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Generalized Bajraktarević means

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written by Richárd Grünwald
supervised by Prof. Dr. Zsolt Páles

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Debrecen, 2024.

Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at Debrecen University.
The results published in the thesis are not reported in any other PhD theses.

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Hereby I confirm that Richárd Grünwald candidate conducted his studies with my supervision within the Mathematical analysis, functional equations and inequalities Doctoral Program of the Doctoral School of Doctoral School of Mathematical and Computational Sciences between 2020 and 2024. The independent studies and research work of the candidate significantly contributed to the results published in the thesis.
I also declare that the results published in the thesis are not reported in any other theses.
I support the acceptance of the thesis.

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signature of the supervisor

GENERALIZED BAJRAKTAREVIĆ MEANS

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Dissertation advisor: Dr. Zsolt Páles

The official opponents of the dissertation:

	Dr.	....
	Dr.	....

The evaluation committee:

chairperson:	Dr.	....
members:	Dr.	....
	Dr.	....
	Dr.	....
	Dr.	....

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Contents

Notations

The following list outlines various symbols that will be utilized later in the document. In this list I denotes a nonempty open real interval.

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of positive real numbers, i.e., the set $\mathbb{R} \cap]0, +\infty[$.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{Z}$	The set of integer numbers.
$\mathbb{N}$	The set of positive integers.
$\{m, \ldots, \ell\}$	The set of integer numbers between m and ℓ, i.e., the set $\mathbb{Z} \cap [m, \ell]$ whenever $m, \ell \in \mathbb{Z}$ with $m \leq \ell$
S^n	The set of those n-dimensional column vectors whose entries are all from the set S, i.e., the Cartesian product $\underbrace{S \times \cdots \times S}_{n \text{ times}}$.
$\mathbb{R}^{n \times k}$	The set of $(n \times k)$-type real matrices, i.e., $\big((\mathbb{R}^n)^T\big)^k$, where $n, k \in \mathbb{N}$.
$[x]^T$	The transpose of a matrix $x \in \mathbb{R}^{n \times k}$, where $n, k \in \mathbb{N}$.
X^T	The set of the transposes of the matrices belonging to the set $X \subseteq \mathbb{R}^{n \times k}$, i.e., the set $\{[x]^T \mid x \in X \subseteq \mathbb{R}^{n \times k}\}$.
x_i	The ith row of a matrix $x \in \mathbb{R}^{n \times k}$, where $n, k \in \mathbb{N}, i \in \{1, \ldots, n\}$, in particular, if $k = 1$ then the ith element of a column vector $x \in \mathbb{R}^n$.
x^i	The ith column of a matrix $x \in \mathbb{R}^{n \times k}$, where $n, k \in \mathbb{N}, i \in \{1, \ldots, k\}$, in particular, if $n = 1$ then the ith element of a row vector $x \in (\mathbb{R}^k)^T$.
$\operatorname{diag}(S^n)$	The diagonal of the set $S^n \subseteq \mathbb{R}^n$, i.e., $\{(x, \ldots, x) \in \mathbb{R}^n \mid x \in S\}$.
$\Delta_n(x)$	$(x, \ldots, x) \in S^n$, where $x \in S \subseteq \mathbb{R}, n \in \mathbb{N}$.
$G^\Delta(x)$	$G(\Delta_n(x))$, where $x \in S \subseteq \mathbb{R}, n \in \mathbb{N}$.

$\Delta_n^k(y)$	The $n \times k$ matrix whose jth column equals $\Delta_n(y_j)$ for $j \in \{1, \dots, k\}$, where $y \in S^k \subseteq \mathbb{R}^k, n, k \in \mathbb{N}$.
$\mathcal{A}^{[n]}$	The n-variable arithmetic mean.
$\mathcal{G}^{[n]}$	The n-variable geometric mean.
$\mathcal{H}^{[n]}$	The n-variable harmonic mean.
$H_r^{[n]}$	The n-variable rth power mean, where $r \in \mathbb{R}$.
$H_{r;\lambda}^{[n]}$	The n-variable weighted rth power mean with weight vector $\lambda \in \mathbb{R}_+^n$, $\lambda_1 + \dots + \lambda_n = 1$, where $r \in \mathbb{R}$.
$G_{r,s}^{[n]}$	The n-variable Gini mean with pair of parameters $(r, s) \in \mathbb{R}^2$.
$G_{r,s;\lambda}^{[n]}$	The n-variable weighted Gini mean with pair of parameters $(r, s) \in \mathbb{R}^2$ and weight vector $\lambda \in \mathbb{R}_+^n$, $\lambda_1 + \dots + \lambda_n = 1$.
$A_f^{[n]}$	The n-variable quasi-arithmetic mean with generator function $f\colon I \to \mathbb{R}$.
$B_{f,p}^{[n]}$	The n-variable Bajraktarević mean with generator function $f\colon I \to \mathbb{R}$ and weight function $p\colon I \to \mathbb{R}_+$.
$A_{f,p}^{[n]}$	The n-variable generalized Bajraktarević mean with generator function $f\colon I \to \mathbb{R}$ and weight function $p\colon I \to \mathbb{R}_+^n$.
$\mathrm{conv}(S)$	The convex hull of the set $S \subseteq \mathbb{R}$.
$f^{(-1)}$	The generalized left inverse of the strictly monotone function $f\colon I \to \mathbb{R}$.
p_0	The sum of the coordinate functions of a weight function $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$, i.e., $p_1 + \dots + p_n$, where $n \in \mathbb{N}$.
$\delta_{i_1,\dots,i_k}$	The extended Kronecker symbol, i.e., for $k \in \mathbb{N}, k \geq 2, i_1, \dots, i_k \in \mathbb{N}$, $$\delta_{i_1,\dots,i_k} := \begin{cases} 1 & \text{if } i_1 = \dots = i_k \\ 0 & \text{otherwise.} \end{cases}$$
e_i	The ith vector of the standard base of $\mathbb{R}^n$, i.e., $(\delta_{ij})_{j=1}^n \in \mathbb{R}^n$, where $n \in \mathbb{N}$.

Introduction

The theory of means holds significant importance in the study of functional equations and inequalities due to its ability to bridge various mathematical concepts. Means provide a framework for understanding the behavior and relationships between different mathematical objects, offering insights into functional relationships and inequalities. By analyzing the properties and characteristics of means, mathematicians can develop techniques to solve complex functional equations and inequalities, making the theory of means an indispensable tool in mathematical analysis and problem-solving. Moreover, means serve as essential tools for establishing connections between different mathematical domains, facilitating interdisciplinary research and applications across various fields of mathematics. It is still actively researched nowadays. Without wishing to be exhaustive, we mention the books [5, 6, 15] as some of the most fundamental, comprehensive works on the subject.

In this introduction, let $I \subseteq \mathbb{R}$ denote a nonempty open interval. For $n \in \mathbb{N}$, a function $M\colon I^n \to \mathbb{R}$ is said to be an *n-variable mean* if

$$\min(x) \leq M(x) \leq \max(x)$$

holds for all $x \in I^n$. We say that M is a *strict n-variable mean* if it is an n-variable mean and the above inequalities are strict whenever $\min(x) < \max(x)$ holds.

Let $n \in \mathbb{N}$. Some of the most classical means are the *n-variable arithmetic mean* $\mathcal{A}^{[n]}\colon \mathbb{R}^n \to \mathbb{R}$, the *$n$-variable geometric mean* $\mathcal{G}^{[n]}\colon \mathbb{R}_+^n \to \mathbb{R}_+$, and the *$n$-variable harmonic mean* $\mathcal{H}^{[n]}\colon \mathbb{R}_+^n \to \mathbb{R}_+$, which are defined as follows

$$\mathcal{A}^{[n]}(x) := \frac{x_1 + \cdots + x_n}{n}, \quad \mathcal{G}^{[n]}(x) := \sqrt[n]{x_1 \cdots x_n}, \quad \mathcal{H}^{[n]}(x) := \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

A notion which subsumes these means is the concept of power means. For $n \in \mathbb{N}$ and $r \in \mathbb{R}$, the *n-variable rth power or rth Hölder mean* $H_r^{[n]}\colon \mathbb{R}_+^n \to \mathbb{R}_+$ is defined by

$$H_r^{[n]}(x) := \begin{cases} \left(\dfrac{x_1^r + \cdots + x_n^r}{n}\right)^{\frac{1}{r}} & \text{if } r \neq 0 \\ \sqrt[n]{x_1 \cdots x_n} & \text{if } r = 0. \end{cases}$$

The aforementioned means are indeed power means by choosing $r = 1, r = 0$, and $r = -1$, respectively. The theory of power means has extensive literature, most of the details of their theory can be found in the monographs [5,6,15,29,30]. The class of power means has been extended in numerous ways. One of the first extensions was introduced by Gini in the paper [11], who for a pair of parameters $(r, s) \in \mathbb{R}^2$ defined the mean $G_{r,s}^{[n]}\colon \mathbb{R}_+^n \to \mathbb{R}_+$ by

$$G_{r,s}^{[n]}(x) := \begin{cases} \left(\dfrac{x_1^r + \cdots + x_n^r}{x_1^s + \cdots + x_n^s}\right)^{\frac{1}{r-s}} & \text{if } r \neq s \\ \exp\left(\dfrac{x_1^r \ln(x_1) + \cdots + x_n^r \ln(x_n)}{x_1^r + \cdots + x_n^r}\right) & \text{if } r = s. \end{cases}$$

This mean is nowadays called the *Gini mean*. The power means form a subclass of Gini means. Indeed, for $r \in \mathbb{R}$, we have $H_r^{[n]} = G_{r,0}^{[n]} = G_{0,r}^{[n]}$. It is important to mention the class of *quasi-arithmetic means* (Definition 1.1.4), too, which, when symmetrically weighted with one weight function, give the definition of *Bajraktarević means* (Definition 1.1.5). These can be generalized even further by considering possibly different weight functions and omitting the continuity of the generator (Definition 1.1.6).

In this thesis, whenever a regularity property is assumed to be valid at each point of the domain of a function, then we do not emphasize the set on which the property in question holds.

Let $n, k \in \mathbb{N}$. In the sequel, the ith entry of a real vector

$$x := (x_i)_{i \in \{1,\ldots,n\}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

will be denoted by x_i, and analogously, the ith row and jth column of a real matrix

$$x := (x_i^j)_{(i,j) \in \{1,\ldots,n\} \times \{1,\ldots,k\}} = \begin{pmatrix} x_1^1 & \cdots & x_1^k \\ \vdots & & \vdots \\ x_n^1 & \cdots & x_n^k \end{pmatrix} \in \mathbb{R}^{n \times k}$$

will be denoted by x_i and x^j, respectively. For convenience, we identify $\mathbb{R}^{n \times k}$ by $\big((\mathbb{R}^n)^T\big)^k$ in the standard manner, where the transpose of a set of matrices (in particular, vectors) is introduced below. We define the *transpose* $[x]^T \in \mathbb{R}^{k \times n}$ *of the matrix* $x \in \mathbb{R}^{n \times k}$ by

$$[x]^T := (x_j^i)_{(i,j) \in \{1,\ldots,k\} \times \{1,\ldots,n\}} = \begin{pmatrix} x_1^1 & \cdots & x_n^1 \\ \vdots & & \vdots \\ x_1^k & \cdots & x_n^k \end{pmatrix}.$$

More generally, for a subset of matrices $X \subseteq \mathbb{R}^{n \times k}$, the *transpose* X^T *of* X denotes the set $\{[x]^T \mid x \in X\}$. For $S \subseteq \mathbb{R}$, the *diagonal* $\operatorname{diag}(S^n)$ *of* S^n is defined by

$$\operatorname{diag}(S^n) := \{(x, \ldots, x) \in \mathbb{R}^n \mid x \in S\}.$$

Let us introduce, the *diagonal map* $\Delta_n\colon S\to \operatorname{diag}(S^n)$ by

$$\Delta_n(x) := (x,\dots,x)\in S^n,$$

and, for an n-variable function $G\colon S^n\to\mathbb{R}$, the function $G^\Delta\colon S\to\mathbb{R}$ by

$$G^\Delta(x) := G(\Delta_n(x)) \qquad (x\in S).$$

More generally, we can define the map $\Delta_n^k\colon S^k\to(\operatorname{diag}(S^n))^k\subseteq\mathbb{R}^{n\times k}$: if $y\in S^k$, then let $\Delta_n^k(y)$ denote the $(n\times k)$-type matrix whose jth column equals $\Delta_n(y_j)$ for $j\in\{1,\dots,k\}$. Lastly, for $k\in\mathbb{N}, k\geq 2, i_1,\dots,i_k\in\mathbb{N}$, we define the *extended Kronecker symbol* by

$$\delta_{i_1,\dots,i_k} := \begin{cases} 1 & \text{if } i_1=\dots=i_k\\ 0 & \text{otherwise.}\end{cases}$$

Let $n\in\mathbb{N}, \emptyset\neq S\subseteq\mathbb{R}$ and let us consider the functional inequality

$$F(x)\leq G(x)$$

for the functions $F,G\colon S^n\to\mathbb{R}$. We say that the above *inequality holds in the local sense*, or F *is locally smaller than or equal to* G *(*F *is locally nongreater than* G*)*, or G *is locally greater than or equal to* F *(*G *is locally nonsmaller than* F*)*, if there exists a nonempty open set $U\subseteq S^n$ which contains $\operatorname{diag}(S^n)$ and the above inequality is satisfied for all $x\in U$. Similarly, we say that the above *inequality holds in the global sense*, or F *is globally smaller than or equal to* G *(*F *is globally nongreater than* G*)*, or G *is globally greater than or equal to* F *(*G *is globally nonsmaller than* F*)*, if the inequality in question holds true for all $x\in S^n$. Observe that, the local and the global validity of the functional equation $F=G$ is included in this definition. Clearly, the global validity of a functional inequality or equation implies its local validity.

After clarifying the basic concepts, let us review the structure of the dissertation. It is based on the papers [12–14] and consists of 4 main chapters. In the first one we introduce the definition of the aforementioned generalized Bajraktarević mean and compute its partial derivatives up to third-order. Then, separated into 3 further chapters, we investigate some important functional equations and inequalities related to the class of means in question.

In the second one we deal with the equality problem of n-variable generalized Bajraktarević means, i.e., the functional equation

$$f^{(-1)}\left(\frac{p_1(x_1)f(x_1)+\dots+p_n(x_n)f(x_n)}{p_1(x_1)+\dots+p_n(x_n)}\right) = g^{(-1)}\left(\frac{q_1(x_1)g(x_1)+\dots+q_n(x_n)g(x_n)}{q_1(x_1)+\dots+q_n(x_n)}\right),$$

where $n\in\mathbb{N}, n\geq 2$ is fixed, $x\in I^n$, the unknown functions $f,g\colon I\to\mathbb{R}$ are strictly monotone, furthermore, $p=(p_1,\dots,p_n)\colon I\to\mathbb{R}_+^n$ and $q=(q_1,\dots,q_n)\colon I\to\mathbb{R}_+^n$ are also unknown functions. The functions $f^{(-1)}$ and $g^{(-1)}$ denote the generalized left inverse of f and g, respectively, which will be defined precisely in Definition 1.1.2.

In the third chapter we investigate the local and global comparison of two n-variable generalized Bajraktarević means, i.e., we establish necessary as well as sufficient conditions in terms of the unknown functions $f, g, p_1, \dots, p_n, q_1, \dots, q_n$ for the comparison inequality

$$f^{-1}\left(\frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)}\right) \\ \leq g^{-1}\left(\frac{q_1(x_1)g(x_1) + \cdots + q_n(x_n)g(x_n)}{q_1(x_1) + \cdots + q_n(x_n)}\right)$$

to be valid in the local and in the global sense. Here the elements $x_1, \dots, x_n \in I$ are arbitrary, $f, g\colon I \to \mathbb{R}$ are assumed to be continuous, strictly monotone and the functions $p_1, \dots, p_n, q_1, \dots, q_n\colon I \to \mathbb{R}_+$ are positive valued.

The aim of the fourth chapter is to investigate inequalities that are analogous to the Hölder and Minkowski inequalities by replacing the addition and the multiplication by a more general operation, and instead of using power means, generalized Bajraktarević means, and in particular, weighted Gini means, are considered. A further aim is to characterize such inequalities both in the local and in the global sense.

Lastly, unnumbered chapters within the dissertation encompass summaries in both English and Hungarian, bibliography, and a compilation of scientific talks related to the topics of the dissertation.

Chapter 1

Basic notions and results

The symbol I stands for a nonempty open real interval in this chapter, in which we introduce the definition of the generalized Bajraktarević mean and compute its partial derivatives up to third-order. The results are contained in our paper [12].

1.1 Definition of generalized Bajraktarević means

To extend the notion of Bajraktarević means, we shall need the following lemma about the existence and properties of the left inverse of strictly monotone (but not necessarily continuous) functions.

Proposition 1.1.1. *Let $f\colon I \to \mathbb{R}$ be a strictly monotone function. Then there exists a uniquely determined monotone function $g\colon \operatorname{conv}(f(I)) \to I$ such that g is the left inverse of f, i.e.,*

$$(g \circ f)(x) = x \qquad (x \in I). \tag{1.1}$$

Furthermore, g is monotone in the same sense as f, it is continuous,

$$(f \circ g)(y) = y \qquad (y \in f(I)), \tag{1.2}$$

and

$$\liminf_{x\to g(y)} f(x) \le y \le \limsup_{x\to g(y)} f(x) \qquad (y \in \operatorname{conv}(f(I))). \tag{1.3}$$

Thus, if f is lower (resp. upper) semicontinuous at $g(y)$, then $f \circ g(y) \le y$ (resp. $y \le f \circ g(y)$).

Proof. Without loss of generality, we may assume that $f\colon I \to \mathbb{R}$ is a strictly increasing function. Then $f\colon I \to f(I)$ is a bijection. The interval I is open, therefore, f has a left and a right limit at every point $x \in I$, which will be denoted by $f_-(x)$ and $f_+(x)$, respectively. We introduce the notation $J_x := [f_-(x), f_+(x)]$, where $x \in I$. Then, for all elements $u < x < v \in I$, we have that

$$f_+(u) < f_-(x) \le f(x) \le f_+(x) < f_-(v).$$

From these inequalities, it follows that $f(x) \in J_x$ holds for all $x \in I$ and $J_x \cap J_u = \emptyset$ whenever u is distinct from x.

The convex hull of $f(I)$ is the smallest interval $J \subseteq \mathbb{R}$ containing $f(I)$. The openness of I implies that $\inf f(I), \sup f(I) \notin J$, hence $J :=]\inf f(I), \sup f(I)[$. We show that

$$J = \bigcup_{x \in I} J_x. \tag{1.4}$$

If $x \in I$, then, for all $u < x$, we have $f_-(x) > f_+(u) = \inf_{u<t} f(t) \geq \inf f(I)$. Similarly, $f_+(x) < \sup f(I)$, therefore, $J_x \subseteq J$. This proves the inclusion $\supseteq$ in (1.4). To prove the reversed inclusion in (1.4), let $y \in J$. Define

$$x := \sup\{u \in I \mid f(u) \leq y\}.$$

Then, for all $n \in \mathbb{N}$, there exists $u_n \in I$ such that $x - \frac{1}{n} < u_n$ and $f(u_n) \leq y$. Thus, $u_n \leq x$ and hence u_n tends to x as $n \to \infty$. Therefore,

$$f_-(x) \leq \limsup_{n\to\infty} f(u_n) \leq y.$$

On the other hand, let $u_n \in I$ be an arbitrary sequence converging to x such that $x < u_n$. Then $y < f(u_n)$, whence we obtain

$$y \leq \lim_{n\to\infty} f(u_n) = f_+(x).$$

The above inequalities imply that $y \in J_x$ holds, which completes the proof of the inclusion $\subseteq$ in (1.4).

Let $y \in J = \operatorname{conv}(f(I))$ be an arbitrarily fixed element. Then there exists a uniquely determined element $x \in I$ such that $y \in J_x$, hence we define the function $g\colon J \to I$ by the prescription $g(y) := x$.

Therefore, if $x \in I$ is an arbitrary element, then it is obvious that $f(x) \in J_x$ and hence $g(f(x)) = x$. Thus, equation (1.1) is valid for all $x \in I$.

To see that g is nondecreasing, let $y_1 < y_2$ be arbitrary elements of J. Then there exist elements $x_1, x_2 \in I$ such that $y_i \in J_{x_i}$. If x_2 were strictly smaller than x_1, then we would have

$$y_2 \leq f_+(x_2) < f_-(x_1) \leq y_1.$$

This contradiction shows that $g(y_1) = x_1 \leq x_2 = g(y_2)$.

To prove that g is continuous, let $y \in J$ and choose $\varepsilon > 0$ so that $g(y)-\varepsilon, g(y)+\varepsilon \in I$. Define $W_\varepsilon :=]f_-(g(y)-\varepsilon), f_+(g(y)+\varepsilon)[$. Then

$$f_-(g(y)-\varepsilon) < f_-(g(y)) \leq y \leq f_+(g(y)) < f_+(g(y)+\varepsilon),$$

hence W_ε is neighborhood of y. By the monotonicity of g, for $w \in W_\varepsilon$, we have that

$$g(y) - \varepsilon = g\big(f_-(g(y)-\varepsilon)\big) \leq g(w) \leq g\big(f_+(g(y)+\varepsilon)\big) = g(y) + \varepsilon,$$

which yields that g is continuous at y.

If $y \in f(I)$, then there exists a uniquely determined element $x \in I$ such that $f(x) = y$ and hence, using (1.1), we get that

$$(f \circ g)(y) = f((g \circ f)(x)) = f(x) = y,$$

which shows that (1.2) holds for all $y \in f(I)$.

To see that (1.3) is valid, let $y \in J$. By the definition of $g(y)$, there exists a unique element $v \in I$ such that $y \in J_v$ and $g(y) = v$. Then, for all $x < v = g(y)$, we have

$$f(x) \le f_+(x) < f_-(v) \le y.$$

Therefore, upon taking the left limit $x \to v^-$, we get

$$\liminf_{x \to g(y)} f(x) = \lim_{x \to g(y)-0} f(x) \le y,$$

which proves the left hand side inequality in (1.3). The verification of the right hand side inequality is completely analogous, therefore it is omitted.

Finally, we prove the uniqueness of g. Assume that $h\colon J \to I$ is a nondecreasing function which is the left inverse of f. We are going to show that h coincides with g on J. Let $y \in J$ be arbitrary. Then there exists $x \in I$ such that $f_-(x) \le y \le f_+(x)$ and $g(y) = x$. Let (x_n) be a strictly increasing and (x'_n) be a strictly decreasing sequence converging to x. Then, for all $n \in \mathbb{N}$, we have

$$f(x_n) < f_-(x) \le y \le f_+(x) < f(x'_n).$$

By the monotonicity of h, it follows that

$$x_n = (h \circ f)(x_n) \le h(y) \le (h \circ f)(x'_n) = x'_n.$$

Taking the limit $n \to \infty$, we arrive at

$$x \le h(y) \le x,$$

which proves that $h(y) = x = g(y)$. □

Definition 1.1.2. The function g described in the above lemma is called the *generalized left inverse of the strictly monotone function* $f\colon I \to \mathbb{R}$ and is denoted by $f^{(-1)}$.

Remark 1.1.3. From (1.1) and (1.2), we get that the restriction of $f^{(-1)}$ to $f(I)$ is the inverse of f in the standard sense. Therefore, $f^{(-1)}$ is the continuous and monotone extension of the inverse of f to the smallest interval containing the range of f.

As discussed in the Introduction, a fairly general mean, which includes numerous important means as particular cases, is the quasi-arithmetic mean, which is defined as follows.

Definition 1.1.4. Let $n \in \mathbb{N}$. Given a strictly monotone, continuous function $f\colon I \to \mathbb{R}$, let us introduce the *n-variable quasi-arithmetic mean* $A_f^{[n]}\colon I^n \to I$ by the following formula

$$A_f^{[n]}(x) := f^{-1}\left(\frac{f(x_1) + \cdots + f(x_n)}{n}\right) \qquad (x \in I^n).$$

We say that the function f is the *generator of* $A_f^{[n]}$.

The following weighted extension of the quasi-arithmetic mean was introduced and investigated by Bajraktarević in [2] and [3].

Definition 1.1.5. Let $n \in \mathbb{N}$. Given a strictly monotone, continuous function $f\colon I \to \mathbb{R}$ and a positive function $p\colon I \to \mathbb{R}_+$, we define the *$n$-variable Bajraktarević mean* $B_{f,p}^{[n]}\colon I^n \to I$ by

$$B_{f,p}^{[n]}(x) := f^{-1}\left(\frac{p(x_1)f(x_1) + \cdots + p(x_n)f(x_n)}{p(x_1) + \cdots + p(x_n)}\right) \qquad (x \in I^n). \tag{1.5}$$

We say that the function f is the *generator*, while p *is the weight function of* $B_{f,p}^{[n]}$.

Let us consider a possibly nonsymmetric generalization of the Bajraktarević mean.

Definition 1.1.6. Let $n \in \mathbb{N}$. Given a strictly monotone function $f\colon I \to \mathbb{R}$ and an n-tuple of positive valued functions $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$, we introduce the *$n$-variable generalized Bajraktarević mean* $A_{f,p}^{[n]}\colon I^n \to I$ by the following formula

$$A_{f,p}^{[n]}(x) := f^{(-1)}\left(\frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)}\right) \qquad (x \in I^n), \tag{1.6}$$

and, to simplify the notations, we will use the following definitions

$$R_{f,p}^{[n]}(x) := \frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)} \qquad \text{and} \qquad p_0 := p_1 + \cdots + p_n. \tag{1.7}$$

Similarly to the previous definitions, we say that the function f is the *generator*, while p *is the weight function of* $A_{f,p}^{[n]}$.

Remark 1.1.7. We emphasize that, in the case when f is continuous, $A_{f,p}^{[n]}$ is indeed a not necessarily symmetric generalization of the symmetric mean $B_{f,p}^{[n]}$, since choosing all the weight functions to be the same, i.e., $p := p_1 = \cdots = p_n$ in (1.6), we get back (1.5).

Proposition 1.1.8. *Let $n \in \mathbb{N}, f\colon I \to \mathbb{R}$ be a strictly monotone function and $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$. Then the function $A_{f,p}^{[n]}\colon I^n \to I$ given by* (1.6) *is well-defined and it is a mean, i.e.,*

$$\min(x) \le A_{f,p}^{[n]}(x) \le \max(x) \qquad (x \in I^n). \tag{1.8}$$

Proof. We may assume that f is strictly increasing (in the decreasing case the proof is completely similar). To show that, for all $x \in I^n$, the formula for $A_{f,p}^{[n]}(x)$ is well-defined and (1.8) holds, consider the ratio $R_{f,p}^{[n]}(x)$.

Due to the positivity of the values of $p_i(x_i)$, we can see that $R_{f,p}^{[n]}(x)$ is a convex combination of the values $f(x_1), \ldots, f(x_n)$, therefore, it is an element of $\operatorname{conv}(f(I))$, which is the domain of $f^{(-1)}$ and hence $A_{f,p}^{[n]}(x) = f^{(-1)}(R_{f,p}^{[n]}(x))$ is well-defined. On the other hand, this also means that

$$\begin{aligned} f(\min(x)) = \min(f(x_1), \ldots, f(x_n)) &\leq R_{f,p}^{[n]}(x) \\ &\leq \max(f(x_1), \ldots, f(x_n)) = f(\max(x)). \end{aligned} \tag{1.9}$$

Furthermore, using that $f^{(-1)}$ is nondecreasing and is the left inverse of f, the inequalities in (1.9) yield

$$\min(x) = f^{(-1)}(f(\min(x))) \leq f^{(-1)}(R_{f,p}^{[n]}(x)) \leq f^{(-1)}(f(\max(x))) = \max(x). \tag{1.10}$$

This proves the mean value inequalities stated in (1.8). □

1.2 Partial derivatives of generalized Bajraktarević means

In the next result we determine the partial derivatives of the generalized Bajraktarević mean up to third-order on the diagonal of the domain under tight regularity assumptions. For instance, as stated below in assertions (1), (2b), (3c), for $m \in \mathbb{N}$, we prove the existence of partial derivatives of the form ∂_i^m only assuming $(m-1)$ times continuous differentiability of p_i. The partial derivatives play an essential role in the subsequent chapters as one of the tools that we use to prove the main theorems is solving the differential equations obtained by differentiating the functional equation on the diagonal of the domain in question.

Theorem 1.2.1. *Let $n \in \mathbb{N}, \ell \in \{1, 2, 3\}$, let $f\colon I \to \mathbb{R}$ be an ℓ times differentiable function with a nonvanishing first derivative, and let $p = (p_1, \ldots, p_n)\colon I \to \mathbb{R}_+^n$. Then we have the following assertions.*

(1) If $\ell = 1$, $i \in \{1, \ldots, n\}$, and p_i is continuous, then the first-order partial derivative $\partial_i A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ and

$$\left(\partial_i A_{f,p}^{[n]}\right)^{\Delta} = \frac{p_i}{p_0}.$$

(2a) If $\ell = 2$, $i, j \in \{1, \ldots, n\}$ with $i \neq j$, furthermore, p_i and p_j are differentiable, then the second-order partial derivative $\partial_i \partial_j A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ and

$$\left(\partial_i \partial_j A_{f,p}^{[n]}\right)^{\Delta} = -\frac{(p_i p_j)'}{p_0^2} - \frac{p_i p_j}{p_0^2} \cdot \frac{f''}{f'}.$$

(2b) If $\ell = 2$, $i \in \{1,\dots,n\}$, and p_i is continuously differentiable, then the second-order partial derivative $\partial_i^2 A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ *and*

$$\left(\partial_i^2 A_{f,p}^{[n]}\right)^{\Delta} = 2\frac{p_i'(p_0 - p_i)}{p_0^2} + \frac{p_i(p_0 - p_i)}{p_0^2} \cdot \frac{f''}{f'}.$$

(3a) If $\ell = 3$, $i, j, k \in \{1,\dots,n\}$ with $i \neq j \neq k \neq i$, furthermore, p_i, p_j, and p_k are differentiable, then the third-order partial derivative $\partial_i\partial_j\partial_k A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ *and*

$$\begin{aligned}\left(\partial_i\partial_j\partial_k A_{f,p}^{[n]}\right)^{\Delta} &= 2\frac{p_ip_j'p_k' + p_i'p_jp_k' + p_i'p_j'p_k}{p_0^3} + 2\frac{(p_ip_jp_k)'}{p_0^3} \cdot \frac{f''}{f'} \\ &\quad + \frac{p_ip_jp_k}{p_0^3}\left(3\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right).\end{aligned}$$

(3b) If $\ell = 3$, $i, j \in \{1,\dots,n\}$ with $i \neq j$, furthermore, p_i is twice differentiable and p_j is differentiable, then the third-order partial derivative $\partial_i^2\partial_j A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ *and*

$$\begin{aligned}\left(\partial_i^2\partial_j A_{f,p}^{[n]}\right)^{\Delta} &= \frac{2p_i'p_j'(2p_i - p_0) + p_j(2(p_i')^2 - p_i''p_0)}{p_0^3} \\ &\quad + \frac{(2p_i'p_j + p_ip_j')(2p_i - p_0)}{p_0^3} \cdot \frac{f''}{f'} \\ &\quad + \frac{p_ip_j}{p_0^3}\left((3p_i - p_0)\left(\frac{f''}{f'}\right)^2 - p_i\frac{f'''}{f'}\right).\end{aligned}$$

(3c) If $\ell = 3$, $i \in \{1,\dots,n\}$ and p_i is twice continuously differentiable, then the third-order partial derivative $\partial_i^3 A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ *and*

$$\begin{aligned}\left(\partial_i^3 A_{f,p}^{[n]}\right)^{\Delta} &= 3\frac{(p_0 - p_i)\big(p_0p_i'' - 2(p_i')^2\big)}{p_0^3} + 3\frac{p_i'(p_0 - 2p_i)(p_0 - p_i)}{p_0^3} \cdot \frac{f''}{f'} \\ &\quad - \frac{p_i(p_0 - p_i)}{p_0^3}\left(3p_i\left(\frac{f''}{f'}\right)^2 - (p_0 + p_i)\frac{f'''}{f'}\right).\end{aligned}$$

Proof. Let $n \in \mathbb{N}, \ell \in \{1,2,3\}$. Assume that $f\colon I \to \mathbb{R}$ is an ℓ times differentiable function with a nonvanishing first derivative. We have the following formulas for the derivatives of f^{-1}:

$$\big(f^{-1}\big)' = \frac{1}{f'} \circ f^{-1}, \quad \big(f^{-1}\big)'' = -\frac{f''}{(f')^3} \circ f^{-1}, \quad \big(f^{-1}\big)''' = \frac{3(f'')^2 - f'f'''}{(f')^5} \circ f^{-1}. \tag{1.11}$$

Furthermore, in order to make the calculations shorter, we use the notations $A_{f,p} = A^{[n]}_{f,p}$ and $R := R^{[n]}_{f,p}$, where the function $R^{[n]}_{f,p}$ was defined in (1.7). Then we have

$$A_{f,p} = A^{[n]}_{f,p} = f^{-1} \circ R^{[n]}_{f,p} = f^{-1} \circ R.$$

To compute the partial derivatives of R, we introduce the notations

$$\begin{aligned} P(x_1, \dots, x_n) &:= p_1(x_1) + \cdots + p_n(x_n), \\ Q(x_1, \dots, x_n) &:= p_1(x_1) f(x_1) + \cdots + p_n(x_n) f(x_n). \end{aligned}$$

Then $R \cdot P = Q$ and we have

$$P^\Delta = p_0, \quad Q^\Delta = p_0 f, \quad R^\Delta = f, \quad f^{-1} \circ R^\Delta = \mathrm{id}\,. \tag{1.12}$$

To prove the first assertion of the theorem, let $x \in I$ be fixed. Then, using the continuity of p_i and the differentiability of f at x, we get

$$\begin{aligned} (\partial_i R^\Delta)(x) &= \lim_{y \to x} \frac{R(\Delta_n(x) + (y-x)e_i) - R(\Delta_n(x))}{y-x} \\ &= \lim_{y \to x} \frac{1}{y-x} \left(\frac{(p_0(x) - p_i(x)) f(x) + p_i(y) f(y)}{p_0(x) - p_i(x) + p_i(y)} - f(x) \right) \\ &= \lim_{y \to x} \frac{p_i(y)}{p_0(x) - p_i(x) + p_i(y)} \cdot \frac{f(y) - f(x)}{y-x} = \frac{p_i f'}{p_0}(x). \end{aligned} \tag{1.13}$$

Therefore, using the standard differentiation rules, the last identity in (1.12) and (1.13), we obtain

$$\partial_i A^\Delta_{f,p} = \partial_i \big(f^{-1} \circ R\big)^\Delta = \left(\frac{\partial_i R}{f' \circ f^{-1} \circ R} \right) \circ \Delta_n = \frac{\frac{p_i}{p_0} f'}{f'} = \frac{p_i}{p_0}.$$

This completes the proof of assertion (1).

For the proof of statement (2a), let $i, j \in \{1, \dots, n\}$ with $i \neq j$ be fixed and assume that p_i and p_j are differentiable and f is twice differentiable. Then, for all $\alpha, \beta \in \{i, j\}$ with $\alpha \neq \beta$, the partial derivatives ∂_α and $\partial_\alpha \partial_\beta$ of P and Q (and hence of R) exist at every point in I^n. Furthermore, for all $(x_1, \dots, x_n) \in I^n$, we have that

$$\begin{aligned} \partial_\alpha P(x_1, \dots, x_n) &= p'_\alpha(x_\alpha), & \partial_\alpha Q(x_1, \dots, x_n) &= (p_\alpha f)'(x_\alpha), \\ \partial_\alpha \partial_\beta P(x_1, \dots, x_n) &= 0, & \partial_\alpha \partial_\beta Q(x_1, \dots, x_n) &= 0. \end{aligned} \tag{1.14}$$

Differentiating the identity $R \cdot P = Q$ with respect to the jth and then with respect to the ith variable, in view of the equalities in the second line in (1.14), it follows that

$$\partial_i \partial_j R \cdot P + \partial_j R \cdot \partial_i P + \partial_i R \cdot \partial_j P = 0$$

holds on I^n, whence, using (1.13) and (1.14), we arrive at

$$\begin{aligned}\partial_i\partial_j R^{\Delta} &= \Big(-\frac{\partial_j R\cdot\partial_i P+\partial_i R\cdot\partial_j P}{P}\Big)^{\Delta}\\ &= -\frac{1}{p_0}\Big(\frac{p_j f'}{p_0}\cdot p_i'+\frac{p_i f'}{p_0}\cdot p_j'\Big) = -\frac{(p_ip_j)'f'}{p_0^2}.\end{aligned}\tag{1.15}$$

Applying the chain rule, the first two formulas in (1.11) and then (1.12), (1.13), (1.15), it follows that

$$\begin{aligned}\partial_i\partial_j A_{f,p}^{\Delta} &= \Big(\big((f^{-1})''\circ R\big)\cdot\partial_i R\cdot\partial_j R+\big((f^{-1})'\circ R\big)\cdot\partial_i\partial_j R\Big)^{\Delta}\\ &= -\frac{f''}{(f')^3}\cdot\frac{p_if'}{p_0}\cdot\frac{p_jf'}{p_0}+\frac{1}{f'}\cdot\frac{-(p_ip_j)'f'}{p_0^2} = -\frac{(p_ip_j)'}{p_0^2}-\frac{p_ip_j}{p_0^2}\cdot\frac{f''}{f'}.\end{aligned}$$

To justify assertion (2b), let $x\in I$ be fixed. Let $i\in\{1,\dots,n\}$ and assume that p_i is continuously differentiable and f is twice differentiable. Then the partial derivative ∂_i of P and Q and hence of R exist at every point in I^n. Differentiating the identity $R\cdot P=Q$ with respect to the ith variable, we have that $\partial_i R\cdot P+R\cdot\partial_i P=\partial_i Q$, whence

$$\partial_i R=\frac{\partial_i Q-R\cdot\partial_i P}{P}.$$

Using this, we obtain

$$\begin{aligned}(\partial_i^2 R^{\Delta})(x) &= \lim_{y\to x}\frac{\partial_i R(\Delta_n(x)+(y-x)e_i)-\partial_i R(\Delta_n(x))}{y-x}\\ &= \lim_{y\to x}\frac{1}{y-x}\left(\frac{(p_if)'(y)-\frac{(p_0(x)-p_i(x))f(x)+p_i(y)f(y)}{p_0(x)-p_i(x)+p_i(y)}p_i'(y)}{p_0(x)-p_i(x)+p_i(y)}-\frac{p_i(x)f'(x)}{p_0(x)}\right)\\ &= \lim_{y\to x}\Bigg(\frac{(p_0(x)-p_i(x))p_i'(y)}{(p_0(x)-p_i(x)+p_i(y))^2}\cdot\frac{f(y)-f(x)}{y-x}\\ &\qquad+\frac{1}{y-x}\Big(\frac{(p_if')(y)}{p_0(x)-p_i(x)+p_i(y)}-\frac{(p_if')(x)}{p_0(x)}\Big)\Bigg)\\ &= \frac{(p_0-p_i)p_i'f'}{p_0^2}(x)+\lim_{y\to x}\Bigg(\frac{1}{p_0(x)-p_i(x)+p_i(y)}\cdot\frac{(p_if')(y)-(p_if')(x)}{y-x}\\ &\qquad-\frac{(p_if')(x)}{(p_0(x)-p_i(x)+p_i(y))p_0(x)}\cdot\frac{p_i(y)-p_i(x)}{y-x}\Bigg)\\ &= \Big(2\frac{p_i'(p_0-p_i)f'}{p_0^2}+\frac{p_if''}{p_0}\Big)(x).\end{aligned}\tag{1.16}$$

Applying standard calculus rules, the first two formulas in (1.11) and then (1.12), (1.13), (1.16), we conclude

$$\begin{aligned}\partial_i^2 A_{f,p}^{\Delta} = \partial_i^2\big(f^{-1}\circ R\big)^{\Delta} &= \Big(\big((f^{-1})''\circ R\big)\cdot(\partial_i R)^2 + \big((f^{-1})'\circ R\big)\cdot\partial_i^2 R\Big)^{\Delta}\\ &= -\frac{f''}{(f')^3}\left(\frac{p_i f'}{p_0}\right)^2 + \frac{1}{f'}\left(2\frac{p_i'(p_0-p_i)f'}{p_0^2} + \frac{p_i f''}{p_0}\right)\\ &= 2\frac{p_i'(p_0-p_i)}{p_0^2} + \frac{p_i(p_0-p_i)}{p_0^2}\cdot\frac{f''}{f'}.\end{aligned}$$

To prove assertion (3a), let $i,j,k\in\{1,\dots,n\}$ with $i\neq j\neq k\neq i$ and assume that p_i, p_j and p_k are differentiable. Then, for all $\alpha,\beta,\gamma\in\{i,j,k\}$ with $\alpha\neq\beta\neq\gamma\neq\alpha$, the partial derivatives ∂_α, $\partial_\alpha\partial_\beta$ and $\partial_\alpha\partial_\beta\partial_\gamma$ of P, Q and hence of R exist at every point in I^n. Furthermore, for all $(x_1,\dots,x_n)\in I^n$, we have the equalities in (1.14) and in addition

$$\partial_\alpha\partial_\beta\partial_\gamma P(x_1,\dots,x_n) = 0, \qquad \partial_\alpha\partial_\beta\partial_\gamma Q(x_1,\dots,x_n) = 0. \tag{1.17}$$

Differentiating the identity $R\cdot P = Q$ with respect to the kth variable, then with respect to the jth variable and then with respect to the ith variable, in view of the last two formulas in (1.14) and (1.17), we get

$$\partial_i\partial_j\partial_k R\cdot P + \partial_i\partial_j R\cdot\partial_k P + \partial_i\partial_k R\cdot\partial_j P + \partial_j\partial_k R\cdot\partial_i P = 0.$$

Thus, applying the first formula in (1.15) and (1.14), we arrive at

$$\begin{aligned}\partial_i\partial_j\partial_k R^{\Delta} &= \left(-\frac{\partial_i\partial_j R\cdot\partial_k P + \partial_i\partial_k R\cdot\partial_j P + \partial_j\partial_k R\cdot\partial_i P}{P}\right)^{\Delta}\\ &= \frac{(p_ip_j)'p_k'f' + (p_ip_k)'p_j'f' + (p_jp_k)'p_i'f'}{p_0^3}\\ &= \frac{2(p_ip_j'p_k' + p_i'p_jp_k' + p_i'p_j'p_k)f'}{p_0^3}.\end{aligned}\tag{1.18}$$

Hence, using (1.11) and then (1.12), (1.13), (1.18), (1.15), we obtain

$$\begin{aligned}\partial_i\partial_j\partial_k A_{f,p}^{\Delta} &= \partial_i\partial_j\partial_k\big(f^{-1}\circ R\big)^{\Delta}\\ &= \Big(\big((f^{-1})'''\circ R\big)\cdot\partial_i R\cdot\partial_j R\cdot\partial_k R + \big((f^{-1})'\circ R\big)\cdot\partial_i\partial_j\partial_k R\\ &\qquad + \big((f^{-1})''\circ R\big)\cdot(\partial_i R\cdot\partial_j\partial_k R + \partial_j R\cdot\partial_i\partial_k R + \partial_k R\cdot\partial_i\partial_j R)\Big)^{\Delta}\\ &= \frac{3(f'')^2 - f'f'''}{(f')^5}\cdot\frac{p_if'}{p_0}\cdot\frac{p_jf'}{p_0}\cdot\frac{p_kf'}{p_0} + \frac{1}{f'}\cdot 2\frac{(p_ip_j'p_k' + p_i'p_jp_k' + p_i'p_j'p_k)f'}{p_0^3}\\ &\quad - \frac{f''}{(f')^3}\left(\frac{p_if'}{p_0}\cdot\frac{-(p_jp_k)'f'}{p_0^2} + \frac{p_jf'}{p_0}\cdot\frac{-(p_ip_k)'f'}{p_0^2} + \frac{p_kf'}{p_0}\cdot\frac{-(p_ip_j)'f'}{p_0^2}\right)\end{aligned}$$

$$= 2\frac{p_i p_j' p_k' + p_i' p_j p_k' + p_i' p_j' p_k}{p_0^3}$$
$$+ 2\frac{(p_i p_j p_k)'}{p_0^3} \cdot \frac{f''}{f'} + \frac{p_i p_j p_k}{p_0^3}\left(3\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right).$$

To verify assertion (3b), let $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and assume that p_i is twice and p_j is once differentiable. Then, for all $\alpha, \beta \in \{i, j\}$ with the assumption α and β are not equal to j simultaneously, the partial derivatives ∂_α, $\partial_\alpha \partial_\beta$ and $\partial_i^2 \partial_j$ of P, Q and hence of R exist at every point in I^n. Furthermore, for all $(x_1, \ldots, x_n) \in I^n$, we have (1.14), (1.17), and in addition

$$\begin{aligned} \partial_i^2 P(x_1, \ldots, x_n) &= p_i''(x_i), & \partial_i^2 Q(x_1, \ldots, x_n) &= (p_i f)''(x_i), \\ \partial_i^2 \partial_j P(x_1, \ldots, x_n) &= 0, & \partial_i^2 \partial_j Q(x_1, \ldots, x_n) &= 0. \end{aligned} \tag{1.19}$$

Differentiating the equality $R \cdot P = Q$ with respect to the jth variable, and then with respect to the ith variable twice, using (1.14) and (1.19), we get

$$\partial_i^2 \partial_j R \cdot P + \partial_i^2 R \cdot \partial_j P + 2\partial_i \partial_j R \cdot \partial_i P + \partial_j R \cdot \partial_i^2 P = 0.$$

Thus, applying the formulas (1.16), (1.14), (1.15), (1.13), and (1.19), we arrive at the equation

$$\begin{aligned} \partial_i^2 \partial_j R^\Delta &= \left(-\frac{\partial_i^2 R \cdot \partial_j P + 2\partial_i \partial_j R \cdot \partial_i P + \partial_j R \cdot \partial_i^2 P}{P}\right)^\Delta \\ &= -\left(2\frac{(p_0 - p_i)p_i' f'}{p_0^2} + \frac{p_i f''}{p_0}\right) \cdot \frac{p_j'}{p_0} + 2\frac{(p_i p_j)' f'}{p_0^2} \cdot \frac{p_i'}{p_0} - \frac{p_j f'}{p_0} \cdot \frac{p_i''}{p_0} \\ &= \frac{(2p_i' p_j'(2p_i - p_0) + p_j(2(p_i')^2 - p_i'' p_0))f' - p_0 p_i p_j' f''}{p_0^3}. \end{aligned} \tag{1.20}$$

Therefore, using the identities in (1.11) and then (1.12), (1.13), (1.20), (1.15), and (1.16), we get

$$\begin{aligned} \partial_i^2 \partial_j A_{f,p}^\Delta &= \Big(\big((f^{-1})''' \circ R\big) \cdot (\partial_i R)^2 \cdot \partial_j R + \big((f^{-1})' \circ R\big) \cdot \partial_i^2 \partial_j R \\ &\qquad + \big((f^{-1})'' \circ R\big) \cdot \big(2\partial_i R \cdot \partial_i \partial_j R + \partial_j R \cdot \partial_i^2 R\big)\Big)^\Delta \\ &= \frac{3(f'')^2 - f' f'''}{(f')^5} \cdot \frac{(p_i f')^2}{p_0^2} \cdot \frac{p_j f'}{p_0} \\ &\quad + \frac{1}{f'} \cdot \frac{(2p_i' p_j'(2p_i - p_0) + p_j(2(p_i')^2 - p_i'' p_0))f' - p_0 p_i p_j' f''}{p_0^3} \\ &\quad - \frac{f''}{(f')^3}\left(-2\frac{p_i f'}{p_0} \cdot \frac{(p_i p_j)' f'}{p_0^2} + \frac{p_j f'}{p_0} \cdot \frac{p_0(2p_i' f' + p_i f'') - (p_i p_i)' f'}{p_0^2}\right) \end{aligned}$$

$$= \frac{2p_i'p_j'(2p_i - p_0) + p_j(2(p_i')^2 - p_i''p_0)}{p_0^3} + \frac{(2p_i'p_j + p_ip_j')(2p_i - p_0)}{p_0^3} \cdot \frac{f''}{f'} + \frac{p_ip_j}{p_0^3}\bigg((3p_i - p_0)\bigg(\frac{f''}{f'}\bigg)^2 - p_i\frac{f'''}{f'}\bigg),$$

which completes the proof of case (3b).

To prove assertion (3c), let $i \in \{1, \dots, n\}$ and assume that p_i is twice continuously differentiable. Then the partial derivatives ∂_i, ∂_i^2 of P, Q and hence of R exist at every point in I^n. We have that

$$\partial_i^2 R = \partial_i\bigg(\frac{\partial_i Q - R \cdot \partial_i P}{P}\bigg) = \frac{\partial_i^2 Q \cdot P - Q \cdot \partial_i^2 P - 2\partial_i Q \cdot \partial_i P + 2R \cdot (\partial_i P)^2}{P^2}.$$

Then, for all $x, y \in I$, we get that

$$\begin{aligned}
&\partial_i^2 R(\Delta_n(x) + (y - x)e_i) \\
&\quad= \frac{(p_if)''(y)((p_0 - p_i)(x) + p_i(y)) - \big(((p_0 - p_i)f)(x) + (p_if)(y)\big)p_i''(y)}{((p_0 - p_i)(x) + p_i(y))^2} \\
&\qquad - \frac{2(p_if)'(y)p_i'(y) - 2\frac{((p_0-p_i)f)(x)+(p_if)(y)}{(p_0-p_i)(x)+p_i(y)}(p_i')^2(y)}{((p_0 - p_i)(x) + p_i(y))^2} \\
&\quad= \frac{(p_0 - p_i)(x)\big(((p_0 - p_i)(x) + p_i(y))p_i''(y) - 2(p_i')^2(y)\big)}{((p_0 - p_i)(x) + p_i(y))^3}(f(y) - f(x)) \\
&\qquad + \frac{2(p_0 - p_i)(x)(p_i'f')(y)}{((p_0 - p_i)(x) + p_i(y))^2} + \frac{(p_if'')(y)}{(p_0 - p_i)(x) + p_i(y)}.
\end{aligned} \tag{1.21}$$

Therefore, using (1.21), (1.16), and the twice continuous differentiability of p_i, we obtain

$$\begin{aligned}
&(\partial_i^3 R^\Delta)(x) \\
&\quad= \lim_{y\to x} \frac{\partial_i^2 R(\Delta_n(x) + (y - x)e_i) - \partial_i^2 R(\Delta_n(x))}{y - x} \\
&\quad= \lim_{y\to x} \frac{1}{y - x}\Bigg(\frac{(p_0 - p_i)(x)\big(((p_0 - p_i)(x) + p_i(y))p_i''(y) - 2(p_i')^2(y)\big)}{((p_0 - p_i)(x) + p_i(y))^3}(f(y) - f(x)) \\
&\qquad\qquad + \frac{2(p_0 - p_i)(x)(p_i'f')(y)}{((p_0 - p_i)(x) + p_i(y))^2} + \frac{(p_if'')(y)}{(p_0 - p_i)(x) + p_i(y)} \\
&\qquad\qquad - \bigg(2\frac{p_i'(p_0 - p_i)f'}{p_0^2} + \frac{p_if''}{p_0}\bigg)(x)\Bigg) \\
&\quad= \bigg(\frac{3(p_0 - p_i)\big(p_0p_i'' - 2(p_i')^2\big)f'}{p_0^3} + \frac{3(p_0 - p_i)p_i'f''}{p_0^2} + \frac{p_if'''}{p_0}\bigg)(x).
\end{aligned} \tag{1.22}$$

Hence, applying (1.11), (1.13), (1.22), and (1.16), we conclude

$$\begin{aligned}
\partial_i^3 A_{f,p}^{\Delta} &= \partial_i^3\big(f^{-1}\circ R\big)^{\Delta} \\
&= ((f^{-1})'''\circ R)(\partial_i R)^3 + ((f^{-1})''\circ R)(3\partial_i R\cdot\partial_i^2 R) + ((f^{-1})'\circ R)\cdot\partial_i^3 R \\
&= \frac{3(f'')^2 - f'f'''}{(f')^5}\left(\frac{p_i f'}{p_0}\right)^3 - \frac{f''}{(f')^3}\left(3\frac{p_i f'}{p_0}\left(2\frac{p_i'(p_0-p_i)f'}{p_0^2} + \frac{p_i f''}{p_0}\right)\right) \\
&\quad + \frac{1}{f'}\left(\frac{3(p_0-p_i)\big(p_0p_i'' - 2(p_i')^2\big)f'}{p_0^3} + \frac{3(p_0-p_i)p_i'f''}{p_0^2} + \frac{p_i f'''}{p_0}\right) \\
&= \frac{3(p_0-p_i)\big(p_0p_i'' - 2(p_i')^2\big)}{p_0^3} + 3\frac{p_i'(p_0-2p_i)(p_0-p_i)}{p_0^3}\cdot\frac{f''}{f'} \\
&\quad - \frac{p_i(p_0-p_i)}{p_0^3}\left(3p_i\left(\frac{f''}{f'}\right)^2 - (p_0+p_i)\frac{f'''}{f'}\right),
\end{aligned}$$

which completes the proof of assertion (3c). $\square$

Chapter 2

Equality problem

In the theory of quasi-arithmetic means the characterization of the equality of means with different generators is a basic problem which was completely solved in the book [15]. Using this characterization, the homogeneous quasi-arithmetic means can also be described, they are exactly the power means. In [2] (see also [3]) Bajraktarević introduced a new generalization of quasi-arithmetic means by adding a weight function to the formula. He also described the equality of such means (called Bajraktarević means since then) with fixed number of variables in the at least 3-variable setting assuming three times differentiability. Daróczy and Losonczi in [9], later Daróczy and Páles in [10] arrived at the same conclusion with first-order differentiability and without differentiability, respectively, but assuming the equality for all $n \in \mathbb{N}$. As an application of the characterization of the equality, Aczél and Daróczy in [1] determined the homogeneous Bajraktarević means that include Gini means which were introduced by Gini in [11]. Losonczi in [22] described the equality of 2-variable Bajraktarević means under sixth-order regularity assumptions, which was later improved to first-order restrictions in [38], and an algebraic condition that was later removed in [23]. Using these results, the homogeneous 2-variable means were also determined by Losonczi (see [24], [25]).

The paper [12] contains the results of this chapter, in which we deal with the equality problem of n-variable generalized Bajraktarević means, i.e., we give necessary as well as sufficient conditions in terms of the unknown functions listed below for the functional equation

$$
\begin{aligned}
f^{(-1)}&\left(\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)}\right)\\
&= g^{(-1)}\left(\frac{q_1(x_1)g(x_1)+\cdots+q_n(x_n)g(x_n)}{q_1(x_1)+\cdots+q_n(x_n)}\right)
\end{aligned}
$$

to be valid locally or globally, where $n \in \mathbb{N}, n \geq 2$ is fixed, I is a nonempty open real interval, which will be the case in this entire chapter, the unknown functions $f, g\colon I \to \mathbb{R}$ are strictly monotone, furthermore, $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$ are also unknown functions.

This problem in the symmetric 2-variable case, i.e., when $n = 2$ and $p_1 = p_2$ (see Theorem 2.2.9) had been already investigated and solved under sixth-order regularity assumptions by Losonczi in [22]. Later Páles and Zakaria reduced the severity of the differentiability assumptions and obtained the same result under first-order restrictions in their paper [38]. Apart from the aforementioned theorem, in this context, all other results in the chapter were newly discovered in our paper [12]. More precisely, in the nonsymmetric 2-variable case, assuming three times differentiability of f, g and the existence of $i \in \{1, 2\}$ such that either p_i is twice continuously differentiable and p_{3-i} is continuous, or p_i is twice differentiable and p_{3-i} is once differentiable, we prove that the equality holds if and only if there exist four constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that

$$cf + d > 0, \qquad g = \frac{af + b}{cf + d}, \qquad \text{and} \qquad q_\ell = (cf + d)p_\ell \qquad (\ell \in \{1, \dots, n\}).$$

In the case $n \geq 3$, we obtain the same conclusion under weaker regularity assumptions. Namely, we suppose that f and g are three times differentiable, p is continuous and there exist $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k \neq i$ such that p_i, p_j, p_k are differentiable.

2.1 Sufficient conditions

Theorem 2.1.1. *Let $f\colon I \to \mathbb{R}$ be a strictly increasing function, $n \in \mathbb{N}$, and let further $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$. Then, for all $x \in I^n$, the equality $y = A_{f,p}^{[n]}(x)$ holds if and only if, for $z \in I$,*

$$\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i)) \begin{cases} < 0 & \text{if } z < y \\ > 0 & \text{if } z > y. \end{cases} \tag{2.1}$$

If f is strictly decreasing, then the inequalities in (2.1) hold with reversed inequality sign.

Proof. Assume that $f\colon I \to \mathbb{R}$ is strictly increasing, let $x \in I^n$ and $y := A_{f,p}^{[n]}(x)$. If $z < y$, then $f(z) < R_{f,p}^{[n]}(x)$ because in the opposite case we would have $f(z) \geq R_{f,p}^{[n]}(x)$ which implies $z = f^{(-1)}(f(z)) \geq f^{(-1)}(R_{f,p}^{[n]}(x)) = A_{f,p}^{[n]}(x) = y$, contradicting the choice of z. Rearranging the inequality $f(z) < R_{f,p}^{[n]}(x)$, it easily follows that

$$\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i)) < 0.$$

In the case $z > y$, we get $f(z) > R_{f,p}^{[n]}(x)$, which implies the second inequality in (2.1).

Observe that the function

$$z \mapsto \varphi(z) := \sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i))$$

is strictly increasing. Therefore, it changes sign at at most one point in I. If (2.1) holds for y, then φ changes sign at y. On the other hand, as we have seen it above, φ also changes sign at $A^{[n]}_{f,p}(x)$. Hence $y = A^{[n]}_{f,p}(x)$ must hold. □

Corollary 2.1.2. *Let $f\colon I \to \mathbb{R}$ be continuous, strictly monotone, $n \in \mathbb{N}$, and let further $p = (p_1, \dots, p_n)\colon\ I \to \mathbb{R}^n_+$. Then, for all $x \in I^n$, the value $y = A^{[n]}_{f,p}(x)$ is the unique solution of the equation*

$$\sum_{i=1}^{n} p_i(x_i)(f(y) - f(x_i)) = 0.$$

Proof. The function

$$y \mapsto \varphi(y) := \sum_{i=1}^{n} p_i(x_i)(f(y) - f(x_i))$$

is strictly monotone and continuous. Therefore, it vanishes at most one point in I. Applying Theorem 2.1.1, we obtain that φ changes sign at $y = A^{[n]}_{f,p}(x)$. Thus, using that φ is continuous, φ vanishes at $y = A^{[n]}_{f,p}(x)$. □

The next result establishes a sufficient condition for the equality of the n-variable generalized Bajraktarević means. We will call this situation the *canonical case of the equality*.

Theorem 2.1.3. *Let $f, g\colon I \to \mathbb{R}$ be strictly monotone functions, $n \in \mathbb{N}$, and let further $p = (p_1, \dots, p_n)\colon\ I \to \mathbb{R}^n_+$, $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}^n_+$. If there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that*

$$cf + d > 0, \qquad g = \frac{af + b}{cf + d}, \qquad \text{and} \qquad q_i = (cf + d)p_i \qquad (i \in \{1, \dots, n\}) \tag{2.2}$$

hold on I, then $A^{[n]}_{f,p} = A^{[n]}_{g,q}$ is globally valid.

Proof. Let $x \in I^n$ be arbitrary. Using the formulas (2.2), we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n} q_i(x_i)(g(z) - g(x_i)) \\
&\quad = \sum_{i=1}^{n} (cf(x_i) + d)p_i(x_i) \left(\frac{(af(z) + b)(cf(x_i) + d) - (af(x_i) + b)(cf(z) + d)}{(cf(x_i) + d)(cf(z) + d)} \right) \\
&\quad = \frac{ad - bc}{cf(z) + d} \Big(\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i)) \Big).
\end{aligned}$$

It shows that the function $\sum_{i=1}^{n} q_i(x_i)(g(z) - g(x_i))$ changes sign at y if and only if $\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i))$ changes sign at y. Hence, applying Theorem 2.1.1, $A^{[n]}_{f,p}(x) = A^{[n]}_{g,q}(x)$ holds. The element x being arbitrary in I^n, we get the statement of the theorem. □

2.2 Necessary conditions

With the aid of the following lemma, we can reduce the regularity assumptions in our statements.

Let $n \in \mathbb{N}$. For all $i \in \{1, \dots, n\}$, let $e_i \in \mathbb{R}^n$ denote the ith vector of the standard base of $\mathbb{R}^n$, i.e., let $e_i := (\delta_{i,j})_{j=1}^n$. Given two weight functions $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$, we will also use the following notation:

$$r_0 := \frac{q_0}{p_0} = \frac{q_1 + \cdots + q_n}{p_1 + \cdots + p_n}.$$

Lemma 2.2.1. *Let $f, g\colon I \to \mathbb{R}$ be continuous strictly monotone functions, $n \in \mathbb{N}, n \geq 2$, and $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$, $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$. Assume that $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ is locally valid. Then the following two assertions hold.*

(i) For all $i \in \{1, \dots, n\}$, the function p_i is continuous if and only if the function q_i is continuous.

(ii) Let $k \in \mathbb{N}$. Assume that $f, g\colon I \to \mathbb{R}$ are k times differentiable (resp. k times continuously differentiable) functions with nonvanishing first derivatives. Then, for all $i \in \{1, \dots, n\}$, the function p_i is k times differentiable (resp. k times continuously differentiable) if and only if q_i is k times differentiable (resp. k times continuously differentiable).

Proof. In what follows, we will prove that the regularity properties possessed by p_i are transferred to the corresponding q_i. The reversed statements can similarly be verified.

Assume that $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ holds in the local sense, that is, on some open set $U \subseteq I^n$ which contains $\operatorname{diag}(I^n)$. For $i \in \{1, \dots, n\}$, denote

$$U_i := \{(x, y) \in I^2 \mid \Delta_n(x) + (y - x)e_i \in U\}.$$

Then U_i is an open set containing $\operatorname{diag}(I^2)$. By our assumption, we have that, for all $(x, y) \in U_i$,

$$A_{g,q}^{[n]}(\Delta_n(x) + (y - x)e_i) = A_{f,p}^{[n]}(\Delta_n(x) + (y - x)e_i).$$

This is equivalent to the following equality

$$\frac{(q_0(x) - q_i(x))g(x) + q_i(y)g(y)}{q_0(x) - q_i(x) + q_i(y)} = \big(g \circ f^{-1}\big)\left(\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)}\right), \tag{2.3}$$

where $(x, y) \in U_i$. Observe that, for $x, y \in I$ with $x \neq y$, the inequalities $p_i(x) < p_0(x)$ and $f(x) \neq f(y)$ imply that

$$\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)} \neq f(y).$$

Therefore,

$$(g \circ f^{-1})\left(\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)}\right) \neq g(y).$$

Thus, solving equation (2.3) with respect to $q_i(y)$, we get

$$q_i(y) = (q_0(x) - q_i(x)) \frac{(g \circ f^{-1})\left(\frac{(p_0(x)-p_i(x))f(x)+p_i(y)f(y)}{p_0(x)-p_i(x)+p_i(y)}\right) - g(x)}{g(y) - (g \circ f^{-1})\left(\frac{(p_0(x)-p_i(x))f(x)+p_i(y)f(y)}{p_0(x)-p_i(x)+p_i(y)}\right)}, \tag{2.4}$$

where $(x, y) \in U_i$, $x \neq y$.

Let $x_0 \in I$ be an arbitrarily fixed point. The pair (x_0, x_0) is an interior point of U_i, therefore, there exists $x \in I \setminus \{x_0\}$ such that $(x, x_0) \in U_i$. Then the set

$$V_i := \{y \in I \mid (x, y) \in U_i,\, x \neq y\}$$

is a neighborhood of x_0 on which we have the equality (2.4) for q_i.

Provided that f and g are continuous and p_i is continuous at x_0, it follows that $g \circ f^{-1}$ is continuous on $f(I)$ and hence the mapping

$$y \mapsto (g \circ f^{-1})\left(\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)}\right) \tag{2.5}$$

is continuous at x_0. This shows that the right hand side of (2.4) is a continuous function of y at x_0 and hence q_i is continuous at x_0. This proves the first assertion.

Provided that, for some $k \in \mathbb{N}$, the functions $f, g\colon I \to \mathbb{R}$ are k times differentiable (resp. k times continuously differentiable) with nonvanishing first derivatives and that p_i is k times differentiable (resp. k times continuously differentiable) at x_0, it follows, by the standard calculus rules, that $g \circ f^{-1}$ is k times differentiable (resp. k times continuously differentiable) and hence the mapping (2.5) is also k times differentiable (resp. k times continuously differentiable) at x_0. This implies that the right hand side of (2.4) is a k times differentiable (resp. k times continuously differentiable) function of y at x_0 and hence q_i is k times differentiable (resp. k times continuously differentiable) at x_0. This proves the second statement. □

The following extension theorem is of basic importance since in the proofs of the main theorems, first we can state the validity of the canonical case of the equality only on a subinterval of the domain. Then the next main step of the proof is to extend the validity to the entire underlying interval.

Theorem 2.2.2. *Let $f, g\colon I \to \mathbb{R}$ be continuous, strictly monotone, $n \in \mathbb{N}$, let further $p = (p_1, \dots, p_n)\colon\ I \to \mathbb{R}_+^n$ be a continuous function and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$. Assume that $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ is locally valid and that there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ and a nonempty open subinterval J of I such that (2.2) holds on J. Then q is continuous and (2.2) is also valid on I.*

Proof. First of all, using Lemma 2.2.1 and the continuity of f, g and p, it is clear that q is continuous.

Assume that $A^{[n]}_{f,p} = A^{[n]}_{g,q}$ holds locally, that is, on some open set $U \subseteq I^n$ containing $\operatorname{diag}(I^n)$ and for some constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ there exists a nonempty open subinterval J of I such that (2.2) holds on J. We may assume that J is a maximal subinterval of I with this property. To complete the proof, we have to show that $J = I$. To the contrary, suppose that $J \neq I$. Then one of the strict inequalities

$$\inf I < \inf J =: \alpha \qquad \text{or} \qquad \sup J < \sup I \tag{2.6}$$

must be valid. We may suppose that first inequality in (2.6) holds. Hence, due to the continuity of f, p_1, and q_1 at α, it follows from (2.2) that $q_1(\alpha) = (cf(\alpha) + d)p_1(\alpha)$. Therefore, $q_1(\alpha) > 0$ implies that $cf(\alpha) + d > 0$. Consequently, using the continuity of all functions, for all $x \in \bar{J} := J \cup \{\alpha\}$, by Theorem 2.1.3, we get that

$$cf(x) + d > 0, \qquad g(x) = \frac{af(x) + b}{cf(x) + d},$$

and, for all $i \in \{1, \ldots, n\}$,

$$q_i(x) = (cf(x) + d)p_i(x)$$

are valid. By the continuity of f, there is an element $\bar{\alpha} \in I$ with $\bar{\alpha} < \alpha$ such that $cf(x) + d > 0$ for all $x \in \bar{I} :=]\bar{\alpha}, \alpha] \cup J$. Define the functions $\bar{g}\colon \bar{I} \to \mathbb{R}$ and $\bar{q}\colon \bar{I} \to \mathbb{R}^n_+$ by

$$\bar{g}(x) := \frac{af(x) + b}{cf(x) + d} \qquad \text{and} \qquad \bar{q}_i(x) := (cf(x) + d)p_i(x), \tag{2.7}$$

where $x \in \bar{I}$, $i \in \{1, \ldots, n\}$. Thus, for all $x \in \bar{J}$, the equations

$$g(x) = \bar{g}(x) \qquad \text{and} \qquad q_i(x) = \bar{q}_i(x) \qquad (i \in \{1, \ldots, n\}) \tag{2.8}$$

hold. On the other hand, the maximality property of J implies that there is no $\beta < \alpha$ such that (2.8) is valid for all $x \in]\beta, \alpha] \cup J$. Furthermore, the equality $A^{[n]}_{f,p} = A^{[n]}_{g,q}$ on U and Theorem 2.1.3 applied to the conditions (2.7) yield that

$$A^{[n]}_{g,q}(x) = A^{[n]}_{f,p}(x) = A^{[n]}_{\bar{g},\bar{q}}(x) \tag{2.9}$$

is also valid for all $x \in \bar{U} := (\bar{I})^n \cap U$. The point $(\alpha, \ldots, \alpha)$ is an interior point of $\bar{U}$, therefore, there exists $r > 0$ such that $]\alpha - r, \alpha + r[^n \subseteq \bar{U}$ and hence (2.9) holds for all $x \in]\alpha - r, \alpha + r[^n$.

In what follows, we assume that g is strictly increasing and hence $\bar{g}$ must be also strictly increasing. The functions g and $\bar{g}$ are identical on $[\alpha, \alpha + r[$, therefore, their inverses are also equal on $[g(\alpha), g(\alpha + r)[$.

The following claim is essential for the rest of the proof.

Claim. *If* $x \in]\alpha - r, \alpha + r[^n$ *such that* $\alpha \leq A_{g,q}^{[n]}(x)$*, then*

$$\frac{q_1(x_1)g(x_1) + \cdots + q_n(x_n)g(x_n)}{q_1(x_1) + \cdots + q_n(x_n)} = \frac{\bar{q}_1(x_1)\bar{g}(x_1) + \cdots + \bar{q}_n(x_n)\bar{g}(x_n)}{\bar{q}_1(x_1) + \cdots + \bar{q}_n(x_n)}. \tag{2.10}$$

Indeed, the condition on x implies that $\alpha \leq A_{g,q}^{[n]}(x) \leq \max(x) < \alpha + r$ also holds, hence $g(A_{g,q}^{[n]}(x)) = \bar{g}(A_{g,q}^{[n]}(x))$. On the other hand, in view of (2.9), we have the equality $A_{g,q}^{[n]}(x) = A_{\bar{g},\bar{q}}^{[n]}(x)$. Therefore, $g(A_{g,q}^{[n]}(x)) = \bar{g}(A_{\bar{g},\bar{q}}^{[n]}(x))$, which implies the equation (2.10).

Let $y_0 \in]\alpha, \alpha + r[$ be fixed. Then the inequality $g(\alpha) < g(y_0)$ implies that

$$g(\alpha) < \frac{q_i(\alpha)g(\alpha) + (q_0 - q_i)(y_0)g(y_0)}{q_i(\alpha) + (q_0 - q_i)(y_0)} \qquad (i \in \{1, \ldots, n\}).$$

Now, by the continuity of the functions $g, q_1, \ldots, q_n$, we can find a number $\delta_0 := \delta(y_0) \in]0, \min(y_0 - \alpha, \alpha + r - y_0)[\subsetneq]0, r[$ such that, for all $x \in]\alpha - \delta_0, \alpha]$ and $y \in]y_0 - \delta_0, y_0 + \delta_0[$,

$$g(\alpha) \leq \frac{q_i(x)g(x) + (q_0 - q_i)(y)g(y)}{q_i(x) + (q_0 - q_i)(y)} \qquad (i \in \{1, \ldots, n\}). \tag{2.11}$$

Applying the inverse of g side by side to this inequality, we get $\alpha \leq A_{g,q}^{[n]}(x_1, \ldots, x_n)$, where $x_i := x$ and $x_j := y$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$. Therefore, in view of the Claim above and the equality (2.8), for all $x \in]\alpha - \delta_0, \alpha]$, $y \in]y_0 - \delta_0, y_0 + \delta_0[$, and $i \in \{1, \ldots, n\}$, we have that

$$\frac{q_i(x)g(x) + (q_0 - q_i)(y)g(y)}{q_i(x) + (q_0 - q_i)(y)} = \frac{\bar{q}_i(x)\bar{g}(x) + (q_0 - q_i)(y)g(y)}{\bar{q}_i(x) + (q_0 - q_i)(y)}.$$

This equality can be rewritten as

$$\begin{aligned} 0 = q_i(x)\bar{q}_i(x)(g(x) - \bar{g}(x)) + (q_0 - q_i)(y)(q_i(x)g(x) - \bar{q}_i(x)\bar{g}(x)) \\ + (q_0 - q_i)(y)g(y)(\bar{q}_i(x) - q_i(x)). \end{aligned} \tag{2.12}$$

Consider the sets

$$\begin{aligned} S &:= \{x \in]\alpha - r, \alpha[\mid g(x) \neq \bar{g}(x)\}, \\ S_i &:= \{x \in]\alpha - r, \alpha[\mid q_i(x) \neq \bar{q}_i(x)\} \quad (i \in \{1, \ldots, n\}). \end{aligned}$$

In the next step we show that

$$S \cap]\alpha - \delta_0, \alpha[= S_i \cap]\alpha - \delta_0, \alpha[\qquad (i \in \{1, \ldots, n\}). \tag{2.13}$$

If $x \in]\alpha-\delta_0,\alpha[\setminus S$ then $g(x)=\bar{g}(x)$. Using this, (2.12) simplifies to the product equality
$$(q_0-q_i)(y)\cdot(g(x)-g(y))\cdot(q_i(x)-\bar{q}_i(x))=0.$$
The first factor is not zero because it is the sum of positive terms. Using that $x<\alpha<y_0-\delta_0<y$, the strict monotonicity of g implies that $g(x)<g(y)$, proving that the second factor is also not zero. Therefore, we must have $q_i(x)=\bar{q}_i(x)$, which shows that $x\in]\alpha-\delta_0,\alpha[\setminus S_i$. Conversely, if $x\in]\alpha-\delta_0,\alpha[\setminus S_i$, then $q_i(x)=\bar{q}_i(x)$. In this case (2.12) reduces to the product equality

$$q_i(x)\cdot(q_i(x)+(q_0-q_i)(y))\cdot(g(x)-\bar{g}(x))=0.$$

The first two factors are positive, hence we must have $g(x)=\bar{g}(x)$, which proves that $x\in]\alpha-\delta_0,\alpha[\setminus S$ and completes the proof of equality (2.13). The maximality of the interval J, in view of (2.13), implies that

$$\sup S\cap]\alpha-\delta_0,\alpha[=\sup S_i\cap]\alpha-\delta_0,\alpha[=\alpha \qquad (i\in\{1,\dots,n\}). \tag{2.14}$$

Let $i\in\{1,\dots,n\}$ be fixed and $y_1,y_2\in]y_0-\delta_0,y_0+\delta_0[$ be arbitrary such that $y_1\neq y_2$. Replacing y by y_1 and y_2 in (2.12), and then subtracting the two equations so obtained side by side, we get that

$$\begin{aligned}0=&((q_0-q_i)(y_1)-(q_0-q_i)(y_2))\cdot(q_i(x)g(x)-\bar{q}_i(x)\bar{g}(x))\\&+((q_0-q_i)(y_1)g(y_1)-(q_0-q_i)(y_2)g(y_2))\cdot(\bar{q}_i(x)-q_i(x)).\end{aligned}\tag{2.15}$$

Let $x_1,x_2\in]\alpha-\delta_0,\alpha[$ be arbitrary. Substituting x by x_1 and then x_2 in (2.15), we get a homogeneous linear system of two equations of the form

$$\xi\cdot(q_i(x_i)g(x_i)-\bar{q}_i(x_i)\bar{g}(x_i))+\eta\cdot(\bar{q}_i(x_i)-q_i(x_i))=0 \qquad (i\in\{1,2\}),$$

which is nontrivially solvable with respect to (ξ,η) because the equalities

$$\begin{aligned}\xi&:=(q_0-q_i)(y_1)-(q_0-q_i)(y_2)=0,\\ \eta&:=(q_0-q_i)(y_1)g(y_1)-(q_0-q_i)(y_2)g(y_2)=0\end{aligned}\tag{2.16}$$

cannot be satisfied simultaneously. Indeed, if $\xi=0$, then $(q_0-q_i)(y_1)=(q_0-q_i)(y_2)>0$. This equality together with $\eta=0$ implies that $g(y_1)=g(y_2)$. The strict monotonicity of g then yields $y_1=y_2$, which contradicts the choice of y_1 and y_2. Hence the determinant of the system (2.16) must be equal to zero, that is,

$$\begin{vmatrix}q_i(x_1)g(x_1)-\bar{q}_i(x_1)\bar{g}(x_1) & \bar{q}_i(x_1)-q_i(x_1)\\ q_i(x_2)g(x_2)-\bar{q}_i(x_2)\bar{g}(x_2) & \bar{q}_i(x_2)-q_i(x_2)\end{vmatrix}=0.$$

If $x_1,x_2\in S\cap]\alpha-\delta_0,\alpha[=S_i\cap]\alpha-\delta_0,\alpha[$ are arbitrary, then $\bar{q}_i(x_1)\neq q_i(x_1)$ and $\bar{q}_i(x_2)\neq q_i(x_2)$, therefore, the above determinantal equality can be rewritten as

$$\frac{q_i(x_1)g(x_1)-\bar{q}_i(x_1)\bar{g}(x_1)}{\bar{q}_i(x_1)-q_i(x_1)}=\frac{q_i(x_2)g(x_2)-\bar{q}_i(x_2)\bar{g}(x_2)}{\bar{q}_i(x_2)-q_i(x_2)}.$$

Therefore, there exists a real constant c_i such that

$$c_i = \frac{q_i(x)g(x) - \bar{q}_i(x)\bar{g}(x)}{\bar{q}_i(x) - q_i(x)}$$

holds for all $x \in S \cap]\alpha - \delta_0, \alpha[$. Solving this equation with respect to $\bar{g}(x)$, we obtain that

$$\bar{g}(x) = \frac{q_i(x)}{\bar{q}_i(x)}(g(x) + c_i) - c_i \tag{2.17}$$

is valid for all $x \in S \cap]\alpha - \delta_0, \alpha[$. Substituting the formula (2.17) into (2.12), for all $x \in S \cap]\alpha - \delta_0, \alpha[$ and $y \in]y_0 - \delta_0, y_0 + \delta_0[$, we arrive at the equation

$$(\bar{q}_i(x) - q_i(x)) \cdot \big(q_i(x)(g(x) + c_i) + (q_0 - q_i)(y)(g(y) + c_i)\big) = 0,$$

which simplifies to the identity

$$q_i(x)(g(x) + c_i) = -(q_0 - q_i)(y)(g(y) + c_i),$$

where $x \in S \cap]\alpha - \delta_0, \alpha[$, $y \in]y_0 - \delta_0, y_0 + \delta_0[$. Therefore, there exists a real constant d_i such that

$$q_i(x)(g(x) + c_i) = d_i = -(q_0 - q_i)(y)(g(y) + c_i),$$

where $x \in S \cap]\alpha - \delta_0, \alpha[$, $y \in]y_0 - \delta_0, y_0 + \delta_0[$. Using these equalities on the domain indicated, the inequality (2.11) implies that

$$g(\alpha) \leq \frac{q_i(x)g(x) + (q_0 - q_i)(y)g(y)}{q_i(x) + (q_0 - q_i)(y)} = \frac{d_i - c_i q_i(x) - d_i - c_i(q_0 - q_i)(y)}{q_i(x) + (q_0 - q_i)(y)} = -c_i. \tag{2.18}$$

Therefore, for all $x \in S \cap]\alpha - \delta_0, \alpha[$, we have that $g(x) < g(\alpha) \leq -c_i$, which yields that $d_i < 0$ and $q_i(x) = \frac{d_i}{g(x)+c_i}$. Hence q_i is strictly increasing on $S \cap]\alpha - \delta_0, \alpha[$. As a consequence of this property, it follows that the equality $q_i(x)(g(x) + c_i) = d_i$ uniquely determines the constants c_i and d_i. Indeed, if $q_i(x)(g(x) + c_i') = d_i'$ were also true for all $x \in S \cap]\alpha - \delta_0, \alpha[$ and for some constant c_i' and d_i', then subtracting the two equations side by side, we get $q_i(x)(c_i - c_i') = d_i - d_i'$. If $c_i \neq c_i'$, then this last equality yields that q_i is constant, which contradicts its strict monotonicity. Therefore, $c_i = c_i'$ implying that $d_i = d_i'$ is also valid.

In the final step, instead of a fixed element $y_0 \in]\alpha, \alpha + r[$, we take another arbitrary element $y' \in]\alpha, \alpha + r[$. Repeating the same argument as above, there exists a positive number $\delta^* := \delta(y^*)$ and real constants c_i^*, d_i^* such that

$$q_i(x)(g(x) + c_i^*) = d_i^* = -(q_0 - q_i)(y)(g(y) + c_i^*),$$

where $x \in S \cap]\alpha - \delta^*, \alpha[$, $y \in]y^* - \delta^*, y^* + \delta^*[$. On the set $S \cap]\alpha - \min(\delta^*, \delta_0), \alpha[$, we have both $q_i(x)(g(x) + c_i) = d_i$ and $q_i(x)(g(x) + c_i^*) = d_i^*$. Due to the uniqueness property, it follows that $c_i^* = c_i$ and $d_i^* = d_i$. Therefore,

$$d_i = -(q_0 - q_i)(y)(g(y) + c_i) \tag{2.19}$$

is valid for all $y \in]y^* - \delta^*, y^* + \delta^*[$, in particular, for $y = y^*$. The point y^* being arbitrary, we can see that (2.19) holds for all $y \in]\alpha, \alpha + r[$. Comparing the signs of both sides, we obtain that $g(y) + c_i > 0$ for all $y \in]\alpha, \alpha + r[$. Upon taking the limit $y \to \alpha^+$, it follows that $g(\alpha) + c_i \geq 0$. On the other hand, by (2.18), we also have that $g(\alpha) + c_i \leq 0$, whence $g(\alpha) + c_i = 0$ follows. Using that (2.14) holds, we may also take the limit $x \to \alpha^-$ in the equality

$$q_i(x)(g(x) + c_i) = d_i \qquad (x \in S \cap]\alpha - \delta_0, \alpha[),$$

whence we arrive at $d_i = 0$, which is the desired contradiction. □

The first- and second-order necessary conditions of the equality of two generalized Bajraktarević means are detailed in the following two lemmas, respectively.

Lemma 2.2.3. *Let $n \in \mathbb{N}, n \geq 2$ and $f, g \colon I \to \mathbb{R}$ be differentiable functions with nonvanishing first derivatives and $i \in \{1, \dots, n\}$. Let further $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n) \colon I \to \mathbb{R}_+^n$ such that p_i and q_i are continuous. If $\partial_i A_{f,p}^{[n]} = \partial_i A_{g,q}^{[n]}$ holds on $\operatorname{diag}(I^n)$, then*

$$\frac{q_i}{q_0} = \frac{p_i}{p_0} \tag{2.20}$$

holds on I.

Proof. In view of Theorem 1.2.1, we have

$$\frac{q_i}{q_0} = \left(\partial_i A_{g,q}^{[n]}\right)^{\Delta} = \left(\partial_i A_{f,p}^{[n]}\right)^{\Delta} = \frac{p_i}{p_0}.$$

□

Lemma 2.2.4. *Let $n \in \mathbb{N}, n \geq 2$, $f, g \colon I \to \mathbb{R}$ be twice differentiable functions with non-vanishing first derivatives. Let $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n) \colon I \to \mathbb{R}_+^n$ be continuous functions and assume that, for all $i \in \{1, \dots, n\}$, (2.20) holds on I. Let $j, k \in \{1, \dots, n\}$. Then the following two assertions hold.*

(i) Provided that $j \neq k$ and p_j, p_k, q_j, q_k are differentiable functions, if $\partial_j \partial_k A_{f,p}^{[n]} = \partial_j \partial_k A_{g,q}^{[n]}$ holds on $\operatorname{diag}(I^n)$, then there exists a nonzero constant γ such that, for all $i \in \{1, \dots, n\}$,

$$q_i^2 g' = \gamma p_i^2 f' \tag{2.21}$$

is valid on I.

(ii) Provided that $j = k$ and p_j, q_j are continuously differentiable functions, if $\partial_j^2 A_{f,p}^{[n]} = \partial_j^2 A_{g,q}^{[n]}$ holds on $\operatorname{diag}(I^n)$, then there exists a nonzero constant γ such that, for all $i \in \{1, \dots, n\}$, (2.21) is valid on I.

Proof. From Lemma 2.2.3 we obtain that $q_i = r_0 p_i$ holds for all $i \in \{0, \dots, n\}$. Assume that $j \neq k$. Then, using Theorem 1.2.1, we have that

$$\frac{(p_j p_k)'}{p_0^2} + \frac{p_j p_k}{p_0^2} \cdot \frac{f''}{f'} = -\left(\partial_j \partial_k A_{f,p}^{[n]}\right)^{\Delta} = -\left(\partial_j \partial_k A_{g,q}^{[n]}\right)^{\Delta} = \frac{(r_0^2 p_j p_k)'}{r_0^2 p_0^2} + \frac{r_0^2 p_j p_k}{r_0^2 p_0^2} \cdot \frac{g''}{g'}.$$

Thus, after reduction, we get that

$$\frac{1}{2}\left(\frac{f''}{f'} - \frac{g''}{g'}\right) = \frac{r_0'}{r_0} \tag{2.22}$$

is valid on I. Hence, there exists $\gamma \in \mathbb{R} \setminus \{0\}$ such that

$$r_0 = \sqrt{\gamma \cdot \frac{f'}{g'}} \tag{2.23}$$

holds on I, whence, using Lemma 2.2.3 again, it follows that, for all $i \in \{1, \dots, n\}$, (2.21) is valid.

If $j = k$, then with a similar calculation we arrive at the same differential equation for r_0. □

The following notion and a corresponding well-known lemma play a basic role in the proofs of the subsequent results since, by solving the differential equations obtained by differentiating the functional equation at issue, one of the cases that we get is exactly the equality for which the next lemma gives a necessary condition under assuming only the regularity that is needed for the definition to be correct.

Definition 2.2.5. For a three times differentiable function $f\colon I \to \mathbb{R}$ with a nonvanishing first derivative, we introduce its *Schwarzian derivative* $S_f\colon I \to \mathbb{R}$ by the following formula:

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Lemma 2.2.6. *Let $f, g\colon I \to \mathbb{R}$ be three times differentiable functions with nonvanishing first derivatives. If $S_f = S_g$ is valid on I, then there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that $cf + d$ is positive on I and*

$$g = \frac{af + b}{cf + d} \tag{2.24}$$

holds on I.

Proof. Denote the function $g \circ f^{-1}$ by φ. Then, by the chain rule for the Schwarzian derivative (see [16, Chapter 10]), we have

$$S(g) = S(\varphi \circ f) = (S(\varphi) \circ f) \cdot (f')^2 + S(f).$$

Therefore $S(f) = S(g)$ holds on I if and only if $S(\varphi) = 0$ is satisfied on $J := f(I)$. This latter equality however can be valid (again by [16]) if and only if φ is a Möbius transform on J, i.e., there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that $cx + d > 0$ and $\varphi(x) = \frac{ax+b}{cx+d}$ for $x \in J$. Substituting x by $f(u)$, these properties are equivalent to the positivity of the function $cf + d$ and the equality (2.24) on the interval I, respectively. □

Our first main result is contained in the following theorem. It completely characterizes the equality of two generalized Bajraktarević means with at least three variables.

Theorem 2.2.7. *Let $n \in \mathbb{N}, n \geq 3$ and $f, g \colon I \to \mathbb{R}$ be three times differentiable functions with nonvanishing first derivatives. Let $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$ be a continuous function and $q = (q_1, \dots, q_n) : I \to \mathbb{R}_+^n$. Assume that there exist $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k \neq i$ such that p_i, p_j, p_k are differentiable functions. Then the following assertions are equivalent.*

(i) $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ *holds globally.*

(ii) $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ *holds locally.*

(iii) The function q is continuous, the functions q_i, q_j, q_k are differentiable, and the equalities

$$\begin{aligned} \partial_\ell A_{f,p}^{[n]} &= \partial_\ell A_{g,q}^{[n]} \qquad (\ell \in \{1, \dots, n-1\}), \\ \partial_i \partial_j A_{f,p}^{[n]} &= \partial_i \partial_j A_{g,q}^{[n]}, \\ \partial_i \partial_j \partial_k A_{f,p}^{[n]} &= \partial_i \partial_j \partial_k A_{g,q}^{[n]} \end{aligned}$$

hold on $\operatorname{diag}(I^n)$.

(iv) There exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that

$$g = \frac{af + b}{cf + d} \qquad \text{and} \qquad q_\ell = (cf + d)p_\ell \qquad (\ell \in \{1, \dots, n\})$$

hold on I.

Proof. The implication (i)⇒(ii) is obvious. Applying Lemma 2.2.1, it is also easy to see that assertion (iii) follows from statement (ii). The implication (iv)⇒(i) is a consequence of Theorem 2.1.3. It remains to prove that assertion (iii) implies statement (iv).

Without loss of generality, we can assume that $i = 1$, $j = 2$, and $k = 3$. One can easily see that if $\partial_\ell A_{f,p}^{[n]} = \partial_\ell A_{g,q}^{[n]}$ holds for all $\ell \in \{1, \dots, n-1\}$, then it is also valid for $\ell = n$. Using Lemma 2.2.3, we have that $q_\ell = r_0 p_\ell$ holds for all $\ell \in \{0, \dots, n\}$. Hence, using the equality $q'_\ell = r'_0 p_\ell + r_0 p'_\ell$, we get

$$
\begin{aligned}
&2\frac{p_1p_2'p_3' + p_1'p_2p_3' + p_1'p_2'p_3}{p_0^3} + 2\frac{p_1p_2p_3' + p_1p_2'p_3 + p_1'p_2p_3}{p_0^3}\cdot\frac{f''}{f'}\\
&\qquad + \frac{p_1p_2p_3}{p_0^3}\left(3\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right) = \left(\partial_1\partial_2\partial_3 A_{f,p}^{[n]}\right)^\Delta = \left(\partial_1\partial_2\partial_3 A_{g,q}^{[n]}\right)^\Delta\\
&\quad = 2\frac{r_0^3(p_1p_2'p_3' + p_1'p_2p_3' + p_1'p_2'p_3)}{r_0^3p_0^3} + 4\frac{r_0'r_0^2(p_1p_2p_3' + p_1p_2'p_3 + p_1'p_2p_3)}{r_0^3p_0^3}\\
&\qquad + 6\frac{r_0(r_0')^2p_1p_2p_3}{r_0^3p_0^3} + 6\frac{r_0^2r_0'p_1p_2p_3}{r_0^3p_0^3}\cdot\frac{g''}{g'} + 2\frac{r_0^3(p_1p_2p_3' + p_1p_2'p_2 + p_1'p_2p_3)}{r_0^3p_0^3}\cdot\frac{g''}{g'}\\
&\qquad + \frac{r_0^3p_1p_2p_3}{r_0^3p_0^3}\left(3\left(\frac{g''}{g'}\right)^2 - \frac{g'''}{g'}\right).
\end{aligned}
$$

Thus, applying (2.22) three times, after reduction, it follows that

$$
3\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'} = \frac{3}{2}\left(\frac{f''}{f'} - \frac{g''}{g'}\right)^2 + 3\left(\frac{f''}{f'} - \frac{g''}{g'}\right)\frac{g''}{g'} + 3\left(\frac{g''}{g'}\right)^2 - \frac{g'''}{g'}
$$

is valid on I. Whence we obtain that $S_f = S_g$ holds on I. Therefore, using Lemma 2.2.6, there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that $cf + d$ is positive and (2.24) holds on I. Substituting (2.24) into (2.23), we get that $r_0 = \delta(cf + d)$ is fulfilled on I, where $\delta := \sqrt{\frac{\gamma}{ad-bc}} > 0$. Therefore,

$$
q_\ell = r_0 p_\ell = (\delta cf + \delta d)p_\ell \qquad (\ell \in \{1, \dots, n\}),
$$

and

$$
g = \frac{af + b}{cf + d} = \frac{\delta af + \delta b}{\delta cf + \delta d},
$$

which proves that assertion (iv) holds with the constant vector $(\bar{a}, \bar{b}, \bar{c}, \bar{d}) := \delta \cdot (a, b, c, d)$. $\square$

The second main theorem has two variants concerning the regularity assumptions and characterizes the equality of 2-variable generalized nonsymmetric Bajraktarević means.

Theorem 2.2.8. *Let $f, g\colon I \to \mathbb{R}$ be three times differentiable functions with nonvanishing first derivatives. Let $p = (p_1, p_2)\colon I \to \mathbb{R}_+^2$ and $q = (q_1, q_2)\colon I \to \mathbb{R}_+^2$ such that $p_1 \neq p_2$. Assume that there exists $i \in \{1, 2\}$ such that one of the following regularity conditions is satisfied.*

(a) p_i is twice continuously differentiable and p_{3-i} is continuous.

(b) p_i is twice differentiable and p_{3-i} is once differentiable.

Then the following assertions are pairwise equivalent.

(i) $A_{f,p}^{[2]} = A_{g,q}^{[2]}$ *holds globally.*

(ii) $A_{f,p}^{[2]} = A_{g,q}^{[2]}$ *holds locally.*

(iv) There exist $a, b, c, d \in \mathbb{R}$ *with* $ad \neq bc$ *such that*

$$g = \frac{af+b}{cf+d}, \qquad q_1 = (cf+d)p_1, \qquad \textit{and} \qquad q_2 = (cf+d)p_2$$

hold on I.

In addition, if f *and* p *are* ℓ *times continuously differentiable, then* $A_{f,p}^{[2]}$ *is also* ℓ *times continuously differentiable.*

Proof. The implication (i)⇒(ii) is obvious. The implication (iv)⇒(i) is a consequence of Theorem 2.1.3. It remains to prove that (ii) implies statement (iv) in both regularity settings.

Applying Lemma 2.2.1, one can see that we have the following assertions from statement (ii), under the regularity assumptions (a) and (b) of Theorem 2.2.8, respectively.

(iii) The function q_i is twice continuously differentiable, q_{3-i} is continuous, furthermore

$$\partial_i A_{f,p}^{[2]} = \partial_i A_{g,q}^{[2]}, \qquad \partial_i^2 A_{f,p}^{[2]} = \partial_i^2 A_{g,q}^{[2]}, \qquad \text{and} \qquad \partial_i^3 A_{f,p}^{[2]} = \partial_i^3 A_{g,q}^{[2]}$$

hold on $\operatorname{diag}(I^2)$.

(iii)' The function q_i is twice differentiable, q_{3-i} is once differentiable, furthermore

$$\partial_i A_{f,p}^{[2]} = \partial_i A_{g,q}^{[2]}, \qquad \partial_i^2 A_{f,p}^{[2]} = \partial_i^2 A_{g,q}^{[2]}, \qquad \partial_i^2 \partial_{3-i} A_{f,p}^{[2]} = \partial_i^2 \partial_{3-i} A_{g,q}^{[2]}$$

hold on $\operatorname{diag}(I^2)$.

Without loss of generality, we can assume that $i = 1$. Then, using the first equation of (iii) or (iii)' and Lemma 2.2.3, we have $q_j = r_0 p_j$ for all $j \in \{0, 1, 2\}$. Due to the equality $r_0 = q_i/p_i$, it follows that r_0 is twice differentiable. Furthermore, by the second equation of assertion (iii) or (iii)', we have that (2.22) also holds as we have seen in the proof of Lemma 2.2.4. Observe that, differentiating (2.22), we obtain that

$$\frac{r_0''}{r_0} = \frac{1}{4}\left(2\frac{f'''}{f'} - 2\frac{g'''}{g'} - \left(\frac{f''}{f'}\right)^2 + 3\left(\frac{g''}{g'}\right)^2 - 2\frac{f''}{f'} \cdot \frac{g''}{g'}\right). \tag{2.25}$$

Under the regularity assumption (a) of Theorem 2.2.8, the third equality in condition (iii), and formula (3c) of Theorem 1.2.1 yields that

$$\begin{aligned}
&-\frac{p_2\big(6(p_1')^2-3p_1''(p_1+p_2)\big)}{p_0^3}-3\frac{p_1'p_2(p_1-p_2)}{p_0^3}\cdot\frac{f''}{f'}-3\frac{p_1^2p_2}{p_0^3}\left(\frac{f''}{f'}\right)^2\\
&\quad+\frac{p_1p_2(2p_1+p_2)}{p_0^3}\cdot\frac{f'''}{f'}=\left(\partial_1^3A_{f,p}^{[2]}\right)^\Delta=\left(\partial_1^3A_{g,q}^{[2]}\right)^\Delta\\
&=-\frac{r_0p_2\big(6r_0^2(p_1')^2+12r_0r_0'p_1p_1'+6(r_0')^2p_1^2\big)}{r_0^3p_0^3}\\
&\quad-\frac{r_0p_2\big(-3r_0(p_1+p_2)(r_0p_1''+2r_0'p_1'+r_0''p_1)\big)}{r_0^3p_0^3}\\
&\quad-3\frac{r_0^2p_2(r_0p_1'+r_0'p_1)(p_1-p_2)}{r_0^3p_0^3}\cdot\frac{g''}{g'}-3\frac{r_0^3p_1^2p_2}{r_0^3p_0^3}\cdot\left(\frac{g''}{g'}\right)^2+\frac{r_0^3p_1p_2(2p_1+p_2)}{r_0^3p_0^3}\cdot\frac{g'''}{g'}.
\end{aligned}\tag{2.26}$$

Hence, from (2.26), using (2.22) and (2.25), it follows that

$$\begin{aligned}
0=&-3\frac{p_1'p_2(p_1-p_2)}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)-3\frac{p_1^2p_2}{p_0^3}\left(\left(\frac{f''}{f'}\right)^2-\left(\frac{g''}{g'}\right)^2\right)\\
&+\frac{p_1p_2(2p_1+p_2)}{p_0^3}\left(\frac{f'''}{f'}-\frac{g'''}{g'}\right)+3\frac{p_1'p_2(p_1-p_2)}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)\\
&+\frac{3}{2}\cdot\frac{p_1^2p_2}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)^2\\
&-\frac{3}{4}\cdot\frac{p_1p_2(p_1+p_2)}{p_0^3}\left(2\frac{f'''}{f'}-2\frac{g'''}{g'}-\left(\frac{f''}{f'}\right)^2+3\left(\frac{g''}{g'}\right)^2-2\frac{f''}{f'}\cdot\frac{g''}{g'}\right)\\
&+\frac{3}{2}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)\frac{g''}{g'},
\end{aligned}$$

whence we get

$$0=\frac{1}{2}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\left(\frac{f'''}{f'}-\frac{g'''}{g'}\right)-\frac{3}{4}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\left(\left(\frac{f''}{f'}\right)^2-\left(\frac{g''}{g'}\right)^2\right),$$

which simplifies to

$$0=\frac{1}{2}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\big(S_f-S_g\big).\tag{2.27}$$

Using that $p_1\neq p_2$, by continuity, it follows that there exists an open nonempty subinterval $J\subseteq I$ such that $p_1(x)\neq p_2(x)$ holds for $x\in J$. Therefore, the above equation implies that $S_f=S_g$ holds on J and hence, by Theorem 2.2.2, on I. Thus, using Lemma 2.2.6, there exist $a,b,c,d\in\mathbb{R}$ with $ad\neq bc$ such that $cf+d$ is positive and (2.24) holds on I. Substituting (2.24) into (2.23), we get that $r_0=\delta(cf+d)$ holds on

I, where $\delta := \sqrt{\frac{\gamma}{ad-bc}} > 0$. Therefore, with the same argument as at the end of the proof of Theorem 2.2.7, we can see that assertion (iv) holds with the constant vector $(\bar{a}, \bar{b}, \bar{c}, \bar{d}) := \delta \cdot (a, b, c, d)$.

Under the assumption (b) of Theorem 2.2.8, the third equality of condition (iii)' and formula (3b) of Theorem 1.2.1 imply that

$$\begin{aligned}
&\frac{2p_1'p_2'(p_1-p_2)+p_2(2(p_1')^2-p_1''(p_1+p_2)}{p_0^3} \\
&\quad+\frac{(2p_1'p_2+p_1p_2')(p_1-p_2)}{p_0^3}\cdot\frac{f''}{f'}+\frac{p_1p_2(2p_1-p_2)}{p_0^3}\left(\frac{f''}{f'}\right)^2 \\
&\quad-\frac{p_1^2p_2}{p_0^3}\cdot\frac{f'''}{f'}=\left(\partial_1^2\partial_2A_{f,p}^{[2]}\right)^{\Delta}=\left(\partial_1^2\partial_2A_{g,q}^{[2]}\right)^{\Delta} \\
&=\frac{2r_0(r_0p_1'+r_0'p_1)(r_0p_2'+r_0'p_2)(p_1-p_2)}{r_0^3p_0^3} \\
&\quad+\frac{r_0p_2\big(2\big(r_0^2(p_1')^2+2r_0r_0'p_1p_1'+(r_0')^2p_1^2\big)\big)}{r_0^3p_0^3} \\
&\quad+\frac{r_0p_2\big(-r_0(r_0p_1''+2r_0'p_1'+r_0''p_1)(p_1+p_2)\big)}{r_0^3p_0^3} \\
&\quad+\frac{r_0^2(2p_2(r_0p_1'+r_0'p_1)+p_1(r_0p_2'+r_0'p_2))(p_1-p_2)}{r_0^3p_0^3}\cdot\frac{g''}{g'} \\
&\quad+\frac{r_0^3p_1p_2(2p_1-p_2)}{r_0^3p_0^3}\left(\frac{g''}{g'}\right)^2-\frac{r_0^3p_1^2p_2}{r_0^3p_0^3}\cdot\frac{g'''}{g'}.
\end{aligned} \tag{2.28}$$

Hence, from (2.28), using (2.22) and (2.25), we arrive at

$$\begin{aligned}
0=&\frac{2p_1p_1'p_2-2p_1'p_2^2+p_1^2p_2'-p_1p_2p_2'}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right) \\
&+\frac{p_1p_2(2p_1-p_2)}{p_0^3}\left(\left(\frac{f''}{f'}\right)^2-\left(\frac{g''}{g'}\right)^2\right)-\frac{p_1^2p_2}{p_0^3}\left(\frac{f'''}{f'}-\frac{g'''}{g'}\right) \\
&-\frac{2p_1p_1'p_2-2p_1'p_2^2+p_1^2p_2'-p_1p_2p_2'}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right) \\
&+\frac{1}{4}\cdot\frac{p_1p_2(p_1+p_2)}{p_0^3}\left(2\frac{f'''}{f'}-2\frac{g'''}{g'}-\left(\frac{f''}{f'}\right)^2+3\left(\frac{g''}{g'}\right)^2-2\frac{f''}{f'}\cdot\frac{g''}{g'}\right) \\
&-\frac{1}{2}\cdot\frac{p_1p_2(2p_1-p_2)}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)^2-\frac{3}{2}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)\frac{g''}{g'},
\end{aligned}$$

whence we have that (2.27) holds, thus following a similar train of thought as above, we get assertion (iv). $\square$

As we already mentioned, the symmetric 2-variable case was solved firstly under sixth- and then under first-order regularity assumptions in [22] and in [38], respectively.

Theorem 2.2.9. *Let $f, g\colon I \to \mathbb{R}$ be continuously differentiable functions with nonvanishing first derivatives. Let $p\colon I \to \mathbb{R}_+$ be a continuously differentiable function and $q\colon I \to \mathbb{R}_+$. Then the following assertions are equivalent.*

(i) $A^{[2]}_{f,(p,p)} = A^{[2]}_{g,(q,q)}$ *holds globally.*

(ii) $A^{[2]}_{f,(p,p)} = A^{[2]}_{g,(q,q)}$ *holds locally.*

(iv) Either there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that

$$g = \frac{af + b}{cf + d} \qquad \textit{and} \qquad q = (cf + d)p$$

are valid on I or there exist two polynomials P and Q of at most second degree such that P and Q are positive on $f(I)$ and $g(I)$, respectively, and there exist two constants $\alpha, \beta \in \mathbb{R}$ such that

$$g = G^{-1} \circ (\alpha F \circ f + \beta), \qquad p = P^{-\frac{1}{2}} \circ f, \qquad \textit{and} \qquad q = Q^{-\frac{1}{2}} \circ g$$

hold on I, where F and G denote a primitive function of $1/P$ and $1/Q$, respectively.

Proof. The implication (i)⇒(ii) is obvious. The proof of the implication (iii)⇒(iv) is based on Lemma 2.2.1, the result of Páles and Zakaria [38, Corollary 9], and a recent characterization of the equality of 2-variable (symmetric) Bajraktarević means with two-variable quasi-arithmetic means [37]. The proof of the implication (iv)⇒(i) is also described in the paper [37]. □

Chapter 3

Comparison problem

The global comparison problem of classical Bajraktarević means with arbitrary number of variables was solved by Daróczy and Losonczi in [9]. Their result was extended to the context of deviation means by Daróczy in [8] and by Daróczy and Páles in [10]. The Gini means according to the result of Aczél and Daróczy (see [1]) are precisely the continuous and homogeneous Bajraktarević means on $I = \mathbb{R}_+$. Their comparison was also characterized by Daróczy and Losonczi in [9]. For the setting when I is a proper subinterval of $\mathbb{R}_+$, the comparison problem of Gini means was solved by Losonczi (see [20]) and this result was extended to the setting of Gini means with complex parameters by Páles in [35]. Another generalization of Bajraktarević means, in terms of a measure and two functions, was introduced and their comparison problem was investigated by Losonczi and Páles (see [27]). The notion of local and global comparison in broad classes of means was introduced by Páles and Zakaria in [36], where such comparison problems were in the focus of their research.

Motivated by these preliminaries, the main goal of this chapter is to investigate the local and global comparison problem of two n-variable generalized Bajraktarević means, i.e., to find necessary as well as sufficient conditions in terms of the unknown functions listed below for the functional inequality

$$\begin{aligned} f^{(-1)}&\left(\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)}\right)\\ &\leq g^{(-1)}\left(\frac{q_1(x_1)g(x_1)+\cdots+q_n(x_n)g(x_n)}{q_1(x_1)+\cdots+q_n(x_n)}\right) \end{aligned} \tag{3.1}$$

to be valid locally or globally, where $n \in \mathbb{N}, n \geq 2$ is fixed, I is a nonempty open real interval, which will be the case in this entire chapter unless otherwise stated, the unknown functions $f, g\colon I \to \mathbb{R}$ are strictly monotone, and $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$ are also unknown functions. The results of this section were first proved in our paper [13]. We note that, as we discussed in Chapter 2, the equality of these means in the local and in the global sense are equivalent properties. However, as we shall see, this is not the case for the comparison problem.

3.1 Local comparison

Definition 3.1.1. Let $n \in \mathbb{N}, n \geq 2$ and $U \in \mathbb{R}^n$ be a nonempty open set. A function $f\colon U \to \mathbb{R}$ is called *partially differentiable at $a \in U$ with respect to its ith variable* if the limit

$$\lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}$$

exists and finite. We say that f is *partially differentiable at $a \in U$* if it is partially differentiable with respect to all its variables. Finally, f is called *partially differentiable on U* if it is partially differentiable at each point of U.

In order to investigate inequality (3.1) in the local sense, we recall the following result from [36].

Theorem 3.1.2. *Let $n \in \mathbb{N}, n \geq 2$ and let $M, N\colon I^n \to I$ be n-variable means such that M is locally smaller than N. Assume that M and N are partially differentiable on $\operatorname{diag}(I^n)$. Then, for all $i \in \{1, \dots, n\}$,*

$$\partial_i M^\Delta = \partial_i N^\Delta. \tag{3.2}$$

If, in addition, M and N are twice differentiable on the diagonal $\operatorname{diag}(I^n)$, then the symmetric $(n-1) \times (n-1)$-type matrix

$$\big(\partial_i \partial_j N^\Delta - \partial_i \partial_j M^\Delta\big)_{i,j=1}^{n-1} \tag{3.3}$$

is positive semidefinite.

On the other hand, if the equality (3.2) holds for all $i \in \{1, \dots, n\}$, furthermore, M and N are twice continuously differentiable on the diagonal $\operatorname{diag}(I^n)$ and the symmetric $(n-1) \times (n-1)$-type matrix given by (3.3) is positive definite, then M is locally smaller than N.

Our first main result of this chapter establishes necessary and sufficient conditions for the local comparability of generalized Bajraktarević means.

Theorem 3.1.3. *Let $n \in \mathbb{N}, n \geq 2$ and let $f, g\colon I \to \mathbb{R}$ be differentiable functions with nonvanishing first derivatives and $p, q\colon I \to \mathbb{R}_+^n$ be continuous functions. Assume that $A_{f,p}^{[n]}$ is locally smaller than $A_{g,q}^{[n]}$. Then*

$$\frac{p_i}{p_0} = \frac{q_i}{q_0} \qquad (i \in \{1, \dots, n\}). \tag{3.4}$$

If, in addition, f, g are twice differentiable and p, q are continuously differentiable, then the function

$$\frac{q_0^2 |g'|}{p_0^2 |f'|} \tag{3.5}$$

is increasing.

On the other hand, if f, g and p, q are twice continuously differentiable, (3.4) *holds and the function* (3.5) *has a positive derivative, then $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{g,q}$.*

Proof. In view of Theorem 1.2.1, the differentiability of f, g and the continuity of p, q imply that the partial derivatives of the means $M := A^{[n]}_{f,p}$ and $N := A_{q,q}$ exist at the diagonal points of I^n. Therefore, by Theorem 3.1.2, we have that (3.2) is valid. Thus, using assertion (1) of Theorem 1.2.1, it follows that

$$\frac{p_i}{p_0} = \left(\partial_i A^{[n]}_{f,p}\right)^{\Delta} = \left(\partial_i A^{[n]}_{g,q}\right)^{\Delta} = \frac{q_i}{q_0} \qquad (i \in \{1, \dots, n\}),$$

which shows that (3.4) is valid. In what follows, let r_0 denote the ratio function q_0/p_0. Then, according to (3.4), we have that $q_i = r_0 p_i$.

If, in addition, f, g are twice differentiable and p, q are continuously differentiable, then Theorem 1.2.1 implies that the second-order partial derivatives of the means $M := A^{[n]}_{f,p}$ and $N := A^{[n]}_{g,q}$ exist at the diagonal points of I^n. Therefore, by Theorem 3.1.2, we have that the matrix given by (3.3) is positive semidefinite. This yields that

$$\partial_i^2 \left(A^{[n]}_{g,q} - A^{[n]}_{f,p}\right)^{\Delta} \geq 0. \tag{3.6}$$

Applying assertion (2b) of Theorem 1.2.1 and the equality $q_i = r_0 p_i$, for $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned}
&\partial_i^2 \left(A^{[n]}_{g,q} - A^{[n]}_{f,p}\right)^{\Delta} \\
&\quad = 2\frac{q_i'(q_0 - q_i)}{q_0^2} + \frac{q_i(q_0 - q_i)}{q_0^2} \cdot \frac{g''}{g'} - 2\frac{p_i'(p_0 - p_i)}{p_0^2} - \frac{p_i(p_0 - p_i)}{p_0^2} \cdot \frac{f''}{f'} \\
&\quad = 2\frac{(r_0' p_i + r_0 p_i')(p_0 - p_i)}{r_0 p_0^2} + \frac{p_i(p_0 - p_i)}{p_0^2} \cdot \frac{g''}{g'} - 2\frac{p_i'(p_0 - p_i)}{p_0^2} - \frac{p_i(p_0 - p_i)}{p_0^2} \cdot \frac{f''}{f'} \\
&\quad = \frac{p_i(p_0 - p_i)}{p_0^2}\left(2\frac{r_0'}{r_0} + \frac{g''}{g'} - \frac{f''}{f'}\right).
\end{aligned}$$

The factor $\frac{p_i(p_0 - p_i)}{p_0^2}$ is positive, therefore, the inequality (3.6) simplifies to

$$0 \leq 2\frac{r_0'}{r_0} + \frac{g''}{g'} - \frac{f''}{f'} = \left(\log\left(\frac{q_0^2 |g'|}{p_0^2 |f'|}\right)\right)'. \tag{3.7}$$

This proves that the function (3.5) is increasing.

In the last part of the proof, assume that f, g and p, q are twice continuously differentiable, (3.4) holds and the function (3.5) has a positive derivative. Then the means $M := A^{[n]}_{f,p}$ and $N := A_{q,q}$ are twice continuously differentiable on I^n and (3.7) is valid with strict inequality everywhere on I. We show that the matrix (3.3) is positive definite.

The diagonal entries of this matrix have been computed above. For the entries off the diagonal, i.e., for $i, j \in \{1, \ldots, n\}$ with $i \neq j$, according assertion (2a) of Theorem 1.2.1 and the equalities $q_i = r_0 p_i$ and $q_j = r_0 p_j$, we have

$$\begin{aligned}
\partial_i \partial_j \left(A^{[n]}_{g,q} - A^{[n]}_{f,p} \right)^{\Delta} &= -\frac{(q_i q_j)'}{q_0^2} - \frac{q_i q_j}{q_0^2} \cdot \frac{g''}{g'} + \frac{(p_i p_j)'}{p_0^2} + \frac{p_i p_j}{p_0^2} \cdot \frac{f''}{f'} \\
&= -\frac{2 r_0 r_0' p_i p_j + r_0^2 (p_i p_j)'}{r_0^2 p_0^2} - \frac{p_i p_j}{p_0^2} \cdot \frac{g''}{g'} + \frac{(p_i p_j)'}{p_0^2} + \frac{p_i p_j}{p_0^2} \cdot \frac{f''}{f'} \\
&= -\frac{p_i p_j}{p_0^2} \left(2\frac{r_0'}{r_0} + \frac{g''}{g'} - \frac{f''}{f'} \right).
\end{aligned}$$

Therefore, for $i, j \in \{1, \ldots, n\}$,

$$\partial_i \partial_j \left(A^{[n]}_{g,q} - A^{[n]}_{f,p} \right)^{\Delta} = \frac{p_i (\delta_{i,j} p_0 - p_j)}{p_0^2} \left(2\frac{r_0'}{r_0} + \frac{g''}{g'} - \frac{f''}{f'} \right).$$

In order to prove that the matrix (3.3) is positive definite, according to Sylvester's criterion, it is necessary and sufficient that the $(k \times k)$-type minors of this matrix should have a positive determinant.

For $k \in \{1, \ldots, n\}$, we obtain

$$\begin{aligned}
&\det \left(\left(\partial_i \partial_j (A^{[n]}_{g,q} - A^{[n]}_{f,p})^{\Delta} \right)_{i,j=1}^{k} \right) \\
&\qquad = \det \left(\left(\frac{p_i (\delta_{i,j} p_0 - p_j)}{p_0^2} \left(2\frac{r_0'}{r_0} + \frac{g''}{g'} - \frac{f''}{f'} \right) \right)_{i,j=1}^{k} \right) \\
&\qquad = \left(2\frac{r_0'}{r_0} + \frac{g''}{g'} - \frac{f''}{f'} \right)^{k} \frac{(p_1 \cdots p_k)^2}{p_0^{2k}} \det \left(\left(\frac{\delta_{i,j} p_0}{p_j} - 1 \right)_{i,j=1}^{k} \right).
\end{aligned}$$

Now we compute the last determinant of the above expression:

$$\det \left(\left(\frac{\delta_{i,j} p_0}{p_j} - 1 \right)_{i,j=1}^{k} \right) = \begin{vmatrix}
\frac{p_0}{p_1} - 1 & -1 & \ldots & -1 & -1 \\
-1 & \frac{p_0}{p_2} - 1 & \ldots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \ldots & \frac{p_0}{p_{k-1}} - 1 & -1 \\
-1 & -1 & \ldots & -1 & \frac{p_0}{p_k} - 1
\end{vmatrix}$$

$$= \begin{vmatrix} \frac{p_0}{p_1} & 0 & \dots & 0 & -\frac{p_0}{p_k} \\ 0 & \frac{p_0}{p_2} & \dots & 0 & -\frac{p_0}{p_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{p_0}{p_{k-1}} & -\frac{p_0}{p_k} \\ -1 & -1 & \dots & -1 & \frac{p_0}{p_k} - 1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{p_0}{p_1} & 0 & \dots & 0 & -\frac{p_0}{p_k} \\ 0 & \frac{p_0}{p_2} & \dots & 0 & -\frac{p_0}{p_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{p_0}{p_{k-1}} & -\frac{p_0}{p_k} \\ 0 & 0 & \dots & 0 & \frac{p_0}{p_k}(1 - (\frac{p_1}{p_0} + \dots + \frac{p_k}{p_0})) \end{vmatrix}$$

$$= \frac{p_0^k}{p_1 \cdots p_k}\left(1 - \left(\frac{p_1}{p_0} + \dots + \frac{p_k}{p_0}\right)\right) = \frac{p_0^{k-1}(p_0 - (p_1 + \dots + p_k))}{p_1 \cdots p_k}.$$

Therefore,

$$\det\left(\left(\partial_i\partial_j\left(A_{g,q}^{[n]} - A_{f,p}^{[n]}\right)^{\Delta}\right)_{i,j=1}^{k}\right) = \left(2\frac{r_0'}{r_0} + \frac{g''}{g'} - \frac{f''}{f'}\right)^k \cdot \frac{p_1 \cdots p_k(p_0 - (p_1 + \dots + p_k))}{p_0^{k+1}}.$$

Here, each factor is an everywhere positive function, hence Sylvester's criterion is validated, which then implies that the matrix (3.3) is positive definite. In view of the last assertion of Theorem 3.1.2, it follows that $M = A_{f,p}^{[n]}$ is locally smaller than $N = A_{q,q}$. $\square$

In what follows, we consider the local comparison problem of generalized Bajraktarević means when I is a nonempty open subinterval of $\mathbb{R}_+$ and the weight functions $p_1, \dots, p_n$ and $q_1, \dots, q_n$ are proportional to power functions.

Corollary 3.1.4. *Let $I \subseteq \mathbb{R}_+$ be a nonempty open interval and let $f, g\colon I \to \mathbb{R}$ be differentiable functions with nonvanishing first derivatives. Let $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$, $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and define*

$$p_i(x) := \lambda_i x^{\alpha_i} \qquad \textit{and} \qquad q_i(x) := \mu_i x^{\beta_i} \qquad (i \in \{1, \dots, n\},\ x \in I). \tag{3.8}$$

Assume that $A_{f,p}^{[n]}$ is locally smaller than $A_{g,q}^{[n]}$. Then there exist $\gamma > 0$ and $\delta \in \mathbb{R}$ such that

$$\mu_i = \gamma\lambda_i \qquad \textit{and} \qquad \beta_i = \alpha_i + \delta \qquad (i \in \{1, \dots, n\}). \tag{3.9}$$

If, in addition, f and g are twice differentiable, then the map

$$x \mapsto x^{2\delta} \frac{|g'(x)|}{|f'(x)|} \tag{3.10}$$

is increasing on I.

On the other hand, if f, g are twice continuously differentiable, (3.9) *holds and the function* (3.10) *has a positive derivative, then $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{g,q}$.*

Proof. Assume that $f, g\colon I \to \mathbb{R}$ are differentiable functions with nonvanishing first derivatives and that $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{g,q}$. Then, by Theorem 3.1.3, the first-order necessary condition (3.4) holds. This, for all $i, j \in \{1, \dots, n\}$, implies that

$$\frac{p_i}{p_j} = \frac{q_i}{q_j},$$

that is,

$$\frac{\lambda_i x^{\alpha_i}}{\lambda_j x^{\alpha_j}} = \frac{\mu_i x^{\beta_i}}{\mu_j x^{\beta_j}} \qquad (x \in I).$$

Therefore,

$$x^{\alpha_i - \alpha_j + \beta_j - \beta_i} = \frac{\mu_i \lambda_j}{\mu_j \lambda_i} \qquad (x \in I).$$

The left hand side is a power function which is constant on I, hence $\beta_i - \alpha_i = \beta_j - \alpha_j$ and $\mu_i/\lambda_i = \mu_j/\lambda_j$. Thus, there exist $\gamma > 0$ and $\delta \in \mathbb{R}$ such that (3.9) holds.

If, additionally, $f, g\colon I \to \mathbb{R}$ are twice differentiable functions, then by the second-order necessary condition of Theorem 3.1.3, the function (3.5) is increasing. On the other hand, using (3.4), (3.8), and then (3.9), we have

$$\frac{q_0^2 |g'|}{p_0^2 |f'|}(x) = \frac{q_i^2 |g'|}{p_i^2 |f'|}(x) = \frac{\mu_i^2 x^{2\beta_i} |g'(x)|}{\lambda_i^2 x^{2\alpha_i} |f'(x)|} = \gamma^2 x^{2\delta} \frac{|g'(x)|}{|f'(x)|}. \tag{3.11}$$

Therefore, the function (3.10) must be increasing.

To prove the reversed statement, suppose that f, g are twice continuously differentiable, (3.9) holds and the function (3.10) has a positive derivative on I.

Then, for $i \in \{1, \dots, n\}$ and $x \in I$,

$$\begin{aligned}
\frac{q_i}{q_0}(x) &= \frac{\mu_i x^{\beta_i}}{\mu_1 x^{\beta_1} + \cdots + \mu_n x^{\beta_n}} \\
&= \frac{\gamma \lambda_i x^{\delta + \alpha_i}}{\gamma \lambda_1 x^{\delta + \alpha_1} + \cdots + \gamma \lambda_n x^{\delta + \alpha_n}} = \frac{\lambda_i x^{\alpha_i}}{\lambda_1 x^{\alpha_1} + \cdots + \lambda_n x^{\alpha_n}} = \frac{p_i}{p_0}(x),
\end{aligned}$$

which shows that (3.4) is valid on I. Therefore (3.11) holds and hence we can see that the function (3.5) has a positive derivative. Therefore, in view of the second part of Theorem 3.1.3, it follows that $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{g,q}$. □

As an immediate consequence of the above corollary, we can characterize the local comparison of generalized power means.

Corollary 3.1.5. *Let $I \subseteq \mathbb{R}_+$ be a nonempty open interval and $a, b \in \mathbb{R}$. Define $f, g\colon I \to \mathbb{R}$ by*

$$f(x) := \begin{cases} x^a & \text{if } a \neq 0 \\ \log(x) & \text{if } a = 0 \end{cases} \qquad \text{and} \qquad g(x) := \begin{cases} x^b & \text{if } b \neq 0 \\ \log(x) & \text{if } b = 0. \end{cases} \tag{3.12}$$

Let $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$, $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and define $p, q\colon I \to \mathbb{R}_+^n$ by (3.8). If $A_{f,p}^{[n]}$ is locally smaller than $A_{f,p}^{[n]}$ then there exist $\gamma > 0$ and $\delta \in \mathbb{R}$ such that (3.9) holds and $a \leq b + 2\delta$.

On the other hand, if (3.9) holds and $a < b + 2\delta$, then $A_{f,p}^{[n]}$ is locally smaller than $A_{g,q}^{[n]}$.

Proof. Obviously, Corollary 3.1.4 can be applied and now the functions f and g are infinitely many times differentiable with nonvanishing first derivatives. To show the necessity of the conditions, observe that, for some positive real constant c,

$$x^{2\delta}\frac{|g'(x)|}{|f'(x)|} = cx^{b-a+2\delta} \qquad (x \in I)$$

is valid. Therefore, (3.10) is increasing if and only if $a \leq b + 2\delta$. For the sufficiency, observe that the function (3.10) has a positive derivative if and only if $a < b + 2\delta$. □

3.2 Global comparison

If $A_{f,p}^{[n]}$ is globally smaller than $A_{g,q}^{[n]}$, then $A_{f,p}^{[n]}$ is locally smaller than $A_{g,q}^{[n]}$ and hence, in view of Theorem 3.1.3, the necessity of (3.4) follows and the function (3.5) must be increasing. However, these conditions are not strong enough to imply the global comparability of the aforementioned means. The first theorem of this subsection deals with how the condition (3.4) needs to be augmented to yield global comparability.

Theorem 3.2.1. *Let $f, g\colon I \to \mathbb{R}$ be strictly monotone, differentiable functions with nonvanishing first derivatives, $n \in \mathbb{N}, n \geq 2$, and $p, q\colon I \to \mathbb{R}_+^n$. Assume that (3.4) holds and*

$$\frac{p_0(x)(f(x) - f(y))}{p_0(y)f'(y)} \leq \frac{q_0(x)(g(x) - g(y))}{q_0(y)g'(y)} \qquad (x, y \in I). \tag{3.13}$$

Then $A_{f,p}^{[n]}$ is globally smaller than $A_{g,q}^{[n]}$.

Proof. Multiplying both sides of (3.13) by $\frac{p_i(x)}{p_0(x)} = \frac{q_i(x)}{q_0(x)}$, which is valid according to (3.4), it follows that

$$\frac{p_i(x)(f(x) - f(y))}{p_0(y)f'(y)} \leq \frac{q_i(x)(g(x) - g(y))}{q_0(y)g'(y)} \qquad (x, y \in I,\ i \in \{1, \dots, n\}). \tag{3.14}$$

To show that (3.1) holds, let $x_1, \dots, x_n, y \in I$ be arbitrary. Applying the ith inequality in (3.14) at (x_i, y), then adding up the inequalities so obtained side by side, we get

$$\sum_{i=1}^{n} \frac{p_i(x_i)(f(x_i) - f(y))}{p_0(y)f'(y)} \leq \sum_{i=1}^{n} \frac{q_i(x_i)(g(x_i) - g(y))}{q_0(y)g'(y)}. \tag{3.15}$$

Let us now take the substitution $y := A_{f,p}^{[n]}(x_1, \dots, x_n)$ in the above inequality. Then, by the definition of the mean $A_{f,p}^{[n]}$, we have

$$f(y) = \frac{\sum_{i=1}^{n} p_i(x_i)f(x_i)}{\sum_{i=1}^{n} p_i(x_i)}.$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{p_i(x_i)(f(x_i) - f(y))}{p_0(y)f'(y)} \\
&\quad = \frac{1}{p_0(y)f'(y)} \left(\sum_{i=1}^{n} p_i(x_i)f(x_i) - f(y) \sum_{i=1}^{n} p_i(x_i) \right) \\
&\quad = \frac{1}{p_0(y)f'(y)} \left(\sum_{i=1}^{n} p_i(x_i)f(x_i) - \frac{\sum_{i=1}^{n} p_i(x_i)f(x_i)}{\sum_{i=1}^{n} p_i(x_i)} \sum_{i=1}^{n} p_i(x_i) \right) = 0.
\end{aligned}$$

Thus, the inequality (3.15) simplifies to

$$0 \leq \sum_{i=1}^{n} \frac{q_i(x_i)(g(x_i) - g(y))}{q_0(y)g'(y)}.$$

Assume first that g' is positive everywhere on I. Then g and also its inverse are strictly increasing and the previous inequality is equivalent to

$$0 \leq \sum_{i=1}^{n} q_i(x_i)(g(x_i) - g(y)),$$

which implies that

$$g(y) \sum_{i=1}^{n} q_i(x_i) \leq \sum_{i=1}^{n} q_i(x_i)g(x_i).$$

Dividing this inequality by $\sum_{i=1}^{n} q_i(x_i)$ and then applying g^{-1} to the inequality so obtained side by side, it follows that $y \leq A_{g,q}^{[n]}(x_1, \dots, x_n)$, that is, the inequality (3.1) is fulfilled.

If g' is everywhere negative then g and also its inverse are strictly decreasing and the proof is similar with obvious modifications. $\square$

Our second result discusses how the condition on the increasingness of the function (3.5) needs to be tightened to imply global comparability. It turns out that it is sufficient if all the multiplicative factors of the function (3.5) are increasing.

Theorem 3.2.2. *Let $f, g\colon I \to \mathbb{R}$ be strictly monotone differentiable functions with nonvanishing first derivatives, $n \in \mathbb{N}, n \geq 2$, and $p, q\colon I \to \mathbb{R}_+^n$. Assume that (3.4) holds and the functions*

$$\frac{q_0}{p_0} \qquad \textit{and} \qquad \frac{|g'|}{|f'|} \tag{3.16}$$

are increasing. Then $A_{f,p}^{[n]}$ is globally smaller than $A_{g,q}^{[n]}$.

Proof. To verify this statement, it suffices to show that the increasingness of the functions (3.16) implies the inequality (3.13). We assume that f' and g' are positive (and hence f and g are strictly increasing).

Let $x, y \in I$ be arbitrary. If $x = y$, then (3.13) is trivial. We consider now the case when $x < y$. Then, by the increasingness of the first function in (3.16), we have

$$\frac{q_0(x)}{p_0(x)} \leq \frac{q_0(y)}{p_0(y)}.$$

By the Cauchy Mean Value Theorem, there exists $z \in]x, y[$ such that

$$0 < \frac{g(x) - g(y)}{f(x) - f(y)} = \frac{g'(z)}{f'(z)} \leq \frac{g'(y)}{f'(y)},$$

where, for the inequality, we used that $z < y$ is valid and the function g'/f' is increasing. Multiplying the respective sides of the above inequalities by $q_0(x)/p_0(x)$, it follows that

$$\frac{q_0(x)(g(x) - g(y))}{p_0(x)(f(x) - f(y))} \leq \frac{q_0(y)g'(y)}{p_0(y)f'(y)}.$$

This inequality, using that $f(x) - f(y) < 0$ implies that (3.13) is satisfied if $x < y$. In the case when $y < x$, a completely analogous argument shows that the inequality (3.13) is also true.

The proof when f' or g' is negative is completely similar. □

Remark 3.2.3. In this remark we show that the sufficient condition (3.13) implies that the function (3.5) is increasing. Using the notations of the previous theorem, we may assume that f' and g' are positive (and hence f and g are strictly increasing). Let $x, y \in I$ be arbitrary with $x < y$. Then, (3.13) implies that

$$\frac{p_0(x)q_0(y)g'(y)}{p_0(y)q_0(x)f'(y)} \geq \frac{g(x) - g(y)}{f(x) - f(y)}. \tag{3.17}$$

On the other hand, interchanging the roles of x and y in (3.13), we get

$$\frac{p_0(y)(f(y)-f(x))}{p_0(x)f'(x)} \le \frac{q_0(y)(g(y)-g(x))}{q_0(x)g'(x)}.$$

This implies that

$$\frac{p_0(y)q_0(x)g'(x)}{p_0(x)q_0(y)f'(x)} \le \frac{g(y)-g(x)}{f(y)-f(x)}. \tag{3.18}$$

Combining the inequalities (3.17) and (3.18), it follows that

$$\frac{p_0(y)q_0(x)g'(x)}{p_0(x)q_0(y)f'(x)} \le \frac{g(x)-g(y)}{f(x)-f(y)} \le \frac{p_0(x)q_0(y)g'(y)}{p_0(y)q_0(x)f'(y)}$$

and hence

$$\frac{q_0(x)^2g'(x)}{p_0(x)^2f'(x)} \le \frac{q_0(y)^2g'(y)}{p_0(y)^2f'(y)},$$

which proves that the function (3.5) is increasing. The proof in the cases when at least one of the functions f' and g' is negative is completely analogous.

In what follows, we present two particular cases when (3.13) is equivalent to the increasingness of the function (3.5). In the first setting, the weight functions p and q coincide.

Theorem 3.2.4. *Let $f, g\colon I \to \mathbb{R}$ be strictly monotone twice continuously differentiable functions with nonvanishing first derivatives, $n \in \mathbb{N}, n \ge 2$, and let further $p\colon I \to \mathbb{R}_+^n$ be a continuous function. Then the following conditions are pairwise equivalent.*

(i) $A_{f,p}^{[n]}$ *is globally smaller than* $A_{g,p}^{[n]}$.

(ii) $A_{f,p}^{[n]}$ *is locally smaller than* $A_{g,p}^{[n]}$.

(iii) The function $|g'/f'|$ is increasing.

(iv) The following inequality is valid on I:

$$\frac{f''}{f'} \le \frac{g''}{g'}.$$

(v) Provided that g is increasing (decreasing), the function $g \circ f^{-1}$ is convex (concave) on $f(I)$;

(vi)

$$\frac{f(x)-f(y)}{f'(y)} \le \frac{g(x)-g(y)}{g'(y)} \qquad (x, y \in I).$$

Proof. The implication (i)⇒(ii) is trivial. Using Theorem 3.1.3, (iii) is an immediate consequence of (ii).

Now assume that (iii) is valid, i.e., $|g'/f'|$ is increasing. In the case $g'/f' > 0$ this implies that the function $(g'/f')' = (g'/f')(g''/g' - f''/f')$ is nonnegative, whence (iv) follows. In the case $g'/f' < 0$ an analogous argument yields that (iv) is also true.

Suppose that (iv) holds and g is increasing (the other case can be treated very similarly). Since

$$(g \circ f^{-1})'' = ((g \circ f^{-1})')' = \left(\frac{g'}{f'} \circ f^{-1}\right)' = \left(\frac{g'}{f'^2} \cdot \left(\frac{g''}{g'} - \frac{f''}{f'}\right)\right) \circ f^{-1} \geq 0,$$

the function $g \circ f^{-1}$ is convex.

Assume now that (v) holds and g is increasing (the other case is completely similar). Then, by the convexity of $h := g \circ f^{-1}$, for all $u, v \in f(I)$, we have $h(v) + h'(v) \cdot (u - v) \leq h(u)$, that is,

$$g \circ f^{-1}(v) + \left(\frac{g'}{f'} \circ f^{-1}\right)(v) \cdot (u - v) \leq g \circ f^{-1}(u).$$

Substituting $u := f(x)$ and $v := f(y)$, where $x, y \in I$, it follows that

$$g(y) + \frac{g'(y)}{f'(y)}(f(x) - f(y)) \leq g(x),$$

which shows that (vi) is satisfied.

Finally, suppose that (vi) is valid. Then (3.13) is also true with $q_0 := p_0$. Therefore, according to Theorem 3.2.1, we obtain that $A^{[n]}_{f,p}$ is globally smaller than $A^{[n]}_{g,p}$, i.e., (i) is fulfilled. □

In the second setting, the functions f and g are the same.

Theorem 3.2.5. *Let $f\colon I \to \mathbb{R}$ be strictly monotone twice continuously differentiable function with a nonvanishing first derivative, $n \in \mathbb{N}, n \geq 2$, and let further $p, q\colon I \to \mathbb{R}^n_+$ be continuous functions. Then the following conditions are pairwise equivalent.*

(i) $A^{[n]}_{f,p}$ is globally smaller than $A^{[n]}_{f,q}$.

(ii) $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{f,q}$.

(iii) The condition (3.4) holds and the function q_0/p_0 is increasing.

Proof. The implication (i)⇒(ii) is obvious.

Assume first that (ii) is valid. Then, according to Theorem 3.1.3, the equalities in (3.4) must be satisfied and the function $(q_0^2|g'|)/(p_0^2|f'|) = q_0^2/p_0^2$ is increasing. Therefore, condition (iii) holds true.

Finally, suppose that (iii) is satisfied. Then, with $g := f$, we can see that the functions in (3.16) are increasing. Hence, according to Theorem 3.2.2, $A^{[n]}_{f,p}$ is globally smaller than $A^{[n]}_{g,q}$ and assertion (i) is valid. □

Corollary 3.2.6. *Let $I \subseteq \mathbb{R}_+$ be a nonempty open interval and let $f, g\colon I \to \mathbb{R}$ be differentiable functions with nonvanishing first derivatives. Let $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}^n_+$, $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and define $p, q\colon I \to \mathbb{R}^n_+$ by (3.8). Assume that there exist $\gamma > 0$ and $\delta \in \mathbb{R}$ such that (3.9) is satisfied and*

$$\frac{f(x) - f(y)}{f'(y)} \leq \frac{x^\delta (g(x) - g(y))}{y^\delta g'(y)} \qquad (x, y \in I). \tag{3.19}$$

Then $A^{[n]}_{f,p}$ is globally smaller than $A^{[n]}_{g,q}$.

Proof. Due to the definition of p and q by (3.8) and conditions (3.9), we have that (3.4) is valid. We are going to prove that the inequality (3.13) is also satisfied. First, observe that (3.8) and the conditions in (3.9) imply that $q_0(x) = \gamma p_0(x) x^\delta$ holds for all $x \in I$. Therefore, multiplying the inequality (3.19) by $p_0(x)/p_0(y)$ side by side, we get

$$\frac{p_0(x)(f(x) - f(y))}{p_0(y) f'(y)} \leq \frac{\gamma p_0(x) x^\delta (g(x) - g(y))}{\gamma p_0(y) y^\delta g'(y)} = \frac{q_0(x)(g(x) - g(y))}{q_0(y) g'(y)},$$

which shows that the inequality (3.13) is satisfied. Hence, using Theorem 3.2.1, it follows that $A^{[n]}_{f,p}$ is globally smaller than $A^{[n]}_{g,q}$. $\square$

The following function appears in several subsequent computations and also in the proof of the corollary below.

Definition 3.2.7. For $r, s \in \mathbb{R}$, let us introduce the map $\chi_{r,s}\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\chi_{r,s}(t) := \begin{cases} \dfrac{t^r - t^s}{r - s} & \text{if } r \neq s \\ t^r \ln(t) & \text{if } r = s. \end{cases}$$

Corollary 3.2.8. *Let $I \subseteq \mathbb{R}_+$ be a nonempty open interval, $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}^n_+$, $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$, and let further $a, b \in \mathbb{R}$. Define $p, q\colon I \to \mathbb{R}^n_+$ by (3.8) and $f, g\colon I \to \mathbb{R}$ by (3.12). Assume that there exist $\gamma > 0$ and $\delta \in \mathbb{R}$ such that (3.9) and the inequalities*

$$\min(a, 0) \leq \delta + \min(b, 0) \qquad \textit{and} \qquad \max(a, 0) \leq \delta + \max(b, 0) \tag{3.20}$$

are satisfied. Then $A^{[n]}_{f,p}$ is globally smaller than $A^{[n]}_{g,q}$.

Proof. Due to the definition of p and q by (3.8) and conditions (3.9), we have that (3.4) is valid. We are going to prove that the inequality (3.19) is also true.

The mapping $r \mapsto t^r$ is convex for all $t > 0$. Therefore, if

$$\min(r, s) \leq \min(u, v) \qquad \text{and} \qquad \max(r, s) \leq \max(u, v),$$

then

$$\chi_{r,s}(t) \le \chi_{u,v}(t) \qquad (t \in \mathbb{R}_+).$$

Thus, the inequalities in (3.20) imply that

$$\chi_{a,0}(t) \le \chi_{b+\delta,\delta}(t) \qquad (t \in \mathbb{R}_+). \tag{3.21}$$

If $ab \neq 0$, this inequality is equivalent to

$$\frac{t^a - 1}{a} \le t^\delta \frac{t^b - 1}{b} \qquad (t \in \mathbb{R}_+).$$

For $x, y \in I$, substitute $t := x/y$ into this inequality. After simplifications, we obtain

$$\frac{x^a - y^a}{a y^{a-1}} \le \left(\frac{x}{y}\right)^\delta \frac{x^b - y^b}{b y^{b-1}}.$$

This inequality, according to the definition (3.12) of f and g, can be rewritten as (3.19). In the case when $ab = 0$, a similar argument shows that (3.21) also implies (3.19).

Now we are able to apply Corollary 3.2.6 and hence we can conclude the result. □

Chapter 4

Local and global Hölder- and Minkowski-type inequalities

The celebrated inequalities discovered by Hölder and Minkowski can be formulated in various contexts, for instance, in the setting of power means.

To recall the standard Hölder(–Rogers) inequality, which was discovered by Rogers in 1888 and by Hölder in 1889, let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}_+^n$, the inequality

$$\frac{x_1y_1 + \cdots + x_ny_n}{n} \le \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \left(\frac{y_1^q + \cdots + y_n^q}{n}\right)^{\frac{1}{q}}$$

is valid. In the particular case $p = q = 2$, this inequality reduces to the so-called Cauchy–Bunyakovsky–Schwarz inequality, which in the above form was established by Cauchy in 1821. Given a real parameter $p \ge 1$, the standard Minkowski inequality, that was discovered in 1910, states that the pth power mean is subadditive, i.e., for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}_+^n$, the inequality

$$\left(\frac{(x_1+y_1)^p + \cdots + (x_n+y_n)^p}{n}\right)^{\frac{1}{p}} \le \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} + \left(\frac{y_1^p + \cdots + y_n^p}{n}\right)^{\frac{1}{p}}$$

holds.

Briefly, the aim of this chapter is to investigate inequalities that are analogous to the Hölder and Minkowski inequalities by replacing the addition and the multiplication by a more general operation, and instead of using power means, generalized Bajraktarević means, and in particular, weighted Gini means, are considered. A further aim is to characterize such inequalities both in the local and in the global sense. We are going to derive necessary as well as sufficient conditions for the local as well as for the global validity of the functional inequality

$$M_0(\Phi(x_1), \ldots, \Phi(x_n)) \le \Phi(M_1(x^1), \ldots, M_k(x^k)), \tag{4.1}$$

where, $n \in \mathbb{N}, n \ge 2, k \in \mathbb{N}$, for $\alpha \in \{0, \ldots, k\}$, $I_\alpha \subseteq \mathbb{R}$ is a nonempty open interval, $I := I_1 \times \cdots \times I_k$, $M_\alpha \colon I_\alpha^n \to I_\alpha$ is an n-variable mean and $\Phi \colon I \to I_0$. In line with the terminology used in preceding chapters, if there exists an open set $U \subseteq I^n$ such that $\operatorname{diag}(I^n) \subseteq U$ and (4.1) holds for all $x \in U^T \subseteq \prod_{\alpha=1}^k I_\alpha^n$, then we say that (4.1) *holds in the local sense*. If (4.1) is valid for all $x \in (I^n)^T = \prod_{\alpha=1}^k I_\alpha^n$, then we say that (4.1) *holds in the global sense*. Clearly, the global validity of (4.1) implies its local validity.

After that we consider the particular case of (4.1) when all the means are n-variable generalized Bajraktarević means, i.e., we consider the inequality

$$A_{f_0,p^0}^{[n]}(\Phi(x_1),\dots,\Phi(x_n)) \leq \Phi(A_{f_1,p^1}^{[n]}(x^1),\dots,A_{f_k,p^k}^{[n]}(x^k)), \tag{4.2}$$

where, for $\alpha \in \{0,\dots,k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ is a strictly monotone function, $p^\alpha\colon I_\alpha \to \mathbb{R}_+^n$. We obtain necessary as well as sufficient conditions for its validity in the local and also in the global sense.

We mention some important particular cases of (4.2).

1. If $k = 1$, $I_0 = I_1 =: J$ and $\Phi(x) = x$, then (4.2) reduces to the local and global comparison problem of generalized Bajraktarević means on J.

2. If $k \in \mathbb{N}, k \geq 2$, $I_0 = I_1 = \dots = I_k =: J$, $\Phi(x_1,\dots,x_k) = \frac{1}{k}(x_1+\dots+x_k)$, and $f_0 = f_1 = \dots = f_k =: f$, $p^0 = p^1 = \dots = p^k =: p$, then (4.2) means the Jensen convexity of $A_{f,p}^{[n]}$ on J. In this case, (4.2) is said to be a Jensen-type inequality.

3. If $k \in \mathbb{N}, k \geq 2$, $I_0 = I_1 = \dots = I_k = \mathbb{R}_+$, $\Phi(x_1,\dots,x_k) = x_1+\dots+x_k$, and $f_0 = f_1 = \dots = f_k =: f$, $p^0 = p^1 = \dots = p^k =: p$, then (4.2) expresses the subadditivity of $A_{f,p}^{[n]}$ on $\mathbb{R}_+$, which is often called a Minkowski-type inequality.

4. If $k \in \mathbb{N}, k \geq 2$, $I_0 = I_1 = \dots = I_k = \mathbb{R}_+$, $\Phi(x_1,\dots,x_k) = x_1\cdots x_k$, then (4.2) reduces to a Hölder-type inequality for the means $A_{f_0,p^0}^{[n]}, A_{f_1,p^1}^{[n]},\dots,A_{f_k,p^k}^{[n]}$.

There are many results related to the Hölder- and Minkowski-type inequalities. Without completeness, we mention the following standard sources and the references therein: Hardy–Littlewood–Pólya [15], Beckenbach–Bellmann [4], Bullen–Mitrinović–Vasić [6], and Mitrinović–Pečarić–Fink [30]. We also quote the papers [17–21, 26, 28] by Losonczi and the papers [7, 31–34]. With the exception of Theorem 4.3.11, Theorem 4.3.13, and Theorem 4.3.17, all the results of this chapter were first proved in our paper [14].

4.1 Hölder- and Minkowski-type inequalities in the local sense

In the current and in the next subsection, for $k \in \mathbb{N}$ and $\alpha \in \{0,\dots,k\}$, let $I_\alpha \subseteq \mathbb{R}$ stand for a nonempty open interval and set $I := I_1 \times \dots \times I_k$. For the investigation of inequality (4.1), let us introduce the function $F\colon I_1^n \times \dots \times I_k^n \to \mathbb{R}$ by

$$F(x) = F(x^1,\dots,x^k) := \Phi(M_1(x^1),\dots,M_k(x^k)) - M_0(\Phi(x_1),\dots,\Phi(x_n)). \tag{4.3}$$

Remark 4.1.1. Observe that, for all $y \in I$, we have $F(\Delta_n^k(y)) = 0$. Indeed, by using the mean value property of $M_0, M_1, \dots, M_k$, it follows that

$$\begin{aligned} F(\Delta_n^k(y)) &= F(\Delta_n(y_1), \dots, \Delta_n(y_k)) \\ &= \Phi(M_1(\Delta_n(y_1)), \dots, M_k(\Delta_n(y_k))) - M_0(\Delta_n(\Phi(y))) \\ &= \Phi(M_1^\Delta(y_1), \dots, M_k^\Delta(y_k)) - M_0^\Delta(\Phi(y)) = \Phi(y_1, \dots, y_k) - \Phi(y) = 0. \end{aligned}$$

For the computation of the partial derivatives of F at points of the form $\Delta_n^k(y)$, we formulate the following lemma.

Lemma 4.1.2. *Let $n, k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0, \dots, k\}$, $M_\alpha \colon I_\alpha^n \to I_\alpha$ be an n-variable mean, define $F \colon I_1^n \times \cdots \times I_k^n \to \mathbb{R}$ by (4.3), and let $\Phi \colon I \to I_0$.*

(i) For $\alpha \in \{0, \dots, k\}$, assume that M_α is partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and that Φ is differentiable. Then, for all $i \in \{1, \dots, k\}$, $\ell \in \{1, \dots, n\}$, and $y \in I$,

$$\partial_{\ell+n(i-1)} F(\Delta_n^k(y)) = \partial_i \Phi(y) \big(\partial_\ell M_i^\Delta(y_i) - \partial_\ell M_0^\Delta(\Phi(y)) \big).$$

(ii) For $\alpha \in \{0, \dots, k\}$, assume that M_α is twice partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and that Φ is twice differentiable. Then, for all $i, j \in \{1, \dots, k\}$, $\ell, m \in \{1, \dots, n\}$, and $y \in I$,

$$\begin{aligned} &\partial_{\ell+n(i-1)} \partial_{m+n(j-1)} F(\Delta_n^k(y)) \\ &\qquad = \partial_i \partial_j \Phi(y) \big(\partial_m M_j^\Delta(y_j) \partial_\ell M_i^\Delta(y_i) - \delta_{\ell,m} \partial_m M_0^\Delta(\Phi(y)) \big) \\ &\qquad\quad - \partial_j \Phi(y) \big(\partial_i \Phi(y) \partial_\ell \partial_m M_0^\Delta(\Phi(y)) - \delta_{i,j} \partial_\ell \partial_m M_j^\Delta(y_j) \big). \end{aligned}$$

Proof. (i) Let $i \in \{1, \dots, k\}$, $\ell \in \{1, \dots, n\}$, and $y \in I$ be arbitrary. Then the existence of the partial derivative $\partial_{\ell+n(i-1)} F(\Delta_n^k(y))$ and also the formula for it is a direct consequence of the standard chain rule. More precisely,

$$\begin{aligned} \partial_{\ell+n(i-1)} F(\Delta_n^k(y)) &= \partial_{\ell+n(i-1)} F(\Delta_n(y_1), \dots, \Delta_n(y_k)) \\ &= \partial_i \Phi \big(M_1(\Delta_n(y_1)), \dots, M_k(\Delta_n(y_k)) \big) \partial_\ell M_i(\Delta_n(y_i)) \\ &\quad - \partial_\ell M_0(\Phi(y), \dots, \Phi(y)) \partial_i \Phi(y), \end{aligned}$$

which simplifies to the formula stated in (i).

(ii) For $\alpha \in \{0, \dots, k\}$, there exists an open set $U_\alpha \subseteq I_\alpha^n$ such that it contains $\operatorname{diag}(I_\alpha^n)$, the first-order partial derivatives of M_α exist over U_α and their first-order partial derivatives, that is, the second-order partial derivatives of M_α, exist on $\operatorname{diag}(I_\alpha^n)$. Using the continuity of Φ, by shrinking the open sets $U_1, \dots, U_k$, we can also assume

$$(\Phi(x_1), \dots, \Phi(x_n)) \in U_0 \tag{4.4}$$

provided that $x^1 \in U_1, \dots, x^k \in U_k$.

Let $i, j \in \{1, \dots, k\}$, $\ell, m \in \{1, \dots, n\}$, and $y \in I$ be arbitrary. Computing the partial derivative of F over $U_1 \times \cdots \times U_k$ with respect to its $(m + n(j-1))$th variable, that is, with respect to the variable x_m^j, which is the jth entry of x_m and the mth entry of x^j, we get

$$\begin{aligned}\partial_{m+n(j-1)}F(x) = \partial_j\Phi(M_1(x^1), \dots, M_k(x^k))\partial_m M_j(x^j)\\ - \partial_m M_0(\Phi(x_1), \dots, \Phi(x_n))\partial_j\Phi(x_m)\end{aligned}$$

for all matrices $x \in \mathbb{R}^{n\times k}$ with $x^1 \in U_1, \dots, x^k \in U_k$. Using this equality, we can compute the partial derivative of $\partial_{m+n(j-1)}F$ at $\Delta_n^k(y)$ with respect to its $(\ell + n(i-1))$th variable, that is, with respect to the variable x_ℓ^i, which is the ith entry of x_ℓ and the ℓth entry of x^i, as follows

$$\begin{aligned}&\partial_{\ell+n(i-1)}\partial_{m+n(j-1)}F(\Delta_n^k(y))\\ &= \partial_i\partial_j\Phi\big(M_1(\Delta_n(y_1)), \dots, M_k(\Delta_n(y_k))\big)\partial_\ell M_i(\Delta_n(y_i))\partial_m M_j(\Delta_n(y_j))\\ &\quad + \partial_j\Phi\big(M_1(\Delta_n(y_1)), \dots, M_k(\Delta_n(y_k))\big)\delta_{i,j}\partial_\ell\partial_m M_j(\Delta_n(y_j))\\ &\quad - \partial_\ell\partial_m M_0(\Phi(y), \dots, \Phi(y))\partial_i\Phi(y)\partial_j\Phi(y) - \partial_m M_0(\Phi(y), \dots, \Phi(y))\delta_{\ell,m}\partial_i\partial_j\Phi(y).\end{aligned}$$

Using the mean value property of $M_0, M_1, \dots, M_k$, this equality simplifies to the formula asserted in statement (ii). □

The next two results describe the first- and second-order necessary conditions for the validity of (4.1) in the local sense, respectively.

Theorem 4.1.3. *Let $n, k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0, \dots, k\}$, $M_\alpha \colon I_\alpha^n \to I_\alpha$ be an n-variable mean which is partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and let further $\Phi \colon I \to I_0$ be a surjective and differentiable fucntion with nonvanishing first-order partial derivatives. Assume that inequality (4.1) holds in the local sense. Then there exist constants $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ such that, for all $(y_0, y) \in I_0 \times I$ and $\ell \in \{1, \dots, n\}$,*

$$\lambda_\ell = \partial_\ell M_0^\Delta(y_0) = \partial_\ell M_1^\Delta(y_1) = \cdots = \partial_\ell M_k^\Delta(y_k). \tag{4.5}$$

If, additionally, for some $\alpha \in \{0, \dots, k\}$ and $y_\alpha \in I_\alpha$, the mean M_α is differentiable at $\Delta_n(y_\alpha)$, then $\lambda_1 + \cdots + \lambda_n = 1$ also holds.

Proof. For $\alpha \in \{1, \dots, k\}$, let $U_\alpha \subseteq I_\alpha^n$ be a nonempty open set containing $\operatorname{diag}(I_\alpha^n)$ such that (4.1) holds for all matrices $x \in \mathbb{R}^{n\times k}$ with $x^1 \in U_1, \dots, x^k \in U_k$. Then, according to (4.1), F is nonnegative on $U_1 \times \cdots \times U_k$ and, for all $y \in I$, we have $F(\Delta_n^k(y)) = 0$. Therefore, the first-order partial derivatives of F vanish at the point $\Delta_n^k(y)$. In view of Lemma 4.1.2, for all $\ell \in \{1, \dots, n\}$ and $i \in \{1, \dots, k\}$, the equality $\partial_{\ell+n(i-1)}F(\Delta_n^k(y)) = 0$ implies that

$$0 = \partial_i\Phi(y)\big(\partial_\ell M_i^\Delta(y_i) - \partial_\ell M_0^\Delta(\Phi(y))\big).$$

Using that the partial derivatives of Φ do not vanish on I, for all $\ell \in \{1,\dots,n\}$ and $i \in \{1,\dots,k\}$, it follows that

$$\partial_\ell M_i^\Delta(y_i) = \partial_\ell M_0^\Delta(\Phi(y)). \tag{4.6}$$

Now we prove first, for all $\ell \in \{1,\dots,n\}$, that the function $\partial_\ell M_0^\Delta$ is locally constant on I_0. Without loss of generality, we may assume that Φ is strictly increasing in its first variable. To verify the assertion, let $y_0 \in I_0$ be arbitrary. Then, by the surjectivity of Φ, there exists $y \in I$ such that $y_0 = \Phi(y)$. Let $y_1' < y_1 < y_1''$ be arbitrarily fixed elements of I_1. Then, by the assumed monotonicity of Φ, we have

$$y_0' := \Phi(y_1', y_2, \dots, y_k) < y_0 = \Phi(y_1, y_2, \dots, y_k) < y_0'' := \Phi(y_1'', y_2, \dots, y_k).$$

Let $u \in]y_0', y_0''[$ be arbitrary. Then, by the continuity of the function Φ, there exists $v \in]y_1', y_1''[$ such that $u = \Phi(v, y_2, \dots, y_k)$. Applying equality (4.6) for $i = 2$ and $\ell \in \{1,\dots,n\}$ twice, we get

$$\begin{aligned}\partial_\ell M_0^\Delta(u) &= \partial_\ell M_0^\Delta(\Phi(v, y_2, \dots, y_k)) = \partial_\ell M_2^\Delta(y_2)\\ &= \partial_\ell M_0^\Delta(\Phi(y_1, y_2, \dots, y_k)) = \partial_\ell M_0^\Delta(y_0).\end{aligned}$$

Therefore $\partial_\ell M_0^\Delta$ is constant on $]y_0', y_0''[$, which is a neighborhood of y_0. It proves that $\partial_\ell M_0^\Delta$ is differentiable at y_0 and $(\partial_\ell M_0^\Delta)'(y_0) = 0$. The choice of y_0 in I_0 was arbitrary, hence $(\partial_\ell M_0^\Delta)'$ is identically zero on I_0, which is an open subinterval of $\mathbb{R}$. This implies that $\partial_\ell M_0^\Delta$ is constant on I_0. We will denote this constant by λ_ℓ. Equality (4.6) then implies that the partial derivatives $\partial_\ell M_1^\Delta, \dots, \partial_\ell M_k^\Delta$ are also equal to the constant λ_ℓ on their domains.

Finally, assume that, for some $\alpha \in \{0,\dots,k\}$ and $y_\alpha \in I_\alpha$, the mean M_α is differentiable at $(\Delta_n(y_\alpha))$. Then, by the mean value property of M_α, we have that $M_\alpha^\Delta(y) = y$ for all $y \in I_\alpha$. Differentiating this equality with respect to y at $y = y_\alpha$, we get

$$\partial_1 M_i^\Delta(y_\alpha) + \dots + \partial_n M_i^\Delta(y_\alpha) = 1,$$

which implies $\lambda_1 + \dots + \lambda_n = 1$. □

Theorem 4.1.4. *Let $n, k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0,\dots,k\}$, $M_\alpha \colon I_\alpha^n \to I$ be an n-variable mean which is twice differentiable on $\operatorname{diag}(I_\alpha^n)$ and let $\Phi \colon I \to I_0$ be surjective and twice differentiable with nonvanishing first-order partial derivatives. Assume that inequality (4.1) holds in the local sense. Then there exist constants $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ with $\lambda_1 + \dots + \lambda_n = 1$ such that, for all $(y_0, y) \in I_0 \times I$ and $\ell \in \{1,\dots,n\}$, the equalities in (4.5) hold. In addition, for all $y \in I$, the $((nk) \times (nk))$-type matrix whose $(\ell + n(i-1), m + n(j-1))$th entry, where $i, j \in \{1,\dots,k\}$ and $\ell, m \in \{1,\dots,n\}$, is given by*

$$\begin{aligned}\partial_i\partial_j\Phi(y)(\lambda_m\lambda_\ell - \delta_{\ell,m}\lambda_m) - \partial_i\Phi(y)\partial_j\Phi(y)\partial_\ell\partial_m M_0^\Delta(\Phi(y))\\ + \delta_{i,j}\partial_j\Phi(y)\partial_\ell\partial_m M_j^\Delta(y_j)\end{aligned} \tag{4.7}$$

is positive semidefinite.

Proof. For $\alpha \in \{1,\dots,k\}$, let $U_\alpha \subseteq I_\alpha^n$ be a nonempty open set containing $\operatorname{diag}(I_\alpha^n)$ such that (4.1) and (4.4) hold for all matrices $x \in \mathbb{R}^{n\times k}$ with $x^1 \in U_1,\dots,x^k \in U_k$. Then, using Theorem 4.1.3, condition (4.5) is valid with some nonnegative constants $\lambda_1,\dots,\lambda_n$ satisfying also $\lambda_1+\dots+\lambda_n = 1$.

According to (4.1), F is nonnegative on $U_1\times\dots\times U_k$ and, for all $y \in I$, we have $F(\Delta_n^k(y)) = 0$, that is, F has a (local) minimum at $\Delta_n^k(y)$. Therefore, its second derivative at that point, i.e., the $((nk)\times(nk))$-type matrix $\big(\partial_\alpha\partial_\beta F(\Delta_n^k(y))\big)$ is positive semidefinite. In view of Lemma 4.1.2, for every $y \in I$, the $((nk)\times(nk))$-type matrix whose $(\ell+n(i-1), m+n(j-1))$th entry, where $i,j \in \{1,\dots,k\}$ and $\ell, m \in \{1,\dots,n\}$, is given by

$$\begin{aligned}&\partial_i\partial_j\Phi(y)\big(\partial_m M_j^\Delta(y_j)\partial_\ell M_i^\Delta(y_i) - \delta_{\ell,m}\partial_m M_0^\Delta(\Phi(y))\big)\\ &-\partial_j\Phi(y)\big(\partial_i\Phi(y)\partial_\ell\partial_m M_0^\Delta(\Phi(y)) - \delta_{i,j}\partial_\ell\partial_m M_j^\Delta(y_j)\big)\end{aligned}$$

is positive semidefinite. Applying the equalities from (4.5), the statement follows. □

4.2 Hölder- and Minkowski-type inequalities for generalized Bajraktarević means

Under first- as well as second-order differentiability condition, the following theorem describes what necessarily holds if (4.2) is satisfied in the local sense.

Theorem 4.2.1. *Let $n,k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0,\dots,k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be a differentiable function with a nonvanishing first derivative and let $p^\alpha = (p_1^\alpha,\dots,p_n^\alpha)\colon I_\alpha \to \mathbb{R}_+^n$ be continuous, and set $p_0^\alpha := p_1^\alpha+\dots+p_n^\alpha$. Let $\Phi\colon I \to I_0$ be surjective and differentiable with nonvanishing first-order partial derivatives. Assume that inequality (4.2) holds in the local sense. Then there exist constants $\lambda_1,\dots,\lambda_n \in \mathbb{R}_+$ with $\lambda_1+\dots+\lambda_n = 1$ such that, for all $\alpha \in \{0,\dots,k\}$ and $\ell \in \{1,\dots,n\}$,*

$$p_\ell^\alpha = \lambda_\ell p_0^\alpha \tag{4.8}$$

holds on I_α. If, additionally, for $\alpha \in \{0,\dots,k\}$, f_α is twice differentiable, p^α is continuously differentiable and Φ is twice differentiable, then the $(k\times k)$-type matrix $\Gamma(y)$ given by

$$\begin{aligned}\Gamma(y) := \bigg(&-\partial_i\partial_j\Phi(y) - \partial_j\Phi(y)\partial_i\Phi(y)\bigg(2\frac{(p_0^0)'}{p_0^0} + \frac{f_0''}{f_0'}\bigg)(\Phi(y))\\ &+\delta_{i,j}\partial_j\Phi(y)\bigg(2\frac{(p_0^j)'}{p_0^j} + \frac{f_j''}{f_j'}\bigg)(y_j)\bigg)_{i,j=1}^k\end{aligned} \tag{4.9}$$

is positive semidefinite for all $y \in I$.

Proof. For $\alpha \in \{0,\dots,k\}$, let $M_\alpha = A_{f_\alpha,p^\alpha}^{[n]}$ and apply Theorem 4.1.4 to this setting. Then M_α is partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and inequality (4.1) holds in the local sense. According to the first assertion of Theorem 4.1.4 and by the first statement of

Theorem 1.2.1, there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ such that, for all $(y_0, y) \in I_0 \times I$ and $\ell \in \{1, \dots, n\}$, the equalities in (4.5) hold, i.e., for $\alpha \in \{0, \dots, k\}$,

$$\frac{p_\ell^\alpha}{p_0^\alpha}(y_\alpha) = \left(\partial_\ell A_{f_\alpha, p^\alpha}^{[n]}\right)^\Delta (y_\alpha) = \lambda_\ell.$$

This shows that (4.8) is valid on I_α for all $\alpha \in \{0, \dots, k\}$ and $\ell \in \{1, \dots, n\}$. In view of the definition of p_0^α, these equalities imply that $\lambda_1 + \cdots + \lambda_n = 1$ is also true.

Assume now that, additionally, for $\alpha \in \{0, \dots, k\}$, f_α is twice differentiable, p^α is continuously differentiable and Φ is twice differentiable. Using (4.8), according to assertion (2b) of Theorem 1.2.1, we have

$$\left(\partial_\ell^2 A_{f_\alpha, p^\alpha}^{[n]}\right)^\Delta = 2\frac{(p_\ell^\alpha)'(p_0^\alpha - p_\ell^\alpha)}{(p_0^\alpha)^2} + \frac{p_\ell^\alpha(p_0^\alpha - p_\ell^\alpha)}{(p_0^\alpha)^2} \cdot \frac{f_\alpha''}{f_\alpha'} = \lambda_\ell(1 - \lambda_\ell)\left(2\frac{(p_0^\alpha)'}{p_0^\alpha} + \frac{f_\alpha''}{f_\alpha'}\right),$$

and, for $\ell \neq m$,

$$\left(\partial_\ell \partial_m A_{f_\alpha, p^\alpha}^{[n]}\right)^\Delta = -\frac{(p_\ell^\alpha p_m^\alpha)'}{(p_0^\alpha)^2} - \frac{p_\ell^\alpha p_m^\alpha}{(p_0^\alpha)^2} \cdot \frac{f_\alpha''}{f_\alpha'} = -\lambda_\ell \lambda_m \left(2\frac{(p_0^\alpha)'}{p_0^\alpha} + \frac{f_\alpha''}{f_\alpha'}\right).$$

Therefore, for all $\alpha \in \{0, \dots, k\}$ and $\ell, m \in \{1, \dots, n\}$,

$$\left(\partial_\ell \partial_m A_{f_\alpha, p^\alpha}^{[n]}\right)^\Delta = \lambda_m(\delta_{\ell,m} - \lambda_\ell)\left(2\frac{(p_0^\alpha)'}{p_0^\alpha} + \frac{f_\alpha''}{f_\alpha'}\right). \tag{4.10}$$

In view of the second assertion of Theorem 4.1.4, for all $y \in I$, it follows that the $((nk) \times (nk))$-type matrix whose $(\ell + n(i-1), m + n(j-1))$th entry, where $i, j \in \{1, \dots, k\}$ and $\ell, m \in \{1, \dots, n\}$, is given by (4.7) is positive semidefinite. Using formula (4.10), we can conclude that the matrix whose $(\ell + n(i-1), m + n(j-1))$th entry is given by

$$\lambda_m(\delta_{\ell,m} - \lambda_\ell)\Gamma_{i,j}(y)$$

is positive semidefinite.

If a matrix is positive semidefinite, then every minor of the matrix is also positive semidefinite. Therefore, the $(k \times k)$-type submatrix with entries $(1 + n(i-1), 1 + n(j-1))$th, where $i, j \in \{1, \dots, k\}$, is also positive semidefinite, which implies the statement. □

In the next result, we reformulate the positive semidefiniteness condition from the above theorem in terms of a convexity property.

Theorem 4.2.2. *Let $n, k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0, \dots, k\}$, $f_\alpha \colon I_\alpha \to \mathbb{R}$ be a twice differentiable function with a nonvanishing first derivative and let $p^\alpha = (p_1^\alpha, \dots, p_n^\alpha) \colon I_\alpha \to \mathbb{R}_+^n$ be continuously differentiable, set $p_0^\alpha := p_1^\alpha + \cdots + p_n^\alpha$. Let $\Phi \colon I \to I_0$ be surjective and twice differentiable with nonvanishing first-order partial derivatives.*

Assume that inequality (4.2) holds in the local sense. Finally, for $\alpha \in \{0, \dots, k\}$, define the function $\varphi_\alpha \colon I_\alpha \to \mathbb{R}$ and then $\varphi \colon I \to \mathbb{R}^k$ by

$$\varphi_\alpha := \int (p_0^\alpha)^2 f'_\alpha \qquad \textit{and} \qquad \varphi(y) := (\varphi_1(y_1), \dots, \varphi_k(y_k)).$$

Then $\varphi_1, \dots, \varphi_k$ and φ are twice differentiable and invertible functions and the map $\Psi \colon \varphi(I) \to \mathbb{R}$ defined by

$$\Psi(u) := \varphi_0(\Phi(\varphi^{-1}(u)))$$

is concave if $f'_0 > 0$ and convex if $f'_0 < 0$.

Proof. In view of Theorem 4.2.1, our assumptions imply that the matrix-valued map $\Gamma \colon I \to \mathbb{R}^{k \times k}$ defined by (4.9) has positive semidefinite values.

Without loss of generality, we can assume $f'_0 > 0$. Let $\alpha \in \{0, \dots, k\}$. Then the integrand in the definition of φ_α is either positive everywhere or negative everywhere, therefore φ_α is a twice differentiable function with a nonvanishing first derivative, hence it is strictly monotone and it has a twice differentiable inverse $\varphi_\alpha^{-1} \colon \varphi_\alpha(I_\alpha) \to I_\alpha$. Furthermore, we have that, for all $\alpha \in \{0, \dots, k\}$,

$$\frac{\varphi''_\alpha}{\varphi'_\alpha} = \frac{((p_0^j)^2 f'_\alpha)'}{(p_0^j)^2 f'_\alpha} = \frac{2p_0^j (p_0^j)' f'_\alpha + (p_0^j)^2 f''_\alpha}{(p_0^j)^2 f'_\alpha} = 2\frac{(p_0^j)'}{p_0^j} + \frac{f''_\alpha}{f'_\alpha}. \tag{4.11}$$

From the definition of φ, it follows that

$$\varphi^{-1}(u) = (\varphi_1^{-1}(u_1), \dots, \varphi_k^{-1}(u_k)) \qquad (u \in \varphi_1(I_1) \times \dots \times \varphi_k(I_k)).$$

Thus, it is clear that φ and its inverse are also twice differentiable maps.

In order to show that Ψ is concave, we will prove that Ψ'' is negative semidefinite on $\varphi(I)$. First, we compute the first and then the second-order partial derivatives of Ψ. For $i, j \in \{0, \dots, k\}$ and $u \in \varphi(I)$, using standard calculus rules, we obtain

$$\partial_j \Psi(u) = \varphi'_0(\Phi(\varphi^{-1}(u))) \cdot \partial_j \Phi(\varphi^{-1}(u)) \cdot \frac{1}{\varphi'_j(\varphi_j^{-1}(u_j))},$$

and

$$\begin{aligned}
\partial_i\partial_j\Psi(u) &= \varphi_0''(\Phi(\varphi^{-1}(u)))\cdot\partial_i\Phi(\varphi^{-1}(u))\cdot\partial_j\Phi(\varphi^{-1}(u))\frac{1}{\varphi_i'(\varphi_i^{-1}(u_i))}\cdot\frac{1}{\varphi_j'(\varphi_j^{-1}(u_j))}\\
&\quad+\varphi_0'(\Phi(\varphi^{-1}(u)))\cdot\partial_i\partial_j\Phi(\varphi^{-1}(u))\cdot\frac{1}{\varphi_i'(\varphi_i^{-1}(u_i))}\cdot\frac{1}{\varphi_j'(\varphi_j^{-1}(u_j))}\\
&\quad-\delta_{i,j}\varphi_0'(\Phi(\varphi^{-1}(u)))\cdot\partial_j\Phi(\varphi^{-1}(u))\cdot\frac{\varphi_j''(\varphi_j^{-1}(u_j))}{\varphi_j'(\varphi_j^{-1}(u_j))^3}\\
&=\frac{\varphi_0'(\Phi(\varphi^{-1}(u)))}{\varphi_i'(\varphi_i^{-1}(u_i))\cdot\varphi_j'(\varphi_j^{-1}(u_j))}\Big(\partial_i\Phi(\varphi^{-1}(u))\cdot\partial_j\Phi(\varphi^{-1}(u))\\
&\qquad\times\frac{\varphi_0''}{\varphi_0'}(\Phi(\varphi^{-1}(u)))+\partial_i\partial_j\Phi(\varphi^{-1}(u))\\
&\qquad-\delta_{i,j}\partial_j\Phi(\varphi^{-1}(u))\cdot\frac{\varphi_j''}{\varphi_j'}(\varphi_j^{-1}(u_j))\Big).
\end{aligned}$$

Now using the equalities in (4.11) and (4.9), it follows that

$$\partial_i\partial_j\Psi(u)=\frac{\varphi_0'(\Phi(\varphi^{-1}(u)))}{\varphi_i'(\varphi_i^{-1}(u_i))\cdot\varphi_j'(\varphi_j^{-1}(u_j))}\cdot(-\Gamma_{i,j}(\varphi^{-1}(u))).$$

Therefore, for all $u\in\varphi(I)$, we obtain that $\Psi''(u)=(\partial_i\partial_j\Psi(u))_{i,j=1}^k$ is negative semidefinite. This implies that Ψ is concave on $\varphi(I)$. □

Remark 4.2.3. It can be seen from the above argument that the concavity of the auxiliary function Ψ is not merely a consequence of the positive semidefiniteness of the matrix-valued function Γ but, in fact, it is equivalent to it. On the other hand, if all the weight functions are equal to constant 1, then $\varphi_\alpha=f_\alpha$ and, in this case, according to the theory of quasiarithmetic means (see [15]), the concavity of the function Ψ is also sufficient for inequality (4.2) to be valid in the global sense.

The following two theorems establish sufficient conditions for inequality (4.2) to be valid in the local as well as in the global sense.

Theorem 4.2.4. *Let $k\in\mathbb{N}$ and, for $\alpha\in\{0,\dots,k\}$, $f_\alpha\colon I_\alpha\to\mathbb{R}$ be differentiable with a nonvanishing derivative, and $p_0^\alpha\colon I_\alpha\to\mathbb{R}_+$. Furthermore, let $\Phi\colon I\to I_0$ be partially differentiable. Assume that the following inequality is valid in the local sense on I^2, that is, there exists a nonempty open set $V\subseteq I^2$ containing $\operatorname{diag}(I^2)$ such that, for all $(u,y)\in V$,*

$$\frac{p_0^0(\Phi(y))(f_0(\Phi(y))-f_0(\Phi(u)))}{p_0^0(\Phi(u))f_0'(\Phi(u))}\le\sum_{j=1}^k\partial_j\Phi(u)\frac{p_0^j(y_j)(f_j(y_j)-f_j(u_j))}{p_0^j(u_j)f_j'(u_j)}\tag{4.12}$$

holds. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality

$$A^{[n]}_{f_0,p_0^0\lambda}(\Phi(x_1),\dots,\Phi(x_n)) \le \Phi(A^{[n]}_{f_1,p_0^1\lambda}(x^1),\dots,A^{[n]}_{f_k,p_0^k\lambda}(x^k)) \tag{4.13}$$

is valid in the local sense.

Proof. Let $n \in \mathbb{N}$, $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$ be fixed and construct the set $U \subseteq I^n$ as follows:

$$U := \bigcap_{i=1}^{n} \{x \in \mathbb{R}^{n\times k} \mid [\,x\,]^T \in I^n, (A^{[n]}_{f_1,p_0^1\lambda}(x^1),\dots,A^{[n]}_{f_k,p_0^k\lambda}(x^k), x_i^1,\dots,x_i^k) \in V\}. \tag{4.14}$$

Then, due to the continuity of the mean $A_{f_\alpha,p_0^\alpha\lambda}$, each member of the intersection is open and hence so is U. On the other hand, if $[\,x\,]^T \in \operatorname{diag}(I^n)$, then, for all $\alpha \in \{1,\dots,k\}$, we have $x_1^\alpha = \cdots = x_n^\alpha = A_{f_\alpha,p_0^\alpha\lambda}(x^\alpha)$, whence, by the properties of V, it follows that $(A^{[n]}_{f_1,p_0^1\lambda}(x^1),\dots,A^{[n]}_{f_k,p_0^k\lambda}(x^k), x_i^1,\dots,x_i^k) \in V$ holds for all $i \in \{1,\dots,n\}$. This shows that U contains $\operatorname{diag}(I^n)$.

We now prove that, for all $x \in \mathbb{R}^{n\times k}$ with $[\,x\,]^T \in U$, inequality (4.13) is valid. Let us define, for $\alpha \in \{1,\dots,k\}$,

$$y_\alpha := A_{f_\alpha,p_0^\alpha\lambda}(x^\alpha)$$

and let $u \in I$ be arbitrary. As a consequence of this definition, it follows that

$$\sum_{i=1}^{n} \lambda_i p_0^\alpha(x_i^\alpha)(f_\alpha(x_i^\alpha) - f_\alpha(u_\alpha)) = 0 \qquad (\alpha \in \{1,\dots,k\}). \tag{4.15}$$

On the other hand, for all $\alpha \in \{1,\dots,k\}$ and $i \in \{1,\dots,n\}$, we have that $(u_1,\dots,u_k, x_i^1,\dots,x_i^k) \in V$. Therefore, we can apply the inequality (4.12) with the substitution $(u_1,\dots,u_k,y_1,\dots,y_k) := (u_1,\dots,u_k,x_i^1,\dots,x_i^k)$. Then multiplying this inequality by λ_i side by side and summing up the inequalities so obtained, finally, also using the previous equalities in (4.15), we get

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\lambda_i p_0^0(\Phi(x_i))(f_0(\Phi(x_i)) - f_0(\Phi(u)))}{p_0^0(\Phi(u))f_0'(\Phi(u))}\\
&\qquad \le \sum_{i=1}^{n}\sum_{\alpha=1}^{k} \partial_\alpha\Phi(u)\frac{\lambda_i p_0^\alpha(x_i^\alpha)(f_\alpha(x_i^\alpha) - f_\alpha(u_\alpha))}{p_0^\alpha(u_\alpha)f_\alpha'(u_\alpha)}\\
&\qquad = \sum_{\alpha=1}^{k} \frac{\partial_\alpha\Phi(u)}{p_0^\alpha(u_\alpha)f_\alpha'(u_\alpha)} \sum_{i=1}^{n} \lambda_i p_0^\alpha(x_i^\alpha)(f_\alpha(x_i^\alpha) - f_\alpha(u_\alpha)) = 0,
\end{aligned}$$

that is,

$$\sum_{i=1}^{n} \frac{\lambda_i p_0^0(\Phi(x_i))(f_0(\Phi(x_i)) - f_0(\Phi(u)))}{p_0^0(\Phi(u))f_0'(\Phi(u))} \le 0.$$

Assume that f_0' is positive. Then f_0 is strictly increasing and the above inequality is equivalent to

$$\sum_{i=1}^{n} \lambda_i p_0^0(\Phi(x_i))(f_0(\Phi(x_i)) - f_0(\Phi(u))) \leq 0. \tag{4.16}$$

Rearranging this inequality, we obtain

$$\frac{\sum_{i=1}^{n} \lambda_i p_0^0(\Phi(x_i)) f_0(\Phi(x_i))}{\sum_{i=1}^{n} \lambda_i p_0^0(\Phi(x_i))} \leq f_0(\Phi(u)). \tag{4.17}$$

Applying f_0^{-1} side by side and using that f_0^{-1} is strictly increasing, we conclude that

$$\begin{aligned} A^{[n]}_{f_0,p_0^0\lambda}(\Phi(x_1),\dots,\Phi(x_n)) &= f_0^{-1}\left(\frac{\sum_{i=1}^{n} \lambda_i p_0^0(\Phi(x_i)) f_0(\Phi(x_i))}{\sum_{i=1}^{n} \lambda_i p_0^0(\Phi(x_i))}\right) \\ &\leq \Phi(u) = \Phi(A^{[n]}_{f_1,p_0^1\lambda}(x^1),\dots,A^{[n]}_{f_k,p_0^k\lambda}(x^k)), \end{aligned}$$

which completes the proof of inequality (4.13). In the case when f_0' is everywhere negative, the inequalities (4.16) and (4.17) are reversed, however f_0^{-1} is strictly decreasing, therefore we arrive at the same conclusion. □

Remark 4.2.5. In view of Theorem 4.2.1, the weight functions of the generalized Bajraktarević means appearing in (4.2) are necessarily of the form given by (4.8). Therefore, the local as well as the global validity of (4.2) immediately follows from the local as well as the global validity of (4.13), respectively.

Theorem 4.2.6. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0,\dots,k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be differentiable with a nonvanishing derivative, $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ and let $\Phi\colon I \to I_0$ be partially differentiable. Assume that* (4.12) *is satisfied in the global sense on I^2, that is, for all $u, y \in I$. Then, for all $n \in \mathbb{N}$, $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, inequality* (4.13) *holds in the global sense.*

Proof. If (4.12) is satisfied for all $u, y \in I$, then the condition of the previous theorem is validated with the open set $V := I^2$ and the open set U constructed by (4.14) equals I^n. Hence inequality (4.13) holds for all $x \in \mathbb{R}^{n\times k}$ with $[\,x\,]^T \in U$, i.e., it holds in the global sense. □

The next result establishes a necessary condition for (4.12) to be satisfied in the local sense.

Theorem 4.2.7. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0,\dots,k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be twice differentiable with a nonvanishing first derivative and $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ be twice differentiable. In addition, let $\Phi\colon I \to I_0$ be twice differentiable. Assume that inequality* (4.12) *is satisfied in the local sense, that is, there exists an open set $V \subseteq I^2$ with $\operatorname{diag}(I^2) \subseteq V$ such that it holds true for all $(u, y) \in V$. Then the matrix-valued function $\Gamma\colon I \to \mathbb{R}^{k\times k}$ defined by* (4.9) *takes positive semidefinite values.*

Proof. If inequality (4.12) is satisfied for all $(u, y) \in V$ then, for all fixed $y \in I$, the map $\Psi_y\colon I \to \mathbb{R}$ defined as

$$\Psi_y(u) := \sum_{\alpha=1}^{k} \partial_\alpha \Phi(y) \frac{p_0^\alpha(u_\alpha)(f_\alpha(u_\alpha) - f_\alpha(y_\alpha))}{p_0^\alpha(y_\alpha) f_\alpha'(y_\alpha)} - \frac{p_0^0(\Phi(u))(f_0(\Phi(u)) - f_0(\Phi(y)))}{p_0^0(\Phi(y)) f_0'(\Phi(y))} \tag{4.18}$$

has a local minimum at $u = y$. This function is twice differentiable according to our assumptions. Therefore, the second derivative matrix of it at $u = y$ is positive semidefinite. For $i \in \{1, \dots, k\}$, we have

$$\begin{aligned}\partial_i \Psi_y(u) = {} & \frac{\partial_i \Phi(y)}{p_0^i(y_i) f_i'(y_i)} \big((p_0^i)'(u_i)(f_i(u_i) - f_i(y_i)) + p_0^i(u_i) f_i'(u_i)\big) \\ & - \frac{\partial_i \Phi(u)}{p_0^0(\Phi(y)) f_0'(\Phi(y))} \\ & \times \big((p_0^0)'(\Phi(u))(f_0(\Phi(u)) - f_0(\Phi(y))) + p_0^0(\Phi(u)) f_0'(\Phi(u))\big).\end{aligned} \tag{4.19}$$

Thus, for $i, j \in \{1, \dots, k\}$, we obtain

$$\begin{aligned}&\partial_i \partial_j \Psi_y(u) \\ &\quad = \delta_{ij} \frac{\partial_j \Phi(y)}{p_0^j(y_j) f_j'(y_j)} \big((p_0^j)''(u_j)(f_j(u_j) - f_j(y_j)) + 2(p_0^j)'(u_j) f_j'(u_j) + p_0^j(u_j) f_j''(u_j)\big) \\ &\qquad - \frac{\partial_i \partial_j \Phi(u)}{p_0^0(\Phi(y)) f_0'(\Phi(y))} \big((p_0^0)'(\Phi(u))(f_0(\Phi(u)) - f_0(\Phi(y))) + p_0^0(\Phi(u)) f_0'(\Phi(u))\big) \\ &\qquad - \frac{\partial_i \Phi(u) \partial_j \Phi(u)}{p_0^0(\Phi(y)) f_0'(\Phi(y))} \big((p_0^0)''(\Phi(u))(f_0(\Phi(u)) - f_0(\Phi(y))) \\ &\qquad\qquad + 2(p_0^0)'(\Phi(u)) f_0'(\Phi(u)) + p_0^0(\Phi(u)) f_0''(\Phi(u))\big).\end{aligned} \tag{4.20}$$

Hence, after substituting $u := y$ in the above equality, we get

$$\begin{aligned}\partial_i \partial_j \Psi_y(y) = {} & \delta_{ij} \partial_j \Phi(y) \left(2 \frac{(p_0^j)'}{p_0^j} + \frac{f_j''}{f_j'}\right)(y_j) - \partial_i \partial_j \Phi(y) \\ & - \partial_i \Phi(y) \partial_j \Phi(y) \left(2 \frac{(p_0^0)'}{p_0^0} + \frac{f_0''}{f_0'}\right)(\Phi(y)) = \Gamma_{i,j}(y).\end{aligned} \tag{4.21}$$

This shows the pointwise positive semidefiniteness of the function $\Gamma\colon I \to \mathbb{R}^{k \times k}$. □

The converse of the above statement is generally not valid. The conditions must be tightened, namely, positive definiteness must be assumed instead of positive semidefiniteness.

Theorem 4.2.8. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0, \dots, k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be twice continuously differentiable with a nonvanishing first derivative, $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ be twice continuously* *differentiable. Furthermore, let $\Phi\colon I \to I_0$ be twice continuously differentiable. Assume, for all $y \in I$, that the $(k \times k)$-type matrix $\Gamma(y)$ defined by (4.9) is positive definite. Then, for all $n \in \mathbb{N}$, $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, inequality (4.13) holds in the local sense.*

Proof. For all fixed $y \in I$, define the function $\Psi_y\colon I \to \mathbb{R}$ by the formula (4.18). This function is twice continuously differentiable in view of our assumptions and $\Psi_y(y) = 0$ holds for all $y \in I$. After simple computations, for $i, j \in \{1, \dots, k\}$ and $u \in I$, we obtain that the equalities (4.19) and (4.20) hold.

Putting $u := y$ into equality (4.19), we can see that

$$\partial_i \Psi_y(y) = 0 \qquad (i \in \{1, \dots, k\}),$$

that is, $\Psi'_y(y) = \big(\partial_i \Psi_y(y)\big)_{i=1}^k = 0$ holds for all $y \in I$.

Substituting $u := y$ into equality (4.20), we can conclude that (4.21) is valid for all $y \in I$. According to the positive definiteness of the matrix-valued function $\Gamma\colon I \to \mathbb{R}^{k\times k}$ defined by (4.9), it follows that $\Psi''_y(y) := \big(\partial_i\partial_j \Psi_y(y)\big)_{i,j=1}^k$ is positive definite for all $y \in I$.

For $u, y \in I$, let us denote the smallest eigenvalue of the $(k \times k)$ symmetric matrix $\Psi''_y(u) := \big(\partial_i\partial_j \Psi_y(u)\big)_{i,j=1}^k$ by $\psi(u, y)$. Considering the assumptions of twice continuous differentiability, it follows that the map $(u, y) \mapsto \Psi''_y(u)$ is continuous, thereby ensuring continuity for the map $(u, y) \mapsto \psi(u, y)$ as well. On the other hand, for all $y \in I$, we have that $\Psi''_y(y) = \Gamma(y)$ is positive definite, which implies $\psi(y, y) > 0$. Therefore, there exists an open set $W \subseteq I^2$ containing $\operatorname{diag}(I^2)$ on which ψ is positive.

By the Taylor Mean Value Theorem, for all $u, y \in I$, there exists $t \in [0, 1]$ such that

$$\begin{aligned}\Psi_y(u) &= \Psi_y(y) + \Psi'_y(y)(y-u) + \frac{1}{2}(y-u)^T \Psi''_y(tu + (1-t)y)(y-u)\\ &= \frac{1}{2}(y-u)^T \Psi''_y(tu + (1-t)y)(y-u) \geq \frac{1}{2}\psi(tu + (1-t)y, y)\|y-u\|^2.\end{aligned} \tag{4.22}$$

Define

$$V := \{(u, y) \in I^2 \mid [y, u] \times \{y\} = [(y, y), (u, y)] \subseteq W\}.$$

We will show that V is an open subset of I^2. To see this, let $(u, y) \in V$ be arbitrary. Since the segment $[(y, y), (u, y)]$ is a compact subset of W, it follows that it is disjoint from $W^c := \mathbb{R}^{2k} \setminus W$, which is the complement of W and hence it is a closed set. Therefore, there exists a positive number r so that the distance of every point of the segment $[(y, y), (u, y)]$ from W^c is at least r. Let $v, x \in \mathbb{R}^k$ such that $\|v - u\| < r/2$ and $\|x - y\| < r/2$ hold. Let (w, x) be an arbitrary point of the segment $[(x, x), (v, x)]$. Then there exists $t \in [0, 1]$ such that $w = tv + (1-t)x$. Therefore,

$$\|w - (tu + (1-t)y)\| \leq t\|v - u\| + (1-t)\|x - y\| < r/2,$$

which implies that

$$\begin{aligned}
&\|(w,x)-(tu+(1-t)y,y)\| \\
&\qquad \le \|(w,x)-(tu+(1-t)y,x)\| + \|(tu+(1-t)y,x)-(tu+(1-t)y,y)\| \\
&\qquad = \|w-(tu+(1-t)y)\| + \|x-y\| < r.
\end{aligned}$$

In other words, any point of the segment $[(x,x),(v,x)]$ is closer to some point of the segment $[(y,y),(u,y)]$ than r. This yields $[(x,x),(v,x)] \subseteq W$, i.e., $(v,x) \in V$ whenever $\|v-u\| < r/2$ and $\|x-y\| < r/2$ hold, consequently (u,y) is an interior point of V. This completes the proof of the openness of V. On the other hand, it is obvious that V also contains $\operatorname{diag}(I^2)$.

Using that ψ is positive on W, (4.22) yields that $\Psi_y(u) \ge 0$ for all $(u,y) \in V$. Therefore, (4.12) is valid and by applying Theorem 4.2.4, it follows that (4.13) holds in the local sense. □

4.3 Hölder- and Minkowski-type inequalities for weighted Gini means

In this section, for $k \in \mathbb{N}$, $\alpha \in \{1,\dots,k\}$, let I_α denote a nonempty open subinterval of $\mathbb{R}_+$, let $I := I_1 \times \cdots \times I_k$, and we apply the above results to some important particular cases of the inequality (4.1). First, we deal with inequalities obtained by specializing the means M_α in (4.1) for all $\alpha \in \{0,\dots,k\}$ and then we draw some conclusions by choosing the function Φ in (4.1) to be the k-variable addition and multiplication.

Definition 4.3.1. Let $n \in \mathbb{N}$ and, for $r \in \mathbb{R}$, let us recall the definition of the *n-variable weighted rth power or rth Hölder mean with weight vector $\lambda \in \mathbb{R}_+^n$, $\lambda_1+\cdots+\lambda_n = 1$* and, for $(r,s) \in \mathbb{R}^2$, the *n-variable weighted Gini mean with pair of parameters $(r,s) \in \mathbb{R}^2$ and weight vector $\lambda \in \mathbb{R}_+^n$, $\lambda_1+\cdots+\lambda_n = 1$*:

$$H^{[n]}_{r;\lambda}(x_1,\dots,x_n) := \begin{cases} \left(\lambda_1 x_1^r + \cdots + \lambda_n x_n^r\right)^{\frac{1}{r}} & \text{if } r \ne 0 \\ x_1^{\lambda_1}\cdots x_n^{\lambda_n} & \text{if } r = 0, \end{cases}$$

$$G^{[n]}_{r,s;\lambda}(x_1,\dots,x_n) := \begin{cases} \left(\dfrac{\lambda_1 x_1^r + \cdots + \lambda_n x_n^r}{\lambda_1 x_1^s + \cdots + \lambda_n x_n^s}\right)^{\frac{1}{r-s}} & \text{if } r \ne s \\ \exp\left(\dfrac{\lambda_1 x_1^r \ln(x_1) + \cdots + \lambda_n x_n^r \ln(x_n)}{\lambda_1 x_1^r + \cdots + \lambda_n x_n^r}\right) & \text{if } r = s. \end{cases}$$

Remark 4.3.2. It is clear that in the particular case $s = 0$, the mean $G^{[n]}_{r,s;\lambda}$ simplifies to $H^{[n]}_{r;\lambda}$.

Remark 4.3.3. We also note that, for $n \in \mathbb{N}$ and $\ell \in \{1,\dots,n\}$, with $p_\ell(t) := \lambda_\ell t^s$ and $f(t) := t^{r-s}$ if $r \ne s$ or $f(t) := \ln(t)$ if $r = s$, we can see that $A^{[n]}_{f,p} = G^{[n]}_{r,s;\lambda}$. Using

Theorem 1.2.1, it follows that

$$\Big(\partial_\ell G^{[n]}_{r,s;\lambda}\Big)^{\Delta}(t) = \lambda_\ell \qquad (\ell \in \{1,\dots,n\},\, t \in \mathbb{R}_+),$$

and

$$\Big(\partial_\ell\partial_m G^{[n]}_{r,s;\lambda}\Big)^{\Delta}(t) = \lambda_m(\delta_{\ell,m} - \lambda_\ell)\frac{r+s-1}{t} \qquad (\ell, m \in \{1,\dots,n\},\, t \in \mathbb{R}_+). \tag{4.23}$$

Furthermore,

$$\frac{p_0(y)(f(y)-f(u))}{p_0(u)f'(u)} = u\,\chi_{r,s}\Big(\frac{y}{u}\Big) \qquad (u, y \in \mathbb{R}_+), \tag{4.24}$$

where the function $\chi_{r,s}\colon \mathbb{R}_+ \to \mathbb{R}$ is defined in Definition 3.2.7.

The next result consists of a necessary as well as a sufficient condition for a general inequality corresponding to weighted Gini means to be valid in the local sense.

Theorem 4.3.4. *Let $n, k \in \mathbb{N}, k \geq 2$, $\lambda \in \mathbb{R}^n_+$, and, for $\alpha \in \{0,\dots,k\}$, $(r_\alpha, s_\alpha) \in \mathbb{R}^2$ and $\Phi\colon I \to \mathbb{R}_+$ be twice differentiable with nonvanishing first-order partial derivatives. Then, for the inequality*

$$G^{[n]}_{r_0,s_0;\lambda}(\Phi(x_1),\dots,\Phi(x_n)) \leq \Phi\big(G^{[n]}_{r_1,s_1;\lambda}(x^1),\dots,G^{[n]}_{r_k,s_k;\lambda}(x^k)\big) \tag{4.25}$$

to be valid in the local sense it is necessary that the $(k \times k)$-type matrix $\Gamma(y)$ given by

$$\Gamma(y) := \bigg(-\partial_i\partial_j\Phi(y) - \partial_j\Phi(y)\partial_i\Phi(y)\frac{r_0+s_0-1}{\Phi(y)} + \delta_{i,j}\partial_j\Phi(y)\frac{r_j+s_j-1}{y_j}\bigg)_{i,j=1}^{k}$$

be positive semidefinite for all $y \in I$. Conversely, if this matrix is positive definite for all $y \in I$, then (4.25) holds in the local sense on I.

Proof. The necessity of the positive semidefiniteness of $\Gamma(y)$, for all $y \in I$, is an immediate consequence of Theorem 4.2.4, Theorem 4.2.7, and formula (4.23). The other direction is also obvious due to Theorem 4.2.8. □

The theorem below gives a sufficient condition for the inequality (4.25) to be valid in the global sense.

Theorem 4.3.5. *Let $k \in \mathbb{N}, k \geq 2$, for $\alpha \in \{0,\dots,k\}$, $(r_\alpha, s_\alpha) \in \mathbb{R}^2$ and let $\Phi\colon I \to \mathbb{R}_+$ be partially differentiable. Assume that the inequality*

$$\Phi(u)\chi_{r_0,s_0}\bigg(\frac{\Phi(y)}{\Phi(u)}\bigg) \leq \sum_{j=1}^{k}\partial_j\Phi(u)u_j\chi_{r_j,s_j}\Big(\frac{y_j}{u_j}\Big) \tag{4.26}$$

is valid in the global sense on I^2, that is, for all $(u, y) \in I^2$. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}^n_+$ with $\lambda_1 + \dots + \lambda_n = 1$, the inequality (4.25) holds in the global sense on I.

Proof. The statement directly follows from Theorem 4.2.6 since inequality (4.12) turns out to be equivalent to (4.26) by applying formula (4.24). □

For the investigation of the particular cases when Φ is the sum and the product function, we will need the following auxiliary result.

Lemma 4.3.6. *Let $k \in \mathbb{N}$, for $i \in \{0, \dots, k\}$, $c_i \in \mathbb{R}$. Then the matrix*

$$C := \big(\delta_{i,j} c_i + c_0\big)_{i,j=1}^{k}$$

is positive semidefinite if and only if either $c_i \geq 0$ for all $i \in \{0, \dots, k\}$, or there exists $i \in \{0, \dots, k\}$ such that $c_i < 0$ and $c_j > 0$ for all $j \in \{0, \dots, k\} \setminus \{i\}$ and

$$\frac{1}{c_0} + \frac{1}{c_1} + \dots + \frac{1}{c_k} \leq 0. \tag{4.27}$$

Furthermore, C is positive definite if and only if either $c_i \geq 0$ for all $i \in \{0, \dots, k\}$ and $c_i = 0$ can hold for at most one index $i \in \{0, \dots, k\}$ or there exists $i \in \{0, \dots, k\}$ such that $c_i < 0$ and $c_j > 0$ for all $j \in \{0, \dots, k\} \setminus \{i\}$ and (4.27) is valid with a strict inequality.

Proof. The quadratic form $Q\colon \mathbb{R}^k \to \mathbb{R}$ generated by C is given by

$$Q(x) := c_0 \left(\sum_{\ell=1}^{k} x_\ell \right)^2 + \sum_{\ell=1}^{k} c_\ell x_\ell^2 \qquad (x \in \mathbb{R}^k).$$

Assume first that C is positive semidefinite. This means that $Q(x) \geq 0$ for all $x \in \mathbb{R}^k$. With the notation $x_0 := -\sum_{\ell=1}^{k} x_\ell$, this inequality can be rewritten as

$$\sum_{\ell=0}^{k} c_\ell x_\ell^2 \geq 0 \tag{4.28}$$

for all $(x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1}$ with $x_0 + x_1 + \dots + x_k = 0$. To prove the necessity of the condition, assume that $\min(c_0, c_1, \dots, c_k) < 0$. Choose $i \in \{0, \dots, k\}$ such that $c_i = \min(c_0, c_1, \dots, c_k) < 0$. For a fixed $j \in (\{0, \dots, k\}) \setminus \{i\}$, define the vector $x = (x_0, x_1, \dots, x_k)$ by $x_\ell := \delta_{i,\ell} - \delta_{j,\ell}$ for $\ell \in \{0, \dots, k\}$, that is, $x_\ell := 0$ if $\ell \in (\{0, \dots, k\}) \setminus \{i, j\}$ and $x_i := 1$, $x_j := -1$. Thus $x_0 + x_1 + \dots + x_k = 0$ holds, which, by (4.28), yields that $\sum_{\ell=0}^{k} c_\ell x_\ell^2 \geq 0$, that is, $c_i + c_j \geq 0$. This implies that $c_j > 0$ for all $j \in (\{0, \dots, k\}) \setminus \{i\}$.

In order to show that the inequality (4.27) is also valid, we substitute the vector $x = (x_0, x_1, \dots, x_k)$ given by

$$x_j := \frac{1}{c_j} \quad (j \in (\{0, \dots, k\}) \setminus \{i\}) \qquad \text{and} \qquad x_i := -\sum_{j \in (\{0,\dots,k\}) \setminus \{i\}} \frac{1}{c_j}. \tag{4.29}$$

Then, obviously, $x_0 + x_1 + \cdots + x_k = 0$, which, again by (4.28), yields $\sum_{\ell=0}^{k} c_\ell x_\ell^2 \geq 0$, that is,

$$\sum_{j \in (\{0,\ldots,k\}) \setminus \{i\}} c_j \left(\frac{1}{c_j} \right)^2 + c_i \left(- \sum_{j \in (\{0,\ldots,k\}) \setminus \{i\}} \frac{1}{c_j} \right)^2 \geq 0. \tag{4.30}$$

After simplifications, we obtain

$$1 + c_i \left(\sum_{j \in (\{0,\ldots,k\}) \setminus \{i\}} \frac{1}{c_j} \right) \geq 0, \tag{4.31}$$

which, using $c_i < 0$, shows that the inequality (4.27) is valid.

Assume that C is positive definite. Then $Q(x) > 0$ holds for all $x \in \mathbb{R}^k \setminus \{0\}$. With the notation $x_0 := -\sum_{\ell=1}^{k} x_\ell$, this inequality can be rewritten as

$$\sum_{\ell=0}^{k} c_\ell x_\ell^2 > 0 \tag{4.32}$$

for all $(x_0, x_1, \ldots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\}$ with $x_0 + x_1 + \cdots + x_k = 0$. Since the positive definiteness of Q implies its positive semidefiniteness, there are two possible cases:

(a) $c_i \geq 0$ holds for all $i \in \{0, \ldots, k\}$;

(b) there exists $i \in \{0, \ldots, k\}$ such that $c_i < 0$ and $c_j > 0$ for all $j \in (\{0, \ldots, k\}) \setminus \{i\}$ and (4.27) holds.

Assume first that case (a) holds. If $c_i = c_j = 0$ were valid for some $i, j \in \{0, \ldots, k\}$ with $i \neq j$, then substituting $x_\ell := \delta_{i,\ell} - \delta_{j,\ell}$ for $\ell \in \{0, \ldots, k\}$ into (4.32), we would get that $0 < \sum_{\ell=0}^{k} c_\ell x_\ell^2 = c_i + c_j = 0$. This contradiction shows that $c_i = 0$ can hold only for at most one index $i \in \{0, \ldots, k\}$.

Consider now the case (b). Then there exists $i \in \{0, \ldots, k\}$ such that $c_i < 0$ and $c_j > 0$ for all $j \in (\{0, \ldots, k\}) \setminus \{i\}$. Substituting the vector $x = (x_0, x_1, \ldots, x_k)$ given by (4.29) into (4.32), it follows that $\sum_{\ell=0}^{k} c_\ell x_\ell^2 > 0$. This implies that the inequalities (4.30) and then (4.31) are strict. Therefore, (4.27) is also satisfied with a strict inequality.

Now we show the sufficiency of the aforementioned conditions. In the first case, i.e., when $c_i \geq 0$ for all $i \in \{0, \ldots, k\}$, the inequality (4.28) is obvious and hence Q is positive semidefinite. If, in addition, $c_i = 0$ holds for at most one index $i \in \{0, \ldots, k\}$, then (4.32) is also valid, that is, Q is positive definite in this case.

Consider the other case when $c_i < 0$ and $c_j > 0$ for all $j \in (\{0, \ldots, k\}) \setminus \{i\}$ is valid for some $i \in \{0, \ldots, k\}$ and (4.27) holds. Then, from (4.27) it follows that

$$c_i \geq - \left(\sum_{j \in (\{0,\ldots,k\}) \setminus \{i\}} \frac{1}{c_j} \right)^{-1}. \tag{4.33}$$

Let $(x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1}$ with $x_0 + x_1 + \dots + x_k = 0$. Applying the Cauchy–Schwarz inequality to the vectors

$$\left(\frac{1}{\sqrt{c_j}}\right)_{j\in(\{0,\dots,k\})\setminus\{i\}} \qquad \text{and} \qquad \left(-\sqrt{c_j}x_j\right)_{j\in(\{0,\dots,k\})\setminus\{i\}},$$

we obtain that

$$\left(\sum_{j\in(\{0,\dots,k\})\setminus\{i\}} \frac{1}{c_j}\right)\left(\sum_{j\in(\{0,\dots,k\})\setminus\{i\}} c_j x_j^2\right) \geq \left(\sum_{j\in(\{0,\dots,k\})\setminus\{i\}} -x_j\right)^2 = x_i^2.$$

Therefore, combining this inequality with (4.33), we can conclude that

$$\sum_{j=0}^{k} c_j x_j^2 \geq \sum_{j\in(\{0,\dots,k\})\setminus\{i\}} c_j x_j^2 - \left(\sum_{j\in(\{0,\dots,k\})\setminus\{i\}} \frac{1}{c_j}\right)^{-1} x_i^2 \geq 0. \tag{4.34}$$

Thus, we have proved that (4.28) is valid and hence Q is positive semidefinite. If in this case (4.27) is valid with a strict inequality, then (4.33) is also strict. Then, for a nonzero vector $(x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1}$ with $x_0 + x_1 + \dots + x_k = 0$ the first inequality in (4.34) is strict provided that $x_i \neq 0$ and the second inequality is strict if $x_i = 0$. Thus, (4.32) holds true, which shows that Q is positive definite. □

Our next result characterizes the Minkowski-type inequality for weighted Gini means in the local sense.

Theorem 4.3.7. *Let $n, k \in \mathbb{N}, k \geq 2$ and $\lambda \in \mathbb{R}_+^n$. For $i \in \{0, \dots, k\}$, let further $(r_i, s_i) \in \mathbb{R}^2$ and $\gamma_i := r_i + s_i - 1$. For the inequality*

$$G^{[n]}_{r_0,s_0;\lambda}(x_1^1 + \dots + x_1^k, \dots, x_n^1 + \dots + x_n^k) \leq G^{[n]}_{r_1,s_1;\lambda}(x^1) + \dots + G^{[n]}_{r_k,s_k;\lambda}(x^k) \tag{4.35}$$

to hold in the local sense on I, it is necessary that exactly one of the following cases be valid:

(i)

$$\gamma_0 \leq 0 \leq \min(\gamma_1, \dots, \gamma_k); \tag{4.36}$$

(ii) $\gamma_i > 0$ *for all* $i \in \{0, \dots, k\}$ *and*

$$\sum_{i\in J_+} \left(\frac{1}{\gamma_i} - \frac{1}{\gamma_0}\right) \sup I_i \leq \sum_{i\in J_-} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i}\right) \inf I_i; \tag{4.37}$$

(iii) $\gamma_0 < 0$ *and there exists* $i \in \{1, \dots, k\}$ *such that* $\gamma_i < 0$, *for all* $j \in \{1, \dots, k\} \setminus \{i\}$, $\gamma_j > 0$, *and inequality* (4.37) *is also valid,*

where, for the last two cases, we define

$$J_+ := \left\{ i \in \{1, \dots, k\} \colon \frac{1}{\gamma_i} > \frac{1}{\gamma_0} \right\} \qquad \textit{and} \qquad J_- := \left\{ i \in \{1, \dots, k\} \colon \frac{1}{\gamma_0} > \frac{1}{\gamma_i} \right\}.$$

Conversely, if either (4.36) is valid and $\gamma_i = 0$ can hold for at most one $i \in \{0, \dots, k\}$, or $\gamma_0 \neq \gamma_\ell$ for some $\ell \in \{1, \dots, k\}$ and one of the conditions (ii) or (iii) hold, then (4.35) is valid in the local sense on I.

Proof. We apply Theorem 4.3.4 with the setting $I_0 := \mathbb{R}_+$ and $\Phi \colon I \to I_0$ defined by $\Phi(y) := y_1 + \cdots + y_k$. According to Theorem 4.3.4, for the validity of (4.35) in the local sense, it is necessary that the values of the function $\Gamma \colon I \to \mathbb{R}^{k \times k}$ defined by

$$\Gamma(y) := \left(\delta_{i,j} \frac{\gamma_i}{y_i} - \frac{\gamma_0}{y_1 + \cdots + y_k} \right)_{i,j=1}^{k}$$

be positive semidefinite matrices. By Lemma 4.3.6, this property is characterized by the following system of conditions: either

$$c_0 := -\frac{\gamma_0}{y_1 + \cdots + y_k} \geq 0 \qquad \text{and} \qquad c_i := \frac{\gamma_i}{y_i} \geq 0 \qquad (i \in \{1, \dots, k\}), \tag{4.38}$$

or there exists $i \in \{0, \dots, k\}$ such that $c_i < 0$, for all $j \in \{0, \dots, k\} \setminus \{i\}$, $c_j > 0$, and (4.27) holds, i.e.,

$$\frac{1}{c_0} + \frac{1}{c_1} + \cdots + \frac{1}{c_k} \leq 0.$$

Observe that $\operatorname{sign}(\gamma_0) = -\operatorname{sign}(c_0)$ and $\operatorname{sign}(\gamma_i) = \operatorname{sign}(c_i)$ for all $i \in \{1, \dots, k\}$. Therefore, the first alternative can hold if and only if (4.36) is satisfied. The second alternative can be valid if and only if either $\gamma_i > 0$ for all $i \in \{0, \dots, k\}$, or $\gamma_0 < 0$ and there exists $i \in \{1, \dots, k\}$ such that $\gamma_i < 0$, for all $j \in \{1, \dots, k\} \setminus \{i\}$, $\gamma_j > 0$, and

$$0 \leq \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_1} \right) y_1 + \cdots + \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_k} \right) y_k \qquad (y \in I).$$

One can easily see that this inequality can be rewritten as (4.37) and hence the necessity of the other two alternatives has been established.

To prove the reversed implication of the theorem, consider first the case when (4.36) is valid and $\gamma_i = 0$ can hold for at most one $i \in \{0, \dots, k\}$. Then, for every $y \in I$, the numbers $c_0, c_1, \dots, c_k$ defined in (4.38) are nonnegative and $c_i = 0$ can hold for at most one $i \in \{0, \dots, k\}$. Thus, in view of the second assertion of Lemma 4.3.6, it follows that $\Gamma(y)$ is positive definite.

Now consider the second case when, for some $\ell \in \{1, \dots, k\}$, we have that $\gamma_0 \neq \gamma_\ell$ and either $\gamma_i > 0$ for all $i \in \{0, \dots, k\}$, or $\gamma_0 < 0$ and there exists $i \in \{1, \dots, k\}$ such that $\gamma_i < 0$, for all $j \in \{1, \dots, k\} \setminus \{i\}$, $\gamma_j > 0$ and (4.37) is also valid. Let $y \in I$ be fixed. If $\ell \in J_+$, i.e., $\frac{1}{\gamma_\ell} > \frac{1}{\gamma_0}$, then

$$\Big(\frac{1}{\gamma_\ell} - \frac{1}{\gamma_0}\Big) y_\ell < \Big(\frac{1}{\gamma_\ell} - \frac{1}{\gamma_0}\Big) \sup I_\ell,$$

while if $\ell \in J_-$, i.e., $\frac{1}{\gamma_0} > \frac{1}{\gamma_\ell}$, then

$$\Big(\frac{1}{\gamma_0} - \frac{1}{\gamma_\ell}\Big) \inf I_\ell < \Big(\frac{1}{\gamma_0} - \frac{1}{\gamma_\ell}\Big) y_\ell.$$

Therefore, (4.37) implies that

$$\sum_{i \in J_+} \Big(\frac{1}{\gamma_i} - \frac{1}{\gamma_0}\Big) y_i < \sum_{i \in J_-} \Big(\frac{1}{\gamma_0} - \frac{1}{\gamma_i}\Big) y_i,$$

which then yields

$$0 < \Big(\frac{1}{\gamma_0} - \frac{1}{\gamma_1}\Big) y_1 + \cdots + \Big(\frac{1}{\gamma_0} - \frac{1}{\gamma_k}\Big) y_k.$$

Hence, (4.27) is valid with a strict inequality sign. On the other hand, with the exception of one index, the numbers $c_0, c_1, \dots, c_k$ are positive. Thus, in view of the second assertion of Lemma 4.3.6, it follows that $\Gamma(y)$ is positive definite in this case as well. □

Corollary 4.3.8. *Let $n, k \in \mathbb{N}, k \geq 2$ and $\lambda \in \mathbb{R}_+^n$. For $i \in \{0, \dots, k\}$, let further $(r_i, s_i) \in \mathbb{R}^2$. For $\alpha \in \{1, \dots, k\}$, assume that the nonempty open interval $I_\alpha \subseteq \mathbb{R}_+$ fulfills $\inf I_\alpha = 0$. Then, in order that the inequality (4.35) be valid in the local sense on I, it is necessary that*

$$\max(1, r_0 + s_0) \leq \min(r_1 + s_1, \dots, r_k + s_k). \tag{4.39}$$

Conversely, if this inequality is strict, then (4.35) holds in the local sense on I.

Proof. The proof is based on Theorem 4.3.7. Denote $\gamma_i := r_i + s_i - 1$ for $i \in \{0, \dots, k\}$. If condition (4.36) holds, then (4.39) is obvious because

$$\begin{aligned} \max(1, r_0 + s_0) &= 1 + \max(0, \gamma_0) = 1 \\ &\leq 1 + \min(\gamma_1, \dots, \gamma_k) = \min(r_1 + s_1, \dots, r_k + s_k). \end{aligned}$$

In the remaining two cases (4.37) is valid. However, due to our assumptions on the intervals, the right hand side of (4.37) is equal to 0. Therefore, the left hand side of this equality must be an empty sum, i.e., $J_+ = \emptyset$, which means

$$\frac{1}{\gamma_0} \geq \frac{1}{\gamma_i} \qquad (i \in \{1, \dots, k\}) \tag{4.40}$$

should be valid. In the case when $\gamma_i > 0$ holds true for all $i \in \{0, \dots, k\}$, that is, $1 \le \min(r_0 + s_0, r_1 + s_1, \dots, r_k + s_k)$, then the inequalities in (4.40) hold if and only if (4.39) is satisfied. In the case when $\gamma_0 < 0$, then for at least one $j \in \{1, \dots, k\}$, we have that $\gamma_j > 0$, and hence (4.40) cannot hold.

Assume now that (4.39) is satisfied with a strict inequality. Then $\gamma_i > 0$ for all $i \in \{1, \dots, k\}$. If $\gamma_0 \le 0$, then (4.39) implies that (4.36) is valid with a strict inequality and thus the first alternative of the sufficiency of Theorem 4.3.7 holds. If $\gamma_0 > 0$, then the strict version of (4.39) shows that $\gamma_0 < \gamma_i$ for all $i \in \{1, \dots, k\}$, therefore, the left and the right hand sides of (4.37) are equal to zero and the second alternative of the sufficiency of Theorem 4.3.7 holds. The third alternative of the sufficiency of Theorem 4.3.7 cannot happen if (4.39) is valid. □

For the global validity of the Minkowski-type inequality the theorem below establishes the following sufficient condition.

Theorem 4.3.9. *Let $k \in \mathbb{N}, k \ge 2$ and, for $i \in \{0, \dots, k\}$, $(r_i, s_i) \in \mathbb{R}^2$. Assume that, for all $(u, y) \in I^2$, the inequality*

$$\chi_{r_0,s_0}\Big(\frac{y_1 + \dots + y_k}{u_1 + \dots + u_k}\Big) \le \sum_{j=1}^{k} \frac{u_j}{u_1 + \dots + u_k} \chi_{r_j,s_j}\Big(\frac{y_j}{u_j}\Big) \tag{4.41}$$

holds. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \dots + \lambda_n = 1$, the inequality (4.35) holds in the global sense on I.

Proof. With $\Phi(y_1, \dots, y_k) := y_1 + \dots + y_k$, the condition (4.26) turns out to be equivalent to (4.41) and hence the result follows from Theorem 4.3.5. □

By specializing the underlying intervals, we can formulate a more specific statement.

Corollary 4.3.10. *Let $k \in \mathbb{N}, k \ge 2$ and, for $i \in \{0, \dots, k\}$, $(r_i, s_i) \in \mathbb{R}^2$. Assume that, for all $z \in \mathbb{R}_+^k$ and $t_1, \dots, t_k \in [0, 1]$ with $t_1 + \dots + t_k = 1$, the following inequality is valid*

$$\chi_{r_0,s_0}(t_1 z_1 + \dots + t_k z_k) \le \sum_{j=1}^{k} t_j \chi_{r_j,s_j}(z_j). \tag{4.42}$$

Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \dots + \lambda_n = 1$, the inequality (4.35) holds in the global sense on $\mathbb{R}_+^k$.

Proof. Let $(u, y) \in (\mathbb{R}_+^k)^2$ be arbitrary. Then, with the substitutions

$$z_j := \frac{y_j}{u_j} \qquad \text{and} \qquad t_j := \frac{u_j}{u_1 + \dots + u_k} \qquad (j \in \{1, \dots, k\})$$

inequality (4.42) implies the validity of (4.41) on $(\mathbb{R}_+^k)^2$ and, according to Theorem 4.3.9, this condition yields that the inequality (4.35) holds in the global sense on $\mathbb{R}_+^k$. □

In order to compare our results above to existing ones, we recall two theorems related to the global validity of the Minkowski-type inequalities. In the setting of two-variable Gini means the Minkowski inequality was characterized by Czinder and Páles in [7, Theorem 5] (see also [26] for a particular case).

Theorem 4.3.11. *Let $k \in \mathbb{N}, k \geq 2$ and, for $i \in \{0, \dots, k\}$, $(r_i, s_i) \in \mathbb{R}^2$. Then the inequality*

$$G^{[2]}_{r_0,s_0}(x_1 + \cdots + x_k, y_1 + \cdots + y_k) \leq G^{[2]}_{r_1,s_1}(x_1, y_1) + \cdots + G^{[2]}_{r_k,s_k}(x_k, y_k) \quad (4.43)$$

is valid for all $x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{R}_+$ if and only if

(i) $0 \leq \min(r_1, s_1, \dots . r_k, s_k)$,

(ii) $\min(r_0, s_0) \leq \min(1, r_1, s_1, \dots . r_k, s_k)$,

(iii) $\max(1, r_0 + s_0) \leq \min(r_1 + s_1, \dots, r_k + s_k)$.

Remark 4.3.12. Observe that the third condition in the above theorem is the necessary condition for the local validity of (4.43) on $\mathbb{R}_+$. The conditions (i) and (ii) are, however, not necessary for the local validity of (4.43) on $\mathbb{R}_+$.

Necessary and sufficient conditions for the global validity of the Minkowski-type inequality for Gini means with arbitrary number of variables was established by Páles in [31, Theorem 3.1].

Theorem 4.3.13. *Let $k \in \mathbb{N}, k \geq 2$ and, for $i \in \{0, \dots, k\}$, $(r_i, s_i) \in \mathbb{R}^2$. Then the inequality*

$$G^{[n]}_{r_0,s_0}(x^1_1 + \cdots + x^k_1, \dots, x^1_n + \cdots + x^k_n) \leq G^{[n]}_{r_1,s_1}(x^1) + \cdots + G^{[n]}_{r_k,s_k}(x^k) \quad (4.44)$$

is valid for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^{n \times k}_+$ if and only if

(i) $0 \leq \min(r_1, s_1, \dots, r_k, s_k)$,

(ii) $\min(r_0, s_0) \leq \min(1, r_1, s_1, \dots, r_k, s_k)$,

(iii) $\max(1, r_0, s_0) \leq \min(\max(r_1, s_1), \dots, \max(r_k, s_k))$.

Remark 4.3.14. Observe that conditions (i) and (ii) of the above two theorems are identical, therefore they may be necessary for the global validity of (4.44) for any fixed $n \in \mathbb{N}$. The form of the third condition related to any fixed $n \in \mathbb{N}$ is not known. We also note that conditions (i)-(iii) of Theorem 4.3.13 are also necessary and sufficient for the validity of the inequality (4.42) on the domain indicated in Corollary 4.3.10.

Our next results characterize Hölder-type inequalities for Gini means in the local and in the global sense.

Theorem 4.3.15. *Let $n, k \in \mathbb{N}, k \geq 2$ and $\lambda \in \mathbb{R}_+^n$. Let $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$, $\gamma_i := r_i + s_i$ for $i \in \{0, \dots, k\}$. Then, in order that the inequality*

$$G^{[n]}_{-r_0,-s_0;\lambda}(x_1^1 \cdots x_1^k, \dots, x_n^1 \cdots x_n^k) \leq G^{[n]}_{r_1,s_1;\lambda}(x^1) \cdots G^{[n]}_{r_k,s_k;\lambda}(x^k) \tag{4.45}$$

be valid in the local sense on $\mathbb{R}_+^k$ it is necessary that either $\gamma_i \geq 0$ for all $i \in \{0, \dots, k\}$, or there exists $i \in \{0, \dots, k\}$ such that $\gamma_i < 0$, for all $j \in (\{0, \dots, k\}) \setminus \{i\}$, $\gamma_j > 0$ and

$$\frac{1}{\gamma_0} + \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_k} \leq 0 \tag{4.46}$$

holds. Conversely, if either $\gamma_i \geq 0$ for all $i \in \{0, \dots, k\}$ and $\gamma_i = 0$ can hold for at most one index $i \in \{0, \dots, k\}$, or there exists $i \in \{0, \dots, k\}$ such that $\gamma_i < 0$, for all $j \in \{1, \dots, k\} \setminus \{i\}$, $\gamma_j > 0$ and (4.46) is valid with a strict inequality, then (4.45) holds in the local sense on $\mathbb{R}_+^k$.

Proof. In the first step, we apply Theorem 4.3.4 with the function $\Phi\colon I \to \mathbb{R}_+$ defined by $\Phi(y) := y_1 \cdots y_k$. Then for the validity of (4.45) in the local sense it is necessary (and sufficient) that the values of the function $\Gamma\colon \mathbb{R}_+^k \to \mathbb{R}^{k \times k}$ defined by

$$\begin{aligned}\Gamma(y) &:= \left((\delta_{i,j} - 1)\frac{1}{y_i y_j}\prod_{\ell=1}^k y_\ell - (-\gamma_0 - 1)\frac{1}{y_i y_j}\prod_{\ell=1}^k y_\ell + \delta_{i,j}(\gamma_j - 1)\frac{1}{y_j^2}\prod_{\ell=1}^k y_\ell\right)_{i,j=1}^k \\ &= \left(\frac{1}{y_i y_j}\prod_{\ell=1}^k y_\ell\big(\delta_{i,j}\gamma_j + \gamma_0\big)\right)_{i,j=1}^k\end{aligned}$$

be positive semidefinite (positive definite) matrices for all $y \in I$. However, this property holds if and only if the scalar matrix

$$\Gamma^* := \big(\delta_{i,j}\gamma_j + \gamma_0\big)_{i,j=1}^k$$

is positive semidefinite (positive definite). The statement now follows from Lemma 4.3.6 with $c_i := \gamma_i \; \big(i \in \{0, \dots, k\}\big)$. □

Theorem 4.3.16. *Let $k \in \mathbb{N}, k \geq 2$ and, for $i \in \{0, \dots, k\}$, $(r_i, s_i) \in \mathbb{R}^2$. Assume that, for all $z_1 \in (I_1/I_1), \dots, z_k \in (I_k/I_k)$, the inequality*

$$\chi_{-r_0,-s_0}(z_1 \cdots z_k) \leq \sum_{j=1}^k \chi_{r_j,s_j}(z_j) \tag{4.47}$$

holds. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality (4.45) holds in the global sense on I.

Proof. With the function $\Phi\colon I \to \mathbb{R}_+$ defined by $\Phi(y_1,\dots,y_k) := y_1 \cdots y_k$, condition (4.26) turns out to be equivalent to

$$\chi_{-r_0,-s_0}\left(\frac{y_1\cdots y_k}{u_1\cdots u_k}\right) \le \sum_{j=1}^{k} \chi_{r_j,s_j}\left(\frac{y_j}{u_j}\right).$$

Introducing the new variables $z_i := y_i/u_i$ for $i \in \{1,\dots,k\}$, we can conclude that (4.47) is valid for all $z_1 \in (I_1/I_1),\dots,z_k \in (I_k/I_k)$ if and only if the above inequality holds for all $(y,u) \in (I_1 \times \cdots \times I_k)^2$. Hence the result follows from Theorem 4.3.5. □

The global validity of (4.45) with an unfixed number of variables was characterized by Páles in [32, 33].

Theorem 4.3.17. *Let $k \in \mathbb{N}, k \ge 2$ and, for $i \in \{0,\dots,k\}$, $(r_i,s_i) \in \mathbb{R}^2$. Then the inequality*

$$G^{[n]}_{-r_0,-s_0}(x_1^1\cdots x_1^k,\dots,x_n^1\cdots x_n^k) \le G^{[n]}_{r_1,s_1}(x^1)\cdots G^{[n]}_{r_k,s_k}(x^k)$$

is valid for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_+^{n\times k}$ if and only if

(i) for all $i \in \{0,\dots,k\}$, $\max(s_i,r_i) \ge 0$, and

(ii) for all $i \in \{0,\dots,k\}$ with $\min(s_i,r_i) < 0$, we have $\max(s_j,r_j) > 0$ for all $j \in (\{0,\dots,k\}) \setminus \{i\}$, and

$$\frac{1}{\min(s_i,r_i)} + \sum_{\substack{j=0\\ j\ne i}}^{k} \frac{1}{\max(s_j,r_j)} \le 0.$$

Remark 4.3.18. We note that the conditions (i) and (ii) are necessary and sufficient for the validity of inequality (4.47) for all $z_1,\dots,z_k \in \mathbb{R}_+$ (cf. [33]).

Summary

Summary of Chapter 1 – Basic notions and results

In this chapter, I stands for a nonempty open real interval and we introduce the definition of the generalized Bajraktarević mean and compute its partial derivatives up to third-order. The results are contained in our paper [12].

Definition of generalized Bajraktarević means

To extend the notion of Bajraktarević means, we shall need the following lemma about the existence and properties of the left inverse of strictly monotone (but not necessarily continuous) functions.

Proposition. *Let $f\colon I \to \mathbb{R}$ be a strictly monotone function. Then there exists a uniquely determined monotone function $g\colon \operatorname{conv}(f(I)) \to I$ such that g is the left inverse of f, i.e.,*

$$(g \circ f)(x) = x \qquad (x \in I). \tag{1}$$

Furthermore, g is monotone in the same sense as f, continuous,

$$(f \circ g)(y) = y \qquad (y \in f(I)), \tag{2}$$

and

$$\liminf_{x \to g(y)} f(x) \le y \le \limsup_{x \to g(y)} f(x) \qquad (y \in \operatorname{conv}(f(I))).$$

Thus, if f is lower (resp. upper) semicontinuous at $g(y)$, then $f \circ g(y) \le y$ (resp. $y \le f \circ g(y)$).

Definition. The function g described in the above lemma is called the *generalized left inverse of the strictly monotone function* $f\colon I \to \mathbb{R}$ and is denoted by $f^{(-1)}$.

Remark. From (1) and (2), we get that the restriction of $f^{(-1)}$ to $f(I)$ is the inverse of f in the standard sense. Therefore, $f^{(-1)}$ is the continuous and monotone extension of the inverse of f to the smallest interval containing the range of f.

A fairly general mean, which includes numerous important means as particular cases, is the quasi-arithmetic mean, which is defined as follows.

Definition. Let $n \in \mathbb{N}$. Given a strictly monotone, continuous function $f\colon I \to \mathbb{R}$, let us introduce the *n-variable quasi-arithmetic mean* $A_f^{[n]}\colon I^n \to I$ by the following formula

$$A_f^{[n]}(x) := f^{-1}\left(\frac{f(x_1) + \cdots + f(x_n)}{n}\right) \qquad (x \in I^n).$$

We say that the function f is the *generator of* $A_f^{[n]}$.

The following weighted extension of the quasi-arithmetic mean was introduced and investigated by Bajraktarević in [2] and [3].

Definition. Let $n \in \mathbb{N}$. Given a strictly monotone, continuous function $f\colon I \to \mathbb{R}$ and a positive valued function $p\colon I \to \mathbb{R}_+$, let us define the *$n$-variable Bajraktarević mean* $B_{f,p}^{[n]}\colon\ I^n \to I$ by

$$B_{f,p}^{[n]}(x) := f^{-1}\left(\frac{p(x_1)f(x_1) + \cdots + p(x_n)f(x_n)}{p(x_1) + \cdots + p(x_n)}\right) \qquad (x \in I^n). \tag{3}$$

We say that the function f is the *generator*, while p *is the weight function of* $B_{f,p}^{[n]}$.

Let us consider a possibly nonsymmetric generalization of the Bajraktarević mean.

Definition. Let $n \in \mathbb{N}$. Given a strictly monotone function $f\colon I \to \mathbb{R}$ and an n-tuple of positive valued functions $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$, we introduce the *$n$-variable generalized Bajraktarević mean* $A_{f,p}^{[n]}\colon I^n \to I$ by the following formula

$$A_{f,p}^{[n]}(x) := f^{(-1)}\left(\frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)}\right) \qquad (x \in I^n), \tag{4}$$

and, to simplify the notations, we will use the following definitions

$$R_{f,p}^{[n]}(x) := \frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)} \qquad \text{and} \qquad p_0 := p_1 + \cdots + p_n.$$

Similarly to the previous definitions, we say that the function f is the *generator*, while p *is the weight function of* $A_{f,p}^{[n]}$

Remark. We emphasize that $A_{f,p}^{[n]}$ is indeed a not necessarily symmetric generalization of the symmetric mean $B_{f,p}^{[n]}$, since, in the case when f is continuous, choosing all the weight functions to be the same, i.e., $p := p_1 = \cdots = p_n$ in (4), we get back (3).

Proposition. *Let $n \in \mathbb{N}$, $f\colon I \to \mathbb{R}$ be a strictly monotone function and $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$. Then the function $A_{f,p}^{[n]}\colon I^n \to I$ given by* (4) *is well-defined and it is a mean, i.e.,*

$$\min(x) \le A_{f,p}^{[n]}(x) \le \max(x) \qquad (x \in I^n).$$

Partial derivatives of generalized Bajraktarević means

In the next result we determine the partial derivatives of the generalized Bajraktarević means, up to third-order at the diagonal points of the domain under tight regularity assumptions. For instance, as stated below in assertions (1), (2b), (3c), for $m \in \mathbb{N}$, we prove the existence of partial derivatives of the form ∂_i^m only assuming $(m-1)$ times continuous differentiability of p_i. The partial derivatives play an essential role in the subsequent chapters as one of the tools that we use to prove the main theorems is solving the differential equations obtained by differentiating the functional equation on the diagonal of the domain in question.

Theorem. *Let $n \in \mathbb{N}, \ell \in \{1,2,3\}$, let $f\colon I \to \mathbb{R}$ be an ℓ times differentiable function with a nonvanishing first derivative, and let $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$. Then we have the following assertions.*

(1) If $\ell = 1$, $i \in \{1, \dots, n\}$, and p_i is continuous, then the first-order partial derivative $\partial_i A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ and

$$\left(\partial_i A_{f,p}^{[n]}\right)^{\Delta} = \frac{p_i}{p_0}.$$

(2a) If $\ell = 2$, $i, j \in \{1, \dots, n\}$ with $i \neq j$, furthermore, p_i and p_j are differentiable, then the second-order partial derivative $\partial_i \partial_j A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ and

$$\left(\partial_i \partial_j A_{f,p}^{[n]}\right)^{\Delta} = -\frac{(p_i p_j)'}{p_0^2} - \frac{p_i p_j}{p_0^2} \cdot \frac{f''}{f'}.$$

(2b) If $\ell = 2$, $i \in \{1, \dots, n\}$, and p_i is continuously differentiable, then the second-order partial derivative $\partial_i^2 A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ and

$$\left(\partial_i^2 A_{f,p}^{[n]}\right)^{\Delta} = 2\frac{p_i'(p_0 - p_i)}{p_0^2} + \frac{p_i(p_0 - p_i)}{p_0^2} \cdot \frac{f''}{f'}.$$

(3a) If $\ell = 3$, $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k \neq i$, furthermore, p_i, p_j, and p_k are differentiable, then the third-order partial derivative $\partial_i \partial_j \partial_k A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ and

$$\begin{aligned}\left(\partial_i \partial_j \partial_k A_{f,p}^{[n]}\right)^{\Delta} &= 2\frac{p_i p_j' p_k' + p_i' p_j p_k' + p_i' p_j' p_k}{p_0^3} + 2\frac{(p_i p_j p_k)'}{p_0^3} \cdot \frac{f''}{f'} \\ &\quad + \frac{p_i p_j p_k}{p_0^3}\left(3\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right).\end{aligned}$$

(3b) If $\ell = 3$, $i, j \in \{1, \dots, n\}$ with $i \neq j$, furthermore, p_i is twice differentiable and p_j is differentiable, then the third-order partial derivative $\partial_i^2 \partial_j A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ *and*

$$\left(\partial_i^2 \partial_j A_{f,p}^{[n]}\right)^\Delta = \frac{2p_i' p_j' (2p_i - p_0) + p_j (2(p_i')^2 - p_i'' p_0)}{p_0^3} + \frac{(2p_i' p_j + p_i p_j')(2p_i - p_0)}{p_0^3} \cdot \frac{f''}{f'} + \frac{p_i p_j}{p_0^3}\left((3p_i - p_0)\left(\frac{f''}{f'}\right)^2 - p_i \frac{f'''}{f'}\right).$$

(3c) If $\ell = 3$, $i \in \{1, \dots, n\}$ and p_i is twice continuously differentiable, then the third-order partial derivative $\partial_i^3 A_{f,p}^{[n]}$ exists on $\operatorname{diag}(I^n)$ *and*

$$\left(\partial_i^3 A_{f,p}^{[n]}\right)^\Delta = 3\frac{(p_0 - p_i)\big(p_0 p_i'' - 2(p_i')^2\big)}{p_0^3} + 3\frac{p_i'(p_0 - 2p_i)(p_0 - p_i)}{p_0^3} \cdot \frac{f''}{f'} - \frac{p_i(p_0 - p_i)}{p_0^3}\left(3p_i\left(\frac{f''}{f'}\right)^2 - (p_0 + p_i)\frac{f'''}{f'}\right).$$

Summary of Chapter 2 – Equality problem

The paper [12] contains the results of this chapter, in which we deal with the equality problem of n-variable generalized Bajraktarević means, i.e., we give necessary as well as sufficient conditions in terms of the unknown functions listed below for the functional equation

$$f^{(-1)}\left(\frac{p_1(x_1)f(x_1) + \dots + p_n(x_n)f(x_n)}{p_1(x_1) + \dots + p_n(x_n)}\right) = g^{(-1)}\left(\frac{q_1(x_1)g(x_1) + \dots + q_n(x_n)g(x_n)}{q_1(x_1) + \dots + q_n(x_n)}\right)$$

to be valid locally or globally, where $n \in \mathbb{N}, n \geq 2$ is fixed, I is a nonempty open real interval, which will be the case in this entire chapter, the unknown functions $f, g\colon I \to \mathbb{R}$ are strictly monotone, furthermore, $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$ are also unknown functions.

This problem in the symmetric 2-variable case, i.e., when $n = 2$ and $p_1 = p_2$, was already investigated and solved under sixth-order regularity assumptions by Losonczi in [22]. Later Páles and Zakaria reduced the severity of the differentiability assumptions and obtained the same result under first-order restrictions in their paper [38]. Apart from the aforementioned case, in this context, all other results in the chapter were newly discovered in our paper [12]. More precisely, in the nonsymmetric 2-variable case, assuming three times differentiability of f, g and the existence of $i \in \{1, 2\}$ such that either

p_i is twice continuously differentiable and p_{3-i} is continuous, or p_i is twice differentiable and p_{3-i} is once differentiable, we prove that the equality holds if and only if there exist four constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that

$$cf + d > 0, \qquad g = \frac{af + b}{cf + d}, \qquad \text{and} \qquad q_\ell = (cf + d)p_\ell \qquad (\ell \in \{1, \dots, n\}).$$

In the case $n \geq 3$, we obtain the same conclusion under weaker regularity assumptions. Namely, we suppose that f and g are three times differentiable, p is continuous and there exist $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k \neq i$ such that p_i, p_j, p_k are differentiable.

Sufficient conditions

Theorem. *Let $f\colon I \to \mathbb{R}$ be a strictly increasing function, $n \in \mathbb{N}$, and let further $p = (p_1, \dots, p_n)\colon\ I \to \mathbb{R}_+^n$. Then, for all $x \in I^n$, the equality $y = A_{f,p}^{[n]}(x)$ holds if and only if, for $z \in I$,*

$$\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i)) \begin{cases} < 0 & \text{if } z < y \\ > 0 & \text{if } z > y. \end{cases} \tag{5}$$

If f is strictly decreasing, then the inequalities in (5) *hold with reversed inequality sign.*

Corollary. *Let $f\colon I \to \mathbb{R}$ be continuous, strictly monotone, $n \in \mathbb{N}$, and let further $p = (p_1, \dots, p_n)\colon\ I \to \mathbb{R}_+^n$. Then, for all $x \in I^n$, the value $y = A_{f,p}^{[n]}(x)$ is the unique solution of the equation*

$$\sum_{i=1}^{n} p_i(x_i)(f(y) - f(x_i)) = 0.$$

The next result establishes a sufficient condition for the equality of the n-variable generalized Bajraktarević means. We will call this situation the *canonical case of the equality*.

Theorem. *Let $f, g\colon I \to \mathbb{R}$ be strictly monotone functions, $n \in \mathbb{N}$, and let further $p = (p_1, \dots, p_n)\colon\ I \to \mathbb{R}_+^n$, $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$. If there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that*

$$cf + d > 0, \qquad g = \frac{af + b}{cf + d}, \qquad \textit{and} \qquad q_i = (cf + d)p_i \qquad (i \in \{1, \dots, n\}) \tag{6}$$

hold on I, then $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ is globally valid.

Necessary conditions

With the aid of the following lemma, we can reduce the regularity assumptions in our statements.

Let $n \in \mathbb{N}$. For all $i \in \{1, \dots, n\}$, let $e_i \in \mathbb{R}^n$ denote the ith vector of the standard base of $\mathbb{R}^n$, i.e., let $e_i := (\delta_{i,j})_{j=1}^n$. Given two weight functions $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$, we will also use the following notation:

$$r_0 := \frac{q_0}{p_0} = \frac{q_1 + \dots + q_n}{p_1 + \dots + p_n}.$$

Lemma. *Let $f, g\colon I \to \mathbb{R}$ be continuous strictly monotone functions, $n \in \mathbb{N}, n \geq 2$, and $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$, $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$. Assume that $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ is locally valid. Then the following two assertions hold.*

(i) For all $i \in \{1, \dots, n\}$, the function p_i is continuous if and only if the function q_i is continuous.

(ii) Let $k \in \mathbb{N}$. Assume that $f, g\colon I \to \mathbb{R}$ are k times differentiable (resp. k times continuously differentiable) functions with nonvanishing first derivatives. Then, for all $i \in \{1, \dots, n\}$, the function p_i is k times differentiable (resp. k times continuously differentiable) if and only if q_i is k times differentiable (resp. k times continuously differentiable).

The following extension theorem is of basic importance since in the proofs of the main theorems, first we can state the validity of the canonical case of the equality only on a subinterval of the domain. Then the next main step of the proof is to extend the validity to the entire underlying interval.

Theorem. *Let $f, g\colon I \to \mathbb{R}$ be continuous, strictly monotone, $n \in \mathbb{N}$, let further $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ be a continuous function and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$. Assume that $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ is locally valid and that there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ and a nonempty open subinterval J of I such that (6) holds on J. Then q is continuous and (6) is also valid on I.*

The first- and second-order necessary conditions of the equality of two generalized Bajraktarević means are detailed in the following two lemmas, respectively.

Lemma. *Let $n \in \mathbb{N}, n \geq 2$ and $f, g\colon I \to \mathbb{R}$ be differentiable functions with nonvanishing first derivatives and $i \in \{1, \dots, n\}$. Let further $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$ such that p_i and q_i are continuous. If $\partial_i A_{f,p}^{[n]} = \partial_i A_{g,q}^{[n]}$ holds on $\operatorname{diag}(I^n)$, then*

$$\frac{q_i}{q_0} = \frac{p_i}{p_0} \tag{7}$$

holds on I.

Lemma. *Let $n \in \mathbb{N}, n \geq 2$ and $f, g\colon I \to \mathbb{R}$ be twice differentiable functions with nonvanishing first derivatives. Let $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}_+^n$ be continuous functions and assume that, for all $i \in \{1, \dots, n\}$, (7) holds on I. Let $j, k \in \{1, \dots, n\}$. Then the following two assertions hold.*

(i) Provided that $j \neq k$ and p_j, p_k, q_j, q_k are differentiable functions, if $\partial_j \partial_k A_{f,p}^{[n]} = \partial_j \partial_k A_{g,q}^{[n]}$ holds on $\operatorname{diag}(I^n)$, then there exists a nonzero constant γ such that, for all $i \in \{1, \dots, n\}$,

$$q_i^2 g' = \gamma p_i^2 f' \tag{8}$$

is valid on I.

(ii) Provided that $j = k$ and p_j, q_j are continuously differentiable functions, if $\partial_j^2 A_{f,p}^{[n]} = \partial_j^2 A_{g,q}^{[n]}$ holds on $\operatorname{diag}(I^n)$, then there exists a nonzero constant γ such that, for all $i \in \{1, \dots, n\}$, (8) is valid on I.

The following notion and a corresponding lemma play a basic role in the proofs of the subsequent results since, by solving the differential equations obtained by differentiating the functional equation at issue, one of the cases that we get is exactly the equality for which the next lemma gives a necessary condition under assuming only the regularity that is needed for the definition to be correct.

Definition. For a three times differentiable function $f\colon I \to \mathbb{R}$ with a nonvanishing first derivative, we introduce its *Schwarzian derivative* $S_f\colon I \to \mathbb{R}$ by the following formula:

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Lemma. *Let $f, g\colon I \to \mathbb{R}$ be three times differentiable functions with nonvanishing first derivatives. If $S_f = S_g$ is valid on I, then there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that $cf + d$ is positive on I and*

$$g = \frac{af + b}{cf + d}$$

holds on I.

Our first main result is contained in the following theorem. It completely characterizes the equality of two generalized Bajraktarević means with at least three variables.

Theorem. *Let $n \in \mathbb{N}, k \geq 3$ and $f, g\colon I \to \mathbb{R}$ be three times differentiable functions with nonvanishing first derivatives. Let $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$ be a continuous function and $q = (q_1, \dots, q_n) : I \to \mathbb{R}_+^n$. Assume that there exist $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k \neq i$ such that p_i, p_j, p_k are differentiable functions. Then the following assertions are equivalent.*

(i) $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ holds globally.

(ii) $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ *holds locally.*

(iii) The function q *is continuous, the functions* q_i, q_j, q_k *are differentiable, and the equalities*

$$\begin{aligned} \partial_\ell A_{f,p}^{[n]} &= \partial_\ell A_{g,q}^{[n]} \qquad (\ell \in \{1,\dots,n-1\}), \\ \partial_i \partial_j A_{f,p}^{[n]} &= \partial_i \partial_j A_{g,q}^{[n]}, \\ \partial_i \partial_j \partial_k A_{f,p}^{[n]} &= \partial_i \partial_j \partial_k A_{g,q}^{[n]} \end{aligned}$$

hold on $\operatorname{diag}(I^n)$.

(iv) There exist $a, b, c, d \in \mathbb{R}$ *with* $ad \neq bc$ *such that*

$$g = \frac{af+b}{cf+d} \qquad \text{and} \qquad q_\ell = (cf+d)p_\ell \qquad (\ell \in \{1,\dots,n\})$$

hold on I.

The second main theorem has two variants concerning the regularity assumptions and characterizes the equality of 2-variable generalized nonsymmetric Bajraktarević means.

Theorem. *Let* $f, g\colon I \to \mathbb{R}$ *be three times differentiable functions with nonvanishing first derivatives. Let* $p = (p_1, p_2)\colon I \to \mathbb{R}_+^2$ *and* $q = (q_1, q_2)\colon I \to \mathbb{R}_+^2$ *such that* $p_1 \neq p_2$. *Assume that there exists* $i \in \{1, 2\}$ *such that one of the following regularity conditions is satisfied.*

(a) p_i *is twice continuously differentiable and* p_{3-i} *is continuous.*

(b) p_i *is twice differentiable and* p_{3-i} *is once differentiable.*

Then the following assertions are pairwise equivalent.

(i) $A_{f,p}^{[2]} = A_{g,q}^{[2]}$ *holds globally.*

(ii) $A_{f,p}^{[2]} = A_{g,q}^{[2]}$ *holds locally.*

(iv) There exist $a, b, c, d \in \mathbb{R}$ *with* $ad \neq bc$ *such that*

$$g = \frac{af+b}{cf+d}, \qquad q_1 = (cf+d)p_1, \qquad \text{and} \qquad q_2 = (cf+d)p_2$$

hold on I.

In addition, if f *and* p *are* ℓ *times continuously differentiable, then* $A_{f,p}^{[2]}$ *is also* ℓ *times continuously differentiable.*

As we already mentioned, the symmetric 2-variable case was solved firstly under sixth- and then under first-order regularity assumptions in [22] and in [38], respectively.

Theorem. *Let $f, g\colon I \to \mathbb{R}$ be continuously differentiable functions with nonvanishing first derivatives. Let $p\colon I \to \mathbb{R}_+$ be a continuously differentiable function and $q\colon I \to \mathbb{R}_+$. Then the following assertions are equivalent.*

(i) $A^{[2]}_{f,(p,p)} = A^{[2]}_{g,(q,q)}$ *holds globally.*

(ii) $A^{[2]}_{f,(p,p)} = A^{[2]}_{g,(q,q)}$ *holds locally.*

(iii) The function q is three times differentiable and the equalities

(iv) Either there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that

$$g = \frac{af + b}{cf + d} \qquad \text{and} \qquad q = (cf + d)p$$

are valid on I or there exist two polynomials P and Q of at most second degree such that P and Q are positive on $f(I)$ and $g(I)$, respectively, and there exist two constants $\alpha, \beta \in \mathbb{R}$ such that

$$g = G^{-1} \circ (\alpha F \circ f + \beta), \qquad p = P^{-\frac{1}{2}} \circ f, \qquad \text{and} \qquad q = Q^{-\frac{1}{2}} \circ g$$

hold on I, where F and G denote a primitive function of $1/P$ and $1/Q$, respectively.

Summary of Chapter 3 – Comparison problem

The main goal of this chapter is to investigate the local and global comparison problem of two n-variable generalized Bajraktarević means, i.e., to find necessary as well as sufficient conditions in terms of the unknown functions listed below for the functional inequality

$$\begin{aligned} f^{(-1)}&\left(\frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)}\right) \\ &\leq g^{(-1)}\left(\frac{q_1(x_1)g(x_1) + \cdots + q_n(x_n)g(x_n)}{q_1(x_1) + \cdots + q_n(x_n)}\right) \end{aligned} \tag{9}$$

to be valid locally or globally, where $n \in \mathbb{N}, n \geq 2$ is fixed, I is a nonempty open real interval, which will be the case in this entire chapter unless otherwise stated, the unknown functions $f, g\colon I \to \mathbb{R}$ are strictly monotone, and $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}^n_+$ and $q = (q_1, \dots, q_n)\colon I \to \mathbb{R}^n_+$ are also unknown functions. The results of this section were first proved in our paper [13]. We note that, as we discussed in Chapter 2, the equality of these means in the local and in the global sense are equivalent properties. However, as we shall see, this is not the case for the comparison problem.

Concerning the global comparison problem, the main result of the chapter states that if f, g are differentiable functions with nonvanishing first derivatives and, for all $i \in \{1, \dots, n\}$,

$$\frac{p_i}{p_0} = \frac{q_i}{q_0} \qquad \text{and} \qquad \frac{p_0(x)(f(x) - f(y))}{p_0(y)f'(y)} \leq \frac{q_0(x)(g(x) - g(y))}{q_0(y)g'(y)} \qquad (x, y \in I)$$

are satisfied (where $p_0 := p_1 + \cdots + p_n$ and $q_0 := q_1 + \cdots + q_n$), then the above comparison inequality holds for all $x_1, \dots, x_n \in I$.

Local comparison

Definition. Let $n \in \mathbb{N}, n \geq 2$ and $U \in \mathbb{R}^n$ be a nonempty open set. A function $f\colon U \to \mathbb{R}$ is called *partially differentiable at $a \in U$ with respect to its ith variable* if the limit

$$\lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}$$

exists and finite. We say that f is *partially differentiable at $a \in U$* if it is partially differentiable with respect to all its variables. Finally, f is called *partially differentiable on U* if it is partially differentiable at each point of U.

In order to investigate inequality (9) in the local sense, we recall the following result from [36].

Theorem. *Let $n \in \mathbb{N}, n \geq 2$ and let $M, N\colon I^n \to I$ be n-variable means such that M is locally smaller than N. Assume that M and N are partially differentiable on* $\operatorname{diag}(I^n)$*. Then, for all $i \in \{1, \dots, n\}$,*

$$\partial_i M^\Delta = \partial_i N^\Delta. \tag{10}$$

If, in addition, M and N are twice differentiable on the diagonal $\operatorname{diag}(I^n)$*, then the symmetric $((n-1) \times (n-1))$-type matrix*

$$\big(\partial_i\partial_j N^\Delta - \partial_i\partial_j M^\Delta\big)_{i,j=1}^{n-1} \tag{11}$$

is positive semidefinite.

On the other hand, if (10) holds for all $i \in \{1, \dots, n\}$, furthermore, M and N are twice continuously differentiable on $\operatorname{diag}(I^n)$ *and the symmetric $((n-1) \times (n-1))$-type matrix given by (11) is positive definite, then M is locally smaller than N.*

Our first main result of this chapter establishes necessary and sufficient conditions for the local comparability of generalized Bajraktarević means.

Theorem. *Let $n \in \mathbb{N}, n \geq 2$ and let $f, g\colon I \to \mathbb{R}$ be differentiable functions with nonvanishing first derivatives and $p, q\colon I \to \mathbb{R}_+^n$ be continuous functions. Assume that $A_{f,p}^{[n]}$ is locally smaller than $A_{g,q}^{[n]}$. Then*

$$\frac{p_i}{p_0} = \frac{q_i}{q_0} \qquad (i \in \{1, \dots, n\}). \tag{12}$$

If, in addition, f, g are twice differentiable and p, q are continuously differentiable, then the function

$$\frac{q_0^2|g'|}{p_0^2|f'|} \tag{13}$$

is increasing.

On the other hand, if f, g and p, q are twice continuously differentiable, (12) *holds and the function* (13) *has a positive derivative, then $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{g,q}$.*

In what follows, we consider the local comparison problem of generalized Bajraktarević means when I is a nonempty open subinterval of $\mathbb{R}_+$ and the weight functions $p_1, \dots, p_n$ and $q_1, \dots, q_n$ are proportional to power functions.

Corollary. *Let $I \subseteq \mathbb{R}_+$ be a nonempty open interval and let $f, g\colon I \to \mathbb{R}$ be differentiable functions with nonvanishing first derivatives. Let $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}^n_+$, $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and define*

$$p_i(x) := \lambda_i x^{\alpha_i} \qquad \textit{and} \qquad q_i(x) := \mu_i x^{\beta_i} \qquad (i \in \{1, \dots, n\},\, x \in I). \tag{14}$$

Assume that $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{g,q}$. Then there exist $\gamma > 0$ and $\delta \in \mathbb{R}$ such that

$$\mu_i = \gamma \lambda_i \qquad \textit{and} \qquad \beta_i = \alpha_i + \delta \qquad (i \in \{1, \dots, n\}). \tag{15}$$

If, in addition, f and g are twice differentiable, then the map

$$x \mapsto x^{2\delta} \frac{|g'(x)|}{|f'(x)|} \tag{16}$$

is increasing on I.

On the other hand, if f, g are twice continuously differentiable, (15) *holds and the function* (16) *has a positive derivative, then $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{g,q}$.*

As an immediate consequence of the above corollary, we can characterize the local comparison of generalized power means.

Corollary. *Let $I \subseteq \mathbb{R}_+$ be a nonempty open interval and $a, b \in \mathbb{R}$. Define $f, g\colon I \to \mathbb{R}$ by*

$$f(x) := \begin{cases} x^a & \textit{if } a \neq 0 \\ \log(x) & \textit{if } a = 0 \end{cases} \qquad \textit{and} \qquad g(x) := \begin{cases} x^b & \textit{if } b \neq 0 \\ \log(x) & \textit{if } b = 0. \end{cases} \tag{17}$$

Let $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}^n_+$, $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and define $p, q\colon I \to \mathbb{R}^n_+$ by (14)*. If $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{f,p}$ then there exist $\gamma > 0$ and $\delta \in \mathbb{R}$ such that* (15) *holds and $a \leq b + 2\delta$.*

On the other hand, if (15) *holds and $a < b + 2\delta$, then $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{g,q}$.*

Global comparison

If $A^{[n]}_{f,p}$ is globally smaller than $A^{[n]}_{g,q}$, then $A^{[n]}_{f,p}$ is locally smaller than $A^{[n]}_{g,q}$ and hence the necessity of (12) follows and the function (13) must be increasing. However, these conditions are not strong enough to imply the global comparability of the aforementioned means. The first theorem of this subsection deals with how the condition (12) needs to be augmented to yield global comparability.

Theorem. *Let $f, g\colon I \to \mathbb{R}$ be strictly monotone, differentiable functions with nonvanishing first derivatives, $n \in \mathbb{N}, n \geq 2$, and $p, q\colon I \to \mathbb{R}^n_+$. Assume that* (12) *holds and*

$$\frac{p_0(x)(f(x) - f(y))}{p_0(y)f'(y)} \leq \frac{q_0(x)(g(x) - g(y))}{q_0(y)g'(y)} \qquad (x, y \in I). \tag{18}$$

Then $A^{[n]}_{f,p}$ is globally smaller than $A^{[n]}_{g,q}$.

Our second result discusses how the condition on the increasingness of the function (13) needs to be tightened to imply global comparability. It turns out that it is sufficient if all the multiplicative factors of the function (13) are increasing.

Theorem. *Let $f, g\colon I \to \mathbb{R}$ be strictly monotone differentiable functions with nonvanishing first derivatives, $n \in \mathbb{N}, n \geq 2$, and $p, q\colon I \to \mathbb{R}^n_+$. Assume that* (12) *holds and the functions*

$$\frac{q_0}{p_0} \qquad \text{and} \qquad \frac{|g'|}{|f'|}$$

are increasing. Then $A^{[n]}_{f,p}$ is globally smaller than $A^{[n]}_{g,q}$.

We note that the sufficient condition (18) implies that the function (13) is increasing. In general, the converse is not valid, however, in what follows, we present two particular cases when it is. In the first setting, the weight functions p and q coincide.

Theorem. *Let $f, g\colon I \to \mathbb{R}$ be strictly monotone twice continuously differentiable functions with nonvanishing first derivatives, $n \in \mathbb{N}, n \geq 2$, and let further $p\colon I \to \mathbb{R}^n_+$ be a continuous function. Then the following conditions are pairwise equivalent.*

(i) $A^{[n]}_{f,p}$ *is globally smaller than* $A^{[n]}_{g,p}$.

(ii) $A^{[n]}_{f,p}$ *is locally smaller than* $A^{[n]}_{g,p}$.

(iii) The function $|g'/f'|$ is increasing.

(iv) The following inequality is valid on I

$$\frac{f''}{f'} \leq \frac{g''}{g'}.$$

(v) *Provided that g is increasing (decreasing), the function $g \circ f^{-1}$ is convex (concave) on $f(I)$;*

(vi)

$$\frac{f(x)-f(y)}{f'(y)} \leq \frac{g(x)-g(y)}{g'(y)} \qquad (x, y \in I).$$

In the second setting, the functions f and g are the same.

Theorem. *Let $f\colon I \to \mathbb{R}$ be strictly monotone twice continuously differentiable function with a nonvanishing first derivative, $n \in \mathbb{N}, n \geq 2$, and let further $p, q\colon I \to \mathbb{R}_+^n$ be continuous functions. Then the following conditions are pairwise equivalent.*

(i) $A_{f,p}^{[n]}$ *is globally smaller than* $A_{f,q}^{[n]}$.

(ii) $A_{f,p}^{[n]}$ *is locally smaller than* $A_{f,q}^{[n]}$.

(iii) *The condition* (12) *holds and the function q_0/p_0 is increasing.*

Corollary. *Let $I \subseteq \mathbb{R}_+$ be a nonempty open interval and let $f, g\colon I \to \mathbb{R}$ be differentiable functions with nonvanishing first derivatives. Let $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$, $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and define $p, q\colon I \to \mathbb{R}_+^n$ by* (14). *Assume that there exist $\gamma > 0$ and $\delta \in \mathbb{R}$ such that* (15) *is satisfied and*

$$\frac{f(x)-f(y)}{f'(y)} \leq \frac{x^\delta(g(x)-g(y))}{y^\delta g'(y)} \qquad (x, y \in I).$$

Then $A_{f,p}^{[n]}$ *is globally smaller than* $A_{g,q}^{[n]}$.

Corollary. *Let $I \subseteq \mathbb{R}_+$ be a nonempty open interval, $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$, $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$, and let further $a, b \in \mathbb{R}$. Define $p, q\colon I \to \mathbb{R}_+^n$ by* (14) *and $f, g\colon I \to \mathbb{R}$ by* (17). *Assume that there exist $\gamma > 0$ and $\delta \in \mathbb{R}$ such that* (15) *and the inequalities*

$$\min(a, 0) \leq \delta + \min(b, 0) \qquad \text{and} \qquad \max(a, 0) \leq \delta + \max(b, 0)$$

are satisfied. Then $A_{f,p}^{[n]}$ *is globally smaller than* $A_{g,q}^{[n]}$.

Summary of Chapter 4 – Local and global Hölder- and Minkowski-type inequalities

The celebrated inequalities discovered by Hölder and Minkowski can be formulated in various contexts, for instance, in the setting of power means.

To recall the standard Hölder(–Rogers) inequality, which was discovered by Rogers in 1888 and by Hölder in 1889, let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}_+^n$, the inequality

$$\frac{x_1 y_1 + \cdots + x_n y_n}{n} \leq \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \left(\frac{y_1^q + \cdots + y_n^q}{n}\right)^{\frac{1}{q}}$$

is valid. In the particular case $p = q = 2$, this inequality reduces to the so-called Cauchy–Bunyakovsky–Schwarz inequality, which in the above form was established by Cauchy in 1821. Given a real parameter $p \geq 1$, the standard Minkowski inequality, that was discovered in 1910, states that the pth power mean is subadditive, i.e., for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}_+^n$, the inequality

$$\left(\frac{(x_1 + y_1)^p + \cdots + (x_n + y_n)^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} + \left(\frac{y_1^p + \cdots + y_n^p}{n}\right)^{\frac{1}{p}}$$

holds.

Briefly, the aim of this chapter is to investigate inequalities that are analogous to the Hölder and Minkowski inequalities by replacing the addition and the multiplication by a more general operation, and instead of using power means, generalized Bajraktarević means, and in particular, weighted Gini means, are considered. A further aim is to characterize such inequalities both in the local and in the global sense. We are going to derive necessary as well as sufficient conditions for the local as well as for the global validity of the functional inequality

$$M_0(\Phi(x_1), \ldots, \Phi(x_n)) \leq \Phi(M_1(x^1), \ldots, M_k(x^k)), \tag{19}$$

where, $n \in \mathbb{N}, n \geq 2, k \in \mathbb{N}$, for $\alpha \in \{0, \ldots, k\}$, $I_\alpha \subseteq \mathbb{R}$ is a nonempty open interval, $I := I_1 \times \cdots \times I_k$, $M_\alpha \colon I_\alpha^n \to I_\alpha$ is an n-variable mean and $\Phi \colon I \to I_0$. In line with the terminology used in preceding chapters, if there exists an open set $U \subseteq I^n$ such that $\operatorname{diag}(I^n) \subseteq U$ and (19) holds for all $x \in U^T \subseteq \prod_{\alpha=1}^k I_\alpha^n$, then we say that (19) *holds in the local sense*. If (19) is valid for all $x \in (I^n)^T = \prod_{\alpha=1}^k I_\alpha^n$, then we say that (19) *holds in the global sense*. Clearly, the global validity of (19) implies its local validity.

After that we consider the particular case of (19) when all the means are n-variable generalized Bajraktarević means, i.e., we consider the inequality

$$A_{f_0,p^0}^{[n]}(\Phi(x_1), \ldots, \Phi(x_n)) \leq \Phi(A_{f_1,p^1}^{[n]}(x^1), \ldots, A_{f_k,p^k}^{[n]}(x^k)), \tag{20}$$

where, for $\alpha \in \{0, \ldots, k\}$, $f_\alpha \colon I_\alpha \to \mathbb{R}$ is a strictly monotone function, $p^\alpha \colon I_\alpha \to \mathbb{R}_+^n$. We obtain necessary as well as sufficient conditions for its validity in the local and also in the global sense.

We mention some important particular cases of (20).

1. If $k = 1$, $I_0 = I_1 =: J$ and $\Phi(x) = x$, then (20) reduces to the local and global comparison problem of generalized Bajraktarević means on J.

2. If $k \in \mathbb{N}, k \geq 2$, $I_0 = I_1 = \cdots = I_k =: J$, $\Phi(x_1, \ldots, x_k) = \frac{1}{k}(x_1 + \cdots + x_k)$, and $f_0 = f_1 = \cdots = f_k =: f$, $p^0 = p^1 = \cdots = p^k =: p$, then (20) means the Jensen convexity of $A_{f,p}^{[n]}$ on J. In this case, (20) is said to be a Jensen-type inequality.

3. If $k \in \mathbb{N}, k \geq 2$, $I_0 = I_1 = \cdots = I_k = \mathbb{R}_+$, $\Phi(x_1, \ldots, x_k) = x_1 + \cdots + x_k$, and $f_0 = f_1 = \cdots = f_k =: f$, $p^0 = p^1 = \cdots = p^k =: p$, then (20) expresses the subadditivity of $A_{f,p}^{[n]}$ on $\mathbb{R}_+$, which is often called a Minkowski-type inequality.

4. If $k \in \mathbb{N}, k \geq 2$, $I_0 = I_1 = \cdots = I_k = \mathbb{R}_+$, $\Phi(x_1, \ldots, x_k) = x_1 \cdots x_k$, then (20) reduces to a Hölder-type inequality for the means $A_{f_0,p^0}^{[n]}, A_{f_1,p^1}^{[n]}, \ldots, A_{f_k,p^k}^{[n]}$.

With the exception three theorems, all the results of this chapter were first proved in our paper [14]. Two of three theorems deals with the characterization of the global validity of the Minkowski-type inequality for Gini means in the 2-variable case and with unfixed number of variables. The third one gives a necessary and sufficient system of conditions for the global validity of the Hölder-type inequality for Gini means with unfixed number of variables. Each of these three theorems is preceded by an emphasis on the author(s) and publications in which the result is found.

Hölder- and Minkowski-type inequalities in the local sense

In the current and in the next subsection, for $k \in \mathbb{N}$ and $\alpha \in \{0, \ldots, k\}$, let $I_\alpha \subseteq \mathbb{R}$ stand for a nonempty open interval and set $I := I_1 \times \cdots \times I_k$. For the investigation of inequality (19), let us introduce the function $F\colon I_1^n \times \cdots \times I_k^n \to \mathbb{R}$ by

$$F(x) = F(x^1, \ldots, x^k) := \Phi(M_1(x^1), \ldots, M_k(x^k)) - M_0(\Phi(x_1), \ldots, \Phi(x_n)). \quad (21)$$

Remark. Observe that, for all $y \in I$, we have $F(\Delta_n^k(y)) = 0$. Indeed, by using the mean value property of $M_0, M_1, \ldots, M_k$, it follows that

$$\begin{aligned} F(\Delta_n^k(y)) &= F(\Delta_n(y_1), \ldots, \Delta_n(y_k)) \\ &= \Phi(M_1(\Delta_n(y_1)), \ldots, M_k(\Delta_n(y_k))) - M_0(\Delta_n(\Phi(y))) \\ &= \Phi(M_1^\Delta(y_1), \ldots, M_k^\Delta(y_k)) - M_0^\Delta(\Phi(y)) = \Phi(y_1, \ldots, y_k) - \Phi(y) = 0. \end{aligned}$$

For the computation of the partial derivatives of F at points of the form $\Delta_n^k(y)$, we formulate the following lemma.

Lemma. *Let $n, k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0, \ldots, k\}$, $M_\alpha\colon I_\alpha^n \to I_\alpha$ be an n-variable mean, define $F\colon I_1^n \times \cdots \times I_k^n \to \mathbb{R}$ by (21), and let $\Phi\colon I \to I_0$.*

(i) For $\alpha \in \{0, \ldots, k\}$, assume that M_α is partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and that Φ is differentiable. Then, for all $i \in \{1, \ldots, k\}$, $\ell \in \{1, \ldots, n\}$, and $y \in I$,

$$\partial_{\ell+n(i-1)} F(\Delta_n^k(y)) = \partial_i \Phi(y)\big(\partial_\ell M_i^\Delta(y_i) - \partial_\ell M_0^\Delta(\Phi(y))\big).$$

(ii) For $\alpha \in \{0,\dots,k\}$, assume that M_α is twice partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and that Φ is twice differentiable. Then, for all $i,j \in \{1,\dots,k\}$, $\ell,m \in \{1,\dots,n\}$, and $y \in I$,

$$\begin{aligned}&\partial_{\ell+n(i-1)}\partial_{m+n(j-1)}F(\Delta_n^k(y))\\ &\qquad= \partial_i\partial_j\Phi(y)\big(\partial_m M_j^\Delta(y_j)\partial_\ell M_i^\Delta(y_i) - \delta_{\ell,m}\partial_m M_0^\Delta(\Phi(y))\big)\\ &\qquad\quad - \partial_j\Phi(y)\big(\partial_i\Phi(y)\partial_\ell\partial_m M_0^\Delta(\Phi(y)) - \delta_{i,j}\partial_\ell\partial_m M_j^\Delta(y_j)\big).\end{aligned}$$

The next two results describe the first- and second-order necessary conditions for the validity of (19) in the local sense, respectively.

Theorem. *Let $n,k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0,\dots,k\}$, $M_\alpha\colon I_\alpha^n \to I_\alpha$ be an n-variable mean which is partially differentiable on $\operatorname{diag}(I_\alpha^n)$ and let further $\Phi\colon I \to I_0$ be a surjective and differentiable function with nonvanishing first-order partial derivatives. Assume that inequality (19) holds in the local sense. Then there exist constants $\lambda_1,\dots,\lambda_n \in \mathbb{R}_+$ such that, for all $(y_0,y) \in I_0 \times I$ and $\ell \in \{1,\dots,n\}$,*

$$\lambda_\ell = \partial_\ell M_0^\Delta(y_0) = \partial_\ell M_1^\Delta(y_1) = \cdots = \partial_\ell M_k^\Delta(y_k). \tag{22}$$

If, additionally, for some $\alpha \in \{0,\dots,k\}$ and $y_\alpha \in I_\alpha$, the mean M_α is differentiable at $\Delta_n(y_\alpha)$, then $\lambda_1 + \cdots + \lambda_n = 1$ also holds.

Theorem. *Let $n,k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0,\dots,k\}$, $M_\alpha\colon I_\alpha^n \to I$ be an n-variable mean which is twice differentiable on $\operatorname{diag}(I_\alpha^n)$ and let $\Phi\colon I \to I_0$ be surjective and twice differentiable with nonvanishing first-order partial derivatives. Assume that inequality (19) holds in the local sense. Then there exist constants $\lambda_1,\dots,\lambda_n \in \mathbb{R}_+$ with $\lambda_1+\cdots+\lambda_n = 1$ such that, for all $(y_0,y) \in I_0 \times I$ and $\ell \in \{1,\dots,n\}$, the equalities in (22) hold. In addition, for all $y \in I$, the $((nk)\times(nk))$-type matrix whose $(\ell+n(i-1), m+n(j-1))$th entry, where $i,j \in \{1,\dots,k\}$ and $\ell,m \in \{1,\dots,n\}$, is given by*

$$\begin{aligned}&\partial_i\partial_j\Phi(y)(\lambda_m\lambda_\ell - \delta_{\ell,m}\lambda_m) - \partial_i\Phi(y)\partial_j\Phi(y)\partial_\ell\partial_m M_0^\Delta(\Phi(y))\\ &\qquad\qquad + \delta_{i,j}\partial_j\Phi(y)\partial_\ell\partial_m M_j^\Delta(y_j)\end{aligned}$$

is positive semidefinite.

Hölder- and Minkowski-type inequalities for generalized Bajraktarević means

Under first- as well as second-order differentiability condition, the following theorem describes what necessarily holds if (20) is satisfied in the local sense.

Theorem. *Let $n,k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0,\dots,k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be a differentiable function with a nonvanishing first derivative and let $p^\alpha = (p_1^\alpha,\dots,p_n^\alpha)\colon I_\alpha \to \mathbb{R}_+^n$ be* *continuous, and set $p_0^\alpha := p_1^\alpha + \cdots + p_n^\alpha$. Let $\Phi\colon I \to I_0$ be surjective and differentiable with nonvanishing first-order partial derivatives. Assume that inequality (20) holds in the local sense. Then there exist constants $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$ with $\lambda_1 + \cdots + \lambda_n = 1$ such that, for all $\alpha \in \{0, \ldots, k\}$ and $\ell \in \{1, \ldots, n\}$,*

$$p_\ell^\alpha = \lambda_\ell p_0^\alpha \tag{23}$$

holds on I_α. If, additionally, for $\alpha \in \{0, \ldots, k\}$, f_α is twice differentiable, p^α is continuously differentiable and Φ is twice differentiable, then the $(k \times k)$-type matrix $\Gamma(y)$ given by

$$\begin{aligned}\Gamma(y) := \bigg(- \partial_i\partial_j\Phi(y) - \partial_j\Phi(y)\partial_i\Phi(y)\bigg(2\frac{(p_0^0)'}{p_0^0} + \frac{f_0''}{f_0'}\bigg)(\Phi(y)) \\ + \delta_{i,j}\partial_j\Phi(y)\bigg(2\frac{(p_0^j)'}{p_0^j} + \frac{f_j''}{f_j'}\bigg)(y_j)\bigg)_{i,j=1}^k\end{aligned} \tag{24}$$

is positive semidefinite for all $y \in I$.

In the next result, we reformulate the positive semidefiniteness condition from the above theorem in terms of a convexity property.

Theorem. *Let $n, k \in \mathbb{N}, k \geq 2$ and, for $\alpha \in \{0, \ldots, k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be a twice differentiable function with a nonvanishing first derivative and let $p^\alpha = (p_1^\alpha, \ldots, p_n^\alpha)\colon I_\alpha \to \mathbb{R}_+^n$ be continuously differentiable, set $p_0^\alpha := p_1^\alpha + \cdots + p_n^\alpha$. Let $\Phi\colon I \to I_0$ be surjective and twice differentiable with nonvanishing first-order partial derivatives. Assume that inequality (20) holds in the local sense. Finally, for $\alpha \in \{0, \ldots, k\}$, define the function $\varphi_\alpha\colon I_\alpha \to \mathbb{R}$ and then $\varphi\colon I \to \mathbb{R}^k$ by*

$$\varphi_\alpha := \int (p_0^\alpha)^2 f_\alpha' \qquad \textit{and} \qquad \varphi(y) := (\varphi_1(y_1), \ldots, \varphi_k(y_k)).$$

Then $\varphi_1, \ldots, \varphi_k$ and φ are twice differentiable and invertible functions and the map $\Psi\colon \varphi(I) \to \mathbb{R}$ defined by

$$\Psi(u) := \varphi_0(\Phi(\varphi^{-1}(u)))$$

is concave if $f_0' > 0$ and convex if $f_0' < 0$.

Remark. We note that the concavity of the auxiliary function Ψ is not merely a consequence of the positive semidefiniteness of the matrix-valued function Γ but, in fact, it is equivalent to it. On the other hand, if all the weight functions are equal to constant 1, then $\varphi_\alpha = f_\alpha$ and, in this case, according to the theory of quasiarithmetic means (see [15]), the concavity of the function Ψ is also sufficient for inequality (20) to be valid in the global sense.

The following two theorems establish sufficient conditions for inequality (20) to be valid in the local as well as in the global sense.

Theorem. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0,\dots,k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be differentiable with a nonvanishing derivative, and $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$. Furthermore, let $\Phi\colon I \to I_0$ be partially differentiable. Assume that the following inequality is valid in the local sense on I^2, that is, there exists a nonempty open set $V \subseteq I^2$ containing $\operatorname{diag}(I^2)$ such that, for all $(u,y) \in V$,*

$$\frac{p_0^0(\Phi(y))(f_0(\Phi(y)) - f_0(\Phi(u)))}{p_0^0(\Phi(u))f_0'(\Phi(u))} \le \sum_{j=1}^{k} \partial_j\Phi(u)\frac{p_0^j(y_j)(f_j(y_j) - f_j(u_j))}{p_0^j(u_j)f_j'(u_j)} \tag{25}$$

holds. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality

$$A^{[n]}_{f_0,p_0^0\lambda}(\Phi(x_1),\dots,\Phi(x_n)) \le \Phi(A^{[n]}_{f_1,p_0^1\lambda}(x^1),\dots,A^{[n]}_{f_k,p_0^k\lambda}(x^k)) \tag{26}$$

is valid in the local sense.

Remark. Observe that the weight functions of the generalized Bajraktarević means appearing in (20) are necessarily of the form given by (23). Therefore, the local as well as the global validity of (20) immediately follows from the local as well as the global validity of (26), respectively.

Theorem. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0,\dots,k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be differentiable with a nonvanishing derivative, $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ and let $\Phi\colon I \to I_0$ be partially differentiable. Assume that (25) is satisfied in the global sense on I^2, that is, for all $u, y \in I$. Then, for all $n \in \mathbb{N}$, $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, inequality (26) holds in the global sense.*

The next result establishes a necessary condition for (25) to be satisfied in the local sense.

Theorem. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0,\dots,k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be twice differentiable with a nonvanishing first derivative and $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ be twice differentiable. In addition, let $\Phi\colon I \to I_0$ be twice differentiable. Assume that inequality (25) is satisfied in the local sense, that is, there exists an open set $V \subseteq I^2$ with $\operatorname{diag}(I^2) \subseteq V$ such that it holds true for all $(u,y) \in V$. Then the matrix-valued function $\Gamma\colon I \to \mathbb{R}^{k\times k}$ defined by (24) takes positive semidefinite values.*

The converse of the above statement is generally not valid. The conditions must be tightened, namely, positive definiteness must be assumed instead of positive semidefiniteness.

Theorem. *Let $k \in \mathbb{N}$ and, for $\alpha \in \{0,\dots,k\}$, $f_\alpha\colon I_\alpha \to \mathbb{R}$ be twice continuously differentiable with a nonvanishing first derivative, $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ be twice continuously differentiable. Furthermore, let $\Phi\colon I \to I_0$ be twice continuously differentiable. Assume, for all $y \in I$, that the $(k \times k)$-type matrix $\Gamma(y)$ defined by (24) is positive definite. Then, for all $n \in \mathbb{N}$, $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, inequality (26) holds in the local sense.*

Hölder- and Minkowski-type inequalities for weighted Gini means

In this section, for $k \in \mathbb{N}, \alpha \in \{1, \dots, k\}$, let I_α denote a nonempty open subinterval of $\mathbb{R}_+$, let $I := I_1 \times \cdots \times I_k$, and we apply the above results to some important particular cases of the inequality (19). First, we deal with inequalities obtained by specializing the means M_α in (19) for all $\alpha \in \{0, \dots, k\}$ and then we draw some conclusions by choosing the function Φ in (19) to be the k-variable addition and multiplication.

Definition. Let $n \in \mathbb{N}$ and, for $r \in \mathbb{R}$, let us recall the definition of the *n-variable weighted rth power or rth Hölder mean with weight vector* $\lambda \in \mathbb{R}_+^n, \lambda_1 + \cdots + \lambda_n = 1$ and, for $(r, s) \in \mathbb{R}^2$, the *n-variable weighted Gini mean with pair of parameters* $(r, s) \in \mathbb{R}^2$ *and weight vector* $\lambda \in \mathbb{R}_+^n, \lambda_1 + \cdots + \lambda_n = 1$:

$$H_{r;\lambda}^{[n]}(x_1, \dots, x_n) := \begin{cases} \left(\lambda_1 x_1^r + \cdots + \lambda_n x_n^r\right)^{\frac{1}{r}} & \text{if } r \neq 0 \\ x_1^{\lambda_1} \cdots x_n^{\lambda_n} & \text{if } r = 0, \end{cases}$$

$$G_{r,s;\lambda}^{[n]}(x_1, \dots, x_n) := \begin{cases} \left(\dfrac{\lambda_1 x_1^r + \cdots + \lambda_n x_n^r}{\lambda_1 x_1^s + \cdots + \lambda_n x_n^s}\right)^{\frac{1}{r-s}} & \text{if } r \neq s \\ \exp\left(\dfrac{\lambda_1 x_1^r \ln(x_1) + \cdots + \lambda_n x_n^r \ln(x_n)}{\lambda_1 x_1^r + \cdots + \lambda_n x_n^r}\right) & \text{if } r = s. \end{cases}$$

Remark. It is clear that in the particular case $s = 0$, the mean $G_{r,s;\lambda}^{[n]}$ simplifies to $H_{r;\lambda}^{[n]}$. We also note that, for $n \in \mathbb{N}$ and $\ell \in \{1, \dots, n\}$, with $p_\ell(t) := \lambda_\ell t^s$ and $f(t) := t^{r-s}$ if $r \neq s$ or $f(t) := \ln(t)$ if $r = s$, we can see that $A_{f,p}^{[n]} = G_{r,s;\lambda}^{[n]}$.

The next result consists of a necessary as well as a sufficient condition for a general inequality corresponding to weighted Gini means to be valid in the local sense.

Theorem. *Let $n, k \in \mathbb{N}, k \geq 2$, $\lambda \in \mathbb{R}_+^n$, and, for $\alpha \in \{0, \dots, k\}$, $(r_\alpha, s_\alpha) \in \mathbb{R}^2$ and $\Phi \colon I \to \mathbb{R}_+$ be twice differentiable with nonvanishing first-order partial derivatives. Then, for the inequality*

$$G_{r_0,s_0;\lambda}^{[n]}(\Phi(x_1), \dots, \Phi(x_n)) \leq \Phi\big(G_{r_1,s_1;\lambda}^{[n]}(x^1), \dots, G_{r_k,s_k;\lambda}^{[n]}(x^k)\big) \tag{27}$$

to be valid in the local sense it is necessary that the $(k \times k)$-type matrix $\Gamma(y)$ given by

$$\Gamma(y) := \left(-\partial_i \partial_j \Phi(y) - \partial_j \Phi(y) \partial_i \Phi(y) \frac{r_0 + s_0 - 1}{\Phi(y)} + \delta_{i,j} \partial_j \Phi(y) \frac{r_j + s_j - 1}{y_j} \right)_{i,j=1}^k$$

be positive semidefinite for all $y \in I$. Conversely, if this matrix is positive definite for all $y \in I$, then (27) holds in the local sense on I.

The theorem below gives a sufficient condition for the inequality (27) to be valid in the global sense.

Theorem. *Let $k \in \mathbb{N}, k \geq 2$, for $\alpha \in \{0, \dots, k\}$, $(r_\alpha, s_\alpha) \in \mathbb{R}^2$ and let $\Phi \colon I \to \mathbb{R}_+$ be partially differentiable. Assume that the inequality*

$$\Phi(u)\chi_{r_0,s_0}\left(\frac{\Phi(y)}{\Phi(u)}\right) \leq \sum_{j=1}^{k} \partial_j \Phi(u) u_j \chi_{r_j,s_j}\left(\frac{y_j}{u_j}\right)$$

is valid in the global sense, that is, for all $(u, y) \in I^2$. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}^n_+$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality (27) holds in the global sense on I.

For the investigation of the particular cases when Φ is the sum and the product function, we will need the following auxiliary result.

Lemma. *Let $k \in \mathbb{N}$, for $i \in \{0, \dots, k\}$, $c_i \in \mathbb{R}$. Then the matrix*

$$C := \big(\delta_{i,j} c_i + c_0\big)_{i,j=1}^{k}$$

is positive semidefinite if and only if either $c_i \geq 0$ for all $i \in \{0, \dots, k\}$, or there exists $i \in \{0, \dots, k\}$ such that $c_i < 0$ and $c_j > 0$ for all $j \in \{0, \dots, k\} \setminus \{i\}$ and

$$\frac{1}{c_0} + \frac{1}{c_1} + \cdots + \frac{1}{c_k} \leq 0. \tag{28}$$

Furthermore, C is positive definite if and only if either $c_i \geq 0$ for all $i \in \{0, \dots, k\}$ and $c_i = 0$ can hold for at most one index $i \in \{0, \dots, k\}$ or there exists $i \in \{0, \dots, k\}$ such that $c_i < 0$ and $c_j > 0$ for all $j \in \{0, \dots, k\} \setminus \{i\}$ and (28) is valid with a strict inequality.

Our next result characterizes the Minkowski-type inequality for weighted Gini means in the local sense.

Theorem. *Let $n, k \in \mathbb{N}, k \geq 2$ and $\lambda \in \mathbb{R}^n_+$. For $i \in \{0, \dots, k\}$, let further $(r_i, s_i) \in \mathbb{R}^2$ and $\gamma_i := r_i + s_i - 1$. For the inequality*

$$G^{[n]}_{r_0,s_0;\lambda}(x_1^1 + \cdots + x_1^k, \dots, x_n^1 + \cdots + x_n^k) \leq G^{[n]}_{r_1,s_1;\lambda}(x^1) + \cdots + G^{[n]}_{r_k,s_k;\lambda}(x^k) \tag{29}$$

to hold in the local sense on I, it is necessary that exactly one of the following cases be valid:

(i)

$$\gamma_0 \leq 0 \leq \min(\gamma_1, \dots, \gamma_k); \tag{30}$$

(ii) $\gamma_i > 0$ for all $i \in \{0, \dots, k\}$ and

$$\sum_{i \in J_+} \left(\frac{1}{\gamma_i} - \frac{1}{\gamma_0}\right) \sup I_i \leq \sum_{i \in J_-} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i}\right) \inf I_i; \tag{31}$$

(iii) $\gamma_0 < 0$ *and there exists* $i \in \{1, \dots, k\}$ *such that* $\gamma_i < 0$*, for all* $j \in \{1, \dots, k\} \setminus \{i\}$*,* $\gamma_j > 0$*, and inequality* (31) *is also valid,*

where, for the last two cases, we define

$$J_+ := \left\{ i \in \{1, \dots, k\} \mid \frac{1}{\gamma_i} > \frac{1}{\gamma_0} \right\} \qquad \text{and} \qquad J_- := \left\{ i \in \{1, \dots, k\} \colon \frac{1}{\gamma_0} > \frac{1}{\gamma_i} \right\}.$$

Conversely, if either (30) *is valid and* $\gamma_i = 0$ *can hold for at most one* $i \in \{0, \dots, k\}$*, or* $\gamma_0 \neq \gamma_\ell$ *for some* $\ell \in \{1, \dots, k\}$ *and one of the conditions (ii) or (iii) hold, then* (29) *is valid in the local sense on* I*.*

Corollary. *Let* $n, k \in \mathbb{N}, k \geq 2$ *and* $\lambda \in \mathbb{R}_+^n$*. For* $i \in \{0, \dots, k\}$*, let further* $(r_i, s_i) \in \mathbb{R}^2$*. For* $\alpha \in \{1, \dots, k\}$*, assume that the nonempty open interval* $I_\alpha \subseteq \mathbb{R}_+$ *fulfills* $\inf I_\alpha = 0$*. Then, in order that the inequality* (29) *be valid in the local sense on* I*, it is necessary that*

$$\max(1, r_0 + s_0) \leq \min(r_1 + s_1, \dots, r_k + s_k).$$

Conversely, if this inequality is strict, then (29) *holds in the local sense on* I*.*

For the global validity of the Minkowski-type inequality the theorem below establishes the following sufficient condition.

Theorem. *Let* $k \in \mathbb{N}, k \geq 2$ *and, for* $i \in \{0, \dots, k\}$*,* $(r_i, s_i) \in \mathbb{R}^2$*. Assume that, for all* $(u, y) \in I^2$*, the inequality*

$$\chi_{r_0,s_0}\left(\frac{y_1 + \cdots + y_k}{u_1 + \cdots + u_k}\right) \leq \sum_{j=1}^{k} \frac{u_j}{u_1 + \cdots + u_k} \chi_{r_j,s_j}\left(\frac{y_j}{u_j}\right)$$

holds. Then, for all $n \in \mathbb{N}$ *and* $\lambda \in \mathbb{R}_+^n$ *with* $\lambda_1 + \cdots + \lambda_n = 1$*, the inequality* (29) *holds in the global sense on* I*.*

By specializing the underlying intervals, we can formulate a more specific statement.

Corollary. *Let* $k \in \mathbb{N}, k \geq 2$ *and, for* $i \in \{0, \dots, k\}$*,* $(r_i, s_i) \in \mathbb{R}^2$*. Assume that, for all* $z \in \mathbb{R}_+^k$ *and* $t_1, \dots, t_k \in [0, 1]$ *with* $t_1 + \cdots + t_k = 1$*, the following inequality is valid*

$$\chi_{r_0,s_0}(t_1 z_1 + \cdots + t_k z_k) \leq \sum_{j=1}^{k} t_j \chi_{r_j,s_j}(z_j). \tag{32}$$

Then, for all $n \in \mathbb{N}$ *and* $\lambda \in \mathbb{R}_+^n$ *with* $\lambda_1 + \cdots + \lambda_n = 1$*, the inequality* (29) *holds in the global sense on* $\mathbb{R}_+^k$*.*

In order to compare our results above to existing ones, we recall two theorems related to the global validity of the Minkowski-type inequalities. In the setting of two-variable Gini means the Minkowski inequality was characterized by Czinder and Páles in [7, Theorem 5] (see also [26] for a particular case).

Theorem. *Let $k \in \mathbb{N}, k \geq 2$ and, for $i \in \{0, \dots, k\}$, $(r_i, s_i) \in \mathbb{R}^2$. Then the inequality*

$$G^{[2]}_{r_0,s_0}(x_1 + \cdots + x_k, y_1 + \cdots + y_k) \leq G^{[2]}_{r_1,s_1}(x_1, y_1) + \cdots + G^{[2]}_{r_k,s_k}(x_k, y_k) \tag{33}$$

is valid for all $x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{R}_+$ if and only if

(i) $0 \leq \min(r_1, s_1, \dots .r_k, s_k)$,

(ii) $\min(r_0, s_0) \leq \min(1, r_1, s_1, \dots .r_k, s_k)$,

(iii) $\max(1, r_0 + s_0) \leq \min(r_1 + s_1, \dots, r_k + s_k)$.

Remark. Observe that the third condition in the above theorem is the necessary condition for the local validity of (33) on $\mathbb{R}_+$. The conditions (i) and (ii) are, however, not necessary for the local validity of (33) on $\mathbb{R}_+$.

Necessary and sufficient conditions for the global validity of the Minkowski-type inequality for Gini means with arbitrary number of variables was established by Páles in [31, Theorem 3.1].

Theorem. *Let $k \in \mathbb{N}, k \geq 2$ and, for $i \in \{0, \dots, k\}$, $(r_i, s_i) \in \mathbb{R}^2$. Then the inequality*

$$G^{[n]}_{r_0,s_0}(x^1_1 + \cdots + x^k_1, \dots, x^1_n + \cdots + x^k_n) \leq G^{[n]}_{r_1,s_1}(x^1) + \cdots + G^{[n]}_{r_k,s_k}(x^k) \tag{34}$$

is valid for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^{n\times k}_+$ if and only if

(i) $0 \leq \min(r_1, s_1, \dots, r_k, s_k)$,

(ii) $\min(r_0, s_0) \leq \min(1, r_1, s_1, \dots, r_k, s_k)$,

(iii) $\max(1, r_0, s_0) \leq \min(\max(r_1, s_1), \dots, \max(r_k, s_k))$.

Remark. Observe that conditions (i) and (ii) of the above two theorems are identical, therefore they may be necessary for the global validity of (34) for any fixed $n \in \mathbb{N}$. The form of the third condition related to any fixed $n \in \mathbb{N}$ is not known. We also note that conditions (i)-(iii) of the above theorem are also necessary and sufficient for the validity of the inequality (32) for all $z \in \mathbb{R}^k_+$ and $t_1, \dots, t_k \in [0, 1]$ with $t_1 + \cdots + t_k = 1$.

Our next results characterize Hölder-type inequalities for Gini means in the local and in the global sense.

Theorem. *Let $n, k \in \mathbb{N}, k \geq 2$ and $\lambda \in \mathbb{R}^n_+$. Let further $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$, $\gamma_i := r_i + s_i$ for $i \in \{0, \dots, k\}$. Then, in order that the inequality*

$$G^{[n]}_{-r_0,-s_0;\lambda}(x^1_1 \cdots x^k_1, \dots, x^1_n \cdots x^k_n) \leq G^{[n]}_{r_1,s_1;\lambda}(x^1) \cdots G^{[n]}_{r_k,s_k;\lambda}(x^k) \tag{35}$$

be valid in the local sense on $\mathbb{R}_+^k$ it is necessary that either $\gamma_i \geq 0$ for all $i \in \{0,\dots,k\}$, or there exists $i \in \{0,\dots,k\}$ such that $\gamma_i < 0$, for all $j \in (\{0,\dots,k\}) \setminus \{i\}$, $\gamma_j > 0$ and

$$\frac{1}{\gamma_0} + \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_k} \leq 0 \tag{36}$$

holds. Conversely, if either $\gamma_i \geq 0$ for all $i \in \{0,\dots,k\}$ and $\gamma_i = 0$ can hold for at most one index $i \in \{0,\dots,k\}$, or there exists $i \in \{0,\dots,k\}$ such that $\gamma_i < 0$, for all $j \in \{1,\dots,k\} \setminus \{i\}$, $\gamma_j > 0$ and (36) is valid with a strict inequality, then (35) holds in the local sense on $\mathbb{R}_+^k$.

Theorem. *Let $k \in \mathbb{N}, k \geq 2$ and, for $i \in \{0,\dots,k\}$, $(r_i, s_i) \in \mathbb{R}^2$. Assume that, for all $z_1 \in (I_1/I_1), \dots, z_k \in (I_k/I_k)$, the inequality*

$$\chi_{-r_0,-s_0}(z_1 \cdots z_k) \leq \sum_{j=1}^{k} \chi_{r_j,s_j}(z_j) \tag{37}$$

holds. Then, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the inequality (35) holds in the global sense on I.

The global validity of (35) with an unfixed number of variables was characterized by Páles in [32, 33].

Theorem. *Let $k \in \mathbb{N}, k \geq 2$ and, for $i \in \{0,\dots,k\}$, $(r_i, s_i) \in \mathbb{R}^2$. Then the inequality*

$$G^{[n]}_{-r_0,-s_0}(x_1^1 \cdots x_1^k, \dots, x_n^1 \cdots x_n^k) \leq G^{[n]}_{r_1,s_1}(x^1) \cdots G^{[n]}_{r_k,s_k}(x^k)$$

is valid for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_+^{n\times k}$ if and only if

(i) for all $i \in \{0,\dots,k\}$, $\max(s_i, r_i) \geq 0$, and

(ii) for all $i \in \{0,\dots,k\}$ with $\min(s_i, r_i) < 0$, we have $\max(s_j, r_j) > 0$ for all $j \in (\{0,\dots,k\}) \setminus \{i\}$, and

$$\frac{1}{\min(s_i, r_i)} + \sum_{\substack{j=0\\ j\neq i}}^{k} \frac{1}{\max(s_j, r_j)} \leq 0.$$

Remark. We note that the conditions (i) and (ii) are necessary and sufficient for the validity of inequality (37) for all $z_1, \dots, z_k \in \mathbb{R}_+$ (cf. [33]).

Summary in Hungarian – Összefoglaló

Az 1. Fejezet összefoglalója – Alapvető fogalmak és eredmények

Ebben a fejezetben I egy nemüres, nyílt, valós intervallumot. Bevezetjük az általánosított Bajraktarević-közép fogalmát és harmadrendig kiszámítjuk a parciális deriváltjait. Az eredmények megtalálható a [12] dolgozatunkban.

Az általánosított Bajraktarević-közép definíciója

A Bajraktarević-közép fogalmának kiterjesztéséhez szükségünk lesz az alábbi lemmára, amely szigorúan monoton (de nem feltétlenül folytonos) függvények általánosított bal inverzének létezéséről és tulajdonságairól szól.

Állítás. *Legyen $f\colon I \to \mathbb{R}$ egy szigorúan monoton függvény. Ekkor létezik egy egyértelműen meghatározott $g\colon \operatorname{conv}(f(I)) \to I$ monoton függvény, amely bal inverze f-nek, azaz*

$$(g \circ f)(x) = x \qquad (x \in I). \tag{38}$$

Továbbá g folytonos, ugyanabban az értelemben monoton, mint f,

$$(f \circ g)(y) = y \qquad (y \in f(I)), \tag{39}$$

és

$$\liminf_{x\to g(y)} f(x) \le y \le \limsup_{x\to g(y)} f(x) \qquad (y \in \operatorname{conv}(f(I))).$$

Így, ha f alulról (felülről) félig folytonos $g(y)$-ban, akkor $f \circ g(y) \le y$ ($y \le f \circ g(y)$).

Definíció. A fenti lemmában jellemzett g függvényt a *szigorúan monoton $f\colon I \to \mathbb{R}$ függvény általánosított bal inverzének* nevezünk és az $f^{(-1)}$ szimbólummal jelöljük.

Megjegyzés. Az (38) és (39) egyenletekből világos, hogy $f^{(-1)}$ leszűkítése $f(I)$-re az f függvény klasszikus inverze. Tehát $f^{(-1)}$ az f inverzének a folytonos és monoton kiterjesztése a legszűkebb olyan intervallumra, amely tartalmazza f értékkészletét.

Egy meglehetősen általános közép, amely számos nevezetes közepet speciális esetként magában foglal, a kváziaritmetikai közép, amelyet az alábbiak szerint definiálunk.

Definíció. Legyen $n \in \mathbb{N}$. Adott $f\colon I \to \mathbb{R}$ szigorúan monoton, folytonos függvény esetén az $A_f^{[n]}\colon I^n \to I$ módon jelölt *n-változós kváziaritmetikai közepet* az alábbi képlettel definiáljuk:

$$A_f^{[n]}(x) := f^{-1}\left(\frac{f(x_1)+\cdots+f(x_n)}{n}\right) \qquad (x \in I^n).$$

Azt mondjuk, hogy f az $A_f^{[n]}$ *generátorfüggvénye*.

Bajraktarević [2]-ben, illetve [3]-ban a kváziaritmetikai közép alábbi súlyozott kiterjesztését vezette be és vizsgálta.

Definíció. Legyen $n \in \mathbb{N}$. Adott $f\colon I \to \mathbb{R}$ szigorúan monoton, folytonos és $p\colon I \to \mathbb{R}_+$ pozitív értékű függvény esetén a $B_{f,p}^{[n]}\colon\ I^n \to I$ módon jelölt *n-változós Bajraktarević-közepet* az alábbi képlettel definiáljuk:

$$B_{f,p}^{[n]}(x) := f^{-1}\left(\frac{p(x_1)f(x_1)+\cdots+p(x_n)f(x_n)}{p(x_1)+\cdots+p(x_n)}\right) \qquad (x \in I^n). \tag{40}$$

Azt mondjuk, hogy f a $B_{f,p}^{[n]}$ *generátorfüggvénye*, míg p a $B_{f,p}^{[n]}$ *súlyfüggvénye*.

Tekintsük a Bajraktarević-közép egy esetlegesen aszimmetrikus általánosítását.

Definíció. Legyen $n \in \mathbb{N}$. Adott $f\colon I \to \mathbb{R}$ szigorúan monoton függvény és $p = (p_1,\dots,p_n)\colon I \to \mathbb{R}_+^n$ pozitív koordinátájú vektorértékű függvény esetén az $A_{f,p}^{[n]}\colon I^n \to I$ szimbólummal jelölt *n-változós általánosított Bajraktarević-közepet* az alábbi képlettel definiáljuk:

$$A_{f,p}^{[n]}(x) := f^{(-1)}\left(\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)}\right) \qquad (x \in I^n), \tag{41}$$

és, a jelölések egyszerűsítése végett, az alábbi definíciókat is használni fogjuk:

$$R_{f,p}^{[n]}(x) := \frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)} \qquad \text{és} \qquad p_0 := p_1+\cdots+p_n.$$

Hasonlóan az előző definíciókhoz, azt mondjuk f az $A_{f,p}^{[n]}$ *generátorfüggvénye*, míg p az $A_{f,p}^{[n]}$ *súlyfüggvénye*.

Megjegyzés. Kihangsúlyozzuk, hogy $A_{f,p}^{[n]}$ valóban egy nem szükségképpen szimmetrikus általánosítása $B_{f,p}^{[n]}$-nek, hiszen, abban az esetben, amikor f folytonos, az összes súlyfüggvényt egyformának választva, azaz a $p := p_1 = \cdots = p_n$ helyettesítést végrehajtva (41)-ben, visszakapjuk (40)-mat.

Állítás. *Legyen $n \in \mathbb{N}$, $f\colon I \to \mathbb{R}$ egy szigorúan monoton függvény és legyen $p = (p_1,\dots,p_n)\colon\ I \to \mathbb{R}_+^n$. Ekkor a (41) képlettel definiált $A_{f,p}^{[n]}\colon I^n \to I$ függvény jóldefiniált és egy közép, azaz*

$$\min(x) \le A_{f,p}^{[n]}(x) \le \max(x) \qquad (x \in I^n).$$

Az általánosított Bajraktarević-közép parciális deriváltjai

Az alábbi tételben, feszes regularitási feltételek mellett, I^n diagonális pontjaiban, harmadrendig meghatározzuk általánosított Bajraktarević-közép parciális deriváltjait. Feszes regularitási feltételek alatt azt értjük, hogy például, ahogy alább az (1), (2b), (3c) állításokban látható, $m \in \mathbb{N}$ esetén csupán $(m-1)$-szeri folytonos differenciálhatóságot feltételezünk p_i-ről ahhoz, hogy bizonyítsuk a ∂_i^m alakú parciális deriváltak létezését és kiszámítsuk azokat. A parciális deriváltak kulcsszerepet játszanak a későbbi fejezetekben, hiszen az egyik eszköz, amelyet a fő tételek bizonyításához használunk az nem más, mint a szóban forgó függvényegyenlet értelmezési tartományának diagonálisán történő differenciálása által nyert differenciálegyenletek megoldása.

Tétel. *Legyen $n \in \mathbb{N}, \ell \in \{1,2,3\}$, $f\colon I \to \mathbb{R}$ egy ℓ-szer differenciálható, nemnulla első deriválttal rendelkező függvény, és legyen $p = (p_1, \dots, p_n)\colon I \to \mathbb{R}_+^n$. Ekkor az alábbi állítások érvényesek.*

(1) Ha $\ell = 1$, $i \in \{1, \dots, n\}$, és p_i folytonos, akkor a $\partial_i A_{f,p}^{[n]}$ elsőrendű parciális derivált létezik a $\operatorname{diag}(I^n)$ diagonálison, és

$$\left(\partial_i A_{f,p}^{[n]}\right)^{\Delta} = \frac{p_i}{p_0}.$$

(2a) Ha $\ell = 2$, $i, j \in \{1, \dots, n\}$, hogy $i \neq j$, továbbá p_i és p_j differenciálhatók, akkor a $\partial_i \partial_j A_{f,p}^{[n]}$ másodrendű parciális derivált létezik a $\operatorname{diag}(I^n)$ diagonálison, és

$$\left(\partial_i \partial_j A_{f,p}^{[n]}\right)^{\Delta} = -\frac{(p_i p_j)'}{p_0^2} - \frac{p_i p_j}{p_0^2} \cdot \frac{f''}{f'}.$$

(2b) Ha $\ell = 2$, $i \in \{1, \dots, n\}$, és p_i folytonosan differenciálható, a $\partial_i^2 A_{f,p}^{[n]}$ másodrendű parciális derivált létezik a $\operatorname{diag}(I^n)$ diagonálison, és

$$\left(\partial_i^2 A_{f,p}^{[n]}\right)^{\Delta} = 2\frac{p_i'(p_0 - p_i)}{p_0^2} + \frac{p_i(p_0 - p_i)}{p_0^2} \cdot \frac{f''}{f'}.$$

(3a) Ha $\ell = 3$, $i, j, k \in \{1, \dots, n\}$, hogy $i \neq j \neq k \neq i$, továbbá p_i, p_j és p_k differenciálhatók, akkor a $\partial_i \partial_j \partial_k A_{f,p}^{[n]}$ harmadrendű parciális derivált létezik a $\operatorname{diag}(I^n)$ diagonálison, és

$$\begin{aligned}\left(\partial_i \partial_j \partial_k A_{f,p}^{[n]}\right)^{\Delta} &= 2\frac{p_i p_j' p_k' + p_i' p_j p_k' + p_i' p_j' p_k}{p_0^3} + 2\frac{(p_i p_j p_k)'}{p_0^3} \cdot \frac{f''}{f'} \\ &\quad + \frac{p_i p_j p_k}{p_0^3}\left(3\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right).\end{aligned}$$

(3b) Ha $\ell = 3$, $i, j \in \{1, \dots, n\}$, hogy $i \neq j$, továbbá p_i kétszer differenciálható és p_j differenciálható, akkor a $\partial_i^2 \partial_j A_{f,p}^{[n]}$ harmadrendű parciális derivált létezik a $\operatorname{diag}(I^n)$ diagonálison, és

$$\begin{aligned}\left(\partial_i^2 \partial_j A_{f,p}^{[n]}\right)^\Delta &= \frac{2p_i'p_j'(2p_i - p_0) + p_j(2(p_i')^2 - p_i''p_0)}{p_0^3} \\ &\quad + \frac{(2p_i'p_j + p_ip_j')(2p_i - p_0)}{p_0^3} \cdot \frac{f''}{f'} \\ &\quad + \frac{p_ip_j}{p_0^3}\left((3p_i - p_0)\left(\frac{f''}{f'}\right)^2 - p_i\frac{f'''}{f'}\right).\end{aligned}$$

(3c) Ha $\ell = 3$, $i \in \{1, \dots, n\}$, és p_i kétszer folytonosan differenciálható, akkor a $\partial_i^3 A_{f,p}^{[n]}$ harmadrendű parciális derivált létezik a $\operatorname{diag}(I^n)$ diagonálison, és

$$\begin{aligned}\left(\partial_i^3 A_{f,p}^{[n]}\right)^\Delta &= 3\frac{(p_0 - p_i)\big(p_0p_i'' - 2(p_i')^2\big)}{p_0^3} + 3\frac{p_i'(p_0 - 2p_i)(p_0 - p_i)}{p_0^3} \cdot \frac{f''}{f'} \\ &\quad - \frac{p_i(p_0 - p_i)}{p_0^3}\left(3p_i\left(\frac{f''}{f'}\right)^2 - (p_0 + p_i)\frac{f'''}{f'}\right).\end{aligned}$$

A 2. Fejezet összefoglalója – Egyenlőségi probléma

A [12] dolgozat tartalmazza az eredményeket ebből a fejezetből, amelyben az n-változós általánosított Bajraktarević-közép egyenlőségi problémájával foglalkozunk, azaz szükséges, illetve elégséges feltételeket adunk az alább felsorolt ismeretlen függvényekre nézve ahhoz, hogy az

$$\begin{aligned}f^{(-1)}&\left(\frac{p_1(x_1)f(x_1) + \dots + p_n(x_n)f(x_n)}{p_1(x_1) + \dots + p_n(x_n)}\right) \\ &= g^{(-1)}\left(\frac{q_1(x_1)g(x_1) + \dots + q_n(x_n)g(x_n)}{q_1(x_1) + \dots + q_n(x_n)}\right)\end{aligned} \tag{42}$$

függvényegyenlet fennálljon lokálisan vagy globálisan, ahol $n \in \mathbb{N}, n \geq 2$ rögzített, I egy nemüres, nyílt, valós intervallumot jelöl, amely az egész fejezetben így lesz, az $f, g \colon I \to \mathbb{R}$ ismeretlen függvények szigorúan monotonak, és $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$, illetve $q = (q_1, \dots, q_n) \colon I \to \mathbb{R}_+^n$ szintén ismeretlen függvények.

Ezt a kérdést a szimmetrikus 2-változós esetben, azaz amikor $n = 2$ és $p_1 = p_2$, Losonczi [22]-ben már vizsgálta, és meg is oldotta hatodrendű regularitási feltételek mellett. Később Páles és Zakaria [38]-ben megjavította ezt az eredményt, sikerült ugyanazt igazolniuk elsőrendű megszorítások mellett. Az előbb említett eset kivételével, ebben a formában, a fejezet minden más eredményét mi publikáltuk először [12]-ben. Pontosabban, az aszimmetrikus 2-változós esetben, azt feltételezve, hogy f, g háromszor differenciálható, és hogy létezik $i \in \{1, 2\}$, hogy vagy p_i kétszer folytonosan differenciálható és p_{3-i} folytonos, vagy p_i kétszer differenciálható és p_{3-i} differenciálható, azt bizonyítjuk, hogy az egyenlőség pontosan akkor teljesül, ha léteznek $a, b, c, d \in \mathbb{R}$ konstansok, hogy $ad \neq bc$, és

$$cf + d > 0, \qquad g = \frac{af + b}{cf + d} \qquad \text{és} \qquad q_\ell = (cf + d)p_\ell \qquad (\ell \in \{1, \dots, n\}).$$

Az $n \geq 3$ esetben ugyanezt a konklúziót kapjuk gyengébb regularitási feltételek mellett. Nevezetesen, azt tesszük fel, hogy f és g háromszor differenciálható, p folytonos és léteznek $i, j, k \in \{1, \dots, n\}$, hogy $i \neq j \neq k \neq i$ és p_i, p_j, p_k differenciálhatók.

Elégséges feltételek

Tétel. *Legyen $f: I \to \mathbb{R}$ egy szigorúan monoton növekedő függvény, $n \in \mathbb{N}$, és legyen $p = (p_1, \dots, p_n): I \to \mathbb{R}_+^n$. Ekkor az $y = A_{f,p}^{[n]}(x)$ egyenlet pontosan akkor áll fenn minden $x \in I^n$-re, ha, $z \in I$ esetén,*

$$\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i)) \begin{cases} < 0 & \textit{if } z < y \\ > 0 & \textit{if } z > y. \end{cases} \tag{43}$$

Ha f szigorúan monoton csökkenő, akkor a (43)-beli egyenlőtlenségek fordított relációs jellel teljesülnek.

Következmény. *Legyen $f: I \to \mathbb{R}$ folytonos, szigorúan monoton, $n \in \mathbb{N}$, és legyen $p = (p_1, \dots, p_n): I \to \mathbb{R}_+^n$. Ekkor, minden $x \in I^n$ esetén, az $y = A_{f,p}^{[n]}(x)$ érték egyértelmű megoldása az alábbi egyenletnek:*

$$\sum_{i=1}^{n} p_i(x_i)(f(y) - f(x_i)) = 0.$$

A következő eredmény egy elégséges feltételt ad két n-változós általánosított Bajraktarević-közép egyenlőségére. Ezt az esetet az *egyenlőség kanonikus esetének* fogjuk hívni.

Tétel. *Legyenek $f, g: I \to \mathbb{R}$ szigorúan monoton függvények, $n \in \mathbb{N}$, és legyenek $p = (p_1, \dots, p_n): I \to \mathbb{R}_+^n$, $q = (q_1, \dots, q_n): I \to \mathbb{R}_+^n$. Ha léteznek $a, b, c, d \in \mathbb{R}$ konstansok, hogy $ad \neq bc$, és*

$$cf + d > 0, \qquad g = \frac{af + b}{cf + d} \qquad \textit{és} \qquad q_i = (cf + d)p_i \qquad (i \in \{1, \dots, n\}) \tag{44}$$

teljesülnek I-n, akkor $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ globálisan érvényes.

Szükséges feltételek

Az alábbi lemma segítségével csökkenteni tudjuk a regularitási feltételeket állításainkban.

Legyen $n \in \mathbb{N}$. Minden $i \in \{1, \dots, n\}$ esetén jelölje $\mathbb{R}^n$ természetes bázisának i-edik vektorát $e_i \in \mathbb{R}^n$, azaz legyen $e_i := (\delta_{i,j})_{j=1}^n$. Két adott $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$ és $q = (q_1, \dots, q_n) \colon I \to \mathbb{R}_+^n$ súlyfüggvény esetén az alábbi jelölést is használni fogjuk:

$$r_0 := \frac{q_0}{p_0} = \frac{q_1 + \dots + q_n}{p_1 + \dots + p_n}.$$

Lemma. *Legyenek $f, g \colon I \to \mathbb{R}$ szigorúan monoton, folytonos függvények, $n \in \mathbb{N}, n \geq 2$, és legyenek $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$, $q = (q_1, \dots, q_n) \colon I \to \mathbb{R}_+^n$. Tegyük fel, hogy $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ lokálisan teljesül. Ekkor az alábbi két állítás érvényes.*

(i) Minden $i \in \{1, \dots, n\}$ esetén a p_i függvény pontosan akkor folytonos, ha q_i folytonos.

(ii) Legyen $k \in \mathbb{N}$. Tegyük fel, hogy $f, g \colon I \to \mathbb{R}$ k-szor differenciálható (k-szor folytonosan differenciálható) függvények nemnulla első deriválttal. Ekkor, minden $i \in \{1, \dots, n\}$ esetén, a p_i függvény pontosan akkor k-szor differenciálható (k-szor folytonosan differenciálható), ha q_i k-szor differenciálható (k-szor folytonosan differenciálható)

A következő kiterjesztési tétel alapvető fontosságú, hiszen a fő tételek bizonyításaiban először csak az értelmezési tartomány egy részintervallumán tudjuk garantálni az egyenlőség kanonikus esetének fennállását. Aztán a következő fő lépés a bizonyításokban az az, hogy kiterjesszük ezt az egész intervallumra.

Tétel. *Legyenek $f, g \colon I \to \mathbb{R}$ szigorúan monoton, folytonos függvények, $n \in \mathbb{N}$, $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$ egy folytonos függvény, és legyen $q = (q_1, \dots, q_n) \colon I \to \mathbb{R}_+^n$. Tegyük fel, hogy $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ lokálisan fennáll, és hogy léteznek $a, b, c, d \in \mathbb{R}$ konstansok, melyekre $ad \neq bc$ érvényes, továbbá, hogy létezik I-nek egy J nemüres, nyílt részintervalluma, amelyen (2.2) érvényes. Ekkor q folytonos és (2.2) az I intervallumon is teljesül.*

Két általánosított Bajraktarević-közép egyenlőségére vonatkozó első- és másodrendű szükséges feltételeket az alábbi két lemmában részletezzük.

Lemma. *Legyen $n \in \mathbb{N}, n \geq 2$ és $f, g \colon I \to \mathbb{R}$ differenciálható függvények nemnulla első deriválttal, és legyen $i \in \{1, \dots, n\}$. Legyenek továbbá $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$ és $q = (q_1, \dots, q_n) \colon I \to \mathbb{R}_+^n$ olyan függvények, hogy p_i és q_i folytonosak. Ha $\partial_i A_{f,p}^{[n]} = \partial_i A_{g,q}^{[n]}$ fennáll a $\operatorname{diag}(I^n)$ diagonálison, akkor*

$$\frac{q_i}{q_0} = \frac{p_i}{p_0} \tag{45}$$

teljesül az I intervallumon.

Lemma. *Legyen $n \in \mathbb{N}, n \geq 2$ és $f, g \colon I \to \mathbb{R}$ kétszer differenciálható függvények nemnulla első deriválttal. Legyenek továbbá $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$ és $q = (q_1, \dots, q_n) \colon I \to \mathbb{R}_+^n$ folytonos függvények, és tegyük fel, hogy, minden $i \in \{1, \dots, n\}$ esetén, a (45) egyenlőség fennáll az I intervallumon. Legyen $j, k \in \{1, \dots, n\}$. Ekkor az alábbi két állítás érvényes.*

(i) Feltéve, hogy $j \neq k$ és a p_j, p_k, q_j, illetve q_k függvények differenciálhatók, ha $\partial_j \partial_k A_{f,p}^{[n]} = \partial_j \partial_k A_{g,q}^{[n]}$ teljesül a $\operatorname{diag}(I^n)$ diagonálison, akkor létezik egy olyan nemnulla γ konstans, hogy minden $i \in \{1, \dots, n\}$ esetén

$$q_i^2 g' = \gamma p_i^2 f' \tag{46}$$

érvényes az I intervallumon.

(ii) Feltéve, hogy $j = k$ és a p_j, q_j függvények folytonosan differenciálhatók, ha $\partial_j^2 A_{f,p}^{[n]} = \partial_j^2 A_{g,q}^{[n]}$ teljesül a $\operatorname{diag}(I^n)$ diagonálison, akkor létezik egy olyan nemnulla γ konstans, hogy minden $i \in \{1, \dots, n\}$ esetén (46) érvényes az I intervallumon.

A következő fogalom és egy kapcsolódó lemma kulcsszerepet játszik a fejezet későbbi bizonyításaiban hiszen, megoldva a szóban forgó függvényegyenlet differenciálásával kapott differenciálegyenleteket, az egyik eset, ami adódik az pontosan az az egyenlet, amire a következő lemma ad egy szükséges feltételt, csupán annyi regularitást feltételezve, ami jóldefiniáltsághoz elengedhetetlen.

Definíció. Egy $f \colon I \to \mathbb{R}$ háromszor differenciálható, nemnulla első deriválttal rendelkező függvény esetén definiáljuk a függvény $S_f \colon I \to \mathbb{R}$ módon jelölt *Schwarz-deriváltját* az alábbi képlettel:

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Lemma. *Legyenek $f, g \colon I \to \mathbb{R}$ háromszor differenciálható, nemnulla első deriválttal rendelkező függvények. Ha $S_f = S_g$ fennáll az I intervallumon, akkor léteznek $a, b, c, d \in \mathbb{R}$ konstansok, hogy $ad \neq bc$, $cf + d$ pozitív I-n, és*

$$g = \frac{af + b}{cf + d}$$

teljesül az I intervallumon.

A fejezet első fő tétele következik, amely teljes mértékben karakterizálja két legalább 3-változós általánosított Bajraktarević-közép egyenlőségét.

Tétel. *Legyen $n \in \mathbb{N}, n \geq 3$ és $f, g \colon I \to \mathbb{R}$ háromszor differenciálható, nemnulla első deriválttal rendelkező függvények. Legyen továbbá $p = (p_1, \dots, p_n) \colon I \to \mathbb{R}_+^n$ egy folytonos függvény és legyen $q = (q_1, \dots, q_n) : I \to \mathbb{R}_+^n$. Tegyük fel, hogy léteznek $i, j, k \in \{1, \dots, n\}$ indexek, hogy $i \neq j \neq k \neq i$ és a p_i, p_j, p_k függvények differenciálhatók. Ekkor az alábbi állítások páronként ekvivalensek.*

(i) $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ *teljesül globálisan.*

(ii) $A_{f,p}^{[n]} = A_{g,q}^{[n]}$ *teljesül lokálisan.*

(iii) *A* q *függvény folytonos, a* q_i, q_j, q_k *függvények differenciálhatók, és a*

$$\begin{aligned} \partial_\ell A_{f,p}^{[n]} &= \partial_\ell A_{g,q}^{[n]} \qquad (\ell \in \{1, \dots, n-1\}), \\ \partial_i \partial_j A_{f,p}^{[n]} &= \partial_i \partial_j A_{g,q}^{[n]}, \\ \partial_i \partial_j \partial_k A_{f,p}^{[n]} &= \partial_i \partial_j \partial_k A_{g,q}^{[n]} \end{aligned}$$

egyenletek teljesülnek a $\operatorname{diag}(I^n)$ *diagonálison.*

(iv) *Léteznek* $a, b, c, d \in \mathbb{R}$ *konstansok, hogy* $ad \neq bc$ *és*

$$g = \frac{af+b}{cf+d} \qquad \text{és} \qquad q_\ell = (cf+d)p_\ell \qquad (\ell \in \{1, \dots, n\})$$

teljesülnek az I *intervallumon.*

A második fő tételnek két változata van a regularitási feltételek szempontjából, és ez karakterizálja a 2-változós általánosított aszimmetrikus Bajraktarević-közepek egyenlőségét.

Tétel. *Legyenek* $f, g\colon I \to \mathbb{R}$ *háromszor differenciálható, nemnulla első deriválttal rendelkező függvények. Legyenek* $p = (p_1, p_2)\colon I \to \mathbb{R}_+^2$ *és* $q = (q_1, q_2)\colon I \to \mathbb{R}_+^2$*, hogy* $p_1 \neq p_2$*. Tegyük fel, hogy létezik* $i \in \{1, 2\}$*, hogy az alábbi regularitási feltételek egyike teljesül.*

(a) p_i *kétszer folytonosan differenciálható és* p_{3-i} *folytonos.*

(b) p_i *kétszer differenciálható és* p_{3-i} *differenciálható.*

Ekkor az alábbi állítások páronként ekvivalensek.

(i) $A_{f,p}^{[2]} = A_{g,q}^{[2]}$ *teljesül globálisan.*

(ii) $A_{f,p}^{[2]} = A_{g,q}^{[2]}$ *teljesül lokálisan.*

(iv) *Léteznek* $a, b, c, d \in \mathbb{R}$ *konstansok, hogy* $ad \neq bc$ *és*

$$g = \frac{af+b}{cf+d}, \qquad q_1 = (cf+d)p_1, \qquad \text{és} \qquad q_2 = (cf+d)p_2$$

teljesülnek az I *intervallumon.*

Ha ráadásul az f *és* p *függvények* ℓ*-szer folytonosan differenciálhatók, akkor* $A_{f,p}^{[2]}$ *is* ℓ*-szer folytonosan differenciálható.*

Ahogy korábban is említettük, a szimmetrikus 2-változós esetet már megoldották először hatod-, majd pedig elsőrendű regularitási feltételek mellett a [22], illetve [38] dolgozatokban.

Tétel. *Legyenek $f, g\colon I \to \mathbb{R}$ folytonosan differenciálható, nemnulla első deriválttal rendelkező függvények. Legyen továbbá $p\colon I \to \mathbb{R}_+$ egy folytonosan differenciálható függvény és $q\colon I \to \mathbb{R}_+$. Ekkor az alábbi állítások ekvivalensek.*

(i) $A^{[2]}_{f,(p,p)} = A^{[2]}_{g,(q,q)}$ *teljesül globálisan.*

(ii) $A^{[2]}_{f,(p,p)} = A^{[2]}_{g,(q,q)}$ *teljesül lokálisan.*

(iv) *Vagy léteznek $a, b, c, d \in \mathbb{R}$ konstansok, hogy $ad \neq bc$ és*

$$g = \frac{af+b}{cf+d} \qquad \textit{és} \qquad q = (cf+d)p$$

teljesülnek az I intervallumon, vagy léteznek P és Q legfeljebb másodfokú polinomok, hogy P, illetve Q pozitívak $f(I)$-n, illetve $g(I)$-n, és léteznek $\alpha, \beta \in \mathbb{R}$ konstansok, hogy

$$g = G^{-1} \circ (\alpha F \circ f + \beta), \qquad p = P^{-\frac{1}{2}} \circ f, \qquad \textit{és} \qquad q = Q^{-\frac{1}{2}} \circ g$$

fennállnak az I intervallumon, ahol F, illetve G az $1/P$, illetve $1/Q$ egy-egy primitív függvényét jelölik.

A 3. Fejezet összefoglalója – Összehasonlítási probléma

Ennek a fejezetnek az a fő célja, hogy vizsgáljuk két n-változós általánosított Bajraktarević-közép lokális és globális összehasonlítási problémáját, azaz, hogy szükséges, illetve elégséges feltételeket adjunk az alább felsorolt ismeretlen függvényekre nézve ahhoz, hogy az

$$\begin{aligned} &f^{(-1)}\left(\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)}\right) \\ &\qquad \leq g^{(-1)}\left(\frac{q_1(x_1)g(x_1)+\cdots+q_n(x_n)g(x_n)}{q_1(x_1)+\cdots+q_n(x_n)}\right) \end{aligned} \tag{47}$$

függvényegyenlőtlenség érvényes legyen lokálisan vagy globálisan, ahol $n \in \mathbb{N}, n \geq 2$ rögzített, I egy nemüres, nyílt, valós intervallum, amely az egész fejezetben így lesz, ha csak mást nem mondunk, az $f, g\colon I \to \mathbb{R}$ ismeretlen függvények szigorúan monotonak, illetve $p = (p_1, \ldots, p_n)\colon I \to \mathbb{R}_+^n$ és $q = (q_1, \ldots, q_n)\colon I \to \mathbb{R}_+^n$ is ismeretlen függvények. Ezen szekció eredményeit a [13] dolgozatunkban bizonyítottuk. Megjegyezzük, hogy, ahogyan a 2. Fejezetben is láttuk, ezen közepek egyenlőségének lokális és globális fennállása ekvivalens. Viszont, ahogyan ebben a fejezetben olvasható, az összehasonlítási probléma esetében nem ez a helyzet.

A globális összehasonlítási problémát illetően a fejezet fő eredménye azt mondja ki, hogy ha az f, g függvények differenciálhatók és nemnulla első deriválttal rendelkeznek, és minden $i \in \{1, \dots, n\}$ esetén

$$\frac{p_i}{p_0} = \frac{q_i}{q_0} \qquad \text{és} \qquad \frac{p_0(x)(f(x)-f(y))}{p_0(y)f'(y)} \leq \frac{q_0(x)(g(x)-g(y))}{q_0(y)g'(y)} \qquad (x, y \in I)$$

teljesülnek (ahol $p_0 := p_1 + \cdots + p_n$ és $q_0 := q_1 + \cdots + q_n$), akkor a fenti összehasonlíthatósági egyenlőtlenség fennáll minden $x_1, \dots, x_n \in I$ esetén.

Lokális összehasonlíthatóság

Definíció. Legyen $n \in \mathbb{N}, n \geq 2$ és $U \in \mathbb{R}^n$ egy nemüres, nyílt halmaz. Az $f\colon U \to \mathbb{R}$ függvényt *az $a \in U$ pontban az i-edik változójára nézve parciálisan differenciálhatónak* hívjuk, ha a

$$\lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}$$

határérték létezik és véges. Azt mondjuk, hogy f *az $a \in U$ pontban parciálisan differenciálható*, ha minden változójára nézve parciálisan differenciálható. Végezetül, azzal a szóhasználattal élünk, hogy f *parciálisan differenciálható az U halmazon*, ha U minden pontjában parciálisan differenciálható.

A (47) összehasonlíthatósági egyenlőtlenség lokális vizsgálatához felelevenítjük [36] alábbi eredményét.

Tétel. *Legyen $n \in \mathbb{N}, n \geq 2$ és legyenek $M, N\colon I^n \to I$ olyan n-változós közepek, hogy M lokálisan kisebb, mint N. Tegyük fel, hogy M és N parciálisan differenciálhatók a $\operatorname{diag}(I^n)$ diagonálison. Ekkor minden $i \in \{1, \dots, n\}$ esetén*

$$\partial_i M^\Delta = \partial_i N^\Delta. \tag{48}$$

Ha ráadásul M és N kétszer differenciálhatók a $\operatorname{diag}(I^n)$ diagonálison, akkor a

$$\big(\partial_i\partial_j N^\Delta - \partial_i\partial_j M^\Delta\big)_{i,j=1}^{n-1} \tag{49}$$

szimmetrikus $((n-1) \times (n-1))$-típusú mátrix pozitív szemidefinit.

Másfelől, ha a (48) egyenlet érvényes minden $i \in \{1, \dots, n\}$ esetén, továbbá M és N kétszer folytonosan differenciálhatók a $\operatorname{diag}(I^n)$ diagonálison és a (49) által adott szimmetrikus $((n-1) \times (n-1))$-típusú mátrix pozitív definit, akkor M lokálisan kisebb, mint N.

E fejezet első fő eredménye szükséges, illetve elégséges feltételeket ad az általánosított Bajraktarević-közepek lokális összehasonlíthatóságára.

Tétel. *Legyen $n \in \mathbb{N}, n \geq 2$, $f, g\colon I \to \mathbb{R}$ differenciálható, nemnulla első deriválttal rendelkező függvények, továbbá legyenek $p, q\colon I \to \mathbb{R}_+^n$ folytonos függvények. Tegyük fel, hogy $A_{f,p}^{[n]}$ lokálisan kisebb, mint $A_{g,q}^{[n]}$. Ekkor*

$$\frac{p_i}{p_0} = \frac{q_i}{q_0} \qquad (i \in \{1, \dots, n\}). \tag{50}$$

Ha ráadásul f, g kétszer differenciálhatók és p, q folytonosan differenciálhatók, akkor a

$$\frac{q_0^2|g'|}{p_0^2|f'|} \tag{51}$$

függvény monoton növekedő.

Másfelől, ha f, g és p, q kétszer folytonosan differenciálhatók, (50) *teljesül és a* (51)*-ben látott függvény deriváltja pozitív, akkor $A_{f,p}^{[n]}$ lokálisan kisebb, mint $A_{g,q}^{[n]}$.*

A következőkben az általánosított Bajraktarević-közepek lokális összehasonlíthatósági problémájának azt az esetét tekintjük, amikor I egy nemüres, nyílt részintervalluma $\mathbb{R}_+$-nak, és a $p_1, \dots, p_n$, illetve $q_1, \dots, q_n$ súlyfüggvények hatványfüggvények pozitív számszorosai.

Következmény. *Legyen $I \subseteq \mathbb{R}_+$ egy nemüres, nyílt intervallum, $f, g\colon I \to \mathbb{R}$ differenciálható, nemnulla első deriválttal rendelkező függvények. Legyen $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$, $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$, és a súlyfüggvényeket a*

$$p_i(x) := \lambda_i x^{\alpha_i} \qquad \text{és} \qquad q_i(x) := \mu_i x^{\beta_i} \qquad (i \in \{1, \dots, n\},\ x \in I) \tag{52}$$

formulákkal definiáljuk. Tegyük fel, hogy $A_{f,p}^{[n]}$ lokálisan kisebb, mint $A_{g,q}^{[n]}$. Ekkor léteznek $\gamma > 0$ és $\delta \in \mathbb{R}$ konstansok úgy, hogy

$$\mu_i = \gamma \lambda_i \qquad \text{és} \qquad \beta_i = \alpha_i + \delta \qquad (i \in \{1, \dots, n\}). \tag{53}$$

Ha ráadásul f és g kétszer differenciálhatók, akkor az

$$x \mapsto x^{2\delta} \frac{|g'(x)|}{|f'(x)|} \tag{54}$$

leképezés monoton növekedő az I intervallumon.

Másfelől, ha f, g kétszer folytonosan differenciálhatók, (53) *teljesül és a* (54)*-ban látott függvény deriváltja pozitív, akkor $A_{f,p}^{[n]}$ lokálisan kisebb, mint $A_{g,q}^{[n]}$.*

A fenti eredmény egy azonnali következményeként karakterizálni tudjuk az általánosított hatványközepek lokális összehasonlíthatóságát.

Következmény. *Legyen $I \subseteq \mathbb{R}_+$ egy nemüres, nyílt intervallum, $a, b \in \mathbb{R}$ pedig konstansok. Definiáljuk az $f, g\colon I \to \mathbb{R}$ függvényeket az alábbi képletekkel:*

$$f(x) := \begin{cases} x^a & \text{ha } a \neq 0 \\ \log(x) & \text{ha } a = 0 \end{cases} \qquad \text{és} \qquad g(x) := \begin{cases} x^b & \text{ha } b \neq 0 \\ \log(x) & \text{ha } b = 0. \end{cases} \tag{55}$$

Legyen $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$, $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$, és definiáljuk a $p, q\colon I \to \mathbb{R}_+^n$ függvényeket (52) szerint. Ha $A_{f,p}^{[n]}$ lokálisan kisebb, mint $A_{f,p}^{[n]}$, akkor léteznek $\gamma > 0$ és $\delta \in \mathbb{R}$ konstansok, hogy (53) teljesül és $a \leq b + 2\delta$.

Másfelől, ha (53) fennáll és $a < b + 2\delta$, akkor $A_{f,p}^{[n]}$ lokálisan kisebb, mint $A_{g,q}^{[n]}$.

Globális összehasonlíthatóság

Ha $A_{f,p}^{[n]}$ globálisan kisebb, mint $A_{g,q}^{[n]}$, akkor $A_{f,p}^{[n]}$ lokálisan is kisebb, mint $A_{g,q}^{[n]}$, és ezért (50) szükségessége azonnal adódik, és a (51)-ben látott függvény monoton növekedő kell legyen. Viszont ezek a feltételek nem elég erősek ahhoz, hogy maguk után vonják az előbb említett közepek globális összehasonlíthatóságát. Az alfejezet első tétele azzal foglalkozik, hogy az (50) feltételt mivel kell kiegészíteni ahhoz, hogy adódjon a globális összehasonlíthatóság.

Tétel. *Legyenek $f, g\colon I \to \mathbb{R}$ szigorúan monoton, differenciálható, nemnulla első deriválttal rendelkező függvények, $n \in \mathbb{N}, n \geq 2$, és legyenek $p, q\colon I \to \mathbb{R}_+^n$. Tegyük fel, hogy (50) teljesül, és*

$$\frac{p_0(x)(f(x) - f(y))}{p_0(y) f'(y)} \leq \frac{q_0(x)(g(x) - g(y))}{q_0(y) g'(y)} \qquad (x, y \in I). \tag{56}$$

Ekkor $A_{f,p}^{[n]}$ globálisan kisebb, mint $A_{g,q}^{[n]}$.

A második eredményünk pedig azzal, hogy az (51) függvény monoton növekedésére vonatkozó feltételt hogyan kell szigorítani ahhoz, hogy az maga után vonja a globális összehasonlíthatóságot. Az derül ki, hogy ha az (51) függvényt alkotó szorzótényezők mindegyike monoton növekedő, akkor az már elegendő.

Tétel. *Legyenek $f, g\colon I \to \mathbb{R}$ szigorúan monoton, differenciálható, nemnulla első deriválttal rendelkező függvények, $n \in \mathbb{N}, n \geq 2$, és $p, q\colon I \to \mathbb{R}_+^n$. Tegyük fel, hogy (50) teljesül, illetve*

$$\frac{q_0}{p_0} \qquad \text{és} \qquad \frac{|g'|}{|f'|}$$

monoton növekedők. Ekkor $A_{f,p}^{[n]}$ globálisan kisebb, mint $A_{g,q}^{[n]}$.

Megjegyezzük, hogy az (56) elégséges feltétel maga után vonja az (51) függvény monoton növekedését. Általánosságban a megfordítás nem igaz, azonban az alábbiakban bemutatunk két speciális esetet, amikor az. Az első esetben p és q azonosak.

Tétel. *Legyenek $f,g\colon I\to\mathbb{R}$ szigorúan monoton, kétszer differenciálható, nemnulla első deriválttal rendelkező függvények, $n\in\mathbb{N}, n\geq 2$, és legyen $p\colon I\to\mathbb{R}_+^n$ egy folytonos függvény. Ekkor az alábbi állítások páronként ekvivalensek.*

(i) $A_{f,p}^{[n]}$ *globálisan kisebb, mint* $A_{g,p}^{[n]}$.

(ii) $A_{f,p}^{[n]}$ *lokálisan kisebb, mint* $A_{g,p}^{[n]}$.

(iii) A $|g'/f'|$ *függvény monoton növekedő.*

(iv) Az alábbi egyenlőtlenség teljesül az I *intervallumon:*

$$\frac{f''}{f'}\leq\frac{g''}{g'}.$$

(v) Feltéve, hogy g *monoton növekedő (csökkenő), a* $g\circ f^{-1}$ *függvény konvex (konkáv)* $f(I)$*-n;*

(vi)

$$\frac{f(x)-f(y)}{f'(y)}\leq\frac{g(x)-g(y)}{g'(y)}\qquad(x,y\in I).$$

A második esetben az f és g függvények egyeznek meg.

Tétel. *Legyen $f\colon I\to\mathbb{R}$ egy szigorúan monoton, kétszer folytonosan differenciálható, nemnulla első deriválttal rendelkező függvény, $n\in\mathbb{N}, n\geq 2$, és legyenek $p,q\colon I\to\mathbb{R}_+^n$ folytonos függvények. Ekkor az alábbi állítások páronként ekvivalensek.*

(i) $A_{f,p}^{[n]}$ *globálisan kisebb, mint* $A_{f,q}^{[n]}$.

(ii) $A_{f,p}^{[n]}$ *lokálisan kisebb, mint* $A_{f,q}^{[n]}$.

(iii) A (50) *feltétel teljesül és a* q_0/p_0 *függvény monoton növekedő.*

Következmény. *Legyen $I\subseteq\mathbb{R}_+$ egy nemüres, nyílt intervallum, $f,g\colon I\to\mathbb{R}$ differenciálható, nemnulla első deriválttal rendelkező függvények. Legyen továbbá $n\in\mathbb{N}, n\geq 2$, $(\lambda_1,\dots,\lambda_n)$, $(\mu_1,\dots,\mu_n)\in\mathbb{R}_+^n$, $(\alpha_1,\dots,\alpha_n)$, $(\beta_1,\dots,\beta_n)\in\mathbb{R}^n$ és definiáljuk a $p,q\colon I\to\mathbb{R}_+^n$ függvényeket* (52) *szerint. Tegyük fel, hogy léteznek $\gamma>0$ és $\delta\in\mathbb{R}$ konstansok, hogy* (53) *fennáll, és*

$$\frac{f(x)-f(y)}{f'(y)}\leq\frac{x^{\delta}(g(x)-g(y))}{y^{\delta}g'(y)}\qquad(x,y\in I).$$

Ekkor $A_{f,p}^{[n]}$ *globálisan kisebb, mint* $A_{g,q}^{[n]}$.

Következmény. *Legyen* $I \subseteq \mathbb{R}_+$ *egy nemüres, nyílt intervallum,* $n \in \mathbb{N}, n \geq 2$, $(\lambda_1, \dots, \lambda_n)$, $(\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$, $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$, *és legyenek* $a, b \in \mathbb{R}$ *konstansok. Definiáljuk a* $p, q\colon I \to \mathbb{R}_+^n$ *függvényeket* (52)*, a* $f, g\colon I \to \mathbb{R}$ *függvényeket pedig* (55) *szerint. Tegyük fel, hogy léteznek* $\gamma > 0$ *és* $\delta \in \mathbb{R}$ *konstansok, hogy* (53)*, illetve a*

$$\min(a, 0) \leq \delta + \min(b, 0) \qquad \text{és} \qquad \max(a, 0) \leq \delta + \max(b, 0)$$

egyenlőtlenségek teljesülnek. Ekkor $A_{f,p}^{[n]}$ *globálisan kisebb, mint* $A_{g,q}^{[n]}$.

A 4. Fejezet összefoglalója – Lokális és globális Hölder- és Minkowski-típusú egyenlőtlenségek

A Hölder, illetve Minkowski által felfedezett híres egyenlőtlenségek számos formában felírhatók, például hatványközepekre vonatkozóan is. Ahhoz, hogy felidézzük a Rogers által 1888-ban és Hölder által 1889-ben kitalált klasszikus Hölder(–Rogers) egyenlőtlenséget, legyen $p, q > 1$ olyanok, hogy $\frac{1}{p} + \frac{1}{q} = 1$. Ekkor minden $n \in \mathbb{N}$ és $x, y \in \mathbb{R}_+^n$ esetén az

$$\frac{x_1 y_1 + \dots + x_n y_n}{n} \leq \left(\frac{x_1^p + \dots + x_n^p}{n}\right)^{\frac{1}{p}} \left(\frac{y_1^q + \dots + y_n^q}{n}\right)^{\frac{1}{q}}$$

egyenlőtlenség teljesül. A $p = q = 2$ speciális esetben ez az egyenlőtlenség az úgynevezett Cauchy–Bunyakovsky–Schwarz egyenlőtlenségre redukálódik, amelyet a fenti formában Cauchy fedezett fel 1821-ben. Egy adott $p \geq 1$ valós paraméter esetén az 1910-ben publikált klasszikus Minkowski-egyenlőtlenség azt állítja, hogy a p-edik hatványközép szubadditív, azaz minden $n \in \mathbb{N}$ és $x, y \in \mathbb{R}_+^n$ esetén az

$$\left(\frac{(x_1 + y_1)^p + \dots + (x_n + y_n)^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{x_1^p + \dots + x_n^p}{n}\right)^{\frac{1}{p}} + \left(\frac{y_1^p + \dots + y_n^p}{n}\right)^{\frac{1}{p}}$$

egyenlőtlenség fennáll.

Tömören, ennek a fejezetnek az a célja, hogy az összeadást, illetve a szorzást egy általánosabb műveletre és a hatványközepeket pedig általánosított Bajraktarević-közepekre, speciálisan súlyozott Gini-közepekre cserélve, a Hölder- és Minkowski-egyenlőtlenségekkel analóg egyenlőtlenségeket vizsgáljunk. Ezenfelül az is célkitűzésünk, hogy karakterizáljuk ezeket az egyenlőtlenségeket lokális és globális értelemben is. Szükséges, illetve elégséges feltételeket fogunk levezetni az

$$M_0(\Phi(x_1), \dots, \Phi(x_n)) \leq \Phi(M_1(x^1), \dots, M_k(x^k)), \tag{57}$$

függvényegyenlőtlenség lokális, illetve globális teljesülésére, ahol $n \in \mathbb{N}, n \geq 2, k \in \mathbb{N}$, minden $\alpha \in \{0, \dots, k\}$ esetén $I_\alpha \subseteq \mathbb{R}$ egy nemüres, nyílt intervallum, $I := I_1 \times \dots \times I_k$, $M_\alpha\colon I_\alpha^n \to I_\alpha$ egy n-változós közép, és $\Phi\colon I \to I_0$ adott leképezés. Összhangban a korábbi fejezetekben használt terminológiával, ha létezik olyan $U \subseteq I^n$ nyílt halmaz, amely tartalmazza a $\mathrm{diag}(I^n)$ diagonálist, és a (57) egyenlőtlenség teljesül minden $x \in U^T \subseteq \prod_{\alpha=1}^k I_\alpha^n$ esetén, akkor azt mondjuk, hogy (57) *lokálisan érvényes*. Ha (57) fennáll minden $x \in (I^n)^T = \prod_{\alpha=1}^k I_\alpha^n$ esetén, akkor azt mondjuk, hogy (57) *globálisan érvényes*. Világos, hogy (57) globális érvényessége maga után vonja annak lokális érvényességét.

Ezt követően azt a speciális esetét tekintjük a (57) egyenlőtlenségnek, amikor az összes közép n-változós általánosított Bajraktarević-közép, azaz az

$$A_{f_0,p^0}^{[n]}(\Phi(x_1),\dots,\Phi(x_n)) \leq \Phi(A_{f_1,p^1}^{[n]}(x^1),\dots,A_{f_k,p^k}^{[n]}(x^k)), \tag{58}$$

függvényegyenlőtlenséggel foglalkozunk, ahol minden $\alpha \in \{0,\dots,k\}$ esetén az ismeretlen $f_\alpha\colon I_\alpha \to \mathbb{R}$ függvény szigorúan monoton, és $p^\alpha\colon I_\alpha \to \mathbb{R}_+^n$ is ismeretlen függvény. Szükséges, illetve elégséges feltételeket kapunk a lokális, illetve a globális érvényességére.

Megemlítjük (58) néhány fontos speciális esetét.

1. Ha $k=1$, $I_0 = I_1 =: J$ és $\Phi(x) = x$, akkor (58) az általánosított Bajraktarević-közepek J intervallumon való lokális és globális összehasonlíthatóságára egyszerűsödik.

2. Ha $k \in \mathbb{N}, k \geq 2$, $I_0 = I_1 = \dots = I_k =: J$, $\Phi(x_1,\dots,x_k) = \frac{1}{k}(x_1+\dots+x_k)$, és $f_0 = f_1 = \dots = f_k =: f$, $p^0 = p^1 = \dots = p^k =: p$, akkor (58) az $A_{f,p}^{[n]}$ közép J-n való Jensen-konvexitását jelenti. Ebben az esetben (58)-mat Jensen-típusú egyenlőtlenségnek mondjuk.

3. Ha $k \in \mathbb{N}, k \geq 2$, $I_0 = I_1 = \dots = I_k = \mathbb{R}_+$, $\Phi(x_1,\dots,x_k) = x_1+\dots+x_k$, és $f_0 = f_1 = \dots = f_k =: f$, $p^0 = p^1 = \dots = p^k =: p$, akkor (58) az $A_{f,p}^{[n]}$ közép $\mathbb{R}_+$ halmazon való szubadditivitását fejezi ki, amelyet gyakran Minkowski-típusú egyenlőtlenségnek is mondunk.

4. Ha $k \in \mathbb{N}, k \geq 2$, $I_0 = I_1 = \dots = I_k = \mathbb{R}_+$, $\Phi(x_1,\dots,x_k) = x_1 \cdots x_k$, akkor (58) egy az $A_{f_0,p^0}^{[n]}, A_{f_1,p^1}^{[n]},\dots,A_{f_k,p^k}^{[n]}$ közepekre vonatkozó Hölder-típusú egyenlőtlenségre redukálódik.

Három tétel kivételével a fejezet összes eredményét a [14] dolgozatunkban bizonyítottuk először. A három tétel közül kettő a Gini-közepekre vonatkozó Minkowski-típusú egyenlőtlenség globális fennállását karakterizálja a 2-változós esetben, illetve rögzítetlen változószám mellett. A harmadik a Gini-közepekre vonatkozó Hölder-típusú egyenlőtlenség globális teljesülésére ad szükséges és elégséges feltételrendszert rögzítetlen változószám mellett. Mindhárom említett tétel előtt kihangsúlyozzuk, hogy az adott eredmény mely szerző(k)nek köszönhető, és melyik publikációban található meg.

Hölder- és Minkowski-típusú egyenlőtlenségek a lokális értelemben

Ebben és a következő alfejezetben minden $k \in \mathbb{N}$ és $\alpha \in \{0,\dots,k\}$ esetén $I_\alpha \subseteq \mathbb{R}$ egy nemüres, nyílt intervallumot fog jelölni, és az $I := I_1 \times \dots \times I_k$ jelölést is rögzítjük.

A (57) egyenlőtlenség vizsgálatához vezessük be az $F\colon I_1^n \times \cdots \times I_k^n \to \mathbb{R}$ függvényt, amely az alábbi képlettel definiált:

$$F(x) = F(x^1, \ldots, x^k) := \Phi(M_1(x^1), \ldots, M_k(x^k)) - M_0(\Phi(x_1), \ldots, \Phi(x_n)). \quad (59)$$

Megjegyzés. Vegyük észre, hogy minden $y \in I$ esetén $F(\Delta_n^k(y)) = 0$. Valóban, felhasználva az $M_0, M_1, \ldots, M_k$ függvények középérték tulajdonságát, azt kapjuk, hogy

$$\begin{aligned} F(\Delta_n^k(y)) &= F(\Delta_n(y_1), \ldots, \Delta_n(y_k)) \\ &= \Phi(M_1(\Delta_n(y_1)), \ldots, M_k(\Delta_n(y_k))) - M_0(\Delta_n(\Phi(y))) \\ &= \Phi(M_1^\Delta(y_1), \ldots, M_k^\Delta(y_k)) - M_0^\Delta(\Phi(y)) = \Phi(y_1, \ldots, y_k) - \Phi(y) = 0. \end{aligned}$$

Megfogalmazzuk a következő lemmát az F függvény parciális deriváltjainak kiszámításához a $\Delta_n^k(y)$ alakú pontokban.

Lemma. *Legyenek $n, k \in \mathbb{N}, k \geq 2$ és minden $\alpha \in \{0, \ldots, k\}$ esetén $M_\alpha\colon I_\alpha^n \to I_\alpha$ egy n-változós közép, definiáljuk az $F\colon I_1^n \times \cdots \times I_k^n \to \mathbb{R}$ függvényt (59) szerint, és legyen $\Phi\colon I \to I_0$.*

(i) Minden $\alpha \in \{0, \ldots, k\}$ esetén tegyük fel, hogy M_α parciálisan differenciálható a $\operatorname{diag}(I_\alpha^n)$ diagonálison, és hogy Φ differenciálható. Ekkor minden $i \in \{1, \ldots, k\}$, $\ell \in \{1, \ldots, n\}$ és $y \in I$ esetén

$$\partial_{\ell+n(i-1)} F(\Delta_n^k(y)) = \partial_i \Phi(y)\big(\partial_\ell M_i^\Delta(y_i) - \partial_\ell M_0^\Delta(\Phi(y))\big).$$

(ii) Minden $\alpha \in \{0, \ldots, k\}$ esetén tegyük fel, hogy M_α kétszer parciálisan differenciálható a $\operatorname{diag}(I_\alpha^n)$ diagonálison, és hogy Φ kétszer differenciálható. Ekkor minden $i, j \in \{1, \ldots, k\}$, $\ell, m \in \{1, \ldots, n\}$ és $y \in I$ esetén

$$\begin{aligned} &\partial_{\ell+n(i-1)} \partial_{m+n(j-1)} F(\Delta_n^k(y)) \\ &\qquad = \partial_i \partial_j \Phi(y)\big(\partial_m M_j^\Delta(y_j) \partial_\ell M_i^\Delta(y_i) - \delta_{\ell,m} \partial_m M_0^\Delta(\Phi(y))\big) \\ &\qquad\quad - \partial_j \Phi(y)\big(\partial_i \Phi(y) \partial_\ell \partial_m M_0^\Delta(\Phi(y)) - \delta_{i,j} \partial_\ell \partial_m M_j^\Delta(y_j)\big). \end{aligned}$$

Az alábbi két eredmény az első- és másodrendű szükséges feltételeket írja le (57) lokális teljesüléséhez.

Tétel. *Legyenek $n, k \in \mathbb{N}, k \geq 2$ és minden $\alpha \in \{0, \ldots, k\}$ esetén $M_\alpha\colon I_\alpha^n \to I_\alpha$ egy n-változós közép, amely parciálisan differenciálható a $\operatorname{diag}(I_\alpha^n)$ diagonálison, és legyen továbbá $\Phi\colon I \to I_0$ szürjektív és differenciálható, nemnulla elsőrendű parciális deriváltakkal rendelkező függvény. Tegyül fel, hogy a (57) egyenlőtlenség lokálisan érvényes. Ekkor léteznek $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$ konstansok, hogy minden $(y_0, y) \in I_0 \times I$ és $\ell \in \{1, \ldots, n\}$ esetén*

$$\lambda_\ell = \partial_\ell M_0^\Delta(y_0) = \partial_\ell M_1^\Delta(y_1) = \cdots = \partial_\ell M_k^\Delta(y_k). \quad (60)$$

Ha ráadásul néhány $\alpha \in \{0, \ldots, k\}$ és $y_\alpha \in I_\alpha$ esetén az M_α közép differenciálható $\Delta_n(y_\alpha)$-ban, akkor $\lambda_1 + \cdots + \lambda_n = 1$ is fennáll.

Tétel. *Legyenek $n, k \in \mathbb{N}, k \geq 2$ és minden $\alpha \in \{0, \dots, k\}$ esetén $M_\alpha\colon I_\alpha^n \to I$ egy n-változós közép, amely kétszer parciálisan differenciálható a $\operatorname{diag}(I_\alpha^n)$ diagonálison, és legyen $\Phi\colon I \to I_0$ egy szürjektív, kétszer differenciálható, nemnulla elsőrendű parciális deriváltakkal rendelkező függvény. Tegyük fel, hogy az (57) egyenlőtlenség lokálisan érvényes. Ekkor léteznek $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ konstansok, hogy $\lambda_1 + \dots + \lambda_n = 1$, és minden $(y_0, y) \in I_0 \times I$ és $\ell \in \{1, \dots, n\}$ esetén a (60)-beli egyenlőségek fennállnak. Ha ráadásul minden $y \in I$ esetén az $((nk) \times (nk))$-típusú mátrix, amelynek $(\ell + n(i-1), m + n(j-1))$-edik eleme, ahol $i, j \in \{1, \dots, k\}$ és $\ell, m \in \{1, \dots, n\}$, a*

$$\begin{aligned}\partial_i\partial_j\Phi(y)(\lambda_m\lambda_\ell - \delta_{\ell,m}\lambda_m) - \partial_i\Phi(y)\partial_j\Phi(y)\partial_\ell\partial_m M_0^\Delta(\Phi(y))\\ + \delta_{i,j}\partial_j\Phi(y)\partial_\ell\partial_m M_j^\Delta(y_j)\end{aligned}$$

módon adott, pozitív szemidefinit.

Általánosított Bajraktarević-közepekre vonatkozó Hölder- és Minkowski típusú egyenlőtlenségek

Az alábbi tétel azt írja le, első, illetve másodrendű differenciálhatósági feltétel mellett, hogy (58) lokális érvényessége esetén mi teljesül szükségszerűen.

Tétel. *Legyenek $n, k \in \mathbb{N}, k \geq 2$ és minden $\alpha \in \{0, \dots, k\}$ esetén $f_\alpha\colon I_\alpha \to \mathbb{R}$ egy differenciálható, nemnulla első deriválttal rendelkező függvény, legyen továbbá $p^\alpha = (p_1^\alpha, \dots, p_n^\alpha)\colon I_\alpha \to \mathbb{R}_+^n$ folytonos és $p_0^\alpha := p_1^\alpha + \dots + p_n^\alpha$. Legyen $\Phi\colon I \to I_0$ szürjektív, differenciálható, nemnulla elsőrendű parciális deriváltakkal rendelkező függvény. Tegyük fel, hogy a (58) egyenlőtlenség lokálisan érvényes. Ekkor léteznek $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ konstansok, hogy $\lambda_1 + \dots + \lambda_n = 1$ és minden $\alpha \in \{0, \dots, k\}$, illetve $\ell \in \{1, \dots, n\}$ esetén*

$$p_\ell^\alpha = \lambda_\ell p_0^\alpha \tag{61}$$

fennáll az I_α intervallumon. Ha ráadásul minden $\alpha \in \{0, \dots, k\}$ esetén f_α kétszer differenciálható, p^α folytonosan differenciálható és Φ kétszer differenciálható, akkor a

$$\begin{aligned}\Gamma(y) := \Bigg(- \partial_i\partial_j\Phi(y) - \partial_j\Phi(y)\partial_i\Phi(y)\bigg(2\frac{(p_0^0)'}{p_0^0} + \frac{f_0''}{f_0'}\bigg)(\Phi(y))\\ + \delta_{i,j}\partial_j\Phi(y)\bigg(2\frac{(p_0^j)'}{p_0^j} + \frac{f_j''}{f_j'}\bigg)(y_j)\Bigg)_{i,j=1}^k\end{aligned} \tag{62}$$

képlettel adott $(k \times k)$ típusú $\Gamma(y)$ mátrix pozitív szemidefinit minden $y \in I$ esetén.

Az alábbi eredmény egy konvexitási tulajdonság formájában újrafogalmazza a fenti tétel pozitív szemidefinitségi feltételét.

Tétel. *Legyenek $n, k \in \mathbb{N}, k \geq 2$ és minden $\alpha \in \{0, \dots, k\}$ esetén $f_\alpha\colon I_\alpha \to \mathbb{R}$ egy kétszer differenciálható, nemnulla első deriválttal rendelkező függvény, legyen továbbá*

$p^\alpha = (p_1^\alpha, \dots, p_n^\alpha)\colon I_\alpha \to \mathbb{R}_+^n$ *folytonosan differenciálható és* $p_0^\alpha := p_1^\alpha + \dots + p_n^\alpha$. *Legyenk* $\Phi\colon I \to I_0$ *egy szürjektív, kétszer differenciálható, nemnulla elsőrendű parciális deriváltakkal rendelkező függvény. Tegyük fel, hogy a (58) egyenlőtlenség lokálisan érvényes. Végül, minden* $\alpha \in \{0, \dots, k\}$ *esetén definiáljuk a* $\varphi_\alpha\colon I_\alpha \to \mathbb{R}$ *és aztán a* $\varphi\colon I \to \mathbb{R}^k$ *függvényeket az alábbiak szerint:*

$$\varphi_\alpha := \int (p_0^\alpha)^2 f_\alpha' \qquad \text{és} \qquad \varphi(y) := (\varphi_1(y_1), \dots, \varphi_k(y_k)).$$

Ekkor $\varphi_1, \dots, \varphi_k$ *és* φ *kétszer differenciálható, invertálható függvények, és a*

$$\Psi(u) := \varphi_0(\Phi(\varphi^{-1}(u)))$$

formulával definiált $\Psi\colon \varphi(I) \to \mathbb{R}$ *leképezés konkáv, ha* $f_0' > 0$, *és konvex, ha* $f_0' < 0$.

Megjegyzés. Megjegyezzük, hogy a Ψ segédfüggvény konkávitása nem csupán egy következménye a Γ mátrixértékű függvény pozitív szemidefinitségének, hanem igazából ekvivalens azzal. Másfelől, ha minden súlyfüggvény a konstans 1 függvény, akkor $\varphi_\alpha = f_\alpha$, és a kváziaritmetikai közepek elmélete szerint (ld. [15]) ebben az esetben Ψ konkávitása elégséges is az (58) egyenlőtlenség globális teljesüléséhez.

Az alábbi két tétel elégséges feltételeket ad az (58) egyenlőtlenség lokális, illetve globális fennállására.

Tétel. *Legyen* $k \in \mathbb{N}$ *és minden* $\alpha \in \{0, \dots, k\}$ *esetén* $f_\alpha\colon I_\alpha \to \mathbb{R}$ *egy differenciálható, nemnulla első deriválttal rendelkező függvény,* $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$, *és legyen továbbá* $\Phi\colon I \to I_0$ *parciálisan differenciálható. Tegyük fel, hogy az alábbi egyenlőtlenség lokálisan teljesül az* I^2 *halmazon, azaz létezik egy* $V \subseteq I^2$ *nemüres, nyílt halmaz, amely tartalmazza a* $\operatorname{diag}(I^2)$ *diagonálist, és minden* $(u, y) \in V$ *esetén*

$$\frac{p_0^0(\Phi(y))(f_0(\Phi(y)) - f_0(\Phi(u)))}{p_0^0(\Phi(u))f_0'(\Phi(u))} \le \sum_{j=1}^{k} \partial_j \Phi(u) \frac{p_0^j(y_j)(f_j(y_j) - f_j(u_j))}{p_0^j(u_j) f_j'(u_j)} \tag{63}$$

fennáll. Ekkor minden $n \in \mathbb{N}$ *és olyan* $\lambda \in \mathbb{R}_+^n$ *konstansok esetén, amelyekre* $\lambda_1 + \dots + \lambda_n = 1$ *érvényes, az*

$$A^{[n]}_{f_0, p_0^0 \lambda}(\Phi(x_1), \dots, \Phi(x_n)) \le \Phi(A^{[n]}_{f_1, p_0^1 \lambda}(x^1), \dots, A^{[n]}_{f_k, p_0^k \lambda}(x^k)) \tag{64}$$

egyenlőtlenség lokálisan teljesül.

Megjegyzés. Vegyük észre, hogy az (58)-ben előforduló általánosított Bajraktarević-közepek súlyfüggvényei szükségszerűen olyan alakúak, mint (61)-ben. Ezért a (58) egyenlőtlenség lokális, illetve globális teljesülése azonnal következik a (64) egyenlőtlenség lokális, illetve globális érvényességéből.

Tétel. *Legyen $k \in \mathbb{N}$ és minden $\alpha \in \{0, \dots, k\}$ esetén $f_\alpha\colon I_\alpha \to \mathbb{R}$ egy differenciálható, nemnulla első deriválttal rendelkező függvény, $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$, és legyen $\Phi\colon I \to I_0$ parciálisan differenciálható. Tegyük fel, hogy (63) globálisan, azaz minden $u, y \in I$ esetén teljesül az I^2 halmazon. Ekkor minden $n \in \mathbb{N}$ és olyan $\lambda \in \mathbb{R}_+^n$ konstansok esetén, amelyekre $\lambda_1 + \dots + \lambda_n = 1$ érvényes, a (64) egyenlőtlenség globálisan teljesül.*

Az alábbi eredmény egy szükséges feltételt ad a (63) egyenlőtlenség lokális fennállására.

Tétel. *Legyen $k \in \mathbb{N}$ és minden $\alpha \in \{0, \dots, k\}$ esetén $f_\alpha\colon I_\alpha \to \mathbb{R}$ egy kétszer differenciálható, nemnulla első deriválttal rendelkező függvény, $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ pedig egy kétszer differenciálható függvény. Legyen továbbá $\Phi\colon I \to I_0$ kétszer differenciálható. Tegyük fel, hogy a (63) egyenlőtlenség lokálisan teljesül, azaz létezik olyan $V \subseteq I^2$ nemüres, nyílt halmaz, amely tartalmazza a $\mathrm{diag}(I^2) \subseteq V$ diagonálist, és (63) fennáll minden $(u, y) \in V$ esetén. Ekkor a (62) által definiált $\Gamma\colon I \to \mathbb{R}^{k\times k}$ mátrixértékű függvény értékei pozitív szemidefinit mátrixok.*

A fenti állítása megfordítása általánosságban nem igaz. A feltételeken szigorítani kell, nevezeten pozitív definitséget kell feltételezni pozitív szemidefinitség helyett.

Tétel. *Legyen $k \in \mathbb{N}$ és minden $\alpha \in \{0, \dots, k\}$ esetén $f_\alpha\colon I_\alpha \to \mathbb{R}$ egy kétszer folytonosan differenciálható, nemnulla első deriválttal rendelkező függvény, $p_0^\alpha\colon I_\alpha \to \mathbb{R}_+$ pedig egy kétszer folytonosan differenciálható függvény. Legyen továbbá $\Phi\colon I \to I_0$ kétszer folytonosan differenciálható. Tegyük fel, hogy minden $y \in I$ esetén a (62) szerint adott $(k \times k)$-típusú $\Gamma(y)$ mátrix pozitív definit. Ekkor minden $n \in \mathbb{N}$ és olyan $\lambda \in \mathbb{R}_+^n$ esetén, amelyekre $\lambda_1 + \dots + \lambda_n = 1$ érvényes, a (64) egyenlőtlenség lokálisan teljesül.*

Súlyozott Gini-közepekre vonatkozó Hölder- és Minkowski-típusú egyenlőtlenségek

Ebben a fejezetben $k \in \mathbb{N}, \alpha \in \{1, \dots, k\}$ esetén I_α az $\mathbb{R}_+$ halmaz egy nemüres, nyílt részintervallumát jelöli, legyen $I := I_1 \times \dots \times I_k$, és a fenti eredményeket alkalmazzuk a (57) egyenlőtlenség néhány fontos speciális esetére. Először olyan egyenlőtlenségekkel foglalkozunk, amelyeket úgy kapunk (57)-ből, hogy, minden $\alpha \in \{0, \dots, k\}$ esetén, az M_α közepeket konkretizáljuk, aztán pedig az olyan esetekben vonunk le következtetéseket, amikor (57)-ben a Φ függvényt a k-változós összeadásnak, illetve szorzásnak választjuk.

Definíció. Legyen $n \in \mathbb{N}$ és elevenítsük fel adott $r \in \mathbb{R}$ esetén az *n-változós $\lambda \in \mathbb{R}_+^n, \lambda_1 + \dots + \lambda_n = 1$ súlyvektorral súlyozott r-edig hatvány vagy r-edik Hölder közép*, adott $(r, s) \in \mathbb{R}^2$ esetén pedig az *n-változós $\lambda \in \mathbb{R}_+^n, \lambda_1 + \dots + \lambda_n = 1$ súlyvektorral* *súlyozott, $(r,s)\in\mathbb{R}^2$ paraméterű Gini-közép* fogalmát:

$$H^{[n]}_{r;\lambda}(x_1,\dots,x_n) := \begin{cases} \big(\lambda_1 x_1^r + \cdots + \lambda_n x_n^r\big)^{\frac{1}{r}} & \text{ha } r\neq 0 \\ x_1^{\lambda_1}\cdots x_n^{\lambda_n} & \text{ha } r=0, \end{cases}$$

$$G^{[n]}_{r,s;\lambda}(x_1,\dots,x_n) := \begin{cases} \left(\dfrac{\lambda_1 x_1^r + \cdots + \lambda_n x_n^r}{\lambda_1 x_1^s + \cdots + \lambda_n x_n^s}\right)^{\frac{1}{r-s}} & \text{ha } r\neq s \\ \exp\left(\dfrac{\lambda_1 x_1^r \ln(x_1) + \cdots + \lambda_n x_n^r \ln(x_n)}{\lambda_1 x_1^r + \cdots + \lambda_n x_n^r}\right) & \text{ha } r=s. \end{cases}$$

Megjegyzés. Világos, hogy a $s=0$ speciális esetben $G^{[n]}_{r,s;\lambda}$ a $H^{[n]}_{r;\lambda}$ közepet adja vissza. Azt is megjegyezzük, hogy minden $n\in\mathbb{N}$ és $\ell\in\{1,\dots,n\}$ esetén a $p_\ell(t):=\lambda_\ell t^s$ és $f(t):=t^{r-s}$, ha $r\neq s$ vagy $f(t):=\ln(t)$, ha $r=s$ választásokkal $A^{[n]}_{f,p}=G^{[n]}_{r,s;\lambda}$ adódik.

Az alábbi állítás egy, a súlyozott Gini-közepekre vonatkozó általános egyenlőtlenség lokális fennállására ad szükséges, illetve elégséges feltételt.

Tétel. *Legyenek $n,k\in\mathbb{N}, k\geq 2$, $\lambda\in\mathbb{R}^n_+$ és minden $\alpha\in\{0,\dots,k\}$ esetén legyen $(r_\alpha,s_\alpha)\in\mathbb{R}^2$. Legyen továbbá $\Phi\colon I\to\mathbb{R}_+$ kétszer differenciálható, nemnulla elsőrendű parciális deriváltakkal rendelkező függvény. Ekkor a*

$$G^{[n]}_{r_0,s_0;\lambda}(\Phi(x_1),\dots,\Phi(x_n)) \leq \Phi\big(G^{[n]}_{r_1,s_1;\lambda}(x^1),\dots,G^{[n]}_{r_k,s_k;\lambda}(x^k)\big) \tag{65}$$

egyenlőtlenség lokális teljesüléséhez szükséges, hogy a

$$\Gamma(y) := \left(-\partial_i\partial_j\Phi(y) - \partial_j\Phi(y)\partial_i\Phi(y)\frac{r_0+s_0-1}{\Phi(y)} + \delta_{i,j}\partial_j\Phi(y)\frac{r_j+s_j-1}{y_j}\right)_{i,j=1}^k$$

formulával definiált $(k\times k)$-típusú $\Gamma(y)$ mátrix pozitív szemidefinit legyen minden $y\in I$ esetén. Megfordítva, ha ez a mátrix pozitív definit minden $y\in I$ esetén, akkor (65) lokálisan fennáll I-n.

Az alábbi tétel egy elégséges feltételt ad a (65) egyenlőtlenség globális érvényességére.

Tétel. *Legyen $k\in\mathbb{N}, k\geq 2$ és minden $\alpha\in\{0,\dots,k\}$ esetén legyen $(r_\alpha,s_\alpha)\in\mathbb{R}^2$. Legyen továbbá $\Phi\colon I\to\mathbb{R}_+$ egy parciálisan differenciálható függvény. Tegyük fel, hogy*

$$\Phi(u)\chi_{r_0,s_0}\left(\frac{\Phi(y)}{\Phi(u)}\right) \leq \sum_{j=1}^k \partial_j\Phi(u)u_j\chi_{r_j,s_j}\left(\frac{y_j}{u_j}\right)$$

globálisan, azaz minden $(u,y)\in I^2$-re teljesül. Ekkor minden $n\in\mathbb{N}$ és olyan $\lambda\in\mathbb{R}^n_+$ esetén, amelyre $\lambda_1+\cdots+\lambda_n=1$ fennáll, a (65) egyenlőtlenség globálisan érvényes I-n.

Azon speciális esetek vizsgálatához, amikor Φ az összeg-, illetve a szorzatfüggvény, szükségünk lesz az alábbi segédállításra.

Lemma. *Legyen $k \in \mathbb{N}$ és minden $i \in \{0, \dots, k\}$ esetén $c_i \in \mathbb{R}$. Ekkor a*

$$C := \big(\delta_{i,j} c_i + c_0\big)_{i,j=1}^{k}$$

mátrix pontosan akkor pozitív szemidefinit, ha vagy $c_i \geq 0$ minden $i \in \{0, \dots, k\}$ esetén, vagy létezik $i \in \{0, \dots, k\}$, hogy $c_i < 0$ és $c_j > 0$ minden $j \in \{0, \dots, k\} \setminus \{i\}$ indexre, és

$$\frac{1}{c_0} + \frac{1}{c_1} + \cdots + \frac{1}{c_k} \leq 0. \tag{66}$$

Továbbá C pontosan akkor pozitív definit, ha vagy $c_i \geq 0$ minden $i \in \{0, \dots, k\}$ esetén, és $c_i = 0$ legfeljebb egy $i \in \{0, \dots, k\}$ indexre teljesül, vagy létezik $i \in \{0, \dots, k\}$, hogy $c_i < 0$ és $c_j > 0$ minden $j \in \{0, \dots, k\} \setminus \{i\}$ esetén és (66) szigorú egyenlőtlenséggel áll fenn.

Az alábbi eredmény karakterizálja a súlyozott Gini-közepekre vonatkozó Minkowski-típusú egyenlőtlenséget a lokális esetben.

Tétel. *Legyenek $n, k \in \mathbb{N}, k \geq 2$ és $\lambda \in \mathbb{R}_+^n$. Minden $i \in \{0, \dots, k\}$ esetén legyen továbbá $(r_i, s_i) \in \mathbb{R}^2$ és $\gamma_i := r_i + s_i - 1$. A*

$$G^{[n]}_{r_0,s_0;\lambda}(x_1^1 + \cdots + x_1^k, \dots, x_n^1 + \cdots + x_n^k) \leq G^{[n]}_{r_1,s_1;\lambda}(x^1) + \cdots + G^{[n]}_{r_k,s_k;\lambda}(x^k) \tag{67}$$

egyenlőtlenség I-n való lokális fennállásához szükséges, hogy az alábbi esetek közül pontosan egy teljesüljön:

(i)

$$\gamma_0 \leq 0 \leq \min(\gamma_1, \dots, \gamma_k); \tag{68}$$

(ii) $\gamma_i > 0$ *minden* $i \in \{0, \dots, k\}$ *indexre, és*

$$\sum_{i \in J_+} \left(\frac{1}{\gamma_i} - \frac{1}{\gamma_0}\right) \sup I_i \leq \sum_{i \in J_-} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i}\right) \inf I_i; \tag{69}$$

(iii) $\gamma_0 < 0$ *és létezik* $i \in \{1, \dots, k\}$, *hogy* $\gamma_i < 0$, *minden* $j \in \{1, \dots, k\} \setminus \{i\}$ *esetén* $\gamma_j > 0$, *és a (69) egyenlőtlenség fennáll,*

ahol, az utolsó két esethez definiáljuk az alábbi halmazokat:

$$J_+ := \left\{ i \in \{1, \dots, k\} \mid \frac{1}{\gamma_i} > \frac{1}{\gamma_0} \right\} \qquad \text{és} \qquad J_- := \left\{ i \in \{1, \dots, k\} \colon \frac{1}{\gamma_0} > \frac{1}{\gamma_i} \right\}.$$

Megfordítva, ha vagy (68) teljesül és emellett $\gamma_i = 0$ legfeljebb egy $i \in \{0, \dots, k\}$ index esetén teljesül, vagy $\gamma_0 \neq \gamma_\ell$ valamilyen $\ell \in \{1, \dots, k\}$ indexre és (ii) vagy (iii) közül az egyik feltétel teljesül, akkor (67) lokálisan érvényes I-n.

Következmény. *Legyenek $n, k \in \mathbb{N}, k \geq 2$ és $\lambda \in \mathbb{R}_+^n$. Minden $i \in \{0, \dots, k\}$ esetén legyen továbbá $(r_i, s_i) \in \mathbb{R}^2$. Minden $\alpha \in \{1, \dots, k\}$ index esetén tegyük fel, hogy az $I_\alpha \subseteq \mathbb{R}_+$ nemüres, nyílt intervallum kielégíti az $\inf I_\alpha = 0$ megszorítást. Ekkor ahhoz, hogy a (67) egyenlőtlenség lokálisan fennálljon az I intervallumon, szükséges az, hogy*

$$\max(1, r_0 + s_0) \leq \min(r_1 + s_1, \dots, r_k + s_k).$$

Megfordítva, ha ez az egyenlőtlenség szigorú, akkor (67) lokálisan teljesül I-n.

Az következő állítás a Minkowski-típusú egyenlőtlenség globális érvényességére ad egy szükséges feltételt.

Tétel. *Legyen $k \in \mathbb{N}, k \geq 2$ és minden $i \in \{0, \dots, k\}$ esetén legyen $(r_i, s_i) \in \mathbb{R}^2$. Tegyük fel, hogy minden $(u, y) \in I^2$ esetén a*

$$\chi_{r_0,s_0}\left(\frac{y_1 + \cdots + y_k}{u_1 + \cdots + u_k}\right) \leq \sum_{j=1}^{k} \frac{u_j}{u_1 + \cdots + u_k} \chi_{r_j,s_j}\left(\frac{y_j}{u_j}\right)$$

egyenlőtlenség igaz. Ekkor minden $n \in \mathbb{N}$ és olyan $\lambda \in \mathbb{R}_+^n$ esetén, amelyre $\lambda_1 + \cdots + \lambda_n = 1$ teljesül, a (67) egyenlőtlenség globálisan fennáll az I intervallumon.

A mögöttes intervallumot specializálva egy konkrétabb állítást is meg tudunk fogalmazni.

Következmény. *Legyen $k \in \mathbb{N}, k \geq 2$ és minden $i \in \{0, \dots, k\}$ index esetén legyen $(r_i, s_i) \in \mathbb{R}^2$. Tegyük fel, hogy minden $z \in \mathbb{R}_+^k$ és olyan $t_1, \dots, t_k \in [0, 1]$ esetén, amelyre $t_1 + \cdots + t_k = 1$ teljesül, az alábbi egyenlőtlenség fennáll:*

$$\chi_{r_0,s_0}(t_1 z_1 + \cdots + t_k z_k) \leq \sum_{j=1}^{k} t_j \chi_{r_j,s_j}(z_j). \tag{70}$$

Ekkor minden $n \in \mathbb{N}$ és olyan $\lambda \in \mathbb{R}_+^n$ esetén, amelyre $\lambda_1 + \cdots + \lambda_n = 1$ teljesül, a (67) egyenlőtlenség globálisan érvényes az $\mathbb{R}_+^k$ halmazon.

Ahhoz, hogy az eredményeinket összehasonlítsuk a már meglévőekkel, felelevenítünk két, a Minkowski-típusú egyenlőtlenség globális fennállására vonatkozó tételt. A 2-változós Gini-közepek esetében a Minkowski-típusú egyenlőtlenséget Czinder és Páles karakterizálta (ld. [7, Theorem 5] és egy speciális esetért [26]).

Tétel. *Legyen $k \in \mathbb{N}, k \geq 2$ és minden $i \in \{0, \dots, k\}$ esetén legyen $(r_i, s_i) \in \mathbb{R}^2$. Ekkor a*

$$G_{r_0,s_0}^{[n]}(x_1 + \cdots + x_k, y_1 + \cdots + y_k) \leq G_{r_1,s_1}^{[n]}(x_1, y_1) + \cdots + G_{r_k,s_k}^{[n]}(x_k, y_k) \tag{71}$$

egyenlőtlenség pontosan akkor teljesül minden $x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{R}_+$ esetén, ha *(i)* $0 \le \min(r_1, s_1, \dots .r_k, s_k)$,

(ii) $\min(r_0, s_0) \le \min(1, r_1, s_1, \dots .r_k, s_k)$,

(iii) $\max(1, r_0 + s_0) \le \min(r_1 + s_1, \dots, r_k + s_k)$.

Megjegyzés. Vegyük észre, hogy a fenti tétel harmadik feltétele nem más, mint a (71) egyenlőtlenség $\mathbb{R}_+$-on való lokális fennállásának szükséges feltétele. Az (i) és (ii) feltételek azonban nem szükséges feltételei (71) $\mathbb{R}_+$-on való lokális érvényességének.

Rögzítetlen változószám mellett a Gini-közepekkel kapcsolatos Minkowski-típusú egyenlőtlenség globális teljesülésére vonatkozó szükséges és elegendő feltételeket Páles határozta meg (ld. [31, Theorem 3.1]).

Tétel. *Legyen $k \in \mathbb{N}, k \ge 2$ és minden $i \in \{0, \dots, k\}$ esetén legyen $(r_i, s_i) \in \mathbb{R}^2$. Ekkor a*

$$G^{[n]}_{r_0,s_0}(x^1_1 + \dots + x^k_1, \dots, x^1_n + \dots + x^k_n) \le G^{[n]}_{r_1,s_1}(x^1) + \dots + G^{[n]}_{r_k,s_k}(x^k) \tag{72}$$

egyenlőtlenség pontosan akkor áll fenn minden $n \in \mathbb{N}$ és $x \in \mathbb{R}^{n\times k}_+$ esetén, ha

(i) $0 \le \min(r_1, s_1, \dots, r_k, s_k)$,

(ii) $\min(r_0, s_0) \le \min(1, r_1, s_1, \dots, r_k, s_k)$,

(iii) $\max(1, r_0, s_0) \le \min(\max(r_1, s_1), \dots, \max(r_k, s_k))$.

Megjegyzés. Vegyük észre, hogy az előző két tétel (i) és (ii) feltételezi megegyeznek, ezért ezek szükséges feltételei lehetnek (72) globális teljesüléséhez minden rögzített $n \in \mathbb{N}$ esetén. A harmadik feltétel alakja nem ismert rögzített $n \in \mathbb{N}$ esetén. Azt is megjegyezzük, hogy a fenti tétel (i)-(iii) feltételrendszere szükséges és elégséges ahhoz, hogy a (70) egyenlőtlenség fennálljon minden $z \in \mathbb{R}^k_+$ és olyan $t_1, \dots, t_k \in [0, 1]$ esetén, amelyre $t_1 + \dots + t_k = 1$ teljesül.

Az alábbiakban Gini-közepekre vonatkozó Hölder-típusú egyenlőtlenségeket karakterizálunk mind a lokális, mind pedig a globális esetben.

Tétel. *Legyenek $n, k \in \mathbb{N}, k \ge 2$ és $\lambda \in \mathbb{R}^n_+$. Legyenek $(r_0, s_0), \dots, (r_k, s_k) \in \mathbb{R}^2$ és $\gamma_i := r_i + s_i$ minden $i \in \{0, \dots, k\}$ esetén. Ekkor ahhoz, hogy a*

$$G^{[n]}_{-r_0,-s_0;\lambda}(x^1_1 \cdots x^k_1, \dots, x^1_n \cdots x^k_n) \le G^{[n]}_{r_1,s_1;\lambda}(x^1) \cdots G^{[n]}_{r_k,s_k;\lambda}(x^k) \tag{73}$$

egyenlőtlenség lokálisan fennálljon $\mathbb{R}^k_+$-n, szükséges az, hogy vagy $\gamma_i \ge 0$ minden $i \in \{0, \dots, k\}$-ra, vagy létezik $i \in \{0, \dots, k\}$, hogy $\gamma_i < 0$, minden $j \in (\{0, \dots, k\}) \setminus \{i\}$ esetén $\gamma_j > 0$ és

$$\frac{1}{\gamma_0} + \frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_k} \le 0 \tag{74}$$

teljesül. Megfordítva, ha vagy $\gamma_i \geq 0$ minden $i \in \{0, \ldots, k\}$ esetén és emellett $\gamma_i = 0$ legfeljebb egy $i \in \{0, \ldots, k\}$ index esetén áll fenn, vagy létezik $i \in \{0, \ldots, k\}$ hogy $\gamma_i < 0$, minden $j \in \{1, \ldots, k\} \setminus \{i\}$ esetén $\gamma_j > 0$ és (74) szigorú egyenlőtlenséggel teljesül, akkor (73) érvényes lokálisan az $\mathbb{R}_+^k$ halmazon.

Tétel. *Legyen $k \in \mathbb{N}, k \geq 2$ és minden $i \in \{0, \ldots, k\}$ esetén legyen $(r_i, s_i) \in \mathbb{R}^2$. Tegyük fel, hogy minden $z_1 \in (I_1/I_1), \ldots, z_k \in (I_k/I_k)$ esetén a*

$$\chi_{-r_0,-s_0}(z_1 \cdots z_k) \leq \sum_{j=1}^{k} \chi_{r_j,s_j}(z_j) \tag{75}$$

fennáll. Ekkor minden $n \in \mathbb{N}$ és olyan $\lambda \in \mathbb{R}_+^n$ esetén, amelyre $\lambda_1 + \cdots + \lambda_n = 1$ teljesül, a (73) egyenlőtlenség globálisan érvényes I-n.

Rögzítetlen változószám mellett a globális esetben a (73) egyenlőtlenséget Páles karakterizálta a [32, 33] dolgozatokban.

Tétel. *Legyen $k \in \mathbb{N}, k \geq 2$ és minden $i \in \{0, \ldots, k\}$ esetén legyen $(r_i, s_i) \in \mathbb{R}^2$. Ekkor a*

$$G_{-r_0,-s_0}^{[n]}(x_1^1 \cdots x_1^k, \ldots, x_n^1 \cdots x_n^k) \leq G_{r_1,s_1}^{[n]}(x^1) \cdots G_{r_k,s_k}^{[n]}(x^k)$$

egyenlőtlenség pontosan akkor teljesül minden $n \in \mathbb{N}$ és $x \in \mathbb{R}_+^{n \times k}$ esetén, ha

(i) minden $i \in \{0, \ldots, k\}$ esetén $\max(s_i, r_i) \geq 0$, és

(ii) minden olyan $i \in \{0, \ldots, k\}$ esetén, amelyre $\min(s_i, r_i) < 0$ teljesül, $\max(s_j, r_j) > 0$ érvényes minden $j \in (\{0, \ldots, k\}) \setminus \{i\}$ esetén és

$$\frac{1}{\min(s_i, r_i)} + \sum_{\substack{j=0 \\ j \neq i}}^{k} \frac{1}{\max(s_j, r_j)} \leq 0.$$

Megjegyzés. Megjegyezzük, hogy az (i) és (ii) feltételek szükségesek és elégségesek ahhoz, hogy a (75) egyenlőtlenség fennálljon minden $z_1, \ldots, z_k \in \mathbb{R}_+$ esetén (ld. [33]).

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- *Characterization of the equality of generalized Bajraktarević means*
  The 18th International Conference on Functional Equations and Inequalities
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