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CHARACTERIZATIONS OF CONVEXITY VIA HADAMARD'S INEQUALITY

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Abstract. The classical Hermite–Hadamard inequality, under some weak regularity conditions, characterizes convexity. The aim of the present paper is to give analogous result for the case of generalized convexity induced by two dimensional Chebyshev systems. The basic tool of the proofs is a characterization theorem of continuous, non-convex functions.

1. Introduction

As it is well known, the classical Hermite–Hadamard inequality (see [9] and [12] for interesting historical remarks) is not merely the consequence of convexity but, under some weak regularity assumptions, characterizes it ([11, Excersice 8, p. 205]). More precisely, the following statements remain true:

THEOREM A. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a convex function. Then, f is continuous and, for all elements $x < y$ of I, satisfies the inequality*

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt \leqslant \frac{f(x)+f(y)}{2}.$$

Conversely, if a function $f : I \to \mathbb{R}$ is continuous and, for all elements $x < y$ of I, satisfies either the inequality

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

or

$$\frac{1}{y-x}\int_x^y f(t)dt \leqslant \frac{f(x)+f(y)}{2},$$

then it is convex.

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Moreover, Jensen's inequality, that is obtained by "dropping" the integral average of the function from the Hermite–Hadamard inequality, also characterizes continuous convex functions.

Most of the results of the theory of convexity and itself the notion of convexity, too, are based on the geometric properties of affine functions. In fact, two of the properties are crucial: each affine function is continuous and every two points of the plain with distinct first coordinates can be interpolated by a unique affine function. Keeping these properties, Beckenbach generalized the notion of convexity with the help of two parameter interpolation families or, as it is named after him, *Beckenbach families* (see [1]). In this paper we restrict our investigations only to the special case when the Beckenbach family of the induced convexity notion has a *linear structure*.

DEFINITION. Let $I \subset \mathbb{R}$ be an interval. We say that a pair of functions (ω_1, ω_2) is a *positive Chebyshev system* or simply a *regular pair* over I, if $\omega_1, \omega_2 : I \to \mathbb{R}$ are continuous and, for all elements $x < y$ of I, fulfill the inequality

$$\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} > 0.$$

Given a regular pair (ω_1, ω_2) over I, a function $f : I \to \mathbb{R}$ is said to be *generalized convex with respect to* (ω_1, ω_2) or shortly (ω_1, ω_2)*-convex* if, for all elements $x < y < z$ of I, it satisfies the inequality

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} \geqslant 0.$$

Clearly, in the particular case when the members of the underlying regular pairs are the special functions $\omega_1(x) := 1$ and $\omega_2(x) := x$, the definition leads to the notion of "classical" convexity. For simplicity, in the followings this setting is cited as *standard setting* and the induced convexity notion as *standard convexity*.

In an earlier paper [4] we investigated some properties of regular pairs and generalized convexity induced by regular pairs. Among others it turned out that, on open intervals, claiming positivity on the first component of a regular pair is not an essential restriction. Therefore, in many further theorems, we assume that the regular pair (ω_1, ω_2) is a *positive* one, which means that ω_1 is positive on its domain. It can easily be checked that (ω_1, ω_2) is a positive regular pair on an open interval I if and only if the functions ω_1, ω_2 are continuous, ω_1 is positive, and the function ω_2/ω_1 is strictly monotone increasing on I.

Also in the mentioned paper, Hermite–Hadamard-type inequalities were presented for generalized convex functions which, in the standard setting, reduce to the classical Hermite–Hadamard inequality. The result reads as follows (for a generalization, see [7]):

THEOREM B. *Let (ω_1, ω_2) be a positive regular pair on an open interval I and let $\rho : I \to \mathbb{R}$ be a positive integrable function. Define, for all elements $x < y$ of I,* *the points* $\xi(x,y)$ *and the coefficients* $c(x,y), c_1(x,y), c_2(x,y)$ *by the formulae*

$$\xi(x,y)=\left(\frac{\omega_2}{\omega_1}\right)^{-1}\left(\frac{\int_x^y \omega_2\rho}{\int_x^y \omega_1\rho}\right), \qquad c(x,y)=\frac{\int_x^y \omega_1\rho}{\omega_1(\xi)} \tag{1}$$

and

$$c_1(x,y)=\frac{\begin{vmatrix} \int_x^y \omega_1\rho & \omega_1(y) \\ \int_x^y \omega_2\rho & \omega_2(y)\end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y)\end{vmatrix}}, \qquad c_2(x,y)=\frac{\begin{vmatrix} \omega_1(x) & \int_x^y \omega_1\rho \\ \omega_2(x) & \int_x^y \omega_2\rho\end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y)\end{vmatrix}}. \tag{2}$$

If $f: I \to \mathbb{R}$ *is an* (ω_1,ω_2)*-convex function, then it is continuous and, for all elements* $x<y$ *of* I*, satisfies the following Hermite–Hadamard-type inequality:*

$$c(x,y)f(\xi(x,y)) \leqslant \int_x^y f\rho \leqslant c_1(x,y)f(x)+c_2(x,y)f(y).$$

The aim of the present paper is to give an analogous result to Theorem A in the case of generalized convexity induced by regular pairs verifying the converse assertion of Theorem B. To do this, first we investigate some geometrical properties of *generalized lines* (see below). Then, we give characterization theorems for generalized convexity and generalized non-convexity, respectively. Both theorems turns out to be very important in the proofs of the main results concerning the characterization of generalized convexity via Hadamard's inequality.

2. Auxiliary tools

Given a regular pair (ω_1,ω_2) over an interval I, denote the set of all linear combinations of the base functions ω_1 and ω_2 by $Ł(\omega_1,\omega_2)$. A function $\omega: I \to \mathbb{R}$ is said to be a *generalized line* if it belongs to the linear hull $Ł(\omega_1,\omega_2)$. Some results concerning generalized lines were presented in [4]; one of the most important of them states the existence of a generalized line "parallel" to the x-axis.

LEMMA 1. *If* (ω_1,ω_2) *is a regular pair on an interval* I*, then there exists a generalized line* ω *that is positive on the interior of* I*.*

In fact, the statement of the lemma can be improved: on compact subintervals of the domain, generalized lines are uniformly away from zero. As another important property, let us mention that pointwise convergence is not only a necessary but also a sufficient condition for the uniform convergence of sequences of generalized lines.

The notion of convexity can be characterized via the geometric properties of the interpolation chords and support lines. Namely, a function is convex if and only if its graph is "under" the chord joining any two points of the graph; further on, a function is convex if and only if, in each point, its graph is "above" the supporting line. The following result generalizes these properties for generalized convex functions and is the basic tool in verifying Theorem B.

THEOREM 1. *Let* (ω_1, ω_2) *be a regular pair over an open interval* I. *The following statements are equivalent:*

(i) $f : I \to \mathbb{R}$ *is* (ω_1, ω_2)*-convex;*

(ii) *for all* $x_0 \in I$ *there exist* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\alpha\omega_1(x_0) + \beta\omega_2(x_0) = f(x_0),$$
$$\alpha\omega_1(x) + \beta\omega_2(x) \leqslant f(x) \qquad (x \in I);$$

(iii) *for all elements* $x < p < y$ *of* I,

$$f(p) \leqslant \alpha\omega_1(p) + \beta\omega_2(p)$$

where the coefficients α, β *are the solutions of the system of linear equations*

$$f(x) = \alpha\omega_1(x) + \beta\omega_2(x),$$
$$f(y) = \alpha\omega_1(y) + \beta\omega_2(y).$$

Proof. Hint. The equivalence of the first and second assertions has already been shown in [4], therefore we shall verify, for example, that the (ω_1, ω_2)-convexity of a function f is equivalent to the third assertion.

For this purpose, express the unknowns α and β from the system of linear equations of (iii) by Cramer's rule. Then, multiplying both sides by the common positive denominator of α and β, the inequality $f(p) \leqslant \alpha\omega_1(p) + \beta\omega_2(p)$ can be rewritten into the form

$$\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} f(p) \leqslant \begin{vmatrix} f(x) & f(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} \omega_1(p) + \begin{vmatrix} f(y) & f(x) \\ \omega_1(y) & \omega_1(x) \end{vmatrix} \omega_2(p),$$

or equivalently, interchanging the columns of the coefficient of $\omega_2(p)$ and applying the expansion theorem "backwards"

$$0 \leqslant \begin{vmatrix} f(x) & f(p) & f(y) \\ \omega_1(x) & \omega_1(p) & \omega_1(y) \\ \omega_2(x) & \omega_2(p) & \omega_2(y) \end{vmatrix}.$$

Under the assumption of continuity, functions that are *not* generalized convex can be characterized via very similar properties. It turns out that these kind of functions are necessarily locally strictly concave at some point of the domain. The obtained theorem generalizes the analogous result known for convex functions (see [13], [14], [15], [8]) and also plays the key role in proving the main results of the paper.

THEOREM 2. *Let* (ω_1, ω_2) *be a regular pair on an open interval* I, *furthermore let* $f : I \to \mathbb{R}$ *be a continuous function. Then, the following assertions are equivalent:*

(i) f *is not* (ω_1, ω_2)*-convex;*

(ii) *there exist elements* $x < y$ *of* I *such that* $\omega < f$ *on* $]x, y[$ *where* ω *is the generalized line determined by the properties*

$$\omega(x) = f(x), \qquad \omega(y) = f(y);$$

(iii) *there exist elements* $x < p < y$ *of* I *and a generalized line* ω *such that* $\omega \geqslant f$ *on* $[x, y]$, *furthermore*

$$f(x) < \omega(x), \quad f(p) = \omega(p), \quad f(y) < \omega(y);$$

(iv) *there exists* $p \in I$ *such that* f *is locally strictly* (ω_1, ω_2)*-concave at* p, *that is, there exist elements* $x < p < y$ *of* I *such that, for all* $x < u < p < v < y$, *the following inequality holds:*

$$\begin{vmatrix} f(u) & f(p) & f(v) \\ \omega_1(u) & \omega_1(p) & \omega_1(v) \\ \omega_2(u) & \omega_2(p) & \omega_2(v) \end{vmatrix} < 0.$$

Proof. $(i) \Rightarrow (ii)$. If f is not (ω_1, ω_2)-convex, then, by assertion (iii) of Theorem 1, there exist elements $x_0 < p < y_0$ of I such that $\omega(p) < f(p)$, where ω is the generalized line determined by the properties $\omega(x_0) = f(x_0)$ and $\omega(y_0) = f(y_0)$. Define the function $F : [x_0, y_0] \to \mathbb{R}$ by $F := f - \omega$, furthermore the elements x and y by the formulae

$$x := \sup\{\, t \mid F(t) = 0,\ x_0 \leqslant t < p \,\},$$
$$y := \inf\{\, t \mid F(t) = 0,\ p < t \leqslant y_0 \,\}.$$

Clearly, $x_0 \leqslant x < p < y \leqslant y_0$ hold; moreover, $F(x) = F(y) = 0$ and $F > 0$ on $]x, y[$ due to the continuity of F. That is, $\omega(x) = f(x)$, $\omega(y) = f(y)$ and $f(t) > \omega(t)$ for all $t \in]x, y[$.

$(ii) \Rightarrow (iii)$. Take the elements $x < y$ of I such that $\omega < f$ hold on $]x, y[$ where ω is the generalized line determined by the interpolation properties $\omega(x) = f(x)$ and $\omega(y) = f(y)$. Denote the generalized line that is positive on I by ω_0 and define $t_0 \in \mathbb{R}$ by the formula

$$t_0 := \max_{[x,y]} \frac{f - \omega}{\omega_0}.$$

Clearly, $t_0 > 0$ since $\omega < f$ on $]x, y[$. We show that the generalized line $\omega^* := \omega + t_0\omega_0$ satisfies the required properties. Indeed, by the positivity of $t_0\omega_0$ and the interpolation properties of ω, we get

$$\omega^*(x) := \omega(x) + t_0\omega_0(x) > \omega(x) = f(x),$$
$$\omega^*(y) := \omega(y) + t_0\omega_0(y) > \omega(x) = f(y).$$

On the other hand, the function $(f - \omega)/\omega_0$ is continuous on $[x, y]$ hence it takes its maximum at some $p \in]x, y[$:

$$t_0 = \frac{f(p) - \omega(p)}{\omega_0(p)}.$$

Therefore,

$$\omega^*(p) := \omega(p) + t_0\omega_0(p) = f(p).$$

$(iii) \Rightarrow (iv)$. Due to the continuity of the functions f and ω, we may assume that p is the minimal element of $]x,y[$ fulfilling the properties of the assertion. Then, $f(u) < \omega(u)$ if $x < u < p$ and $f(v) \leqslant \omega(v)$ if $p < v < y$. Therefore,

$$\begin{vmatrix} f(u) & f(p) & f(v) \\ \omega_1(u) & \omega_1(p) & \omega_1(v) \\ \omega_2(u) & \omega_2(p) & \omega_2(v) \end{vmatrix} < \begin{vmatrix} \omega(u) & \omega(p) & \omega(v) \\ \omega_1(u) & \omega_1(p) & \omega_1(v) \\ \omega_2(u) & \omega_2(p) & \omega_2(v) \end{vmatrix}$$

since the adjoint determinants of $f(u)$ and $f(v)$ are positive, furthermore, f and ω coincide at p. However, ω is a linear combination of ω_1 and ω_2 hence the left hand side of the previous inequality equals zero.

$(iv) \Rightarrow (i)$. Trivial.

As a direct consequence of the theorem above, in the standard setting, we get the characterization properties of continuous, non convex functions:

COROLLARY 1. *Let I be an open interval and let $f : I \to \mathbb{R}$ be a continuous function. The following statements are equivalent:*

(i) *f is not convex;*

(ii) *there exists $x < y$ elements of I and an affine function $\phi : I \to \mathbb{R}$ such that $\phi(t) < f(t)$ on $]x,y[$, furthermore $\phi(x) = f(x)$ and $\phi(y) = f(y)$ hold;*

(iii) *there exists $x < p < y$ elements of I and an affine function $\phi : I \to \mathbb{R}$ such that $\phi(t) \geqslant f(t)$ on $]x,y[$, furthermore $f(x) < \phi(x)$, $f(p) = \phi(p)$ and $f(y) < \phi(y)$ hold;*

(iv) *there exists $p \in I$ such that f is locally strictly concave at p, that is, there exist elements $x < p < y$ of I such that, for all $x < u < p < v < y$, the following inequality holds:*

$$\begin{vmatrix} f(u) & f(p) & f(v) \\ 1 & 1 & 1 \\ u & p & v \end{vmatrix} < 0.$$

For another application of Theorem 2, we get immediately that generalized convexity, similarly to the standard one, is a *localizable* property.

COROLLARY 2. *Let (ω_1, ω_2) be a regular pair over an open interval I, furthermore $f : I \to \mathbb{R}$ be a given function. Then, the following assertions are equivalent:*

(i) *f is (ω_1, ω_2)-convex;*

(ii) *f is locally (ω_1, ω_2)-convex, that is, each element of the domain has a neighborhood where it is (ω_1, ω_2)-convex;*

(iii) *f is continuous and, for all $p \in I$, there exist elements $x < p < y$ of I such that*

$$\begin{vmatrix} f(u) & f(p) & f(v) \\ \omega_1(u) & \omega_1(p) & \omega_1(v) \\ \omega_2(u) & \omega_2(p) & \omega_2(v) \end{vmatrix} \geqslant 0$$

for all $x < u < p < v < y$ (i. e., f is locally convex at each point).

Proof. Hint. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial. For the implication $(iii) \Rightarrow (i)$, assume that a continuous function $f : I \to \mathbb{R}$ is not (ω_1, ω_2)-convex. Then, by the last assertion of Theorem 2, it is locally strictly (ω_1, ω_2)-concave at some $p \in I$, hence it cannot be locally (ω_1, ω_2)-convex at p.

3. The main results

The main results are presented in the subsequent three theorems. The first and the second ones concern the left and right hand side inequalities of Theorem B independently, while the third one is analogous to the classical Jensen inequality. In the proof of each theorem, we shall deal only with the "necessity" part since the "sufficiency" is due to Theorem B.

THEOREM 3. *Let (ω_1, ω_2) be a positive regular pair on an open interval I and let $\rho : I \to \mathbb{R}$ be a positive integrable function. Define, for all elements $x < y$ of I, the functions $\xi(x, y)$ and $c(x, y)$ by the formulae in* 1*. Then, a continuous function $f : I \to \mathbb{R}$ is generalized convex with respect to (ω_1, ω_2) if and only if, for all elements $x < y$ of I, it satisfies the inequality*

$$c(x, y)f\big(\xi(x, y)\big) \leqslant \int_x^y f\rho.$$

Proof. Observe first that the mapping $(x, y) \mapsto \xi(x, y)$ is continuous in each variable and takes its value between x and y since it is a Lagrange-type mean-value. Further on, $c(x, y)$ and $\xi(x, y)$ are constructed so that all generalized lines are the solutions of the functional equation

$$c(x, y)\omega\big(\xi(x, y)\big) = \int_x^y \omega\rho \qquad (x < y). \tag{3}$$

Assume that f satisfies the inequality above and, indirectly, is not (ω_1, ω_2)-convex. Then, by assertion (iii) of Theorem 2, there exist elements $x < p < y$ of I and a generalized line ω such that $f \leqslant \omega$ on $[x, y]$ and

$$f(x) < \omega(x), \qquad f(p) = \omega(p), \qquad f(y) < \omega(y).$$

If, for example, $p \leqslant \xi(x, y)$, then there exists $u \in]p, y]$ such that $p = \xi(x, u)$ since ξ is a Lagrange-type mean-value. The inequality $f(x) < \omega(x)$ and the continuity of f implies that $f < \omega$ on a right neighborhood of x hence, applying 3, we get the contradiction

$$c(x, u)f(p) = c(x, u)f\big(\xi(x, u)\big) \leqslant \int_x^u f\rho < \int_x^u \omega\rho = c(x, u)\omega\big(\xi(x, u)\big) = c(x, u)f(p).$$

The other case, when $\xi(x, y) \leqslant p$, can similarly be checked and also leads to a contradiction.

THEOREM 4. *Let (ω_1, ω_2) be a positive regular pair over an open interval I and let $\rho : I \to \mathbb{R}$ be a positive integrable function. Define, for all elements $x < y$ of I,* *the functions* $c_1(x,y)$ *and* $c_2(x,y)$ *by the formulae in* 2*. Then, a continuous function* $f: I \to \mathbb{R}$ *is generalized convex with respect to* (ω_1, ω_2) *if and only if, for all elements* $x < y$ *of* I*, it satisfies the inequality*

$$\int_x^y f\rho \leqslant c_1(x,y)f(x) + c_2(x,y)f(y).$$

Proof. Note first that $c_1(x,y)$ and $c_2(x,y)$ are constructed such that all generalized lines are the solutions of the functional equation

$$\int_x^y \omega\rho = c_1(x,y)\omega(x) + c_2(x,y)\omega(y). \tag{4}$$

Assume indirectly that f is not (ω_1, ω_2)-convex. Then, by assertion (ii) of Theorem 2, there exist elements $x < y$ of I and a generalized line ω such that $\omega(x) = f(x)$, $\omega(y) = f(y)$ and $\omega < f$ on $]x,y[$. Therefore,

$$\int_x^y \omega\rho < \int_x^y f\rho \leqslant c_1(x,y)f(x) + c_2(x,y)f(y) = c_1(x,y)\omega(x) + c_2(x,y)\omega(y)$$

which contradicts 4.

Let us mention that, instead of the side condition of continuity, the function f might be assumed only upper semicontinuous in Theorem 3; similarly, it suffices to require that f is lower semicontinuous in Theorem 4.

THEOREM 5. *Let* (ω_1, ω_2) *be a positive regular pair on an open interval* I *and* $f: I \to \mathbb{R}$ *be a continuous function. Keeping the notations of Theorem B,* f *is* (ω_1, ω_2)*-convex if and only if, for all elements* $x < y$ *of* I*, it satisfies the inequality*

$$c(x,y)f\big(\xi(x,y)\big) \leqslant c_1(x,y)f(x) + c_2(x,y)f(y).$$

Proof. Observe first that the functions c, c_1, c_2 and ξ are constructed so that all the generalized lines are the solutions of the functional equation

$$c(x,y)\omega\big(\xi(x,y)\big) = c_1(x,y)\omega(x) + c_2(x,y)\omega(y) \qquad (x < y)$$

since both sides have the common value $\int_x^y \omega\rho$. Assume indirectly that $f: I \to \mathbb{R}$ is not generalized convex with respect to (ω_1, ω_2). Then, by Theorem 2, there exist elements $x < y$ of I and a generalized line ω fulfilling the conditions

$$\omega(x) = f(x), \qquad \omega|_{]x,y[} < f|_{]x,y[}, \qquad \omega(y) = f(y).$$

Therefore, taking the observation above into consideration, one can immediately get that

$$\begin{aligned} c(x,y)f\big(\xi(x,y)\big) &\leqslant c_1(x,y)f(x) + c_2(x,y)f(y) = c_1(x,y)\omega(x) + c_2(x,y)\omega(y) \\ &= c(x,y)\omega\big(\xi(x,y)\big) < c(x,y)f\big(\xi(x,y)\big), \end{aligned}$$

which is a contradiction.

In the proof we used a similar method that is applied in Theorem 4. However, an alternative approach can also be followed using the idea of the proof of Theorem 3. This shows that Theorem 5 remains true requiring either upper or lower semicontinuity on the function f.

To give a unified view, the previous results are combined in the next corollary. This corollary completes the assertions of Theorem 1 and we get a much more comprehensive characterization of generalized convex functions.

COROLLARY 3. *Let* (ω_1, ω_2) *be a positive regular pair on an open interval* I *and let* $\rho : I \to \mathbb{R}$ *be a positive integrable function. Keeping the notations of Theorem B, the following assertions are equivalent for any function* $f : I \to \mathbb{R}$*:*
(i) f *is generalized convex with respect to* (ω_1, ω_2)*;*
(ii) f *is continuous and, for all elements* $x < y$ *of* I*, satisfies the inequality*

$$c(x, y)f\big(\xi(x, y)\big) \leqslant \int_x^y f\rho;$$

(iii) f *is continuous and, for all elements* $x < y$ *of* I*, satisfies the inequality*

$$\int_x^y f\rho \leqslant c_1(x, y)f(x) + c_2(x, y)f(y);$$

(iv) f *is continuous and, for all elements* $x < y$ *of* I*, satisfies the inequality*

$$c(x, y)f\big(\xi(x, y)\big) \leqslant c_1(x, y)f(x) + c_2(x, y)f(y).$$

To demonstrate the meaning of the above Corollary, we specify its result to (cosh, sinh)-convexity. Then, for any continuous function $f : I \to \mathbb{R}$, we have the equivalence of the following four properties (an analogous characterization of (cos, sin)-convexity is left to the reader):
(i) $f : [a, b] \to \mathbb{R}$ is a (cosh, sinh)-convex function;
(ii) for all $x < y$ in I, f satisfies

$$2\sinh\Big(\frac{y-x}{2}\Big)f\Big(\frac{x+y}{2}\Big) \leqslant \int_x^y f(t)dt;$$

(iii) for all $x < y$ in I, f satisfies

$$\int_x^y f(t)dt \leqslant \tanh\Big(\frac{y-x}{2}\Big)\big(f(x) + f(y)\big);$$

(iv) for all $x < y$ in I, f satisfies

$$f\Big(\frac{x+y}{2}\Big) \leqslant \frac{f(x) + f(y)}{2\cosh\big(\frac{y-x}{2}\big)}.$$

In fact, the convexity notion induced by regular pairs are the very special case of the convexity notion induced by *Chebyshev systems* (see [10]). Hadamard's classical inequality can also be extended for this setting (consult the papers [2] and [3, 4, 5, 6]) hence the question arises, quite evidently, *whether Hermite–Hadamard-type inequalities* *also characterize generalized convexity in the general case or not.* To give an affirmative answer, even in the case when the underlying Chebyshev system of the induced convexity notion is the *polynomial* one (see [16] and [11]), remains an open problem and could be the subject of further investigations.

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