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ON THE BEST HARDY CONSTANT FOR QUASI–ARITHMETIC MEANS AND HOMOGENEOUS DEVIATION MEANS

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Abstract. The aim of this paper is to characterize the so-called Hardy means, i.e., those means $M\colon \bigcup_{n=1}^{\infty}\mathbb{R}_+^n \to \mathbb{R}_+$ that satisfy the inequality

$$\sum_{n=1}^{\infty} M(x_1,\dots,x_n) \leqslant C \sum_{n=1}^{\infty} x_n$$

for all positive sequences (x_n) with some finite positive constant C. The smallest constant C satisfying this property is called the Hardy constant of the mean M.

In this paper we determine the Hardy constant in the cases when the mean M is either a concave quasi-arithmetic or a concave and homogeneous deviation mean.

1. Introduction

Hardy's, Landau's, Carleman's and Knopp's celebrated inequalities in an equivalent and unified form state that

$$\sum_{n=1}^{\infty} \mathscr{P}_p(x_1,\dots,x_n) \leqslant C(p) \sum_{n=1}^{\infty} x_n, \tag{1}$$

for every sequences $(x_n)_{n=1}^{\infty}$ with positive terms, where $\mathscr{P}_p$ denotes the p-th *power mean* (extended to the limiting cases $p=\pm\infty$) and

$$C(p) := \begin{cases} 1 & p=-\infty, \\ (1-p)^{-1/p} & p\in(-\infty,0)\cup(0,1), \\ e & p=0, \\ +\infty & p\in[1,\infty], \end{cases}$$

and this constant is sharp, i.e., it cannot be diminished. First result of this type with nonoptimal constant was established by Hardy in the seminal paper [14]. Later it was

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improved and extended by Landau [18], Knopp [15], and Carleman [4] whose results are summarized in inequality (1). More about the history of the developments related to Hardy type inequalities is sketched in catching surveys by Pečarić–Stolarsky [37], Duncan–McGregor [12], and in a recent book of Kufner–Maligranda–Persson [17].

In a more general setting, for a given mean $M\colon \bigcup_{n=1}^{\infty} I^n \to I$ (where I is a real interval with $\inf I = 0$), let $\mathscr{H}(M)$ denote the smallest nonnegative extended real number, called the *Hardy constant of* M, such that

$$\sum_{n=1}^{\infty} M(x_1,\dots,x_n) \leqslant \mathscr{H}(M) \sum_{n=1}^{\infty} x_n$$

for all sequences $(x_n)_{n=1}^{\infty}$ belonging to I. If $\mathscr{H}(M)$ is finite, then we say that M is a *Hardy mean*. In this setup, a p-th power mean is a Hardy mean if and only if $p \in [-\infty, 1)$ and $\mathscr{H}(\mathscr{P}_p) = C(p)$ for all $p \in [-\infty, +\infty]$.

For investigating the Hardy property of means, we recall several notions that have been partly introduced and used in the paper [35]. Let $I \subseteq \mathbb{R}$ be an interval and let $M\colon \bigcup_{n=1}^{\infty} I^n \to I$ be an arbitrary mean.

We say that M is *symmetric*, *(strictly) increasing*, and *Jensen convex (concave)* if, for all $n \in \mathbb{N}$, the n-variable restriction $M|_{I^n}$ is a symmetric, (strictly) increasing in each of its variables, and Jensen convex (concave) on I^n, respectively. It is worth mentioning that means are locally bounded functions, therefore, the so-called Bernstein–Doetsch theorem implies that Jensen convexity (concavity) is equivalent to ordinary convexity (concavity) (cf. [3]). If $I = \mathbb{R}_+$, we can analogously define the notion of homogeneity of M. Finally, the mean M is called *repetition invariant* if, for all $n, m \in \mathbb{N}$ and $(x_1,\dots,x_n) \in I^n$, the following identity is satisfied

$$M(\underbrace{x_1,\dots,x_1}_{m\text{-times}},\dots,\underbrace{x_n,\dots,x_n}_{m\text{-times}}) = M(x_1,\dots,x_n).$$

Having all these definitions, let us recall the two main theorems of the paper [35]. The first result provides a lower estimation of Hardy constant.

THEOREM 1.1. *Let $I \subset \mathbb{R}_+$ be an interval with $\inf I = 0$ and $M\colon \bigcup_{n=1}^{\infty} I^n \to I$ be a mean. Then, for all non-summable sequences $(x_n)_{n=1}^{\infty}$ in I,*

$$\mathscr{H}(M) \geqslant \liminf_{n\to\infty} x_n^{-1} \cdot M(x_1, x_2, \dots, x_n).$$

In particular,

$$\mathscr{H}(M) \geqslant \sup_{y\in I} \liminf_{n\to\infty} \frac{n}{y} \cdot M\left(\frac{y}{1}, \frac{y}{2}, \dots, \frac{y}{n}\right).$$

Under stronger assumptions for the mean M, the latter lower estimate obtained above becomes equality by the following result.

THEOREM 1.2. *Let $M\colon \bigcup_{n=1}^{\infty} \mathbb{R}_+^n \to \mathbb{R}_+$ be an increasing, symmetric, repetition invariant, and Jensen concave mean. Then*

$$\mathscr{H}(M) = \sup_{y>0} \liminf_{n\to\infty} \frac{n}{y} \cdot M\left(\frac{y}{1}, \frac{y}{2}, \dots, \frac{y}{n}\right). \tag{2}$$

If, in addition, M is also homogeneous, then

$$\mathscr{H}(M) = \lim_{n\to\infty} n \cdot M\left(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}\right),$$

in particular this limit exists.

Upon taking M to be a power mean in the above theorem, the Hardy–Landau–Knopp–Carleman inequality (1) can easily be deduced. For the details, see [35].

The purpose of our paper is to find explicit formulas for the constant on the right hand side of equation (2) in two important classes of means: quasi-arithmetic means and homogeneous deviation means. On the other hand we will present some iff conditions for these means to satisfy assumptions of theorem above.

2. Families of means

From now on our consideration will go twofold. First, we will introduce and recall the most important results concerning quasi-arithmetic means. Later, in Section 2.2, we will do the same with the family of deviation means. This splitting will appear in the next section too.

2.1. Quasi-arithmetic means

Idea of quasi-arithmetic means first only glimpsed in a pioneering paper by Knopp [15]. Their theory was somewhat later axiomatized in a series of three independent but nearly simultaneous papers by De Finetti [11], Kolmogorov [16], and Nagumo [27] at the beginning of 1930s.

Let I be an interval and $f\colon I \to \mathbb{R}$ be a continuous, strictly monotone function. For $n \in \mathbb{N}$ and for a given vector $x = (x_1, \dots, x_n) \in I^n$, set

$$\mathscr{A}_f(x) := f^{-1}\left(\frac{f(x_1) + \cdots + f(x_n)}{n}\right).$$

The mean $\mathscr{A}_f\colon \bigcup_{n=1}^{\infty} I^n \to I$ defined this way is called the *quasi-arithmetic mean generated by the function f*. Quasi-arithmetic means are a natural generalization of power means. Indeed, whenever $I = \mathbb{R}_+$ and $f = \pi_p$, where $\pi_p(x) := x^p$ if $p \neq 0$ and $\pi_0(x) := \ln x$, then mean $\mathscr{A}_f$ coincides with p-th power mean (this is what was noticed by Knopp [15]). These means share most of the properties of power means. In particular, it is easy to verify that they are symmetric, strictly increasing, and repetition invariant. In fact, they admit even more properties of power means (cf. [16], [1]).

Immediately after the formal definition, Mulholland [26] characterized the Hardy property in the class of quasi-arithmetic means. Namely, he proved that

$$\mathscr{A}_f \text{ is a Hardy mean} \iff \begin{pmatrix} \text{there exist parameters } q < 1 \text{ and } C > 0 \\ \text{such that } \mathscr{A}_f(x) \leqslant C \cdot \mathscr{P}_q(x) \text{ for all } x \end{pmatrix}. \tag{3}$$

Later, in 1948, Mikusiński [25] proved that the comparability problem within this family can be (under natural smoothness assumptions) boiled down to pointwise comparability of the mapping $f \mapsto \frac{f''}{f'}$ (negative of this operator is called the *Arrow–Pratt index of absolute risk aversion*; cf. [2, 38]). More precisely, he proved

PROPOSITION 2.1. *Let $I \subset \mathbb{R}$ be an interval, $f, g\colon I \to \mathbb{R}$ be twice differentiable functions having nowhere vanishing first derivative. Then the following two conditions are equivalent*

(i) $\mathscr{A}_f(x_1,\dots,x_n) \leqslant \mathscr{A}_g(x_1,\dots,x_n)$ *for all $n \in \mathbb{N}$ and vector $(x_1,\dots,x_n) \in I^n$;*

(ii) $\frac{f''(x)}{f'(x)} \leqslant \frac{g''(x)}{g'(x)}$ *for all $x \in I$.*

Obviously, if $I \subseteq \mathbb{R}_+$, then condition (ii) can be equivalently written as

$$\kappa_f(x) := \frac{xf''(x)}{f'(x)} + 1 \leqslant \frac{xg''(x)}{g'(x)} + 1 =: \kappa_g(x) \qquad (x \in I).$$

It is easy to verify (with the notation π_p introduced above), that the equality $\kappa_{\pi_p} \equiv p$ holds for all $p \in \mathbb{R}$. Therefore, in view of Proposition 2.1, we have

$$\mathscr{P}_q = \mathscr{A}_{\pi_q} \leqslant \mathscr{A}_f \leqslant \mathscr{A}_{\pi_p} = \mathscr{P}_p,$$

where $q := \inf_I \kappa_f$ and $p := \sup_I \kappa_f$, moreover these parameters are sharp. In other words, the operator $\kappa_{(\cdot)}$ could be applied to embed quasi-arithmetic means into the scale of power means (cf. [36]). As a trivial consequence we obtain a natural estimations of Hardy constants

$$C(q) \leqslant \mathscr{H}(\mathscr{A}_f) \leqslant C(p). \tag{4}$$

In the next result we characterize Jensen concave quasi-arithmetic means. Some results in this direction have recently been obtained in the paper [5].

THEOREM 2.2. *Let $f\colon I \to \mathbb{R}$ be a twice continuously differentiable function with a nonvanishing first derivative. Then the quasi-arithmetic mean $\mathscr{A}_f$ is Jensen concave if and only if either f'' is identically zero or f'' is nowhere zero and the ratio function $\frac{f'}{f''}$ is a convex and negative function on I.*

Proof. Without loss of generality, we may assume that f is strictly increasing, then $f' > 0$ holds on I. The Jensen concavity of the mean $\mathscr{A}_f$ is equivalent to its concavity, that is, for all $n \in \mathbb{N}$ and for all $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in I^n$, $t \in [0,1]$, we have

$$\mathscr{A}_f(tx + (1-t)y) \geqslant t\mathscr{A}_f(x) + (1-t)\mathscr{A}_f(y),$$

in more detailed form,

$$f^{-1}\left(\frac{f(tx_1+(1-t)y_1)+\cdots+f(tx_n+(1-t)y_n)}{n}\right)$$
$$\geqslant tf^{-1}\left(\frac{f(x_1)+\cdots+f(x_n)}{n}\right)+(1-t)f^{-1}\left(\frac{f(y_1)+\cdots+f(y_n)}{n}\right).$$

Applying f to this inequality side by side and introducing the new variables $u_i := f(x_i)$ and $v_i := f(y_i)$, we get that the above inequality is equivalent to

$$\frac{f(tf^{-1}(u_1)+(1-t)f^{-1}(v_1))+\cdots+f(tf^{-1}(u_n)+(1-t)f^{-1}(v_n))}{n}$$
$$\geqslant f\left(tf^{-1}\left(\frac{u_1+\cdots+u_n}{n}\right)+(1-t)f^{-1}\left(\frac{v_1+\cdots+v_n}{n}\right)\right)$$

for all $u_1,\ldots,u_n,v_1,\ldots,v_n \in f(I) =: J$. The meaning of this inequality is exactly the Jensen convexity of the function

$$F_t(u,v) := f(tf^{-1}(u)+(1-t)f^{-1}(v))$$

on J^2. By the continuity of F_t, this property is equivalent to the convexity of F_t. In view of the regularity assumptions of the theorem, this property is satisfied if and only if the second derivative matrix $F_t''(u,v)$ is positive semidefinite for all $u,v \in J$ and $t \in (0,1)$. This is equivalent to the positive semidefiniteness of $F_t''(f(x),f(y))$ for all $x,y \in I$ and $t \in (0,1)$. By the Sylvester determinant test, this 2×2 matrix is positive semidefinite if and only if

$$\partial_1^2 F_t(f(x),f(y)) \geqslant 0, \qquad \partial_2^2 F_t(f(x),f(y)) \geqslant 0 \tag{5}$$

and

$$\partial_1^2 F_t(f(x),f(y))\partial_2^2 F_t(f(x),f(y)) - \partial_1\partial_2 F_t(f(x),f(y))^2 \geqslant 0. \tag{6}$$

We can easily obtain that

$$\begin{aligned}
\partial_1^2 F_t(f(x),f(y)) &= \frac{t^2 f''(tx+(1-t)y)}{f'(x)^2} - \frac{tf'(tx+(1-t)y)f''(x)}{f'(x)^3},\\
\partial_2^2 F_t(f(x),f(y)) &= \frac{(1-t)^2 f''(tx+(1-t)y)}{f'(y)^2} - \frac{(1-t)f'(tx+(1-t)y)f''(y)}{f'(y)^3},\\
\partial_1\partial_2 F_t(f(x),f(y)) &= \frac{t(1-t)f''(tx+(1-t)y)}{f'(x)f'(y)}.
\end{aligned}$$

Therefore, the first inequality in (5), is equivalent to

$$\frac{tf''(tx+(1-t)y)}{f'(tx+(1-t)y)} \geqslant \frac{f''(x)}{f'(x)}.$$

Putting $x=y$, it follows that $f''(x) \leqslant 0$ for all $x \in I$. If, for some $x \in I$, $f''(x)$ were zero, then this inequality and the nonpositivity of f'' implies that $f''(tx+(1-t)y)=0$ for all $y \in I$ and $t \in (0,1)$. Letting t tend to zero, we get that $f''(y)=0$ for all $y \in I$. Therefore, from now on, we may assume that f'' is strictly negative on I.

The inequality in (6) can now be rewritten in the following form:

$$t\frac{f'(x)}{f''(x)} + (1-t)\frac{f'(y)}{f''(y)} \geqslant \frac{f'(tx+(1-t)y)}{f''(tx+(1-t)y)} \qquad (x,y \in I, t \in (0,1)). \tag{7}$$

Thus, we have proved that the ratio function f'/f'' is convex and negative.

For the reversed implication, assume that either $f''=0$, or f'' is nowhere zero and f'/f'' is convex and negative. If $f''=0$, then $f(x)=ax+b$ for some real constants a,b, hence $\mathscr{A}_f$ equals the arithmetic mean, which is trivially Jensen concave (and also Jensen convex).

Without loss of generality, we again may assume that f' is positive. Suppose now that f'' is negative and f'/f'' is convex. As we have seen above, the negativity of the function f'' and the convexity of f'/f'', which is expressed by inequality (7) imply that (6) is satisfied. Using that f'' is negative (recall that f' is positive), inequality (7) yields that, for all $x,y\in I$, $t\in(0,1)$,

$$t\frac{f'(x)}{f''(x)}\geqslant\frac{f'(tx+(1-t)y)}{f''(tx+(1-t)y)}\quad\text{and}\quad(1-t)\frac{f'(y)}{f''(y)}\geqslant\frac{f'(tx+(1-t)y)}{f''(tx+(1-t)y)}.$$

A simple computation now shows that the two inequalities in (5) are also fulfilled. Therefore, the second derivative matrix of F_t is positive semidefinite on J^2, which implies the convexity of F_t for all $t\in(0,1)$. However, this property is equivalent to the concavity of the quasi-arithmetic mean $\mathscr{A}_f$. $\square$

2.2. Deviation means

Given a function $E\colon I\times I\to\mathbb{R}$ vanishing on the diagonal of $I\times I$, continuous and strictly decreasing with respect to the second variable, we can define a mean $\mathscr{D}_E\colon\bigcup_{n=1}^{\infty}I^n\to I$ in the following manner (cf. Daróczy [6]). For every $n\in\mathbb{N}$ and for every vector $x=(x_1,\dots,x_n)\in I^n$, the *deviation mean (or Daróczy mean)* $\mathscr{D}_E(x)$ is the unique solution y of the equation

$$E(x_1,y)+\ldots+E(x_n,y)=0.$$

By [28] deviation means are symmetric and repetition invariant. The increasingness of a deviation mean $\mathscr{D}_E$ is equivalent to the increasingness of the deviation E in its first variable. All these properties and characterizations are consequences of the general results obtained in a series of papers by Losonczi [19, 20, 22, 21, 23, 24] (for Bajraktarević means and Gini means) and by Daróczy [6, 7], Daróczy–Losonczi [8], Daróczy–Páles [9, 10] (for deviation means) and by Páles [28, 29, 30, 31, 32, 33, 34] (for deviation and quasi-deviation means).

Observe that if $E(x,y)=f(x)-f(y)$ for some continuous, strictly monotone function $f\colon I\to\mathbb{R}$, then the deviation mean $\mathscr{D}_E$ reduces to the quasi-arithmetic mean $\mathscr{A}_f$. Therefore, deviation means include quasi-arithmetic means. One can also notice that Bajraktarević means and Gini means also form subclasses of deviation means.

It is known [34] that a deviation mean generated by a continuous deviation function $E\colon\mathbb{R}_+^2\to\mathbb{R}$ is homogeneous if and only if E is of the form $E(x,y)=g(y)f(\frac{x}{y})$ for some continuous functions $f,g\colon\mathbb{R}_+\to\mathbb{R}$ such that f vanishes at 1 and g is positive. Clearly, the deviation mean generated by E is determined only by the function f, therefore, as we are going to deal with homogeneous deviation means only, let $\mathscr{E}_f$ denote the corresponding deviation mean. In the next section, we will determine the

Hardy constant for the homogeneous deviation mean $\mathscr{E}_f$ under general circumstances for f. The following result will be instrumental for our considerations.

THEOREM 2.3. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a strictly increasing concave function with $f(1)=0$. Then the function $E\colon \mathbb{R}_+^2 \to \mathbb{R}$ defined by $E(x,y) := f\big(\frac{x}{y}\big)$ is a deviation and the corresponding deviation mean $\mathscr{E}_f := \mathscr{D}_E$ is homogeneous, continuous, increasing and concave.*

Proof. It is easy to check, using the continuity (which is a consequence of concavity) and strict increasingness of f that the function E is a deviation function. The homogeneity of the mean $\mathscr{E}_f$ is obvious. We have that E is strictly increasing in its first variable, hence, one can show, that the mean $\mathscr{E}_f$ is also strictly increasing. The continuity of f implies the continuity of the mean directly. We only prove its concavity.

Let $x=(x_1,\dots,x_n)\in\mathbb{R}_+^n$ and $u=(u_1,\dots,u_n)\in\mathbb{R}_+^n$ and denote $\mathscr{E}_f(x)$ and $\mathscr{E}_f(u)$ by y and v, respectively. Then we have that

$$f\Big(\frac{x_1}{y}\Big)+\cdots+f\Big(\frac{x_n}{y}\Big)=0 \qquad\text{and}\qquad f\Big(\frac{u_1}{v}\Big)+\cdots+f\Big(\frac{u_n}{v}\Big)=0.$$

These equations and the convexity of f follow that

$$\begin{aligned}
0&=\frac{y}{y+v}\bigg(f\Big(\frac{x_1}{y}\Big)+\cdots+f\Big(\frac{x_n}{y}\Big)\bigg)+\frac{v}{y+v}\bigg(f\Big(\frac{u_1}{v}\Big)+\cdots+f\Big(\frac{u_n}{v}\Big)\bigg)\\
&=\bigg(\frac{y}{y+v}f\Big(\frac{x_1}{y}\Big)+\frac{v}{y+v}f\Big(\frac{u_1}{v}\Big)\bigg)+\cdots+\bigg(\frac{y}{y+v}f\Big(\frac{x_n}{y}\Big)+\frac{v}{y+v}f\Big(\frac{u_n}{v}\Big)\bigg)\\
&\leqslant f\Big(\frac{y}{y+v}\frac{x_1}{y}+\frac{v}{y+v}\frac{u_1}{v}\Big)+\cdots+f\Big(\frac{y}{y+v}\frac{x_n}{y}+\frac{v}{y+v}\frac{u_n}{v}\Big)\\
&=f\Big(\frac{x_1+u_1}{y+v}\Big)+\cdots+f\Big(\frac{x_n+u_n}{y+v}\Big).
\end{aligned}$$

Therefore,

$$\frac{\mathscr{E}_f(x)+\mathscr{E}_f(u)}{2}=\frac{y+v}{2}\leqslant\mathscr{E}_f\Big(\frac{x+u}{2}\Big),$$

which proves that $\mathscr{E}_f$ is Jensen concave. Being also continuous, this property implies the concavity of $\mathscr{E}_f$. $\square$

In fact one could strengthen the result of the above theorem, by showing that the concavity of f is not only sufficient but also necessary in order that the mean $\mathscr{E}_f$ be concave.

3. Main results

We will apply results contained in [35] to both quasi-arithmetic and homogeneous deviation means. Let us notice that the intersection of these families are power means, which are usually treated as a trivial case in a consideration of Hardy property. (See inequality (1).)

3.1. Quasi-arithmetic means

Let us begin this section with our main result, which allows us to compute the Hardy constant for a large class of quasi-arithmetic means.

Theorem 3.1. *Let I be a real interval with $\inf I = 0$ and let $f\colon I \to \mathbb{R}$ be a twice continuously differentiable function with nowhere vanishing first derivative. Define*

$$q := \liminf_{x\to 0^+} \kappa_f(x) \leqslant \limsup_{x\to 0^+} \kappa_f(x) =: p. \tag{8}$$

Then, for all $x \in I$,

$$C(q) \leqslant \liminf_{n\to\infty} \frac{n}{x} \mathscr{A}_f\big(\tfrac{x}{1}, \tfrac{x}{2}, \dots, \tfrac{x}{n}\big) \leqslant \limsup_{n\to\infty} \frac{n}{x} \mathscr{A}_f\big(\tfrac{x}{1}, \tfrac{x}{2}, \dots, \tfrac{x}{n}\big) \leqslant C(p) \tag{9}$$

and, consequently,

$$C(q) \leqslant \mathscr{H}(\mathscr{A}_f). \tag{10}$$

Observe that inequality (9) is the strengthening of (4).

Proof. We prove the right hand side inequality of (9) only, the proof of the left hand side inequality is completely analogous, therefore, its detailed proof is left to the reader.

If $p \in [1, +\infty]$, then $C(p) = +\infty$, therefore there is nothing to prove. Assume that $p \in [-\infty, 1)$. Consider any real number $r \in (p, 1) \setminus \{0\}$. Then there exists $\delta > 0$ such that

$$\kappa_f(x) \leqslant r \qquad \text{for all} \quad x \in (0, \delta). \tag{11}$$

Let $\varphi\colon I \to \mathbb{R}$ be a C^2 function with a non-vanishing first derivative such that, for all $x \in I$,

$$\kappa_\varphi(x) = \max\big(r, \kappa_f(x)\big). \tag{12}$$

(This equation for φ is a second-order differential equation, which has such solutions, see e.g. [36].) By (11), we have:

$$\kappa_\varphi(x) = r \qquad \text{for all} \quad x \in (0, \delta). \tag{13}$$

Therefore, there exists constants α, β such that $\varphi(x) = \alpha x^r + \beta$ for all $x \in (0, \delta)$.

Define $\varphi_0 := \frac{1}{\alpha}\varphi - \beta$. Then $\kappa_\varphi = \kappa_{\varphi_0}$, hence equality (12) holds with φ_0, too, furthermore $\mathscr{A}_\varphi = \mathscr{A}_{\varphi_0}$. Therefore, with no loss of generality, we may assume that φ and φ_0 coincide, i.e.

$$\varphi(x) = x^r, \qquad x \in (0, \delta]. \tag{14}$$

As a consequence, for the inverse function of φ, we have the formula

$$\varphi^{-1}(y) = y^{1/r} \qquad \text{for} \quad y \in \begin{cases} (0, \delta^r] & \text{if } r > 0, \\ [\delta^r, +\infty) & \text{if } r < 0. \end{cases} \tag{15}$$

Due to (12), we have $\kappa_f \leqslant \kappa_\varphi$ so, by Mikusiński's theorem, we get $\mathscr{A}_f \leqslant \mathscr{A}_\varphi$.

Fix now $x \in I$. We have

$$\limsup_{n\to\infty} \frac{n}{x}\mathscr{A}_f(\tfrac{x}{1},\tfrac{x}{2},\ldots,\tfrac{x}{n}) \leqslant \limsup_{n\to\infty} \frac{n}{x}\mathscr{A}_\varphi(\tfrac{x}{1},\tfrac{x}{2},\ldots,\tfrac{x}{n}) = \limsup_{n\to\infty} \frac{n}{x}\varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi\left(\frac{x}{i}\right)\right).$$

Now our argument splits into two similar cases depending on the sign of r.

Consider first the case $r>0$. By the increasingness of φ, the sequence $\left(\varphi(\frac{x}{n})\right)$ converges to zero, therefore there exists $n_0 \in \mathbb{N}$ such that, for all $n>n_0$,

$$\frac{1}{n}\sum_{i=1}^{n}\varphi\left(\frac{x}{i}\right) < \delta^r.$$

It also implies, for all $n>n_0$, that $\frac{x}{n}<\delta$, therefore, applying the construction of n_0 with (14) and (15), we get

$$\begin{aligned}\frac{n}{x}\varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi\left(\frac{x}{i}\right)\right) &= \frac{n}{x}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi\left(\frac{x}{i}\right)\right)^{1/r} = \left(\frac{n^{r-1}}{x^r}\sum_{i=1}^{n}\varphi\left(\frac{x}{i}\right)\right)^{1/r}\\ &= \left(\frac{n^{r-1}}{x^r}\sum_{k=1}^{n_0}\left(\varphi\left(\frac{x}{k}\right)-\left(\frac{x}{k}\right)^r\right)+\frac{n^{r-1}}{x^r}\sum_{k=1}^{n}\left(\frac{x}{k}\right)^r\right)^{1/r}.\end{aligned}$$

As $r-1<0$ we obtain that the first term tends to 0 as $n\to\infty$. The second term equals

$$\frac{n^{r-1}}{x^r}\sum_{k=1}^{n}\left(\frac{x}{k}\right)^r = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{-r} \longrightarrow \int_0^1 t^{-r}dt = \frac{1}{1-r}.$$

Thus

$$\limsup_{n\to\infty} \frac{n}{x}\mathscr{A}_f(\tfrac{x}{1},\tfrac{x}{2},\ldots,\tfrac{x}{n}) \leqslant \limsup_{n\to\infty} \frac{n}{x}\mathscr{A}_\varphi(\tfrac{x}{1},\tfrac{x}{2},\ldots,\tfrac{x}{n}) = (1-r)^{-1/r} = C(r).$$

With appropriate changes in the above argument, we can obtain that the same inequality holds also in the case $r<0$. Finally, upon passing the limit $r\to p$, we obtain

$$\limsup_{n\to\infty} \frac{n}{x}\mathscr{A}_f(\tfrac{x}{1},\tfrac{x}{2},\ldots,\tfrac{x}{n}) \leqslant C(p).$$

Completely analogous considerations lead to the inequality

$$\liminf_{n\to\infty} \frac{n}{x}\mathscr{A}_f(\tfrac{x}{1},\tfrac{x}{2},\ldots,\tfrac{x}{n}) \geqslant C(q).$$

Finally, using this inequality and the second inequality of Theorem 1.1, it follows that (10) is also valid. Thus the proof is complete. □

Combining Proposition 2.1 with Theorem 3.1 we immediately obtain

THEOREM 3.2. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a twice continuously differentiable function with nowhere vanishing first derivative. If the limit $p := \lim_{x\to 0^+}\kappa_f(x)$ exists and $\kappa_f(x) \leqslant p$ for every $x>0$, then $\mathscr{H}(\mathscr{A}_f) = C(p)$. In particular, this mean is Hardy if and only if $p<1$.*

Proof. By (4) we get $\mathscr{H}(\mathscr{A}_f) \leqslant C(\sup \kappa_f(x)) = C(p)$. The converse inequality is implied by Theorem 3.1 – more precisely by inequality (10). □

The following result strengthens Mulholland's theorem [26] in the class of concave quasi-arithmetic means (compare (3) and the implication (i) ⇒ (ii) below).

THEOREM 3.3. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a twice continuously differentiable function with nowhere vanishing first and second derivatives such that f'/f'' is convex and negative. Then κ_f is a decreasing function. Furthermore, the following assertions are equivalent:*

(i) $\mathscr{A}_f$ is a Hardy mean;

(ii) There exists a parameter $q<1$ such that $\mathscr{A}_f \leqslant \mathscr{P}_q$;

(iii) $p := \lim_{x\to 0^+} \kappa_f(x) < 1$.

And, in each of the above cases, $\mathscr{A}_f \leqslant \mathscr{P}_p$, and $\mathscr{H}(\mathscr{A}_f) = C(p)$.

Proof. By the convexity and negativity of the function f'/f'', the mapping $x \mapsto f'(x)/(x\cdot f''(x))$ is negative and increasing. Therefore, κ_f is decreasing, whence the right limit p of κ_f at zero exists and $\kappa_f \leqslant p$. Then, by Proposition 2.1, it follows that $\mathscr{A}_f \leqslant \mathscr{P}_p$.

To prove $(i) \Rightarrow (ii)$, assume that $\mathscr{A}_f$ is a Hardy mean. Then Theorem 3.2 implies that $p<1$, hence (ii) holds with $q=p$. The implication $(ii) \Rightarrow (iii)$ is obvious.

Finally, we prove $(iii) \Rightarrow (i)$. In view of Theorem 2.2, the convexity and negativity of f'/f'' implies that $\mathscr{A}_f$ is a concave quasi-arithmetic mean. Therefore, by Theorems 1.2 and 3.1,

$$\begin{aligned}\mathscr{H}(\mathscr{A}_f) &= \sup_{y>0} \liminf_{n\to\infty} \frac{n}{y}\cdot \mathscr{A}_f\left(\frac{y}{1},\frac{y}{2},\ldots,\frac{y}{n}\right)\\ &\leqslant \sup_{y>0} \limsup_{n\to\infty} \frac{n}{y}\cdot \mathscr{A}_f\left(\frac{y}{1},\frac{y}{2},\ldots,\frac{y}{n}\right) \leqslant C(p),\end{aligned}$$

where $p<1$, hence $\mathscr{A}_f$ is a Hardy mean.

To complete the proof, observe that, by the convexity and negativity of f'/f'', the mean $\mathscr{A}_f$ is concave, therefore Theorem 3.2 implies the equality $\mathscr{H}(\mathscr{A}_f) = C(p)$. □

3.2. Homogeneous deviation means

THEOREM 3.4. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a strictly increasing concave function with $f(1)=0$. Then the homogeneous deviation mean $\mathscr{E}_f$ is a Hardy mean if and only if*

$$\int_0^1 f\left(\frac{1}{t}\right)dt < +\infty \tag{16}$$

and, if the above inequality holds, then its Hardy constant is the unique positive solution c of the equation

$$\int_0^c f\left(\frac{1}{t}\right)dt = 0.$$

Proof. By Theorem 2.3, the deviation mean $\mathscr{E}_f$ is homogeneous, continuous, increasing and concave. Therefore, applying the second assertion of Theorem 1.2, we have that

$$\mathscr{H}(\mathscr{E}_f) = \lim_{n\to\infty} \mathscr{E}_f\left(\frac{n}{1}, \frac{n}{2}, \ldots, \frac{n}{n}\right).$$

Assume first that $\mathscr{E}_f$ possesses the Hardy property. Then $\mathscr{H}(\mathscr{E}_f)$ is finite, and hence the sequence $\left(\mathscr{E}_f\left(\frac{n}{1}, \frac{n}{2}, \ldots, \frac{n}{n}\right)\right)$ is bounded. Thus, there exists $K > 0$ such that, for all $n \in \mathbb{N}$,

$$\mathscr{E}_f\left(\frac{n}{1}, \frac{n}{2}, \ldots, \frac{n}{n}\right) \leqslant K, \tag{17}$$

which is equivalent to

$$f\left(\frac{n}{K}\right) + f\left(\frac{n}{2K}\right) + \cdots + f\left(\frac{n}{nK}\right) \leqslant 0.$$

For the sake of brevity define the function $f^*\colon \mathbb{R}_+ \to \mathbb{R}$ by $f^*(t) := f\left(\frac{1}{t}\right)$. Then f^* is a continuous strictly decreasing function with $f^*(1) = 0$. Hence, the above inequality is equivalent to

$$\frac{K}{n}\left(f^*\left(\frac{K}{n}\right) + f^*\left(\frac{2K}{n}\right) + \cdots + f^*\left(\frac{nK}{n}\right)\right) \leqslant 0. \tag{18}$$

By the decreasingness of f^*, for all $i \in \{1, 2, \ldots, n\}$, we have that

$$\int_{\frac{iK}{n}}^{\frac{(i+1)K}{n}} f^*(t)dt \leqslant \frac{K}{n} f^*\left(\frac{iK}{n}\right).$$

After adding up these inequalities side by side and using (18), we get

$$\int_{\frac{K}{n}}^{\frac{(n+1)K}{n}} f^*(t)dt \leqslant 0. \tag{19}$$

Hence, for all $n \geqslant K$,

$$\int_{\frac{K}{n}}^{1} f^*(t)dt \leqslant -\int_{1}^{\frac{(n+1)K}{n}} f^*(t)dt.$$

Upon taking the limit $n \to \infty$, we arrive at the inequality

$$(0 \leqslant) \int_0^1 f^*(t)dt \leqslant -\int_1^K f^*(t)dt,$$

which proves that condition (16) must be valid.

Now assume that (16) holds. Define the function $F\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$F(x) := \int_0^x f\left(\frac{1}{t}\right)dt = \int_0^x f^*(t)dt.$$

Obviously, F is continuous, strictly increasing on $(0,1)$, and strictly decreasing on $(1,\infty)$. To show that it has a zero on $(1,\infty)$, it suffices to prove that $F(x)$ tends to $-\infty$ as $x \to +\infty$.

By the concavity of f, there exists a positive constant $a \in \mathbb{R}$ such that, for all $x \in \mathbb{R}_+$,

$$f(x) = f(x) - f(1) \leqslant a(x-1).$$

Therefore, for $x \geqslant 1$, we get

$$\begin{aligned} F(x) = \int_0^x f\Big(\frac{1}{t}\Big)dt &\leqslant \int_0^1 f\Big(\frac{1}{t}\Big)dt + a\int_1^x \Big(\frac{1}{t} - 1\Big)dt \\ &\leqslant \int_0^1 f\Big(\frac{1}{t}\Big)dt + a(\ln(x) - x + 1). \end{aligned}$$

The right hand side estimate tends to $-\infty$ as $x \to +\infty$, therefore F also has this property. This ensures that F has a unique zero, denoted by c, in the interval $(1,\infty)$.

In the rest of the proof, we show that $c = \mathscr{H}(\mathscr{E}_f)$. Let $K > \mathscr{H}(\mathscr{E}_f)$ be an arbitrary number. Then there exists n_0 such that, for all $n \geqslant n_0$, we have the inequality (17). Repeating the same argument as above, this inequality implies (19) for all $n \geqslant n_0$. Upon taking the limit $n \to \infty$, we obtain that

$$F(K) = \int_0^K f^*(t)dt \leqslant 0.$$

Consequently, $c \leqslant K$. Taking now the limit $K \to \mathscr{H}(\mathscr{E}_f)$, we get that $c \leqslant \mathscr{H}(\mathscr{E}_f)$.

The proof of the reversed inequality $c \geqslant \mathscr{H}(\mathscr{E}_f)$ is completely analogous, therefore the equality $c = \mathscr{H}(\mathscr{E}_f)$ holds as desired. □

In order to formulate a corollary of this theorem for Gini means (cf. [13]), we introduce the following notation: Given two real numbers $p,q \in \mathbb{R}$, define the function $\chi_{p,q}\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\chi_{p,q}(x) := \begin{cases} \dfrac{x^p - x^q}{p-q} & \text{if } p \neq q, \\ x^p \ln(x) & \text{if } p = q. \end{cases}$$

In this case, the function $E_{p,q}\colon \mathbb{R}_+^2 \to \mathbb{R}$ defined by

$$E_{p,q}(x,y) := y^p \chi_{p,q}\Big(\frac{x}{y}\Big)$$

is a deviation function on $\mathbb{R}_+$. The deviation mean generated by $E_{p,q}$ will be denoted by $\mathscr{G}_{p,q}$ and called the Gini mean of parameter p,q (cf. [13]). One can easily see that $\mathscr{G}_{p,q}$ has the following explicit form:

$$\mathscr{G}_{p,q}(x_1,\dots,x_n) := \begin{cases} \left(\dfrac{x_1^p + \cdots + x_n^p}{x_1^q + \cdots + x_n^q}\right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \exp\left(\dfrac{x_1^p \ln(x_1) + \cdots + x_n^p \ln(x_n)}{x_1^p + \cdots + x_n^p}\right) & \text{if } p = q. \end{cases} \tag{20}$$

Clearly, in the particular case $q = 0$, the mean $\mathscr{G}_{p,q}$ reduces to the pth Hölder mean $\mathscr{P}_p$. It is also obvious that $\mathscr{G}_{p,q} = \mathscr{G}_{q,p}$.

COROLLARY 3.5. *Gini means $\mathscr{G}_{p,q}$ is increasing and concave if and only if*

$$\min(p,q) \leqslant 0 \leqslant \max(p,q) \leqslant 1. \tag{21}$$

In this case, $\mathscr{G}_{p,q}$ is a Hardy mean if and only if $\max(p,q) < 1$ *and then*

$$\mathscr{H}(\mathscr{G}_{p,q}) = \begin{cases} \left(\dfrac{1-p}{1-q}\right)^{\frac{1}{q-p}} & p \neq q, \\ e & p = q = 0. \end{cases}$$

Proof. By the results of Losonczi [20], [22], $\mathscr{G}_{p,q}$ is concave if and only if (21) holds. On the other hand, $\mathscr{G}_{p,q}$ is increasing if and only if the first two inequalities in (21) are satisfied, that is, if 0 is between p and q.

Now, for $p \neq q$, in view of Theorem 3.4, with $f = \chi_{p,q}$, we get that $\mathscr{G}_{p,q}$ is a Hardy mean if and only if

$$\int_0^1 \chi_{p,q}\left(\frac{1}{t}\right) dt < +\infty.$$

For $p \neq q$, we have

$$\int_0^1 \chi_{p,q}\left(\frac{1}{t}\right) dt = \frac{1}{p-q} \int_0^1 (t^{-p} - t^{-q})\, dt,$$

which is finite if and only if $\max(p,q) < 1$. In that case, the Hardy constant c of the mean $\mathscr{G}_{p,q}$ satisfies

$$0 = \int_0^c \chi_{p,q}\left(\frac{1}{t}\right) dt = \frac{1}{p-q} \int_0^c (t^{-p} - t^{-q})\, dt = \frac{1}{p-q}\left(\frac{1}{1-p}c^{1-p} - \frac{1}{1-q}c^{1-q}\right).$$

Solving this equation with respect to c, we obtain that

$$c = \left(\frac{1-p}{1-q}\right)^{\frac{1}{q-p}}.$$

In the case $p = q = 0$, we get

$$\int_0^1 \chi_{0,0}\left(\frac{1}{t}\right) dt = -\int_0^1 \ln t\, dt = 1 < +\infty,$$

proving that $\mathscr{G}_{0,0}$ is a Hardy mean. For its Hardy constant c it remains to find the positive solution of the equation

$$0 = \int_0^c \chi_{0,0}\left(\frac{1}{t}\right) dt = -\int_0^c \ln t\, dt = c(1 - \ln c),$$

which results the solution $c = e$. $\square$

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