangle \vee \mathcal{R}^\triangledown \subseteq \{X^2\}^\partial$;*

(4) *$\big(\mathcal{R}^\vartriangle \vee \mathcal{R}^\triangledown\big)^\infty \subseteq \{X^2\}$.*

By using Theorem 3.12.7, and some basic properties of the families $\mathcal{E}_\mathcal{R}$ and $\mathcal{D}_\mathcal{R}$, we can also easily prove the following two theorems.

Theorem 3.13.5. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is hyperconnected;*

(2) *$A^c \notin \mathcal{E}_\mathcal{R}$ for all $A \in \mathcal{E}_\mathcal{R}$;*

(3) *$A \cap B \neq \emptyset$ for all $A, B \in \mathcal{E}_\mathcal{R}$.*

Theorem 3.13.6. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is hyperconnected;*

(2) *$A \in \mathcal{D}_\mathcal{R}$ or $A^c \in \mathcal{D}_\mathcal{R}$ for all $A \subseteq X$;*

(3) *$A \in \mathcal{D}_\mathcal{R}$ or $B \in \mathcal{D}_\mathcal{R}$ whenever $X = A \cup B$.*

Proof. For instance if (3) does not hold, then there exist $A, B \subseteq X$ such that $X = A \cup B$, but $A \notin \mathcal{D}_\mathcal{R}$ and $B \notin \mathcal{D}_\mathcal{R}$. Hence, by using Theorem 1.3.14, we can infer that $A^c \in \mathcal{E}_\mathcal{R}$ and $B^c \in \mathcal{E}_\mathcal{R}$. Moreover, we can also note that $A^c \cap B^c = (A \cup B)^c = X^c = \emptyset$. Therefore, by Theorem 3.13.5, assertions (1) does not also holds. This shows that (1) implies (3).

3.14 Some Particular Theorems on Minimal and Connected Relators

In addition to Theorem 3.4.3, Corollary 3.1.6 and Theorem 3.1.7, we can also prove

Theorem 3.14.1. *If $\mathcal{R}$ is weakly proximal, then the following assertions are equivalent:*
(1) *$\mathcal{R}$ is paratopologically minimal;*
(2) *$\mathcal{R}$ is quasi-proximally minimal;*
(3) *$\mathcal{R}$ is quasi-topologically minimal.*

Proof. From Theorems 3.4.3 and 3.1.3, we know that (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2) even if $\mathcal{R}$ is not supposed to be weakly proximal. Therefore, we need only prove that now (2) also implies (1).

For this, note that if (1) does not hold, then by Theorem 3.5.1 we have $\mathcal{R} \not\subseteq \{X^2\}$. Therefore, there exists $R \in \mathcal{R}$ such that $R \neq X^2$. Thus, there exist $x, y \in X$ such that $(x, y) \notin R$. Hence, we can infer that $y \notin R(x)$, and thus $R(x) \neq X$. Moreover, since $\mathcal{R}$ is weakly proximal, there exists $A \in \tau_{\mathcal{R}}$ such that $x \in A \subseteq R(x)$, and thus $\emptyset \neq A \neq X$. This shows that $\tau_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$, and thus (2) does not also hold. Therefore, (2) implies (1).

Quite similarly, we can also prove the following theorem which will now be rather proved as a consequence of the above theorem.

Theorem 3.14.2. *If $\mathcal{R}$ is topological, then the following assertions are equivalent:*
(1) *$\mathcal{R}$ is paratopologically minimal;*
(2) *$\mathcal{R}$ is quasi-topologically minimal.*

Proof. If $\mathcal{R}$ is topological, then from Theorem 1.10.10 we can see that $\mathcal{R}^{\wedge}$ is proximal. Thus, by Theorem 3.14.1, the following assertions are equivalent:
(a) $\mathcal{R}^{\wedge}$ is paratopologically minimal; (b) $\mathcal{R}^{\wedge}$ is quasi-topologically minimal.
Moreover, from Theorems 3.5.3 and 3.1.10 we can see that (a) is equivalent to (1), and (b) is equivalent to (2). Therefore, (1) and (2) are also equivalent.

Remark 3.14.3. Now, for an easy illustration of Theorems 3.14.2 and 3.5.7, one can note that if in particular $\mathcal{T}$ is a topology on X, then under the notations $\operatorname{int}(A) = \bigcup \mathcal{T} \cap \mathcal{P}(A)$ and $\mathcal{E} = \{ A \subseteq X : \operatorname{int}(A) \neq \emptyset \}$, the following assertions are equivalent:
(1) $\mathcal{E} = \{X\}$; (2) $\mathcal{T} = \{\emptyset, X\}$; (3) $\operatorname{int}(A) = \emptyset$ for $A \in \mathcal{P}(X) \setminus \{X\}$.

However, it is now more important to note that, in addition to Theorem 3.14.2, we can also prove the following

Theorem 3.14.4. *If $\mathcal{R}$ is topological, then the following assertions are equivalent:*
(1) *$\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{D}_{\mathcal{R}}$;*
(2) *$\mathcal{R}$ is hyperconnected;* (3) *$\mathcal{R}$ is quasi-topologically hyperconnected.*

Proof. From Theorem 3.12.3, we know that (2) always implies (3). Moreover, if (2) holds, then by Definition 3.12.1 we have $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$. Hence, by using that $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, we can see that (1) also holds even if $\mathcal{R}$ is not supposed to be topological.

On the other hand, if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 1.9.6, there exists $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subseteq A$. Hence, if (1) holds we can infer that $V \in \mathcal{D}_{\mathcal{R}}$. Now, since $\mathcal{D}_{\mathcal{R}}$ is ascending, we can also state that $A \in \mathcal{D}_{\mathcal{R}}$. Therefore, $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$, and thus (2) also holds.

Quite similarly, if $A, B \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 1.9.6 we can state that there exist $V, W \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such $V \subseteq A$ and $W \subseteq B$. Therefore, if (3) holds, then $V \cap W \neq \emptyset$, and thus $A \cap B \neq \emptyset$ is also true. Now, by Theorem 3.13.5, we can see that (2) also holds.

The following two theorems show that quasi-ultraconnected relators are less important than the quasi-hyperconnected ones.

Theorem 3.14.5. *If $\mathcal{R}$ is T_1–separating and $\operatorname{card}(X) > 1$, then $\mathcal{R}$ is not quasi-topologically ultraconnected.*

Proof. By the assumption, for any $x, y \in X$, with $x \neq y$, there exists $R \in \mathcal{R}$ such that $x \notin R(y)$, and thus $R(y) \cap \{x\} = \emptyset$. Hence, by Definition 1.3.2, we can see that $y \notin \operatorname{cl}_{\mathcal{R}}(\{x\})$, and thus $\operatorname{cl}_{\mathcal{R}}(\{x\}) \subseteq \{x\}$. Therefore, $\{x\} \in \mathcal{F}_{\mathcal{R}}$, and thus also $\{x\} \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ for all $x \in X$. Thus, if $\mathcal{R}$ is quasi-topologically ultraconnected, i. e., the family $\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ has the binary intersection property, then the family $\{\{x\}\}_{x \in X}$ also has the binary intersection property. Therefore, $\{x\} \cap \{y\} \neq \emptyset$ for all $x, y \in X$. Hence, we can infer that X is either the empty set or a singleton, and thus $\operatorname{card}(X) \leq 1$. This contradiction shows that $\mathcal{R}$ cannot be quasi-topologically ultraconnected.

Theorem 3.14.6. *If $\mathcal{R}$ is weakly topological, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topologically ultraconnected;*

(2) $\operatorname{cl}_{\mathcal{R}}(x) \cap \operatorname{cl}_{\mathcal{R}}(y) \neq \emptyset$ *for all* $x, y \in X$;

(3) $\operatorname{cl}_{\mathcal{R}}(A) \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *for all* $\emptyset \neq A, B \subseteq X$.

Proof. By Remark 1.9.4 and Theorem 1.8.3 for any $x \in X$ we have

$$\operatorname{cl}_{\mathcal{R}}(x) = \operatorname{cl}_{\mathcal{R}}(\{x\}) \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}.$$

Moreover, if (1) holds, then the family $\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ has the binary intersection property. Thus, in this case, the family $\{\operatorname{cl}_{\mathcal{R}}(x)\}_{x \in X}$ also has the binary intersection property. Therefore, (2) also holds.

On the other hand, if (2) holds, then by using that $\operatorname{cl}_{\mathcal{R}}(x) = \operatorname{cl}_{\mathcal{R}}(\{x\}) \subseteq \operatorname{cl}_{\mathcal{R}}(A)$ for all $x \in A \subseteq X$, we can at once see that (3) also holds. While, if (3) holds and $x, y \in X$, then by taking $A = \{x\}$ and $B = \{y\}$, we can at once see that (2) also holds.

Therefore, it remains to show only that (2) also implies (1). For this, note that if $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$, then by taking $x \in A$ and $y \in B$, we have

$$\operatorname{cl}_{\mathcal{R}}(x) = \operatorname{cl}_{\mathcal{R}}(\{x\}) \subseteq \operatorname{cl}_{\mathcal{R}}(A) \subseteq A \quad \text{and} \quad \operatorname{cl}_{\mathcal{R}}(y) = \operatorname{cl}_{\mathcal{R}}(\{y\}) \subseteq \operatorname{cl}_{\mathcal{R}}(B) \subseteq B.$$

Moreover, if (2) holds, then $\operatorname{cl}_{\mathcal{R}}(x) \cap \operatorname{cl}_{\mathcal{R}}(y) \neq \emptyset$, and thus $A \cap B \neq \emptyset$. Therefore, (1) also holds.

Remark 3.14.7. Note that the implications (3) $\Longleftrightarrow$ (2) $\Longrightarrow$ (1) do not require any extra assumptions on $\mathcal{R}$.

Moreover, instead of the weak-topologicalness of $\mathcal{R}$, it is enough to assume only that $\mathcal{R}$ is weakly quasi-topological and $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1}$ is non-partial.

3.15 Quasi-Door, Quasi-Superset and Quasi-Submaximal Relators

Analogously to the definition of a door topology by Kelley [61], we may naturally introduce the following

Definition 3.15.1. The relator $\mathcal{R}$ will be called

(1) a *quasi-proximally door relator* if $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$;

(2) a *quasi-topologically door relator* if $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$.

Now, by using this definition, we can easily establish the following two theorems.

Theorem 3.15.2. *The following assertions are equivalent :*

(1) $\mathcal{R}$ *is quasi-proximally door ;*

(2) $\mathcal{P}(X) \setminus \tau_{\mathcal{R}} \subseteq \tau\!\!\!-_{\mathcal{R}}$; (3) $\mathcal{P}(X) \setminus \tau\!\!\!-_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$.

Theorem 3.15.3. *The following assertions are equivalent :*

(1) $\mathcal{R}$ *is quasi-topologically door ;*

(2) $\mathcal{P}(X) \setminus \mathcal{T}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$; (3) $\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.

Proof. To prove the implication (2) $\Longrightarrow$ (1), note that if (2) holds, then we have $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \big(\mathcal{P}(X) \setminus \mathcal{T}_{\mathcal{R}}\big) \subseteq \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$. Therefore, $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$, and thus (1) also holds.

Remark 3.15.4. Now, for instance, we can also easily see that $\mathcal{R}$ is quasi-topologically door if and only if, for any $A \subseteq X$, we have either $A \in \mathcal{T}_{\mathcal{R}}$ or $A^c \in \mathcal{T}_{\mathcal{R}}$.

Namely, if for instance $\mathcal{R}$ is quasi-topologically door, then by Theorem 3.15.3 we have $\mathcal{P}(X) \setminus \mathcal{T}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$. Therefore, if $A \subseteq X$ such that $A \notin \mathcal{T}_{\mathcal{R}}$, then we necessarily we have $A \in \mathcal{F}_{\mathcal{R}}$. Hence, by Theorem 1.4.8, it follows that $A^c \in \mathcal{T}_{\mathcal{R}}$.

Because of a reformulation of the definition of a superset topology by Levine [74], we may also naturally introduce the following

Definition 3.15.5. The relator $\mathcal{R}$ will be called

(1) *quasi-proximally superset relator* if $\mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$;

(2) *quasi-topologically superset relator* if $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.

Thus, we can easily prove the following two theorems.

Theorem 3.15.6. *The following assertions are equivalent :*

(1) $\mathcal{R}$ *is quasi-proximally superset ;*

(2) $\mathcal{E}_{\mathcal{R}} \setminus \tau_{\mathcal{R}} = \emptyset$; (3) $\mathcal{P}(X) = \tau\!\!\!-_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{R}}$;

(4) $\mathcal{P}(X) \setminus \tau\!\!\!-_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$; (5) $\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \subseteq \tau\!\!\!-_{\mathcal{R}}$.

Theorem 3.15.7. *The following assertions are equivalent :*

(1) $\mathcal{R}$ *is quasi-topologically superset ;*

(2) $\mathcal{E}_{\mathcal{R}} \setminus \mathcal{T}_{\mathcal{R}} = \emptyset$; (3) $\mathcal{P}(X) = \mathcal{F}_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{R}}$;

(4) $\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$; (5) $\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Proof. It is clear that the inclusion $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$ is equivalent to the property $\mathcal{E}_{\mathcal{R}} \setminus \mathcal{T}_{\mathcal{R}} = \emptyset$. Therefore, assertions (1) and (2) are equivalent.

Moreover, if (3) does not hold, then there exists $A \subseteq X$ such that $A \notin \mathcal{F}_{\mathcal{R}} \cup \mathcal{D}_{\mathcal{R}}$, and thus $A \notin \mathcal{F}_{\mathcal{R}}$ and $A \notin \mathcal{D}_{\mathcal{R}}$. Hence, by using the equality $A = (A^c)^c$ and Theorems 1.3.14 and 1.4.8, we can infer that $A^c \in \mathcal{E}_{\mathcal{R}}$ and $A^c \notin \mathcal{T}_{\mathcal{R}}$. Therefore, $\mathcal{E}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}(1)$, thus (1) does not also hold. Consequently, (1) implies (3).

The converse implication $(3) \Longrightarrow (1)$ can be proved quite similarly. Therefore, assertions (1) and (3) are also equivalent. Moreover, analogously to Theorem 3.15.3, it is clear that assertions (3), (4) and (5) are also equivalent.

Concerning superset relators, we can also easily prove the following

Theorem 3.15.8. *If $\mathcal{R}$ is non-partial, then the following assertions hold:*

(1) *$\mathcal{R}$ is quasi-proximally superset if and only if $\mathcal{E}_{\mathcal{R}} = \tau_{\mathcal{R}} \setminus \{\emptyset\}$;*

(2) *$\mathcal{R}$ is quasi-topologically superset if and only if $\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.*

Proof. By Theorems 1.4.11 and 1.4.13, we have $\tau_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$ for any relator $\mathcal{R}$. Moreover, if $\mathcal{R}$ is non-partial, then by Theorem 1.8.9 we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$. Therefore, in this case, $\mathcal{R}$ is quasi-proximally (quasi-topologically) superset if and only if $\mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}} \setminus \{\emptyset\}$ $\big(\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}\big)$.

Analogously to the definition of a submaximal topology by Bourbaki [12], we may also naturally introduce the following

Definition 3.15.9. The relator $\mathcal{R}$ will be called

(1) *quasi-proximally submaximal* if $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$;

(2) *quasi-topologically submaximal* if $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.

Thus, analogously to Theorems 3.15.6 and 3.15.7, we can also easily prove the following two theorems.

Theorem 3.15.10. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-proximally submaximal;*

(2) $\mathcal{D}_{\mathcal{R}} \setminus \tau_{\mathcal{R}} = \emptyset$; (3) $\mathcal{P}(X) = \tau\!\!\!-_{\mathcal{R}} \cup \mathcal{E}_{\mathcal{R}}$;

(4) $\mathcal{P}(X) \setminus \tau\!\!\!-_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$; (5) $\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}} \subseteq \tau\!\!\!-_{\mathcal{R}}$.

Theorem 3.15.11. *The following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topologically submaximal;*

(2) $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{T}_{\mathcal{R}} = \emptyset$; (3) $\mathcal{P}(X) = \mathcal{F}_{\mathcal{R}} \cup \mathcal{E}_{\mathcal{R}}$;

(4) $\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$; (5) $\mathcal{P}(X) \setminus \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.

3.16 Relationships Among Door, Superset and Submaximality

Now, in contrast to Theorems 3.1.3, 3.6.5, 3.10.7 and 3.11.6, we have the following

Theorem 3.16.1. *If $\mathcal{R}$ is quasi-proximally door, superset, resp. submaximal, then $\mathcal{R}$ is quasi-topologically door, superset, resp. submaximal.*

Proof. By Theorem 1.4.11, we have $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$ and $\tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$ for any $\mathcal{R}$. Hence, by the corresponding definitions, it is clear that the required implications are true.

For instance, if $\mathcal{R}$ is quasi-proximally door, then by Definition 3.15.1 and the above observation, we have $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$, and thus also $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$. Therefore, by Definition 3.15.1, $\mathcal{R}$ is a quasi-topologically door relator.

Theorem 3.16.2. *If $\mathcal{R}$ is nonvoid and quasi-proximally (quasi-topologically) door, then $\mathcal{R}$ is quasi-proximally (quasi-topologically) submaximal.*

Proof. Suppose first that $\mathcal{R}$ is quasi-topologically door and $A \in \mathcal{D}_{\mathcal{R}}$. Then, by the corresponding definitions, we have $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$ and $X = \operatorname{cl}_{\mathcal{R}}(A)$.

Now, if $A \notin \mathcal{T}_{\mathcal{R}}$, then because of $X \in \mathcal{T}_{\mathcal{R}}$ we can state that $A \neq X$. Moreover, because of $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$, we can state that $A \in \mathcal{F}_{\mathcal{R}}$, and thus $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$. Hence, since $X = \operatorname{cl}_{\mathcal{R}}(A)$, we can infer that $X \subseteq A$, and thus $A = X$. This contradiction proves that $A \in \mathcal{T}_{\mathcal{R}}$. Therefore, $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus $\mathcal{R}$ is quasi-topologically submaximal.

Next, suppose that $\mathcal{R}$ is quasi-proximally door and $A \in \mathcal{D}_{\mathcal{R}}$. Then, by Definitions 3.15.1, 1.3.2 and Theorem 1.3.10, we have $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$ and $X = R^{-1}[A]$ for all $R \in \mathcal{R}$.

Now, if $A \notin \tau_{\mathcal{R}}$, then because of $X \in \tau_{\mathcal{R}}$ we can state that $A \neq X$. Moreover, because of $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$, we can state that $A \in \tau\!\!\!-_{\mathcal{R}}$. Hence, by using Theorem 1.4.4, we can infer that $A \in \tau_{\mathcal{R}^{-1}}$. Therefore, by Definitions 1.4.1 and 1.3.2, there exists $R \in \mathcal{R}$ such that $R^{-1}[A] \subseteq A$. Hence, since $X = R^{-1}[A]$, we can infer that $X \subseteq A$, and thus $A = X$. This contradiction proves that $A \in \tau_{\mathcal{R}}$. Therefore, $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$, and thus $\mathcal{R}$ is quasi-proximally submaximal.

Remark 3.16.3. Note that if $\mathcal{R}$ is quasi-proximally door, then because of $\tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}} = \mathcal{P}(X)$ and $\tau_{\emptyset} \cup \tau\!\!\!-_{\emptyset} = \emptyset$ we necessarily have $\mathcal{R} \neq \emptyset$.

While, if $\mathcal{R}$ is quasi-topologically door, then by using $\mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}} = \mathcal{P}(X)$ and $\mathcal{T}_{\emptyset} \cup \mathcal{F}_{\emptyset} = \{\emptyset, X\}$ we can only prove $\mathcal{R} \neq \emptyset$ if $\operatorname{card}(X) > 1$.

Theorem 3.16.4. *If $\mathcal{R}$ is hyperconnected, quasi-proximally (quasi-topologically) submaximal, then*

(1) *$\mathcal{R}$ is quasi-proximally (quasi-topologically) door;*

(2) *$\mathcal{R}$ is quasi-proximally (quasi-topologically) superset.*

Proof. By Definitions 3.12.1 and 3.15.9, we have $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$ $(\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}})$. Therefore, $\mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$ $(\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}})$, and thus assertion (2) is true. Therefore, actually we need only prove assertion (1).

For this, suppose that $\mathcal{R}$ is hyperconnected, quasi-proximally submaximal. Then, by Definitions 3.12.1 and 3.15.9, we have $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$. Now, if $A \in \mathcal{D}_{\mathcal{R}}$, then because of $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$, we also have $A \in \tau_{\mathcal{R}}$. While, if $A \in \mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}$, then by Theorem 1.3.14 we have $A^c \in \mathcal{E}_{\mathcal{R}}$. Hence, by using that $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$, we can infer that $A^c \in \mathcal{D}_{\mathcal{R}}$. Thus, again by $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$, we also have $A^c \in \tau_{\mathcal{R}}$. Hence, by Theorem 1.4.8, we can infer that $A \in \tau\!\!\!-_{\mathcal{R}}$. Therefore, in both cases, we have $A \in \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$. This proves that $\mathcal{P}(X) \subseteq \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$, and thus $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \tau\!\!\!-_{\mathcal{R}}$. Therefore, $\mathcal{R}$ is quasi-proximally door.

Thus, we have proved the first statement of (1). The second statement of (1) can be proved quite similarly.

Now, as an immediate consequence of Theorems 3.16.2 and 3.16.4, we can also state

Corollary 3.16.5. *If $\mathcal{R}$ is nonvoid and hyperconnected, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-proximally (quasi-topologically) door ;*

(2) *$\mathcal{R}$ is quasi-proximally (quasi-topologically) submaximal.*

Remark 3.16.6. Note that the implication (2) $\Longrightarrow$ (1) does not require the extra condition that $\mathcal{R} \neq \emptyset$. However, $\mathcal{D}_{\emptyset} = \mathcal{P}(X)$, but $\mathcal{T}_{\emptyset} = \{\emptyset\}$. Therefore, $\emptyset$ is a topologically submaximal relator if and only if $X = \emptyset$.

Concerning quasi-topologically superset relators, we can also easily prove the following two theorems.

Theorem 3.16.7. *If $\mathcal{R}$ is quasi-topologically superset, then $\mathcal{R}$ is strongly quasi-topological.*

Proof. Now, for any $x \in X$ and $R \in \mathcal{R}$, we have $R(x) \in \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, by Remark 1.9.4, the required assertion is true.

Theorem 3.16.8. *If $\mathcal{R}$ is quasi-topologically filtered and quasi-topologically superset such that $\mathcal{R}$ is not quasi-topologically maximal, then $\mathcal{R}$ is quasi-topologically hyperconnected.*

Proof. Assume on the contrary that $\mathcal{R}$ is not quasi-topologically hyperconnected. Then by Definition 3.10.1, there exist $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $A \cap B = \emptyset$. Thus, for any $x \in X$, we have $\{x\} = \big(A \cup \{x\} \big) \cap \big(B \cup \{x\} \big)$.

Moreover, because of $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, we also have $A, B \in \mathcal{E}_{\mathcal{R}}$. Thus, since $\mathcal{E}_{\mathcal{R}}$ is ascending, we can also state that $A \cup \{x\}$, $B \cup \{x\} \in \mathcal{E}_{\mathcal{R}}$. Hence, by using that $\mathcal{R}$ is quasi-topologically superset, and thus $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we infer that $A \cup \{x\}$, $B \cup \{x\} \in \mathcal{T}_{\mathcal{R}}$. Now, since $\mathcal{R}$ is quasi-topologically filtered, and thus $\mathcal{T}_{\mathcal{R}}$ is closed under binary intersection, we can also state that $\big(A \cup \{x\} \big) \cap \big(B \cup \{x\} \big) \in \mathcal{T}_{\mathcal{R}}$. Therefore, $\{x\} \in \mathcal{T}_{\mathcal{R}}$, and thus $\{x\} \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.

Hence, by using that $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, we can infer that $\{x\} \in \mathcal{E}_{\mathcal{R}}$. Now, since $x \in X$ was arbitrary and $\mathcal{E}_{\mathcal{R}}$ is ascending, it is clear that $\mathcal{P}(X) \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$. Hence, by using again that $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can infer that $\mathcal{P}(X) \setminus \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, we actually have $\mathcal{T}_{\mathcal{R}} = \mathcal{P}(X)$, and thus $\mathcal{R}$ is quasi-topologically maximal. This contradiction proves that $\mathcal{R}$ is quasi-topologically hyperconnected.

3.17 Resolvable and Irresolvable Relators

Because of a reformulation of the definition of a resolvable topology by Hewitt [51], we may also naturally introduce the following

Definition 3.17.1. The relator $\mathcal{R}$ will be called *resolvable* if

$$\mathcal{D}_{\mathcal{R}} \not\subseteq \mathcal{E}_{\mathcal{R}}.$$

The importance of this definition can easily be clarified by the following

Example 3.17.2. If X and $\mathcal{R}$ are as in Example 2.8.6, then $\mathcal{R}$ is a resolvable tolerance relator.

To prove the resolvability of $\mathcal{R}$, note that $R_n(x) =]\, x - n^{-1}, \ x + n^{-1} \,[$ for all $x \in X$ and $n \in \mathbb{N}$. Moreover, recall that every nonvoid, open interval in $\mathbb{R}$ contains both rational and irrational numbers. Therefore, $\mathbb{Q} \in \mathcal{D}_{\mathcal{R}}$ and $\mathbb{Q} \notin \mathcal{E}_{\mathcal{R}}$ and thus $\mathbb{Q} \in \mathcal{D}_{\mathcal{R}} \not\subseteq \mathcal{E}_{\mathcal{R}}$.

By using Theorem 1.3.14, Definition 3.17.1 can be reformulated in the following

Theorem 3.17.3. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is resolvable;*

(2) $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} \neq \emptyset$; (3) *there exists* $A \in \mathcal{D}_{\mathcal{R}}$ *such that* $A^c \in \mathcal{D}_{\mathcal{R}}$.

Now, by calling the relator $\mathcal{R}$ *irresolvable* if it is not resolvable, we can also easily establish the following

Theorem 3.17.4. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is irresolvable;* (2) $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}$; (3) $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} = \emptyset$.

Hence, by Definition 3.12.1, it is clear that in particular we also have

Corollary 3.17.5. *The following assertions are equivalent:*

(1) $\mathcal{D}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}}$; (2) $\mathcal{R}$ *is irresolvable and hyperconnected.*

Moreover, by using Theorem 3.17.4 and Definitions 3.15.5 and 3.15.9, we can also easily establish the following

Theorem 3.17.6. *If* $\mathcal{R}$ *is irresolvable, quasi-proximally (quasi-topologically) superset, then* $\mathcal{R}$ *is quasi-proximally (quasi-topologically) submaximal.*

Proof. By Theorem 3.17.4 and Definition 3.15.5, we have

$$\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}} \text{ and } \mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}} \ (\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}).$$

Therefore, $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$ ($\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$), thus by Definition 3.15.9 the required assertion is true.

Now, by using Definition 3.17.1 and Theorem 3.17.3, we can also easily prove following counterpart of Theorem 3.13.6.

Theorem 3.17.7. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is irresolvable;*

(2) $A^c \notin \mathcal{D}_{\mathcal{R}}$ *for all* $A \in \mathcal{D}_{\mathcal{R}}$; (3) $A \cap B \neq \emptyset$ *for all* $A, B \in \mathcal{D}_{\mathcal{R}}$.

Proof. If (2) does not hold, then there exists $A \in \mathcal{D}_{\mathcal{R}}$ such that $A^c \in \mathcal{D}_{\mathcal{R}}$. Therefore, by Theorem 3.17.3, $\mathcal{R}$ is resolvable, and thus (1) does not hold. Consequently, (1) implies (2).

While, if (3) does not hold, then there exists $A, B \in \mathcal{D}_{\mathcal{R}}$ such that $A \cap B = \emptyset$. Hence, we can infer that $B \subseteq A^c$, and thus $A^c \in \mathcal{D}_{\mathcal{R}}$. Therefore, (2) does not also hold. Consequently, (2) implies (3).

Finally, if (1) does not hold, then by Theorem 3.17.3 there exists $A \in \mathcal{D}_{\mathcal{R}}$ such that $A^c \in \mathcal{D}_{\mathcal{R}}$. Thus, since $A \cap A^c = \emptyset$, assertion (3) does not also hold. Consequently, (3) also implies (1).

Moreover, analogously to Theorems 3.13.6 and 3.13.1, we can also easily prove the following two theorems.

Theorem 3.17.8. *The following assertions are equivalent :*

(1) *$\mathcal{R}$ is irresolvable ;*

(2) *$A \in \mathcal{E}_{\mathcal{R}}$ or $A^c \in \mathcal{E}_{\mathcal{R}}$ for all $A \subseteq X$;*

(3) *$A \in \mathcal{E}_{\mathcal{R}}$ or $B \in \mathcal{E}_{\mathcal{R}}$ whenever $X = A \cup B$.*

Theorem 3.17.9. *The relator $\mathcal{R}$ is resolvable (irresolvable) if and only if any one of the relators $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$ and $\mathcal{R}^\triangle$ is resolvable (irresolvable).*

3.18 A Few Illustrating Examples and a Diagram

The following example, given by Pataki [94], will show that even a very particular quasi-proximally minimal relator need not be topologically minimal. Thus, the converse of Theorem 3.1.3 is not true.

Example 3.18.1. If $X = \{1, 2, 3\}$ and $R_1, R_2 \subseteq X^2$ such that

$R_1(1) = X$, $R_1(2) = \{1, 2\}$, $R_1(3) = \{1, 3\}$,

$R_2(1) = \{1, 2\}$, $R_2(2) = X$, $R_2(3) = \{2, 3\}$,

then $\mathcal{R} = \{R_1, R_2\}$ is a tolerance relator on X such that :

(1) $\mathcal{R}$ is quasi-proximally minimal ;

(2) $\mathcal{R}$ is both irresolvable and hyperconnected ;

(3) $\mathcal{R}$ is neither paratopologically nor quasi-topologically minimal ;

(4) $\mathcal{R}$ is neither quasi-proximally nor quasi-topologically door, superset and submaximal ;

(5) $\mathcal{R}$ is both quasi-proximally and quasi-topologically connected, hyperconnected and ultraconnected.

It can be easily seen that R_1 and R_2 are reflexive and symmetric relations on X. Therefore, $\mathcal{R}$ is a tolerance relator on X. Moreover, by using Theorem 3.12.7, we can easily see that $\mathcal{R}$ is hyperconnected. Thus, by Corollary 3.12.4 and Theorem 3.12.3, $\mathcal{R}$ is both quasi-proximally and quasi-topologically connected and hyperconnected.

On the other hand, by using Theorem 1.4.7, we can easily see that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, \{1, 2\}, X\}$. Therefore, $\mathcal{T}_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$, and thus $\mathcal{R}$ is not quasi-topologically minimal. Hence, by Theorem 3.4.3, it follows that $\mathcal{R}$ is also not paratopologically minimal. (This is also immediate from the fact that $\{1, 2\} = R_1(2) \in \mathcal{E}_{\mathcal{R}}$.)

Now, by using Theorem 1.4.8, we can also note that $\mathcal{F}_{\mathcal{R}} = \{\emptyset, \{3\}, X\}$. Therefore, $\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ also has the binary intersection property, and thus $\mathcal{R}$ is quasi-topologically ultraconnected. Hence, by Theorem 3.11.6, it follows that $\mathcal{R}$ is quasi-proximally ultraconnected.

On the other hand, concerning the set $A = \{1, 2\}$, we can also easily see that

$$R_i[A] = R_i(1) \cup R_i(2) = X \not\subseteq A$$

for all $i = 1, 2$, and thus $A \notin \tau_{\mathcal{R}}$. Hence, by using that $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can already infer that $\tau_{\mathcal{R}} = \{\emptyset, X\}$, and thus $\mathcal{R}$ is quasi-proximally minimal.

Now, we can also note that

$$\mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}} = \{\emptyset, \{3\}, \{1, 2\}, X\} \neq \mathcal{P}(X).$$

Therefore, $\mathcal{R}$ is not quasi-topologically door. Moreover, by using Theorems 1.3.17 and 1.3.14, we can also easily see that

$$\mathcal{E}_{\mathcal{R}} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, X\} = \mathcal{D}_{\mathcal{R}}.$$

Therefore, $\mathcal{E}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}$, and thus $\mathcal{R}$ is not also quasi-topologically superset and submaximal. Hence, by Theorem 3.16.1, we can see that $\mathcal{R}$ is also not quasi-proximally door, superset and submaximal. Moreover, since $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} = \emptyset$, we can also state that $\mathcal{R}$ is not resolvable.

Remark 3.18.2. In connection with the above relator $\mathcal{R}$, it is also noteworthy that

$$(R_i \circ R_j)(x) = R_i[R_j(x)] = \bigcup_{y \in R_j(x)} R_i(y) = X$$

for all $x \in X$ and $i, j = 1, 2$. Therefore, $R_i \circ R_j = X^2$ for all $i = 1, 2$, and thus

$$\mathcal{R} \circ \mathcal{R} = \{R \circ S : \ R, S \in \mathcal{R}\} = \{X^2\}.$$

Hence, in particular we can see that $\mathcal{R}^2 = \{R^2 : \ R \in \mathcal{R}\} = \{X^2\}$, and thus $\mathcal{R}$ is 2–well-chained in a natural sense.

Moreover, if $\mathcal{R}$ is as in Example 3.18.1, then by Theorem 3.1.7 and Corollary 3.1.6, it is clear that $\mathcal{R}$ cannot be proximally simple and topologically fine. However, by using direct arguments, we can prove some much better assertions.

Example 3.18.3. If $\mathcal{R}$ is as in Example 3.18.1, then

(1) $\mathcal{R}$ is not uniformly, proximally and topologically simple;

(2) $\mathcal{R}$ is quasi-proximally, quasi-topologically and paratopologically simple.

Now, by using the preorder relations $U = X^2$ and $V = A^2 \cup (A^c \times X)$ with $A = \{1, 2\}$, we can easily see that

$$\tau_{\mathcal{R}} = \{\emptyset, X\} = \tau_{\{U\}} \qquad \text{and} \qquad \mathcal{T}_{\mathcal{R}} = \{\emptyset, A, X\} = \mathcal{T}_{\{V\}}.$$

Hence, by using Theorem 1.5.4, we can already infer that

$$\mathcal{R}^{\#\infty} = \{U\}^{\#\infty} \qquad \text{and} \qquad \mathcal{R}^{\wedge\infty} = \{V\}^{\wedge\infty}.$$

Therefore, $\mathcal{R}$ is both quasi-proximally and quasi-topologically simple.

Moreover, if W is a relation on X such that

$$W(1) = \{1, 2\}, \qquad W(2) = \{2, 3\}, \qquad W(3) = \{1, 3\},$$

then by using Theorem 1.3.17 we can easily see that $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\{W\}}$. Hence, by using Theorem 1.5.4, we can infer that $\mathcal{R}^{\triangle} = \{W\}^{\triangle}$. Therefore, $\mathcal{R}$ is also paratopologically simple.

Next, we show that $\mathcal{R}$ is not topologically simple. For this, assume on the contrary that $\mathcal{R}$ is topologically simple. Then, there exists a relation S on X such that $\mathcal{R}^{\wedge} = \{S\}^{\wedge}$. Hence, by using that $\wedge$ is extensive, we can infer that $R_1, R_2 \in \{S\}^{\wedge}$ and $S \in \mathcal{R}^{\wedge}$. Thus, in particular by the definition of $\wedge$, we have both $S(3) \subseteq R_1(3)$ and $S(3) \subseteq R_2(3)$, and moreover either $R_1(3) \subseteq S(3)$ or $R_2(3) \subseteq S(3)$. Hence, by using that $R_1(3) = \{1, 3\}$ and $R_2(3) = \{2, 3\}$, we can infer that either $\{1, 3\} \subseteq \{3\}$ or $\{2, 3\} \subseteq \{3\}$. This contradiction shows that $\mathcal{R}$ cannot be topologically simple.

Hence, it is clear that $\mathcal{R}$ cannot also be $\Box$–simple for any operation $\Box$ for relators with $\Box \wedge = \wedge$. Thus, in particular, $\mathcal{R}$ cannot also be uniformly and proximally simple.

Remark 3.18.4. Concerning the relator $\mathcal{R}$, considered in Example 3.18.1, we can also note that $X^2 \notin \mathcal{R}$, and thus $\mathcal{R}$ cannot be $\Box$–fine for any operation $\Box$ for relators with $X^2 \in \mathcal{R}^{\Box}$.

Recall that the relator considered in Example 3.18.1 is quasi-topologically connected. Therefore, to see that the converse of Theorem 3.6.5 is also not true, we have to consider another example.

The following somewhat more difficult example, given also by Pataki [94], will show that even a very particular quasi-proximally connected relator need not be quasi-topologically connected.

Example 3.18.5. If $X = \{i\}_{i=1}^{4}$ and $R_i \subseteq X^2$ for all $i \in X$ such that

$$\begin{array}{llll} R_1(1) = \{1, 2\}, & R_1(2) = X, & R_1(3) = R_1(4) = \{2, 3\ 4\}, & \\ R_2(1) = X, & R_2(2) = \{1, 2\}, & R_2(3) = R_2(4) = \{1, 3, 4\}, & \\ R_3(1) = R_3(2) = \{1, 2, 4\}, & & R_3(3) = \{3, 4\}, & R_3(4) = X, \\ R_4(1) = R_4(2) = \{1, 2, 3\}, & & R_4(3) = X, & R_4(4) = \{3, 4\}, \end{array}$$

then $\mathcal{R} = \{R_i\}_{i=1}^{4}$ is a tolerance relator on X such that:

(1) $\mathcal{R}$ is not resolvable, hyperconnected and paratopologically minimal;
(2) $\mathcal{R}$ is quasi-proximally minimal, connected, hyperconnected and ultraconnected;
(3) $\mathcal{R}$ is neither quasi-proximally nor quasi-topologically door, superset and submaximal;
(4) $\mathcal{R}$ is not quasi-topologically minimal, connected, hyperconnected and ultraconnected.

It can again be easily seen that each R_i is a reflexive and symmetric relation on X. Therefore, $\mathcal{R}$ is a tolerance relator on X.

Moreover, we can at once see that $R_1(1) \cap R_3(3) = \emptyset$. Therefore, by Theorem 3.12.7, we can state that $\mathcal{R}$ is not hyperconnected. Hence, by using Theorem 3.12.5, we can infer that $\mathcal{R}$ is not paratopologically minimal. (This statement is also immediate from the fact that $\{1, 2\} = R_1(1) \in \mathcal{E}_{\mathcal{R}}$.)

On the other hand, by using Theorems 1.4.7 and 1.4.8, we can see that

$$\mathcal{T}_{\mathcal{R}} = \{\emptyset, \{1, 2\}, \{3, 4\}, X\} = \mathcal{F}_{\mathcal{R}}.$$

Therefore, $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}} \not\subseteq \{\emptyset, X\}$, and thus $\mathcal{R}$ is not quasi-topologically connected. Hence, by using Theorems 3.6.3, 3.10.5 and 3.11.5, we can infer that $\mathcal{R}$ is also not quasi-topologically minimal, hyperconnected and ultraconnected. (The latter statements are now also quite obvious by the corresponding definitions.)

On the other hand, concerning the sets $A = \{1, 2\}$ and $B = \{3, 4\}$ we can also easily see that $R_i[A] = R_i(1) \cup R_i(2) \not\subseteq A$ and $R_i[B] = R_i(3) \cup R_i(4) \not\subseteq B$ for all $i \in X$, and thus $A, B \notin \tau_{\mathcal{R}}$. Hence, by using that $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can infer that $\tau_{\mathcal{R}} = \{\emptyset, X\}$, and thus $\mathcal{R}$ is quasi-proximally minimal. Hence, by using Theorems 3.6.3, 3.10.3 and 3.11.3, we can infer that $\mathcal{R}$ is also quasi-proximally connected, hyperconnected and ultraconnected. (The latter statements are now also quite obvious by the corresponding definitions.)

Now, we can also note that $\mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}} \neq \mathcal{P}(X)$. Therefore, $\mathcal{R}$ is not a quasi-topologically door relator. Moreover, by Theorems 1.3.17 and 1.3.14, we can easily see that

$$\mathcal{E}_{\mathcal{R}} = \{\{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X\}$$
$$\text{and } \mathcal{D}_{\mathcal{R}} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X\}.$$

Therefore, $\mathcal{E}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}} \not\subseteq \mathcal{T}_{\mathcal{R}}$, and thus $\mathcal{R}$ is not also a quasi-topologically superset and submaximal relator. Hence, by Theorem 3.16.1, we can see that $\mathcal{R}$ is also not a quasi-proximally door, superset and submaximal relator. Moreover, since $\mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}} = \emptyset$, we can also state that $\mathcal{R}$ is not resolvable.

Remark 3.18.6. In connection with the above relator $\mathcal{R}$, it is also noteworthy that

$$\mathcal{R}^2 = \{R^2 : R \in \mathcal{R}\} = \{X^2\}, \qquad \text{but} \qquad \mathcal{R} \circ \mathcal{R} = \{R \circ S : R, S \in \mathcal{R}\} \not\subseteq \{X^2\}.$$

Namely, for instance, we have $R_1[R_3(3)] = R_1[\{3, 4\}] = \{2, 3, 4\} \neq X$.

Moreover, if $\mathcal{R}$ is as in Example 3.18.5, then again by Theorem 3.1.7 and Corollary 3.1.6, it is clear that $\mathcal{R}$ cannot be proximally simple and topologically fine. However, by using direct arguments, we can again prove some better assertions.

Example 3.18.7. If $\mathcal{R}$ is as in Example 3.18.5, then

(1) $\mathcal{R}$ is not uniformly and proximally simple;

(2) $\mathcal{R}$ is quasi-proximally, quasi-topologically, topologically and paratopologically simple.

By taking $U = X^2$, we can note that U is an equivalence relation on X such that $\tau_{\mathcal{R}} = \{\emptyset, X\} = \tau_{\{U\}}$. Hence, by using Theorem 1.5.4, we can infer that $\mathcal{R}^{\#\infty} = \{U\}^{\#\infty}$, and thus $\mathcal{R}$ is quasi-proximally simple.

Moreover, by taking $V = A^2 \cup B^2$, with $A = \{1, 2\}$ and $B = \{3, 4\}$, we can note that V is an equivalence relation on X such that $V(1) = V(2) = \{1, 2\}$ and $V(3) = V(4) = \{3, 4\}$. Hence, it is clear that in addition to $\mathcal{R} \subseteq \{V\}^{\wedge}$, we also have $V \in \mathcal{R}^{\wedge}$. Therefore, $\mathcal{R}^{\wedge} = \{V\}^{\wedge}$, and thus $\mathcal{R}$ is topologically simple. Hence, it is clear that $\mathcal{R}$ is also quasi-topologically simple. Moreover, since $\wedge\, \vartriangle = \vartriangle$, we can also state that $\mathcal{R}$ is also paratopologically simple.

Next, we show directly that $\mathcal{R}$ is not proximally simple. For this, assume on the contrary that $\mathcal{R}$ is proximally simple. Then, there exists a relation S on X such that $\mathcal{R}^{\#} = \{S\}^{\#}$. Then, by using that $\#$ is extensive, we can infer that $\mathcal{R} \subseteq \{S\}^{\#}$ and $S \in \mathcal{R}^{\#}$. Thus, in particular we have $S(3) \subseteq R_1(3)$ and $S(3) \subseteq R_2(3)$, and thus $S(3) \subseteq R_1(3) \cap R_2(3) = \{3, 4\}$. Moreover, quite similarly we can also see that $S(4) \subseteq \{3, 4\}$. Therefore, for the set $A = \{3, 4\}$, we have $S[A] \subseteq A$. On the other hand, since $S \in \mathcal{R}^{\#}$, we have $R_i[A] \subseteq$

$S[A]$, and thus $R_i[A] \subseteq A$ for some $i \in X$. However, this is a contradiction since $\operatorname{card}(A) = 2$, while $\operatorname{card}\big(R_i[A]\big) \geq 3$ for all $i \in X$. Therefore, $\mathcal{R}$ is not proximally simple. Hence, since $*\# = \#$, it is clear that $\mathcal{R}$ cannot also be uniformly simple.

Remark 3.18.8. Concerning the relator $\mathcal{R}$, considered in Example 3.18.5, we can also note that $X^2 \notin \mathcal{R}$, and thus $\mathcal{R}$ cannot be $\square$–fine for any operation $\square$ for relators with $X^2 \in \mathcal{R}^{\square}$.

Remark 3.18.9. Simple and quasi-simple relators have formerly been intensively investigated by Száz and Mala [116, 79, 83, 84, 81].

However, the characterization of paratopologically simple relators and the existence of non-paratopologically simple relators were serious problems.

They were first established by J. Deák and G. Pataki. (See [92].) In particular, Pataki has constructed a non-paratopologically simple equivalence relator.

This justified an old conjecture of Árpád Száz that, in addition to preordered nets, multi-preordered nets also have to be intensively investigated.

Now the following example, suggested probably also by Pataki [93], will show that even some very particular quasi-topologically minimal relators need not be paratopologically minimal. Thus, in particular, the converse of Theorem 3.4.3 is not true.

Example 3.18.10. If $X = \mathbb{R}$ and R is a relation on X such that

$$R(x) = \{x-1\} \cup [x, +\infty[$$

for all $x \in X$, then $\mathcal{R} = \{R\}$ is a reflexive relator on X such that:

(1) $\mathcal{R}$ is not paratopologically minimal;

(2) $\mathcal{R}$ is both resolvable and hyperconnected;

(3) $\mathcal{R}$ is neither quasi-proximally nor quasi-topologically door, superset and submaximal;

(4) $\mathcal{R}$ is both quasi-proximally and quasi-topologically minimal, connected, hyperconnected and ultraconnected.

It is clear that R is a reflexive relation on X, and thus $\mathcal{R}$ is a reflexive relator on X. Moreover, we can at once see that $R(x) \cap R(y) \neq \emptyset$ for all $x, y \in X$. Thus, by Theorem 3.12.7, $\mathcal{R}$ is hyperconnected.

On the other hand, we can at once see that $R \neq X^2$, and thus $\mathcal{R} \not\subseteq \{X^2\}$. Therefore, by Theorem 3.5.1, $\mathcal{R}$ is not paratopologically minimal. Moreover, we can also note that $\mathbb{N} \cap R(x) \neq \emptyset$ and $R(x) \not\subseteq \mathbb{N}$ for all $x \in X$. Therefore, by Theorem 1.3.17, $\mathbb{N} \in \mathcal{D}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$, and thus $\mathcal{R}$ is resolvable.

Now, actually it remains only to show that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$, and thus $\mathcal{R}$ is quasi-topologically minimal. Namely, in this case, by Theorems 3.1.3, 3.6.3, 3.10.3 and 3.11.3, the remaining parts of assertion (4) are also true. Moreover, by Definitions 3.15.1, 3.15.5 and 3.15.9, assertion (3) is also true.

For the proof of $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$, note that if $A \in \mathcal{T}_{\mathcal{R}}$, then by Theorem 1.4.7, for any $a \in A$, we have $R(a) \subseteq A$, and thus $\{a-1\} \cup [a, +\infty[\subseteq A$. Therefore, if $x \in A$, then $\{x-1\} \cup [x, +\infty[\subseteq A$, and thus in particular $x-1 \in A$. Therefore, $\{x-2\} \cup [x-1, +\infty[\subseteq A$, and thus in particular $x-2 \in A$. Hence, is clear that we can only have either $A = \emptyset$ or $A = X$. Therefore, $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$.

Remark 3.18.11. If $\mathcal{R}$ is as in Example 3.18.10, then it is also worth noticing that

$$R(x) = \{x-1\} \cup [\, x,\, +\infty\, [\, \in \mathcal{E}_{\mathcal{R}}\,,$$

but

$$\mathrm{cl}_{\mathcal{R}}(x) = R^{-1}(x) = \,]-\infty\,,\, x\,] \cup \{x+1\} \notin \{\emptyset,\, X\} = \mathcal{F}_{\mathcal{R}}$$

for all $x \in X$.

Therefore, despite of $\mathcal{T}_{\mathcal{R}} = \{\emptyset,\, X\}$, $\mathcal{E}_{\mathcal{R}}$ is quite a large subfamily of $\mathcal{P}(X)$. Moreover, the relator $\mathcal{R}$ is very far from being even weakly quasi-topological.

The following example will show that, despite of the close resemblance of Definitions 3.10.1 and 3.11.1, quasi-proximal and quasi-topological ultraconnectedness properties are quite independent from the corresponding hyperconnectedness ones.

Example 3.18.12. If $X = \{1,\, 2,\, 3\}$ and $R_1,\, R_2 \subseteq X^2$ such that

$R_1(1) = \{1\}$, $\quad R_1(2) = X$, $\quad R_1(3) = X$,
$R_2(1) = X$, $\quad R_2(2) = \{2\}$, $\quad R_2(3) = X$,

then $\mathcal{R} = \{R_1,\, R_2\}$ is a preorder relator on X such that:

(1) $\mathcal{R}$ is both quasi-proximally and quasi-topologically ultraconnected;

(2) $\mathcal{R}$ is neither quasi-proximally nor quasi-topologically hyperconnected.

For this, note that $R_1 = \{1\}^2 \cup (\{1\}^c \times X)$ and $R_2 = \{2\}^2 \cup (\{2\}^c \times X)$. Therefore, by a basic property of Pervin relations, R_1 and R_2 are preorder (reflexive and transitive) relations on X. Thus, $\mathcal{R}$ is a preorder relator on X.

Moreover, by using some further basic properties of Pervin relations, we can see that

$$\tau_{\mathcal{R}} = \{\emptyset,\, \{1\},\{2\},\, X\} \qquad \text{and} \qquad \mathcal{T}_{\mathcal{R}} = \{\emptyset,\, \{1\},\{2\},\, \{1,\, 2\},\, X\},$$

and thus

$$\digamma_{\mathcal{R}} = \{\emptyset,\, \{1,\, 3\},\, \{2,\, 3\},\, X\} \quad \text{and} \quad \mathcal{F}_{\mathcal{R}} = \{\emptyset, \{3\}, \{1,3\}, \{2,3\}, X\}.$$

Therefore, the families $\digamma_{\mathcal{R}} \setminus \{\emptyset\}$ and $\mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$ have the binary intersection property, but the families $\tau_{\mathcal{R}} \setminus \{\emptyset\}$ and $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ do not have the binary intersection property.

Remark 3.18.13. By using the equality $R_A^{-1} = R_{A^c}$, we can quite easily see that

$R_1^{-1}(1) = X$, $\quad R_1^{-1}(2) = \{2,\, 3\}$, $\quad R_1^{-1}(3) = \{2,\, 3\}$,
$R_2^{-1}(1) = \{1,\, 3\}$, $\quad R_2^{-1}(2) = X$, $\quad R_2^{-1}(3) = \{1,\, 3\}$.

Hence, by some other basic properties of Pervin relations, it is clear that

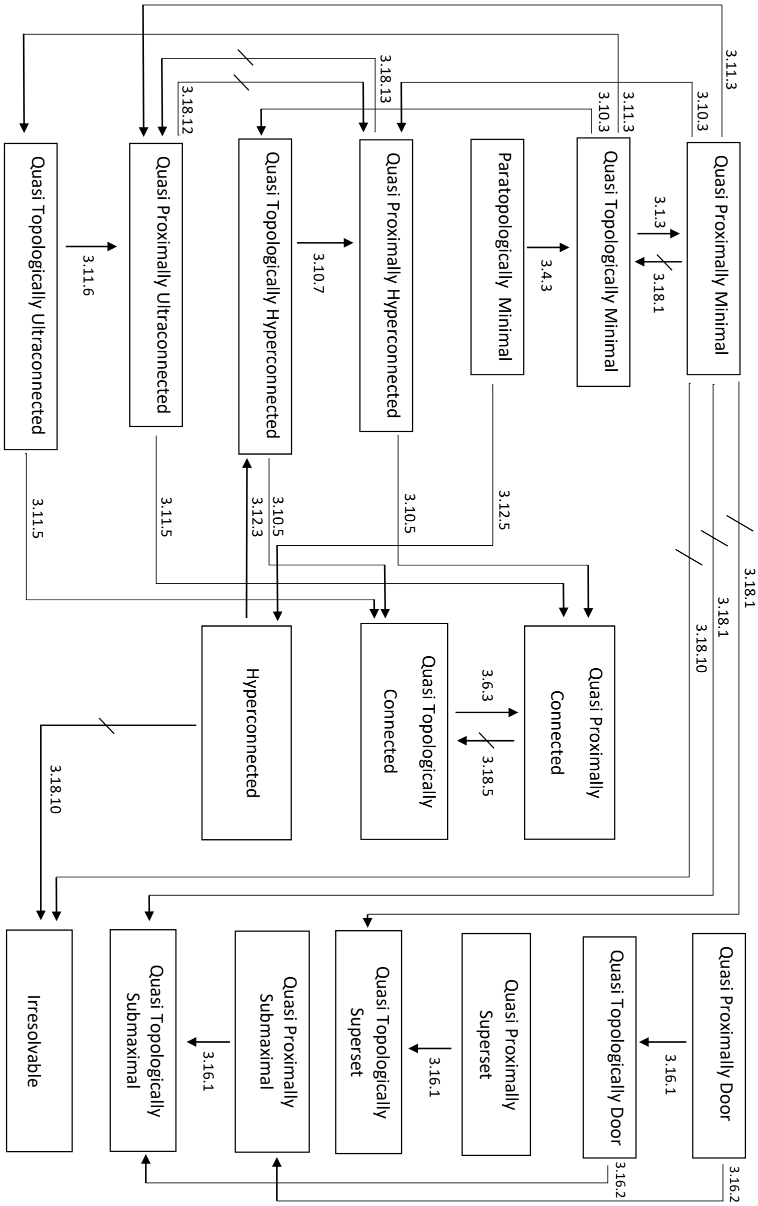
$$\tau_{\mathcal{R}^{-1}} = \{\emptyset, \{1,3\}, \{2,3\}, X\} \quad \text{and} \quad \mathcal{T}_{\mathcal{R}^{-1}} = \{\emptyset, \{1,3\}, \{2,3\}, X\},$$

and thus

$$\digamma_{\mathcal{R}^{-1}} = \{\emptyset,\, \{1\},\{2\},\, X\} \qquad \text{and} \qquad \mathcal{F}_{\mathcal{R}^{-1}} = \{\emptyset,\, \{1\},\{2\},\, X\}.$$

Therefore, we can also state that $\mathcal{R}^{-1}$ is a both quasi-proximally and quasi-topologically hyperconnected preorder relator on X such that $\mathcal{R}^{-1}$ is neither quasi-proximally nor quasi-topologically ultraconnected.

Diagram 3.18.14. For a nonvoid relator $\mathcal{R}$ on a nonvoid set X, the following implications are true:



Summary

The purpose of this PhD dissertation is to investigate, in relator spaces, counterparts of several types of generalised open sets and connectedness properties which have been studied by a great number of authors in topological spaces.

The Introduction contains several historical facts on the investigations of these two enormous topics. Moreover, it indicates that by using *relators* (families of relations) instead of topologies we can get some substantial generalizations.

Actually, by the results of Pervin [95] and Száz [126], each minimal structure and generalized topology can be easily derived from families of preorder relations. Thus, in contrast to a common belief, they should not also be studied separately.

Chapter 1 is devoted to collect some relevant facts on relators and their induced basic tools, such as proximal and topological interiors, open and fat sets, for instance. Moreover, here some primary classifications of relators are also included.

In Chapter 2, ten types of generalized open sets are introduced and investigated. For instance, a subset A of a relator space $X(\mathcal{R})$ is called *semi-open* if $A \subseteq \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)$, and *quasi-open* if $V \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(V)$ for some open subset V of $X(\mathcal{R})$.

Thus, for instance, it is shown that if in particular the relator $\mathcal{R}$ is topological, then A is a semi-open (quasi-open) subset of $X(\mathcal{R})$ if and only if there exist an open subset V of $X(\mathcal{R})$ and a subset B of $\mathrm{res}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(A) \setminus A$ such that $A = V \cup B$.

While, in Chapter 3, a relator $\mathcal{R}$ on X is, for instance, called *quasi-proximally minimal* if $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$, and *quasi-topologically connected* if $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{\emptyset, X\}$, where $\tau_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ denote the families of all proximally and topologically open subsets of $X(\mathcal{R})$, respectively.

There, for instance, it is shown that $\mathcal{R}$ is quasi-topologically connected if the relator $\mathcal{R}^{\wedge} \vee \mathcal{R}^{\wedge -1} = \{ R \cup S^{-1} : \ \mathcal{R}, S \in \mathcal{R}^{\wedge} \}$, where $\mathcal{R}^{\wedge} = \{ S \subseteq X^2 : \ \forall\, x \in X : \ x \in \mathrm{int}_{\mathcal{R}}\big(S(x)\big) \}$, is quasi-proximally minimal.

The latter statement shows that the properties of quasi-topologically connected relators can, in principle, be immediately derived from those of the quasi-proximally minimal ones. Hence, it can be seen that connectedness is a particular case of well-chainedness.

At the end of the dissertation, several possibilities for some further, more general investigations are suggested. The results of this dissertation have been published in two papers [101, 107] and one chapter [102].

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator on* X *to* Y, and the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space*. For the origins of this notion see [114, 123] and the references in [114].

If in particular $\mathcal{R}$ is a relator on X to itself, then $\mathcal{R}$ is simply called a *relator on* X. Thus, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ instead of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$.

Relator spaces of this simpler homogeneous type are already substantial generalizations of the various *ordered sets* [27] and *uniform spaces* [40]. However, they are insufficient for

some important purposes. (See, for instance, [44] and [122].)

A relator $\mathcal{R}$ on X to Y, or the relator space $(X, Y)(\mathcal{R})$, is called *simple* if $\mathcal{R} = \{R\}$ for some relation R on X to Y. Simple relator spaces $(X, Y)(R)$ and $X(R)$ were called *formal contexts* and *gosets* (generalized ordered sets) in [44] and [134], respectively.

In the dissertation, we shall mainly be considering relators on X. A relator $\mathcal{R}$ on X, or the relator space $X(\mathcal{R})$, will, for instance, be called *reflexive* if each member of $\mathcal{R}$ is reflexive on X. Thus, we may also naturally speak of *preorder, tolerance, and equivalence relators*.

For instance, for a family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{ R_A : \ A \in \mathcal{A}\}$, where $R_A = A^2 \cup (A^c \times X)$, is an important preorder relator on X. Such relators were first used by Pervin [95], and later also by Levine [77] and Száz [126].

While, for a family $\mathcal{D}$ of *pseudo-metrics* on X, the family $\mathcal{R}_{\mathcal{D}} = \{B_r^d : r > 0, \ d \in \mathcal{D}\}$, where $B_r^d = \{(x, y) : \ d(x, y) < r\}$, is an important tolerance relator on X. Such relators were first considered by Weil [146].

Moreover, if $\mathfrak{S}$ is a family of *covers (partitions)* of X, then the family $\mathcal{R}_{\mathfrak{S}} = \{ S_{\mathcal{A}} : \mathcal{A} \in \mathfrak{S} \}$, where $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$, is a tolerance (equivalence) relator on X. Equivalence relators were first investigated by Levine [76].

If $\mathcal{R}$ is a relator on X, then in Chapter 1, for any $A, B \subseteq X$ and $x, y \in X$, we define

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subseteq B$ for some $R \in \mathcal{R}$;
(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$;
(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$; (4) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$;

and moreover

(5) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$; (6) $A \in \tau\!\!\!-_{\mathcal{R}}$ if $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$;
(7) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$; (8) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$;
(9) $A \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(A) \neq \emptyset$; (10) $A \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) = X$.

The relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ are called *the proximal and topological interiors* generated by $\mathcal{R}$, respectively. While, the members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ are called the *proximally open, topologically open and fat subsets* of the relator space $X(\mathcal{R})$, respectively.

The relators

$$\mathcal{R}^{*} = \{ S \subseteq X^2 : \quad \exists \ R \in \mathcal{R} : \quad R \subseteq S \};$$
$$\mathcal{R}^{\#} = \{ S \subseteq X^2 : \quad \forall \ A \subseteq X : \quad \exists \ R \in \mathcal{R} : \quad R[A] \subseteq S[A] \};$$
$$\mathcal{R}^{\wedge} = \{ S \subseteq X^2 : \quad \forall \ x \in X : \quad \exists \ R \in \mathcal{R} : \quad R(x) \subseteq S(x) \};$$
$$\mathcal{R}^{\vartriangle} = \{ S \subseteq X^2 : \quad \forall \ x \in X : \quad \exists \ u \in X : \quad \exists \ R \in \mathcal{R} : \quad R(u) \subseteq S(x) \}.$$

are called the *uniform, proximal, topological and paratopological closures (or refinements)* of the relator $\mathcal{R}$, respectively.

Thus, $\mathcal{R}^{\#}$, $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\vartriangle}$ are the largest relators on X such that $\operatorname{Int}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^{\#}}$, $\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^{\wedge}}$ and $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{\vartriangle}}$. However, in general there is no largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$. This is a serious disadvantage of the topologically open sets compared to the fat and proximally open ones.

In Chapter 2, motivated by some similar definitions in topological spaces, a subset A of the relator space $X(\mathcal{R})$ is called *topologically*

(1) *preopen* if $A \subseteq \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$;
(2) *semi-open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)$;
(3) *regular open* if $A = \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$;
(4) *α–open* if $A \subseteq \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big)\big)$;
(5) *β–open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)\big)$;
(6) *a–open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \cap \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$;
(7) *b–open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \cup \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A)\big)$;
(8) *quasi-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}}(V)$;
(9) *pseudo-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq \operatorname{cl}_{\mathcal{R}}(A)$.

The families of the above sets A are denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with
$\kappa = p$, s, r, α, β, a, b, q and ps, respectively.
Moreover, the set A is called *topologically*
(10) *γ–open* if there exists $V \in \mathcal{T}_{\mathcal{R}}^{s}$ such that $A \subseteq V \subseteq \operatorname{cl}_{\mathcal{R}}(A)$;
(11) *δ–open* if there exists $V \in \mathcal{T}_{\mathcal{R}}^{p}$ such that $V \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}}(V)$.
And, the families of the latter sets are denoted by $\mathcal{T}_{\mathcal{R}}^{\gamma}$ and $\mathcal{T}_{\mathcal{R}}^{\delta}$.

In Chapter 3, motivated by some similar definitions in topological spaces, a relator $\mathcal{R}$ on X is called

(1) *quasi-proximally minimal* if $\tau_{\mathcal{R}} \subseteq \{\emptyset, X\}$;
(2) *quasi-topologically minimal* if $\mathcal{T}_{\mathcal{R}} \subseteq \{\emptyset, X\}$;
(3) *paratopologically minimal* if $\mathcal{E}_{\mathcal{R}} \subseteq \{X\}$;
(4) *quasi-proximally connected* if $\tau_{\mathcal{R}} \cap \digamma_{\mathcal{R}} \subseteq \{\emptyset, X\}$;
(5) *quasi-topologically connected* if $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{\emptyset, X\}$;
(6) *hyperconnected* if $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$;
(7) *quasi-proximally hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \tau_{\mathcal{R}} \setminus \{\emptyset\}$;
(8) *quasi-topologically hyperconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$
(9) *quasi-proximally ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \digamma_{\mathcal{R}} \setminus \{\emptyset\}$;
(10) *quasi-topologically ultraconnected* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}_{\mathcal{R}} \setminus \{\emptyset\}$;
(11) *quasi-proximally door relator* if $\mathcal{P}(X) = \tau_{\mathcal{R}} \cup \digamma_{\mathcal{R}}$;
(12) *quasi-topologically door relator* if $\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$;
(13) *quasi-proximally superset relator* if $\mathcal{E}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$;
(14) *quasi-topologically superset relator* if $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$;
(15) *quasi-proximally submaximal* if $\mathcal{D}_{\mathcal{R}} \subseteq \tau_{\mathcal{R}}$;
(16) *quasi-topologically submaximal* if $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$;
(17) *resolvable* if $\mathcal{D}_{\mathcal{R}} \not\subseteq \mathcal{E}_{\mathcal{R}}$.

In the dissertation, we are mainly dealing with characterizations of these 28 properties and establishing the relationships among some of them.

Possibilities for Some Further, More General Investigations

1. A Stacked Three Relator Space

Most of the results of the present dissertation can be unified and generalized by using some particular cases of a *non-conventional, stacked three relator space*

$$(X, Y)(\mathcal{R}, \mathcal{U}, \mathcal{V})(\mathcal{D}) = \big((X, Y), (\mathcal{R}, \mathcal{U}, \mathcal{V}), \mathcal{D}\big)$$

suggested by Száz [142], where

(1) $\mathcal{R}$ is a *relator* on X to Y, i. e., a set of relations on X to Y;

(2) $\mathcal{U}$ is a *super relator* on X to Y, i. e., a relator on $\mathcal{P}(X)$ to Y;

(3) $\mathcal{V}$ is a *hyper relator* on X to Y, i. e., a relator on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$;

(4) $\mathcal{D}$ is a *stack* on Y, i. e., $\mathcal{D} \subseteq \mathcal{P}(Y)$ such that $D \in \mathcal{D}$ and $D \subseteq E \subseteq Y$ imply $E \in \mathcal{D}$.

Here, by Száz [126], we may assume, without any loss of generality, that

$$\mathcal{D} = \mathcal{D}_{\mathcal{S}} = \{ B \subseteq Y : \quad Y = \mathrm{cl}_{\mathcal{S}}(B) \} = \{ B \subseteq Y : \quad \forall\ S \in \mathcal{S} : \quad Y = S^{-1}[B] \}$$

for some preorder relator $\mathcal{S}$ on Y.

Therefore, instead of the above non-conventional, stacked three relator space, we may, equally well, consider a *non-conventional, four relator space*

$$(X, Y)(\mathcal{R}, \mathcal{U}, \mathcal{V})(\mathcal{S}) = (X, Y)(\mathcal{R}, \mathcal{U}, \mathcal{V})(\mathcal{D}_{\mathcal{S}}),$$

where $\mathcal{S}$ is a relator on Y, or on an arbitrary set to Y.

Note that if in particular $Y \neq \emptyset$ and $S_0 = Y^2$, then

$$\mathcal{D}_0 = \mathcal{D}_{S_0} = \mathcal{D}_{\{S_0\}} = \mathcal{P}(Y) \setminus \{\emptyset\} = \{ B \subseteq Y : \quad B \neq \emptyset \}.$$

Thus, this will be the most important particular case of our subsequent treatment.

Namely, by using (2) and (4), for any $B \subseteq Y$ we may naturally define

$$\mathrm{Cl}_{(\mathcal{U}, \mathcal{D})}(B) = \{ A \subseteq X : \quad \forall\ U \in \mathcal{U} : \quad U(A) \cap B \in \mathcal{D} \}$$

and

$$\mathrm{Int}_{(\mathcal{U}, \mathcal{D})}(B) = \mathrm{Cl}_{(\mathcal{U}, \mathcal{D})}(B^c)^c = \mathcal{P}(X) \setminus \mathrm{Cl}_{(\mathcal{U}, \mathcal{D})}(Y \setminus B).$$

Thus, we can easily prove that

$$\mathrm{Int}_{(\mathcal{U}, \mathcal{D})}(B) = \{ A \subseteq X : \quad \exists\ U \in \mathcal{U} : \quad U(A) \setminus B \notin \mathcal{D} \}.$$

Moreover, by using the dual

$$\mathcal{E} = \{ E \subseteq Y : \quad E^c \notin \mathcal{D} \} = \{ E \subseteq Y : \quad \forall\ D \in \mathcal{D} : \quad E \cap D \neq \emptyset \}$$

of the stack $\mathcal{D}$, we can easily establish that

$$\mathrm{Int}_{(\mathcal{U}, \mathcal{D})}(B) = \{ A \subseteq X : \quad \exists\ U \in \mathcal{U} : \quad U(A)^c \cup B \in \mathcal{E} \}.$$

Now, by using the above hyper relations, we may also naturally define

$$\mathrm{cl}_{(\mathcal{U}, \mathcal{D})}(B) = \{ x \in X : \quad \{x\} \in \mathrm{Cl}_{(\mathcal{U}, \mathcal{D})}(B) \}$$

and

$$\mathrm{int}_{(\mathcal{U}, \mathcal{D})}(B) = \{ x \in X : \quad \{x\} \in \mathrm{Int}_{(\mathcal{U}, \mathcal{D})}(B) \}.$$

Moreover, by using the latter super relations, we may also naturally define

$$\mathcal{D}_{(\mathcal{U},\mathcal{D})} = \{ D \subseteq Y : \quad X = \text{cl}_{(\mathcal{U},\mathcal{D})}(D) \}$$

and

$$\mathcal{E}_{(\mathcal{U},\mathcal{D})} = \{ E \subseteq Y : \quad \text{int}_{(\mathcal{U},\mathcal{D})}(E) \neq \emptyset \}.$$

The hyper and super relations $\text{Int}_{(\mathcal{U},\mathcal{D})}$ and $\text{int}_{(\mathcal{U},\mathcal{D})}$ are called the *proximal and topological interiors* generated by the super relator $\mathcal{U}$ and the stack $\mathcal{D}$, respectively. While, the members of the family $\mathcal{E}_{(\mathcal{U},\mathcal{D})}$ are called the *fat subsets* of the *stacked super relator space* $(X, Y)(\mathcal{U})(\mathcal{D})$.

Now, for instance, we may also naturally define

$$\text{Cl}_{\mathcal{U}} = \text{Cl}_{(\mathcal{U},\mathcal{D}_0)} \qquad \text{and} \qquad \text{Cl}_{(\mathcal{R},\mathcal{D})} = \text{Cl}_{(\mathcal{R}^{\triangleright},\mathcal{D})},$$

where

$$\mathcal{R}^{\triangleright} = \{ R^{\triangleright} : \quad R \in \mathcal{R} \}, \qquad \text{with} \qquad R^{\triangleright}(A) = R[A]$$

for all $A \subseteq X$.

In addition to the super relator $\mathcal{R}^{\triangleright}$, now we may also naturally consider the ordinary relator

$$\mathcal{U}^{\triangleleft} = \{ U^{\triangleleft} : \quad U \in \mathcal{U} \}, \qquad \text{with} \qquad U^{\triangleleft}(x) = U\big(\{x\}\big)$$

for all $x \in X$. Moreover, we may also naturally consider the super relator

$$\mathcal{U}^{\circ} = \{ U^{\circ} : \quad U \in \mathcal{U} \}, \qquad \text{with} \qquad U^{\circ} = U^{\triangleleft\triangleright}.$$

Note that $R^{\triangleright\triangleleft} = R$ for all $R \in \mathcal{R}$, and thus $\mathcal{R}^{\triangleright\triangleleft} = \{ R^{\triangleright\triangleleft} : \quad R \in \mathcal{R} \} = \mathcal{R}$.

Concerning the latter relators, we can easily prove that

$$\text{Cl}_{(\mathcal{U}^{\triangleleft},\mathcal{D})} = \text{Cl}_{(\mathcal{U}^{\circ},\mathcal{D})} \qquad \text{and} \qquad cl_{(\mathcal{U}^{\triangleleft},\mathcal{D})} = \text{cl}_{(\mathcal{U}^{\circ},\mathcal{D})} = \text{cl}_{(\mathcal{U},\mathcal{D})}.$$

Therefore, by using super relators instead of the ordinary ones, the structures cl and $\mathcal{D}$ cannot be generalized.

However if for instance $X = \{1, 2\}$ and U is a super relation on X such that, for any $A \subseteq X$, we have

$$U(A) = A \quad \text{if} \quad A \neq X, \qquad \text{and} \qquad U(A) = \{1\} \quad \text{if} \quad A = X,$$

then it can be shown that $\text{Cl}_U = \text{Cl}_{\{U\}} \neq \text{Cl}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X.

Now, in accordance with our former definitions, the super relator $\mathcal{U}$ may, for instance, be naturally called

(a) *$\mathcal{D}$–irresolvable* if $\mathcal{D}_{(\mathcal{U},\mathcal{D})} \subseteq \mathcal{E}_{(\mathcal{U},\mathcal{D})}$;

(b) *$\mathcal{D}$–hyperconnected* if $\mathcal{E}_{(\mathcal{U},\mathcal{D})} \subseteq \mathcal{D}_{(\mathcal{U},\mathcal{D})}$;

(c) *$\mathcal{D}$–paratopologically minimal* if $\mathcal{E}_{(\mathcal{U},\mathcal{D})} \subseteq \{Y\}$.

Recall that $\mathcal{E}_{(\mathcal{U},\mathcal{D})} = \mathcal{E}_{(\mathcal{U}^{\triangleleft},\mathcal{D})}$, and thus $\mathcal{E}_{\mathcal{U}} = \mathcal{E}_{(\mathcal{U},\mathcal{D}_0)} = \mathcal{E}_{(\mathcal{U}^{\triangleleft},\mathcal{D}_0)} = \mathcal{E}_{\mathcal{U}^{\triangleleft}}$. Therefore, the particular cases $\mathcal{D} = \mathcal{D}_0$ of the above definitions are only some slight generalizations of our former similar definitions.

Moreover, if for instance $X = \mathbb{R}$ and

$$R_n = \{ (x, y) \in X^2 : \quad d(x, y) < n^{-1} \}$$

for all $n \in \mathbb{N}$, then $\mathcal{R} = \{R_n\}_{n \in \mathbb{N}}$ is an important tolerance relator on X, with several useful additional properties, which does not have the above properties.

However, despite this, the above curious properties, in the particular case $X = Y$ and $\mathcal{D} = \mathcal{D}_0$, have several interesting characterizations and applications.

2. The Homogeneous Particular Case

Now, to simplify our preceding considerations, we shall assume that:
(1) $\mathcal{R}$ is a *relator* on X;
(2) $\mathcal{U}$ is a *super relator* on X;
(3) $\mathcal{V}$ is a *hyper relator* on X;
(4) $\mathcal{D}$ is a *stack* on X, and thus in particular $\mathcal{D}_0 = \mathcal{P}(X) \setminus \{\emptyset\}$.

That is, instead of the possibly inhomogeneous, non-conventional, stacked three relator space $\big(X, Y\big)\big(\mathcal{R}, U, \mathcal{V}\big)(\mathcal{D})$, we shall now consider the homogeneous, non-conventional, stacked three relator space

$$X\big(\mathcal{R}, \mathcal{U}, \mathcal{V}\big)(\mathcal{D}) = \big(X, X\big)\big(\mathcal{R}, \mathcal{U}, \mathcal{V}\big)(\mathcal{D})$$

In this particular case, in addition to the corresponding particular cases of our former basic tools, we may also naturally define the families

$$\tau_{(\mathcal{U},\mathcal{D})} = \big\{A \subseteq X : \ A \in \operatorname{Int}_{(\mathcal{U},\mathcal{D})}(A)\big\}; \qquad \tau\!\!\!-_{(\mathcal{U},\mathcal{D})} = \big\{A \subseteq X : \ A^c \notin \operatorname{Cl}_{(\mathcal{U},\mathcal{D})}(A)\big\};$$
$$\mathcal{T}_{(\mathcal{U},\mathcal{D})} = \big\{A \subseteq X : \ A \subseteq \operatorname{int}_{(\mathcal{U},\mathcal{D})}(A)\big\}; \qquad \mathcal{F}_{(\mathcal{U},\mathcal{D})} = \big\{A \subseteq X : \ \operatorname{cl}_{(\mathcal{U},\mathcal{D})}(A) \subseteq A\big\}.$$

The members of the families, $\tau_{(\mathcal{U},\mathcal{D})}$ and $\mathcal{T}_{(\mathcal{U},\mathcal{D})}$ are called the *proximally open and topologically open subsets* of the stacked super relator space $X(\mathcal{U})(\mathcal{D})$, respectively.

We have seen that the open sets are usually less important tools than the interior relations and the fat sets. For instance, it may occur that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{(\mathcal{R}^{\triangleright},\mathcal{D}_0)} = \{\emptyset, X\}$, but $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{(\mathcal{R}^{\triangleright},\mathcal{D}_0)}$ is a quite large subfamily of $\mathcal{P}(X)$. Moreover, for instance, by using the fat and dense sets, we may naturally define the convergence and adherence of nets to nets and points even in an inhomogeneous relator space $(X, Y)(\mathcal{R})$. Thus, the relations $\operatorname{Lim}_{\mathcal{R}}$ and $\operatorname{Adh}_{\mathcal{R}}$ are usually more powerful tools, than the relation $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$.

Now, in accordance with our former definitions, the super relator $\mathcal{U}$ may, for instance, be naturally called

(1) *$\mathcal{D}$–quasi-proximally minimal* if $\tau_{(\mathcal{U},\mathcal{D})} \subseteq \{\emptyset, X\}$;
(2) *$\mathcal{D}$–quasi-topologically connected* if $\mathcal{T}_{(\mathcal{U},\mathcal{D})} \cap \mathcal{F}_{(\mathcal{U},\mathcal{D})} \subseteq \{\emptyset, X\}$.

Again, the particular cases $\mathcal{D} = \mathcal{D}_0$ of these definitions are only slight generalizations of our former similar definitions.

Moreover, we can also prove that

$$\tau\!\!\!-_{\mathcal{U}} = \big\{A \subseteq X : \quad \exists\, U \in \mathcal{U} : \quad A \subseteq U^{\star}(A)\big\}$$

and

$$\tau\!\!\!-_{\mathcal{U}^{\star}} = \big\{A \subseteq X : \quad \exists\, U \in \mathcal{U} : \quad A \subseteq U(A)\big\},$$

where

$$\mathcal{U}^{\star} = \big\{U^{\star} : \quad U \in \mathcal{U}\big\}, \qquad \text{with} \qquad U^{\star}(A) = U\,(A^c)^c$$

for all $A \subseteq X$.

Therefore, if for instance U is a super relation on X such that

$$U(A) = \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big)$$

for all $A \subseteq X$, then

$$\mathcal{T}_U = \big\{ A \subseteq X : \quad A \subseteq \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)\big) \big\}$$

and

$$\mathcal{T}_{U^\star} = \big\{ A \subseteq X : \quad A \subseteq \mathrm{cl}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}(A)\big) \big\}.$$

Thus, $\mathcal{T}_U$ and $\mathcal{T}_{U^\star}$ are just the families $\mathcal{T}_{\mathcal{R}}^{p}$ and $\mathcal{T}_{\mathcal{R}}^{s}$ of all topologically preopen and semi-open subsets of the relator space $X(\mathcal{R})$.

Now, for any $\mathcal{A} \subseteq \mathcal{P}(X)$, we may also naturally define

$$\mathcal{A}^{k} = \mathcal{A}^{k_{\mathcal{V}}} = \mathrm{cl}_{\mathcal{V}}(\mathcal{A}) \qquad \text{and} \qquad \mathcal{A}^{\ell} = \mathcal{A}^{\ell_{\mathcal{V}}} = \mathrm{cl}_{\mathcal{V}^{-1}}(\mathcal{A}).$$

Thus, by Theorem 1.3.10, we can at once state that

$$\mathcal{A}^{\ell} = \textstyle\bigcap_{V \in \mathcal{V}} V[\mathcal{A}] \qquad \text{and} \qquad \mathcal{A}^{k} = \textstyle\bigcap_{V \in \mathcal{V}} V^{-1}[\mathcal{A}].$$

Moreover, we can easily establish that

$$\mathcal{A}^{\ell} = \big\{ B \subseteq X : \quad \forall\, V \in \mathcal{V} : \quad \exists\, A \in \mathcal{A} : \quad B \in V(A) \big\}$$

and

$$\mathcal{A}^{k} = \big\{ B \subseteq X : \quad \forall\, V \in \mathcal{V} : \quad \exists\, A \in \mathcal{A} : \quad A \in V(B) \big\}.$$

Therefore, if for instance V is a hyper relation on X such that

$$V(A) = \big\{ B \subseteq X : \quad A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A) \big\}$$

for all $A \subseteq X$, then for any $\mathcal{A} \subseteq \mathcal{P}(X)$ we have

$$\mathcal{A}^{\ell_V} = \big\{ B \subseteq X : \quad \exists\, A \in \mathcal{A} : \quad A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A) \big\}$$

and

$$\mathcal{A}^{k_V} = \big\{ B \subseteq X : \quad \exists\, A \in \mathcal{A} : \quad B \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(B) \big\}.$$

Thus, $\mathcal{T}_{\mathcal{R}}^{\ell_V}$ and $\mathcal{T}_{\mathcal{R}}^{k_V}$ are just the families $\mathcal{T}_{\mathcal{R}}^{q}$ and $\mathcal{T}_{\mathcal{R}}^{ps}$ of all topologically quasi-open and pseudo-open subsets of the relator space $X(\mathcal{R})$.

Bibliography

[1] M. E. Abd El-Monsef, S.N. El-Deeb R. A. Mahmoud, *β–open sets and β–continuous mappings*, Bull. Fac. Sci. Assiut Univ., **12** (1983), 77–90.

[2] S. Acharjee, M. Th. Rassias and Á. Száz, *Upper and Lower Semicontinuous Relations in Relator Spaces*, In: Th. M. Rassiass and P. M. Pardalos(Eds.), Mathematical Analysis, Optimization, Approximation and Applications, World Scientific, 2022, to appear.

[3] P. Alexandroff, *Zur Begründung der n-dimensionalen mengentheorischen Topologie*, Math. Ann., **94** (1925), 296–308.

[4] T. M. Al-shami, *On supra semi open sets and some applications on topological spaces*, J. Adv. Stud. Topol., **8** (2017), 144–153.

[5] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24–32.

[6] D. Andrijević, *On SPO–equivalent topologies*, Suppl. Rend. Circ. Mat. Palermo, **29** (1992), 317–328.

[7] D. Andrijević, *A note on α–equivalent topologies*, Mat. Vesnik, **45** (1993), 65–69.

[8] D. Andrijević, *On b–open sets*, Mat. Vesnik, **48** (1996), 59–64.

[9] D. Andrijević, M. Ganster, *On PO–equivalent topologies*, Suppl. Rend. Circ. Mat. Palermo, **24** (1990), 251–256.

[10] H. Arianpoor, *Preorder relators and generalized topologies*, J. Lin. Top. Algebra, **5** (2016), 271–277

[11] J. Ávila, F. Molina, *Generalized weak structures*, Int. Math. Forum **7** (2012), 2589–2595.

[12] N. Bourbaki, *General Topology, Chap 1– 4*, Springer-Verlag, Berlin, 1989.

[13] N. Bourbaki, *Éléments de Mathématique, Algébre Commutative, Chap. 1–4*, Springer, Berlin, 2006.

[14] E. Čech, *Topological Spaces*, Academia, Prague, 1966.

[15] H. H. Crossley, E. Michael, *Metrizability of certain countable unions*, Illinois J. Math, **8** (1964), 351–360.

[16] S. G. Crossley, *A note on semitopological classes*, Proc. Amer. Math. Soc, **43** (1974), 416–420.

[17] S. G. Crossley, S. K. Hildebrandt, *Semi-closure*. Texas J. Sci., **22** (1971), 99–112.

[18] S. G. Crossley, S. K. Hildebrandt, *Semi-topological properties*. Fund. Math., **74** (1972), 233–254.

[19] Á. Császár, *Foundations of General Topology*, Pergamon Press, London, 1963.

[20] Á. Császár, *Generalized open sets*, Acta Math. Hungar., **75** (1997), 65–87.

[21] Á. Császár, *On the γ–interior and γ–closure of a set*, Acta Math. Hungar., **80** (1998), 89–93.

[22] Á. Császár, *γ–quasi-open sets*, Studia Sci. Math. Hungar., **38** (2001), 171–176.

[23] Á. Császár, *Remarks on γ-quasi-open sets*, Studia Sci. Math. Hungar., **39** (2002), 137–141.

[24] Á. Császár, *Further remarks on the formula for the γ–interior*, Acta Math. Hungar., **113** (2006), 325–332.

[25] Á. Császár, *Remarks on quasi-topologies*, Acta Math. Hungar., **119** (2008), 197–200.

[26] Á. Császár, *Weak structures*, Acta Math. Hungar. **131** (2011), 193–195.
[27] B. Davey and H. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 2002.
[28] A. S. Davis, *Indexed systems of neighbordoods for general topological spaces*, Amer. Math. Monthly, **68** (1961), 886–894.
[29] K. Dlaska, N. Ergun, M. Ganster, *On the topology generated by semi-regular sets*, Indian J. Pure Appl. Math., **25** (1994), 1163–1170.
[30] D. Doičinov, *A unified theory of topological spaces, proximity spaces and uniform spaces*, Dokl. Acad. Nauk SSSR, **156** (1964), 21–24. (Russian)
[31] J. Dontchev, *On superconnected spaces*, Serdica, **20** (1994), 345–350.
[32] J. Dontchev, *The characterization of spaces and maps via semi-preopen sets*, Indian J. Pure Appl. Math., **25** (1994), 939–947.
[33] J. Dontchev, *Survey on preopen sets, Meetings on Topological Spaces, Theory and Applications*, Yatsushiro College of Technology, Kumamoto, Japan, 1998, 18 pp.
[34] Z. Duszyński, T. Noiri, *Semi-open, semi-closed sets and semi-continuity of functions*, Math. Pannon., **23** (2012), 195–200.
[35] V. A. Efremovič, *The geometry of proximity*, Mat. Sb., **31** (1952), 189–200. (Russian)
[36] V. A. Efremović and A. S. Švarc, *A new definition of uniform spaces, Metrization of proximity spaces*, Dokl. Acad. Nauk. SSSR, **89** (1953), 393–396. (Russian)
[37] E. Ekici, *On weak structures due to Császár*, Acta Math. Hungar., **134** (2012), 565–570.
[38] N. Elez, O. Papaz, *The new operators in topological spaces*, Math. Moravica, **17** (2013), 63–68.
[39] R. Engelking, *General Topology*, Polish Scientific Publishers, Warszawa, 1977.
[40] P. Fletcher and W. F. Lindgren, *Quasi-Uniform Spaces*, Marcel Dekker, New York, 1982.
[41] G. B. Folland, *A tale of topology*, Amer. Math. Monthly, **117** (2010), 663–672.
[42] M. Ganster, I. L. Reilly and M. K. Vamanamurthy, *Dense sets and irresolvable spaces*, Ricerche Mat., **36** (1987), 163–170.
[43] M. Ganster, D. Andrijević, *On some questions concerning semi-preopen sets*, J. Inst. Math. Com. Sci., **1** (1988), 65–72.
[44] B. Ganter and R. Wille, *Formal Concept Analysis*, Springer-Verlag, Berlin, 1999.
[45] M. Ganster, M. Steiner, *On some questions about b–open sets*, Quest. Answers in Gen. Topology, **25** (2007), 45–52.
[46] S. Givant, P. Halmos, *Introduction to Boolean Algebras* Springer-Verlag, Berlin, 2009.
[47] T. Glavosits, *Generated preorders and equivalences*, Acta Acad. Paed. Agrienses, Sect. Math., **29** (2002), 95–103.
[48] W. H. Gottschalk, *Intersection and closure*. Proc. Amer. Math. Soc. **4** (1953), 470–473.
[49] F. Hausdorff, *Grundzüge der Mengenlehre*, (German) Chelsea Publishing Company, New York, N. Y., 1949.
[50] H. Herrlich, *Topological Structures*, Math. Centre Tracts, **52** (1974), 59–122.
[51] E. Hewitt, *A problem of set-theoretic topology*, Duke Math. J, **10** (1943), 309–333.
[52] W. Hunsaker and W. Lindgren, *Construction of quasi-uniformities*, Math. Ann., **188** (1970), 39–42.
[53] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A, **27** (1941), 222–224.
[54] J. R. Isbell, *Uniform Spaces*, Amer. Math. Soc., Providence, 1964.

[55] Y. Isomichi, *New concept in the theory of topological spaces–Supercondensed set, subcondensed set, and condensed set*, Pacific J. Math., **38** (1971), 657–668.

[56] B. M. Ittanagi, S. S. Benchalli, *On paraopen sets and maps in topological spaces*, Kyungpook Math. J., **56** (2016), 301–310.

[57] R. Jamunarani, P. Jeyanthi, T. Noiri, *On generalized weak structures*, Journal of Algorithms and Computation, **47** (2016), 21–26.

[58] Y. B. Jun, S. W. Jeong, H. j. Lee, J. W. Lee, *Applications of pre-open sets*, Appl. Gen. Top., **9** (2008), 213–228.

[59] S.-M. Jung, *Interiors and cosures of sets and applications*, Int. J. Pure Math. **3** (2016), 41–45.

[60] S.-M. Jung, D. Nam, *Some properties of interior and closure in general topology*, Mathematics, **7** (2019), 10 pp.

[61] J. L. Kelley, *General Topology*, Van Nostrand Reinhold Company, New York, 1955.

[62] H. Kenyon, *Two theorems on relations*, Trans. Amer. Math. Soc., **107** (1963), 1–9 .

[63] H. J. Kowalsky, *Topologische Räumen*, Birkhäuser, Basel, 1960.

[64] K. Kuratowski, *Sur l'opération* $\overline{A}$ *de l'analysis situs*, Fund. Math., **3**(1922), 182–199. (An English translation: On the operation $\overline{A}$ in analysis situs, prepared by M. Bowron in 2010, is available on the Internet.)

[65] K. Kuratowski, *Topologie I*, Revised and augmented eddition: Topology I, Academic Press, New York, 1966.

[66] J. Kurdics, *A note on connection properties*, Acta Math. Acad. Paedagog. Nyházi., **12**,(1990), 57–59.

[67] J. Kurdics and Á. Száz, *Well-chained relator spaces*, Kyungpook Math. J., **32** (1992), 263–271.

[68] J. Kurdics and Á. Száz, *Well-chainedness characterizations of connected relators*, Math. Pannon., **4** (1993), 37–45.

[69] J. E. Leuschen and B. T. Sims, *Stronger forms of connectivity*, Rend. Circ. Mat. Palermo, **21** (1972), 255–266.

[70] N. Levine, *On the commutivity of the closure and interior operators in topological spaces*, Amer. Math. Montly, **68** (1961), 474–477.

[71] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.

[72] N. Levine, *Some remarks on the closure operator in topological spaces*, Amer. Math. Monthly, **70** (1963), p. 553.

[73] N. Levine, *Strongly connected sets in topology*, Amer. Math. Monthly, **72** (1965), 1098–1101.

[74] N. Levine, *The superset topology*, Amer. Math. Monthly, **75** (1968), 745–746.

[75] N. Levine, *Dense topologies*, Amer. Math. Monthly, **75** (1968), 847–852.

[76] N. Levine, *On uniformities generated by equivalence relations*, Rend. Circ. Mat. Palermo, **18** (1969), 62–70.

[77] N. Levine, *On Pervin's quasi uniformity*, Math. J. Okayama Univ., **14** (1970), 97–102.

[78] N. Levine, *Well-chained uniformities*, Kyungpook Math. J., **11** (1971), 143–149.

[79] J. Mala, *An equation for families of relations*, Pure Math. Apll., Ser. B, **1** (1990), 185–188.

[80] J. Mala, *Relators generating the same generalized topology*, Acta Math. Hungar., **60** (1992), 291–297.

[81] J. Mala, *Finitely generated quasi-proximities*, Period. Math. Hungar., **35** (1997), 193–197.

[82] J. Mala, Á. Száz, *Modifications of relators*, Acta Math. Hungar. **77**(1997), 69–81.

[83] J. Mala and Á. Száz, *Equations for families of relations can also be solved*, C. R. Math. Rep. Acad. Sci. Canada, **12** (1990), 109–112.

[84] J. Mala and Á. Száz, *Properly topologically conjugated relators*, Pure Math. Appl., Ser. B, **3** (1992), 119–136.

[85] Z. P. Mamuzić, *Introduction to General Topology*, Noordhoff, Groningen, 1963.

[86] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt, **53** (1982), 47–53.

[87] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press, Cambridge, 1970.

[88] H. Nakano and K. Nakano, *Connector theory*, Pacific J. Math., **56** (1975), 195–213.

[89] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math., **15** (1965), 195–213.

[90] T. Noiri, *On semi-continuous mappings*, Lincei-Rend. Sci. Fis. Mat. Nat., **54** (1973), 210–214.

[91] T. Noiri, *Hyperconnectedness and preopen sets*, Rev. Roum. Math. Pures Appl., **29** (1984), 329–334.

[92] G. Pataki, *Supplementary notes to the theory of simple relators*, Radovi Mat., **9** (1999), 101–118.

[93] G. Pataki, *On the extensions, refinements and modifications of relators*, Math. Balk., **15** (2001), 155–186.

[94] G. Pataki and Á. Száz, *A unified treatment of well-chainedness and connectedness properties*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), **19** (2003), 101–165.

[95] W. J. Pervin, *Quasi-uniformization of topological spaces*, Math. Ann., **147** (1962), 316–317.

[96] W. J. Pervin, *Quasi-proximities for topological spaces*, Math. Ann., **150** (1963), 325–326.

[97] V. Pipitone, G. Russo, *Spazi semiconnessi e spazi semiaperty*, Rend. Circ. Mat. Palermo, **24** (1975), 273–285.

[98] M. Th. Rassias and Á. Száz, *Basic tools and continuity-like properties in relator spaces*, Contrib. Math., **3** (2021), 77–106

[99] Th. M. Rassias, Á. Száz, *A General Framework for Studying Certain Generalized Topologically Open Sets in Relator Spaces*, In: Th. M. Rassias (Ed.), Nonlinear Analysis, Differential Equations, and Applications, Springer Optimization and Its Applications 173, Springer Nature Switzerland AG, **173** , 2021, 415–491.

[100] Th. M. Rassias and Á. Száz, *Ordinary, Super and Hyper Relators Can Be Used to Treat the Various Generalized Open Sets in a Unified Way*, In: N. J. Daras and Th. M. Rassias (Eds.), Approximation and Computation in Science and Engineering, Springer Optimizations and Applications 180, Springer Nature Switzerland AG, to appear.

[101] Th. M. Rassias, M. Salih, Á. Száz, *Characterizations of generalized topologically open sets in relator spaces*, Montes Taurus J. Pure Appl. Math., **3** (2021), 39–94.

[102] Th. M. Rassias, M. Salih, Á. Száz, *Set-theoretic Properties of Generalized Topologically Open Sets in Relator Spaces*, In: I. N. Parasidis, E. Providas and Th. M. Rassias (Eds.), Mathematical Analysis in Interdisciplinary Research, Springer Optimizations and Its Applications 179, Springer Nature Switzerland AG, 2022, to appear.

[103] Th. M. Rassias, M. Salih, Á. Száz, *Characterizations and Set Theoretic Properties of Some Generalized Open and Fat sets in Relator Spaces*, In: Th. M. Rassiass and P. M. Pardalos(Eds.), Mathematical Analysis, Optimization, Approximation and Applications, World Scientific, 2022, to appear.

[104] I. L. Reilly, M. K. Vamanamurthy, *On α-continuity in topological spaces*, Acta Math. Hungar., **45** (1985), 27–32.

[105] F. Riesz, *Die Genesis der Raumbegriffs*, Math. Naturwiss. Ber. Ungarn, **24** (1906), 309–353.

[106] F. Riesz, *Stetigkeitsbegriff und abstrakte Mengenlehre*, Atti IV Congr. Intern.Mat., Roma, **II** (1908), 18–24.

[107] M. Salih, Á. Száz, *Generalizations of some ordinary and extreme connectedness properties of topological spaces to relator spaces*, Elec. Res. Arch., **28** (2020), 471–548.

[108] M. S. Sarsak, *On some properties of generalized open sets in generalized topological spaces*, Demonstr. Math., **46** (2013), 415–427.

[109] W. Sierpinski, *General Topology*, Mathematical Expositions, University of Toronto Press, Toronto, **7**, 1952.

[110] P. Sivagami, *Remarks on γ–interior*, Acta Math. Hungar., **119** (2008), 81–94.

[111] Yu. M. Smirnov, *On proximity spaces*, Math. Sb., **31** (1952), 543–574. (Russian.)

[112] L. A. Steen and J. A. Seebach, *Counterexamples in Topology*, Springer-Verlag, New York, 1970.

[113] M. H. Stone, *Application of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., **41** (1937), 374–481.

[114] Á. Száz, *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar., **50** (1987), 177–201.

[115] Á. Száz, *Directed, topological and transitive relators*, Publ. Math. Debrecen, **35** (1988), 179–196.

[116] Á. Száz, *Relators, Nets and Integrals*, Unfinished doctoral thesis, Debrecen, 1991.

[117] Á. Száz, *Inverse and symmetric relators*, Acta Math. Hungar., **60** (1992), 157–176.

[118] Á. Száz, *Structures derivable from relators*, Singularité, **3** (1992), 14–30.

[119] Á. Száz, *Refinements of relators*, Tech. Rep., Inst. Math., Univ. Debrecen **76** (1993), 19 pp.

[120] Á. Száz, *Cauchy nets and completeness in relator spaces*, Colloq. Math. Soc. János Bolyai **55** (1993), 479–489.

[121] Á. Száz, *Neighbourhood relators*, Bolyai Soc. Math. Stud. **4** (1995), 449–465

[122] Á. Száz, *Somewhat continuity in a unified framework for continuities of relations*, Tatra Mt. Math. Publ., **24** (2002), 41–56.

[123] Á. Száz, *Upper and lower bounds in relator spaces*, Serdica Math. J., **29** (2003), 239–270.

[124] Á. Száz, *Rare and meager sets in relator spaces*, Tatra Mt. Math. Publ., **28** (2004), 75–95.

[125] Á. Száz, *Galois-type connections on power sets and their applications to relators*, Tech. Rep., Inst. Math., Univ. Debrecen, **2** (2005), 38 pp.

[126] Á. Száz, *Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities*, Filomat, **21** (2007), 87–97.

[127] Á. Száz, *Applications of fat and dense sets in the theory of additive functions*, Tech. Rep., Inst. Math., Univ. Debrecen, **3** (2007), 29 pp.

[128] Á. Száz, *Galois type connections and closure operations on preordered sets*, Acta Math. Univ. Comen., **78** (2009), 1–21.

[129] Á. Száz, *Lower semicontinuity properties of relations in relator spaces*, Adv. Stud. Contemp. Math., (Kyungshang), **23** (2013), 107–158.

[130] Á. Száz, *Inclusions for compositions and box products of relations*, J. Int. Math. Virt. Inst., **3** (2013), 97–125.

[131] Á. Száz, *A particular Galois connection between relations and set functions*, Acta Univ. Sapientiae, Math., **6** (2014), 73–91.

[132] Á. Száz, *Generalizations of Galois and Pataki connections to relator spaces*, J. Int. Math. Virt. Inst., **4** (2014), 43–75.

[133] Á. Száz, *Remarks and problems at the conference on inequalities and applications*, Hajdúszoboszló, Hungary, Tech. Rep., Inst. Math., Univ. Debrecen, **5** (2014), 12 pp.

[134] Á. Száz, *Basic Tools, Increasing Functions, and Closure Operations in Generalized Ordered Sets*, In: P. M. Pardalos and Th. M. Rassias (Eds.), Contributions in Mathematics and Engineering: In Honor of Constantion Caratheodory, Springer, 2016, 551–616.

[135] Á. Száz, *Four general continuity properties, for pairs of functions, relations and relators, whose particular cases could be investigated by hundreds of mathematicians*, Tech. Rep., Inst. Math., Univ. Debrecen, **1** (2017), 17 pp.

[136] Á. Száz, *The closure-interior Galois connection and its applications to relational equations and inclusions*, J. Int. Math. Virt. Inst., **8** (2018), 181–224.

[137] Á. Száz, *A unifying framework for studying continuity, increasingness, and Galois connections*, MathLab J., **1** (2018), 154–173.

[138] Á. Száz, *Corelations are More Powerful Tools Than Relations*, In: Th. M. Rassias (Ed.), Applications of Nonlinear Analysis, Springer Optimization and Its Applications, **134**, 2018, 711–779.

[139] Á. Száz, *Galois and Pataki connections on generalized ordered sets*, Earthline J. Math. Sci., **2** (2019), 283–323.

[140] Á. Száz, *Super and hyper products of super relations*, Tatra Mt. Math. Publ. **78** (2021), 85-118.

[141] Á. Száz, *Contra continuity properties of relations in relator spaces*, In: S. Acharjee and G. Bosi (Eds.), Advances in Topology and Their Interdisciplinary Application, Springer, to appear.

[142] Á. Száz, *Non-Conventional Stacked Three Relator Spaces* In: S. Acharjee, C. Özel, J. F. Peters and G. Bosi (Eds.), Advances in Topology and its Structures: Foundations, Theory and Application, John Wiley and Sons, submitted.

[143] W. J. Thron, *Topological Structures*, Holt, Rinehart and Winston, New York, 1966.

[144] H. Tietze, *Beiträge zur allgemeinen Topologie I. Axiome für verschiedene Fassungen des Umgebungsbegriffs*, Math. Ann., **88** (1923), 290–312.

[145] J. W. Tukey, *Convergence and Uniformity in Topology*, Princeton Univ. Press, Princeton, 1940.

[146] A. Weil, *Sur les Espaces á Structure Uniforme et sur la Topologie Générale*, Actual. Sci. Ind. 551, Herman and Cie, Paris, 1937.

[147] R. L. Wilder, *Evolution of the topological concept of connected*, Amer. Math. Monthly, **85** (1978), 720–726.

List of Talks

1. *Two Important Operations in Generalized Uniformities*, 2^{nd} International Hazar Scientific Researches Conference, Khazar University, Baku, Azerbaijan, April 10-12, 2021.
2. *Minimality Properties of the Family* $\mathcal{T}^{p}_{\mathcal{R}}$ *in Relator Spaces*, 64^{th} Annual Online Meeting of the Australian Mathematical Society, University of New England, Australia, December 8–11, 2020.
3. *Closure and Interior Operations in Relator Spaces*, 9^{th} Interdisciplinary Doctoral Conference, Doctoral Student Association of the University of Pécs, Hungary, November 27–28, 2020.
4. *An Important Property of Topologically Semi-open and Preopen sets*, 34^{th} International Summer Conference on Real Functions Theory, Slovak Academy of Sciences, Slovakia, September 7–13, 2020.
5. *Two Important Operations in Relator Spaces*, Síkfőkút Seminar of the Department of Analysis, University of Debrecen, Síkfőkút, Hungary, August 28–30, 2020.
6. *Generalized Open Sets Should Not Also Be Studied Without Generalized Uniformities*, The 16^{th} International Students' Conference on Analysis, Síkfőkút, Hungary, February 1–4, 2020.
7. *Ordinary and Extreme Connectedness Properties of Relator Spaces*, Analysis Research Seminar, Institute of Mathematics, University of Debrecen, Hungary, March 20, 2019.
8. *A New Type of Weakly Commutative Groups*, The 2^{nd} International Scientific Conference, The University of Zakho, Kurdistan Region of Iraq, April 18 – 20, 2017.

UNIVERSITY of DEBRECEN

UNIVERSITY AND NATIONAL LIBRARY
UNIVERSITY OF DEBRECEN
H-4002 Egyetem tér 1, Debrecen
Phone: +3652/410-443, email: publikaciok@lib.unideb.hu

Registry number: DEENK/101/2022.PL
Subject: PhD Publication List

Candidate: Muwafaq Mahdi Salih
Doctoral School: Doctoral School of Mathematical and Computational Sciences
MTMT ID: 10076180

List of publications related to the dissertation

Foreign language international book chapters (1)

1. Rassias, T. M., **Salih, M. M.**, Száz, Á.: Set-theoretic Properties of Generalized Topologically Open Sets in Relator Spaces.
   In: Mathematical Analysis in Interdisciplinary Research, Springer, Natural Switzeland, [Accepted by publisher], 1-70, 2022.

Foreign language scientific articles in international journals (2)

2. Rassias, T. M., **Salih, M. M.**, Száz, Á.: Characterizations of Generalized Topologically Open Sets in Relator Spaces.
   Montes Taurus J. Pure Appl. Math. 3 (3), 39-94, 2021. ISSN: 2687-4814.

3. **Salih, M. M.**, Száz, Á.: Generalizations of some ordinary and extreme connectedness properties of topological spaces to relator spaces.
   Electron. Res. Arch. 28 (1), 471-548, 2020. EISSN: 2688-1594.
   DOI: http://dx.doi.org/10.3934/era.2020027
   IF: 1.833

List of other publications

Foreign language international book chapters (1)

4. Rassias, T. M., **Salih, M. M.**, Száz, Á.: Characterizations and Set Theoretic Properties of Some Generalized Open and Fat sets in Relator Spaces.
   In: Mathematical Analysis, Optimization, Approximation and Applications, World Scientific, Singapore, [Accepted by publisher], 1-96, 2022.

UNIVERSITY of DEBRECEN

UNIVERSITY AND NATIONAL LIBRARY
UNIVERSITY OF DEBRECEN
H-4002 Egyetem tér 1, Debrecen
Phone: +3652/410-443, email: publikaciok@lib.unideb.hu

Foreign language scientific articles in international journals (3)

5. Abdullah, H. N., Ahmed, D., **Salih, M. M.**: Using fibonacci number to integrate 2x2 and 3x3 matrices.
   The Journal of Duhok University. 21 (1), 1-5, 2018. ISSN: 1812-7568.
   DOI: http://dx.doi.org/10.26682/sjuod.2018.21.1.1
6. Ibrahim, H. Z., **Salih, M. M.**: A New Type of Weakly Commutative Groups.
   Science Journal of University of Zakho. 5 (2), 228-231, 2017. ISSN: 2410-7549.
   DOI: http://dx.doi.org/10.25271/2017.5.2.373
7. **Salih, M. M.**, Ahmed, D.: α-Q-fuzzy Subgroups.
   Acad J. Nawroz Univ. 6 (3), 26-31, 2017. EISSN: 2520-789X.
   DOI: http://dx.doi.org/10.25007/ajnu.v6n3a74

Total IF of journals (all publications): 1,833
Total IF of journals (publications related to the dissertation): 1,833

The Candidate's publication data submitted to the iDEa Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

25 February, 2022